On the Growth of Polynomials and Entire Functions of Exponential Type

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On the Growth of Polynomials and Entire Functions of Exponential Type

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Lisa Ann Harden, daughter of William and Rebecca Harden, was born October 31, 1978, in Columbus, Ohio. She graduated from Austin High School in Decatur, Alabama, in 1997. She then attended Jacksonville State University in Jacksonville, Alabama, for four years and graduated magna cum laude with a Bachelor of Science degree in Secondary Education in April, 2001. After continuing as a graduate student at Jacksonville State University for one year, she entered Graduate School, Auburn Universtiy, in August, 2002.

Thesis Abstract

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Concerning the growth of a polynomial and its derivative, the following inequalities are well known as Bernstein Inequalities.

$$\max_{|z|=R}|p(z)| \leq \max_{|z|=1}|p(z)|R^n, \quad \text{for} \quad R \geq 1, \tag{1}$$

$$\max_{|z|=1}|p'(z)| \leq \max_{|z|=1}|p(z)|n, \tag{2}$$

$$\max_{|z|=\rho}|p(z)| \geq \max_{|z|=1}|p(z)|\rho^n, \quad \text{for} \quad 0 < \rho \leq 1. \tag{3}$$

All the above inequalities are best possible and are of great importance both from a theoretical point of view and for applications.

The thesis consists of three chapters. In Chapter 1, we provide a brief history of these inequalities and provide the proof of the known fact that all three inequalities above are equivalent in the sense that they can be derived from each other.

Also, this chapter contains proof of inequality (1), some of its generalizations, and its sharpening when the polynomial does not have a zero at $z = 0$.

In Chapter 2, we study inequality (1) for polynomials having no zeros in $\{z : |z| < 1\}$, and then for polynomials having no zeros inside the circle $\{z : |z| = K\}$, $K > 0$, by providing proofs of several results known in this direction. If $p(z)$ is a polynomial of degree n then, as can be easily verified, the function $f(z) = p(e^{iz})$ is an entire function of exponential type n, and thus the results for entire functions of exponential type can be considered as generalizations of the corresponding results for polynomials.

In Chapter 3 we study the generalizations for entire functions of exponential type of inequality (1) and of some other inequalities studied in Chapter 2. Also in this chapter, we provide a partially different proof of a well known result concerning polynomials having no zeros inside the unit circle. Finally, the proof of a known result that sharpens a well known result of R. P. Boas has been provided.

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Style manual or journal used Transactions of the American Mathematical Society (together with the style known as "auphd"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

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Table of Contents

Chapter 1

Introduction

We denote the set of real numbers by $\mathbb{R}$ and the field of complex numbers by $\mathbb{C}$. Any element of $\mathbb{C}$ can be thought of as a point in the complex plane. We define the extended complex plane by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Then for any $z \in \mathbb{C}$, we denote a polynomial by $p(z) := \sum_{v=0}^{n} a_v z^v$ for $a_v \in \mathbb{C}$ unless otherwise noted. The derivative of $p(z)$ is denoted by $p'(z)$, and we let $M(p;r) := \max_{|z|=r} |p(z)|$.

If p is a polynomial of degree at most n, then the following inequalitites are well known as Bernstein inequalities.

$$M(p;R) \leq M(p;1)R^n, \quad \text{for} \quad R \geq 1 \tag{1.1}$$

$$M(p';1) \leq M(p;1)n \tag{1.2}$$

$$M(p;\rho) \geq M(p;1)\rho^n, \quad \text{for} \quad 0 < \rho \leq 1. \tag{1.3}$$

Each of the inequalities above is best possible, and each attains equality only for polynomials of the form $p(z) = M(p;1)e^{i\gamma}z^n$, $\gamma \in \mathbb{R}$. Clearly, if $p(z) = M(p;1)e^{i\gamma}z^n$, $\gamma \in \mathbb{R}$, then for $R > 1$,

$$\begin{aligned} M(p;R) &= \max_{|z|=R} |M(p;1)e^{i\gamma}z^n| \\ &= M(p;1)|e^{i\gamma}|R^n \\ &= M(p;1)R^n. \end{aligned}$$

Also,

$$\begin{aligned} M(p';1) &= \max_{|z|=1} |p'(z)| \\ &= \max_{|z|=1} |M(p;1)e^{i\gamma} n z^{n-1}| \\ &= M(p;1)n, \end{aligned}$$

and finally, for $0 < \rho \leq 1$,

$$\begin{aligned} M(p;\rho) &= \max_{|z|=\rho} |M(p;1)e^{i\gamma} z^n| \\ &= M(p;1)|e^{i\gamma}|\rho^n \\ &= M(p;1)\rho^n. \end{aligned}$$

The development of inequality (1.2), known as Bernstein's inequality, actually began with a question raised by the famous Russian chemist Mendeleev who was studying the specific gravity of a solution as a function of the percentage of the dissolved substance. There is some practical importance of this function. It is used in testing beer and wine for alcoholic content, and it is also used in analyzing the cooling system of an automobile for concentration of antifreeze. However, physical chemists today do not seem to find it as interesting as Mendeleev did.

Mendeleev was able to approximate the curves which resulted from the functions with successions of quadratic arcs, but the approximations contained corners where the arcs joined. Naturally, he wished to know whether or not these corners were caused by errors of measurement. For this he needed to know, for a quadratic polynomial $P(x) = px^2 + qx + r$ where $|P(x)| \leq 1$ for $-1 \leq x \leq 1$, how large $|P'(x)|$ could be on $-1 \leq x \leq 1$. Mendeleev found that $|P'(x)| \leq 4$ and that this inequality is best possible since for $P(x) = 1 - 2x^2$, we see that $\max_{-1 \leq x \leq 1} |P(x)| = 1$, and $|P'(\pm 1)| = |-4(\pm 1)| = 4$. Making use of this inequality, Mendeleev was convinced that the corners did not occur due to errors of measurement, but were genuine. As to how the bound for $\max |P'(x)|$ for $-1 \leq x \leq 1$ in terms of $\max |P(x)|$ for $-1 \leq x \leq 1$ helped Mendeleev to obtain an answer to this question about the corners in the curve approximations, we refer the reader to the paper of R. P. Boas [4, p. 165].

Mendeleev passed on his problem to the famous Russian mathematician A. A. Markov who studied the problem for polynomials of degree n and proved the following theorem, known as Markov's theorem [12, p. 351].

Theorem 1.1. *If* $p(x) := \sum_{v=0}^{n} a_v x^v$ *is a real polynomial of degree* n *and* $|p(x)| \leq 1$ *on* $[-1, 1]$*, then* $|p'(x)| \leq n^2$ *for* $-1 \leq x \leq 1$*. This inequality is best possible and equality results at only* $x = \pm 1$ *when* $p(x) = T_n(x)$ *where* $T_n(x) = \cos(n \arccos x)$ *is a Chebyshev polynomial of degree* n*.*

The Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$ are actually algebraic polynomials, a fact which is not obvious. This is explained by Rahman and Schmeisser [20, p. 23-24]. Define $T_n(x) := \cos n\theta$ on the interval $[-1, 1]$ where $\theta := \arccos x$. So, $T_n(x) = \cos(n \arccos x)$, which gives $T_0(x) = \cos(0) = 1$, which implies $T_0(x) = 1$. Also, $T_1(x) = \cos(\arccos x) = x$, which implies $T_1(x) = x$.

Next,

$$\begin{aligned}
T_2(x) &= \cos(2\arccos x) \\
&= \cos(2\theta) \\
&= 2\cos^2\theta - 1 \\
&= 2\cos^2(\arccos x) - 1 \\
&= 2x^2 - 1
\end{aligned}$$

which implies $T_2(x) = 2x^2 - 1$. Also,

$$\begin{aligned}
T_3(x) &= \cos(3\arccos x) \\
&= \cos(3\theta) \\
&= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&= 4\cos^3(\arccos x) - 3\cos(\arccos x) \\
&= 4x^3 - 3x
\end{aligned}$$

which implies $T_3(x) = 4x^3 - 3x$. Also, note that $\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta$. To verify this, consider

$$\begin{aligned}
2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta &= \cos(\theta + (n-1)\theta) + \cos(\theta - (n-1)\theta) \\
&\quad - \cos(n-2)\theta \\
&= \cos(\theta + n\theta - \theta) + \cos(\theta - n\theta + \theta) \\
&\quad - \cos(n-2)\theta \\
&= \cos(n\theta) + \cos(2\theta - n\theta) - \cos(n-2)\theta \\
&= \cos(n\theta) + \cos(-(n-2)\theta) - \cos(n-2)\theta \\
&= \cos(n\theta) + \cos(n-2)\theta - \cos(n-2)\theta \\
&= \cos(n\theta).
\end{aligned}$$

Putting together the statements $T_n(x) = \cos n\theta$ where $\theta = \arccos x$ and $\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta$, we get the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad \text{for} \quad n = 2, 3, \ldots$$

from which follows the fact that $T_n(x) = \cos n\theta$ where $\theta = \arccos x$ are algebraic polynomials of degree n. In fact, using the well known trigonometric identity

$$\begin{aligned}
2\cos n\theta &= (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{1\cdot 2}(2\cos\theta)^{n-4} \\
&\quad - \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}(2\cos\theta)^{n-6} + \ldots \quad \ldots (2)
\end{aligned}$$

where the last term is $(-1)^{\frac{n-1}{2}} n(2\cos\theta)$ or $(-1)^{\frac{n}{2}} \cdot 2$ according to whether n is odd or even, respectively, one can easily express $T_n(x)$ as an algebraic polynomial.

Around 1926, the Russian mathematician Serge Bernstein became interested in the analogue of Markov's Theorem for the unit disk in the complex plane instead of the interval $[-1, 1]$. He wished to know the maximum value of $|P'(z)|$ for $|z| \leq 1$ when $P(z)$ is a polynomial of degree at most n with $|P(z)| \leq 1$ for $|z| \leq 1$. In these connections the following inequality is known as Bernstein's inequality.

Theorem 1.2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree at most* n*, then* $\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|$. *This result is best possible, and equality is attained when* $p(z) = \lambda z^n$, $\lambda \in \mathbb{C}$.

Bernstein proved the above inequality with $2n$ in place of n. For more details we refer to the paper of Govil and Mohapatra [12, p. 351-352].

For the proof of inequality (1.1) we will need the maximum modulus principle for unbounded domains. First, we will list the necessary terminology and state the maximum modulus principal for bounded domains, all of which can be found, for example, in the book of Rahman and Schmeisser [20, p. 1-2].

A subset ε of a topological space is connected if it cannot be expressed as the union of two non-empty, disjoint sets O_1, O_2, open in ε. A region is a non-empty, open, connected subset of $\hat{\mathbb{C}}$. A region Ω is simply connected if either $\Omega = \hat{\mathbb{C}}$ or $\hat{\mathbb{C}} \backslash \Omega$ is connected.

A set which is a region, or is obtained from a region Ω by subjoining some or all of the boundary points of Ω, is a domain. A region and its closure are both domains.

An arc γ in $\hat{\mathbb{C}}$ is a continuous mapping of a closed, non-degenerate interval $[a, b]$ into $\hat{\mathbb{C}}$. The range of the mapping is a set of points which is called the trace of γ. By a point on an arc we will mean a point on its trace, and by a function on an arc we will mean a function on its trace. The arc γ is called a Jordan arc if the mapping is one-to-one. If distinct points of $[a, b)$ are mapped onto distinct points of $\hat{\mathbb{C}}$ and the image of b is the same as that of a, then γ is said to be a simple, closed curve or a Jordan curve. In other words, a Jordan curve is a homeomorphism in $\hat{\mathbb{C}}$ of the unit circle. The Jordan curve theorem says that the complement of the trace of a Jordan curve γ with respect to $\hat{\mathbb{C}}$ has precisely two components. One of these two components is bounded in the Euclidean metric if the trace of γ lies in $\mathbb{C}$. That component is called the inside of γ, while the other component is the outside. A Jordan curve $\gamma : [a, b] \to \mathbb{C}$ is said to be positively oriented if the inside of the curve lies on the left of the moving point $\gamma(t)$ as t increases from a to b.

An arc $\gamma : [a, b] \to \mathbb{C}$ is said to be rectifiable if it has finite length. This means that, for some finite number L, $\sum_{\nu=1}^{n} |\gamma(t_\nu) - \gamma(t_\nu - 1)| \leq L$ for every partition $\{a = t_0 < t_1 < \cdots < t_n = b\}$.

An arc $\gamma : [a, b] \to \mathbb{C}$ is piecewise continuously differentiable if there exists a partition $\{a = t_0 < t_1 < \ldots < t_n = b\}$ such that, in each of the intervals $[t_{v-1}, t_v]$ for

$v = 1, \ldots, n$ the functions $\Re\gamma$ and $\Im\gamma$ are continuously differentiable, having one-sided derivatives at the end points, where $\Re$ indicates the real part and $\Im$ indicates the imaginary part. An arc $\gamma : [a, b] \to \mathbb{C}$ is analytic if, about each point $t_0 \in (a, b)$, the function γ can be expanded in a power series $\gamma(t) = c_0 + c_1(t - t_0) + \ldots$, $(c_1 \neq 0)$, which converges in some interval $| \, t - t_0 \, | < \delta$.

Now that we have established the required terminology we will state the Maximum Modulus Principle [20, Theorem 1.6.11].

Theorem 1.3. *Let f be analytic in a region Ω (not necessarily bounded). Then $| \, f(z) \, |$ cannot have a local maximum in Ω unless f is constant in Ω.*

Rahman and Schmeisser also give another formulation of essentially the same theorem which we state now.

Theorem 1.4. *Let f be analytic in a bounded region Ω, and continuous in the closure $\overline{\Omega}$. Suppose, in addition, that $|f(z)| \leq M$ for all z on the boundary of Ω. Then the same inequality holds for all $z \in \Omega$. Moreover, $|f(z)| = M$ for some $z \in \Omega$, only if f is a constant.*

This theorem can be extended for unbounded domains. We state it as follows, and for the proof, we refer to Rahman and Schmeisser [20, Corollary 1.6.13].

Theorem 1.5. *Let $z = z(t)$, $\alpha \leq t \leq \beta$ define a Jordan curve Γ with its trace in $\mathbb{C}$, and denote the inside of Γ by Ω. Also, let φ be a function which is analytic in $\mathbb{C} \backslash \{\Gamma \cup \Omega\}$ and continuous on $\mathbb{C} \backslash \Omega$ such that $| \, \varphi(z) \, | \leq 1$ for all $z \in \Gamma$. Suppose, in* *addition, that* $\varphi(z)$ *tends to a finite limit* l *as* z *tends to infinity, and set* $\varphi(\infty) = l$. *Then* $|\varphi(z)| < 1$ *for all* z *in* $\hat{\mathbb{C}}\backslash\{\Gamma \cup \Omega\}$ *unless* φ *is a constant.*

Let $f(z)$ and $g(z)$ be polynomials such that the degree of $f(z)$ is n and the degree of $g(z)$ is m, where $n \leq m$. Furthermore, suppose that $g(z)$ has all its zeros in the closure of the inside of a Jordan curve, γ, in $\mathbb{C}$, and that $|f(z)| \leq |g(z)|$ on γ. If we take $\varphi(z) = \dfrac{f(z)}{g(z)}$, then $\varphi(z)$ is analytic in $\mathbb{C}\backslash\{$the closure of the inside of $\gamma\}$ and is continuous on $\mathbb{C}\backslash\{$the inside of $\gamma\}$. We can make the statement about continuity because if $g(z)$ has a zero at a point z_0 on γ, then in view of the inequality $|f(z)| \leq |g(z)|$, $f(z)$ also has a zero at z_0. This means that the factors of $f(z)$ and $g(z)$ that make $g(z)$ equal to zero on γ will cancel, and $\varphi(z)$ is thus continuous on γ. Also, since $|f(z)| \leq |g(z)|$ on γ, we know that $\varphi(z) = \dfrac{|f(z)|}{|g(z)|} \leq 1$ on γ. Next note that since the degree of $f(z)$ is less than or equal to the degree of $g(z)$, then $\varphi(z) = \dfrac{f(z)}{g(z)}$ tends to a finite limit l as z tends to infinity. We have now shown that our assumptions satisfy the hypothesis of Theorem 1.5. Thus, we get that $|\varphi(z)| = \dfrac{|f(z)|}{|g(z)|} < 1$ for all z in $\mathbb{C}\backslash\{\gamma \cup$ the inside of $\gamma\}$, that is, $|f(z)| < |g(z)|$ on γ and the outside of γ. We will now state the preceeding information as a theorem which is given in Rahman and Schmeisser [20, Theorem 1.3.6].

