Fractional Domination, Fractional Packings, and

Fractional Isomorphisms of Graphs

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Fractional Domination, Fractional Packings, and Fractional Isomorphisms of Graphs

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Vita

Roberto Ramon Rubalcaba was born on December 13, 1975 in San Diego, California, the only child of a single mother, Rosalie Rubalcaba. From a family of hard-working carpenters, his mother encouraged his studies at an early age.

He attended San Diego City College summer of 1994. He encountered a mentor, Fabio Martinez, who encouraged him not to stop until he had obtained a doctorate. It was also there that he found aspirations of becoming a teacher, as he spent countless hours as a volunteer tutor. He graduated in the fall of 1996.

He later attended San Diego State University where he majored in Mathematics and minored in Geography. In the summer of 1999, he attended a Research Experience for Undergraduates at Auburn University, thanks in part to Robert Grone at SDSU (formerly at AU). It was there that he found the joy of research which resulted in his first paper [15]. He graduated with distinction in August of 1999 and was awarded with the Leadership Award as he encouraged high standards among his peers.

He then attended graduate school at Auburn University in the Department of Discrete and Statistical Sciences. In the summer of 2002, he became the first in his family to receive a Masters degree. While at Auburn University he published three more papers [119], [120], [121].

Dissertation Abstract

Fractional Domination, Fractional Packings, and Fractional Isomorphisms of Graphs

Roberto Ramon Rubalcaba

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The fractional analogues of domination and packing in a graph form an interesting pair of dual linear programs, in that the feasible vectors for both LPs have interpretations as functions from the vertices of the graph to the unit interval. The relationships between the solution sets of these dual problems are investigated. Another pair of dual linear programs, the fractional analogues of total domination and open packing in a graph, also both have interpretations as functions from the vertices to the unit interval. The relationships between the solution sets of these dual problems are also investigated. The fractional analogue of graph isomorphism plays a role in both investigations. Finally, various military strategies are discussed, as well as their fractional analogues.

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Style manual or journal used Discrete Mathematics (together with the style known as "auphd"). Bibliography follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX 2_ε) together with the departmental style-file `auphd.sty`.

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Chapter 1

Introduction to Domination, Fractional Graph Theory, and Linear Programming

1.1 Introduction

Consider the following facilities location problem, where we are trying to find the best locations of, for example, military bases, with the property that each base can handle threats at its location and any neighboring location. We say that a set of locations is a solution to the facilities location problem if threats can be handled at each location (for each location, either there is a base at this location or there is a neighboring location with a base). Suppose also that handling a threat at any location requires 90 troops. See Figure 1.1.

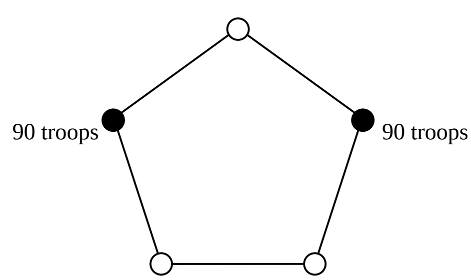


Figure 1.1: A solution to the facilities location problem for C_5, using 2 bases (180 troops).

Instead of having each station manned with 90 troops, we can spread those resources out over the locations. If additional bases do not cost much to build, then we may be able to save money (at least in the long run) by building more bases and relaxing the resource requirement at each base. Figure 1.2 gives a *fractional* solution to the problem above, using only 150 troops, with the same ability to handle a threat at any location. In this solution, we have a base at each location, but each has 1/3 of the resources. One can check that if there were a threat at any location, there would be enough resources from its base and the two neighboring bases to handle the threat. We call this solution *optimal* since there is no solution using less resources.

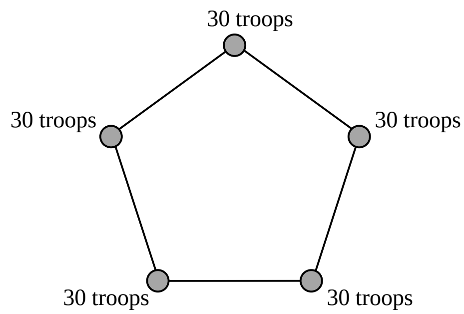
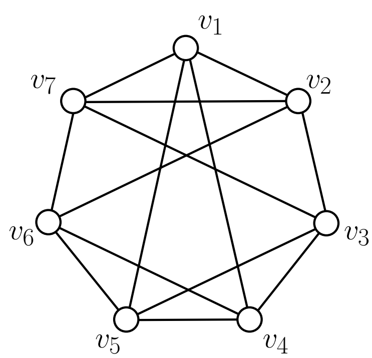


Figure 1.2: A fractional solution of the facilites location problem for C_5, using 5 bases (150 troops).

1.2 Definitions

A graph $G = (V, E)$ consists of a set $V(G)$ of vertices (sometimes called nodes) and a set $E(G)$ of edges which are two-subsets of V. Elements of E, $\{u, v\}$ are denoted as uv. Let $G = (V, E)$ be a simple finite graph of order $|V| = n$, without loops or multiple edges. Two distinct vertices u and v are said to be *adjacent* if $uv \in E(G)$. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of vertices v is adjacent to. When the graph is clear from context, we write $d(v)$ instead. We denote the number of edges in a graph $|E|$ by ε. The *maximum degree* of a graph G, denoted by $\Delta(G)$, is the maximum value of $d_G(v)$ taken over all vertices $v \in V(G)$. The *minimum degree* is denoted by $\delta(G)$. In notation and terminology, we try to follow [94], [95], [30] and [157]; for instance, C_n is the *cycle* on n vertices and P_n is the *path* on n vertices. As notation is not yet standard in Graph Theory, a complete list of notation used in this dissertation, can be found in Appendix A.

A graph can be completely determined by its vertex set and the knowledge of which pairs of vertices are adjacent. This same information can be stored in a matrix. If we order the vertices of the graph G by $\{v_1, \ldots, v_n\}$, the *adjacency matrix* (with respect to this ordering) is the $n \times n$ matrix $A(G) = [a_{i,j}]$ where $a_{ij} = 1$ if v_i is adjacent to v_j in G and 0 otherwise. The *(closed) neighborhood matrix*, denoted by $N(G)$, is defined by $N(G) = A(G) + I$, where I is the n by n identity matrix. When the graph is clear from context, we write A and N for the adjacency matrix and neighborhood matrix respectively.



$$A = \begin{bmatrix} 0&1&0&1&1&0&1 \\ 1&0&1&0&0&1&1 \\ 0&1&0&1&1&0&1 \\ 1&0&1&0&1&1&0 \\ 1&0&1&1&0&1&0 \\ 0&1&0&1&1&0&1 \\ 1&1&1&0&0&1&0 \end{bmatrix}$$

Figure 1.3: A labeled graph and its adjacency matrix.

There are other matrices which store the information from a graph. The vertex-edge incidence matrix is a $n \times \varepsilon$ matrix B with $b_{i,j} = 1$ if vertex v_i is incident with the edge e_j and 0 otherwise. If we keep the ordering of the vertices of the graph G in Figure 1.3 and then order the edges as $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_1v_7, v_1v_4, v_4v_6, v_6v_2, v_2v_5, v_2v_6, v_3v_6, v_3v_5, v_5v_1\}$, then we can form its vertex-edge incidence matrix B displayed below.

$$B = \begin{bmatrix} 1&0&0&0&0&0&1&1&0&0&0&0&0&1 \\ 1&1&0&0&0&0&0&0&0&1&1&0&0&0 \\ 0&1&1&0&0&0&0&0&0&0&0&1&1&0 \\ 0&0&1&1&0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0&1&0&0&1&1 \\ 0&0&0&0&1&1&0&0&1&0&1&1&0&0 \\ 0&0&0&0&0&1&1&0&0&0&0&0&0&0 \end{bmatrix}$$

Given a graph G and a subset of vertices $S \subset V(G)$, the induced subgraph $H = G[S]$ is the graph formed using the vertices in S, and whenever two of these vertices are adjacent in G, they are adjacent in H.

The *open neighborhood* of a vertex $v \in V(G)$ is defined as $N_G(V) = \{u \in V | uv \in E\}$, the set of all vertices adjacent to v. Note that $d_G(v) = |N_G(v)|$ for all $v \in V(G)$. The *closed neighborhood* of a vertex $v \in V(G)$ is defined as $N_G[v] = \{v\} \cup N_G(v)$. We denote the open and closed neighborhood respectively as $N(v)$ and $N[v]$ when the graph G is clear from context. For a set $S \subseteq V$, let $N(S) = \bigcup_{u \in S} N(u)$ and let $N[S] = \bigcup_{u \in S} N[u]$.

1.2.1 Domination and variations on domination

We say that a vertex "dominates" itself and all of its neighbors. A set of vertices $S \subseteq V$ is called a *dominating set* iff every vertex $v \in V$ is either an element of S or is adjacent to some element of S. That is, a set of vertices $S \subseteq V$ is dominating iff $N[S] = V$.

Domination began as a problem on a chessboard, when a question was posed in [118] as to the minimum number of queen pieces that can be placed on a chessboard so that every square is either occupied by a queen, or can be occupied by one of the queens in a single move. It was conjectured that the solution would consist of 5 queens and became known as the "Five Queens problem". We present two well-known solutions, depicted in Figure 1.4(a) and (b). Solution (a) to the five queens problem also has the property that no two queens can attack each other in a single move, thus this solution is an independent dominating set. Solution (b) to the five queens problem also has the property that any two queens can attack each other in a single move.

When a vertex is unable to dominate itself, we have a variation of domination, introduced by Cockayne, Dawes and Hedetniemi [39]. A set of vertices $S \subseteq V$ is called a *total dominating set* iff every vertex $v \in V$ is adjacent to some element of S. That is, a set of vertices $S \subseteq V$ is total dominating iff $N(S) = V$. The size of the smallest such set is the

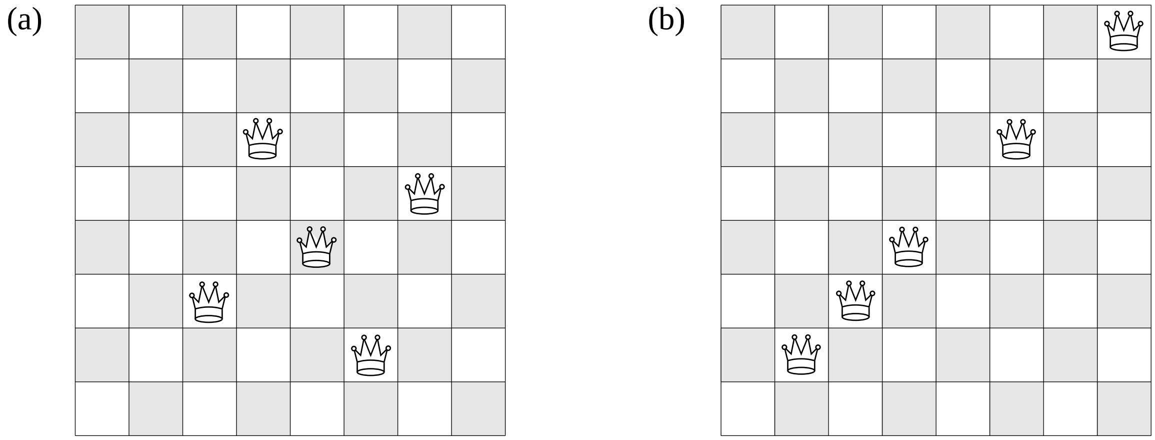


Figure 1.4: Dominating sets of queens on a standard chessboard.

total domination number, denoted as γ_t. If G has vertices of degree 0, called *isolates*, then no such set exists. As observed in [85], another way to define a total dominating set is a dominating set for which the induced subgraph, $G[S]$, contains no isolates.

In a dominating set, every vertex in V is dominated at least once. If we require that every vertex be dominated at least twice, we have a *double dominating* set ([85]). The size of a smallest such set is denoted as $dd(G)$ (or as $\gamma_{\times 2}(G)$).

If we think of our vertices in our dominating set S as computer servers, in communication with vertices (or computers) in $V - S$, then what happens if one server fails? To protect the network, we can have every vertex in $V - S$ be dominated twice, that is, every $v \in V - S$ is adjacent to at least two distinct vertices in S. The size of the smallest such 2-dominating set is denoted as γ_2. This is different than a double dominating set, since members of S need not be dominated twice. The generalization of 2-domination is called k-domination ([94]); in which S is dominating and every $v \in V - S$ is adjacent to at least k distinct members of S. The size of the smallest such k-dominating set is denoted as γ_k.

1.2.2 Let's get fuzzy!

Fractional graph theory has its roots in coloring, starting with independent results from [109], [159], and [38], where the fractional chromatic number was explored. Fractional domination was first defined in [60] and [102].

A *dominating function* on a graph G is a function $g : V \to \{0,1\}$ such that $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all vertices $v \in V$. The *characteristic function* φ_S defined by $\varphi_S(v) = 1$ when $v \in S$ and 0 otherwise is a dominating function iff S is a dominating set. A *minimum* dominating function on a graph G, naturally enough, is a dominating function g which attains the minimum value of $|g| = \sum_{v \in V} g(v)$. This minimum is denoted by $\gamma(G)$ and called the *domination number* of G.

Every function $\varphi : V \to [0,1]$ has a *vector representation* $\vec{\varphi} = (\varphi(v_1), ..., \varphi(v_n))^T$ for any fixed ordering $v_1, ..., v_n$ of the vertices of G. Throughout this dissertation, we shall often refer to a function and its vector interchangeably.

In the following, we represent a vector $\vec{x}$ by $\boldsymbol{x}$. Two vectors satisfy $\boldsymbol{x} > \boldsymbol{y}$ if and only if $x_i > y_i$ for all i. Likewise, $\boldsymbol{x} < \boldsymbol{y}$ if and only if $x_i < y_i$ for all i; $\boldsymbol{x} \leq \boldsymbol{y}$ and $\boldsymbol{x} \geq \boldsymbol{y}$ are defined similarly. For any function $g : V(G) \to \mathcal{A} \subset \boldsymbol{R}$, we call the value $g(v) \in \mathcal{A}$ the *weight* of the vertex. We will often define functions by assignments of weights. We call $|g|$ the *total weight* of the function. When g is defined by its vector $\boldsymbol{g}$, the weight of any vertex $v_i \in V$ is the i^{th} coordinate of the vector $\boldsymbol{g}$ (for some fixed ordering of the vertices).

The vector $\boldsymbol{f}$ of any dominating function f satisfies the constraint $N\boldsymbol{f} \geq \mathbf{1}$. A function $g : V \to [0,1]$ whose vector satisfies this inequality shall be called a *fractional dominating function*, henceforth FDF. A *minimum fractional dominating function* (MFDF) is an FDF g such that the value $|g| = \sum_{v \in V} g(v)$ is as small as possible. This minimum value is the *fractional domination number* of G, denoted by $\gamma_f(G)$.

A set $S \subseteq V$ is called a (*closed*) *neighborhood packing* if for any vertex $x \in G$, $|S \cap N[x]| \leq 1$. This set is sometimes referred to as a 2-packing, since for all $u, v \in S$ the distance from u to v is at least 3. A function $h : V \to \{0, 1\}$ is called a *packing function* if it is the characteristic function of some neighborhood packing. Note that any packing function f satisfies the matrix inequality $N\boldsymbol{f} \leq \mathbf{1}$. A maximum packing function on a graph G is a packing function h which attains the maximum value of $|h| = \sum_{v \in V} h(v)$, denoted by $\pi(G)$ and called the *packing number* of G (the packing number is the same as the 2-packing number, $P_2(G)$).

A function $h : V \to [0, 1]$ is a *fractional packing function* (FPF) provided that $h(N[v]) \leq 1$ for all $v \in V$. Just as for integer packing functions, the vector $\boldsymbol{h}$ of any such FPF h satisfies the constraint $N\boldsymbol{h} \leq \mathbf{1}$. A *maximum fractional packing function* (MFPF) is an FPF h such that the value attained by $|h| = \sum_{v \in V} h(v)$ is as large as possible. This maximum is the *fractional* (*closed neighborhood*) *packing number* of G and is denoted by $\pi_f(G)$.

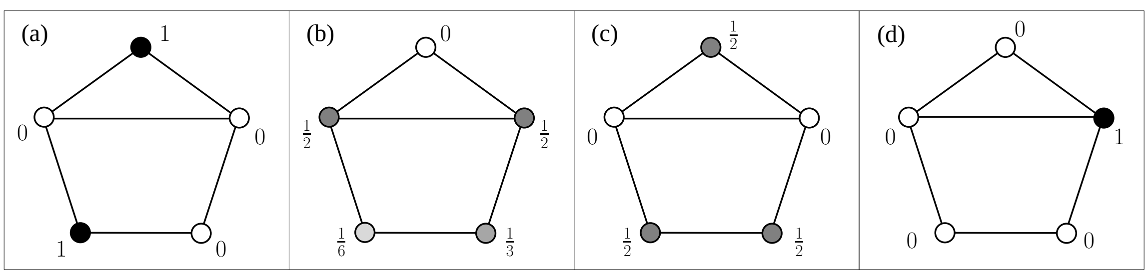


Figure 1.5: (a) A minimum dominating function, (b) an MFDF, (c) an MFPF, and (d) a maximum packing of C_5 with a chord.

Bange, Barkhauskas, and Slater ([6]) called f an *efficient dominating function*, if $f(N[v]) = 1$ for every $v \in V$. It is possible for some graphs to have no efficient dominating function (as with the graph in Figure 1.5). This lead to the definition of the efficient domination number (see [164]). We start with a maximal packing S (not necessarily maximum), and look at how much domination gets done, $|N[S]|$. The maximum value of $|N[S]|$, taken over all maximal packings S, is called the *efficient domination number*, denoted as $F(G)$. If a graph G has an efficient dominating function, then $F(G) = n$, since there is a packing which is also dominating. If no such function exists, then $F(G) < n$.

Alternatively, we can define the efficient domination number as the maximum value of $\sum_{i=1}^{n} (1 + d(v_i))\, g_i$, taken over all packing functions g. From this we can define the *efficient fractional domination number* as the maximum value of $\sum_{i=1}^{n} (1 + d(v_i))\, g_i$, taken over all (maximal) FPFs g. This maximum value is denoted by $F_f(G)$. If a graph G has an efficient fractional dominating function, then $F_f(G) = n$, since there is a fractional packing which is also fractional dominating. If no such function exists, then $F_f(G) < n$.

A *total dominating function* on a graph G without isolates is a function $g : V \to \{0, 1\}$ such that $g(N(v)) = \sum_{u \in N(v)} g(u) \geq 1$ for all vertices $v \in V$. The *characteristic function* φ_S defined by $\varphi_S(v) = 1$ when $v \in S$ and 0 otherwise is a total dominating function iff S is a total dominating set. A *minimum* total dominating function on a graph G is a total dominating function g which attains the minimum value of $|g| = \sum_{v \in V} g(v)$. This minimum is denoted by $\gamma_t(G)$ and called the *total domination number* of G. If G has isolates, then we say $\gamma_t = \infty$. In [157] and [67], the authors use Γ for this parameter; however, we will reserve this notation for the *upper domination number*, the size of a largest *minimal* dominating set.

The vector $\boldsymbol{f}$ of any total dominating function f satisfies the constraint $A\boldsymbol{f} \geq \mathbf{1}$. A function $g : V \to [0, 1]$ whose vector satisfies this inequality shall be called a *fractional total dominating function*, henceforth FTDF. A *minimum fractional total dominating function* (MFTDF) is an FTDF g such that the value $|g| = \sum_{v \in V} g(v)$ is as small as possible. This minimum value is the *fractional total domination number* of G, denoted by $\gamma_f^{\circ}(G)$.

A set $S \subseteq V$ is called an *open neighborhood packing* if for any vertex $x \in G$, $|S \cap N(x)| \leq 1$. A function $h : V \to \{0, 1\}$ is called an *open packing function* if it is the function of some neighborhood packing. Note that any packing function f satisfies the matrix inequality

$A\boldsymbol{f} \leq \mathbf{1}$, where $\boldsymbol{f}$ represents the vector of f. A maximum open packing function on a graph G is an open packing function h which attains the maximum value of $|h| = \sum_{v \in V} h(v)$, denoted by $\pi_t(G)$ and called the *open packing number* of G.

A function $h : V \to [0, 1]$ is a *fractional open packing function* (FOPF) provided that $h(N(v)) \leq 1$ for all $v \in V$. Just as for integer open packing functions, the vector $\boldsymbol{h}$ of any such FOPF h satisfies the constraint $A\boldsymbol{h} \leq \mathbf{1}$. A *maximum fractional open packing function* (MFOPF) is an FOPF h such that the value attained by $|h| = \sum_{v \in V} h(v)$ is as large as possible. This maximum is the *fractional open* (*neighborhood*) *packing number* of G and is denoted by $\pi_f^\circ(G)$.

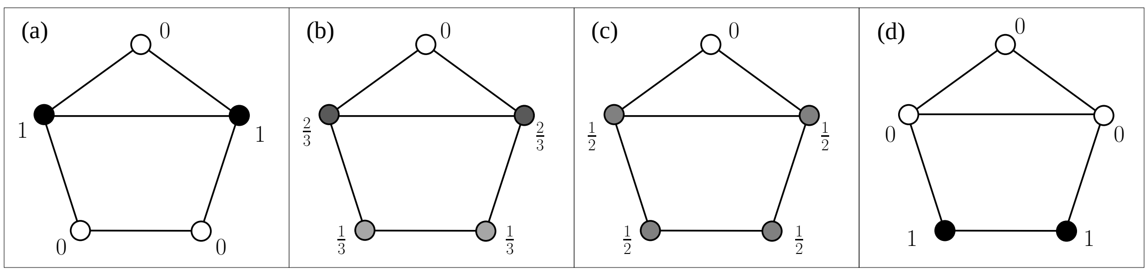


Figure 1.6: (a) A minimum total dominating function, (b) an MTFDF, (c) an MFOPF, and (d) a maximum open packing of C_5 with a chord.

A set of edges $M \subset E(G)$ is called a *matching* if no two edges in M are incident. The matching number μ is the size of a maximum matching. A function $g : E \to [0, 1]$ is a *fractional matching function* provided that for each vertex $v \in V$, $\sum_{uv \in E} g(uv) \leq 1$. The fractional matching number μ_f is the maximum of $\sum_{e \in E} g(e) = |g|$ taken over all fractional matching functions on G. If we restrict the values of $g(e)$ to be only 0 or 1, then g is the characteristic function of a matching.

A function $g : E \to \{0, 1\}$ is a *matching function* provided that for each vertex $v \in V$, $\sum_{uv \in E} g(uv) \leq 1$. Here we have that the matching number is the largest value of $\sum_{e \in E} g(e) = |g|$ taken over all matching functions f on G.

1.3 Integer and linear programming

A *linear program* is an optimization problem where we are maximizing or minimizing a function subject to some constraints. Let M be a real k by m matrix and $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ be real column vectors of the appropriate sizes. For our purposes, linear programs (or LPs) can be expressed in the following two forms:

$$\text{maximize } \boldsymbol{c}^T\boldsymbol{x} \text{ subject to } M\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0} \tag{1.1}$$

$$\text{minimize } \boldsymbol{b}^T\boldsymbol{y} \text{ subject to } M^T\boldsymbol{y} \geq \boldsymbol{c},\ \boldsymbol{y} \geq \boldsymbol{0} \tag{1.2}$$

The linear program in (1.2) is called the (*linear programming*) *dual* of the linear program in (1.1). The expression $\boldsymbol{c}^T\boldsymbol{x}$ in (1.1) is called the *objective function* and any vector $\boldsymbol{x}$ satisfying the constraints $M\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is called a *feasible solution*. The maximum (respectively minimum) value of the objective function taken over all feasible solutions is called the "value" of the LP. Any feasible solution to the LP on which the objective function attains the value is called an *optimal solution*.

If we require, in addition, that the optimal solutions be *integer* valued, then the above two linear programs are called *integer programs* (or IPs). When we start with an integer program and then remove or drop the constraint that the optimal solutions need to be integer valued, we obtain the *linear relaxation* of the IP. We now state a few fundamental theorems from linear programming.

Theorem 1.3.1 ([69] Strong Duality Theorem) *A linear program and its dual have the same value.*

A very important result from the theory of linear programming gives a condition which the optimal vectors of an LP and its linear dual must obey (we do not have a source for the original proof).

Theorem 1.3.2 ([141] Principle of complementary slackness) *Let $\boldsymbol{x'}$ be any optimal solution to the linear program: maximize $\boldsymbol{c}^T\boldsymbol{x}$ subject to $M\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, and let $\boldsymbol{y'}$ be any optimal solution to the dual linear program: minimize $\boldsymbol{b}^T\boldsymbol{y}$ subject to $M^T\boldsymbol{y} \geq \boldsymbol{c}$, $\boldsymbol{y} \geq \boldsymbol{0}$. Then:*

$$\boldsymbol{x'} \cdot (M^T\boldsymbol{y'} - \boldsymbol{c}) = \boldsymbol{y'} \cdot (M\boldsymbol{x'} - \boldsymbol{b}) = \boldsymbol{0}.$$

Many problems in graph theory can be formulated as integer programs. In fractional graph theory, many fractional parameters can be defined by the value of a relaxed linear program. If the matrix and vectors of an LP all have rational entries, then the value will be rational, hence, the reason the term "fractional" instead of real in (1.4) ([141]).

The problem of determining the domination number can be formulated as an integer program using the neighborhood matrix N; $\gamma(G)$ is the value of the integer program:

$$\text{minimize } \boldsymbol{1}^T\boldsymbol{y} \text{ subject to: } N\boldsymbol{y} \geq \boldsymbol{1},\ \boldsymbol{y} \geq \boldsymbol{0},\ y_i \in \mathbb{Z}^+ \tag{1.3}$$

From this, we can define fractional domination number as the value of the linear programming relaxation of the above integer program (1.3); γ_f is the value of the linear program:

$$\text{minimize } \boldsymbol{1}^T\boldsymbol{y} \text{ subject to: } N\boldsymbol{y} \geq \boldsymbol{1},\ \boldsymbol{y} \geq \boldsymbol{0} \tag{1.4}$$

Determining $\pi_f(G)$ can be likewise formulated in LP terms:

$$\text{maximize } \boldsymbol{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \leq \boldsymbol{1},\ \boldsymbol{x} \geq \boldsymbol{0} \tag{1.5}$$

Each of the LP's (1.4), (1.5) is the other's dual linear problem, since N is a symmetric matrix; therefore, $\gamma_f(G) = \pi_f(G)$ for all graphs G. Determining the packing number can be formulated in IP terms, by adding the additional constraint to (1.5) that the optimal solution needs to be integer valued; $\pi(G)$ is the value of the integer program:

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0},\ x_i \in \mathbb{Z}^+ \tag{1.6}$$

By the theory of linear relaxations, $\pi(G) \leq \pi_f(G)$ and $\gamma_f(G) \leq \gamma(G)$.

The problem of determining the total domination number can be formulated as an integer program using the adjacency matrix A; $\gamma_t(G)$ is the value of the integer program:

$$\text{minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to: } A\boldsymbol{y} \geq \mathbf{1},\ \boldsymbol{y} \geq \mathbf{0},\ y_i \in \mathbb{Z}^+ \tag{1.7}$$

From this, we can define fractional total domination number as the value of the linear programming relaxation of the above integer program (1.7); γ_f° is the value of the linear program:

$$\text{minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to: } A\boldsymbol{y} \geq \mathbf{1},\ \boldsymbol{y} \geq \mathbf{0} \tag{1.8}$$

Determining $\pi_f^\circ(G)$ can be likewise formulated in LP terms:

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } A\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0} \tag{1.9}$$

Each of the LP's (1.8), (1.9) is the other's dual linear problem, since A is a symmetric matrix; therefore, $\gamma_f^\circ(G) = \pi_f^\circ(G)$ for all graphs G. Determining the packing number can be formulated in IP terms, by adding the additional constraint to (1.9) that the optimal solution needs to be integer valued; $\pi_t(G)$ is the value of the integer program:

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0},\ x_i \in \mathbb{Z}^+ \tag{1.10}$$

By the theory of linear relaxations, $\pi_t(G) \leq \pi_f^\circ(G)$ and $\gamma_f^\circ(G) \leq \gamma_t(G)$.

Putting these inequalities and equalities together we get the well-known string of inequalities for all graphs G

$$\pi(G) \leq \pi_f(G) = \gamma_f(G) \leq \gamma(G) \tag{1.11}$$

$$\pi_t(G) \leq \pi_f^{\circ}(G) = \gamma_f^{\circ}(G) \leq \gamma_t(G) \tag{1.12}$$

As proved in [157], equality holds in (1.11) for strongly chordal graphs (see section 3.4.5 for a definition) and equality holds in (1.12) for chordal bipartite graphs.

Given a graph G, the problem of finding the efficient domination number $F(G)$ can be formulated as in IP (see [164]), where $d_i = d(v_i)$:

$$\text{maximize } (\boldsymbol{d}+\mathbf{1})^T\boldsymbol{x} \text{ subject to } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0},\ x_i \in \mathbb{Z}^+ \tag{1.13}$$

Relax (1.13) to obtain the LP formulation of efficient fractional domination. F_f is the value of the LP:

$$\text{maximize } (\boldsymbol{d}+\mathbf{1})^T\boldsymbol{x} \text{ subject to } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0} \tag{1.14}$$

An alternative LP formulation of efficient fractional domination:

$$\text{maximize } \mathbf{1}^T N\boldsymbol{x} \text{ subject to } N\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0} \tag{1.15}$$

The problem of finding the fractional matching number can be formulated as a linear program, where B is the vertex-edge incidence matrix of G (with respect to some fixed ordering of the vertices and edges); $\mu_f(G)$ is the value of the linear program:

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to } B\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0} \tag{1.16}$$

1.4 New graphs from old

As in many areas of mathematics, such as group theory, new objects are often obtained from old by considering sums and products.