Theorem 1.6. *Let* f *and* g *be polynomials with the degree of* f *less than or equal to the degree of* g, *and let* γ *be a Jordan curve in* $\mathbb{C}$. *Suppose that* g *has all of its zeros in the closure of the inside of* γ *and that* $|f(z)| \leq |g(z)|$ *on* γ. *Then* $|f(z)| \leq |g(z)|$ *on the outside of* γ. *Moreover, equality is attained at a point of the outside of* γ *if and only if* $f(z) \equiv e^{i\theta} g(z)$ *for some* $\theta \in \mathbb{R}$.

We will now give a proof for inequality (1.1). For this, let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree at most n. We will now apply Theorem 1.6, and for this we take $f(z) = \frac{p(z)}{M(p;1)}$, $g(z) = z^n$, and for γ the circle $|z| = 1$. Then clearly, the hypothesis of Theorem 1.6 is satisfied and therefore $\frac{|p(z)|}{|z|^n} \leq M(p;1)$, for $|z| \geq 1$, that is, $\frac{|p(Re^{i\theta})|}{R^n} \leq M(p;1)$, for $R \geq 1$, $0 \leq \theta \leq 2\pi$ implying that $|p(Re^{i\theta})| \leq M(p;1)R^n$, for $R \geq 1$, $0 \leq \theta \leq 2\pi$ which is equivalent to $\max_{|z|=R} |p(z)| \leq M(p;1)R^n$, for $R \geq 1$, that is, $M(p;R) \leq M(p;1)R^n$, for $R \geq 1$, and thus we have proved inequality (1.1). $\square$

Note that, by inequality (1.1), if $M(p;1) = 1$, then $M(p;R) \leq R^n$ with equality only for $p(z) = \lambda z^n$, where $|\lambda| = 1$.

Now we prove that inequality (1.1) implies inequality (1.3), and for this we consider the function $P(z) = p(\rho z)$, for $0 < \rho \leq 1$, which is a polynomial of degree at most n. Let $R = \frac{1}{\rho} \geq 1$, and therefore we get

$$\begin{aligned} M(P;R) &= \max_{0 \leq \theta < 2\pi} |P(Re^{i\theta})| \\ &= \max_{0 \leq \theta < 2\pi} |p(\rho Re^{i\theta})| \\ &= \max_{0 \leq \theta < 2\pi} |p(e^{i\theta})| \\ &= \max_{|z|=1} |p(z)| \\ &= M(p;1), \end{aligned}$$

which implies

$$\begin{aligned}
M(p;1) &= M(P;R) \\
&\leq M(P;1)R^n, \quad \text{by inequality (1.1)} \\
&= \max_{0\leq\theta<2\pi} |P(e^{i\theta})| \left(\frac{1}{\rho}\right)^n \\
&= \max_{0\leq\theta<2\pi} |p(\rho e^{i\theta})| \frac{1}{\rho^n} \\
&= M(p;\rho)\frac{1}{\rho^n},
\end{aligned}$$

and which clearly is equivalent to $M(p;1) \leq M(p;\rho)\dfrac{1}{\rho^n}$, that is, $M(p;\rho) \geq M(p;1)\rho^n$ which is inequality (1.3). □

In order to prove that inequality (1.3) implies inequality (1.1) we consider the function $P(z) = p(Rz)$ for $R \geq 1$, and we let $\rho = \dfrac{1}{R}$ which is clearly less than or equal to one. Then,

$$\begin{aligned}
M(P;\rho) &= \max_{0\leq\theta<2\pi} |P(\rho e^{i\theta})| \\
&= \max_{0\leq\theta<2\pi} |p(\rho R e^{i\theta})| \\
&= \max_{0\leq\theta<2\pi} |p(e^{i\theta})| \\
&= \max_{|z|=1} |p(z)| \\
&= M(p;1).
\end{aligned}$$

So,

$$\begin{aligned}
M(p;1) &= M(P;\rho) \\
&\geq M(P;1)\rho^n \text{ by inequality (1.3)} \\
&= \max_{0\leq\theta<2\pi} |P(e^{i\theta})| \frac{1}{R^n} \\
&= \max_{0\leq\theta<2\pi} |p(Re^{i\theta})| \frac{1}{R^n} \\
&= M(p;R)\frac{1}{R^n}
\end{aligned}$$

which implies $M(p;1) \geq M(p;R)\frac{1}{R^n}$, that is, $M(p;R) \leq M(p;1)R^n$ which is inequality (1.1). $\square$

Hence, we have proved that inequality (1.1) is equivalent to inequality (1.3).

Govil, Qazi, and Rahman [13, p. 453] mention that another proof for inequality (1.1) is equivalent to inequality (1.3) can be obtained by observing that p is a polynomial of degree at most n if and only if $q(z) := z^n\overline{p(1/\overline{z})}$ is, and that $M(q;r) = r^n M\left(p;\frac{1}{r}\right)$ for $0 < r < \infty$.

It is well known that inequality (1.1) implies inequality (1.2). This fact was observed by Bernstein [13, p. 453] himself. However, for the sake of completeness we provide the proof.

Let $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. If $|z| \geq 1$, then

$$\begin{aligned}
|p(z) - \lambda M(p;1)z^n| &\geq |\lambda M(p;1)z^n| - |p(z)| \\
&\geq |\lambda|M(p;1)|z^n| - M(p;1)|z^n| \text{ by inequality (1.1).}
\end{aligned}$$

$$= \quad (|\lambda| - 1)M(p;1)|z^n|$$

which is clearly greater than zero, and so $|p(z) - \lambda M(p;1)z^n| > 0$, for $|z| \geq 1$, implying that $p(z) - \lambda M(p;1)z^n$ has all its roots in $|z| < 1$. By the Gauss-Lucas Theorem, $p'(z) - \lambda n M(p;1)z^{n-1}$ also has all its roots in $|z| < 1$. So, if $|\lambda| > 1$, then

$$p'(z) - \lambda M(p;1)nz^{n-1} \neq 0, \quad \text{for} |z| \geq 1. \tag{1.4}$$

So, $|p'(z)| \leq M(p;1)n|z|^{n-1}$ for $|z| = R \geq 1$. To see this, suppose otherwise. Then there would exist a point z_0, $|z_0| \geq 1$, such that $|p'(z_0)| > M(p;1)n|z_0|^{n-1}$. Take $\lambda = \dfrac{p'(z_0)}{M(p;1)nz_0^{n-1}}$. Then we see that

$$\begin{aligned} p'(z_0) - \lambda M(p;1)nz_0^{n-1} &= p'(z_0) - \frac{p'(z_0)}{M(p;1)nz_0^{n-1}} M(p;1)nz_0^{n-1} \\ &= p'(z_0) - p'(z_0) \\ &= 0. \end{aligned}$$

Thus, we have taken $\lambda = \dfrac{p'(z_0)}{M(p;1)nz_0^{n-1}}$ where $|\lambda| > 1$, and shown that the left hand side of (1.4) vanishes at z_0 where $|z_0| \geq 1$, which contradicts (1.4). Hence,

$$|p'(z)| \quad \leq \quad M(p;1)n|z|^{n-1}, \quad \text{for} |z| = R \geq 1.$$

The inequality (1.2) is a special case of the above inequality when $|z| = 1$. $\square$

It is also true that inequality (1.2) implies inequality (1.1). Govil, Qazi, and Rahman proved this [13, p. 453-454], and we give their proof below.

Let $p(z) \not\equiv M(p;1)e^{i\gamma}z^n$, for all $\gamma \in \mathbb{R}$. Consider $M(p';1)$ for the polynomial $p(\rho z)$, where $0 < \rho \leq 1$. Then we see that

$$\begin{aligned} M(p';1) &= \max_{|z|=1} |p'(\rho z)| \\ &= \max_{|z|=1} |\rho p'(\rho z)| \quad \text{by the chain rule} \\ &\geq \rho |p'(\rho z)|. \end{aligned}$$

Note that

$$\begin{aligned} M(p;1)n &= \max_{|z|=1} |p(\rho z)| n \\ &= \max_{|z|=\rho} |p(z)| n \\ &= M(p;\rho)n. \end{aligned}$$

So, by inequality (1.2), $M(p';1) \leq M(p;1)n$, which implies $\rho|p'(\rho z)| \leq M(p;\rho)n$. For any given $R \geq 1$, let $M(p;R) = |p(Re^{i\phi})|$ where $0 \leq \phi < 2\pi$. Then

$$\begin{aligned} M(p;R) &= |p(e^{i\phi}) + p(Re^{i\phi}) - p(e^{i\phi})| \\ &= \left| p(e^{i\phi}) + \int_1^R p'(\rho e^{i\phi}) d\rho \right| \\ &\leq M(p;1) + \int_1^R \frac{n}{\rho} M(p;\rho) d\rho. \end{aligned}$$

Now, let $\phi(R) = M(p;1) + \int_1^R \frac{n}{\rho} M(p;\rho) d\rho$. Then, by the Fundamental Theorem of Calculus $\phi'(r) = \frac{n}{R} M(p;R)$, which implies that

$$\begin{aligned}
\frac{R}{n}\phi'(R) &= M(p;R) \\
&\leq M(p;1) + \int_1^R \frac{n}{\rho} M(p;\rho) d\rho \\
&= \phi(R).
\end{aligned}$$

Thus, we have that $\frac{R}{n}\phi'(R) \leq \phi(R)$, which is equivalent to $\phi'(R) - \frac{n}{R}\phi(R) \leq 0$. So, we can now see that

$$\begin{aligned}
\frac{d}{dR}\{R^{-n}\phi(R)\} &= R^{-n}\phi'(R) - nR^{-n-1}\phi(R) \\
&= R^{-n}\left(\phi'(R) - \frac{n}{R}\phi(R)\right) \\
&\leq 0,
\end{aligned}$$

for $R \geq 1$, and $R^{-n}\phi(R)$ is a decreasing function of R for $R \geq 1$. In particular, $M(p;R) \leq \phi(R) \leq \phi(1)R^n = M(p;1)R^n$, which implies $M(p;R) \leq M(p;1)R^n$. Thus we have shown that inequality (1.2) implies inequality (1.1). $\square$

Hence, we have shown that inequality (1.1) and inequality (1.2) are equivalent, and we now see that inequalities (1.1), (1.2), and (1.3) are all equivalent.

Since the equality in inequality (1.1) holds when the polynomial $p(z)$ has all its zeros at $z = 0$, if we exclude polynomials that have zeros at $z = 0$, we should be able to improve upon the bound in (1.1). This fact was observed by Frappier, Rahman, and Ruscheweyh [8, p. 70], who proved

Theorem 1.7. *Let $p(z)$ be a polynomial of degree at most n, $n \geq 2$, then for $R \geq 1$*

$$M(p;R) \;\; \leq \;\; R^n M(p;1) - |p(0)| \left(R^n - R^{n-2}\right).$$

The coefficient of $|p(0)|$ is the best possible for each R.

Other similar inequalities are discussed in a paper by Frappier and Rahman [7, p. 932, 934]. However, rather than looking at the maximum modulus of a complex polynomial on a circle, they look at the maximum modulus on an elipse. We will state some of the generalizations given by them but will not state the proofs which can be found in their paper [7, p. 932, 934].

Let $R > 1$ and denote by $\mathcal{E}_R$ the elipse
$\left\{ z = x + iy : \dfrac{x^2}{\left(\frac{R+R^{-1}}{2}\right)^2} + \dfrac{y^2}{\left(\frac{R-R^{-1}}{2}\right)^2} = 1 \right\}.$

Theorem 1.8. *If P_n is a polynomial of degree at most n such that $\max\limits_{-1 \leq x \leq 1} |P_n(x)| \leq 1$, then $\max\limits_{z \in \mathcal{E}_R} |P_n(z)| \leq R^n$.*

This inequality can be further refined as seen in the next theorem.

Theorem 1.9. *If P_n is a polynomial of degree at most n such that $\max\limits_{-1 \leq x \leq 1} |P_n(x)| \leq 1$, then $\max\limits_{z \in \mathcal{E}_R} |P_n(z)| \leq \dfrac{1}{2} R^n + \dfrac{5 + \sqrt{17}}{4} R^{n-2}$.*

Again, this inequality can be improved.

Theorem 1.10. *If P_n is a polynomial of degree at most n such that* $\max_{-1\le x\le 1} |P_n(x)| \le 1,$ *then* $\max_{z\in\mathcal{E}_R} |P_n(z)| < \frac{1}{2}(R^n + R^{n-2}) + \frac{11}{4}R^{n-4}.$

The purpose of this thesis is to further study inequality (1.1) by looking at its generalizations and extensions. We will first examine (1.1) under the condition that the polynomial, p, has no zeros inside the unit circle, then under the condition that p has no zeros inside a disk of prescribed radius, and finally we will look at the generalization of (1.1) in terms of entire functions of exponential type.

Chapter 2

Results Involving Polynomials with no Zeros Inside a Disk of Prescribed Radius

Recall from Chapter 1 the following theorem.

Theorem 2.1. *If $p(z)$ is a polynomial of degree n such that $M(p;1) = 1$, then $M(p;R) \leq R^n$, $R > 1$ with equality only for $p(z) = \lambda z^n$, where $|\lambda| = 1$.*

Ankeny and Rivlin [1, p. 849] show that this upper bound can be made smaller if we restrict ourselves to polynomials of degree n which have no zeros inside the unit circle. They state and prove the following theorem.

Theorem 2.2. *If $p(z)$ is a polynomial of degree n such that $M(p;1) = 1$ and $p(z)$ has no zeros inside the unit circle, then for $R > 1$, $M(p;R) \leq \dfrac{1+R^n}{2}$ with equality only for $p(z) = \dfrac{\lambda + \mu z^n}{2}$ where $|\lambda| = |\mu| = 1$.*

To prove Theorem 2.2, Ankeny and Rivlin [1, p. 849] use the following conjecture of Erdös which was proved by Lax [16, p. 509-513].

Theorem 2.3. *If $p(z)$ is a polynomial of degree n such that $M(p;1) = 1$ and $p(z)$ has no zeros inside the unit circle, then $M(p';1) \leq \dfrac{n}{2}$.*

The proof that Ankeny and Rivlin give for Theorem 2.2 is stated below.