1.4.1 Graph sums and products

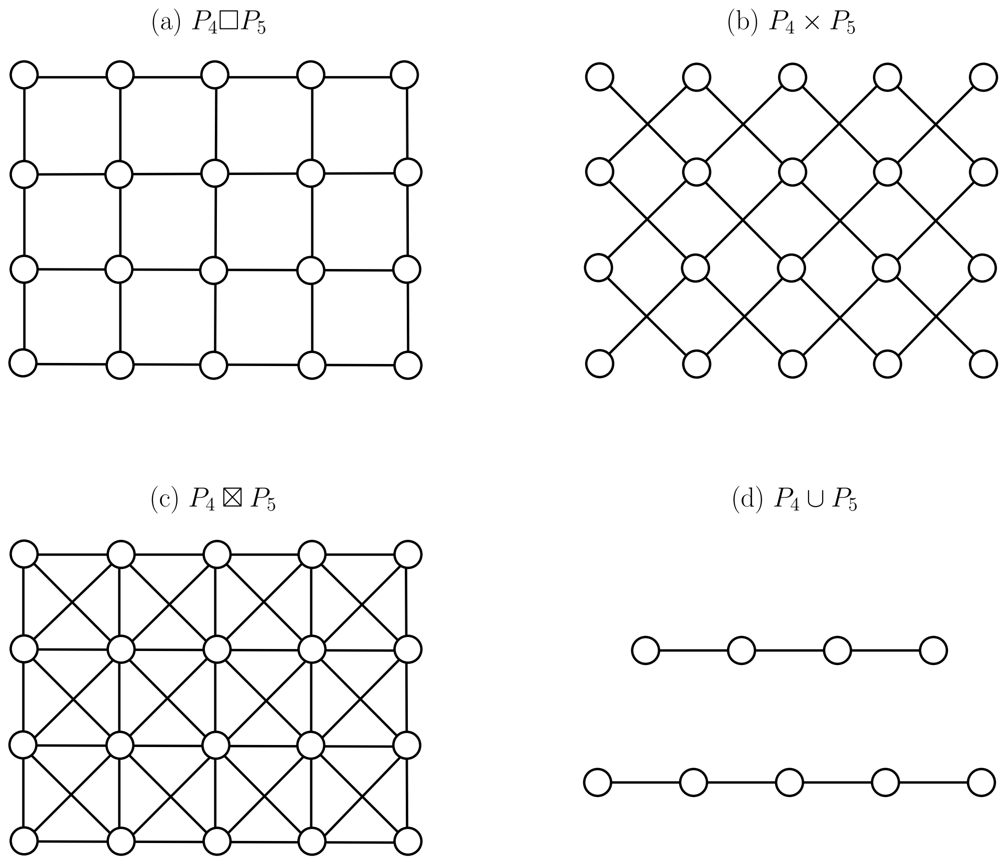


Figure 1.7: Graph products: the (a) Cartesian product $P_4 \square P_5$, (b) categorical product $P_4 \times P_5$, (c) strong direct product $P_4 \boxtimes P_5$, and (d) disjoint union $P_4 \cup P_5$.

The *Cartesian product* of G and H is denoted by $G \square H$; the vertices are the ordered pairs $\{(x, y) | x \in V(G), y \in V(H)\}$, and two vertices (u, v) and (x, y) are adjacent if and only if one of the following is true: $u = x$ and v is adjacent to y in H; or $v = y$ and u is adjacent to x in G. When G is the path on m vertices and H is the path on n vertices, $G\Box H$ is called the m by n *grid graph*, denoted as $G_{m,n}$.

The *categorical product* of G and H is denoted by $G \times H$. The vertices are the ordered pairs $\{(x,y)|x \in V(G), y \in V(H)\}$, and two distinct vertices (u,v) and (x,y) are adjacent if and only if $u \in N_G(x)$ and $v \in N_H(y)$. This product has been called many other names in the literature (often with different notation as well), like conjunctive product, weak direct product, direct product, cardinal product, or even just product.

The *strong direct product* of G and H is denoted by $G \boxtimes H$. The vertices are the ordered pairs $\{(x,y)|x \in V(G), y \in V(H)\}$, and two distinct vertices (u,v) and (x,y) are adjacent if and only if $u \in N_G[x]$ and $v \in N_H[y]$.

Next we look at a graph sum. The *disjoint union* of graphs G and H, denoted by $G \cup H$, is defined by the vertex and edge sets $V = V(G) \cup V(H)$ and $E = E(G) \cup E(H)$, where $V(G) \cap V(H) = \varnothing$.

1.4.2 Graph constructions

There are also several ways to get new graphs from old, such as taking the complement of a graph, taking the line graph, etc. In Chapter 3, we will investigate a well-known graph construction, called the Mycielski construction.

In [68], Frucht and Harary define the *corona* of two graphs G and H as the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H where the ith vertex of G is adjacent to every vertex in the ith copy of H (see Figure 1.8).

Given a graph G with vertices $\{v_1, \ldots, v_n\}$, Mycielski (in [139]) constructed a new graph $Y(G)$ with vertices $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \cup \{z\}$. Whenever $v_j v_k$ is an edge in G, each of $x_j x_k$, $x_j y_k$ and $x_k y_j$ are edges in $Y(G)$. Finally, each of the zy_i are edges in $Y(G)$. (See [130], [139] and [157]). This construction is primarily investigated in graph colorings; however, there has been at least one paper on fractional domination which uses

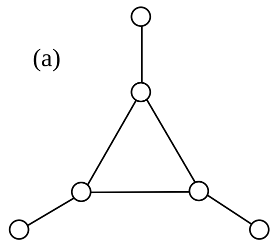


Figure 1.8: (a) The corona $K_3 \circ K_1$ and (b) the corona $K_1 \circ K_3$.

the construction, [67]. We call the sequence of graphs $Y_0 = P_2, Y_1 = Y(P_2)$, $Y_2 = Y(Y(P_2))$, $\ldots, Y_k = Y^k(P_2)$ the *Mycielski graphs*.

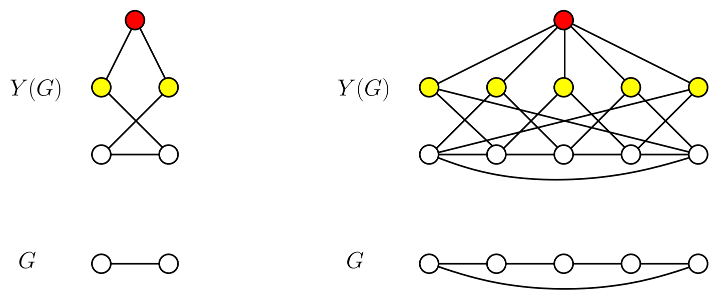


Figure 1.9: $Y(P_2)$ and $Y(Y(P_2)) = Y(C_5)$.

Motivated by [157] and [22], we define a *trampoline* $T(K_n)$ on $2n$ vertices ($n \geq 3$) as follows: begin with a complete graph on the vertices $\{v_1, \ldots, v_n\}$, add the vertices $\{u_1, \ldots, u_n\}$ and add the edges u_iv_i and u_iv_{i+1} (with $v_{n+1} = v_1$); see Figure 1.10a. Trampolines are referred to as *n-suns* in [22]. A *partial trampoline* $T_H(G)$ is the graph on $2n$ vertices formed from any Hamiltonian graph G with Hamilton cycle $H = v_1v_2...v_n$. This can be thought of as taking a trampoline and removing edges from "inside" the K_n (see see Figure 1.10b). When there is only one Hamiltonian cycle, the H will be omitted.

As in [8], the *generalized Hajós graph* is the graph $[K_n]$ on $n + \binom{n}{2}$ vertices formed by starting with a clique on three or more vertices, then adding a vertex u_{ij} for each pair of vertices v_i, v_j in K_n add the edges $u_{ij}v_i$ and $u_{ij}v_j$ (see see Figure 1.11a). As with partial

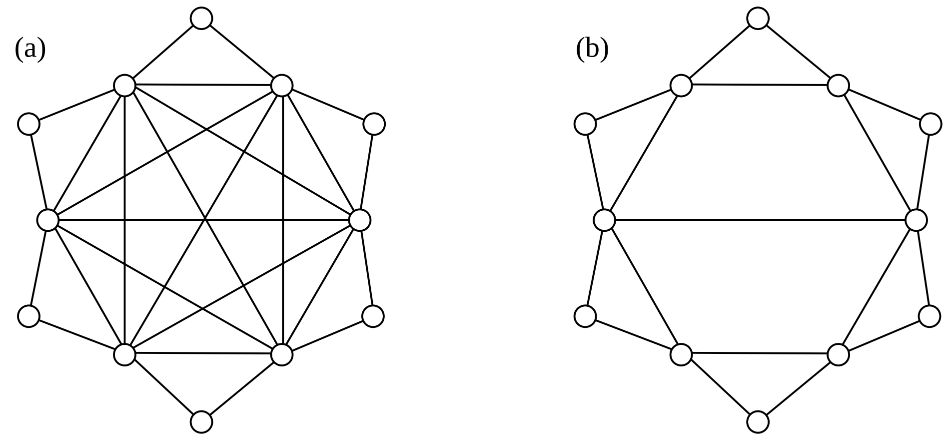


Figure 1.10: (a) The trampoline on 12 vertices $T(K_6)$ and (b) the partial trampoline $T(P_2 \square P_3)$.

trampolines we can start with any Hamiltonian graph G on three or more vertices and then apply the construction on G to obtain the *partial* generalized Hajós graph $[G]$ with $n + \binom{n}{2}$ vertices (see Figure 1.11b).

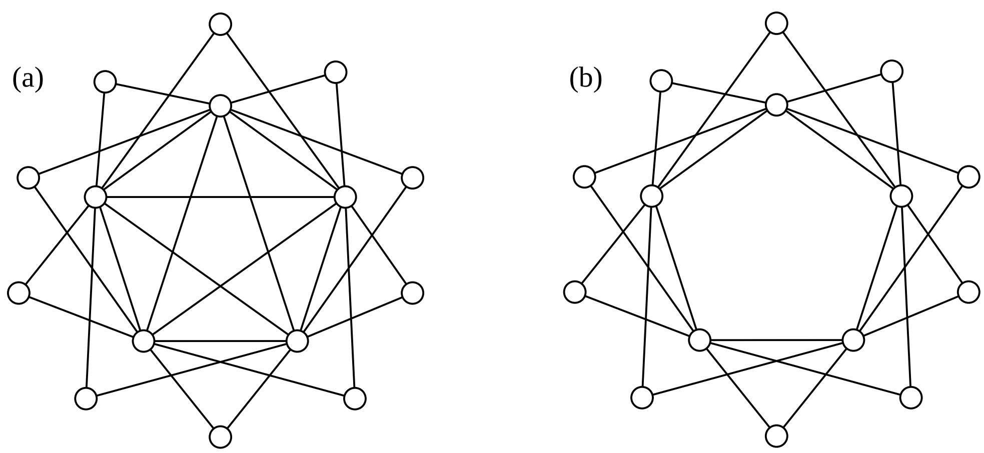


Figure 1.11: (a) The generalized Hajós graph $[K_5]$ and (b) the partial generalized Hajós graph $[C_5]$.

1.5 Notes

Upper and lower bounds on the fractional domination number were found independently in [54] and [80]. For any graph on n vertices, $\frac{n}{\Delta(G)+1} \leq \gamma_f(G) \leq \frac{n}{\delta(G)+1}$. In [52], equality in vizing-like conjectures were found to hold: $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$ and $\gamma_f^{\circ}(G \times H) = \gamma_f^{\circ}(G)\gamma_f^{\circ}(H)$. In [28], Chang found an upper bound for the domination number of the $m \times n$ grid graph (for m and n at least 8): $\gamma(G_{m,n}) \leq \left\lceil \frac{(m+2)(n+2)}{5} \right\rceil - 4$. In [33], it was conjectured that equality holds in the above upper bound for sufficiently large m and n. In [89], Hare found bounds for $\gamma_f(G_{m,n})$ for $m, n > 2$ involving the Fibonacci numbers. There are many other products which can be investigated; see [114] for 256 different graph products.

There are many other variations on domination, which we do not discuss (for an excellent exposition on this topic see [94]). We do investigate a recent variation on domination in Chapter 6 called Roman domination. In this dissertation we only consider a vertex to dominate vertices which are at most distance one away. There is a large area of research where vertices in S can dominate vertices which are distance at most k away from it, called distance-k domination. If vertices are allowed to dominate with different distances, then we have broadcast domination (see [13] and [113]).

In our IP formulations, we first relaxed to its LP, took the dual, then un-relaxed to form the "linear dual" IP. In [24], Bulfin noted, that it is incorrect to speak of *the* dual of an IP. There exist several IPs which are dual to a given IP. For instance, we could take the Lagrangian dual or the surragate dual of the domination IP (1.3) (see [141]).

Chapter 2

Fractional Isomorphisms

2.1 Isomorphisms of graphs

As in any area of mathematics, it is important to know when two objects are the "same" or "different". The numbers 3 and $\frac{6}{2}$ are equal though they are not identical in form; the groups A_3 and Z_3 are isomorphic though not identical. So when are two possibly differently drawn graphs the "same"? If two graphs differ from one another only by the way they are drawn or by the way their vertices (or edges) are labeled, we say they are isomorphic. To be more precise, a graph G is *isomorphic* to H, denoted $G \cong H$, if there exists a one-to-one mapping φ (called an *isomorphism*) from $V(G)$ onto $V(H)$ such that φ preserves adjacency and non–adjacency; that is, $uv \in E(G)$ iff $\varphi(u)\varphi(v) \in E(H)$.

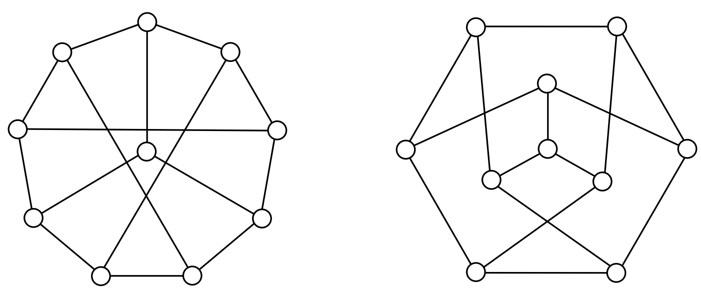


Figure 2.1: Two different drawings of the Petersen graph.

Let G and H be two graphs with adjacency matrices A and B respectively. A *permutation matrix* is a $\{0, 1\}$ matrix with exactly one 1 in each row and column. G and H are isomorphic if and only if there exists a permutation matrix P such that $P^{-1}AP = B$. This permutation matrix acts on the the columns and the rows of A, in a sense relabeling the vertices of G to make H. An equivalent definition of isomorphic graphs is the existence of a permutation matrix P which satisfies $AP = PB$.

2.2 Fractional Isomorphisms of Graphs

The requirement that P is a permutation matrix can be restated as: P is a matrix such that (where $\mathbf{1}$ is the $n \times 1$ matrix of all ones):

$$\begin{array}{ll} (1) & P\mathbf{1} = \mathbf{1} \\ (2) & \mathbf{1}^T P = \mathbf{1} \\ (3) & p_{i,j} \in \{0, 1\} \end{array}$$

Relaxing condition (3), the requirement that P be a $\{0, 1\}$ matrix, amounts to requiring the entries only be nonnegative; however we still want $P\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T P = \mathbf{1}$. Condition (1) gives rise to the use of a *row stochastic matrix*, a non-negative matrix whose row sums are all 1. Condition (2) gives rise to the use of a *column stochastic matrix*, which is the transpose of a row stochastic matrix. A $n \times n$ row stochastic matrix B, which has the property that B^T is also row stochastic, is said to be *doubly stochastic*. Thus, a doubly stochastic matrix S is a matrix whose entries are nonnegative, and whose rows and columns all sum to one; that is $S\mathbf{1} = \mathbf{1}$ and $S^T\mathbf{1} = \mathbf{1}$ (see [111]). Note that S must have non-negative entries, and hence each entry must be in the interval $[0, 1]$.

Let G and H be two graphs with adjacency matrices A and B respectively. We say G and H are *fractionally isomorphic* if and only if there exists a doubly stochastic matrix S so that $AS = SB$; we denote this relationship by $G \cong_f H$. The doubly stochastic matrix S may depend on which adjacency matrices A and B are used, which in turn depend on which orderings of the vertices of G and H are used. It is easy to see, however, that if G and H are fractionally isomorphic with respect to one choice of adjacency matrices, then they are fractionally isomorphic with respect to any other choice of adjacency matrices: suppose $AS = SB$, P and Q are permutation matrices so that $A' = P^T AP$ and $B' = Q^T BQ$; then $A = PA'P^T$ and $B = QB'Q^T$. So $PA'P^T S = SQB'Q^T$ or $A'(P^T SQ) = (P^T SQ)B'$. Further, $P^T SQ$ is doubly stochastic.

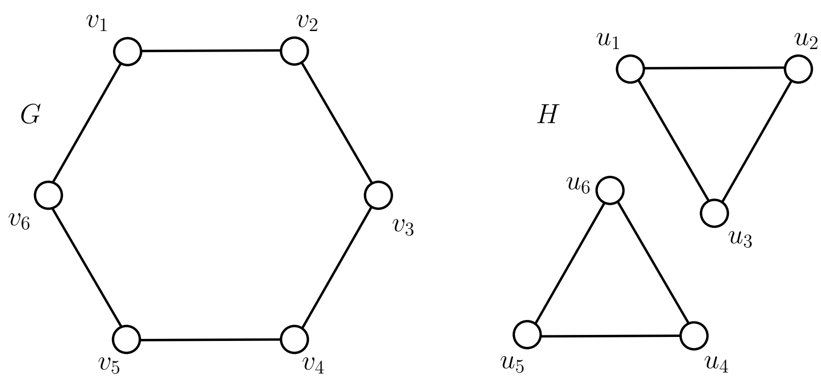


Figure 2.2: Fractionally isomorphic graphs.

As an example, let A be the adjacency matrix of the $G =$ 6-cycle and B be the adjacency matrix of H the disjoint union of two 3-cycles (with respect to the orderings in Figure 2.2). The doubly stochastic matrix $S = \frac{1}{6}J_6$ (where J_6 is the 6×6 matrix of all ones) satisfies $AS = SB$. Thus G and H are fractionally isomorphic; and we write $C_6 \cong_f C_3 \cup C_3$. Note that these two graphs are both 2-regular. In [146], it was proved by taking $S = \frac{1}{n}J_n$ that any two k-regular graphs on n vertices are fractionally isomorphic.

Theorem 2.2.1 ([146]) *If G and H are both regular graphs of degree k on n vertices, then $G \cong_f H$.*

For a non-regular example, consider the two graphs in Figure 2.3. If we let A be the adjacency matrix of G and let B be the adjacency matrix of H (both with respect to the given ordering of V), then the doubly stochastic matrix $S = \frac{1}{2}J_2 \oplus \frac{1}{4}J_4$ satisfies $AS = SB$. Thus, G and H are fractionally isomorphic.

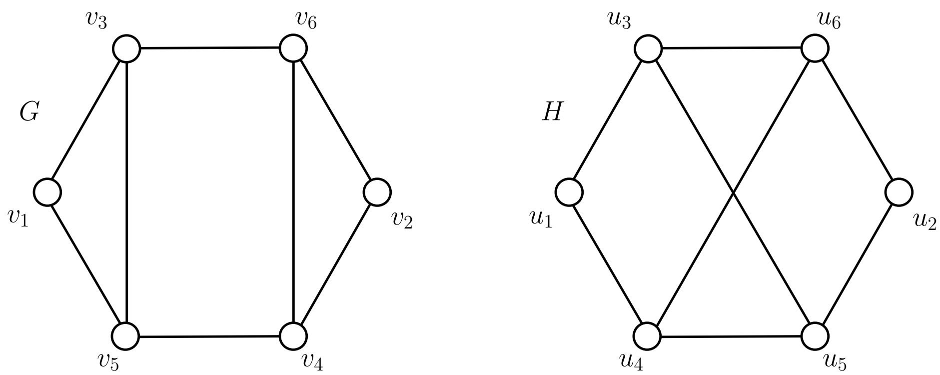


Figure 2.3: Two fractionally isomorphic non-regular graphs.

$$\left[\begin{array}{cc|cccc} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{cc|cccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right] = \left[\begin{array}{cc|cccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right] \left[\begin{array}{cc|cccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

Just as the relation of being isomorphic is an equivalence relation on the set of all unlabeled graphs, and thus differently drawn versions of the same graph are in the same equivalence class, so it is with fractional isomorphism.

Lemma 2.2.2 $\cong_f$ *is an equivalence relation.*

Proof. In the following, let A, B, and C be adjacency matrices of the graphs G, H, and K, respectively.

- Reflexivity: let $S = I_n$ which trivially satisfies $AS = SA$, thus $G \cong_f G$.
- Symmetry: suppose $G \cong_f H$, then there exists a doubly stochastic matrix S which satisfies $AS = SB$. Then take the transpose of both sides to get $S^T A = BS^T$. Since S^T is doubly stochastic, $H \cong_f G$.

- Transitivity: suppose $G \cong_f H$ and $H \cong_f K$. There exist doubly stochastic matrices S and T so that $AS = SB$ and $BT = TC$. Then consider $A(ST) = (AS)T = (SB)T = S(BT) = S(TC) = (ST)C$. Since the product of any two doubly stochastic matrices is doubly stochastic, $G \cong_f K$.

□

Thus the relation of being fractionally isomorphic is an equivalence relation on the set of all unlabeled graphs. So it is natural to wonder what is different or the same about any two fractionally isomorphic graphs. We look at the many differences in the next section. Regular graphs, specifically $n-$cycles and disjoint unions of smaller cycles, will play a crucial role in providing examples.

2.3 Non-invariants of fractional isomorphisms

A parameter ζ is called *invariant* if $\zeta(G) = \zeta(H)$ whenever G and H are fractionally isomorphic. A parameter ζ is called *non–invariant* if $\zeta(G) \neq \zeta(H)$ for two fractionally isomorphic graphs G and H. The terminology comes from isomorphism invariants, which are parameters and properties which do not depend on which labeling of the vertices is used. For example, the adjacency matrix is not invariant (with respect to graph isomorphism).

As we will see, fractionally isomorphic graphs can be quite different (just look at 3-regular graphs on 50 vertices). A list of some non–invariant parameters is given in Appendix B. A list of some properties which can be different is given in Theorem 2.3.5. First we give some necessary conditions for a graph parameter to be invariant (with respect to fractional isomorphism).

Let G be a graph with connected components $\mathscr{C}_1, \ldots, \mathscr{C}_j$. We say that a parameter ζ is *additive* if $\zeta(G) = \zeta(\mathscr{C}_1) + \cdots + \zeta(\mathscr{C}_j)$; *multiplicative* if $\zeta(G) = \zeta(\mathscr{C}_1)\zeta(\mathscr{C}_2)\cdots\zeta(\mathscr{C}_j)$; or *superlative* if $\zeta(G) = \max\{\zeta(\mathscr{C}_1), \ldots, \zeta(\mathscr{C}_j)\}$ or $\zeta(G) = \min\{\zeta(\mathscr{C}_1), \ldots, \zeta(\mathscr{C}_j)\}$. In the following, as with the corona definition, "k copies of G" refers to the disjoint union of G with itself k times, i.e. $kG = \cup_{i=1}^{k} G = \underbrace{G \cup \cdots \cup G}_{k}$.

Lemma 2.3.1 *Every cycle C_n is fractionally isomorphic to the disjoint union of a_1 copies of C_3, a_2 copies of C_4 and a_3 copies of C_5, for some non-negative integers a_1, a_2, and a_3.*

Proof. For any $n \geq 3$, there exist integers $a_i \geq 0$ so that $n = 3a_1 + 4a_2 + 5a_3$. Both C_n and $a_1C_3 \cup a_2C_4 \cup a_3C_5$ are regular of degree two on n vertices, and thus fractionally isomorphic. $\square$

Theorem 2.3.2 *If ζ is an additive invariant parameter, then $\zeta(C_n) = nk$, for some constant k, where k is an integer if ζ is an integer valued parameter.*

Proof. Let $\zeta(C_3) = 3k$. Then since $C_{60} \cong_f 20C_3 \cong_f 15C_4 \cong_f 12C_5$, it follows from additivity that $12\zeta(C_5) = 15\zeta(C_4) = 20\zeta(C_3)$, and so $\zeta(C_4) = \frac{4}{3}\zeta(C_3) = 4k$; and $\zeta(C_5) = \frac{5}{3}\zeta(C_3) = 5k$. Additivity then requires that $\zeta(C_n) = nk$. $\square$

The following is a consequence of Theorem 2.3.2:

Corollary 2.3.3 *If ζ is a multiplicative invariant parameter, then $|\zeta(C_n)| = r^n$, for some constant r.*

Proof. Let $\theta = \log|\zeta|$. Then θ is an additive parameter, which by Theorem 2.3.2 must take the value kn on C_n for some constant k. So $\zeta(C_n) = 10^{\theta(C_n)} = 10^{nk} = r^n$, where $r = 10^k$. $\square$

Theorem 2.3.4 *If ζ is a superlative invariant parameter, then for any m and n, $\zeta(C_n) = \zeta(C_m)$.*

Proof. Assume that ζ is maximized over the connected components of a graph; the proof for a minimizing parameter is similar. Since $C_{60} \cong_f 20C_3 \cong_f 15C_4 \cong_f 12C_5$, we have that

$\zeta(C_{60}) = \max\{\zeta(C_3)\} = \max\{\zeta(C_4)\} = \max\{\zeta(C_5)\}$, and hence $\zeta(C_3) = \zeta(C_4) = \zeta(C_5)$. By Lemma 2.3.1, for any m and n, $\zeta(C_n) = \max\{\zeta(C_3), \zeta(C_4), \zeta(C_5)\} = \zeta(C_m)$. $\square$

The following superlative graph parameters are non–invariant: vertex independence number, clique number, chromatic number, edge chromatic number, matching number, and girth. The fractional chromatic number is also non-invariant, since $\chi_f(C_{2m+1}) = 2 + \frac{1}{m}$ which would, of course, be different for cycles of different order. The fractional clique number, ω_f, turns out to be equal to the fractional chromatic number for all graphs by linear programming strong duality, and thus, ω_f is trivially also non-invariant. The following additive graph parameters are non–invariant: domination number, total domination number, 2-domination number, double domination number, restrained domination number, independent domination number, efficient domination number, packing number, open neighborhood packing number, Roman domination number (see Chapter 6), and the maximum size of a minimal dominating set. The parameter $|Aut(G)|$ is non-invariant. This parameter is almost multiplicative; if no two components of G are isomorphic, then $|Aut(G)| = \prod_H |Aut(H)|$, where the product is taken over the components of G.

The crossing number of a graph $\nu(G)$ is the fewest number of edge crossings taken over all drawings of G in the plane (the minimum can be taken over what is called "good" drawings of G, see [70]). This parameter is non–invariant, since $\nu(\overline{C_6}) = 0 \neq 1 = \nu(\overline{C_3 \cup C_3})$ (see Figure 7 in Appendix B).

According to [157], $\omega_f(G) = \alpha_f(\bar{G})$, and thus with Theorem 2.6.3, α_f, the fractional independence number, is also non-invariant. See Figures 1-18 in Appendix B, where we give examples for each non-invariant parameter mentioned above.

We are unsure of the parameters such as fractional irredundance, but conjecture that this is invariant. In an attempt to describe everything which can be different given two arbitrary fractionally isomorphic graphs, we also investigate which properties are non–invariant.

λ is an *eigenvalue* of a matrix A, if $Ax = \lambda x$ for some nonzero vector x. Eigenvalues can be computed by factoring the determinant of $A - \boldsymbol{It}$. The multiset of eigenvalues of a matrix M is denoted by $\sigma(M)$. The non-increasing sequence of n eigenvalues of an adjacency matrix A of a graph G of order n is called the spectrum of the graph, and is denoted by $Spec(G)$ (see [87]). The eigenvalue of A of maximum absolute value is called the spectral radius, denoted by ρ_G.

Theorem 2.3.5 *The following are non–invariant:*

- spectrum
- maximum eigenvalue of the Laplacian
- Hamiltonicity
- vertex transitivity
- chordality

Proof. Let A and B be adjacency matrices of $G = C_6$ and $H = C_3 \cup C_3$ (both with respect to the orderings used in Figure 2.2). Their spectra are different, since $Spec(G) = \{2, 1, 1, -1, -1, -2\}$, whereas $Spec(H) = \{2, 2, -1, -1, -1, -1\}$. We will see later that although their spectra may be different, they necessarily share the maximum eigenvalue, in this case, 2. Although the spectral radius of the adjacency matrices of any two fractionally isomorphic graphs are the same (see Theorem 2.5.1), the result does not necessarily hold of the Laplacian matrices of any two fractionally isomorphic graphs. The Laplacian of a graph G is the matrix $\mathcal{L} = D - A$, where D is the diagonal matrix of degrees ($D_{i,i} = d(v_i)$ and $D_{i,j} = 0$ whenever $i \neq j$). $\sigma(\mathcal{L}(G)) = \{4, 3, 3, 1, 1, 0\}$, whereas, $\sigma(\mathcal{L}(H)) = \{3, 3, 3, 3, 0, 0\}$.

$\rho_{\mathcal{L}(G)} = 4$, whereas $\rho_{\mathcal{L}(H)} = 3$. C_6 is Hamiltonian, yet the disjoint union of two 3−cycles, $C_3 \cup C_3$ is non–Hamiltonian (see also Figure 2.8 for two connected examples). For vertex–transitivity, see C_7 which is vertex transitive and $C_3 \cup C_4$ which is not. For chordality, C_6 is not chordal, whereas $C_3 \cup C_3$ is chordal.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

□

2.4 Equitable partitions

Before we can discuss what fractionally isomorphic graphs have in common, we need to define a class of special partitions of the vertices (see [75]). We denote the number of edges from a vertex v to a set S of vertices by $d(v, S)$,. A partition of the vertex set V into disjoint sets $V_1, \ldots, V_r$ is *equitable* if for any vertices $x, y \in V_i$, $d(x, V_j) = d(y, V_j)$ for all possible choices of i and j. Thus, the induced subgraph $G[V_i]$ is regular for all i, and the bipartite subgraph $G[V_i, V_j]$ made from only the edges between V_i and V_j is bi-regular (the vertex degrees within each part are equal) for all $i \neq j$.

If $\mathcal{P}$ and $\mathcal{Q}$ are equitable partitions of a common set S, $\mathcal{P}$ is called a *refinement* of $\mathcal{Q}$, provided every cell of $\mathcal{P}$ is contained in a cell of $\mathcal{Q}$. When $\mathcal{P}$ is a refinement of $\mathcal{Q}$, we say that $\mathcal{Q}$ is *coarser* than $\mathcal{P}$. Two vertices u and v are in the same *orbit* if there exists an automorphism φ of the graph such that $\varphi(u) = v$. Although choosing the cells to be orbits under automorphisms yields an equitable partition, for some graphs there exist coarser equitable partitions. Consider the graph in Figure 2.4, the equitable partition $\{\{v_1, v_2, v_3, v_4\}, \{v_5, v_6, v_7, v_8\}\}$ on the left is coarser than the equitable partition formed by the orbits $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6, v_7, v_8\}\}$. Note that v_1 and v_3 are not in the same orbit

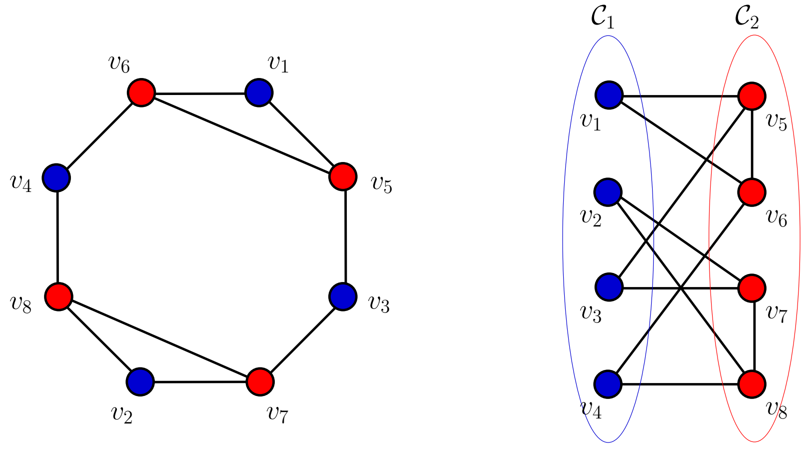


Figure 2.4: A coarsest equitable partition.

since there is no automorphism which would send v_1 to v_3 (v_1 is in a $3-$cycle, while v_3 is not). Every graph has a unique coarsest equitable partition (see [157]).

We shall define the *cell-adjacency matrix* $A^{(\mathcal{C})}$ for an equitable partition $\mathcal{C}$ as follows: the rows and columns of $A^{(\mathcal{C})}$ shall be indexed by the cells of $\mathcal{C}$. The entry $A^{(\mathcal{C})}_{i,j}$ will be equal to $d(x, V_j)$ for any $x \in V_i$. We say that graphs G and H have a common equitable

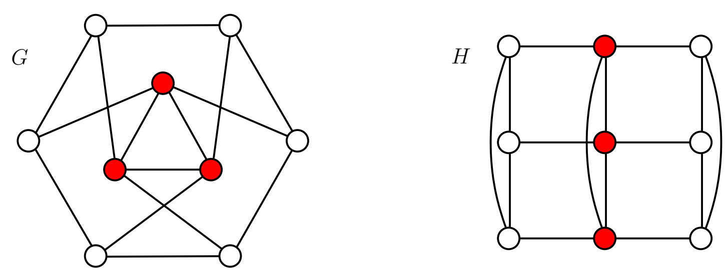


Figure 2.5: Fractionally isomorphic connected graphs.

partition $\mathcal{C}$ if they have the same sizes of cells and $A^{(\mathcal{C})}$ is identical to $B^{(\mathcal{C})}$ (with respect to the same ordering of $\mathcal{C}$). For the equitable partition $\mathcal{Q} = \{red, white\}$ of $V(G)$ and $V(H)$ in Figure 2.5, the cell adjacency matrices (with respect to the ordering: red, white) are the 2×2 matrices:

$$A^{(\mathcal{Q})} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \qquad B^{(\mathcal{Q})} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$$

2.5 What is the same about fractionally isomorphic graphs?

We begin by stating invariant parameters and properties proven in [146]. These serve as necessary conditions for two graphs to be fractionally isomorphic.

Theorem 2.5.1 ([146]) *The following are fractional isomorphism invariants.*

- $n = |V|$
- $\varepsilon = |E|$
- *degree sequences*
- *spectral radius*

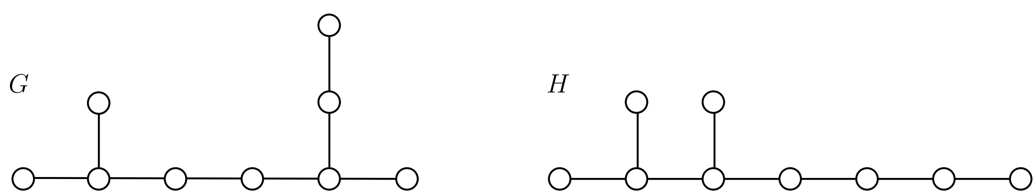


Figure 2.6: Non-fractionally isomorphic graphs with the same degree sequence and graph spectra.

The above necessary conditions for fractionally isomorphic graphs are not sufficient. Figure 2.6, due to Allen Schwenk (see [157]), depicts two graphs with the same degree sequences (thus the same number of vertices and edges), and spectra; yet they are *not* fractionally isomorphic. This is due to a result in [146]: if G is a graph and F is a forest and if $G \cong_f F$ then $G \cong F$ (this is based on a result in McKay's Masters Thesis and article [136]: the only equitable partitions of a forest are those arising from orbits).

The main theorem from [146] states necessary and sufficient conditions for two graphs to be fractionally isomorphic. We refer the reader to [157] for the definition of D^*, the matrix of iterated degree sequences.

Theorem 2.5.2 ([146]) *The following are equivalent:*

- $G \cong_f H$
- *G and H have a common coarsest equitable partition*
- *G and H have some common equitable partition*
- *$D^*(G)$=$D^*(H)$*

2.5.1 An entire class of invariant fractional parameters

If two graphs G and H (with adjacency matrices A and B, respectively) are fractionally isomorphic, then the following infinite class of parameters are invariant:

Theorem 2.5.3 *The* value *of any linear program of the form:* maximize $\boldsymbol{c}^T\boldsymbol{x}$ subject to: $M\boldsymbol{x} \leq \boldsymbol{b}$, $x \geq \mathbf{0}$ *(where $\boldsymbol{b}$ and $\boldsymbol{c}$ are constant vectors and M is any polynomial in A), is invariant.*

Proof. Suppose that G and H are fractionally isomorphic graphs. Let A and B be adjacency matrices of G and H, respectively, and let S be a doubly stochastic matrix such that $AS = SB$. Let $M_1 = p(A)$ and $M_2 = p(B)$, where $p(x)$ is a polynomial, and let $\xi(G)$ and $\xi(H)$ be the values of the linear programs (2.1) and (2.2), respectively. We aim to show that $\xi(G) = \xi(H)$.

$$\max \boldsymbol{c}^T\boldsymbol{x} \text{ subject to: } M_1\boldsymbol{x} \leq \boldsymbol{b}, x \geq \mathbf{0} \tag{2.1}$$

$$\max \boldsymbol{c}^T\boldsymbol{x} \text{ subject to: } M_2\boldsymbol{x} \leq \boldsymbol{b}, x \geq \boldsymbol{0} \tag{2.2}$$

Suppose we can show that multiplication by S maps feasible solutions of (2.2) into feasible solutions of (2.1). It will then follow, since $BS^T = S^TA$, that multiplication by S^T maps feasible solutions of (2.1) into feasible solutions of (2.2). Also, because $\boldsymbol{c}$ is a constant vector, and S is column stochastic, $\boldsymbol{c}(S\boldsymbol{x}) = (\boldsymbol{c}S)\boldsymbol{x} = \boldsymbol{c}^T\boldsymbol{x}$, for any vector $\boldsymbol{x} \geq 0$. It then follows that $\xi(G) \geq \xi(H)$. But then the equality $\xi(H) \geq \xi(G)$ follows as well, by the same argument with S replaced by S^T, and thus $\xi(G) = \xi(H)$.

Suppose that $\boldsymbol{x} \geq \boldsymbol{0}$ satisfies $M_1\boldsymbol{x} \leq \boldsymbol{b}$. Then $S\boldsymbol{x} \geq \boldsymbol{0}$ and $M_1(S\boldsymbol{x}) = p(A)S\boldsymbol{x} = S(p(B)\boldsymbol{x}) = S(M_2\boldsymbol{x}) \leq S\boldsymbol{b} = \boldsymbol{b}$, since $\boldsymbol{b}$ is constant and S is doubly stochastic. □

Thus any fractional parameter ζ_f which is the value of its associated LP (satisfying the above constraints) is invariant. Note that not all fractional parameters have an associated LP where M is a polynomial in A and $\boldsymbol{b}$, $\boldsymbol{c}$ are constant; the LP (1.16) for fractional matching uses the vertex–edge incidence matrix. In fact, it is a consequence of Theorem 2.5.3 that no non-invariant parameter can possibly have an LP formulation as in that theorem. Two parameters whose LPs do satisfy Theorem 2.5.3 are fractional domination (1.4) and fractional total domination (1.8).

Corollary 2.5.4 *γ_f and γ_f° are invariant parameters.*

Proof. Both parameters are the value of a linear program, where the matrix is a polynomial in A, namely $A + I$ and A respectively. The vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are the constant vector $\mathbf{1}$ in both LP's. □

2.6 Graph products

We wonder how fractional isomorphisms interact with graph products. We call a graph product, $\star$, *preserving* if $G \cong_f G'$ and $H \cong_f H'$ implies $G \star H \cong_f G' \star H'$.

Theorem 2.6.1 *$\Box$, $\times$, and $\boxtimes$ are each preserving products.*

Proof. Suppose $G \cong_f G'$ and $H \cong_f H'$. Then G and G' share a coarsest equitable partition $\mathcal{A}$, as do H and H' (call this $\mathcal{B}$). Let $\star$ represent the graph product: $\Box$, $\times$, or $\boxtimes$. Take the equitable partition of the product $G \star H$ by taking the partition $\mathcal{A}$ of $V(G)$ and repeating it for each copy of $V(G)$. Then replace G with G' in $G \star H$, which does not alter the equitable partition of $G \star H$, to obtain $G \star H \cong_f G' \star H$. Next, take the equitable partition of the product $G' \star H$ by taking the partition $\mathcal{B}$ of $V(H)$ and repeating it for each copy of $V(H)$. Then replace H with H' in $G' \star H$, which does not alter the equitable partition of $G' \star H$, to obtain $G' \star H \cong_f G' \star H'$. Do both replacements simultaneously (or at the same time) to obtain $G \star H \cong_f G' \star H'$. $\Box$

Although the corona is not technically a graph product,

Conjecture 2.6.2 *If $G \cong_f G'$ and $H \cong_f H'$ then $G \circ H \cong_f G' \circ H'$ and $H \circ G \cong_f H' \circ G'$.*

Figure 2.7: Fractionally isomorphic Cartesian products.

This together with Corollary 2.5.4, gives us a way to compute the fractional domination and fractional total domination numbers of very large prisms. For example, since $P_3 \Box C_{60} \cong_f 20(P_3 \Box C_3)$, $\gamma_f(P_3 \Box C_{60}) = 20\gamma_f(P_3 \Box C_3) = 20\frac{15}{7}$ and we have $\gamma_f^\circ(P_3 \Box C_{60}) = 20\gamma_f^\circ(P_3 \Box C_3) = 20(3)$. Note also since $P_3 \Box C_{60} \cong_f 20(P_3 \Box C_3) \cong_f 15(P_3 \Box C_4) \cong_f 12(P_3 \Box C_5)$, $\gamma_f(P_3 \Box C_{60}) = 20\frac{15}{7} = 15\frac{20}{7} = 12\frac{25}{7} = \frac{300}{7}$.

Let $m \geq 3$ be any positive integer, with $m = 3a + 4b + 5c$. Then we obtain the following formulas for the fractional domination and fractional total domination numbers of $P_n \Box C_m$

(in the case of $n = 2$, the graph is regular):

$$\gamma_f(P_2 \Box C_m) = \frac{m}{2}$$

$$\gamma_f(P_3 \Box C_m) = a\frac{15}{7} + b\frac{20}{7} + c\frac{25}{7} = \frac{5m}{7}$$

$$\gamma_f(P_4 \Box C_m) = a\frac{30}{11} + b\frac{40}{11} + c\frac{50}{11} = \frac{10m}{11}$$

$$\gamma_f(P_5 \Box C_m) = a\frac{60}{18} + b\frac{80}{18} + c\frac{100}{18} = \frac{10m}{9}$$

$$\gamma_f(P_6 \Box C_m) = a\frac{114}{29} + b\frac{152}{29} + c\frac{190}{29} = \frac{38m}{29}$$

$$\gamma_f(P_7 \Box C_m) = a\frac{213}{47} + b\frac{284}{47} + c\frac{355}{47} = \frac{71m}{47}$$

$$\gamma_f(P_8 \Box C_m) = a\frac{195}{38} + b\frac{130}{19} + c\frac{325}{38} = \frac{65m}{38}$$

$$\gamma_f^\circ(P_2 \Box C_m) = \frac{2m}{3}$$

$$\gamma_f^\circ(P_3 \Box C_m) = a(3) + b(4) + c(5) = m$$

$$\gamma_f^\circ(P_4 \Box C_m) = a\frac{18}{5} + b\frac{24}{5} + c(6) = \frac{6m}{5}$$

$$\gamma_f^\circ(P_5 \Box C_m) = a\frac{9}{2} + b(6) + c\frac{15}{2} = \frac{3m}{2}$$

$$\gamma_f^\circ(P_6 \Box C_m) = a\frac{36}{7} + b\frac{48}{7} + c\frac{60}{7} = \frac{12m}{7}$$

$$\gamma_f^\circ(P_7 \Box C_m) = a(6) + b(8) + c(10) = 2m$$

$$\gamma_f^\circ(P_8 \Box C_m) = a\frac{20}{3} + b\frac{80}{9} + c\frac{100}{9} = \frac{20m}{9}$$

Since $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$, there is no need to use the above technique to compute the fractional domination number of very large strong direct products. However, we can use Theorem 2.6.1 compute the fractional total domination numbers of large strong direct products.

2.6.1 Constructions

We want to know which constructions are preserved by fractional isomorphisms. We call a graph construction $\mathscr{T}$ preserving if $G \cong_f G'$ implies $\mathscr{T}(G) \cong_f \mathscr{T}(G')$.

Theorem 2.6.3 *If $G \cong_f H$, then $\bar{G} \cong_f \bar{H}$.*

Proof. Fix an ordering of the vertices and let A and B be adjacency matrices of G and H respectively (with respect to the fixed ordering). There exists a doubly stochastic matrix S satisfying $AS = SB$. Let J be the $n \times n$ matrix of all ones, then $J - I - A$ is the adjaceny matrix of the complement $\bar{G}$ and $J - I - B$ is the adjaceny matrix of complement $\bar{H}$. Since S commutes with J and the identity, we have $(J - I - A)S = JS - IS - AS = SJ - SI - SB = S(J - I - B)$. Thus $\bar{G}$ and $\bar{H}$ are fractionally isomorphic. □

The Mycielski construction $Y(G)$ defined in Chapter 1, turns out to be a preserving construction.

Theorem 2.6.4 *If $G \cong_f H$, then $Y(G) \cong_f Y(H)$.*

Proof. Let A and B be adjacency matrices of G and H, respectively. As in [20], the adjacency matrix of $Y(G)$ can be written as a $2n + 1 \times 2n + 1$ block matrix as follows:

$$\left[\begin{array}{c|c|c} A & A & \mathbf{0} \\ \hline A & 0_{n\times n} & \mathbf{1} \\ \hline \mathbf{0}^T & \mathbf{1}^T & 0 \end{array}\right]$$

There exists S, doubly stochastic, so that $AS = SB$; and we have

$$\left[\begin{array}{c|c|c} A & A & \mathbf{0} \\ \hline A & 0_{n\times n} & \mathbf{1} \\ \hline \mathbf{0}^T & \mathbf{1}^T & 0 \end{array}\right]\left[\begin{array}{c|c|c} S & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & S & \mathbf{0} \\ \hline \mathbf{0}^T & \mathbf{0}^T & 1 \end{array}\right] = \left[\begin{array}{c|c|c} S & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & S & \mathbf{0} \\ \hline \mathbf{0}^T & \mathbf{0}^T & 1 \end{array}\right]\left[\begin{array}{c|c|c} B & B & \mathbf{0} \\ \hline B & 0_{n\times n} & \mathbf{1} \\ \hline \mathbf{0}^T & \mathbf{1}^T & 0 \end{array}\right]$$

□

In [67], Fisher found that the construction $u(G) = \overline{Y(\bar{G})}$ worked well with fractional domination.

Corollary 2.6.5 *If $G \cong_f H$, then $u(G) \cong_f u(H)$*

Proof. Suppose $G \cong_f H$, then $\bar{G} \cong_f \bar{H}$ by Theorem 2.6.3. Theorem 2.6.4 then gives $Y(\bar{G}) \cong_f Y(\bar{H})$. One more application of Theorem 2.6.3 gives the result. □

2.7 Notes

For a good text on matrix theory including properties and theorems on doubly stochastic matrices, see [111] and [112]. The original work in this chapter was joint work with Walsh [177]. We list several open problems on this topic in Chapter 7.

Most of the counterexamples for non–invariant parameters listed in Appendix B are disconnected. So what if we want to know what is the same or different about any two connected fractionally isomorphic graphs. Do any of the non–invariant parameters suddenly become invariant? Ullman (in [174]) noted that the family of $3-$regular graphs should still provide counterexamples for each non-invariant parameter (except those dealing with connectivity). For Hamiltonicity, we look to the Petersen graph, which is non-Hamiltonian and $3-$regular on 10 vertices. The Tutte Wheel is a $3-$regular graph on 10 vertices formed from a $10-$cycle with five added maximum diameter chords (see Figure 2.8).

If two fractionally isomorphic graphs are connected and if G is Eulerian, then so is H, since degree sequences are preserved. Although, two fractionally isomorphic connected

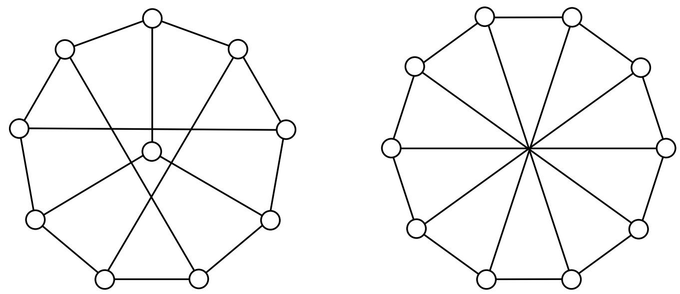


Figure 2.8: Fractionally isomorphic non-Hamiltonian and Hamiltonian graphs.

graphs may not have the same number of duplicated edges required for a minimum Eulerization (see Figure 2.5).

Are the graphs C_6 and $C_3 \cup C_3$ the smallest example of two non–isomorphic yet fractionally isomorphic graphs? One can check that no two non–isomorphic graphs of order $n \leq 4$ have the same degree sequence. If there are two non–isomorphic, fractionally isomorphic graphs of order $n < 6$, they are among the graphs in Figure 2.9. It turns out that our search for a smaller example fails since none of graphs in Figure 2.9 are pairwise fractionally isomorphic. Since trees are only fractionally isomorphic to themselves, the two graphs G26 and G31 cannot be fractionally isomorphic. G36 and G37 do not share the coarsest equitable partition. Lastly, $\gamma_f(G43) = \frac{3}{2}$, whereas $\gamma_f(G44) = \frac{7}{5}$.

The above relaxation of the concept of isomorphism can also be applied to automorphism. Let G be a graph with adjacency matrix A; a doubly stochastic matrix S is a *fractional automorphism* of G iff $AS = SA$. [74] discusses the history of fractional automorphisms (though not by that name) and some of their algebraic properties; [119] and [120] explore the connection between fractional automorphisms and fractional domination.

There are also other relaxations of isomorphism. If we only require P in the isomorphism equation $P^{-1}AP = B$ to be a non-singular matrix, then the graphs G and H are co-spectral (see [146]). Another relaxation of isomorphism is semi-isomorphism. Two graphs

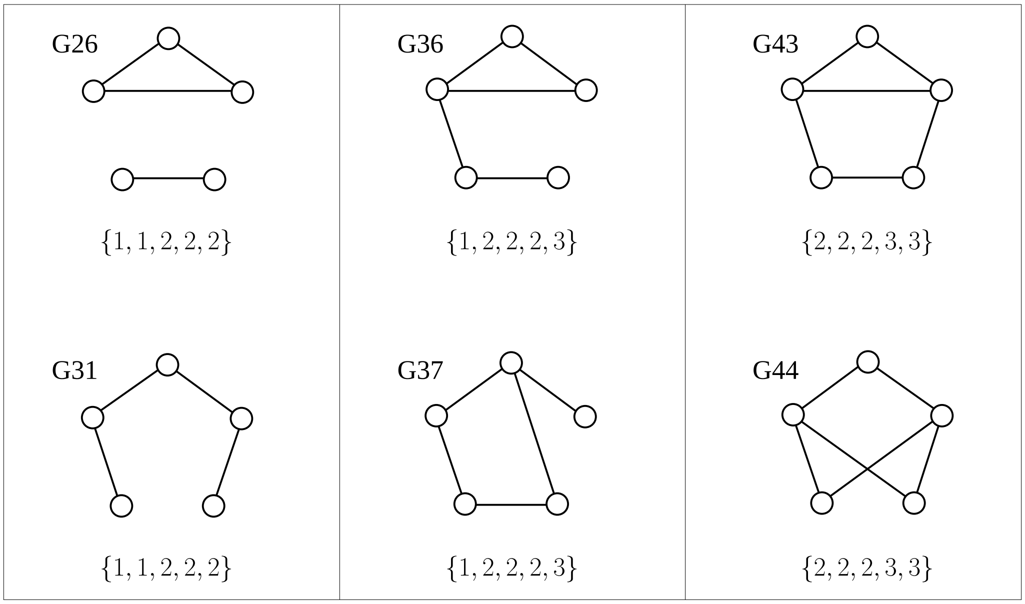


Figure 2.9: Graphs on five or less vertices with the same degree sequence.

are semi-isomorphic (and write $G \cong' H$) if there exists two doubly stochastic matrices P and Q so that $A = QBP$.

In [81] Grone suggested the use of *ortho-stochastic matrices*, which are a subclass of doubly stochastic matrices. Any matrix formed by taking the Hadamard product of a unitary matrix with its conjugate is an ortho-stochastic matrix (see [112]).

Recall that two graphs G and H are *isomorphic* if there exists a one-to-one mapping φ from $V(G)$ onto $V(H)$ such that φ preserves adjacency and non–adjacency; that is, $uv \in E(G)$ iff $\varphi(u)\varphi(v) \in E(H)$. So for two isomorphic graphs there is a one-to-one correspondence with the vertices of G and H. This one-to-one correspondence can be represented as "links" from $V(G)$ to $V(H)$. In the case of fractional isomorphism, this is a bit different. In [146] and [157], an equivalent definition of fractional isomorphism using links is given. In the figure below, G and H are the graphs in Figure 2.3, the links are represented by dashed lines with the weights of $\frac{1}{4}$ for the red links, and $\frac{1}{2}$ for the blue links. The weights come from the doubly stochastic matrix which satisfies $AS = SB$.

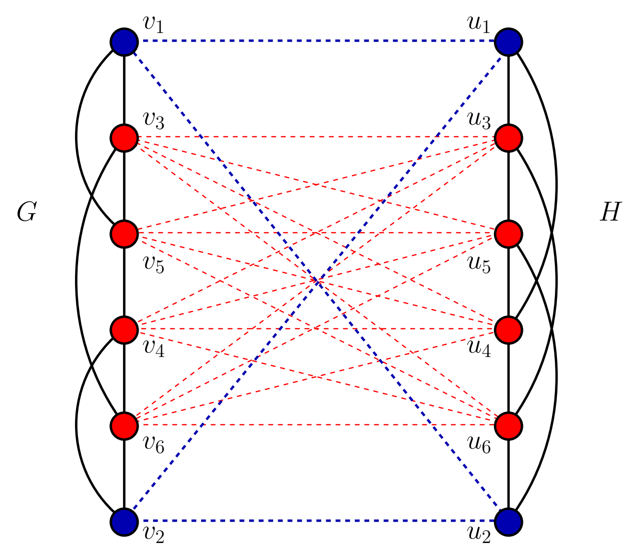


Figure 2.10: Links.

CHAPTER 3

MINIMUM FRACTIONAL DOMINATING FUNCTIONS

AND MAXIMUM FRACTIONAL PACKING FUNCTIONS

3.1 Functions which are both minimum fractional dominating and maximum fractional packing

In Chapter 1, we saw that finding minimum fractional dominating functions and maximum fractional packing functions are dual linear programs. This is a very special dual pair of LPs, though, since the vectors being optimized in both problems can be interpreted as real-valued functions on the set of vertices. Hence, it becomes possible to have a function whose vector simultaneously solves both the fractional domination LP and its dual. We call a function which is both a minimum FDF and a maximum FPF a *fractional dominating-packing function* (FDPF). A function which is both an FDF and FPF is necessarily an MFDF and an MFPF and is therefore an FDPF. We might also refer to such an object as a *(closed neighborhood) fractional partition* on the vertices of G, as it forms a real-valued analogue of a partition of the vertex set of G into closed neighborhoods.

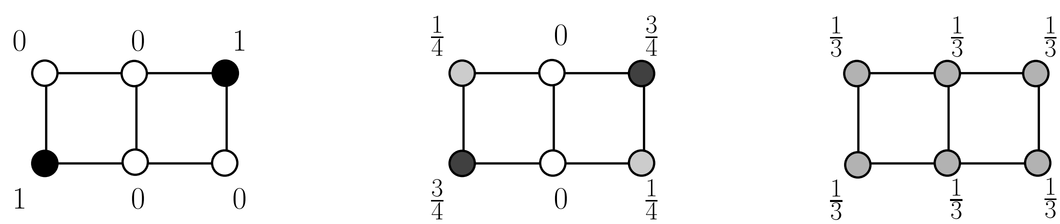


Figure 3.1: (a) A dominating packing function, (b) a fractional dominating packing function, and (c) a minimum fractional dominating function which is not a packing.

3.2 Definition of the classes

We wish to investigate the interactions of MFPFs and MFDFs. Our goal is to classify those graphs in which all MFPFs are FDPFs, or all MFDFs are FDPFs, or when there are no FDPFs at all, as well as any graphs in between. To do this we define the following five classes based on the intersections of the two following sets: Let D_G be the set of all MFDFs on G and let P_G be the set of all MFPFs on G. Every finite simple graph G belongs to exactly one of the classes below:

- $G \in$ Class $\mathcal{N}$ (Null) if $D_G \cap P_G = \varnothing$.
- $G \in$ Class $\mathcal{I}$ (Intersection) if $D_G \cap P_G \neq \varnothing$, $D_G \nsubseteq P_G$ and $P_G \nsubseteq D_G$.
- $G \in$ Class $\mathcal{P}$ (Packing) if $D_G \subsetneq P_G$.
- $G \in$ Class $\mathcal{D}$ (Dominating) if $P_G \subsetneq D_G$.
- $G \in$ Class $\mathcal{E}$ (Equal) if $D_G = P_G$.

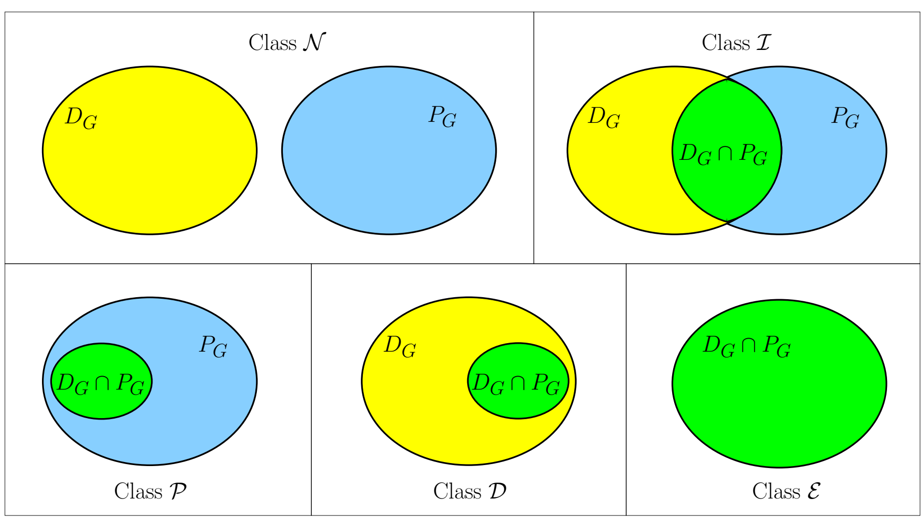


Figure 3.2: The five classes as different intersections of the sets D_G (in yellow) and P_G (in blue); with $D_G \cap P_G$ in green.