Suppose that $p(z)$ is not of the form $\dfrac{\lambda + \mu z^n}{2}$. By Theorem 2.3, $|p'(e^{i\theta})| \leq \dfrac{n}{2}$, $0 \leq \theta < 2\pi$. Take $P(z) = \dfrac{p'(z)}{\frac{n}{2}}$. Then $M(P;1) \leq 1$, and $P(z)$ is clearly not of the form $\dfrac{\lambda+\mu z^n}{2}$. Hence, by Theorem 2.1 when applied to $P(z)=\dfrac{p'(z)}{\frac{n}{2}}$, which is of degree $n-1$, we get

$$M(P;r) \;=\max_{0\le\theta<2\pi}\left|\frac{p'(re^{i\theta})}{\frac{n}{2}}\right| \;\le r^{n-1}, \quad \text{for} \quad r>1,$$

implying

$$|p'(re^{i\theta})| \;<\; \frac{n}{2}r^{n-1}, \quad \text{for} \quad r>1, \quad 0\le\theta<2\pi.$$

Now,

$$\begin{aligned}
\left|p(Re^{i\theta})-p(e^{i\theta})\right| &= \left|\int_1^R e^{i\theta}p'(re^{i\theta})dr\right| \\
&\le \int_1^R |e^{i\theta}p'(re^{i\theta})|dr \\
&< \frac{n}{2}\int_1^R r^{n-1}dr \\
&= \frac{R^n-1}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|p(Re^{i\theta})| &< \frac{R^n-1}{2}+|p(e^{i\theta})| \\
&\le \frac{R^n-1}{2}+1 \\
&= \frac{R^n-1+2}{2} \\
&= \frac{R^n+1}{2}.
\end{aligned}$$

Finally, if $p(z) = \dfrac{\lambda + \mu z^n}{2}$, $|\lambda| = |\mu| = 1$, then for $R > 1$,

$$\begin{aligned} M(p;R) &= \max_{0 \le \theta < 2\pi} \left| \frac{\lambda + \mu R^n e^{in\theta}}{2} \right| \\ &= \frac{1 + R^n}{2}. \end{aligned}$$

□

Another proof for Theorem 2.2, which does not depend on the conjecture of Erdös, is given by K. K. Dewan [6, p. 291-293]. This proof is stated below.

Let $p(z)$ be a polynomial of degree n such that $M(p;1) = 1$, and let $q(z) = z^n(\overline{p(1/\overline{z})})$. Then for $|z| = 1$,

$$\begin{aligned} |q(z)| &= |e^{i\theta}||\overline{p(e^{i\theta})}| \\ &= |p(e^{i\theta})| \\ &= |p(z)|. \end{aligned}$$

So, $|q(z)| = |p(z)|$ for $|z| = 1$. Since $p(z) \neq 0$ for $|z| \le 1$, then $|q(z)| \le |p(z)|$ for $|z| < 1$. If we replace z with $\dfrac{1}{z}$, we see that $|p(z)| \le |q(z)|$ for $|z| > 1$. In particular, $|p(z)| \le |q(z)|$ for $|z| = R > 1$. Now, consider $P(z) = p(z) - \lambda$, for $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Then $P(z) \neq 0$ for $|z| < 1$ and so

$$\begin{aligned} Q(z) &= z^n(\overline{P(1/\overline{z})}) \\ &= z^n(\overline{p(1/\overline{z}) - \lambda}) \\ &= z^n(\overline{p(1/\overline{z})}) - \overline{\lambda} z^n \end{aligned}$$

$$= \quad q(z) - \overline{\lambda} z^n$$

has all its zeros in $|z| \leq 1$. Since for $|z| = 1$,

$$\begin{aligned}
|P(z)| &= |p(z) - \lambda| \\
&= |p(e^{i\theta}) - \lambda| \\
&= |e^{i\theta}||p(e^{i\theta}) - \lambda| \\
&= |e^{in\theta}||\overline{p(e^{i\theta}) - \lambda}| \\
&= |e^{in\theta}\overline{p(e^{i\theta})} - \overline{\lambda} e^{in\theta}| \\
&= |q(z) - \overline{\lambda} z^n| \\
&= |Q(z)|,
\end{aligned}$$

i.e., $|P(z)| = |Q(z)|$ for $|z| = 1$, it follows that $|P(z)| \leq |Q(z)|$ for $|z| > 1$. In particular, $|P(z)| \leq |Q(z)|$ for $|z| = R > 1$. This implies that

$$\begin{aligned}
|P(z)| &= |p(z) - \lambda| \\
&\leq |Q(z)| \\
&= |q(z) - \overline{\lambda} z^n| \\
&= |\overline{\lambda} z^n - q(z)|, \quad \text{for} \quad |z| = R > 1.
\end{aligned}$$

This gives us

$$|p(z)| - |\lambda| \quad \leq |p(z) - \lambda| \quad \leq |\overline{\lambda} z^n - q(z)|, \quad \text{for} \quad |z| = R > 1.$$

Next, if we choose an argument of λ such that $|\overline{\lambda} z^n - q(z)| = |\lambda| R^n - |q(z)|$, $|z| = R > 1$, then we obtain

$$|p(z)| - |\lambda| \leq |\lambda| R^n - |q(z)|, \quad \text{for} \quad |z| = R > 1,$$

which is equivalent to

$$|p(z)| + |q(z)| \leq |\lambda|(1 + R^n), \quad \text{for} \quad |z| = R > 1.$$

Now, if we take the limit as $|\lambda|$ goes to one, we see that

$$|p(z)| + |q(z)| \leq 1 + R^n, \quad \text{for} \quad |z| = R > 1,$$

and if we combine this with $|p(z)| \leq |q(z)|$ for $|z| = R > 1$, we get

$$2|p(z)| \leq 1 + R^n, \quad \text{for every} \quad z \quad \text{on} \quad |z| = R > 1,$$

that is

$$2 \max_{|z|=R} |p(z)| \leq 1 + R^n, \quad \text{for} \quad R > 1,$$

implying

$$M(p; R) \leq \frac{1 + R^n}{2}, \quad \text{for} \quad R > 1,$$

and that Theorem 2.2 is proved. □

Since the equality in Theorem 2.2 holds only for the polynomials $p(z) = \frac{\lambda + \mu z^n}{2}$, $|\lambda| = |\mu| = 1$, that is for polynomials such that |coefficient of $z^n| = \frac{M(p,1)}{2}$, it should be possible to improve upon the bound in Theorem 2.2 if we exclude this class of polynomials, and this was done by Govil [11, p. 80].

Theorem 2.4. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n*, having no zeros in* $|z| < 1$*, then for* $R \geq 1$*, we have*

$$\begin{aligned} M(p;R) &\leq \left(\frac{R^n+1}{2}\right) \| p \| \\ &\quad - \frac{n(\| p \|^2 - 4|a_n|^2)}{2 \| p \|} \left\{ \frac{(R-1) \| p \|}{\| p \| + 2|a_n|} - \ln\left(1 + \frac{(R-1) \| p \|}{\| p \| + 2|a_n|}\right) \right\}, \end{aligned}$$

where $\| p \| = \max_{|z|=1} |p(z)|$. *This inequality becomes equality for the polynomial* $p(z) = (\lambda + \mu z^n)$, $|\lambda| = |\mu|$.

Now, if we let $x = \frac{(R-1)M(p;1)}{M(p;1) + 2|a_n|}$, then the expression in the curly brackets is $\{x - \ln(1+x)\}$ which is positive since $\ln(1+x) < x$ for $x > 0$. Also, since it is well known that $|a_n| \leq \frac{M(p;1)}{2}$ (for example see [10, p. 625]), Theorem 2.4 is surely an improvement over Theorem 2.2 [11, p. 80].

Next, we will discuss polynomials which have no zeros in $|z| < K$, $K \geq 1$. If $p(z)$ has no zeros in $|z| < 1$, then as stated in Theorems 2.2 and 2.3, we have

$$M(p:R) \leq M(p;1)\frac{R^n+1}{2}, \quad \text{for} \quad R \geq 1 \tag{2.1}$$

$$M(p';1) \leq M(p;1)\frac{n}{2}. \tag{2.2}$$

In case $0 \leq \rho < 1$, then we have

$$M(p;\rho) \geq M(p;1)\left(\frac{1+\rho}{2}\right)^n. \tag{2.3}$$

Inequality (2.3) was proved by Rivlin [13, p. 454], and it attains equality for polynomials of the form $p(z) = c(z + e^{i\gamma})^n$, $c \in \mathbb{C}$, $c \neq 0$, $\gamma \in \mathbb{R}$. To check the equality, let $p(z) = c(z + e^{i\gamma})^n$, $c \in \mathbb{C}$, $c \neq 0$, $\gamma \in \mathbb{R}$ and consider

$$\begin{aligned} M(p;\rho) &= \max_{|z|=\rho} |p(z)| \\ &= \max_{|z|=\rho} |c(z + e^{i\gamma})^n| \\ &= \max_{0\leq\theta<2\pi} |c(\rho e^{i\theta} + e^{i\gamma})^n| \\ &= |c|(\rho+1)^n. \end{aligned}$$

Also, consider the right hand side,

$$\begin{aligned} M(p;1)\left(\frac{1+\rho}{2}\right)^n &= \max_{|z|=1} |p(z)| \left(\frac{1+\rho}{2}\right)^n \\ &= \max_{|z|=1} |c(z+e^{i\gamma})^n| \left(\frac{1+\rho}{2}\right)^n \\ &= \max_{0\leq\theta<2\pi} |c(e^{i\theta}+e^{i\gamma})^n| \left(\frac{1+\rho}{2}\right)^n \\ &= |c(2)^n| \left(\frac{1+\rho}{2}\right)^n \\ &= |c|(1+\rho)^n. \end{aligned}$$

Thus, $M(p;\rho) = M(p;1)\left(\frac{1+\rho}{2}\right)^n$ when $p(z) = c(z + e^{i\gamma})^n$, $c \in \mathbb{C}$, $c \neq 0$, $\gamma \in \mathbb{R}$.

Also note that, unlike inequalities (1.1), (1.2), and (1.3), which have been shown to be equivalent, the inequalities (2.1), (2.2), and (2.3) are not equivalent. However, (2.1) can be obtained from (2.2).

R. P. Boas proposed the problem of finding inequalities similar to inequalities (2.1) and (2.2) but for polynomials having no zeros in $|z| < K$, $K > 0$. It was not possible for him to propose an extension of inequality (2.3) because it did not exist at the time. This proposed problem has been studied extensively by many prople, and we wish to present some results related to it in this chapter. Specifically, we wish to present a result of Rahman and Schmeisser [10, p. 624] and some results of Govil, Qazi, and Rahman [13, p. 456-458].

In this direction, we first state an extension of inequality (2.1) which is a special case of a result of Govil and Rahman [15, Theorem 1] (also see Rahman and Schmeisser [20, Theorem 4.23]).

Theorem 2.5. *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then for $1 \leq R \leq K^2$, $M(p; R) \leq \left(\frac{R+K}{1+K}\right)^n M(p; 1)$.*

This result is sharp, with equality holding for $p(z) = (z + K)^n$. To see that equality holds for the mentioned polynomial consider first the left hand side of the inequality.

$$\begin{aligned} M(p; R) &= \max_{|z|=R} |(z + K)^n| \\ &= \max_{0 \leq \theta < 2\pi} |(Re^{i\theta} + K)^n| \\ &= (R + K)^n. \end{aligned}$$

Looking at the right hand side, we see that

$$\begin{aligned}\left(\frac{R+K}{1+K}\right)^n M(p;1) &= \left(\frac{R+K}{1+K}\right)^n \max_{|z|=1} |(z+K)^n| \\ &= \left(\frac{R+K}{1+K}\right)^n \max_{0\le\theta<2\pi} |(e^{i\theta}+K)^n| \\ &= \left(\frac{R+K}{1+K}\right)^n |(1+K)^n| \\ &= (R+K)^n.\end{aligned}$$

Thus, we see that equality holds for $p(z) = (z+K)^n$. However, this result holds only in the range $1 \le R \le K^2$.

While working on extending this range to $R > K^2$, Govil, Qazi, and Rahman [13, p. 456] proved a similar theorem but with a sharper bound which we state now.

Theorem 2.6. *Let* $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ *for* $|z| < K$, *where* $K \ge 1$, *and let* $\lambda = \lambda(K) := \dfrac{Ka_1}{na_0}$. *Then* $M(p;R) \le \left(\dfrac{R^2+2|\lambda|RK+K^2}{1+2|\lambda|K+K^2}\right)^{n/2} M(p;1)$ *for* $1 \le R \le K^2$.

Before giving the proof of this theorem, we will show how it generalizes and sharpens Theorem (2.5) due to Govil and Rahman [15, Theorem 1]. For this, we show that in general

$$\left(\frac{R^2+2|\lambda|RK+K^2}{1+2|\lambda|K+K^2}\right)^{n/2} \le \left(\frac{R+K}{1+K}\right)^n,$$

which is equivalent to showing

$$\left(\frac{R^2+2|\lambda|RK+K^2}{1+2|\lambda|K+K^2}\right) \;\le\; \left(\frac{R+K}{1+K}\right)^2,$$

that is,

$$(R^2+2|\lambda|RK+K^2)(1+K)^2 \;\le\; (1+2|\lambda|K+K^2)(R+K)^2,$$

that is,

$$2R^2K+2|\lambda|RK+2|\lambda|RK^3+2K^3 \;\le\; 2RK+2|\lambda|R^2K+2|\lambda|K^3+2RK^3,$$

which is equivalent to

$$(|\lambda|-1)(R-1)(K^2-R)\le 0$$

which clearly holds if $|\lambda|\le 1$. Hence, $\left(\frac{R^2+2|\lambda|RK+K^2}{1+2|\lambda|K+K^2}\right)^{n/2} \le \left(\frac{R+K}{1+K}\right)^n$ if and only if $|\lambda|\le 1$. We now show that $|\lambda|\le 1$, and for this we use the following theorem of Rahman and Stankiewicz [21, Theorem 2′, p. 180].

Theorem 2.7. *Let* $p_n(z)=\prod_{\nu=1}^{n}(1-z_\nu z)$ *be a polynomial of degree* n *not vanishing in* $|z|<1$ *and let* $p_n'(0)=p_n''(0)=\ldots=p_n^{(l)}(0)=0$. *If* $\phi(z)=\{p_n(z)\}^\epsilon=\sum_{n=0}^{\infty}b_{k,\epsilon}z^k$, *where* $\epsilon=1$ *or* $\epsilon=-1$, *then* $|b_{k,\epsilon}|\le\frac{n}{k}$, $(l+1\le k\le 2l+1)$ *and* $|b_{2l+2,1}|\le\frac{n}{2(l+1)^2}(n+l-1)$, $|b_{2l+2,-1}|\le\frac{n}{2(l+1)^2}(n+l+1)$.

First, note that $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \neq 0$ for $|z| < K$, where $K \geq 1$ is equivalent to $p(Kz) = \sum_{\nu=0}^{n} a_\nu K^\nu z^\nu \neq 0$ for $|z| < 1$. Also note that

$$p(Kz) = \sum_{\nu=0}^{n} a_\nu K^\nu z^\nu = a_0 \sum_{\nu=0}^{n} \frac{a_\nu}{a_0} K^\nu z^\nu.$$

Now consider $\sum_{\nu=0}^{n} \frac{a_\nu}{a_0} K^\nu z^\nu$ which is a polynomial of the desired form since $\sum_{\nu=0}^{n} \frac{a_\nu}{a_0} K^\nu z^\nu \neq 0$ in $|z| < 1$. Then, by Theorem 2.7, if we take $\epsilon = 1$ and $l = 0$, we see $k = 1$ and $|b_{1,1}| = \left|\frac{a_1 K}{a_0}\right| \leq n$ which implies $\frac{|a_1|}{|a_0|} \leq \frac{n}{K}$. Hence, $|\lambda| = \left|\frac{K a_1}{n a_0}\right| \leq 1$.

Unfortunately, Theorem 2.6, although best possible, still only deals with the case where $1 \leq R \leq K^2$ and says nothing where $R > K^2$. However, now that we have $|\lambda| = K\left|\frac{a_1}{n a_0}\right| \leq 1$, then from the inequality in Theorem 2.6 it follows that

$$\begin{aligned}
M(p;K) &\leq \left(\frac{K^2 + 2|\lambda|K^2 + K^2}{1 + 2|\lambda|K + K^2}\right)^{n/2} M(p;1) \\
&= (K^2)^{n/2}\left(\frac{1 + 2|\lambda| + 1}{1 + 2|\lambda|K + K^2}\right)^{n/2} M(p;1) \\
&= K^n\left(\frac{2 + 2|\lambda|}{1 + 2|\lambda|K + K^2}\right)^{n/2} M(p;1) \\
&= K^n\left(\frac{2 + 2\left(K\left|\frac{a_1}{n a_0}\right|\right)}{1 + 2\left(K\left|\frac{a_1}{n a_0}\right|\right)K + K^2}\right)^{n/2} M(p;1) \\
&= K^n\left(\frac{2\left(1 + K\left|\frac{a_1}{n a_0}\right|\right)}{1 + 2\left|\frac{a_1}{n a_0}\right|K^2 + K^2}\right)^{n/2} M(p;1)
\end{aligned}$$

$$
\begin{aligned}
&= K^n \left(\frac{2\left(1 + K\left|\frac{a_1}{na_0}\right|\right)}{1 + K^2\left(2\left|\frac{a_1}{na_0}\right| + 1\right)} \right)^{n/2} M(p;1) \\
&= K^n \left(\frac{\sqrt{2}\sqrt{1 + K\left|\frac{a_1}{na_0}\right|}}{\sqrt{1 + K^2\left(2\left|\frac{a_1}{na_0}\right| + 1\right)}} \right)^{n} M(p;1) \\
&= \left(\frac{K\sqrt{2}\sqrt{1 + K\left|\frac{a_1}{na_0}\right|}}{\sqrt{1 + \left(2\left|\frac{a_1}{na_0}\right| + 1\right)K^2}} \right)^{n} M(p;1) \\
&\leq \left(\frac{K\sqrt{2}\sqrt{2}}{\sqrt{1 + (2+K)K}} \right)^{n} M(p;1) \\
&= \left(\frac{2K}{1 + 2K + K^2} \right)^{n} M(p;1) \\
&= \left(\frac{2K}{(1 + K)^2} \right)^{n} M(p;1) \\
&\leq \left(\frac{2K}{K + 1} \right)^{n} M(p;1).
\end{aligned}
$$

This gives us

$$
M(p;K) \leq \left(\frac{2K}{K + 1} \right)^{n} M(p;1). \tag{2.4}
$$

Now, let $p_K(z) := p(Kz)$. Then, $p_K(z) = \sum_{v=0}^{n} a_v (Kz)^v \neq 0$ for $|z| < 1$, and

$$
\begin{aligned}
M(p_K;1) = \max_{|z|=1} |p(Kz)| &= \max_{0 \leq \theta < 2\pi} |p(Ke^{i\theta})| \\
&= \max_{|z|=K} |p(z)| \\
&= M(p;K).
\end{aligned}
$$

So, if $R > K$, then if we write $R = SK$ where $S := \dfrac{R}{K} > 1$, we may apply (2.1) to p_K, and using the previous estimate for $M(p;K)$ we have

$$\begin{aligned}
M(p;R) &= \max_{0\leq\theta<2\pi} |p(Re^{i\theta})| \\
&= \max_{0\leq\theta<2\pi} \left| p\left(K\left(\frac{R}{K}\right)e^{i\theta}\right)\right| \\
&= \max_{0\leq\theta<2\pi} |p(SKe^{i\theta})| \\
&= \max_{|z|=S} |p(Kz)| \\
&= M(p_k;S) \\
&\leq \left(\frac{S^n+1}{2}\right) M(p_k;1), \quad \text{by} \quad (2.1) \\
&= \left(\frac{S^n+1}{2}\right) M(p;K) \\
&\leq \left(\frac{S^n+1}{2}\right)\left(\frac{2K}{K+1}\right)^n M(p;1), \quad \text{by } (2.4) \\
&= 2^{-1}(S^n+1)2^n\frac{K^n}{(K+1)^n}M(p;1) \\
&= \frac{2^{n-1}(S^n+1)K^n}{(1+K)^n}M(p;1) \\
&= \frac{2^{n-1}\left(\left(\frac{R}{K}\right)^n+1\right)K^n}{(1+K)^n}M(p;1), \quad \text{for} \quad R > K \\
&= \frac{2^{n-1}(R^n+K^n)}{(1+K)^n}M(p;1),
\end{aligned}$$

which gives

$$M(p;R) \leq \frac{2^{n-1}(R^n+K^n)}{(1+K)^n}M(p;1). \tag{2.5}$$

Hence, for any $R > K$, we get $M(p;R) \leq 2^{n-1}\dfrac{R^n + K^n}{(1+K)^n}M(p;1)$, which reduces to (2.1) when $K = 1$. Since for large values of K,

$$2^{n-1}\frac{R^n + K^n}{(1+K)^n}M(p;1) \quad \sim \quad 2^{n-1}\frac{R^n + K^n}{1+K^n}M(p;1) \quad \text{as} \quad K \to \infty,$$

the bound (2.5) does not give a very satisfactory bound because for large values of n, the factor 2^{n-1} may become very large and thus, the factor 2^{n-1} in the previous estimate is out of place [13, p. 456]. The following result of Govil, Qazi, and Rahman [13, Theorem 2] provides an estimate which does not have a factor 2^{n-1}.

Theorem 2.8. *Let* $p(z) := \sum_{v=0}^{n} a_v z^v \neq 0$ *for* $|z| < K$, *where* $K > 1$. *Then,*

$$M(p;R) \leq \frac{R^n}{K^n}\left(\frac{K^n}{K^n+1}\right)^{(R-K^2)/(R+K^2)} M(p;1), \quad \textit{for} \quad R \geq K^2.$$

We will prove Theorem 2.8 later. However, we can now note that for $R = K^2$,

$$\begin{aligned} M(p;R) &\leq \frac{K^{2n}}{K^n}\left(\frac{K^n}{K^n+1}\right)^{(K^2-K^2)/(K^2+K^2)} M(p;1) \\ &= K^{2n-2}\left(\frac{K^n}{K^n+1}\right)^{0} M(p;1) \\ &= K^n M(p;1), \end{aligned}$$

and likewise, by Theorem 2.6, for $R = K^2$,

$$M(p;R) \leq \left(\frac{K^4 + 2|\lambda|K^3 + K^2}{1 + 2|\lambda|K + K^2}\right)^{n/2} M(p;1)$$

$$
\begin{aligned}
&= (K^2)^{n/2}\left(\frac{K^2+2|\lambda|K+1}{1+2|\lambda|K+K^2}\right)^{n/2} M(p;1) \\
&= K^n M(p;1).
\end{aligned}
$$

Thus, for $R=K^2$, Theorem 2.8 reduces to Theorem 2.6.

Note that for $R>K^2$, the quantity $\left(\frac{K^n}{K^n+1}\right)^{(R-K^2)/(R+K^2)}$ lies between 0 and 1, and so, for $R>K^2$, the right hand side of the inequality in Theorem 2.8 is strictly less than $\frac{R^n}{K^n}M(p;1)$.

In fact, Govil, Qazi, and Rahman [13, Remark 1] show that for $R>K^2$, not only

$$
\left(\frac{K^n}{K^n+1}\right)^{(R-K^2)/(R+K^2)} < 1
$$

but

$$
\left(\frac{K^n}{K^n+1}\right)^{(R-K^2)/(R+K^2)} < 1-\left(\frac{R-K^2}{R+K^2}\right)\left(\frac{1}{K^n+1}\right).
$$

This will in fact imply that for $R>K^2$,

$$
M(p;R) < \left(\frac{R^n+K^n}{K^n+1}\right)M(p;1)+\frac{1}{K^n+1}\left\{\frac{2}{K^{n-2}}\cdot\frac{R^n}{R+K^2}-K^n\right\}M(p;1),
$$

and to see this, note that by Theorem 2.8

$$
\begin{aligned}
&M(p;R) \\
&\le \frac{R^n}{K^n}\left(\frac{K^n}{K^n+1}\right)^{(R-K^2)/(R+K^2)} M(p;1), \quad R\ge K^2
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{R^n}{K^n}\left(1-\left(\frac{R-K^2}{R+K^2}\right)\frac{1}{K^n+1}\right)M(p;1), \quad R>K^2\\
&= \left(\frac{R^n}{K^n}-\frac{R^n}{K^n(K^n+1)}\cdot\frac{R-K^2}{R+K^2}\right)M(p;1)\\
&= \left[\frac{R^n(K^n+1)(R+K^2)-R^n(R-K^2)}{K^n(K^n+1)(R+K^2)}\right]M(p;1)\\
&= \left[\frac{(R^nK^n+R^n)(R+K^2)-R^{n+1}+R^nK^2}{K^n(K^n+1)(R+K^2)}\right]M(p;1)\\
&= \left[\frac{R^{n+1}K^n+R^nK^{n+2}+R^{n+1}+R^nK^2-R^{n+1}+R^nK^2}{K^n(K^n+1)(R+K^2)}\right]M(p;1)\\
&= K^2\left[\frac{R^{n+1}K^{n-2}+R^nK^n+2R^n}{K^n(K^n+1)(R+K^2)}\right]M(p;1)\\
&= \left[\frac{R^{n+1}K^{n-2}+R^nK^n+2R^n}{K^{n-2}(K^n+1)(R+K^2)}\right]M(p;1)\\
&= \left[\frac{R^{n+1}K^{n-2}+R^nK^n+RK^{2n-2}+K^{2n}+2R^n-RK^{2n-2}-K^{2n}}{(K^n+1)(K^{n-2})(R+K^2)}\right]M(p;1)\\
&= \left[\frac{(R^nK^{n-2}+K^{2n-2})(R+K^2)+2R^n-K^{2n-2}(R+K^2)}{(K^n+1)(K^{n-2})(R+K^2)}\right]M(p;1)\\
&= \left[\frac{(R^n+K^n)K^{n-2}(R+K^2)+2R^n-K^n(K^{n-2})(R+K^2)}{(K^n+1)(K^{n-2})(R+K^2)}\right]M(p;1)\\
&= \left[\frac{R^n+K^n}{K^n+1}+\frac{1}{K^n+1}\left\{\frac{2}{K^{n-2}}\cdot\frac{R^n}{R+K^2}-K^n\right\}\right]M(p;1).
\end{aligned}
$$

Next, we state an extension of (2.3) to polynomials not vanishing in $|z|<K$, for $K>1$, due to Govil, Qazi, and Rahman [13, Theorem 3].

Theorem 2.9. *Let* $p(z) := \sum_{v=0}^{n} a_v z^v \neq 0$ *for* $|z|<K$, *where* $K\geq 1$, *and let* $\lambda=\lambda(K) := \frac{Ka_1}{na_0}$. *Then* $M(p;\rho)\geq\left(\frac{K^2+2K|\lambda|\rho+\rho^2}{K^2+2K|\lambda|+1}\right)^{n/2}M(p;1)$, *where* $0\leq\rho\leq 1$.

Note that the right hand side of the inequality in Theorem 2.9 is a decreasing function of $|\lambda|$. To see this, take $x = |\lambda|$ and consider

$$\begin{aligned} \frac{d}{dx}\left(\frac{K^2+2K\rho x+\rho^2}{K^2+2Kx+1}\right) &= \frac{(K^2+2Kx+1)(2K\rho)-(K^2+2K\rho x+\rho^2)(2K)}{(K^2+2Kx+1)^2} \\ &= \frac{2K(K^2\rho+2K\rho x+\rho-K^2-2K\rho x-\rho^2)}{(K^2+2Kx+1)^2} \\ &= \frac{2K(K^2\rho+\rho-K^2-\rho^2)}{(K^2+2Kx+1)^2} \\ &= \frac{2K[K^2(\rho-1)-\rho(\rho-1)]}{(K^2+2Kx+1)^2} \\ &= \frac{2K(\rho-1)(K^2-\rho)}{(K^2+2Kx+1)^2} \end{aligned}$$

which is less than or equal to zero since $K > 0$, $\rho \leq 1$, $\rho \leq K^2$, and the denominator is obviously greater than zero. Thus for any n,

$$\left(\frac{K^2+2K|\lambda|\rho+\rho^2}{K^2+2K|\lambda|+1}\right)^{n/2} \geq \left(\frac{K+\rho}{1+K}\right)$$

and therefore, Theorem 2.9 is an improvement of the result that if $p(z)$ is a polynomial of degree n, $p(z) \neq 0$ for $|z| < K$, $K \geq 1$, then for $0 \leq \rho \leq 1$, $M(p,\rho) \geq \left(\frac{\rho+K}{1+K}\right)^n M(p;1)$, where the bound is attained if $p(z) := c(ze^{i\beta}+K)^n$, $c \in \mathbb{C}$, $c \neq 0$, $\beta \in \mathbb{R}$. To see this, consider

$$\begin{aligned} M(p;\rho) &= \max_{0\leq\theta<2\pi} |p(\rho e^{i\theta})| \\ &= \max_{0\leq\theta<2\pi} |c(\rho e^{i\theta}e^{i\beta}+K)^n| \\ &= |c(\rho+K)^n| \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\rho+K}{K+1}\right)^n |c|(K+1)^n \\
&= \left(\frac{\rho+K}{K+1}\right)^n \max_{0\le\beta<2\pi} |c(K+e^{i\beta})^n| \\
&= \left(\frac{\rho+K}{K+1}\right)^n M(p;1), \quad \text{for} \quad c\in\mathbb{C}, \quad c\neq 0, \quad \beta\in\mathbb{R}.
\end{aligned}
$$

Now, we will state a complement to Theorem 2.9 which is also proved by Govil, Qazi, and Rahman [13, Theorem 4]. This theorem will be needed to prove Theorem 2.6.

Theorem 2.10. *Let* $p(z) := \sum_{v=0}^{n} a_v z^n \neq 0$ *for* $|z| < K$*, where* $K \in (0,1]$*, and let* $\lambda = \lambda(K) := \dfrac{Ka_1}{na_0}$*. Then,* $M(p;\rho) \geq \left(\dfrac{K^2+2|\lambda|K\rho+\rho^2}{K^2+2|\lambda|K+1}\right)^{n/2} M(p;1)$*, for* $0 \leq \rho \leq K^2$*.*

For any n, this inequality can be replaced by $M(p;\rho) \geq \left(\dfrac{\rho+K}{K+1}\right)^n M(p;1)$, for $0 \leq \rho \leq K^2$, where the bound is attained if $p(z) := c(ze^{i\beta}+K)^n$, $c \in \mathbb{C}$, $c \neq 0$, $\beta \in \mathbb{R}$.

In order to prove theorems 2.6, 2.8, 2.9, and 2.10, we will need two additional lemmas which we state now. Lemma 2.1 is an extension of (2.3) and is due to Qazi [19, p. 340, Corollary 1] (also see [18, p. 444, Theorem 1.7.6]). Lemma 2.2 is due to Govil, Qazi, and Rahman [13, p. 458].