3.3 The principle of complementary slackness

The principal tool that we shall use in our investigations is the Principle of Complementary Slackness, an important result in the duality theory of linear programming (see Theorem 1.3.2). Recall the neigborhood matrix, $N = A + I$.

Proposition 3.3.1 (Applied principle of complementary slackness) *Let $\boldsymbol{x'}$ be an MFPF, that is any optimal solution to the linear program: maximize $\mathbf{1}^T\boldsymbol{x}$ subject to $N\boldsymbol{x} \leq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$, and let $\boldsymbol{y'}$ be an MFDF, that is any optimal solution to the dual linear program: minimize $\mathbf{1}^T\boldsymbol{y}$ subject to $N^T\boldsymbol{y} \geq \mathbf{1}$, $\boldsymbol{y} \geq \mathbf{0}$. Then*

$$\boldsymbol{x'} \cdot (N^T\boldsymbol{y'} - \mathbf{1}) = \boldsymbol{y'} \cdot (N\boldsymbol{x'} - \mathbf{1}) = \mathbf{0}.$$

We give the two main consequences below:

Corollary 3.3.2 *If f is an MFDF and $v \in V(G)$ for which $f(v) > 0$, then for any MFPF g, $g(N[v]) = 1$.*

Corollary 3.3.3 *If g is an MFPF and $v \in V(G)$ for which $g(v) > 0$, then for any MFDF f, $f(N[v]) = 1$.*

These corollaries, in turn, suffice to establish a number of technical lemmas, which we now state and prove.

Lemma 3.3.4 *If f is an MFDF with $f(v) > 0$ for every vertex $v \in V$, then every MFPF is an MFDF, and thus $P_G \subseteq D_G$.*

Proof. Let f be an MFDF on G with $f(v) > 0$ for each vertex v. Then by Corollary 3.3.2, every MFPF g has the property that $g(N[v]) = 1$ for every vertex v. So g is an MFDF. □

Lemma 3.3.5 *If we can find two functions f and g where f is an MFDF on G with $f(v) > 0$ for each vertex v, and where g is an MFDF on G which is not an FPF, then $G \in$ Class $\mathcal{D}$.*

Proof. The MFDF f gives us $P_G \subseteq D_G$ and the MFDF (non-packing) g gives us $P_G \subsetneq D_G$. $\square$

Lemma 3.3.6 *If f is an MFPF with $f(v) > 0$ for each vertex $v \in V$, then every MFDF is an MFPF, and thus $D_G \subseteq P_G$.*

Proof. Let g be an MFPF on G with $g(v) > 0$ for each vertex v. Then by Corollary 3.3.3, every MFDF f has the property that $f(N[v]) = 1$ for every vertex v. So f is an MFPF. $\square$

Lemma 3.3.7 *If we can find two functions f and g where f is an MFPF on G with $f(v) > 0$ for each vertex v, and where g is an MFPF on G which is not dominating, then $G \in$ Class $\mathcal{P}$.*

Proof. The MFPF f gives us $D_G \subset P_G$ and the MFPF (non-dominating) g gives us $D_G \subsetneq P_G$. $\square$

The results of Lemmas 3.3.5 and 3.3.7 also work if there is a single function which satisfy both properties of f and g simultaneously (or at the same time). This single function can be obtained by taking an appropriate convex combination of the two functions.

If we can find a function which is both an MFDF and an MFPF which has positive weights on each vertex, then combining Lemmas 3.3.4 and 3.3.6 yields:

Corollary 3.3.8 *For a graph G, if there exists an FDPF f with $f(v) > 0$ for each vertex $v \in V$, then $G \in$ Class $\mathcal{E}$.*

3.4 A partial classification

With these preliminaries in place, we are ready to begin sorting families of graphs into our five classes.

3.4.1 Some basic graphs

Theorem 3.4.1 *Every regular graph is Class $\mathcal{E}$.*

Proof. Let G be k-regular; then the function $f(v) = \frac{1}{k+1}$ for all $v \in V$ is an FDPF. Since f is nonzero at each vertex, Corollary 3.3.8 tells us that $G \in$ Class $\mathcal{E}$. □

Theorem 3.4.2 *If $\Delta(G) = n - 1$ and $G \neq K_n$ then $G \in$ Class $\mathcal{P}$.*

Proof. Let S be the set of vertices of maximum degree $n-1$. Since $\pi = \gamma = 1$, the constant function $f(v) = \frac{1}{n}$ is an MFPF. Since $f(N[v]) < 1$ for any $v \in V - S$, f is not dominating. Note that $V - S$ is non-empty since $G \neq K_n$. Thus, by Lemma 3.3.7, $G \in$ Class $\mathcal{P}$. □

It would be nice if we could determine the class of the graph by induced subgraphs. From the above two theorems, we can see this does not work. The star $K_{1,2}$ is Class $\mathcal{P}$ and K_2 is an induced subgraph, however, K_2 is regular and thus Class $\mathcal{E}$.

Theorem 3.4.3 *Let G be the complete r-partite graph with parts of size $n_1, n_2, \ldots, n_r$, $r \geq 2$ and each $n_j \geq 2$. Then $G \in$ Class $\mathcal{E}$.*

Proof. As shown in [80] the function which assigns to each vertex in the jth part the positive weight of

$$\frac{1}{(n_j - 1)\left(\sum_{i=1}^{r} \frac{1}{n_i - 1} + r - 1\right)}$$

is an FDPF. □

3.4.2 Paths and other trees

Theorem 3.4.4 *Let P_n be the path on n vertices for $n \geq 3$. Then:*

$$P_n \in \begin{cases} \textit{Class } \mathcal{P}, & n \equiv 0 \mod 3 \\ \textit{Class } \mathcal{D}, & n \equiv 1 \mod 3 \\ \textit{Class } \mathcal{I}, & n \equiv 2 \mod 3 \end{cases}$$

Proof. Let v_i represent the ith vertex of the path on n vertices. For any positive integer $k \geq 1$, it is easy to check that $\pi(P_{3k}) = \gamma(P_{3k}) = k$ and $\pi(P_{3k+i}) = \gamma(P_{3k+i}) = k + 1$ for $i = 1, 2$. In the following cases, the bracketed blocks of weights are repeated $k - 1$ times.

Case 1: $n = 3k$. Let f be the function which assigns the weight of $\frac{1}{3}$ to each vertex. Since $f(N[v_1]) = \frac{2}{3}$, f is not dominating. So by Lemma 3.3.7, we have $P_{3k} \in$ Class $\mathcal{P}$.

Case 2: $n = 3k + 1$. The function $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}], \ldots, \frac{1}{2}, \frac{1}{2})^T$ is an MFDF with positive weights on every vertex. Since $f(N[v_2]) = \frac{4}{3}$, f is not packing. Therefore $P_{3k+1} \in$ Class $\mathcal{D}$ by Lemma 3.3.5.

Case 3: $n = 3k + 2$. The function $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, [\frac{1}{2}, 0, \frac{1}{2}], \ldots, \frac{1}{2})^T$ is an FDPF. The function $\boldsymbol{g} = (0, 1, 0, 1, [0, 0, 1], \ldots, 0)^T$ is an MFDF which is not packing (since $g(N[v_3]) = 2$). Lastly, the function $\boldsymbol{h} = (1, [0, 0, 1], \ldots, 0, 0, 0, 1)^T$ is an MFPF which is not dominating (since $h(N[v_{3k}]) = 0$). Therefore $P_{3k+2} \in$ Class $\mathcal{I}$. $\square$

Trees in general do not seem as easy to classify; however, certain classes of trees lend themselves easily to analysis. For instance, [40] and [94] define a *healthy spider* $K^*_{1,t}$ as the result of subdividing each edge of a star $K_{1,t}$ into a path of length 3 (see Figure 3.3). Exempting one or more (but not all) of the edges from this subdivision results in a *wounded spider* (see Figure 3.6a). In both of these classes of graphs, the vertex of degree t is referred to as the *head* vertex, and those of degree one the *foot* vertices.

Figure 3.3: (a) A healthy spider: $K^*_{1,6}$ and (b) a wounded spider.

Theorem 3.4.5 $K^*_{1,t} \in$ *Class* $\mathcal{I}$

Proof. The function which assigns the weight of 0 to the head vertex, $\frac{t-1}{t}$ to the foot vertices, and $\frac{1}{t}$ otherwise, is an FDPF. The function which assigns 1 to the foot vertices and 0 otherwise is an MFPF which is not dominating. Lastly, the function which assigns 1 to the vertices of degree two and 0 otherwise, is an MFDF which is not packing. Therefore, $G \in$ Class $\mathcal{I}$. □

Note that for the healthy spider obtained from subdividing both edges of a $K_{1,2}$ we get P_5 which was already shown to be in Class $\mathcal{I}$ by the preceding theorem. The next theorem was stated and proved by Walsh in [153].

Theorem 3.4.6 *Suppose that T is a tree and $T \in$ Class $\mathcal{E}$. Then $|V(T)| \leq 2$.*

Proof. Suppose that T is a Class $\mathcal{E}$ tree with at least three vertices; then we can find two adjacent vertices x, y such that $d(x) = 1$ and $d(y) > 1$. Let f be an FDPF; then $f(N[x]) = f(x) + f(y) = 1$. We shall define two more functions, f_x and f_y, which are equal to f everywhere except on $N[x]$; we set $f_x(x) = 1$ and $f_x(y) = 0$; likewise, $f_y(x) = 0$ and $f_y(y) = 1$. Clearly f_x is an MFPF and f_y is an MFDF, but at least one of them is not a FDPF. □

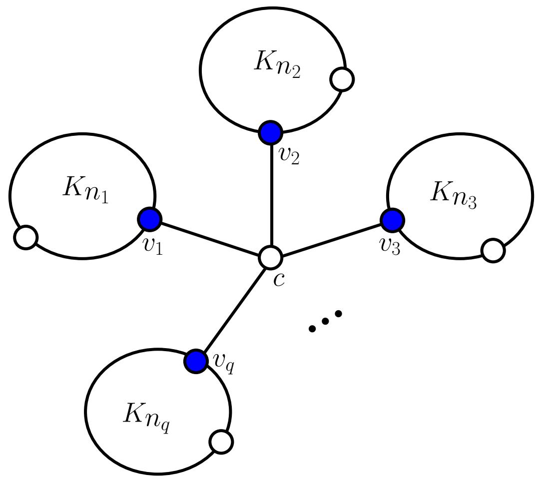


Figure 3.4: A collection of disjoint cliques each connected to a central vertex.

3.4.3 Graphs formed from cliques

If we take a finite collection of $q > 1$ disjoint cliques $\{K_{n_1}, \ldots, K_{n_q}\}$ and for each clique designate a vertex v_i to be adjacent to a vertex c outside of each clique, then we have a graph on $\sum n_i + 1$ vertices. We call c the *central vertex*, each of the vertices in the K_{n_i} which are not adjacent to the central vertex *peripheral vertices*, and the $\{v_i\}$ *juncture vertices*. The central vertex has degree q, the peripheral vertices have degrees n_i, and the juncture vertices have degrees $n_i + 1$.

Theorem 3.4.7 *Let G be constructed from a collection of $q > 1$ disjoint cliques as above. If $n_i \geq 2$ for all i, then $G \in$ Class $\mathcal{I}$.*

Proof. Clearly $\pi = \gamma = q$, so the function which assigns the weight of 1 to each of the juncture vertices and 0 otherwise is an MFDF which is non-packing. The function which assigns the weight of 1 to a single peripheral vertex in each clique and 0 otherwise is an MFPF which is non-dominating. Lastly, take the previous MFPF and move the weight of 1 from the peripheral vertex of just one clique to its juncture vertex. This is an FDPF. Therefore $G \in$ Class $\mathcal{I}$. $\square$

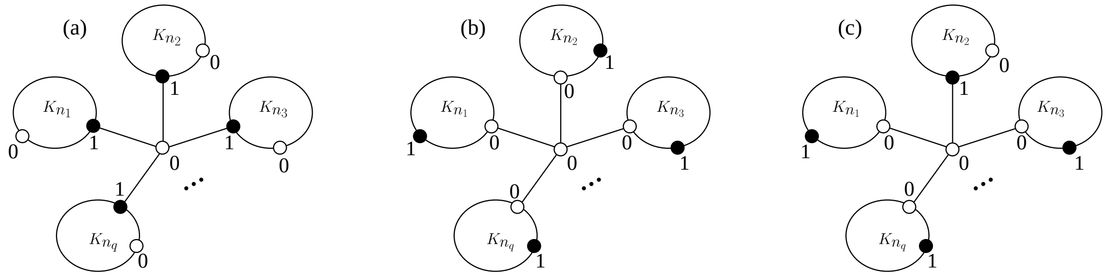


Figure 3.5: (a) An MFDF (non-packing), (b) an MFPF (non-dominating), and (c) an FDPF.

3.4.4 Class $\mathcal{N}$ull graphs

Up until now we have seen examples of graphs in every class except Class $\mathcal{N}$, where no MFDF is an MFPF and no MFPF is an MFDF. That is for any graph in Class $\mathcal{N}$ there are no FDPFs on G. Actually, in this class, it is true that no FDF is a FPF and no FPF is a FDF. There is an easy characterization of Class $\mathcal{N}$ graphs using neighborhood matrices.

Proposition 3.4.8 *G is in Class $\mathcal{N}$ if and only if the system $N\boldsymbol{x} = \boldsymbol{1}$ has no non-negative solutions.*

Proof. If $N\boldsymbol{x} = \mathbf{1}$ has a non-negative solution, then the vector $\boldsymbol{x}$ is an FDPF, thus G is not in Class $\mathcal{N}$. Likewise, if G is not in Class $\mathcal{N}$, then there exists a vector $\boldsymbol{x}$ satisfying $N\boldsymbol{x} = \mathbf{1}$, with $\boldsymbol{x} \geq \mathbf{0}$, that is each x_i is non-negative; thus the system has a non-negative solution. □

The smallest examples of graphs in Class $\mathcal{N}$ are a wounded spider obtained from subdividing one edge of a $K_{1,3}$, a K_3 with two pendant edges, and C_5 with an added chord (depicted in Figure 3.6). In each of these graphs, the red vertex is forced to have a negative weight when solving the system $N\boldsymbol{x} = \mathbf{1}$.

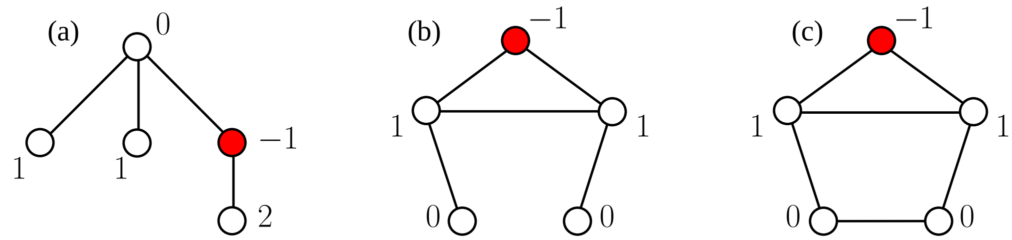


Figure 3.6: Unique solutions to $N\boldsymbol{x} = \mathbf{1}$.

With the above wounded spider (depicted in Figure 3.6a), upon solving $N\boldsymbol{x} = \mathbf{1}$, we find the unique solution is $\boldsymbol{x} = (2, -1, 0, 1, 1)^T$. By Proposition 3.4.8, the above wounded spider is in Class $\mathcal{N}$.

The next Class $\mathcal{N}$ graph is K_3 with two pendant edges (depicted in Figure 3.6b). Upon solving $N\boldsymbol{x} = \mathbf{1}$, we find the unique solution is the function x which assigns a weight of -1 to the vertex of degree two, 1 to each vertex of degree three and 0 to each vertex of degree one. By Proposition 3.4.8, K_3 with two pendant edges is in Class $\mathcal{N}$.

The last Class $\mathcal{N}$ graph on five vertices is C_5 with a chord (see Figure 3.6c). Upon solving $N\boldsymbol{x} = \mathbf{1}$, we find the unique solution is $\boldsymbol{x} = (-1, 1, 0, 0, 1)^T$. By Proposition 3.4.8, this graph is Class $\mathcal{N}$. In [121], we describe all MFDFs of this graph, $\boldsymbol{f} = (0, \frac{1}{2}, t, \frac{1}{2} - t, \frac{1}{2})^T$ where $0 \leq t \leq \frac{1}{2}$. The unique MFPF is $\boldsymbol{g} = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0)^T$. It should be noted that $\boldsymbol{x} = (-1, 1, 0, 0, 1)^T$ could be considered as an efficient $\{-1, 0, 1\}$ dominating function. However, we will restrict our attention to FDFs and FPFs, which by definition, only have weights from the unit interval $[0, 1]$.

We showed above (in Theorems 3.4.1 and 3.4.4) that the other four classes are infinite. We shall now do the same for Class $\mathcal{N}$ using some results from [121], which we restate here using our present terminology.

Lemma 3.4.9 *If f is an MFDF and $v \in V(G)$ for which $f(N[v]) > 1$, then every MFPF g satisfies $g(v) = 0$.*

Proof. Suppose g is an MFPF with $g(v) > 0$. Then by Corollary 3.3.3, every MFDF f satisfies $f(N[v]) = 1$, a contradiction. $\square$

Lemma 3.4.10 *If g is an MFPF and $v \in V(G)$ for which $g(N[v]) < 1$, then every MFDF f satisfies $f(v) = 0$.*

Proof. Suppose f is an MFDF with $f(v) > 0$. Then by Corollary 3.3.2, every MFPF g satisfies $g(N[v]) = 1$, a contradiction. $\square$

In Chapter 1, we defined two graph constructions, the trampoline and the generalized Hajós graph. These constructions are defined for $G = K_n$, $n \geq 3$. We also defined corresponding partial constructions with K_n replaced by any Hamiltonian graph G.

Theorem 3.4.11 *Let G be Hamiltonian, then $T(G)$ is Class $\mathcal{N}$.*

Proof. As noted in [121], the function f defined by $f(u_i) = \frac{1}{2}$ and $f(v_i) = 0$ (for all i) is an FPF; the function g defined by $g(v_i) = \frac{1}{2}$ and $g(u_i) = 0$ (for all i) is an FDF. Since $|f| = |g|$, f is a maximum FPF and g is a minimum FDF. Since $f(N[u_i]) < 1$ for each u_i, then by Lemma 3.4.10, every MFDF h satisfies $h(u_i) = 0$. Since $g(N[v_i]) = \frac{3}{2} > 1$ for each v_i, then by Lemma 3.4.9, every MFPF k satisfies $k(v_i) = 0$. Therefore, no MFPF can be an MFDF. $\square$

Note that the above proof does not depend on which Hamiltonian cycle H is chosen in the construction, hence, the H in $T_H(G)$ is omitted.

Corollary 3.4.12 *All trampolines are in Class $\mathcal{N}$.*

Theorem 3.4.13 *For any Hamiltonian graph G, the partial generalized Hajós graph $[G]$ is Class $\mathcal{N}$.*

Proof. The function $f(u_{ij}) = \frac{1}{n-1}$ (for all $1 \leq i < j \leq n$) and 0 otherwise is an MFPF. The function $g(v_i) = \frac{1}{2}$ (for all i) and 0 otherwise is an MFDF. Since $f(N[u_{ij}]) < 1$ for each u_{ij}, then by Lemma 3.4.10, every MFDF h satisfies $h(u_{ij}) = 0$. Since $g(N[v_i]) \geq \frac{3}{2} > 1$ for each v_i, then by Lemma 3.4.9, every MFPF k satisfies $k(v_i) = 0$. Therefore, no MFPF can be an MFDF. □

3.4.5 Strongly chordal graphs

A well studied class of graphs for which equality holds in (1.11) is the class of strongly chordal graphs. A graph is *strongly chordal* if any cycle on four or more vertices contains a chord and there are no induced trampolines. One might anticipate that if G is a strongly chordal graph, then finding which class G is in would be an easy problem, especially since there are a fair number of papers on strongly chordal graphs in the literature. However, finding which class a strongly chordal graph belongs to is not an easy problem, since trees are strongly chordal and there are examples of trees in each class (see Theorems 3.4.1, 3.4.4 and Figure 3.6a). In fact, nothing can be said about the class of G if $\pi(G) = \gamma(G)$.

3.5 Sums and products of graphs

There is an entire chapter in [95] regarding the domination number of a product of two graphs (see also [114] and [142]). It seems natural, then, to ask whether we can determine the class of a graph product in terms of its ingredients. We will briefly examine a few graph products and disjoint unions in this section using the notation of [67] for the products and [30] for the disjoint union. Again, a complete list of notation can be found in the appendix.

3.5.1 Disjoint unions

Domination and neighborhood packing (in both their integer and fractional forms) interact quite simply with disjoint unions.

Lemma 3.5.1 *The function $f : V(G \cup H) \to [0,1]$ is a fractional dominating (packing) function on $G \cup H$ if and only if $f|_G$ is dominating (packing) on G and $f|_H$ is dominating (packing) on H.*

Using this, we can determine the class of a disjoint union of two graphs, given the classes of the two starting graphs. The results are easy to check, and are summarized in Table 3.5.1.

$\cup$	Class $\mathcal{D}$	Class $\mathcal{E}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{P}$
Class $\mathcal{D}$	Class $\mathcal{D}$	Class $\mathcal{D}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{I}$
Class $\mathcal{E}$	Class $\mathcal{D}$	Class $\mathcal{E}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{P}$
Class $\mathcal{I}$	Class $\mathcal{I}$	Class $\mathcal{I}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{I}$
Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$
Class $\mathcal{P}$	Class $\mathcal{I}$	Class $\mathcal{P}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{P}$

Table 3.1: The class of the disjoint union of two graphs.

Table 3.5.1 suggests a lattice structure on the classes where the lattice join operation is *Class* $(G) \vee$ *Class* $(H) =$ *Class* $(G \cup H)$. To complete the definition of our lattice, we would need to define a meet operation $\wedge$ on the classes. There does exist a meet operation (*Class* $(G) \wedge$ *Class* (H)) on the classes, since there are are a finite number of elements and there is a minimum element, Class $\mathcal{E}$. We present what such a lattice of the classes should look like in Figure 3.7.

3.5.2 Cartesian products

Theorem 3.5.2 *Let G be the 2 by n grid graph $P_2 \square P_n$. Then for $n > 1$ we have:*

$$P_2 \square P_n \in \begin{cases} \text{Class } \mathcal{E}, & n \equiv 0 \mod 2 \\ \text{Class } \mathcal{D}, & n \equiv 1 \mod 2 \end{cases}$$

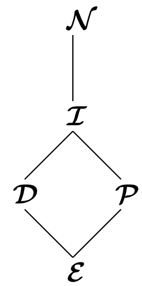


Figure 3.7: The Lattice of Classes, with the join operator: *Class* $(G) \vee$ *Class* $(H) =$ *Class* $(G \cup H)$.

Proof. We consider odd and even values of n separately.

Case 1: $n = 2k$. For $k = 1$ we have C_4 which is regular. For $k > 1$ order the vertices of $P_2 \square P_{2k}$ as $\{v_{1,1}, \ldots, v_{1,2k}; v_{2,1}, \ldots, v_{2,2k}\}$. The function

$$f(v_{i,j}) = \begin{cases} \dfrac{j/2}{2k+1}, & j \equiv 0 \mod 2 \\ \dfrac{k - ((j-1)/2)}{2k+1}, & j \equiv 1 \mod 2 \end{cases}$$

is an FDPF which has positive weights on each vertex so $P_2 \square P_{2k} \in$ Class $\mathcal{E}$ by Corollary 3.3.8.

Case 2: $n = 2k + 1$. For $k \geq 1$ we can find a partition of vertices into closed neighborhoods, that is we can find $k + 1$ vertices $p_1, \ldots, p_{k+1}$ so that each vertex of G is in precisely one closed neighborhood. The vertices p_i are straightforward to find; Figure 3.8 gives a depiction of such a partitioning of $V(P_2 \square P_9)$ into the closed neighborhoods $\{N[p_1], \ldots, N[p_5]\}$,

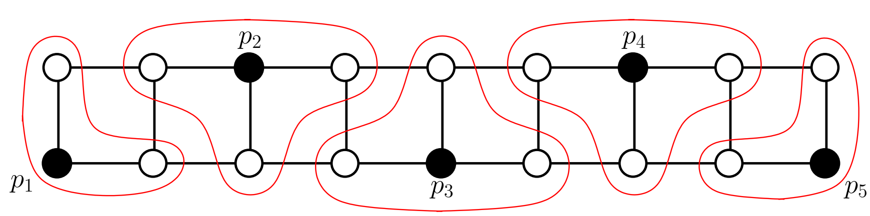


Figure 3.8: A neighborhood partition of $V(P_2 \square P_9)$.

where the p_i are colored black. In fact, there is a formula for finding the p_i based on the ordering used in case 1: $\{p_1, \ldots, p_{k+1}\} = \{v_{1,1}, v_{3,2}, v_{5,1}, \ldots\}$, where p_{k+1} is $v_{2k+1,1}$ if k is even, or $v_{2k+1,2}$ if k is odd.

For $k = 1$ we have a partition using the vertices p_1 and p_2. The function which assigns 1 to each p_i and 0 otherwise is an FDPF. Now consider the constant function which assigns the weight of $\frac{1}{3}$ to each vertex; this function is an MFDF which is not packing. Therefore by Lemma 3.3.5, $P_2 \square P_3 \in$ Class $\mathcal{D}$.

For $k \geq 2$ we have a partition using the vertices p_1, p_{k+1} of degree two and $p_2, \ldots, p_k$ of degree three. The function which assigns 1 to each p_i and 0 otherwise is an FDPF. The function which assigns the weight of 0 to each of $\{p_2, \ldots, p_k\}$ and $\frac{1}{3}$ otherwise is an MFDF which is not packing. Taking a convex combination of these two functions we have an MFDF with positive weights on each vertex. Therefore by Lemma 3.3.5, $P_2 \square P_{2k+1} \in$ Class $\mathcal{D}$. $\square$

This result is somewhat discouraging, in that it suggests the absence of an obvious relationship between the classes of two graphs and that of their Cartesian product: the class of $P_2 \square P_n$ depends only on the parity of n, while the class of P_n depends on the congruence class of n (mod 3). We classify one more grid graph below. See Figure 3.9 to see that $P_4 \square P_4 = G_{4,4}$ is Class $\mathcal{P}$.

3.5.3 Strong direct products

An important product we consider is the *strong direct product* of G and H, denoted by $G \boxtimes H$. (see Figure 1.7d). Here we are a little more fortunate. The interaction between fractional domination and strong direct products is studied in [52]; the following facts are observed there, which we state as lemmas.

Lemma 3.5.3 $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$

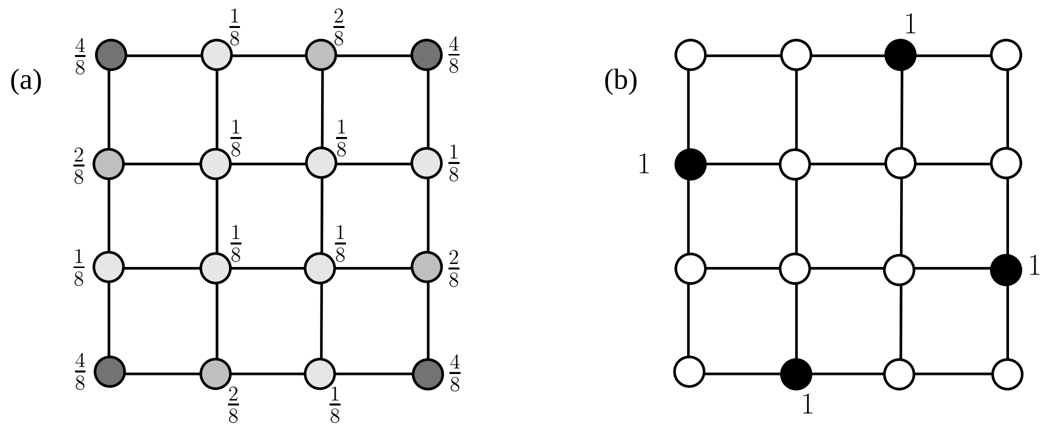


Figure 3.9: (a) An MFPF which is non–dominating and (b) an FDPF of $P_4 \square P_4$.

Lemma 3.5.4 *Let P be an $m \times k$ matrix, Q be an $s \times t$ matrix, x and z be k-vectors, y and w be t-vectors, and $\otimes$ denote the tensor product. Then:*

1. $(P \otimes Q)(x \otimes y) = (Px) \otimes (Qy)$.

2. *If* $x \geq z \geq \mathbf{0}$ *and* $y \geq w \geq \mathbf{0}$, *then* $x \otimes y \geq z \otimes w$.