Lemma 2.1. *Let* $p(z) := \sum_{v=0}^{n} a_v z^v \neq 0$ *in* $D(0;1) := \{z \in \mathbb{C} : |z| < 1\}$ *and let* $\lambda := \dfrac{a_1}{na_0}$*. Then we have* $M(p;\rho_1) \geq \left(\dfrac{1+2|\lambda|\rho_1+\rho_1^2}{1+2|\lambda|\rho_2+\rho_2^2}\right)^{n/2} M(p;\rho_2)$*,* $0 \leq \rho_1 < \rho_2 \leq 1$*.*

Lemma 2.2. *Let p be a polynomial of degree at most n such that $p(z) \neq 0$ in $D(0;\ell) := \{z \in \mathbb{C} : |z| < \ell\}$ for some $\ell > 0$. Then $M(p;\rho) \geq \left(\dfrac{\rho+\ell}{1+\ell}\right)^n M(p;1)$, $0 \leq \rho \leq \min\{1, \ell^2\}$.*

This lemma is also an extension of (2.3). The proof, due to Govil, Qazi, and Rahman [13, p. 458], we state below.

Let $z_v = r_v e^{i\theta_v}$ and $z = \rho e^{i\theta}$ for $0 \leq \theta_\nu < 2\pi$, $0 \leq \theta < 2\pi$. Then,

$$
\begin{aligned}
&\left|\frac{z - z_v}{e^{i\theta} - z_v}\right|^2 \\
&= \frac{(z - z_v)(\overline{z} - \overline{z_v})}{(e^{i\theta} - z_v)(e^{-i\theta} - \overline{z_v})} \\
&= \frac{|z|^2 + |z_v|^2 - 2\Re z\overline{z_v}}{1 + |z_v|^2 - 2\Re e^{i\theta}\overline{z_v}} \\
&= \frac{|\rho e^{i\theta}|^2 + |r_v e^{i\theta_v}|^2 - 2\Re(\rho e^{i\theta}\overline{r_v}e^{-i\theta_v})}{1 + |r_v e^{i\theta_v}|^2 - 2\Re(e^{i\theta}\overline{r_v}e^{-i\theta_v})} \\
&= \frac{\rho^2 + r_v^2 - 2\Re[\rho\overline{r_v}(\cos\theta + i\sin\theta)(\cos\theta_v - i\sin\theta_v)]}{1 + r_v^2 - 2\Re[\overline{r_v}(\cos\theta + i\sin\theta)(\cos\theta_v - i\sin\theta_v)]} \\
&= \frac{\rho^2 + r_v^2 - 2\Re[\rho\overline{r_v}(\cos\theta\cos\theta_v + \sin\theta\sin\theta_v + i(\cos\theta_v\sin\theta - \cos\theta\sin\theta_v))]}{1 + r_v^2 - 2\Re[\overline{r_v}(\cos\theta\cos\theta_v + \sin\theta\sin\theta_v + i(\cos\theta_v\sin\theta - \cos\theta\sin\theta_v))]} \\
&= \frac{\rho^2 + r_v^2 - 2\rho r_v(\cos\theta\cos\theta_v + \sin\theta\sin\theta_v)}{1 + r_v^2 - 2r_v(\cos\theta\cos\theta_v + \sin\theta\sin\theta_v)} \\
&= \frac{\rho^2 + r_v^2 - 2\rho r_v\cos(\theta - \theta_v)}{1 + r_v^2 - 2r_v\cos(\theta - \theta_v)} \\
&= \frac{\rho^2 + 2\rho r_v + r_v^2 - 2\rho r_v - 2\rho r_v\cos(\theta - \theta_v)}{1 + 2r_v + r_v^2 - 2r_v - 2r_v\cos(\theta - \theta_v)} \\
&= \frac{(\rho + r_v)^2 - 2\rho r_v(1 + \cos(\theta - \theta_v))}{(1 + r_v)^2 - 2r_v(1 + \cos(\theta - \theta_v))} \\
&\geq \left(\frac{\rho + r_v}{1 + r_v}\right)^2
\end{aligned}
$$

where the inequality holds only if $(1-\rho)(r_v^2-\rho) \geq 0$. To see that the inequality holds under this condition, consider

$$\frac{(\rho+r_v)^2 - 2\rho r_v(1+\cos(\theta-\theta_v))}{(1+r_v)^2 - 2r_v(1+\cos(\theta-\theta_v))} \quad \geq \quad \left(\frac{\rho+r_v}{1+r_v}\right)^2,$$

which is equivalent to

$$\begin{aligned}
&(\rho+r_v)^2(1+r_v)^2 - 2\rho r_v(1+\cos(\theta-\theta_v))(1+r_v)^2 \\
&\geq (1+r_v)^2(\rho+r_v)^2 - 2r_v(1+\cos(\theta-\theta_v))(\rho+r_v)^2.
\end{aligned}$$

This gives us

$$-\rho(1+r_v)^2 \quad \geq \quad -(\rho+r_v)^2,$$

that is

$$-\rho(1+r_v)^2 + (\rho+r_v)^2 \quad \geq \quad 0.$$

Thus, we have

$$r_v^2 - \rho - \rho r_v^2 + \rho^2 \quad \geq \quad 0,$$

which we can rewrite as

$$(1-\rho)(r_v^2-\rho) \quad \geq \quad 0,$$

since $(1-\rho)(r_v^2-\rho) = r_v^2 - \rho - \rho r_v^2 + \rho^2$. Thus, if $r_v \geq \ell$, then

$$\begin{aligned} \left|\frac{z - z_v}{e^{i\theta} - z_v}\right| &\geq \frac{\rho + r_v}{1 + r_v} \\ &\geq \frac{\rho + \ell}{1 + \ell}, \quad \text{if} \quad 0 \leq \rho \leq \min\{1, \ell^2\} \end{aligned}$$

because $\left(\dfrac{\rho + x}{1 + x}\right)$ is a nondecreasing function of x. Hence, if the polynomial $p(z) := a_m \prod_{v=1}^{m}(z - z_v)$, $a_m \neq 0$, has no zeros in $|z| < \ell$, $\ell > 0$, then $\left|\dfrac{p(\rho e^{i\theta})}{p(e^{i\theta})}\right| \geq \left(\dfrac{\rho + \ell}{1 + \ell}\right)^m$, for $-\pi \leq \theta \leq \pi$, if $0 \leq \rho \leq \min\{1, \ell\}$. Consequently, if θ_0 is such that $|p(e^{i\theta_0})| = M(p; 1)$, then

$$\begin{aligned} |p(\rho e^{i\theta_0})| &\geq |p(e^{i\theta_0})| \left(\frac{\rho + \ell}{1 + \ell}\right)^m \\ &= M(p; 1) \left(\frac{\rho + \ell}{1 + \ell}\right)^m, \end{aligned}$$

if $0 \leq \rho \leq \min\{1, \ell^2\}$, which clearly gives $M(p; \rho) \geq \left(\dfrac{\rho + \ell}{1 + \ell}\right)^m M(p; 1)$. $\square$

Theorem 2.10 is needed to prove Theorem 2.6. Thus, we will now state the proof for Theorem 2.10 as it is given by Govil, Qazi, and Rahman [13, p. 459].

Let $p_K(z) := p(Kz) = a_0 + a_1 K z + \ldots + a_n K^n z^n$. Since $p(z) \neq 0$ in $|z| < K$, we have that $p_K(z) \neq 0$ for $|z| < 1$. Thus, Lemma 2.1 may be applied to p_K, taking $\rho_1 = \dfrac{\rho}{K}$ and $\rho_2 = K$ to obtain

$$\begin{aligned} M(p; \rho) &= M\left(p_K; \frac{\rho}{K}\right) \\ &\geq \left(\frac{1 + 2|\lambda|\frac{\rho}{K} + \left(\frac{\rho}{K}\right)^2}{1 + 2|\lambda| K + K^2}\right)^{n/2} M(p_K; K) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1+2|\lambda|\rho K^{-1}+\rho^2 K^{-2}}{1+2|\lambda|K+K^2}\right)^{n/2} M(p_K;K) \\
&= \left(\frac{K^{-2}(K^2+2|\lambda|\rho K+\rho^2)}{1+2|\lambda|K+K^2}\right)^{n/2} M(p_K;K) \\
&= \left(\frac{K^2+2|\lambda|K\rho+\rho^2}{1+2|\lambda|K+K^2}\right)^{n/2} K^{-n} M(p_K;K) \\
&\geq \left(\frac{K^2+2|\lambda|K\rho+\rho^2}{1+2|\lambda|K+K^2}\right)^{n/2} K^{-n}K^n M(p;1) \quad \text{for} \quad 0 \leq \rho \leq \min\{1,\ell^2\}
\end{aligned}
$$

since $M(p_K;K) = M(p;K^2)$ and $M(p;K^2) \geq \left(\frac{K^2+\ell}{1+\ell}\right)^n M(p;1) \geq K^n M(p;1)$ by Lemma 2.2. Hence, we have $M(p;\rho) \geq \left(\frac{K^2+2|\lambda|K\rho+\rho^2}{1+2|\lambda|K+K^2}\right)^{n/2} M(p;1)$, and the proof of Theorem 2.10 is complete. □

Next, we will state the proof given by Govil, Qazi, and Rahman [13, p. 459] for Theorem 2.6.

First, let $1 \leq R \leq K$. Since $p(z) \neq 0$ for $|z| < K$, the polynomial $p_K(z) := p(Kz) = \sum_{v=0}^{n} a_v K^v z^v \neq 0$ for $|z| < 1$. Besides, $M(p_K;\rho_2) = M(p;R)$ and $M(p_K;\rho_1) = M(p;1)$ where $\rho_1 = \frac{1}{K}$ and $\rho_2 = \frac{R}{K}$. Since $R \in (1,K]$, we see that $0 < \rho_1 < \rho_2 \leq 1$, and from the inequality in Lemma 2.1 we obtain

$$
\begin{aligned}
M(p;1) &= M(p_K;\rho_1) \\
&\geq \left(\frac{1+2|\lambda|K^{-1}+K^{-2}}{1+2|\lambda|RK^{-1}+R^2K^{-2}}\right)^{n/2} M(p_K;\rho_2), \quad \text{for} \quad 1 \leq R \leq K \\
&= \left(\frac{1+2|\lambda|K^{-1}+K^{-2}}{1+2|\lambda|RK^{-1}+R^2K^{-2}}\right)^{n/2} M(p;R), \quad \text{for} \quad 1 \leq R \leq K
\end{aligned}
$$

which is equivalent to the inequality in Theorem 2.6 for $1 \le R \le K$, and thus Theorem 2.6 is established for $1 \le R \le K$.

Now let $K \le R \le K^2$. Then $p_R(z) := p(Rz) \neq 0$ for $|z| < \dfrac{K}{R}$. Since $\dfrac{K}{R} \le 1$ and $\dfrac{1}{R} \le \dfrac{K^2}{R^2}$, we may apply Theorem 2.10 to p_R with $\dfrac{K}{R}$ instead of K and $\rho = \dfrac{1}{R}$ to obtain

$$M(p_R; 1) \;\le\; \left(\frac{\frac{K^2}{R^2} + 2|\lambda|\frac{K}{R} + 1}{\frac{K^2}{R^2} + 2|\lambda|\frac{K}{R} \cdot \frac{1}{R} + \left(\frac{1}{R}\right)^2} \right)^{n/2} M\left(p_R; \frac{1}{R}\right)$$

which is equivalent to

$$M(p; R) \;\le\; \left(\frac{K^2 + 2|\lambda|KR + R^2}{K^2 + 2|\lambda|K + 1} \right)^{n/2} M(p; 1)$$

which is Theorem 2.6 for $K \le R \le K^2$, and thus Theorem 2.6 is completely proved. □

Now we prove Theorem 2.8. The proof here is again due to Govil, Qazi, and Rahman [13, p. 460].

Without loss of generality, we may assume that p is of degree n, and that $M(p; 1) = 1$. From the inequality in Theorem 2.6, it follows that

$$\begin{aligned} M(p; K^2) \;&\le\; \left(\frac{K^4 + 2|\lambda|K^3 + K^2}{1 + 2|\lambda|K + K^2} \right)^{n/2} M(p; 1) \\ &=\; \left[K^2 \left(\frac{K^2 + 2|\lambda|K + 1}{K^2 + 2|\lambda|K + 1} \right) \right]^{n/2} M(p; 1) \\ &=\; K^n M(p; 1) \\ &=\; K^n, \quad \text{since} \quad M(p; a) = 1. \end{aligned}$$

So, $M(p;K^2) \leq K^n$. Hence, if $g(z) := p(K^2 z) = a_0 + K^2 a_1 z + K^4 a_2 z^2 + \ldots + K^{2n} a_n z^n$, then on $|z| = 1$, $|g(z)| = |g(e^{i\theta})| = |p(K^2 e^{i\theta})| \leq M(p;K^2) \leq K^n$, which implies $|g(z)| \leq K^n$ for $|z| = 1$. Besides, since $p(z) \neq 0$ in $|z| < K$, we have that $g(z) = p(K^2 z) \neq 0$ in $|K^2 z| < K$, which implies that $g(z) \neq 0$ in $|z| < \dfrac{1}{K}$. If we set

$$\begin{aligned} G(z) &:= K^{-n} z^n \overline{g(1/\overline{z})} \\ &= K^{-n} z^n \left(\overline{a_0} + K^2 \overline{a_1}(1/z) + \ldots + K^{2n} \overline{a_n}(1/z)^n\right) \\ &= K^{-n} \overline{a_0} z^n + K^{-n+2} \overline{a_1} z^{n-1} + \ldots + K^n \overline{a_n} \end{aligned}$$

then

$$\begin{aligned} |G(z)| &= |G(e^{i\theta})| \\ &= |K^{-n} e^{in\theta} \overline{g(1/e^{-i\theta})}| \\ &= |K^{-n} \overline{g(e^{i\theta})}| \\ &= K^{-n} |g(e^{i\theta})| \\ &= K^{-n} |g(z)|, \quad |z| = 1 \\ &\leq K^{-n}(K^n) \\ &= 1. \end{aligned}$$

So, $|G(z)| \leq 1$ for $|z| = 1$. Also, since $g(z) \neq 0$ for $|z| < \dfrac{1}{K}$, this implies that $G(z) \neq 0$ for $\left|\dfrac{1}{z}\right| < \dfrac{1}{K}$, implying that $G(z) \neq 0$ for $|z| > K$, which gives us $G(z) = 0$ for $|z| \leq K$. Thus, $G(z)$ has all its zeros in the closed disk $|z| \leq K$.

Since $M(p;1) = 1$, that is $\max\limits_{|z|=1} |p(z)| = 1$, it follows from an inequality of Visser [23] that $|a_0| + |a_n| \leq \max\limits_{|z|=1} |p(z)| = 1$. So,

$$|a_0| + |a_n| \leq 1. \tag{2.6}$$

Hence, writing $p(z) := a_n \prod\limits_{v=1}^{n}(z - z_v)$, where $|z_v| \geq K$ for $1 \leq v \leq n$, and $\dfrac{|a_0|}{|a_n|} = |z_1||z_2| \cdots |z_n|$, we see that $\dfrac{|a_0|}{|a_n|} \geq K^n$, implying that $|a_0| \geq K^n|a_n|$. So, from (2.4) we have that $1 \geq |a_0| + |a_n| \geq K^n|a_n| + |a_n| = |a_n|(K^n + 1)$, implying that $|a_n| \leq \dfrac{1}{K^n + 1}$, which implies that

$$|G(0)| = |K^n \overline{a_n}| = |K^n a_n| \leq \frac{K^n}{K^n + 1}. \tag{2.7}$$

To complete the proof, we will rely heavily on the use of Poisson's integral formula [22, p. 124], and for the sake of completeness we state it below.