3. *Let G and H be graphs with adjacency matrices A_G and A_H, respectively; then the adjacency matrix of their product $A_{G \boxtimes H}$ is given by $A_G \otimes A_H$.*

Theorem 3.5.5 *Let x_1 and x_2 be MFDFs on G and H, respectively. Then $x^* = x_1 \otimes x_2$ is an MFDF on $G \boxtimes H$.*

Proof.

$$\begin{aligned} A_{G \boxtimes H} x^* &= (A_G \otimes A_H)(x_1 \otimes x_2) \\ &= (A_G x_1) \otimes (A_H x_2) \\ &\geq \mathbf{1}_{|V(G)|} \otimes \mathbf{1}_{|V(H)|} \\ &= \mathbf{1}_{|V(G \boxtimes H)|} \end{aligned}$$

This shows $x_1 \otimes x_2$ is an FDF on $G \boxtimes H$; $x_1 \otimes x_2$ is an MFDF by Lemma 3.5.3. $\square$

An analogous proof gives us:

Theorem 3.5.6 *Let y_1 and y_2 be MFPFs on G and H, respectively. Then $y^* = y_1 \otimes y_2$ is an MFPF on $G \boxtimes H$.*

This shows that the properties of being dominating and packing are maintained in products; we can also show that the properties of being non-dominating and non-packing are likewise preserved.

Lemma 3.5.7 *If f_1 and f_2 are MFDFs on G and H, respectively, with at least one of f_1 and f_2 not packing; then $f_1 \otimes f_2$ is an MFDF on $G \boxtimes H$ which is not packing.*

Proof. From Theorem 3.5.5 we have that $f_1 \otimes f_2$ is an MFDF. Suppose f_1 is not a packing. To show that $f_1 \otimes f_2$ is not packing, let $u \in V(G)$ such that $f_1(N[u]) > 1$; such a vertex must exist, since otherwise f_1 would be an FPF. Since the weight of a vertex in the strong direct product equals the product of the weights on its component vertices, then by part 3 of Lemma 3.5.4, we can see that $(f_1 \otimes f_2)(N[(u, w)]) = f_1(N[u])f_2(N[w]) > 1$, and hence $f_1 \otimes f_2$ is not packing. □

Lemma 3.5.8 *If f_1 and f_2 are MFPFs on G and H, respectively, with at least one of f_1 and f_2 not dominating; then $f_1 \otimes f_2$ is an MFPF on $G \boxtimes H$ which is not dominating.*

Proof. As above, with the inequalities reversed. □

Together, these give:

Theorem 3.5.9 *The class of $G \boxtimes H$ is determined by the table below, where the first row is the class of G and the first column is the class of H.*

Proof. Theorems 3.5.5 and 3.5.6 show that the tensor product of MFDFs and MFPFs are themselves MFDFs and MFPFs of the product graph, and hence if f_1 and f_2 are FDPFs

$\boxtimes$	Class $\mathcal{D}$	Class $\mathcal{E}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{P}$
Class $\mathcal{D}$	Class $\mathcal{D}$	Class $\mathcal{D}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{I}$
Class $\mathcal{E}$	Class $\mathcal{D}$	Class $\mathcal{E}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{P}$
Class $\mathcal{I}$	Class $\mathcal{I}$	Class $\mathcal{I}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{I}$
Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$	Class $\mathcal{N}$
Class $\mathcal{P}$	Class $\mathcal{I}$	Class $\mathcal{P}$	Class $\mathcal{I}$	Class $\mathcal{N}$	Class $\mathcal{P}$

Table 3.2: The class of the strong direct product of two graphs.

of G and H, respectively, then $f_1 \otimes f_2$ is an FDPF of $G \boxtimes H$. Further, Lemma 3.5.8 can be used to find an MFPF which is not dominating if at least one of G and H is Class $\mathcal{P}$ or Class $\mathcal{I}$. Lemma 3.5.7 can be used to find an MFDF which is not packing, if at least one of G and H is Class $\mathcal{D}$ or Class $\mathcal{I}$. Thus, if one of G and H is Class $\mathcal{I}$ and the other is not Class $\mathcal{N}$, then $G \boxtimes H \in$ Class $\mathcal{I}$. If at least one of G, H is Class $\mathcal{N}$, then $G \boxtimes H \in$ Class $\mathcal{N}$. The remaining cases are left to the reader. $\square$

3.5.4 Categorical products

$P_3 \times P_3$ can be viewed as the disjoint union of $G = C_4$ and $H = K_{1,4}$, (see Figure 3.10). C_4 is Class $\mathcal{E}$ and $K_{1,4}$ is Class $\mathcal{P}$. Thus $C_4 \cup K_{1,4} \cong P_3 \times P_3 \in$ Class $\mathcal{P}$ by Table 3.5.1. $P_3 \times P_4$ can be viewed as the disjoint union of two copies of G96 which can be shown to be Class $\mathcal{N}$, thus G96 $\cup$ G96 $\cong P_3 \times P_4$ is Class $\mathcal{N}$.

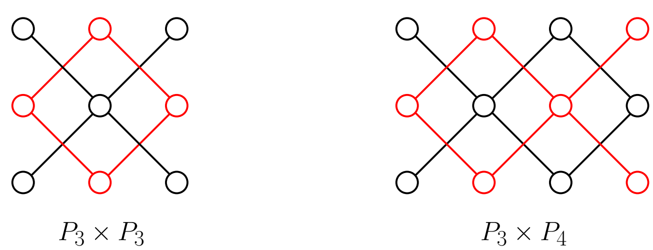


Figure 3.10: The categorical products $P_3 \times P_3 \in$ Class $\mathcal{P}$ and $P_3 \times P_4 \in$ Class $\mathcal{N}$.

3.6 Fractional isomorphisms and equitable partitions

In Chapter 2, we found that any two fractionally isomorphic graphs have the same fractional domination number. Using this, we get a new necessary condition for two graphs to be fractionally isomorphic.

Theorem 3.6.1 *If two graphs G and H are fractionally isomorphic, then they belong to the same class.*

Proof. We proceed by considering the action of the matrix S on a function f; specifically, we shall show that Sf has the property of being minimum fractional dominating (or maximum fractional packing) on G if f has that property on H. Suppose A and B are adjacency matrices of G and H respectively and S is a doubly stochastic matrix such that $AS = SB$. Suppose that f is an MFDF on H; then $(B+I)\boldsymbol{f} = \mathbf{1} + \epsilon$, where $\epsilon \geq \mathbf{0}$. Then:

$$\begin{aligned} N(S\boldsymbol{f}) &= (NS)\boldsymbol{f} \\ &= (AS + IS)\boldsymbol{f} \\ &= (SB + SI)\boldsymbol{f} \\ &= S((B+I)\boldsymbol{f}) \\ &= S(\mathbf{1} + \epsilon) \\ &= \mathbf{1} + S\epsilon \end{aligned}$$

Since both S and ϵ are non-negative, so is their product. Therefore, Sf is an MFDF on G (note that Sf is minimum, since $|Sf| = |f|$ and $\gamma_f(G) = \gamma_f(H)$ as shown in Chapter 2.). Further, $S\epsilon = \mathbf{0}$ if and only if $\epsilon = \mathbf{0}$. Hence, if f is an MFDF but not packing in H, then the same goes for Sf in G.

A similar demonstration will reveal that if f is a maximum fractional packing on H (and thus $(B+I)\boldsymbol{f} = \mathbf{1} - \epsilon$ for some nonnegative vector ϵ), then Sf is a maximum fractional packing on G, and likewise that the property of being non-dominating is preserved.

To complete the proof, note that fractional isomorphism is an equivalence relation, and hence symmetric; specifically, if $AS = SB$, then $BS^T = S^T A$. Hence, S sends D_H into D_G, P_H into P_G and $D_H \cap P_H$ into $D_G \cap P_G$. Further, S^T sends D_G into D_H, P_G into P_H and $D_G \cap P_G$ into $D_H \cap P_H$, hence the two graphs share a class. □

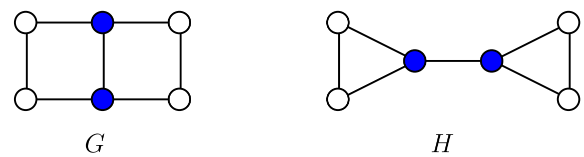


Figure 3.11: G and H are fractionally isomorphic, with $G \in$ Class $\mathcal{D}$, thus $H \in$ Class $\mathcal{D}$.

Although being in the same class is a necessary condition for two graphs to be fractionally isomorphic, it is not sufficient. Both $K_{2,3}$ and C_5 are in Class $\mathcal{E}$, however, they are not fractionally isomorphic to each other (since their degree sequences are different).

Let $\mathcal{C} = \{V_1, \ldots, V_r\}$ be an equitable partition of the vertices $v_1, \ldots, v_n$ of G. Define the matrix $S^{(\mathcal{C})}$ by:

$$S^{(\mathcal{C})}_{i,j} = \begin{cases} 0 \text{ if } v_i \text{ and } v_j \text{ are in different cells of } \mathcal{C} \\ \\ |V_k|^{-1} \text{if } v_i \text{ and } v_j \text{ are both in } V_k \end{cases}$$

Theorem 3.6.2 *Let f be a fractional dominating (or packing) function on G. Then $f_{\mathcal{C}} = S^{(\mathcal{C})} f$ is a fractional dominating (packing) function on G with the property that, if v_i and v_j belong to the same cell of $\mathcal{C}$, then $f_{\mathcal{C}}(v_i) = f_{\mathcal{C}}(v_j)$.*

Proof. First we show that $S^{(C)}$ is a fractional automorphism of G (with adjacency matrix A). To show that $S^{(C)}A = AS^{(C)}$, it suffices to show that either of these products is symmetric. Consider the element $(AS^{(\mathcal{C})})_{i,j} = \sum_k A_{i,k}S^{(\mathcal{C})}_{k,j}$ and its image under transposition. Let us say that $v_i \in V_a$ and $v_j \in V_b$; by the construction of the two matrices, it is clear that $(AS^{(\mathcal{C})})_{i,j} = \frac{d(v_i,V_b)}{|V_b|}$, and similarly $(AS^{(\mathcal{C})})_{j,i} = \frac{d(v_j,V_a)}{|V_a|}$. If $a = b$ then these two quantities are equal, since $G[V_a]$ is regular. If $a \neq b$, then we observe the two quantities to be equal from $d(v_i, V_b)|V_a| = d(v_j, V_a)|V_b|$; this equation results from counting the edges of the bipartite graph $G[V_i, V_j]$ two different ways. Therefore $S^{(\mathcal{C})}$ is a fractional automorphism of G.

It is proved in [120], that if S is a fractional automorphism of G and if f is a fractional dominating or packing function, then so is Sf. Thus, to complete the proof, we only need show that the product function is constant on each cell of the equitable partition. This follows from the observation that, if $V_i = \{v_{i_1}, \ldots, v_{i_m}\}$, then for any $k, 1 \leq k \leq m$ we have $S^{(\mathcal{C})}f(v_{i_k}) = \frac{1}{m}\sum_j f(v_{i_j})$. □

Corollary 3.6.3 *Let $\mathcal{C}$ be an equitable partition of G. If G has an MFDF which is non-packing, an MFPF which is non-dominating, or an FDPF, then it has such a function which is constant on each cell of C.*

Suppose that f is a function on the vertex set of G which is constant on the cells of $\mathcal{C}$, and define a new function $f^{(\mathcal{C})}$ on the cells of $\mathcal{C}$ such that $f^{(\mathcal{C})}(V_i) = f(x_i)$ for $x_i \in V_i$. Then clearly $(A^{(\mathcal{C})} + I)\boldsymbol{f}^{(\mathcal{C})} \geq \mathbf{1}$ if and only if $N\boldsymbol{f} \geq \mathbf{1}$, and likewise if f is a maximum fractional packing or an FDPF.

Note that in the corollary below, the graphs G and H need not have the same order.

Corollary 3.6.4 *Suppose that G and H have identical cell adjacency matrices for some equitable partitions $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then G and H belong to the same class.*

Thus, finding equitable partitions can make discovering fractional dominating and packing functions easier. It should be noted that the natural "reduced" linear program for fractional domination – Minimize $c^T x$ subject to $(A^{(\mathcal{C})} + I)x \geq \mathbf{1}, x \geq \mathbf{0}$ where c is the vector $(|V_1|, \ldots, |V_r|)^T$ – is no longer the dual to the corresponding "reduced" program for fractional packing since the cell-adjacency matrix need not be symmetric.

Corollary 3.6.4, also gives an alternative proof to Theorem 3.6.1, since two graphs are fractionally isomorphic if and only if they share some equitable partition (see Theorem 2.5.2, from [146]).

3.7 Mycielski graphs

As an application of the cell-adjacency matrix, we consider the Mycielski construction, discussed in Chapter 1. The first two Mycielski graphs belong to Class $\mathcal{E}$; P_2 and $Y(P_2) = C_5$ by regularity; the third and fourth by arguments below.

The third Mycielski graph $Y(Y(P_2))$ is called the *Grötzcsh graph.* Using the equitable partition $C = \{X, Y, \{z\}\}$, we solve $(A^{(C)} + I)\boldsymbol{f} = \mathbf{1}$, by augmenting $A^{(C)} + I$ with $\mathbf{1}$ and reducing. After reduction we have the FDPF f which assigns the weight of $\frac{1}{4}$ to each of the x_i, $\frac{1}{8}$ to each y_i and $\frac{3}{8}$ to z. Since f has positive weights on each vertex, $Y^2(P_2)$ is in Class $\mathcal{E}$ by Corollary 3.3.8.

$$\left[\begin{array}{c|c} A^{(C)} + I_3 & \mathbf{1} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 5 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c|c} & \frac{2}{8} \\ I_3 & \frac{1}{8} \\ & \frac{3}{8} \end{array}\right]$$

The fourth Mycielski graph $Y^3(P_2)$ has an equitable partition $D = \{C, C', \{z\}\}$, where C is the equitable partition of $Y^2(P_2)$ used above, and C' is C applied to $\{Y\}$. To solve $(A^{(D)} + I)\boldsymbol{f} = \mathbf{1}$, we augment $A^{(D)} + I$ with $\mathbf{1}$ and reduce. After reduction we have the

FDPF f with positive weights on each cell of D (and thus each vertex), so $Y^3(P_2)$ is in Class $\mathcal{E}$ by Corollary 3.3.8.

$$\left[\begin{array}{c|c} A^{(D)}+I_7 & \mathbf{1} \end{array}\right] = \left[\begin{array}{ccccccc|c} 3 & 2 & 0 & 2 & 2 & 0 & 0 & 1 \\ 2 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 5 & 1 & 0 & 5 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 5 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 & 5 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c|c} & \frac{5}{27} \\ & \frac{3}{27} \\ & \frac{7}{27} \\ I_7 & \frac{2}{27} \\ & \frac{1}{27} \\ & \frac{3}{27} \\ & \frac{9}{27} \end{array}\right]$$

Conjecture 3.7.1 *The kth Mycielski graph is in Class $\mathcal{E}$ for all k.*

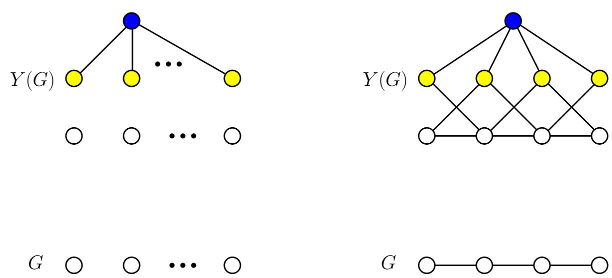


Figure 3.12: $Y(K_1 \cup \cdots \cup K_1)$ and $Y(P_4)$.

Given an arbitrary starting graph G, $Y(G)$ is not necessarily in Class $\mathcal{E}$. If $G = P_4$, the Mycielski $Y(P_4)$ is in Class $\mathcal{N}$ (see Figure 3.12). If G is a collection of isolates, then $Y(G)$ is the disjoint union of the star $K_{1,n}$ with the collection of isolates, which is in Class $\mathcal{P}$ (by Theorem 3.4.2 and Lemma 3.5.1). However, if our starting graph is regular without isolates, we have some results.

Theorem 3.7.2 *Let G be any regular graph without isolates, then $Y(G)$ is in Class $\mathcal{E}$.*

Proof. Each $x \in X$ is adjacent to k of the x_i and, therefore, k of the y_i. Each $y \in Y$ is adjacent to z and k of the x_i. Lastly, z is adjacent to n of the y_i. Therefore, $C = \{X, Y, \{z\}\}$ is an equitable partition. Below we solve $(A^{(C)} + I)\boldsymbol{f} = \mathbf{1}$.

$$\left[\begin{array}{c|c} A^{(C)} + I_3 & \mathbf{1} \end{array}\right] = \left[\begin{array}{ccc|c} k+1 & k & 0 & 1 \\ k & 1 & 1 & 1 \\ 0 & n & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c|c} & \frac{n-1}{k^2+(n-1)k+n-1} \\ I_3 & \frac{k}{k^2+(n-1)k+n-1} \\ & \frac{k(k-1)+n-1}{k^2+(n-1)k+n-1} \end{array}\right]$$

The reduction gives the FDPF f which assigns the weight of $\frac{n-1}{k^2+(n-1)k+n-1}$ to each x_i, the weight of $\frac{k}{k^2+(n-1)k+n-1}$ to each y_i, and the weight of $\frac{k(k-1)+n-1}{k^2+(n-1)k+n-1}$ to z in $Y(G)$. Since G has no isolates, $k > 0$ and $n > 1$; thus each of the three weights are positive. Therefore, $Y(G)$ is in Class $\mathcal{E}$ by Corollary 3.3.8. $\square$

Theorem 3.7.2 may have a natural generalization:

Conjecture 3.7.3 *If G is any regular graph without isolates, then $Y^j(G) \in$ Class $\mathcal{E}$, for all $j \geq 0$.*

We end this section with another conjecture, one whose proof will require more than just finding a convenient equitable partition.

Conjecture 3.7.4 *If G is any Class $\mathcal{E}$ graph without isolates, then $Y(G) \in$ Class $\mathcal{E}$.*

3.8 Miscellaneous graphs

Here we present some interesting graphs whose class membership does not follow immediately from the preceding theory. Figure 3.13 illustrates two Class $\mathcal{E}$ graphs (left), three Class $\mathcal{D}$ graphs (center), and two Class $\mathcal{N}$ graphs (right).

The graph depicted in Figure 3.13(a) is called a pancyclic graph since it has cycles of lengths $3, 4, ..., n$. The function which assigns the weight of $\frac{1}{7}$ to the blue vertices and $\frac{2}{7}$

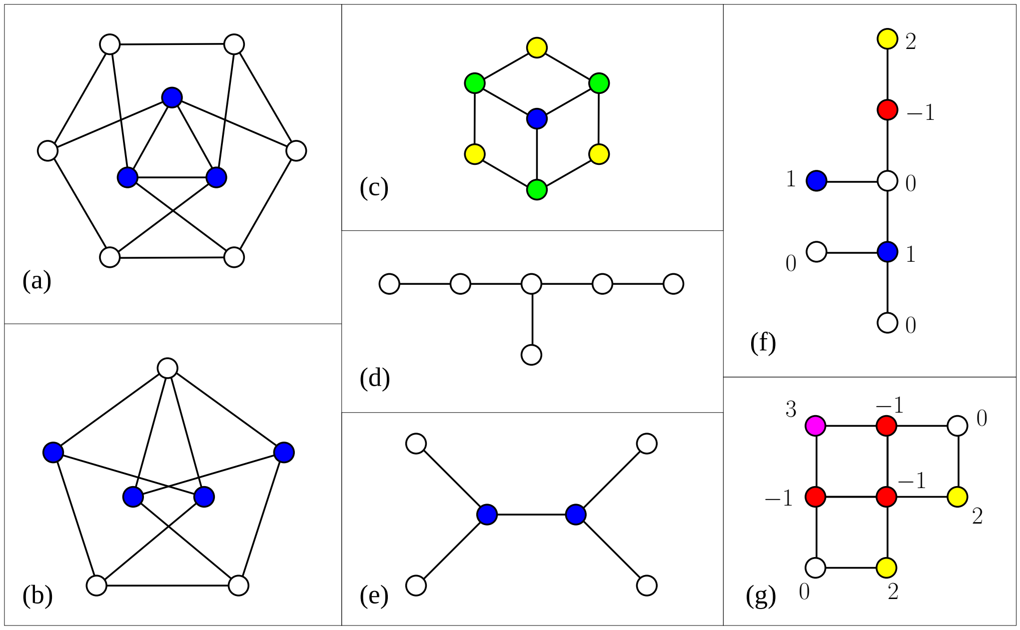


Figure 3.13: (a) A Pancyclic graph, (b) The Moser spindle, (c) A 3-cube with a vertex removed, (d) A wounded spider, (e) A Tree on 6 vertices, (f) A tree on 7 vertices, (g) $G_{3,3}$ minus a vertex of degree two.

otherwise is an FDPF with positive weights on each vertex, so by Corollary 3.3.8, this graph is in Class $\mathcal{E}$. The Moser spindle (also pancyclic) pictured in Figure 3.13(b) is in Class $\mathcal{E}$, since the function which assigns the weight of $\frac{1}{6}$ to the blue vertices and $\frac{1}{3}$ otherwise is an FDPF which has positive weights on each vertex.

Let G be the graph obtained from deleting a vertex of a 3−cube (see Figure 3.13(c)). If we assign the weight of $\frac{5}{12}$ to the green vertices, $\frac{6}{12}$ to the blue vertex and $\frac{2}{12}$ to the yellow vertices we have an MFDF f which which has positive weights on each vertex and is not a packing. Therefore by Lemma 3.3.5, G is in Class $\mathcal{D}$. The wounded spider (Figure 3.13(d)) obtained from subdividing two edges of a $K_{1,3}$ is in Class $\mathcal{D}$, since the constant function of $\frac{1}{2}$ is an MFDF which is not packing. For the tree T on 6 vertices (Figure 3.13(e)), if we assign the weight of $\frac{1}{2}$ to the vertices of degree two, and $\frac{1}{4}$ to the vertices of degree one, then we have an MFDF which which has positive weights on each vertex and is not a packing. Therefore by Lemma 3.3.5, T is in Class $\mathcal{D}$.

The tree on 7 vertices (Figure 3.13(f)), has $\boldsymbol{x} = (2, -1, 0, 1, 0, 1, 0)$ as the unique solution of $N\boldsymbol{x} = \mathbf{1}$, and is thus Class $\mathcal{N}$. The last Class $\mathcal{N}$ graph we give is a 3×3 grid graph with a vertex of degree 2 removed, depicted in Figure 3.13(g). The unique solution to $N\boldsymbol{x} = \mathbf{1}$ is $\boldsymbol{x} = (-1, 2, 2, -1, -1, 3, 0, 0)$.

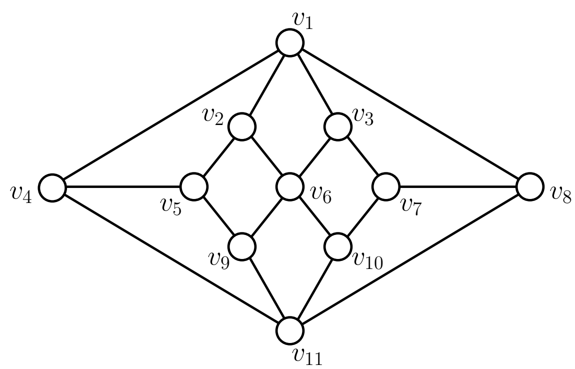


Figure 3.14: The Herschel graph.

The Herschel graph arises in the study of Hamiltonian algorithms (Figure 3.14). An equitable partition of the vertices is $C = \{\{v_1, v_{11}\}, \{v_4, v_8\}, \{v_5, v_7\}, \{v_2, v_3, v_9, v_{10}\}, \{v_6\}\}$. From the reduction below, we see that the function which assigns the weight of $\frac{2}{5}$ to the vertices $\{v_5, v_7\}$ and $\frac{1}{5}$ otherwise is an FDPF. Since each partition receives a positive weight under this FDPF, the Herschel graph is Class $\mathcal{E}$.

$$\left[\begin{array}{c|c} A^{(C)} + I_5 & \mathbf{1} \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c|c} & \frac{1}{5} \\ & \frac{1}{5} \\ I_5 & \frac{2}{5} \\ & \frac{1}{5} \\ & \frac{1}{5} \end{array}\right]$$

3.9 Notes

This chapter was joint work with Walsh [153]. Ideally, the final product of this research program would be a complete classification of, if not all graphs, then at least all major families of graphs into our five classes; or creating an algorithm for finding the class of a graph. There are several more practical avenues to explore, however.

[96] looks at the effects of small perturbations of graphs (the addition and deletion of single vertices or edges) on their domination numbers. We could ask similar questions in this setting: given a graph in a given class, what can we say about the class of the graph that results from deleting an edge or a vertex?

We are particularly interested in the above question for trees. While categorizing all graphs into the five classes may be overly ambitious, we feel that there should be an accessible algorithmic method for determining the class of any tree. One approach which we have been pursuing is to examine which trees are in Class $\mathcal{N}$, and devising measures for quantifying how far a Class $\mathcal{N}$ tree is from being "partitionable". The theory of efficient domination (see [8]), particularly the efficient fractional domination number, should be applicable here.

Recall from Chapter 1, that if there exists an efficient fractional dominating function on a graph G, then the efficient fractional domination number, $F_f(G) = n$. Any fractional efficient dominating function would also be a fractional packing, since by definition, the function g would satisfy $g(N[v]) = 1$ for all $v \in V(G)$. Thus, any graph G on n vertices in Class $\mathcal{I}$, Class $\mathcal{D}$, Class $\mathcal{P}$, or Class $\mathcal{E}$, would have $F_f(G) = n$; and if G is Class $\mathcal{N}$, then $F_f(G) < n$.

We end this chapter with a classification of graphs on 5 or fewer vertices. Of the 52 graphs, only 5 do not follow immediately from the preceeding theory or examples: G36, G37, G41, G44, and G48 (graphs are named using the convention in [147]). We classify these graphs below (see Figure 3.15). The rest follow from Theorems 3.4.1, 3.4.2, 3.4.4, Lemma 3.5.1 and Figure 3.6. We give a complete classification of graphs with five or fewer vertices in Appendix C, class by class for convenience and beauty.

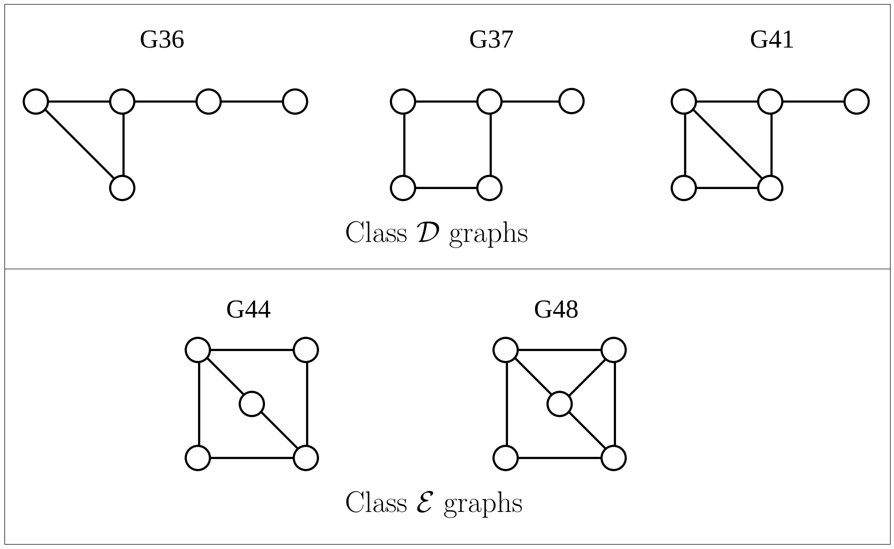


Figure 3.15: Graphs needed to complete the classification of graphs with 5 or fewer vertices.

Chapter 4

Minimum Fractional Total Dominating Functions

and Maximum Fractional Open Packing Functions

4.1 Functions which are both minimum fractional total dominating and maximum fractional open packing

In Chapter 1, we saw that finding fractional total dominating functions and fractional open packing functions are dual linear programs. As with fractional domination and fractional packings in Chapter 3, fractional total domination and fractional open packings are a special dual pair of LPs since the vectors being optimized in both problems can be interpreted as real-valued functions on the set of vertices. Hence, it becomes possible to have a function whose vector simultaneously solves both the fractional total domination LP and its dual. We call a function which is both a minimum FTDF and a maximum FOPF a *fractional total dominating-open packing function* (FTD-OPF). Note that a function which is both an FTDF and FOPF is necessarily an MFTDF and an MFOPF and is therefore an FTD-OPF.

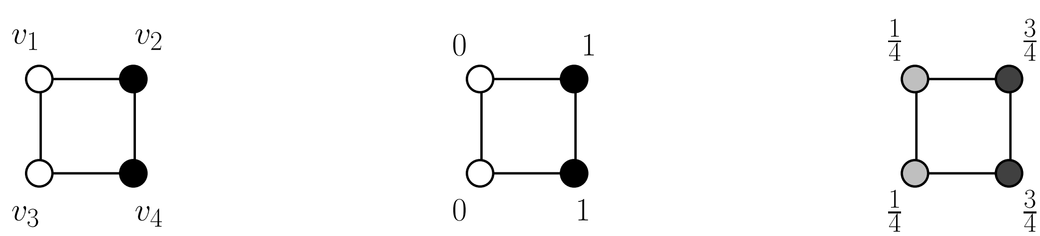


Figure 4.1: (a) The black vertices form a maximum open packing and a minimum total dominating set, (b) its characteristic function, an FTD-OPF; and (c) an FTD-OPF of C_4.

4.2 Definition of the classes

We wish to investigate the interactions of MFOPFs and MFTDFs. Our ultimate goal is to classify those graphs in which all MFOPFs are FTD-OPFs, or all MFTDFs are FTD-OPFs, or when there are no FTD-OPFs at all, as well as any graphs in between. To do this we define the following five classes based on the intersections of the two following sets: Let D^*_G be the set of all MFTDFs on G and let P^*_G be the set of all MFOPFs on G. Every finite simple graph without isolates G belongs to exactly one of the classes below:

- $G \in$ Class $\mathcal{N}^*$ (Null) if $D^*_G \cap P^*_G = \varnothing$.
- $G \in$ Class $\mathcal{I}^*$ (Intersection) if $D^*_G \cap P^*_G \neq \varnothing$, $D^*_G \nsubseteq P^*_G$ and $P^*_G \nsubseteq D^*_G$
- $G \in$ Class $\mathcal{P}^*$ (Packing) if $D^*_G \subsetneq P^*_G$.
- $G \in$ Class $\mathcal{D}^*$ (Dominating) if $P^*_G \subsetneq D^*_G$.
- $G \in$ Class $\mathcal{E}^*$ (Equal) if $D^*_G = P^*_G$.

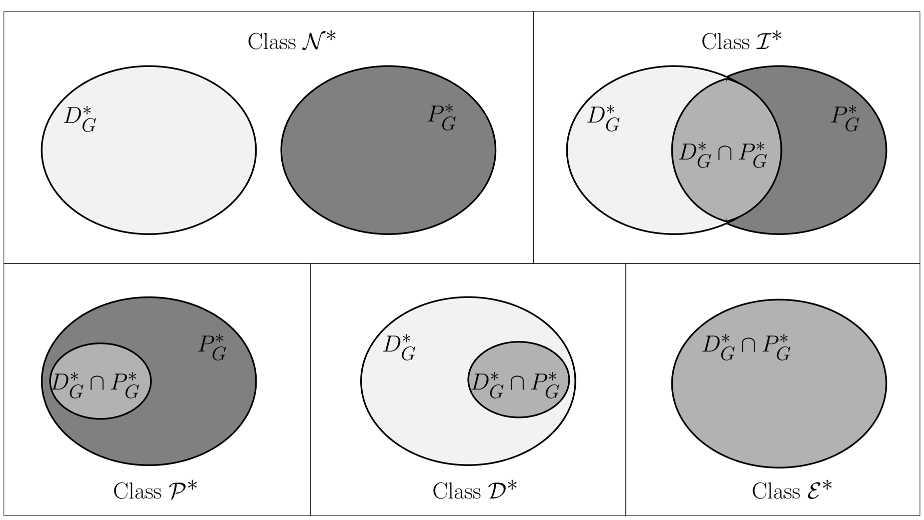


Figure 4.2: The five classes as different intersections of the sets D^*_G (in light gray) and P^*_G (in dark gray); with $D^*_G \cap P^*_G$ in gray.

4.3 The principle of complementary slackness

As in Chapter 3, the principal tool that we shall use in our investigations is the Principle of Complementary Slackness (see Theorem 1.3.2).

Proposition 4.3.1 (Applied principle of complementary slackness) *Let $\boldsymbol{x'}$ be an MFOPF, that is any optimal solution to the linear program: maximize $\mathbf{1}^T\boldsymbol{x}$ subject to $A\boldsymbol{x} \leq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$, and let $\boldsymbol{y'}$ be an MFTDF, that is any optimal solution to the dual linear program: minimize $\mathbf{1}^T\boldsymbol{y}$ subject to $A^T\boldsymbol{y} \geq \mathbf{1}$, $\boldsymbol{y} \geq \mathbf{0}$. Then*

$$\boldsymbol{x'} \cdot (A^T\boldsymbol{y'} - \mathbf{1}) = \boldsymbol{y'} \cdot (A\boldsymbol{x'} - \mathbf{1}) = \mathbf{0}.$$

We give the two main consequences below:

Corollary 4.3.2 *If f is an MFTDF and $v \in V(G)$ for which $f(v) > 0$, then for any MFOPF g, $g(N(v)) = 1$.*

Corollary 4.3.3 *If g is an MFOPF and $v \in V(G)$ for which $g(v) > 0$, then for any MFTDF f, $f(N(v)) = 1$.*

These corollaries, in turn, suffice to establish a number of technical lemmas, which we now state and prove.

Lemma 4.3.4 *If f is an MFTDF with $f(v) > 0$ for every vertex $v \in V$, then every MFOPF is an MFTDF, and thus $P_G^* \subseteq D_G^*$.*

Proof. Let f be an MFTDF on G with $f(v) > 0$ for each vertex v. Then by Corollary 4.3.2, every MFOPF g has the property that $g(N(v)) = 1$ for each vertex v. So g is an MFTDF. □

Lemma 4.3.5 *If we can find two functions f and g where f is an MFTDF on G with $f(v) > 0$ for each vertex v, and where g is an MFTDF on G which is not an open packing, then $G \in$ Class $\mathcal{D}^*$.*

Proof. The MFTDF f gives us $P_G^* \subseteq D_G^*$ and the MFTDF (non-open packing) g gives us $P_G^* \subsetneq D_G^*$. $\square$

Lemma 4.3.6 *If f is an MFOPF with $f(v) > 0$ for each vertex $v \in V$, then every MFTDF is an MFOPF, and thus $D_G^* \subseteq P_G^*$.*

Proof. Let g be an MFOPF on G with $g(v) > 0$ for each vertex v. Then by Corollary 4.3.3, every MFTDF f has the property that $f(N(v)) = 1$ for each v. So f is an MFOPF. $\square$

Lemma 4.3.7 *If we can find two functions f and g where f is an MFOPF on G with $f(v) > 0$ for each vertex v, and where g is an MFOPF on G which is not total dominating, then $G \in$ Class $\mathcal{P}^*$.*

Proof. The MFOPF f gives us $D_G^* \subset P_G^*$ and the MFOPF (non-total dominating) g gives us $D_G^* \subsetneq P_G^*$. $\square$

The results of Lemmas 4.3.5 and 4.3.7 also work if there is a single function which has the properties of f and g simultaneously.

We need two more lemmas to aid in determining Class $\mathcal{N}^*$ graphs using analogues of results from [121].

Lemma 4.3.8 *If f is an MFTDF and $v \in V(G)$ for which $f(N(v)) > 1$, then every MFOPF g satisfies $g(v) = 0$.*

Proof. Suppose g is an MFOPF with $g(v) > 0$. Then by Corollary 4.3.3, every MFTDF f satisfies $f(N[v]) = 1$, a contradiction. $\square$

Lemma 4.3.9 *If g is an MFOPF and $v \in V(G)$ for which $g(N(v)) < 1$, then every MFTDF f satisfies $f(v) = 0$.*

Proof. Suppose f is an MFTDF with $f(v) > 0$. Then by Corollary 4.3.2, every MFOPF g satisfies $g(N[v]) = 1$, a contradiction. □

If we can find a function which is both an MFTDF and an MFOPF which has positive weights on each vertex, then combining Lemmas 4.3.4 and 4.3.6 yields:

Corollary 4.3.10 *For a graph G, if there exists an FTD-OPF f with $f(v) > 0$ for each vertex $v \in V$, then $G \in$ Class $\mathcal{E}^*$.*

4.4 A partial classification

Before we begin, we give necessary and sufficient conditions for a graph to be in Class $\mathcal{N}^*$.

4.4.1 Class $\mathcal{N}$ull* graphs

If G is a Class $\mathcal{N}^*$ graph, then no MFTDF is an MFOPF and no MFOPF is an MFTDF. That is for any graph in Class $\mathcal{N}^*$ there are no FTD-OPFs on G. Therefore, it is true that no FTDF is a FOPF and no FOPF is a FTDF. There is an easy characterization of Class $\mathcal{N}^*$ graphs using its adjacency matrix.

Proposition 4.4.1 *G is in Class $\mathcal{N}^*$ if and only if the system $A\boldsymbol{x} = \boldsymbol{1}$ has no non-negative solutions.*

Proof. If $A\boldsymbol{x} = \mathbf{1}$ has a non–negative solution, then the vector $\boldsymbol{x}$ is an FTD-OPF, thus G is not in Class $\mathcal{N}^*$. Likewise, if G is not in Class $\mathcal{N}^*$, then there exists a vector $\boldsymbol{x}$ satisfying $A\boldsymbol{x} = \mathbf{1}$, with $\boldsymbol{x} \geq \mathbf{0}$ (each x_i is non-negative), thus the system has a non-negative solution. □

If we replace N with A in the LP formulation of efficient fractional domination (1.15), we have the LP formulation of efficient fractional total domination:

$$\text{maximize } \mathbf{1}^T A\boldsymbol{x} \text{ subject to } A\boldsymbol{x} \leq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0} \tag{4.1}$$

Another characterization of Class $\mathcal{N}^*$ graphs is any graph on n vertices (with no isolates) with efficient fractional total domination number strictly less than n.

Proposition 4.4.2 *G of order n is in Class $\mathcal{N}^*$ if and only if the efficient fractional total domination number strictly less than n.*

With these preliminaries in place, we are ready to begin sorting families of graphs into our five classes.

4.4.2 Regular graphs

Theorem 4.4.3 *Every regular graph without isolates is Class $\mathcal{E}^*$.*

Proof. Let G be k-regular with no isolates; then the function $f(v) = \frac{1}{k}$ for all $v \in V$ is an FTD-OPF. Since f is nonzero at each vertex, Corollary 4.3.10 tells us that $G \in$ Class $\mathcal{E}^*$. □

4.4.3 Trees

Theorem 4.4.4 *Every Star is class Class $\mathcal{E}^*$.*

Proof. Let $G = K_{1,n}$. The function which assigns the weight of 1 to the vertex of degree $n-1$ and $\frac{1}{n}$ otherwise is an FTD-OPF. Since this function has positive weights at each vertex G is Class $\mathcal{E}^*$ by Corollary 4.3.10. □

In Chapter 3, we encountered wounded healthy spiders (see Figure 3.3). Healthy spiders are formed from subdividing all of the edges of a $K_{1,t}$ with $t \geq 2$. If we subdivide r edges of a $K_{1,t}$, we have a wounded spider (with $1 < r < t$). We give the proof at the end of Section 4.6.

Theorem 4.4.5 *If G is a spider with $t \geq 2$ and $1 < r \leq t$, then G is Class $\mathcal{N}^*$.*

Theorem 4.4.6 *Every Double Star is Class $\mathcal{P}^*$.*

Proof. Let G be a double star, a tree on $s+t+2$ vertices with a vertex v_s of degree $s+1$ and a vertex v_t of degree $t+1$ both adjacent to each other; where v_s is adjacent to s vertices of degree one and v_t is adjacent to t vertices of degree one. The function which assigns the weight of 1 to v_s and v_t and 0 otherwise is an FTD-OPF. The function which assigns the weight of $\frac{1}{s}$ to the vertices of degree one which are adjacent to v_s and $\frac{1}{t}$ to the vertices of degree one which are adjacent to v_t is an MFOPF which is not total dominating. By taking the average of the two functions, we get an MFOPF which is not total dominating and has positive weights on each vertex. Thus, by Lemma 4.3.7, G is Class $\mathcal{P}^*$. $\square$

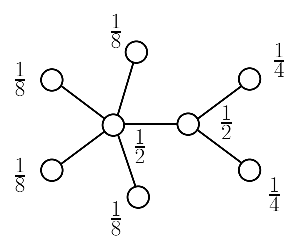


Figure 4.3: An MFOPF which is not total dominating of a double star.

4.4.4 Partial trampolines and generalized Hajós graphs

We revisit two important constructions defined in Chapter 1, partial trampolines and the generalized Hajós graph, $[K_n]$. Recall that both trampolines and $[K_n]$ are defined only for $n \geq 3$. $[K_3] \cong T(K_3)$ which we will see below is Class $\mathcal{P}^*$.

Theorem 4.4.7 *Let G be the partial trampoline $T(C_n)$, then G is Class $\mathcal{P}^*$.*

Proof. The function f which assigns the weight of 0 to the vertices of degree two (called u_i in Chapter 3) and $\frac{1}{2}$ otherwise is an FTD-OPF. The constant function g which assigns the weight of $\frac{1}{4}$ to each vertex is an MFOPF which is not total dominating (since $g(N(u_i)) = \frac{1}{2} < 1$ for each i). Thus, by Lemma 4.3.7, $G \in$ Class $\mathcal{P}^*$. $\square$

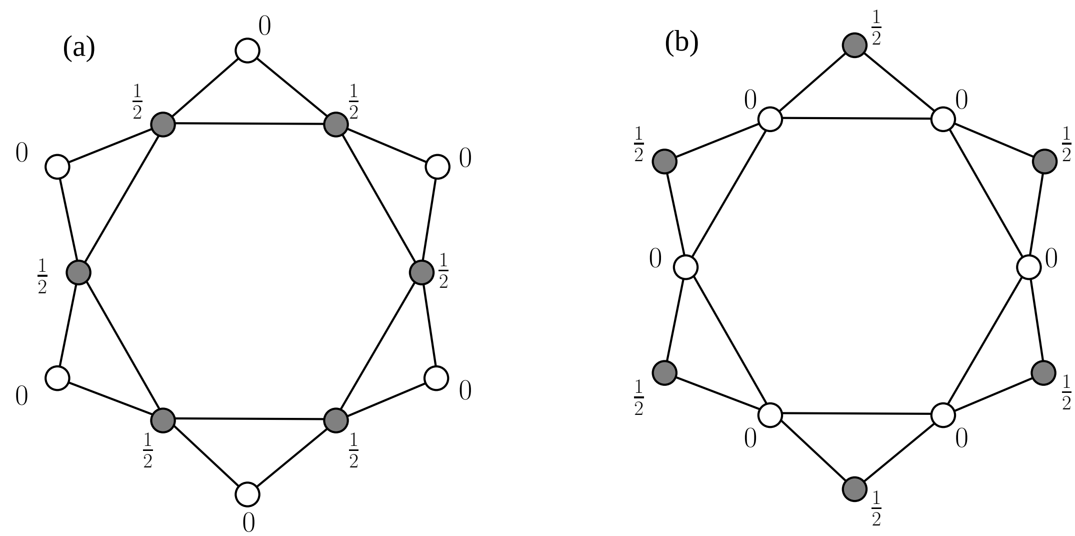


Figure 4.4: (a) An FTD-OPF and (b) an MFOPF which is not total dominating of the partial trampoline $T(C_6)$.

Theorem 4.4.8 *The generalized Hajós graph, $[K_n]$ is Class $\mathcal{N}^*$, if $n \geq 4$.*

Proof. The function f which assigns the weight of 0 to each vertex of degree two (called u_{ij} in Chapter 3) and $\frac{1}{2}$ otherwise is an FTDF which is not an open packing if $n \geq 4$. The function g which assigns the weight of $\frac{1}{n-1}$ to the vertices of degree two and 0 otherwise is a FOPF which is not total dominating. Since $|f| = |g| = \frac{n}{2}$, f is an MFTDF and g is an MFOPF. Since $f(N(v_i)) = \frac{n-1}{2} > 1$ for all i, by Lemma 4.3.8, every MFOPF h satisfies $h(v_i) = 0$ for all i. Since $g(N(u_{ij})) = \frac{1}{n-1} < 1$ then every MFTDF j satisfies $j(u_{ij}) = 0$ for all $1 \leq i < j \leq n$. Therefore, no MFOPF can be an MFTDF, thus $[K_n] \in$ Class $\mathcal{N}^*$. $\square$

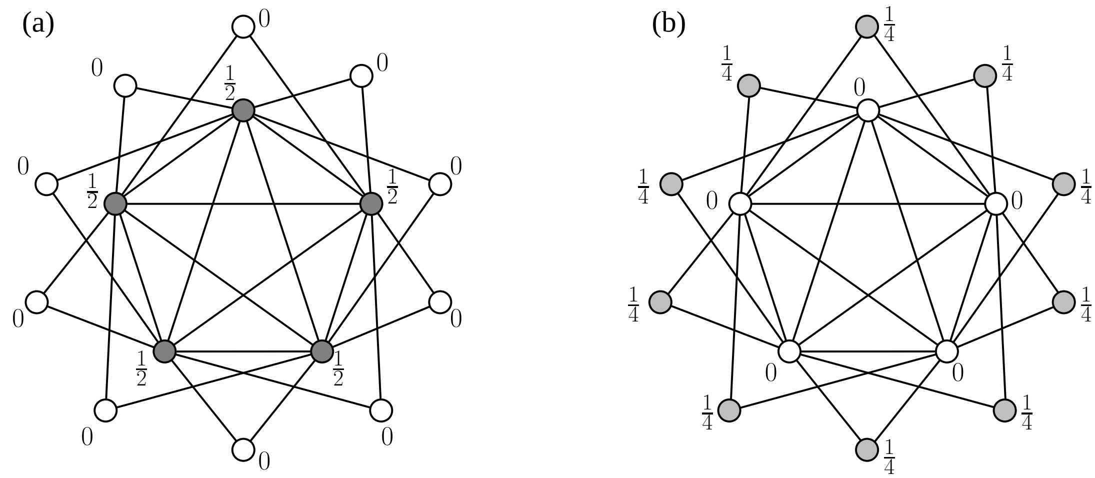


Figure 4.5: (a) An MFTDF which is not an open packing and (b) an MFOPF which is not total dominating of the generalized Hajós graph, $[K_5]$.

4.5 Sums and products of graphs

4.5.1 Disjoint unions

Total domination and open neighborhood packing (in both their integer and fractional forms) interact quite simply with disjoint unions as they did with domination and closed neighborhood packing in Chapter 3.

Lemma 4.5.1 *The function $f : V(G \cup H) \to [0, 1]$ is a fractional total dominating (open packing) function on $G \cup H$ if and only if $f|_G$ is fractional total dominating (open packing) on G and $f|_H$ is fractional total dominating (open packing) on H.*

Using this, we can determine the class of a disjoint union of two graphs, given the classes of the two starting graphs. The results are easy to check, and are summarized as follows:

4.5.2 Cartesian Products

Theorem 4.5.2 *$P_2 \Box P_{3k}$ is Class $\mathcal{P}^*$ for all $k > 0$.*

$\cup$	Class $\mathcal{D}^*$	Class $\mathcal{E}^*$	Class $\mathcal{I}^*$	Class $\mathcal{N}^*$	Class $\mathcal{P}^*$
Class $\mathcal{D}^*$	Class $\mathcal{D}^*$	Class $\mathcal{D}^*$	Class $\mathcal{I}^*$	Class $\mathcal{N}^*$	Class $\mathcal{I}^*$
Class $\mathcal{E}^*$	Class $\mathcal{D}^*$	Class $\mathcal{E}^*$	Class $\mathcal{I}^*$	Class $\mathcal{N}^*$	Class $\mathcal{P}^*$
Class $\mathcal{I}^*$	Class $\mathcal{I}^*$	Class $\mathcal{I}^*$	Class $\mathcal{I}^*$	Class $\mathcal{N}^*$	Class $\mathcal{I}^*$
Class $\mathcal{N}^*$	Class $\mathcal{N}^*$	Class $\mathcal{N}^*$	Class $\mathcal{N}^*$	Class $\mathcal{N}^*$	Class $\mathcal{N}^*$
Class $\mathcal{P}^*$	Class $\mathcal{I}^*$	Class $\mathcal{P}^*$	Class $\mathcal{I}^*$	Class $\mathcal{N}^*$	Class $\mathcal{P}^*$

Table 4.1: Class of the disjoint union of two graphs

Proof. Let $\{p_i\} = \{v_{1,2}, v_{2,2}, \ldots, v_{1,3k-1}, v_{2,3k-1}\}$. This set induces a partition of V(G) into the open neighborhoods $\{N(p_i)\}$. The function which assigns 1 to each p_i and 0 otherwise is an FTD-OPF. The constant function which assigns the weight of $\frac{1}{3}$ to each vertex is a MFOPF, which is not total dominating. Thus, $P_2 \square P_{3k}$ is Class $\mathcal{P}^*$ by Lemma 4.3.7. $\square$

4.6 Fractional isomorphisms and equitable partitions

Theorem 4.6.1 *If two graphs G and H are fractionally isomorphic, then they belong to the same class.*

Proof. We proceed by considering the action of the matrix S on a function f; specifically, we shall show that Sf has the property of being minimum fractional total dominating (or maximum fractional open packing) on G if f has that property on H. Suppose A and B are adjacency matrices of G and H respectively and S is a doubly stochastic matrix such that $AS = SB$. Suppose that f is an MFDF on H; then $B\boldsymbol{f} = \mathbf{1} + \epsilon$, where $\epsilon \geq \mathbf{0}$. Then:

$$
\begin{aligned}
A(S\boldsymbol{f}) &= (AS)\boldsymbol{f} \\
&= (SB)\boldsymbol{f} \\
&= S(B\boldsymbol{f}) \\
&= S(\mathbf{1} + \epsilon) \\
&= \mathbf{1} + S\epsilon
\end{aligned}
$$

Since both S and ϵ are non-negative, so is their product. Therefore, Sf is an MFTDF on G (note that Sf is minimum, since $|Sf| = |f|$ and $\gamma_f^\circ(G) = \gamma_f^\circ(H)$ as shown in Chapter 2.). Further, $S\epsilon = \mathbf{0}$ if and only if $\epsilon = \mathbf{0}$. Hence, if f is an MFTDF but not an open packing in H, then the same goes for Sf in G.

A similar demonstration will reveal that if f is a maximum fractional open packing on H (and thus $B\boldsymbol{f} = \mathbf{1} - \epsilon$ for some nonnegative vector ϵ), then Sf is a maximum fractional packing on G, and likewise, that the property of being not total dominating is preserved.

To complete the proof, note that fractional isomorphism is an equivalence relation, and hence symmetric; specifically, if $AS = SB$ then $BS^T = S^T A$. Hence, S sends D_H^* into D_G^*, P_H^* into P_G^*, and $D_H^* \cap P_H^*$ into $D_G^* \cap P_G^*$. Further, S^T sends D_G^* into D_H^*, P_G^* into P_H^*, and $D_G^* \cap P_G^*$ into $D_H^* \cap P_H^*$, hence the two graphs share a class. $\square$

Theorem 4.6.2 *Let f be a fractional total dominating (or open packing) function on G. Then $f_C = S^{(C)}f$ is a fractional total dominating (open packing) function on G with the property that, if v_i and v_j belong to the same cell of C, then $f_C(v_i) = f_C(v_j)$.*

The proof is similar to the proof of Theorem 3.6.2.

For the graph G in Figure 4.6, the function which assigns the weight of $\frac{1}{4}$ to each of the six blue vertices and $\frac{1}{2}$ to each of the three red vertices is an MFTDF which is not packing,

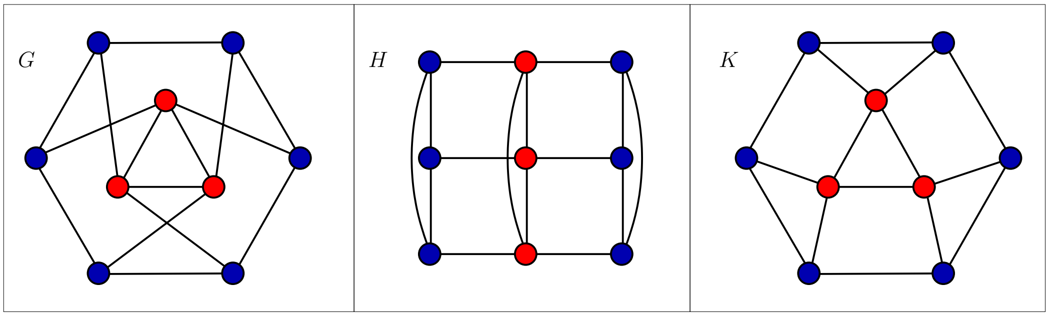


Figure 4.6: Three fractionally isomorphic Class $\mathcal{D}^*$ graphs.

and thus by Lemma 4.3.5, $G \in$ Class $\mathcal{D}^*$. G, H and K are all fractionally isomorphic to each other, and are thus in the same class, Class $\mathcal{D}^*$.

Proof of Theorem 4.4.5. Let r be the number of edges subdivided in a star $K_{1,t}$.

Case 1: $r = t$. G is obtained by subdividing each edge of a $K_{1,t}$. If we let H be the head vertex and B and F be the sets of degree two and foot vertices respectively, then $\{H, B, F\}$ is an equitable partition of G. Let h, b, and f be the weights of the head vertex, degree two and foot vertices respectively. If there existed an FTD-OPF, then there would exist one which satisfied $N(H) = tb = 1$ and $N(F) = b = 1$. This is not possible if $t \geq 2$, therefore the system $A\boldsymbol{x} = \mathbf{1}$ has no non-negative solutions. Thus, G is Class $\mathcal{N}^*$.

Case 2: $n = 3k+1$. Let G be a wounded spider formed from subdividing $1 < r < t$ edges of a $K_{1,t}$. Since we are allowing r subdivisions with $1 < r < t$, we must have $t \geq 3$, and thus the head vertex (of degree t) is well defined. Consider the equitable partition $\{H, B, E, F\}$ where H is the cell containing the head vertex, R is the cell containing vertices of degree two, E is the cell containing foot vertices (degree one) which are adjacent to vertices of degree two, and F is the cell containing foot vertices (degree one) which are adjacent to the head vertex. If there existed an FTD-OPF, then there would exist one which satisfied $N(H) = tb + f = 1$, $N(B) = h + re$, $N(E) = rb$, and $N(F) = h = 1$. This forces each vertex in R to have weight 1, the head vertex to receive the weight of 1, and thus the function would not be an open packing. Therefore, the system $A\boldsymbol{x} = \mathbf{1}$ has no non-negative solutions. Thus, G is Class $\mathcal{N}^*$. $\square$

Theorem 4.4.5 required the number of subdivided edges $1 < r < t$. If $t = 2$ and $r = 1$, then we have a path on four vertices. By assigning the weight of $\frac{1}{4}$ to each vertex of degree two and $\frac{3}{4}$ to the vertices of degree one, we have an MFOPF which is not total dominating with positive weights on each vertex, thus $P_4 \in$ Class $\mathcal{P}^*$.

4.7 The Mycielski construction

Theorem 4.7.1 *Let G be any regular graph of degree $k > 0$, then $Y(G)$ is in Class $\mathcal{E}^*$.*

Proof. Each $x \in X$ is adjacent to k of the x_i and, therefore, k of the y_i. Each $y \in Y$ is adjacent to z and k of the x_i. Lastly, z is adjacent to n of the y_i. Therefore, $C = \{X, Y, \{z\}\}$ is an equitable partition. Below we solve $(A^{(C)})\boldsymbol{f} = \mathbf{1}$.

$$\left[\begin{array}{c|c} A^{(C)} & \mathbf{1} \end{array}\right] = \left[\begin{array}{ccc|c} k & k & 0 & 1 \\ k & 0 & 1 & 1 \\ 0 & n & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{c|c} & \frac{n-k}{nk} \\ I_3 & \frac{1}{n} \\ & \frac{k}{n} \end{array}\right]$$

The reduction gives the FTD-OPF f which assigns the weight of $\frac{n-k}{nk}$ to each x_i, the weight of $\frac{1}{n}$ to each y_i and $\frac{k}{n}$ to z in $Y(G)$. Since $0 < k < n$, each of the weights are positive. Therefore, $Y(G)$ is in Class $\mathcal{E}^*$ by Corollary 4.3.10. $\square$

4.8 Notes

Every MFTDF is necessarily an FDF, however, not necessarily an MFDF. Every MFPF is necessarily an FOPF, but not necessarily an MFOPF. The question of how the sets of all MFTDFs, MFDFs, MFPFs, and MFOPFs interact may be a difficult one. However, we can ask a simpler question of when the sets D_G and P_G interact in the same way as the sets

D_G^* and P_G^* do. This amounts to asking which graphs without isolates are both in Class $\mathcal{X}^*$ and Class $\mathcal{X}$ (where $\mathcal{X}$ is one of: $\{\mathcal{N}, \mathcal{I}, \mathcal{P}, \mathcal{D}, \mathcal{E}\}$). Regular graphs without isolates are both Class $\mathcal{E}^*$ and Class $\mathcal{E}$. The generalized Hajós graph $[K_n]$, for $n \geq 4$, is both Class $\mathcal{N}^*$ and Class $\mathcal{N}$. G15 and G34 are Class $\mathcal{D}^*$, but both are Class $\mathcal{P}$ (see Appendix C). P_5 is Class $\mathcal{N}^*$, however, P_5 is Class $\mathcal{I}$.

In [67], Fisher defined the construction $u(G) = \overline{Y(\bar{G})}$ and found a formula for $\gamma_f^\circ(u(G))$; perhaps this construction could be useful in our investigation of the sets D_G^* and P_G^*.

Just as the strong direct product worked well with MFDFs and MFPFs in Chapter 3, the conjunctive product may work well with MFTDFs and MFOPFs in similar respects.

Chapter 5

Domination Null and Packing Null Vertices

5.1 Introduction

In Chapter 3, we saw that for some graphs, there were vertices which received a weight of zero for some minimum fractional dominating function. If this happens for all MFDFs, then we have the following definition.

We call a vertex $v \in V(G)$ *domination null* iff for every MFDF f on G, $f(v) = 0$. Similarly, a vertex $v \in V(G)$ is *packing null* iff for every MFPF g on G, $g(v) = 0$.