Theorem 2.11. *Let $f(z)$ be analytic in a region including the cirlce $|z| \leq R$, and let $u(r,\theta)$ be its real part. Then for $0 \leq r < R$,*

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} u(R,\phi)\, d\phi.$$

Now, let us suppose that $G(z) \neq 0$ for $|z| \leq 1$. Then applying Poisson's integral formula to $\text{Log}\,|G(z)|$, which is real , we obtain

$$\text{Log}\,|G(re^{i\theta})| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,\text{Log}\,|G(e^{i\phi})|d\phi, \quad \text{for} \quad 0 \leq r < 1.$$

Since $|G(z)| \leq 1$ for $|z| = 1$, we have $\text{Log}\,|G(e^{i\phi})| \leq 0$. So, we conclude that for $0 \leq r \leq 1$ we have

$$\begin{aligned}
\text{Log}\,|G(re^{i\theta})| &\leq \frac{(1+r)(1-r)}{(1+r)^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\text{Log}\,|G(e^{i\phi})|d\phi \\
&= \frac{(1-r)}{(1+r)}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\text{Log}\,|G(e^{i\phi})|d\phi \\
&= \frac{1-r}{1+r}\,\text{Log}\,|G(0)|,
\end{aligned}$$

since $\text{Log}\,|G(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\text{Log}\,|G(e^{i\phi})|d\phi$. So, we have that

$$|G(z)| \leq |G(0)|^{(1-|z|)/(1+|z|)} \quad \text{for} \quad 0 \leq |z| \leq 1,$$

which, when combined with (2.7) gives

$$\begin{aligned}
|G(z)| &\leq |G(0)|^{(1-|z|)/(1+|z|)} \\
&\leq \left(\frac{K^n}{K^n+1}\right)^{(1-|z|)/(1+|z|)}, \quad \text{for} \quad 0 \leq |z| \leq 1.
\end{aligned}$$

Therefore,

$$|G(z)| \leq \left(\frac{K^n}{K^n+1}\right)^{(1-|z|)/(1+|z|)}, \quad \text{for} \quad 0 \leq |z| \leq 1. \tag{2.8}$$

Next we will show that (2.8) remains true even if G has some zeros in $|z| < 1$, say $K^2/\overline{z_1}, \ldots, K^2/\overline{z_m}$. In such a case,

$$\frac{|a_0|}{|a_n|} = |z_1||z_2| \cdots |z_m| \cdots |z_n| \geq \left(\prod_{\mu=1}^{m} |z_\mu| \right) K^{n-m},$$

which implies that $|a_0| \geq |a_n| K^{n-m} \prod_{\mu=1}^{n} |z_\mu|$, and from Visser's inequality, it follows that

$$\begin{aligned} 1 &\geq |a_0| + |a_n| \\ &\geq |a_n| K^{n-m} \prod_{\mu=1}^{m} |z_\mu| + |a_n| \\ &= |a_n| \left(1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu| \right), \end{aligned}$$

which implies that

$$|a_n| \leq \frac{1}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|},$$

and that

$$|G(0)| = |K^n a_n| \leq \frac{K^n}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|}. \tag{2.9}$$

Let

$$
\begin{aligned}
G^*(z) &:= G(z)\prod_{\mu=1}^{m}\frac{K^2 z/z_\mu - 1}{z - K^2/\overline{z_\mu}} \\
&= G(z)\prod_{\mu=1}^{m}\left(\frac{\overline{z_\mu}}{z_\mu}\cdot\frac{K^2 z - z_\mu}{\overline{z_\mu}z - K^2}\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
|G^*(0)| &= \left|G(0)\prod_{\mu=1}^{m}\left(\frac{\overline{z_\mu}}{z_\mu}\cdot\frac{z_\mu}{K^2}\right)\right| \\
&= \left|G(0)\prod_{\mu=1}^{m}\frac{\overline{z_\mu}}{K^2}\right| \\
&\leq \left|\frac{K^n}{1+K^{n-m}\prod_{\mu=1}^{m}|z_\mu|}\prod_{\mu=1}^{m}\left(\frac{\overline{z_\mu}}{K^2}\right)\right|, \quad \text{by} \quad (2.9) \\
&= \frac{K^{n-2m}\prod_{\mu=1}^{m}|z_\mu|}{1+K^{n-m}\prod_{\mu=1}^{m}|z_\mu|},
\end{aligned}
$$

giving us

$$
|G^*(0)| \leq \frac{K^{n-2m}\prod_{\mu=1}^{m}|z_\mu|}{1+K^{n-m}\prod_{\mu=1}^{m}|z_\mu|}. \tag{2.10}
$$

Now,

$$
\begin{aligned}
|G^*(z)| &= \left| G(z) \prod_{\mu=1}^{m} \left(\frac{\overline{z_\mu}}{z_\mu} \cdot \frac{K^2 z - z_\mu}{\overline{z_\mu} z - K^2} \right) \right| \\
&= |G(z)| \prod_{\mu=1}^{m} \left| \frac{\overline{z_\mu}}{z_\mu} \cdot \frac{K^2 z - z_\mu}{\overline{z_\mu} z - K^2} \right| \\
&= |G(z)| \prod_{\mu=1}^{m} \left| \frac{K^2 e^{i\theta} - z_\mu}{\overline{z_\mu} e^{i\theta} - K^2} \right|, \quad \text{for} \quad |z| = 1 \\
&= |G(z)| \prod_{\mu=1}^{m} \left| \frac{z_\mu e^{-i\theta} - K^2}{\overline{z_\mu} e^{i\theta} - K^2} \right| \\
&= |G(z)|.
\end{aligned}
$$

Since $|G(z)| \leq 1$ on $|z| = 1$, then $|G^*(z)| \leq 1$ for $|z| = 1$ and $G^* \neq 0$ for $|z| < 1$. Now, by Poisson's integral formula,

$$
\operatorname{Log} |G^*(re^{i\theta})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^{*2}}{1 - 2r^* \cos(\theta - \phi) + r^{*2}} \operatorname{Log} |G^*(e^{i\phi})| d\phi,
$$

for $0 \leq r^* < 1$. Since $|G^*(z)| \leq 1$ on $|z| = 1$, then for $0 \leq \phi < 2\pi$, we have $\operatorname{Log} |G^*(e^{i\phi})| \leq 0$. Thus, for $0 \leq r^* < 1$,

$$
\begin{aligned}
\operatorname{Log} |G^*(r^* e^{i\theta})| &\leq \frac{(1 + r^*)(1 - r^*)}{(1 + r^*)^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Log} |G^*(e^{i\phi_*})| d\phi_* \\
&= \frac{(1 - r^*)}{(1 + r^*)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Log} |G^*(e^{i\phi_*})| d\phi_* \\
&= \frac{1 - r^*}{1 + r^*} \operatorname{Log} |G^*(0)|,
\end{aligned}
$$

that is, $|G^*(z)| \leq |G^*(0)|^{(1-|z|)/(1+|z|)}$ for $0 \leq |z| < 1$, which when combined with (2.10) gives

$$|G^*(z)| \leq \left(\frac{K^{n-2m} \prod_{\mu=1}^{m} |z_\mu|}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|} \right)^{(1-|z|)/(1+|z|)}, \quad \text{for} \quad |z| < 1.$$

Hence,

$$\left| G(z) \prod_{\mu=1}^{m} \left(\frac{\frac{K^2 z}{z_\mu} - 1}{z - \frac{K^2}{\overline{z_\mu}}} \right) \right| \leq \left(\frac{K^{n-2m} \prod_{\mu=1}^{m} |z_\mu|}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|} \right)^{(1-|z|)/(1+|z|)}, \quad \text{for} \quad |z| < 1,$$

which implies that

$$\begin{aligned} |G(z)| &\leq \left(\frac{K^{n-2m} \prod_{\mu=1}^{m} |z_\mu|}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|} \right)^{(1-|z|)/(1+|z|)} \prod_{\mu=1}^{m} \left| \frac{z - \frac{K^2}{\overline{z_\mu}}}{\frac{K^2 z}{z_\mu} - 1} \right|, \text{ for } |z| < 1 \\ &\leq \left(\frac{K^{n-2m} \prod_{\mu=1}^{m} |z_\mu|}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|} \right)^{(1-|z|)/(1+|z|)} \prod_{\mu=1}^{m} \left(\frac{|z| + \frac{K^2}{|z_\mu|}}{\frac{K^2}{|z_\mu|}|z| + 1} \right), \quad \text{for} \quad |z| < 1. \end{aligned}$$

Setting $t_\mu := \frac{K^2}{|z_\mu|}$ for $1 \leq \mu \leq m$ we see that for $|z| < 1$, we have

$$G(z)| \leq \psi(t_1, t_2, \ldots, t_m)$$

where

$$\psi(t_1, t_2, \ldots, t_m) \;:=\; \left(\frac{K^n}{t_1 \cdot t_2 \cdots t_m + K^{n+m}}\right)^{(1-|z|)/(1+|z|)} \prod_{\mu=1}^{m} \left(\frac{|z| + t_\mu}{t_\mu |z| + 1}\right).$$

To see this, observe that

$$\begin{aligned}
\frac{K^{n-2m} \prod_{\mu=1}^{m} |z_\mu|}{1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|} &= \frac{K^n K^{-2m}}{\prod_{\mu=1}^{m} \frac{1}{|z_\mu|} \left[1 + K^{n-m} \prod_{\mu=1}^{m} |z_\mu|\right]} \\
&= \frac{K^n K^{-2m}}{\prod_{\mu=1}^{m} \frac{1}{|z_\mu|} + K^{n-m}} \\
&= \frac{K^n}{K^{2m} \left[\prod_{\mu=1}^{m} \frac{1}{|z_\mu|} + K^{n-m}\right]} \\
&= \frac{K^n}{\prod_{\mu=1}^{m} \frac{K^2}{|z_\mu|} + K^{n+m}} \\
&= \frac{K^n}{\frac{K^2}{|z_1|} \cdot \frac{K^2}{|z_2|} \cdots \frac{K^2}{|z_m|} + K^{n+m}} \\
&= \frac{K^n}{t_1 \cdot t_2 \cdots t_m + K^{n+m}}.
\end{aligned}$$

Setting $\Lambda := \dfrac{1}{K^{n+m} + t_1 \cdot t_2 \cdots t_m}$ and $A_\nu := (K^n)^{(1-|z|)/(1+|z|)} \prod_{\mu=1, \mu \neq \nu}^{m} \left(\dfrac{|z| + t_\mu}{t_\mu |z| + 1}\right)$ for $1 \leq \nu \leq m$, we have

$$\psi(t_1, t_2, \ldots, t_m) \;=\; \left(\frac{K^n}{t_1 \cdot t_2 \cdots t_m + K^{n+m}}\right)^{(1-|z|)/(1+|z|)} \prod_{\mu=1}^{m} \left(\frac{|z| + t_\mu}{t_\mu |z| + 1}\right)$$

$$= \quad A_\nu \Lambda^{(1-|z|)/(1+|z|)} \left(\frac{|z| + t_\nu}{t_\nu |z| + 1} \right),$$

and, for $\nu \in \{1, \ldots, m\}$, we obtain the partial derivatives

$$\begin{aligned}
\frac{\partial \psi}{\partial t_\nu} &= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \left[\frac{(t_\nu |z| + 1) - (|z| + t_\nu)|z|}{(t_\nu |z| + 1)^2} \right] \\
&\quad + A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{(1 - |z|)}{(1 + |z|)} \cdot \Lambda^{\frac{-2|z|}{1+|z|}} \left(\frac{-t_1 \cdots t_{\nu-1} t_{\nu+1} \cdots t_m}{(K^{n+m} + t_1 \cdots t_m)^2} \right) \\
&= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1 - |z|^2)}{(t_\nu |z| + 1)^2} \\
&\quad - A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{1 - |z|}{1 + |z|} \cdot \Lambda^{\frac{-2|z|}{1+|z|}} \left(\frac{t_1 \cdots t_{\nu-1} t_{\nu+1} \cdots t_m}{(K^{n+m} + t_1 \cdots t_m)^2} \right) \\
&= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1 - |z|^2)}{(t_\nu |z| + 1)^2} \\
&\quad - A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{1 - |z|}{1 + |z|} (K^{n+m} + t_1 \cdots t_m)^{\frac{2|z|}{1+|z|} - 2} (t_1 \cdots t_{\nu-1} t_{\nu+1} \cdots t_m) \\
&= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1 - |z|^2)}{(t_\nu |z| + 1)^2} \\
&\quad - A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{1 - |z|}{1 + |z|} (K^{n+m} + t_1 \cdots t_m)^{\frac{2|z| - 2 - 2|z|}{1+|z|}} \left(\frac{t_1 \cdots t_m}{t_\nu} \right) \\
&= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1 - |z|^2)}{(t_\nu |z| + 1)^2} \\
&\quad - A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{1 - |z|}{1 + |z|} (K^{n+m} + t_1 \cdots t_m)^{\frac{-2}{1+|z|}} \left(\frac{t_1 \cdots t_m}{t_\nu} \right) \\
&= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1 - |z|^2)}{(t_\nu |z| + 1)^2} \\
&\quad - A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{1 - |z|}{1 + |z|} \left(\frac{1}{K^{n+m} + t_1 \cdots t_m} \right)^{\frac{2}{1+|z|}} \left(\frac{t_1 \cdots t_m}{t_\nu} \right) \\
&= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1 - |z|^2)}{(t_\nu |z| + 1)^2} \\
&\quad - A_\nu \cdot \frac{|z| + t_\nu}{t_\nu |z| + 1} \cdot \frac{1 - |z|}{1 + |z|} \Lambda^{\frac{2}{1+|z|}} \left(\frac{t_1 \cdots t_m}{t_\nu} \right).
\end{aligned}$$

So, for $\nu \in \{1, \ldots, m\}$, the partial derivatives

$$\begin{aligned}\frac{\partial \psi}{\partial t_\nu} &= A_\nu \Lambda^{(1-|z|)/(1+|z|)} \frac{(1-|z|^2)}{(t_\nu |z|+1)^2} \\ &\quad - A_\nu \cdot \frac{|z|+t_\nu}{t_\nu|z|+1} \cdot \frac{1-|z|}{1+|z|} \Lambda^{\frac{2}{1+|z|}} \left(\frac{t_1 \cdots t_m}{t_\nu}\right)\end{aligned}$$

are positive if and only if

$$(1+|z|)^2 \;>\; (|z|+t_\nu)(t_\nu|z|+1)\left(\frac{1}{K^{n+m}+t_1\cdots t_m}\right)\left(\frac{t_1\cdots t_m}{t_\nu}\right),$$

which is true since $t_\nu < 1$ for $1 \leq \mu \leq m$ and $K \geq 1$. Hence, $\psi(t_1, \ldots, t_m) \leq \psi(1, \ldots, 1) = \left(\dfrac{K^n}{K^{n+m}+1}\right)^{(1-|z|)/(1+|z|)}$ for $|z| < 1$, and so (2.8) holds even if G has some zeros in the open disk $|z| < 1$.

From (2.8) we therefore conclude that for $0 < |z| \leq 1$,

$$\left|\frac{z^n}{K^n} p\left(\frac{K^2}{\overline{z}}\right)\right| \leq \left(\frac{K^n}{K^n+1}\right)^{(1-|z|)/(1+|z|)}$$

which is equivalent to

$$\left|p\left(\frac{K^2}{\overline{z}}\right)\right| \;\leq\; \frac{K^n}{|z|^n}\left(\frac{K^n}{K^n+1}\right)^{(1-|z|)/(1+|z|)}, \quad \text{for} \;\; 0 < |z| \leq 1$$

which implies that

$$|p(\zeta)| \;\leq\; \frac{|\zeta|^n}{K^n}\left(\frac{K^n}{K^n+1}\right)^{(|\zeta|-K^2)/(|\zeta|+K^2)}, \quad \text{for} \;\; |\zeta| > K^2,$$

which is equivalent to the inequality in Theorem 2.8, and the proof of Theorem 2.8 is thus complete. $\square$

Finally, we state the proof of Theorem 2.9 which is also given by Govil, Qazi, and Rahman [13, p. 462].

Let $p_K(z) = p(Kz) := a_0 + Ka_1 z + \ldots + K^n a_n z^n$. Then $p_K(z) \neq 0$ for $|z| < 1$. Applying Lemma 2.1 to p_K taking $\rho_1 := \dfrac{\rho}{K}$ and $\rho_2 := \dfrac{1}{K}$, we obtain

$$
\begin{aligned}
M(p;\rho) &= M(p_K;\rho_1) \\
&\geq \left(\frac{1 + 2\left|\frac{Ka_1}{na_0}\right|\rho_1 + \rho_1^2}{1 + 2\left|\frac{Ka_1}{na_0}\right|\rho_2 + \rho_2^2} \right)^{n/2} M(p_K;\rho_2) \\
&= \left(\frac{1 + 2|\lambda|\frac{\rho}{K} + \frac{\rho^2}{K^2}}{1 + 2|\lambda|\frac{1}{K} + \frac{1}{K^2}} \right)^{n/2} M(p;1), \quad \text{for} \quad 0 \leq \rho \leq 1 \\
&= \left(\frac{K^2 + 2|\lambda|\rho K + \rho^2}{K^2 + 2|\lambda|K + 1} \right)^{n/2} M(p;1), \quad \text{for} \quad 0 \leq \rho \leq 1,
\end{aligned}
$$

which is the inequality in Theorem 2.9, and thus proves Theorem 2.9. $\square$

Chapter 3

Results Involving Entire Functions of Exponential Type

In this chapter we will study entire functions or exponential type, that is, entire functions with some growth restriction. To begin, we will state some definitions concerning entire functions and entire functions of exponential type which can be found, for example, in the book by Levin [17, p. 1-3], (see also Boas [5, p. 8-12]). An entire function is a function of a complex variable analytic in the entire plane and consequently represented by an everywhere convergent power series $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \ldots$. These functions form a natural generalization of the polynomials, and are close to polynomials in their properties.

The classical investigations of Borel, Hadamard, and Lindelöf dealt with the connection between the growth of an entire function and the distribution of its zeros. The rate of growth of a polynomial as the independent variable goes to infinity is determined, of course, by its degree. On the other hand, the number of roots of a polynomial is equal to its degree. Thus, the more roots a polynomial has, the greater is its growth. This connection between the set of zeros of the function and its growth can be generalized to arbitrary entire functions.

It is well known, and follows trivially from the maximum modulus principle, that $M(f;r) := \max_{|z|=r} |f(z)|$ is an increasing function of r. Also, it is clear that this function is continuous, and to see this let $r_1 < r_2$ and let $0 \le \theta_0 < 2\pi$ be such that

$|f(r_2e^{i\theta_0})| = M(f;r_2)$. Then

$$\begin{aligned} 0 &< M(f;r_2) - M(f;r_1) \\ &= |f(r_2e^{i\theta_0})| - M(f;r_1) \\ &\leq |f(r_2e^{i\theta_0})| - |f(r_1e^{i\theta_0})| \\ &< \epsilon \end{aligned}$$

for $r_2 - r_1 < \delta_\epsilon$ since the function $|f(re^{i\theta})|$ as a function of r, is continuous.

The rate of growth of the function $M(f;r)$ is an important property for the behaviour of an entire function. We first show that for an entire function not a polynomial, $M(f;r)$ grows faster than any positive power of r. The following result, given by Levin [17, p. 2], shows that if a function f does not grow faster than any positive power of r, then f is a polynomial.

Theorem 3.1. *If there exists a positive integer n such that $\liminf_{r\to\infty} \frac{M(f;r)}{r^n} < \infty$, then $f(z)$ is a polynomial of degree at most n.*

We now state the proof of Theorem 3.1 which is also given by Levin [17, p. 2]. Let $n = N$ be a positive integer such that $\liminf_{r\to\infty} \frac{M(f;r)}{r^N} < \infty$. If $f(z) = a_0 + a_1z + \cdots + a_Nz^N + a_{N+1}z^{N+1} + \cdots$ and $P_N(z) = a_0 + a_1z + \cdots + a_Nz^N$, then the function

$$\begin{aligned} \phi(z) &= [f(z) - P_N(z)]z^{-N-1} \\ &= \frac{a_{N+1}z^{N+1} + a_{N+2}z^{N+2} + \cdots}{z^{N+1}} \end{aligned}$$

$$= a_{N+1} + a_{N+2}z + a_{N+3}z^2 + \cdots$$

is entire and tends uniformly to zero on some sequence of circles $|z| = r_n$ where $r_n \to \infty$. To see that $\phi(z)$ tends uniformly to zero on some sequence of circles $|z| = r_n$ where $r_n \to \infty$, first consider

$$\begin{aligned} |P_N(z)| &= |a_0 + a_1 z + \cdots + a_N z^N| \\ &\leq |a_0| + |a_1||z| + |a_z||z^2| + \cdots + |a_N||z^N|, \end{aligned}$$

which implies that

$$\begin{aligned} \max_{|z|=r} |P_N(z)| &\leq |a_0| + |a_1|r + |a_2|r^2 + \cdots + |a_N|r^N \\ &\leq (N+1)M(f;r) \end{aligned}$$

because from Cauchy's Inequality [22, p. 84], $|a_n|r^n \leq M(f;r)$ for all n. Now consider

$$\begin{aligned} \max_{|z|=r} |f(z) - P_N(z)| &\leq \max_{|z|=r} |f(z)| + \max_{|z|=r} |P_N(z)| \\ &\leq M(f;r) + (N+1)M(f;r) \\ &= (N+2)M(f;r). \end{aligned}$$

Since $\liminf_{r\to\infty} \dfrac{M(f;r)}{r^N} < \infty$ implies that $M(f;r) < Lr^N$ on a sequence $\{r_n\} \to \infty$ and for some constant L. So,

$$\begin{aligned}\max_{|z|=r} |\phi(z)| &= \max_{|z|=r} \left| |f(z) - P_N(z)| z^{-N-1} \right| \\ &< \frac{(N+2)Lr^N}{r^{N+1}} \\ &= \frac{(N+2)L}{r}\end{aligned}$$

which approaches zero as r approaches infinity, and thus $\phi(z)$ tends uniformly to zero on some sequence of circles $|z| = r_n$. Thus, it follows from the maximum modulus principle that $\phi(z) \equiv 0$. In other words, $f(z) \equiv P_N(z)$. Hence, $f(z)$ is a polynomial because $P_N(z)$ is a polynomial. □

So, in order to estimate the growth of entire functions which are not polynomials, we must choose comparison functions that grow faster than powers of r. We choose comparison functions of the form e^{r^k}, where $k > 0$.

An entire function $f(z)$ is said to be a function of finite order if there exists a positive constant k such that the inequality $\max_{|z|=r} |f(z)| < e^{r^k}$ is valid for all sufficiently large values of r where $r > r_0(k)$. The greatest lower bound of such numbers k is called the order of the entire function $f(z)$. So, if ρ is the order of the entire function $f(z)$, and if ϵ is an arbitrary positive number, then

$$e^{r^{\rho-\epsilon}} < M(f;r) < e^{r^{\rho+\epsilon}} \tag{3.1}$$

where the inequality on the right is satisfied for all sufficiently large values of r, and the inequality on the left holds for some sequence $\{r_n\}$ of values of r, tending to infinity. From condition (3.1) we define the order ρ of the function to be

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(f;r)}{\log r}.$$

For functions of a given order, a more precise characterization of the growth is given by the type of the function. By the type τ of an entire function $f(z)$ of order ρ we mean the greatest lower bound of positive numbers A for which

$$M(f;r) < e^{Ar^\rho} \tag{3.2}$$

for all sufficiently large values of r. From condition (3.2) we define the type τ of the function to be

$$\tau = \limsup_{r\to\infty} \frac{\log M(f;r)}{r^\rho}.$$

If $\tau = 0$, the function $f(z)$ is said to be of minimal type, if $0 < \tau < \infty$ of normal type, and if $\tau = \infty$ of maximal type.

We shall say that the function $f_2(z)$ is of larger growth than the function $f_1(z)$ if the order of $f_2(z)$ is greater than the order of$f_1(z)$, or if the orders are equal and the type of $f_2(z)$ is larger than the type of $f_1(z)$.

Also note that the order of the sum of two functions is not greater than the larger of the orders of the two summands, and if the orders of the summands and of the sum are all equal, then the type of the sum is not greater than the larger of the types of the two summands. If one of the two functions is of larger growth than the other, then the sum has the same order and type as the function of larger growth.

The rate of growth of an entire function f in different directions can be specified by the Phragmén-Lindelöf indicator function of f defined as

$$h_f(\theta) \quad := \quad \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r}, \quad \text{for} \quad 0 < r < \infty \quad \text{and} \quad 0 \le \theta < 2\pi \ .$$

So, the type of an entire function f is equal to the maximum of the indicator function of f.

According to a fundamental property of the indicator function, if $h_f(\theta_1) \le h_1$ and $h_f(\theta_2) \le h_2$, where $\alpha \le \theta_1 < \theta_2 \le \beta$ and $0 < \theta_2 - \theta_1 < \pi$, then [5, Theorem 5.1.2]

$$h_f(\theta) \quad \le \quad \frac{h_1 \sin(\theta_2 - \theta) + h_2 \sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}, \quad \text{for} \quad \theta_1 \le \theta \le \theta_2.$$

By an entire function of exponential type τ, we will mean an entire function of order less than one and of any type $\tau \ge 0$ or an entire function of order one and of type at most τ. For an entire function f of exponential type, $h_f(\theta) < \infty$ for all θ, and unless $h_f(\theta) \equiv -\infty$, the function h_f is continuous (see [5, Theorem 5.4.1]).

Furthermore, we have [5, Theorem 5.4.4]

$$h_f\left(\theta - \frac{\pi}{2}\right) + h_f\left(\theta + \frac{\pi}{2}\right) \geq 0, \quad \text{for } \ 0 \leq \theta < 2\pi.$$

In this chapter we will be mainly concerned with entire functions of exponential type having no zeros in a half-plane. Our results will include generalizations of the Bernstein type inequalities discussed in chapters one and two. Most of the results that we will be discussing here are given in a recent paper of Govil, Qazi, and Rahman [14].

It is known [5, Theorem 7.8.1] that if ω is an entire function of exponential type such that $\omega(z) \neq 0$ for $\Im z < 0$ and $h_\omega(\alpha) \leq h_\omega(-\alpha)$ for some $\alpha \in (0, \pi)$, then

$$|\omega(z)| \leq |\omega(\bar{z})| \quad \text{for } \ \Im z > 0, \tag{3.3}$$

and so

$$h_\omega(\alpha) \leq h_\omega(-\alpha) \quad \text{for } \ 0 < \alpha < \pi. \tag{3.4}$$

Hence the following result given in a paper of Govil, Qazi, and Rahman [14, Lemma A] holds.

Theorem 3.2. *Let ω be an entire function of exponential type having no zeros in the open lower half-plane. Then (3.3) holds if and only if $h_\omega(\alpha) \leq h_\omega(-\alpha)$ for some $\alpha \in (0, \pi)$. Thus, $h_\omega(\alpha) \leq h_\omega(-\alpha)$ for every $\alpha \in (0, \pi)$, if and only if $h_\omega(\alpha) \leq h_\omega(-\alpha)$ for some $\alpha \in (0, \pi)$.*

If f is analytic and of exponential type in the upper half-plane such that $|f(x)| \leq M$ on the real axis, and $h_f\left(\frac{\pi}{2}\right) \leq a$, then [5, Theorem 6.2.4],

$$|f(z)| \leq Me^{a\Im z} \quad \text{for} \quad \Im z > 0.$$

From this it follows that if f is an entire function of exponential type τ such that $|f(x)| \leq M$ on the real axis, then for $z \in \mathbb{C}$,

$$|f(z)| \leq Me^{\tau|\Im z|}. \tag{3.5}$$

In particular, we get

Theorem 3.3. *If f is an entire function of exponential type τ such that $|f(x)| \leq M$ on the real axis, then $|f(z)| = |f(x+iy)| \leq Me^{\tau|y|}$ for $-\infty < x < \infty$ and $y \leq 0$.*

This result, known as Bernstein's Inequality for functions of exponential type, is mentioned for example, in Govil, Qazi, and Rahman [14, p, 898] and is best possible. Equality holds for $f(z) = \lambda e^{i\tau z}$ where $\lambda \in \mathbb{C}$. To see this, consider

$$\begin{aligned}
|f(z)| &= |f(x+iy)| \\
&= |\lambda e^{i\tau(x+iy)}| \\
&= |\lambda e^{i\tau x}||e^{-\tau y}| \\
&= |\lambda|e^{\tau|y|}, \quad \text{because} \quad y \leq 0 \\
&= \sup_{-\infty<x<\infty} |f(x)|e^{\tau|y|} \\
&= Me^{\tau|y|}.
\end{aligned}$$

The above result, Bernstein's Inequality for functions of exponential type, generalizes the result that if $p(z)$ is a polynomial of degree at most n, then

$$M(p;R) \leq M(p;1)R^n, \quad \text{for} \quad R \geq 1 \tag{3.6}$$

which we have discussed in Chapter 1. This result is best possible.

We will now show that inequality (3.5) generalizes inequality (3.6). If $p(z)$ is a polynomial of degree n, then as is easy to see, the function $f(z) = p(e^{iz})$ is an entire function of exponential type n. Thus,

$$\begin{aligned} \sup_{-\infty<x<\infty} |f(x+iy)| &= \sup_{-\infty<x<\infty} |p(e^{i(x+iy)})| \\ &\leq \sup_{-\infty<x<\infty} |f(x)|e^{n|y|}, \quad \text{for} \quad y \leq 0 \quad \text{by} \quad \text{Theorem 3.3} \end{aligned}$$

which implies that

$$\sup_{-\infty<x<\infty} |p(e^{-y}e^{ix})| \leq \sup_{-\infty<x<\infty} |p(e^{ix})|e^{n|y|}, \quad \text{for} \quad y \leq 0.$$

Now, take $R = e^{-y}$ which is greater than or equal to one because $y \leq 0$. So we have that $y = -\ln R$. Thus, $e^{n|y|} = \left(e^{|y|}\right)^n = \left(e^{\ln R}\right)^n = R^n$, and we have that

$$\begin{aligned} \max_{|z|=R} |p(z)| &\leq \max_{|z|=1} |p(z)|e^{n|y|} \\ &= \max_{|z|=1} |p(z)|R^n, \quad \text{for} \quad R \geq 1. \end{aligned}$$

Hence, inequality (3.5) yields $M(p;R) \le M(p;1)R^n$, which is inequality (3.6), for polynomials of degree at most n and $R \ge 1$.