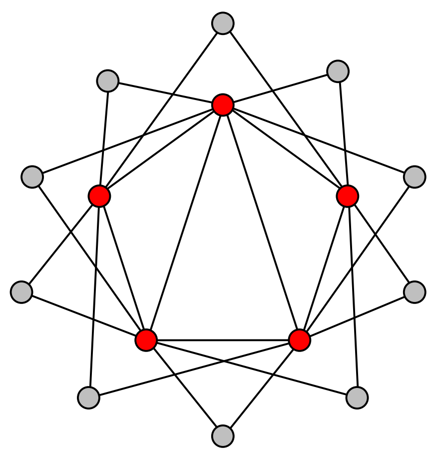


Figure 5.1: Domination null vertices (gray), and packing null vertices (red) of [G47].

As an example, let G be the graph G47 in Figure 22, C_5 with two nonintersecting chords. Then consider partial generalized Hajós graph $[G47]$. The vertices $\{v_i | 1 \leq i \leq n\}$ form a set of packing null vertices in $[G47]$, while $\{u_{ij} | 1 \leq i < j \leq n\}$ is a set of domination null vertices in $[G]$ (with terminology as in Theorem 3.4.13, where the above assertions are proven).

The path on 3 vertices P_3 is the smallest example of a graph with domination null vertices. The path on 4 vertices P_4 is the smallest example of a graph with packing null vertices. The path on 5 vertices P_5 is the smallest graph with domination vertices and packing null vertices. In fact, the middle vertex in P_5 is both domination null and packing null.

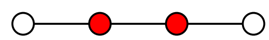


Figure 5.2: Packing null vertices (red) of P_4.

The following consequences of the principle of complementary slackness appear in Chapter 3; we restate them here using the language of this chapter.

Lemma 5.1.1 *If f is an MFDF and $v \in V(G)$ for which $f(N[v]) > 1$, then v is packing null.*

Lemma 5.1.2 *If g is an MFPF and $v \in V(G)$ for which $g(N[v]) < 1$, then v is domination null.*

5.2 Absence of Domination Null and Packing Null Vertices

There are several families of graphs which have no domination null vertices nor packing null vertices. We give partial results on this situation which follow directly from the two Lemmas above.

Lemma 5.2.1 *For a graph G if there exists an FDPF f with $f(v) > 0$ for each vertex $v \in V$ then there are no null vertices of either type in G.*

It follows from the proofs of Theorems 3.4.1, 3.4.3 and 3.7.2 that if G is regular, or G is the complete r-partite graph with parts of size $n_1, n_2, \ldots, n_r$ (with $r \geq 2$ and each $n_j \geq 2$), or if $G = Y(H)$ for some regular graph H with no isolates, then G has no null vertices of either type.

Conjecture 5.2.2 *If $G \in$ Class $\mathcal{E}$, then G has no null vertices (of either type).*

Corollary 3.3.8 can be restated as the following: if G has no null vertices (of either type), then $G \in$ Class $\mathcal{E}$. If Conjecture 5.2.2 is true, then we would have that $G \in$ Class $\mathcal{E}$ iff G has no null vertices (of either type).

Conjecture 5.2.3 *If a graph G is Class $\mathcal{P}$, then there are no packing null vertices.*

Lemma 3.3.7 can be restated as: if G has no packing null vertices, and there exists MFPF which is not dominating, then $G \in$ Class $\mathcal{P}$.

Conjecture 5.2.4 *A graph G is Class $\mathcal{D}$, then there are no domination null vertices.*

Lemma 3.3.5 can be restated as: if G has no domination null vertices, and there exists MFDF which is not packing, then $G \in$ Class $\mathcal{D}$.

Conjecture 5.2.5 *IF graph G is Class $\mathcal{I}$, then there exists a vertex which is both domination and packing null.*

5.3 Total Domination Null and Open Packing Null Vertices

In Chapter 4, we saw that for some graphs, there were vertices which received a weight of zero for some mimimum fractional total dominating function. If this happens for all MFTDFs, then we have the following definition.

We call a vertex $v \in V(G)$ *total domination null* if for every MFTDF f on G, $f(v) = 0$. Similarly, a vertex $v \in V(G)$ *open packing null* if for every MFOPF g on G, $g(v) = 0$.

For P_4, both vertices of degree one are total domination null. For the graph G15 (K_3 with an added edge), the vertices of degree two are open packing null. Recall from the proofs of Theorems 4.4.3 and 4.7.1, that if G is regular or $G = Y(H)$ for some regular graph H with no isolates, then G has no total domination null vertices, nor open packing null vertices.

Not as much is known about Class $\mathcal{I}^*$ graphs at this time to state a conjecture similar to Conjecture 5.2.5, however, we do state conjectures on Class $\mathcal{E}^*$, Class $\mathcal{P}^*$, and Class $\mathcal{D}^*$ graphs.

Conjecture 5.3.1 *If a graph G is Class $\mathcal{E}^*$, then there are no total domination or open packing null vertices.*

Conjecture 5.3.2 *If a graph G is Class $\mathcal{P}^*$, then are no open packing null vertices.*

Conjecture 5.3.3 *If a graph G is Class $\mathcal{D}^*$, then there are no total domination null vertices.*

5.4 Notes

Section 5.1 was joint work with Johnson and Walsh. This chapter was based on [121]. In this paper, several theorems are proved and several questions are asked. To conserve trees, we refer the reader to [121] for a fuller account of domination and packing null vertices.

Chapter 6

Roman Domination

6.1 Introduction

A recent article in *Scientific American* suggested a new variant of domination; see [170]. A few lesser known articles ([149], [150], and [151]) in the John Hopkins Magazine suggested Roman domination a few years earlier. A Roman dominating function on a graph G is a function $f : V \to \{0, 1, 2\}$ which satisfies the property that whenever $f(v) = 0$, there exists a $u \in N(v)$ for which $f(u) = 2$. The total weight of a minimum Roman dominating function is γ_R.

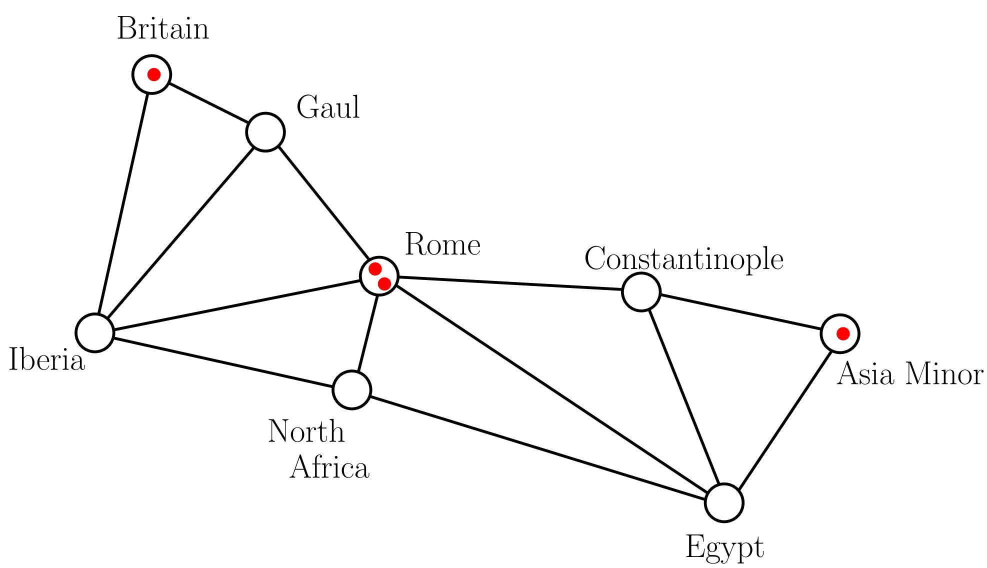


Figure 6.1: The Roman Empire.

The Roman domination number ranges from $\gamma \leq \gamma_R \leq 2\gamma$. The only graph on n vertices with Roman domination number equal to its domination number is $\overline{K_n}$ (see [40]). Graphs with $\gamma_R = 2\gamma$ are called *Roman graphs*.

6.2 Roman domination as an integer program

In [152], ReVelle and Rosing formulate Roman domination as an integer program. For a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, for each v_i in V, define two $\{0, 1\}$ variables X_i and Y_i to be the first and second legions respectively located at v_i. In earlier literature, this program is called the Set Covering Deployment Problem, or SCDP.

$$
\begin{array}{lll}
\text{Minimize} & \sum_{i=1}^{n} (X_i + Y_i) & \\
\text{Subject to:} & & X_i \geq Y_i \text{ for all } i \\
 & & X_i + \displaystyle\sum_{v_i v_j \in E} Y_j \geq 1 \text{ for all } i \\
 & & X_i, Y_i \in \{0, 1\} \text{ for all } i
\end{array} \tag{6.1}
$$

The first constraint guarantees that the first legion is stationed at a vertex before the second. The first and second constraints guarantee that every vertex either has a legion stationed on it or has a neighbor with two legions stationed on it. The third constraint allows for only entire legions to be stationed. From an optimal solution to the SCDP problem, $\{X_1, Y_1, \ldots, X_n, Y_n\}$, we can obtain a minimum Roman dominating function (MRDF) r by letting $r(v_i) = X_i + Y_i$ for all i. Thus, the value of $\sum_{i=1}^{n} (X_i + Y_i)$, for any optimal solution is equal to γ_R.

We wish to translate the above IP into matrix terms. If we let $\boldsymbol{v} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T$ be the $2n \times 1$ matrix $\left[X_1, \ldots, X_n, Y_1, \ldots, Y_n\right]^T$, then (6.1) is equivalent to (6.2) below.

$$\text{Minimize} \quad \begin{bmatrix} \mathbf{1}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$$

$$\text{Subject to:} \quad \begin{bmatrix} I_n & A \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{v} \in \{0,1\} \text{ vector} \tag{6.2}$$

We can relax the condition that $\boldsymbol{v}$ be a $\{0,1\}$ vector, and instead require that the entries be non-negative. Then the integer program (6.2) becomes a linear program. The value of $\mathbf{1}^T\boldsymbol{v}$ for any optimal solution of (6.3) is equal to the *fractional* (open neighborhood) Roman domination number, $\gamma_{R_f^\circ}(G)$.

$$\text{Minimize} \quad \begin{bmatrix} \mathbf{1}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$$

$$\text{Subject to:} \quad \begin{bmatrix} I_n & A \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{v} \geq \mathbf{0} \tag{6.3}$$

The reason for the parenthetical remark "open neighborhood" and the use of "$\circ$" in the notation will become apparent in Section 6.3.

6.2.1 Beamers and buffers

The dual linear program of fractional (open neighborhood) Roman domination (6.3) is given below, where $\begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}^T$ is the $2n \times 1$ matrix with the 1 as the first n entries and 0 as the next n entries, and $\boldsymbol{u}^T$ as the $2n \times 1$ matrix $\begin{bmatrix} \boldsymbol{W} & \boldsymbol{Z} \end{bmatrix}^T$.

$$\begin{aligned} &\text{Maximize} && \begin{bmatrix} \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \\ &\text{Subject to:} && \begin{bmatrix} I_n & I_n \\ A & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \\ &&& \boldsymbol{u} \geq \mathbf{0} \end{aligned} \tag{6.4}$$

The value of the above linear program is the fractional (open neighborhood) Roman domination number. The integer program is then:

$$\begin{aligned} &\text{Maximize} && \begin{bmatrix} \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \\ &\text{Subject to:} && \begin{bmatrix} I_n & I_n \\ A & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \\ &&& \boldsymbol{u} \in \{0,1\} \text{ vector} \end{aligned} \tag{6.5}$$

The value of this integer program will be called the (open neighborhood) beamer buffer number, denoted by π_R°. Thus, we have $\pi_R^\circ(G) \leq \gamma_{R_f^\circ}(G) \leq \gamma_R(G)$, for all graphs G.

There is an interesting story to go along with the dual linear program of Roman domination. On the planet Zelgon, the cities are connected to one another in a network or graph. The beings of Zelgon want to emit as much total light as possible with the following constraints. At any location v_i a light emits W_i units of radiation, not to the inhabitants of its location, but to each of its neighbors. According to a pilot study, it has been determined that the inhabitants of planet Zelgon can handle at most one unit of radiation safely. To enable more light, there is a free buffer material (from the planet's abundant supply of straw) that any location v_i, with $W_i < 1$, may be used to reduce the amount of radiation by Z_i units (with Z_i at most $1 - W_i$).

We illustrate the above with an example, with $G = C_5$. If we let $Y_1 = 1$ (which forces $X_1 = 1$) and $X_3 = X_4 = 1$ and all other X_i and Y_i be zero, then we have $\gamma_R \leq 4$. The function $f(v_i) = X_i + Y_i$ is in fact a minimum Roman dominating function, or an MRDF. To compute $\gamma_{R_f^\circ}$ of C_5, we seek to minimize the sum $\sum_{i=1}^{5}(X_i + Y_i)$ subject to the constraints: $X_i \geq Y_i$ for all i,

$$X_1 + Y_5 + Y_2 \geq 1$$

$$X_2 + Y_1 + Y_3 \geq 1$$

$$X_3 + Y_2 + Y_4 \geq 1$$

$$X_4 + Y_3 + Y_5 \geq 1$$

$$X_5 + Y_4 + Y_1 \geq 1$$

If we let $X_i = Y_i = \frac{1}{3}$, we have $\gamma_{Rf}^{\circ} \leq \frac{10}{3}$. Now we verify by finding an optimal solution to the dual LP: maximize $\sum_{i=1}^{5} W_i$, subject to $W_i + Z_i \leq 1$,

$$W_5 + W_2 - Z_1 \leq 1$$

$$W_1 + W_3 - Z_2 \leq 1$$

$$W_2 + W_4 - Z_3 \leq 1$$

$$W_3 + W_5 - Z_4 \leq 1$$

$$W_4 + W_1 - Z_5 \leq 1$$

If we let $W_i = \frac{2}{3}$ and $Z_i = \frac{1}{3}$, we have $\gamma_{Rf}^{\circ} \geq \frac{10}{3}$. Thus, $\gamma_{Rf}^{\circ} = \frac{10}{3}$, and $f(v) = \left(\frac{1}{3}, \frac{1}{3}\right)$ for all v is an MFRDF of C_5.

In [40], it was shown that $\gamma_R(C_n) = \left\lceil \dfrac{2n}{3} \right\rceil$.

Theorem 6.2.1 $\gamma_{Rf}^{\circ}(C_n) = \dfrac{2n}{3}$.

Proof. Suppose G is a cycle on n vertices. If we let $X_i = Y_i = \frac{1}{3}$, for $1 \leq i \leq n$, this is a solution to (6.3) with total weight $\frac{2n}{3}$. Therefore, $\gamma_{Rf}^{\circ}(C_n) \leq \frac{2n}{3}$. $W_i = \frac{2}{3}$ and $Z_i = \frac{1}{3}$ for $1 \leq i \leq n$, is a solution to the dual LP (6.4), with total weight $\sum_{i=1}^{n} W_i = \frac{2n}{3}$. From this, we have that $\gamma_{Rf}^{\circ}(C_n) \geq \frac{2n}{3}$. Thus, $\gamma_{Rf}^{\circ}(C_n) = \frac{2n}{3}$. □

6.3 Closed neighborhood fractional Roman domination

In the previous section, traveling armies (or portions of) could not defend the location which they were stationed at, a situation which is ambiguous in the integer Roman domination problem. In the integer case, if there is a traveling army stationed at a location, then there is a full home army stationed as well. In the event of an attack on this location, the traveling army need not defend the home. We give the integer programming formulation of closed neighborhood Roman domination below. By the remark above, the value of (6.6) is equal to the value of (6.3); that is, if we replace A in with N in (6.3), the value of the IP is unchanged.

$$\text{Minimize} \quad \begin{bmatrix} \mathbf{1}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$$

$$\text{Subject to:} \qquad \begin{bmatrix} I_n & N \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{v} \in \{0, 1\} \text{ vector} \tag{6.6}$$

We can relax the condition that $\boldsymbol{v}$ be a $\{0, 1\}$ vector, and instead require that the entries be non-negative. Then the integer program (6.6) becomes a linear program. The value of $\mathbf{1}^T\boldsymbol{v}$ for any optimal solution of (6.7) is equal to the *fractional* (closed neighborhood) Roman domination number, $\gamma_{Rf}(G)$.

$$\text{Minimize} \quad \begin{bmatrix} \mathbf{1}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix}$$

$$\text{Subject to:} \qquad \begin{bmatrix} I_n & N \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{v} \geq \mathbf{0} \tag{6.7}$$

6.3.1 Closed neighborhood beamers and buffers

The circumstances for the dual LP also change when closed neighborhoods are used. So now the radiation emitted from a location is felt at that location. The dual linear program of fractional (closed neighborhood) Roman domination (6.7) is given below, where $[\ \mathbf{1} \quad \mathbf{0}\ ]^T$ is the $2n \times 1$ matrix with the 1 as the first n entries and 0 as the next n entries.

$$\text{Maximize} \quad \begin{bmatrix} \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix}$$

$$\text{Subject to:} \qquad \begin{bmatrix} I_n & I_n \\ N & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{u} \geq \mathbf{0} \tag{6.8}$$

The value of the above linear program is the fractional (closed neighborhood) Roman domination number. The integer program is then:

$$\text{Maximize} \quad \begin{bmatrix} \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix}$$

$$\text{Subject to:} \qquad \begin{bmatrix} I_n & I_n \\ N & -I_n \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{u} \in \{0, 1\} \text{ vector} \tag{6.9}$$

The value of this integer program will be called the (closed neighborhood) beamer buffer number, denoted by π_R. Thus, we have $\pi_R(G) \leq \gamma_{Rf}(G) \leq \gamma_R(G)$, for all graphs G.

Theorem 6.3.1 $\gamma_{Rf}(C_n) = \dfrac{n}{2}$.

Proof. Suppose G is a cycle on n vertices. If we let $X_i = Y_i = \frac{1}{4}$, for $1 \leq i \leq n$, this is a solution to (6.7) with total weight $\frac{n}{2}$. Therefore, $\gamma_{Rf}(C_n) \leq \frac{n}{2}$. $W_i = Z_i = \frac{1}{2}$, for $1 \leq i \leq n$, is a solution to the dual LP (6.8), with total weight $\sum_{i=1}^{n} W_i = \frac{n}{2}$. From this, we have that $\gamma_{Rf}(C_n) \geq \frac{n}{2}$. Thus, $\gamma_{Rf}(C_n) = \frac{n}{2}$. $\square$

6.4 Fractional isomorphisms revisited

Roman domination is a non–invariant parameter with respect to fractional isomorphism. Recall that any two $k-$regular graphs on n vertices are fractionally isomorphic, and so $C_9 \cong_f C_4 \cup C_5$. However, $\gamma_R(C_9) = 6$, while $\gamma_R(C_4 \cup C_5) = 7$ (See Figure 14 in Appendix B). We will show that both forms of fractional Roman domination are invariant parameters.

Theorem 6.4.1 *Let S be a fractional isomorphism from G to H and $f(v) = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T$ be a fractional open (respectively closed) Roman dominating function on H. Then $f(v) = \begin{bmatrix} S\boldsymbol{X} & S\boldsymbol{Y} \end{bmatrix}^T$ is a fractional open (respectively closed) Roman dominating function on G.*

Proof. Let A and B be the adjacency matrices of G and H so that $AS = SB$. Let M represent B for fractional open neighborhood Roman domination or $N = B + I$ for fractional closed neighborhood Roman domination on H. Let M' represent A for fractional open neighborhood Roman domination or $N = A + I$ for fractional closed neighborhood Roman domination on G. Let $R = \begin{bmatrix} I_n & M \\ I_n & -I_n \end{bmatrix}$ and $S \oplus S = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$. Suppose $\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T$ is an FRDF on H. Then $R\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T \geq \begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}^T$. This implies that $R\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T = \begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}^T + \boldsymbol{\varepsilon}$, for some non-negative vector $\boldsymbol{\varepsilon}$. Note that $(S \oplus S)\begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}^T = \begin{bmatrix} S\mathbf{1} & S\mathbf{0} \end{bmatrix}^T = \begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}^T$ and $(S \oplus S)\boldsymbol{\varepsilon}$ is non-negative. Furthermore, we have the sum of the entries of $(S \oplus S)\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T$ is the sum of the entries of $\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T$. Hence, if we can show that $(S \oplus S)R = R(S \oplus S)$, then we are done.

$$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_n & M \\ I_n & -I_n \end{bmatrix} = \begin{bmatrix} S & SM \\ S & -S \end{bmatrix} = \begin{bmatrix} S & M'S \\ S & -S \end{bmatrix} = \begin{bmatrix} I_n & M' \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$$

$\square$

Corollary 6.4.2 *$\gamma_{R_f^\circ}$ and γ_{Rf} are invariant parameters.*

Proof. If we let f be a *minimum* FRDF (open or closed), then Theorem 6.4.1 gives $\gamma_{R_f^\circ}(G) \leq \gamma_{R_f^\circ}(H)$ and $\gamma_{Rf}(G) \leq \gamma_{Rf}(H)$. If we replace S with S^T, then we have if $f(v) = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}^T$ is a fractional open (respectively closed) Roman dominating function on G, then $f(v) = \begin{bmatrix} S^T\boldsymbol{X} & S^T\boldsymbol{Y} \end{bmatrix}^T$ is a fractional open (respectively closed) Roman dominating function on H. Thus, if f is a *minimum* FRDF (open or closed), then $\gamma_{R_f^\circ}(G) \geq \gamma_{R_f^\circ}(H)$ and $\gamma_{Rf}(G) \geq \gamma_{Rf}(H)$. $\square$

Corollary 6.4.3 *Let $\mathcal{C}$ be an equitable partition on $V(G)$. Then there exists a minimum fractional (open, respectively closed) Roman dominating function which is constant on each cell of the partition $\mathcal{C}$.*

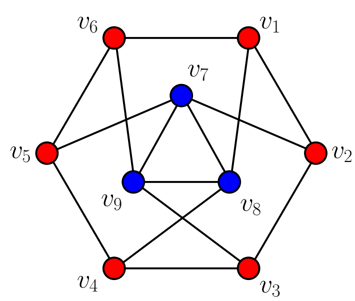


Figure 6.2: A Roman graph G.

As an example, the Roman graph above has nine vertices, thus finding $\gamma_{R_f^\circ}$ would require solving 18 equations in 18 variables X_i and Y_i. We will use only four. We will use the equitable partition of the vertices $\{\{v_1, v_2, v_3, v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\}$, as before. The four variables we will need are: Y_r, X_r, Y_b, X_b, which represent fractions of first and second legions on the red vertices and then blue vertices, respectively. We seek to minimize

$6(X_r+Y_r)+3(X_b+Y_b)$ subject to:

$$X_r \geq Y_r$$

$$X_b \geq Y_b$$

$$X_r + 2Y_r + Y_b \geq 1$$

$$X_b + 2Y_r + 2Y_b \geq 1$$

We can see that $X_r = Y_r = \frac{2}{7}$ and $X_b = Y_b = \frac{1}{7}$ is a solution. To see that it is an optimal solution, we check the dual LP. For the dual, we seek to maximize: $6W_r + 3Z_b$ subject to:

$$W_r + Z_r \leq 1$$

$$W_b + Z_b \leq 1$$

$$2W_r + W_b - Z_r \leq 1$$

$$2W_r + 2W_b - Z_b \leq 1$$

We can see that $W_r = \frac{4}{7}, Z_r = \frac{3}{7}$, $W_b = \frac{2}{7}$, and $Z_b = \frac{5}{7}$ is a solution, with total weight $\frac{30}{7}$. Thus, the function $f(v_i) = \left(\frac{2}{7}, \frac{2}{7}\right)$ for $1 \leq i \leq 6$ and $f(v_i) = \left(\frac{1}{7}, \frac{1}{7}\right)$ for $7 \leq i \leq 9$ is a minimum fractional (open neighborhood) Roman dominating function, with a total weight of $\gamma_{R_f^\circ}(G) = \frac{30}{7}$.

For the fractional (closed neighborhood) Roman domination number of the above Roman graph G we seek to minimize $6(X_r+Y_r)+3(X_b+Y_b)$ subject to:

$$X_r \geq Y_r$$

$$X_b \geq Y_b$$

$$X_r + 3Y_r + Y_b \geq 1$$

$$X_b + 3Y_r + 2Y_b \geq 1$$

We can see that $X_r = Y_r = \frac{2}{9}$ and $X_b = Y_b = \frac{1}{9}$ is a solution. To see that it is an optimal solution, we check the dual LP.

For the dual, we seek to maximize: $6W_r + 3Z_b$ subject to:

$$W_r + Z_r \leq 1$$

$$W_b + Z_b \leq 1$$

$$3W_r + W_b - Z_r \leq 1$$

$$3W_r + 2W_b - Z_b \leq 1$$

We can see that $W_r = \frac{4}{9}, Z_r = \frac{5}{9}$, $W_b = \frac{2}{9}$, and $Z_b = \frac{7}{9}$ is a solution, with total weight $\frac{30}{9}$. Thus, the function $f(v_i) = \left(\frac{2}{9}, \frac{2}{9}\right)$ for $1 \leq i \leq 6$ and $f(v_i) = \left(\frac{1}{9}, \frac{1}{9}\right)$ for $7 \leq i \leq 9$ is a minimum fractional (closed neighborhood) Roman dominating function, with a total weight of $\gamma_{Rf}(G) = \frac{30}{9}$.

On page 28, we saw that the Roman graph G above is fractionally isomorphic to $P_3 \square C_3$. Thus, we have also found $\gamma_{Rf}(P_3 \square C_3) = \frac{30}{7}$ and $\gamma_{R}{}^{\circ}_{f}(P_3 \square C_3) = \frac{30}{9}$. Note that $\gamma_R(P_3 \square C_3) = 5$.

If G is a Roman graph ($\gamma_R = 2\gamma$), then $\gamma_{R}{}^{\circ}_{f}$ is not necessarily twice γ°_{f}, nor is γ_{Rf} necessarily twice γ_f. The above example verifies this, since $\gamma^{\circ}_{f}(G) = 3$. Note that $\gamma_f(G) = \frac{15}{7}$, which is half of the fractional (open neighborhood) Roman domination number of $\frac{30}{7}$.

6.5 Legion mobilization

Is it possible to rearrange the vertices of a dominating set to obtain a different variation of domination? Consider the weights on the vertices in the characteristic function of a dominating set. If we allow the movement of a weight on a vertex to a vertex distance one away, we call this movement a *mobilization* of the weight. If we allow any of the weights to be moved at most once to vertices distance one away, we have a mobilization of the dominating function. In [40], MRDFs are represented by three sets: V_0 is the set of vertices which receive no legions, V_1 is the set of vertices which receive precisely one legion, and V_2 is the set of vertices which receive two legions. The proof of Theorem 6.5.1 is due to Goddard, Hedetniemi ([98]) and Rubalcaba.

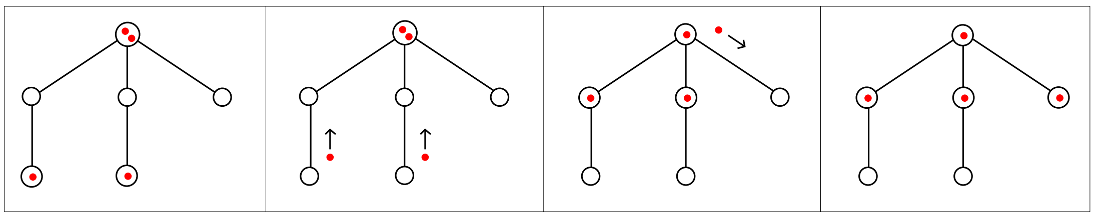


Figure 6.3: Mobilize some of the legions of an MRDF to obtain a *total* dominating set.

Theorem 6.5.1 *If f is a minimum Roman dominating function, having a maximum number of vertices with two legions, on a graph without isolates, then there exists a way to mobilize the legions (by movements of distance at most one) to achieve a total dominating set.*

Proof. Let $f = (V_0, V_1, V_2)$ be an MRDF having a maximum number of vertices in the set V_2. One can assume, therefore, that V_2 dominates the set V_0 and that V_1 is an independent set, every vertex in which is adjacent to at least one vertex in V_0 and no vertices in V_2. Let the vertices of V_1 be $\{u_1, u_2, \ldots, u_k\}$ and let the vertices of V_2 be $\{v_1, v_2, \ldots, v_j\}$.

For $i = 1$ to k do: move the legion at vertex u_i to a neighboring vertex in V_0 which has no legion yet. If no such vertex in V_0 is available, then delete this legion.

For $i = 1$ to j do: move one legion at vertex v_i to a neighboring vertex in V_0 which has no legion yet. If no such vertex in V_0 is available, then delete this legion.

The resulting placement of legions will be a total dominating set, since: each vertex in the original set V_1 now has no legion, but is adjacent to at least one vertex in V_0 having one legion; each vertex in V_0 is adjacent to at least one vertex in V_2 now having one legion; and each vertex in V_2 is adjacent to at least one vertex in V_0 now having one legion. $\square$

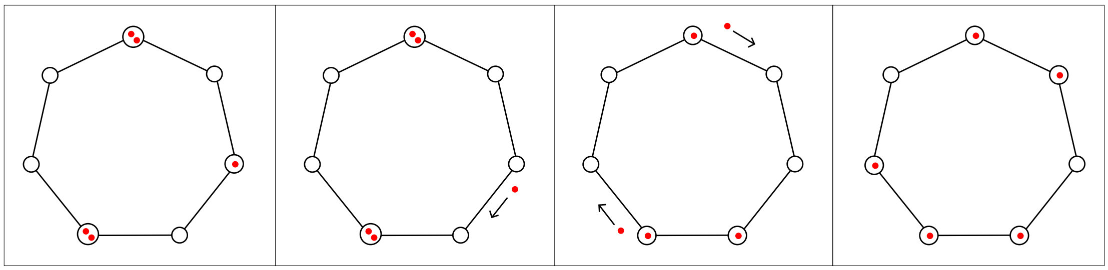


Figure 6.4: Mobilize some of the legions of an MRDF to obtain a *total* dominating set.