If the polynomial $p(z)$ has no zeros in the unit circle, then we have seen in Chapter 2 (see inequality (2.1)) that inequality (3.6) can be replaced by

$$M(p;R) \quad \le \quad M(p;1)\frac{R^n+1}{2}, \quad \text{for} \quad R \ge 1. \tag{3.7}$$

As we know, the above inequality which is due to Ankeney and Riviln [1, p. 849] is best possible with equality holding for $p(z) = \dfrac{\lambda + \mu z^n}{2}$ where $|\lambda| = |\mu| = 1$. The following result of Boas [14, Theorem A] generalizes the above result.

Theorem 3.4. *Let f be an entire function of exponential type τ such that $|f(x)| \le M$, on the real axis. Furthermore, let $f(z) \ne 0$ for $\Im z > 0$, and suppose that $h_f\left(\frac{\pi}{2}\right) = 0$. Then*

$$|f(z)| \quad \le \quad M\frac{e^{\tau|y|}+1}{2}, \quad \textit{for} \;\; y := \Im z \le 0. \tag{3.8}$$

To see how inequality (3.8) generalizes inequality (3.7) first note that for $p(z) := \displaystyle\sum_{\nu=0}^{n} a_\nu z^\nu \ne 0$ for $|z| < 1$, the function $f(z) := p(e^{iz}) \ne 0$ for $\Im z > 0$. Also, the type τ of $f(z) = p(e^{iz})$ which is an entire function of exponential type, is equal to n, and, since $|a_0| = |p(0)| \ne 0$, it is clear that $h_f\left(\frac{\pi}{2}\right) = 0$. Furthermore, $|f(x)| \le M$ on the real axis if $|p(z)| \le M$ on the unit circle. So, we have

$$\sup_{-\infty<x<\infty} |f(x+iy)| \quad = \quad \sup_{-\infty<x<\infty} |p(e^{i(x+iy)})|$$

$$\leq \sup_{-\infty<x<\infty} |f(x)| \frac{e^{n|y|}+1}{2}, \quad \text{for } \; y \leq 0 \; \text{ by Theorem 3.4,}$$

which implies that

$$\sup_{-\infty<x<\infty} |p(e^{-y}e^{ix})| \;\leq\; \sup_{-\infty<x<\infty} |p(e^{ix})| \frac{e^{n|y|}+1}{2}, \quad \text{for } \; y \leq 0.$$

Take $R = e^{-y}$ which is greater or equal to one if $y \leq 0$. Then $y = -\ln R$, which implies that $e^{n|y|} = \left(e^{|y|}\right)^n = \left(e^{\ln R}\right)^n = R^n$. Thus,

$$\begin{aligned} \max_{|z|=R} |p(z)| &\leq \max_{|z|=1} |p(z)| \frac{e^{n|y|}+1}{2} \\ &= \max_{|z|=1} |p(z)| \left(\frac{R^n+1}{2}\right), \quad \text{for } \; R \geq 1. \end{aligned}$$

Hence, inequality (3.8) yields $M(p;R) \leq M(p;1)\dfrac{R^n+1}{2}$, which is inequality (3.7), for polynomials with no zeros inside the unit circle and $R \geq 1$.

Aziz and Dawood [2, p. 307] sharpened inequality (3.7) for the case in which the polynomial p has no zeros on the unit circle. We state their result below.

Theorem 3.5. *Let p be a polynomial of degree at most n having no zeros inside the unit circle, then for $R \geq 1$,*

$$M(p;R) \;\leq\; \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |p(z)| - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |p(z)|. \tag{3.9}$$

Equality holds for $p(z) = \alpha z^n + \beta$ where $|\beta| \geq |\alpha|$.

For the proof of Theorem 3.5 we need the following theorem given by Aziz and Dawood [2, Theorem 2].

Theorem 3.6. *If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z| < 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}.$$

This result is best possible and equality holds for the polynomial $P(z) = \alpha z^n + \beta$, where $|\beta| \geq |\alpha|$.

The proof that we give is partly different from a proof given by Aziz and Dawood [2, p. 309-310]. We give this proof now. Let $M = \max_{|z|=1} |P(z)|$ and $m = \min_{|z|=1} |P(z)|$. Then $m \leq |P(z)|$ for $|z| = 1$. Since all the zeros of $P(z)$ lie in $|z| \geq 1$, therefore, for every complex number α such that $|\alpha| < 1$, it follows (by Rouche's theorem for $m > 0$) that the polynomial $F(z) = P(z) - \alpha m$ does not vanish in $|z| < 1$.

If we define $Q(z) := z^n \overline{P(1/\overline{z})}$ and $G(z) := z^n \overline{F(1/\overline{z})}$, then $G(z)$ has no zeros in $|z| > 1$. This implies that $\dfrac{F(z)}{G(z)}$ is analytic in $|z| \geq 1$, which implies that $\left|\dfrac{F(z)}{G(z)}\right| \leq 1$ for $|z| \geq 1$. Thus, $|F(z)| \leq |G(z)|$ for $|z| \geq 1$, which implies that for every α such that $|\alpha| > 1$, the function $F(z) - \alpha G(z) \neq 0$ in $|z| \geq 1$, that is, $F(z) - \alpha G(z)$ has all its zeros in $|z| < 1$. So, by the Gauss-Lucas Theorem $F'(z) - \alpha G'(z)$ has all its zeros in $|z| < 1$, which implies that $|F'(z)| \leq |\alpha||G'(z)|$

for $|z| \geq 1$. On making $\alpha \to 1$ we get $|F'(z)| \leq |G'(z)|$ for $|z| \geq 1$, which in particular gives us $|F'(e^{i\theta})| \leq |G'(e^{i\theta})|$ where $0 \leq \theta < 2\pi$.

Now recall that $F(z) = P(z) - \alpha m$ and consider

$$\begin{aligned} G(z) &= z^n \overline{F(1/\overline{z})} \\ &= z^n \overline{P(1/\overline{z})} - \overline{\alpha} m z^n \\ &= Q(z) - \overline{\alpha} m z^n. \end{aligned}$$

This implies that $G'(z) = Q'(z) - \overline{\alpha} n m z^{n-1}$. So, we have that

$$\begin{aligned} |P'(z)| &= |F'(z)| \\ &\leq |G'(z)|, \quad \text{for} \quad |z| \geq 1 \\ &= |Q'(z) - \overline{\alpha} n m z^{n-1}|, \quad \text{for} \quad |z| \geq 1. \end{aligned}$$

We can choose the argument of α such that

$$|Q'(z) - \overline{\alpha} n m z^{n-1}| \quad = \quad |Q'(z)| - |\alpha| n m, \quad \text{for} \quad |z| = 1$$

where the right hand side is non-negative since $|Q'(z)| \geq |\alpha| nm$. Making $|\alpha| \to 1$, we have that

$$|P'(z)| \quad \leq \quad |Q'(z)| - nm \quad \text{on} \quad |z| = 1. \tag{3.10}$$

It is well known (see Govil and Rahman [15, p. 511]) that

$$|P'(z)| + |Q'(z)| \leq Mn.$$

Putting this together with inequality (3.10) we get

$$|P'(z)| + |P'(z)| + nm \leq Mn,$$

which implies that

$$2|P'(z)| \leq Mn - mn,$$

giving us

$$|P'(z)| \leq \frac{n}{2}(M - m).$$

Hence, $\max_{|z|=1} |P'(x)| \leq \frac{n}{2}(M - m)$, and Theorem 3.6 is proved. □

We will now use Theorem 3.6 to prove Theorem 3.5. This proof is given by Aziz and Dawood [2, p. 310-311]. Let $M = \max_{|z|=1} |P(z)|$ and $m = \min_{|z|=1} |P(z)|$. Since $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, therefore, by Theorem 3.6 we have

$$|P'(z)| \leq \frac{n}{2}(M - m), \quad \text{for} \quad |z| = 1.$$

Now, $P'(z)$ is a polynomial of degree $n-1$; therefore, it follows from inequality (3.6) that for all $r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} |P'(re^{i\theta})| &\leq \max_{|z|=1} |P'(e^{i\theta})| r^{n-1} \\ &\leq \frac{n}{2} r^{n-1}(M-m). \end{aligned}$$

Also, for each θ where $0 \leq \theta < 2\pi$,

$$P(Re^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(te^{i\theta})dt.$$

This gives

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &\leq \int_1^R |P'(te^{i\theta})|dt \\ &\leq \frac{(M-m)}{2} \int_1^R nt^{n-1}dt \\ &= \frac{1}{2}(R^n - 1)(M-m), \end{aligned}$$

for each θ where $0 \leq \theta < 2\pi$ and $R \geq 1$. Hence

$$\begin{aligned} |P(Re^{i\theta})| &\leq |P(e^{i\theta})| + \frac{1}{2}(R^n - 1)(M-m) \\ &\leq M + \frac{1}{2}(R^n - 1)(M-m) \\ &= \frac{2M}{2} + \frac{MR^n - M}{2} + \frac{-mR^n + m}{2} \\ &= \frac{M + MR^n}{2} + \frac{-m(R^n - 1)}{2} \\ &= \frac{M(R^n + 1)}{2} - \frac{m(R^n - 1)}{2} \end{aligned}$$

for each θ where $0 \leq \theta < 2\pi$ and $R \geq 1$. Thus, we have that

$$\max_{|z|=R} |P(z)| \leq \left(\frac{R^n+1}{2}\right) M - \left(\frac{R^n-1}{2}\right) m$$

for $R \geq 1$. Hence, Theorem 3.5 is proved. $\square$

As Theorem 3.4 is a generalization of inequality (3.7), one would like to obtain a generalization of Theorem 3.5 for entire functions of exponential type, and this is done by Govil, Qazi, and Rahman [14, Theorem 2.1]. We state their result below.

Theorem 3.7. *Let f be an entire function of exponential type τ such that (i) $f(z) \neq 0$ for all z in the open upper half-plane, (ii) $0 \leq \mu \leq |f(x)| \leq M$ for all $x \in \mathbb{R}$, (iii) $h_f\left(\frac{\pi}{2}\right) = 0$. Then*

$$|f(z)| \leq M\left(\frac{e^{\tau|y|}+1}{2}\right) - \mu\left(\frac{e^{\tau|y|}-1}{2}\right), \quad \textit{for } y := \Im z \leq 0.$$

The bound is attained for functions of the form

$$f(z) := \frac{M+\mu}{2} e^{i\alpha} + \frac{M-\mu}{2} e^{i\beta} e^{i\tau z}, \quad \textit{for } \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}.$$

As we have seen, the proof of Theorem 3.5 depends on the proof of Theorem 3.6. So, in order to prove Theorem 3.7 (which is a sharpening of Theorem 3.4), one should first obtain a generalization of Theorem 3.6. This is done by Govil, Qazi, and Rahman [14, Theorem 2.2], and we state their result below.

Theorem 3.8. *Let f be an entire function of exponential type τ such that (i) $f(z) \neq 0$ for all z in the open upper half-plane, (ii) $0 \leq \mu \leq |f(x)| \leq M$ for all $x \in \mathbb{R}$, (iii) $h_f\left(\frac{\pi}{2}\right) = 0$. Then*

$$|f'(x)| \leq \frac{M-\mu}{2}\tau, \quad \text{for } \; x \in \mathbb{R}.$$

The bound is attained for functions of the form

$$f(z) = \frac{M+\mu}{2}e^{i\alpha} + \frac{M-\mu}{2}e^{i\beta}e^{i\tau z}, \quad \text{for } \; \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}.$$

The proof of Theorem 3.8 depends on the following definitions and lemmas.

An entire function ω of exponential type having no zeros for $y := \Im z < 0$ and satisfying one of the conditions (3.3) or (3.4) is said to belong to the class $\mathcal{P}$. An additive homogeneous operator $\mathcal{B}[f(z)]$ which carries entire functions of exponential type into entire functions of exponential type and leaves the class $\mathcal{P}$ invriant is called a $\mathcal{B}$-operator. An operator $\mathcal{B}$ is said to be additive if $\mathcal{B}[f+g] = \mathcal{B}[f] + \mathcal{B}[g]$, and homogeneous if $\mathcal{B}[cf] = c\mathcal{B}[f]$.

Lemma 3.1. *Let $\eta > 0$. The operator $\mathcal{T}_\eta$ which carries the function ω into the function $\omega(z - i\eta)$ is a $\mathcal{B}$-operator.*

The following lemma can be found, for example, in Boas [5, Theorem 11.7.5].

Lemma 3.2. *Differentiation is a $\mathcal{B}$-operator.*

The next lemma can also be found, for example, in Boas [5, Theorem 11.7.2].

Lemma 3.3. *Let Ω be an entire function of class $\mathcal{B}$ and of order 1 type σ. Furthermore, let ω be an entire function of exponential type $\tau \leq \sigma$ such that*

$$|\omega(x)| \;\; \leq \;\; |\Omega(x)|, \quad \textit{for} \;\; x \in \mathbb{R}.$$

Then, for any $\mathcal{B}$-operator $\mathcal{B}$, we have

$$|\mathcal{B}[\omega](x)| \;\; \leq \;\; |\mathcal{B}[\Omega](x)|, \quad \textit{for} \;\; x \in \mathbb{R}.$$

The final lemma is given by Govil, Qazi, and Rahman [14, Theorem 1.1] which we state below.

Lemma 3.4. *Let f be an entire function of exponential type having no zeros in the closed upper half-plane $\overline{H}$, and suppose that $|f(x)| \geq \mu > 0$ on the real axis. Furthermore, let $h_f\left(\frac{\pi}{2}\right) = a$. Then,*

$$|f(x+iy)| \;\; > \;\; \mu e^{ay}, \quad \textit{for} \;\; y > 0, \quad x \in \mathbb{R}$$

except for $f(z) := ce^{-iaz}$ where $c \in \mathbb{C}$ and $|c| = \mu$.

Lemma 3.4 is of interest in itself because it can be seen as a minimum modulus principle for entire functions of exponential type not vanishing in a half-plane. It is in fact, for the proof of Lemma 3.4 that Govil, Qazi, and Rahman [14, Theorem 1.1] needed lemmas 3.1, 3.2, and 3.3, and finally, using Lemma 3.4, they proved Theorem 3.8.

We omit the proof of Theorem 3.8 as it is too technical. However, we will use Theorem 3.8 to prove Theorem 3.7 which is a generalization of Theorem 3.5. We state the proof of Theorem 3.7 below.

Since f' is also an entire function of exponential type τ, it follows from Theorem 3.8 in conjunction with Theorem 3.3 that

$$|f'(x-it)| \leq \frac{M-\mu}{2} re^{\tau t}, \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0.$$

Hence, for any $y > 0$,

$$\begin{aligned}
|f(x-iy)| &\leq |f(x)| + \int_0^y |f'(x+it)| dt \\
&\leq M + \int_0^y \frac{M-\mu}{2} re^{\tau t} dt \\
&= M + \frac{M-\mu}{2}(e^{\tau y}-1) \\
&= \frac{2M + Me^{\tau y} - M - \mu e^{\tau y} + \mu}{2} \\
&= \frac{M + Me^{\tau y} + \mu - \mu e^{\tau y}}{2} \\
&= M\left(\frac{e^{\tau y}+1}{2}\right) - \mu\left(\frac{e^{\tau y}-1}{2}\right),
\end{aligned}$$

which is equivalent to

$$|f(z)| \leq M\left(\frac{e^{\tau|y|}+1}{2}\right) - \mu\left(\frac{e^{\tau|y|}-1}{2}\right), \quad \text{for} \quad y < 0$$

which is the desired result. Thus, Theorem 3.7 is proved. □

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