Observe that starting from right to left in Figure 6.4, we start with a minimum double dominating function, then move three legions (in the reverse direction of the arrows) to obtain a Roman dominating function. This is not always possible, however. The graph in Figure 6.5 shows that $\gamma_R(G)$ can be larger than the double domination number $dd(G)$. In the figure below, $dd([C_5]) = 5$, whereas $\gamma_R([C_5]) = 7$.

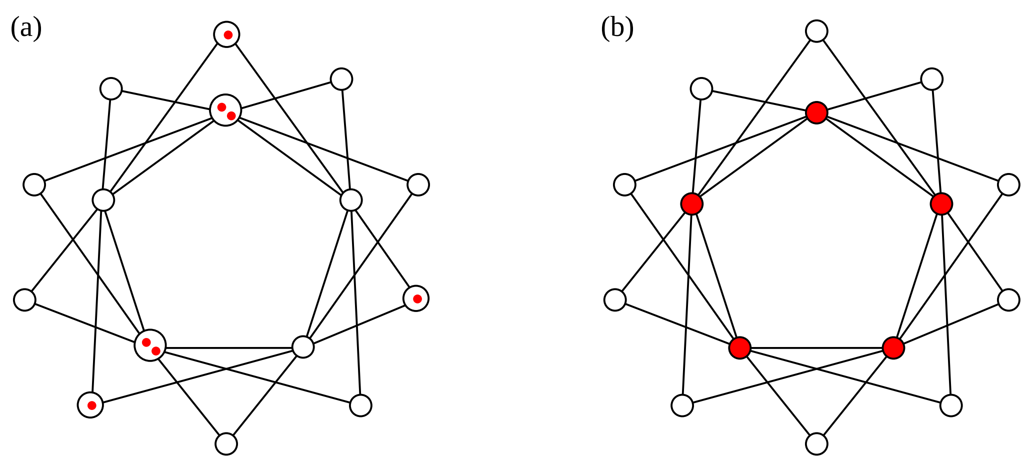


Figure 6.5: (a) The legions form a MRDF and (b) the red vertices form a minimum double dominating set of the partial generalized Hajós graph $[C_5]$

6.5.1 Bounds on γ_R

In [40], upper and lower bounds of the Roman domination number are found in terms of the minimum and maximum degree for any graph without isolates.

$$[40] \qquad \frac{2n}{\Delta+1} \leq \gamma_R \leq n\left(\frac{2+\ln(1+\frac{\delta}{2})}{\delta+1}\right)$$

Corollary 6.5.2 *If G contains no isolates, then $\gamma_t(G) \leq \gamma_R(G)$, furthermore, this bound is sharp.*

Proof. From Theorem 6.5.1, we have $\gamma_t(G) \leq \gamma_R(G)$ for all graphs G without isolates, since certain MRDFs (with a maximum number of vertices with weight 2) can be turned into TDFs with total weight less than or equal to $\gamma_R(G)$. To see that the above bound is sharp, note that $\gamma_t(C_3) = \gamma_R(C_3) = 2$. $\square$

In [94], a $\{k\}-$dominating function is defined as a function $g : V \to \{0, 1, \ldots, k\}$ which satisfies $g(N[v]) \geq k$ for every $v \in V(G)$. The $\{k\}-$domination number, $\gamma_{\{k\}}$ is the minimum weight of a $\{k\}-$dominating function. Figure 6.5 shows that γ_R is not bounded above by $\gamma_{\{2\}}$, since if we let the red vertices get the weight of 1 and 0 otherwise, we have a $\{2\}-$dominating function with total weight $5 < \gamma_R = 7$.

In [55], it was shown that for any graph G and any positive integer $k \geq 2$, $\gamma_f(G) \leq \frac{\gamma_{\{k\}}(G)}{k} \leq \gamma(G)$. In fact, an equivalent definition of the fractional domination number is: $\gamma_f(G) = lim_{k\to\infty}\frac{\gamma_{\{k\}}(G)}{k}$.

Corollary 6.5.3 *If G contains no isolates, then for any integer $k \geq 2$:*

$$\gamma_f(G) \leq \frac{\gamma_{\{k\}}(G)}{k} \leq \gamma(G) \leq \gamma_t(G) \leq \gamma_R(G) \leq 2\gamma(G)$$

A graph parameter $\xi(G)$ is *not comparable* to another parameter $\zeta(G)$ if there exists graphs G and H so that $\xi(G) < \zeta(G)$ and $\xi(H) > \zeta(H)$. There are several parameters which are not comparable to γ_R; in Theorem 6.5.4, we list a few domination parameters which are not comparable to γ_R. In Theorem 6.5.4, $\Gamma(G)$ refers to the maximum cardinality of a minimal dominating set in G; $\gamma_c(G)$ is the minimum cardinality of a connected dominating set; $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set S on G (every vertex outside of S is adjacent to two distinct members of S); and γ_{res} refers to the restrained domination number, the minimum cardinality of a set S of vertices which dominate $V(G)$ and has the property that every vertex outside of S is adjacent to another vertex outside of S (note that γ_r is used for both the restrained domination number and the weak Roman domination number in the literature). Since any connected dominating set with two or more vertices is also a total dominating set, $\gamma_t(G) \leq \gamma_c(G)$ for all G with $\Delta(G) < n-1$ ([94]). However, as we will see below, the connected domination number is not comparable to the Roman domination number.

Theorem 6.5.4 *The domination parameters:* Γ, γ_c, γ_2, dd, $\gamma_{\{2\}}$, *and* γ_{res} *are not comparable to the Roman domination number.*

Proof. Upper domination and Roman domination numbers are not comparable, since $\Gamma(C_6) = 3 < \gamma_R(C_6) = 4$, but $\Gamma(K_{1,3}) = 3 > \gamma_R(K_{1,3}) = 2$. The connected domination and Roman domination numbers are not comparable, since $\gamma_c(P_4) = 2 < \gamma_R(P_4) = 3$, while, $\gamma_c(C_9) = 7 > \gamma_R(C_9) = 6$. The parameters $\gamma_R(G)$ and $\gamma_2(G)$ are not comparable, since $\gamma_2(C_6) = 3 < \gamma_R(C_6) = 4$, whereas $\gamma_2(K_{1,3}) = 4 > \gamma_R(K_{1,4}) = 2$. The double domination number can be quite a bit larger than γ_R, for instance, $dd(K_{1,6}) = 7 > \gamma_R(K_{1,6}) = 2$. Figure 6.5, shows that γ_R can be larger than the double domination number. Since $\gamma_{\{2\}}(G) \leq dd(G)$ (for all graphs G without isolates), the $\{2\}$-domination number can be smaller than γ_R. For the path on four vertices, we have $\gamma_{\{2\}}(P_4) = 4 > \gamma_R(P_4) = 3$.

The restrained domination and Roman domination numbers are not comparable, since $\gamma_{res}(C_5) = 3 < \gamma_R(C_5) = 4$, while, $\gamma_{res}(K_{1,3}) = 4 > \gamma_R(K_{1,3}) = 2$ $\square$

6.6 Notes

The original work in sections 6.1 – 6.4 was joint work with Walsh [177]. Section 6.5 was joint work with Goddard and Hedetniemi [98].

The weak Roman domination number $\gamma_r(G)$ (defined in [99]), is shown to be at most the Roman domination number for any graph G. The total domination number and weak Roman domination number are not comparable, since $\gamma_r(P_5) = \left\lceil \frac{3(5)}{7} \right\rceil = 3 < \gamma_t(P_5) = 4$, whereas $\gamma_r(C_9) = \gamma_R(C_9) = 6 > \gamma_t(C_9) = 5$. Since both total and weak Roman domination numbers are at least the domination number and at most the Roman domination number, γ_r can replace γ_t in the chain in Corollary 6.5.3, and we have for any integer $k \geq 2$:

$$\gamma_f(G) \leq \frac{\gamma_{\{k\}}(G)}{k} \leq \gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G), \text{ for all } G \text{ with } \delta \geq 2$$

Some of the parameters which were not comparable to the Roman domination number may serve as bounds on γ_R, with extra requirements, like minimum degree at least two or higher. Also, $\gamma_{\{2\}}$ was found to be not comparable with γ_R. What if we try to compare $\gamma_{\{k\}}$ for other values of $k > 2$ with γ_R?

Can we show that the total domination number is bounded above by either $\gamma_{R_f^\circ}$ or γ_{R_f}? For which isolate-free graphs G, other than C_3 or C_6, do we have $\gamma_t(G) = \gamma_R(G)$? [21] should be of use here.

Chapter 7

Open Problems

7.1 Open problems

We give several open questions organized by chapter. Johnson offers the prize of "a six-pack of beverages" to any one who can either prove Conjecture 5.2.2 or find a counterexample (see page 83).

7.1.1 Open problems from Chapter 2

- If $G \cong_f H$, then are their upper fractional domination numbers the same? (see [94])
- If $G \cong_f H$, then are their fractional intersection numbers the same? (see [157])
- If G is fractionally Hamiltonian, H is connected, and $G \cong_f H$, then is H necessarily fractionally Hamiltonian? (see [157])
- Which invariant properties or parameters with respect to semi-isomorphism are non-invariant with respect to fractional isomorphism?

7.1.2 Open problems from Chapter 3

- Find an algorithm to find the Class of any graph.

7.1.3 Open problems from Chapter 4

- Find an algorithm to find the Class* of any graph without isolates.

7.1.4 Open problems from Chapter 5

- Is the converse to Theorem 1.3.2 true? If not, is the converse to either Proposition 3.3.1 or Proposition 4.3.1 true?

- Are Conjectures 5.2.2–5.3.3 true?

7.1.5 Open problems from Chapter 6

- We have the following bounds for $\gamma_{R_f^\circ}$ and γ_{Rf}: $\pi_R^\circ \leq \gamma_{R_f^\circ} \leq \gamma_R$ and $\pi_R \leq \gamma_{Rf} \leq \gamma_R$. Can these bounds be improved?

7.2 Conclusion

We have seen the benefits of not only investigating parameters, but also the optimal solutions which yield the value of a parameter (integer or fractional). Much of the research which focuses on the values of a particular parameter for various graphs, bounds on the parameter based on properties of the graph and/or with other parameters, could greatly benefit from this type of investigation. For example, we obtained the new bound $\gamma_t \leq \gamma_R$ (for any graph without isolates) by rearranging weights of optimal solutions of one problem into the other.

At press time of this dissertation, the articles by ReVelle [149], [150], [151], and a note by Peterson [145], were available online at:

[149] http://www.jhu.edu/∼jhumag/0497web/locate3.html

[150] http://www.jhu.edu/∼jhumag/0697web/revelle.html

[151] http://www.jhu.edu/∼jhumag/0997web/roman.html

[145] http://www.maa.org/mathland/mathtrek_9_11_00.html

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Appendix A

Notation

Symbol	Meaning	Page:
$A(G)$	adjacency matrix	3
$A^{(\mathcal{C})}$	cell adjacency matrix	28
$Aut(G)$	automorphism group	25
B	vertex–edge incidence matrix	3
C_n	cycle on n vertices	2
$d(v)$	degree of a vertex v	2
$d(u, S)$	number of edges from u to a set S of vertices	27
dd	double domination number	5
D_G	set of all MFDFs on G	40
D_G^*	set of all MFTDFs on G	68
$E(G)$	edge set of a graph	2
F	efficient domination number	8
F_f	fractional efficient domination number	8
FDPF	fractional dominating-packing function	39
FDT-OPF	fractional total dominating–open packing function	67
MFDF	minimum fractional dominating function	6
MFPF	maximum fractional packing function	7
MFTDF	minimum fractional total dominating function	8
MFOPF	maximum fractional open packing function	9

Symbol	Meaning	Page:
n	order	2
N	neighborhood matrix, $A+I$	3
$N[u]$	closed neighborhood of a vertex v	4
$N(u)$	open neighborhood of a vertex v	4
P_n	path on n vertices	2
P_G	set of all MFPFs on G	40
P_G^*	set of all MFOPFs on G	68
$S \subseteq V$	set of vertices	4
V	vertex set of a graph	2
α_f	fractional independence number	25
Γ	upper domination number	8
γ	domination number.	6
γ_c	connected domination number	100
γ_f	fractional domination number	6
γ_k	k-domination number	5
γ_r	weak Roman domination number	101
$\gamma_{\{k\}}$	$\{k\}$-domination number	99
γ_{res}	restrained domination number	100
γ_R	Roman domination number	85
$\gamma_{R_f^\circ}$	fractional (open) Roman domination number	87
γ_{Rf}	fractional (closed) Roman domination number	91
γ_t	total domination number	8
γ_f°	fractional total domination number	8

Symbol	Meaning	Page:
Δ	maximum vertex degree	2
δ	minimum vertex degree	2
ε	number of edges	2
μ	matching number	9
μ_f	fractional matching number	9
ν	crossing number	25
π	(closed neighborhood) packing number	7
π_t	open (neighborhood) packing number	9
π_R°	(open neighborhood) beamer–buffer number	88
π_R	(closed neighborhood) beamer–buffer number	92
χ	chromatic number	119
χ_f	fractional chromatic number	25
χ'	edge chromatic number	119
ω	clique number	119
$\boldsymbol{f}$	the vector $\vec{f}$	6
$f\|_H$	The function f restricted to the subgraph H	51
$\bar{G}$	complement	34
$[G]$	partial generalized Hajós graph formed from G	17
$G[H]$	The induced subgraph $H \subset V$	4
$G_{m,n}$	$m \times n$ grid graph	15

Symbol	Meaning	Page:
$G \square H$	Cartesian product of G and H	14
$G \times H$	categorical product of G and H	14
$G \boxtimes H$	strong direct product of G and H	14
$G \cup H$	disjoint union of G and H	14
$G \circ H$	corona of G and H	15
$G \cong H$	Graphs G and H are isomorphic	19
$G \cong' H$	Graphs G and H are semi–isomorphic	37
$G \cong_f H$	Graphs G and H are fractionally isomorphic	20
$[K_n]$	generalized Hajós graph	16
K_n	The complete graph; also know as a clique on n vertices	43
$K_{1,t}^*$	Healthy spider	44
$T(K_n)$	trampoline on $2n$ vertices	16
$T_H(G)$	partial trampoline formed from Hamiltonian cycle H	16
$x \otimes y$	tensor product of the vectors x and y	54
$Y(G)$	Mycielski of a graph G	15

Appendix B

Non–Invariants of Fractional Isomorphisms

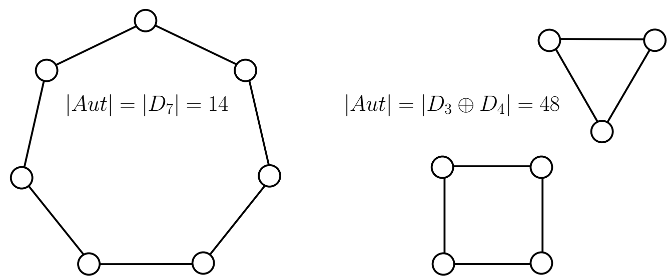

Figure 1: Size of automorphism groups.

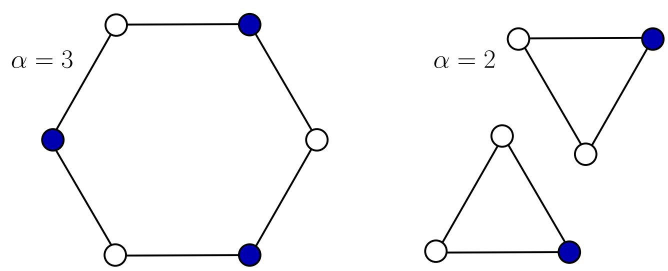

Figure 2: Independent sets of maximum size.

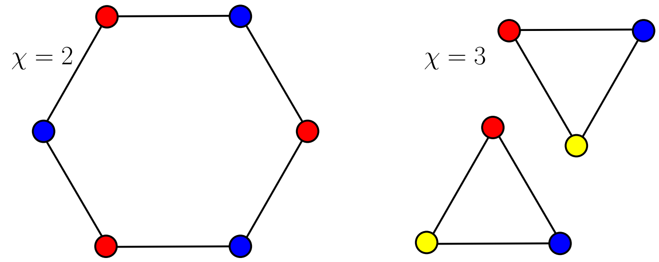

Figure 3: Minimum proper colorings.

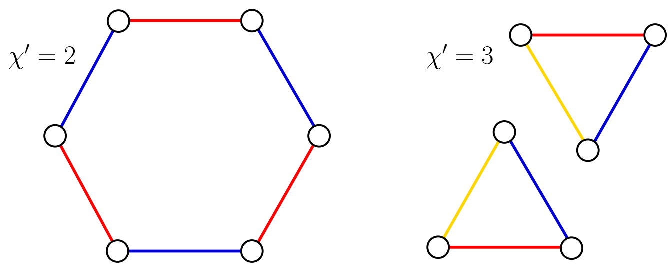

Figure 4: Minimum proper edge colorings.

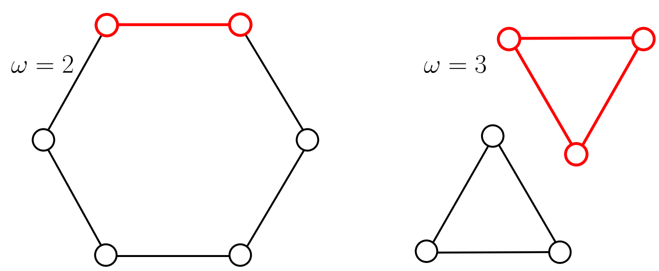

Figure 5: Largest induced cliques.

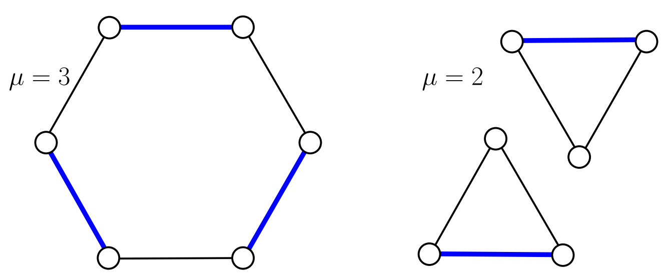

Figure 6: Maximum matchings.

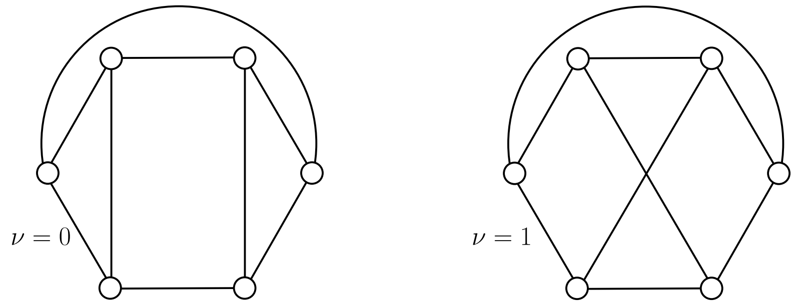

Figure 7: $\overline{C_6}$ and $\overline{C_3 \cup C_3}$ drawn with minimum number of edge crossings in the plane.

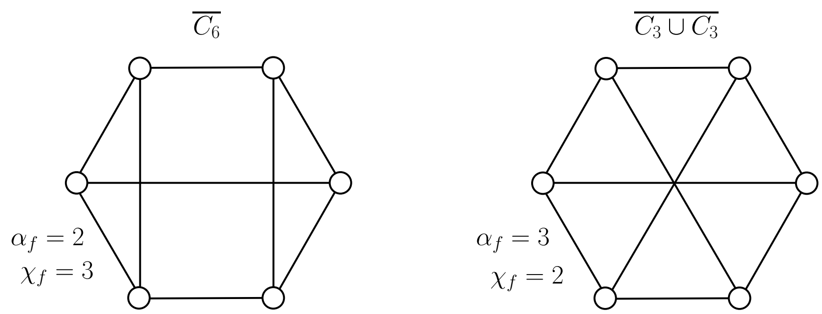

Figure 8: Fractional chromatic number, fractional clique number, and fractional independence number.

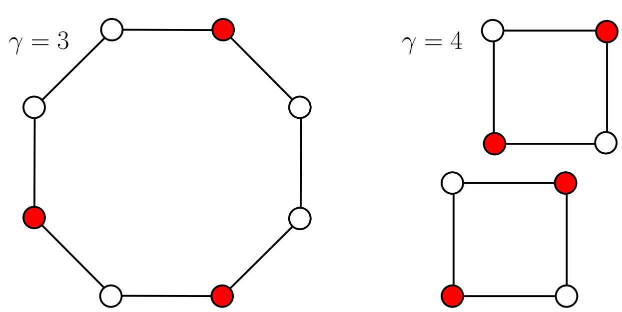

Figure 9: Minimum dominating sets.

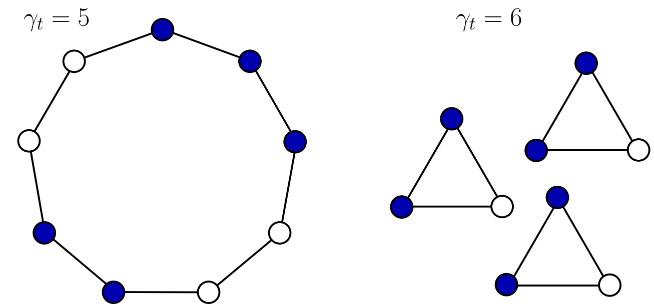

Figure 10: Minimum total dominating sets.

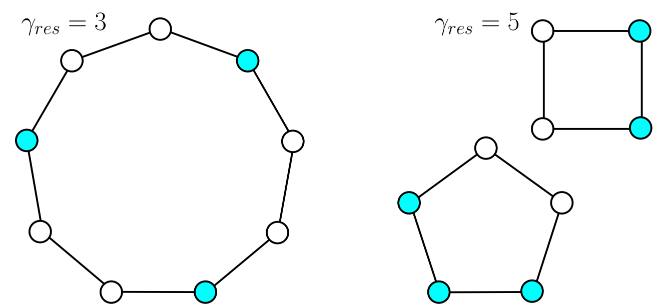

Figure 11: Minimum restrained dominating sets.

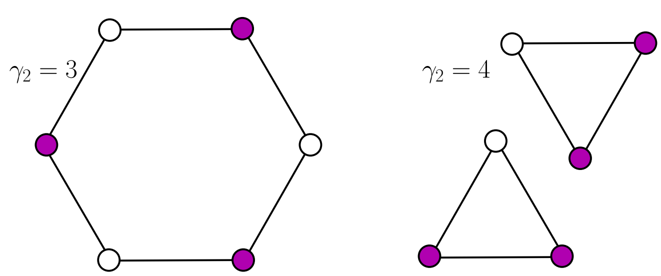

Figure 12: Minimum 2-dominating sets.

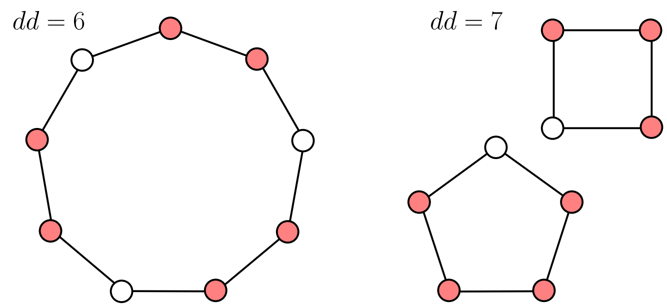

Figure 13: Minimum double dominating sets.

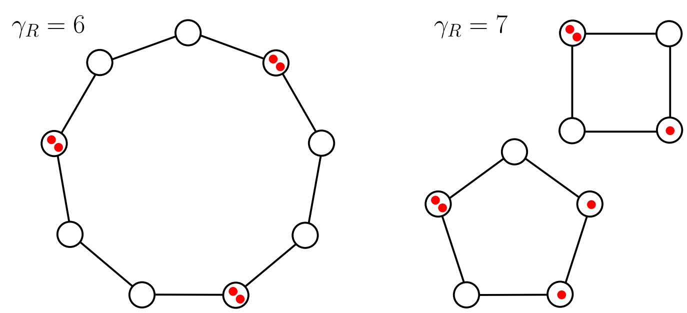

Figure 14: Minimum Roman dominating functions.

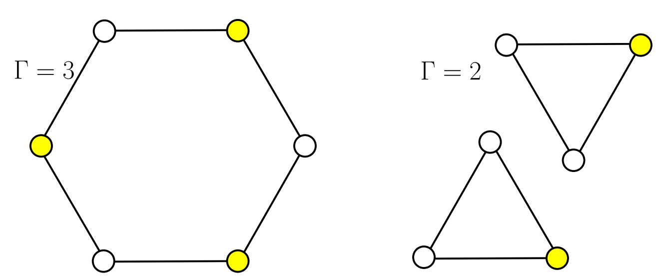

Figure 15: Maximum minimal dominating sets.

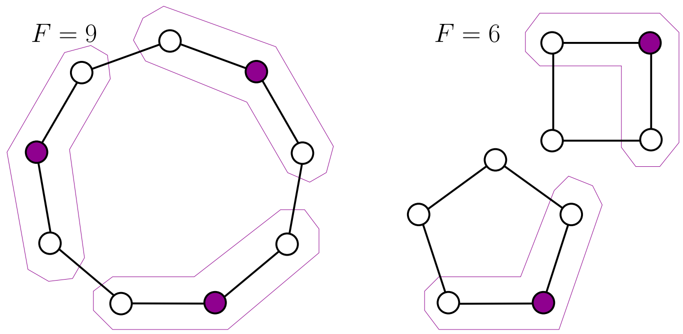

Figure 16: Efficient domination number.

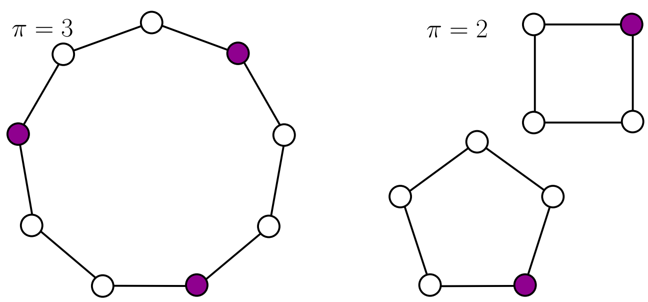

Figure 17: Maximum 2−packings (closed neighborhood packings).

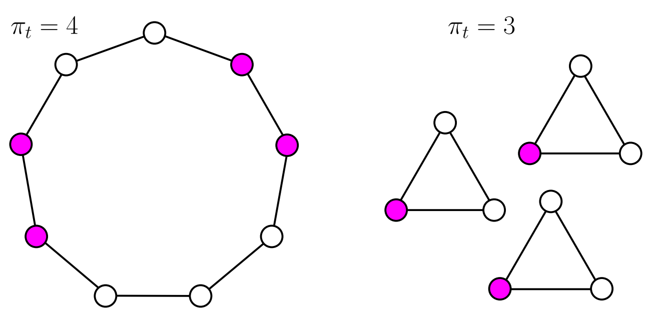

Figure 18: Maximum open neighborhood packings.

Appendix C

Classification of Graphs With 5 or Fewer Vertices

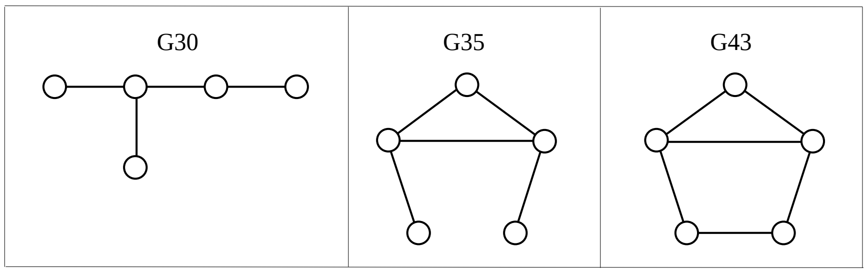

Figure 19: Class $\mathcal{N}$ graphs on 5 or fewer vertices.

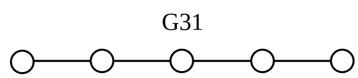

Figure 20: Class $\mathcal{I}$ graphs on 5 or fewer vertices.

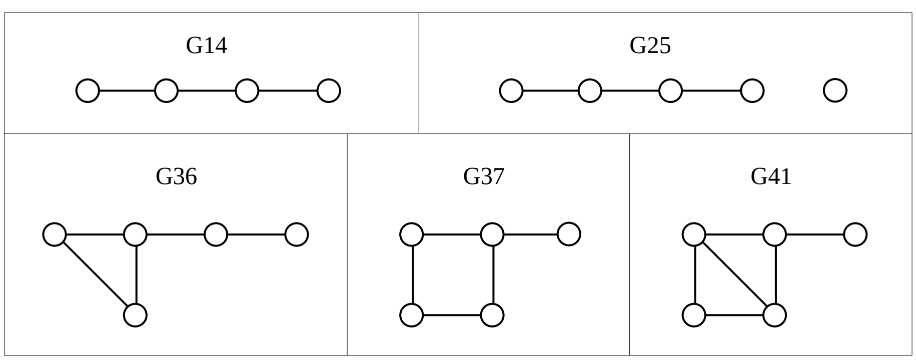

Figure 21: Class $\mathcal{D}$ graphs on 5 or fewer vertices.

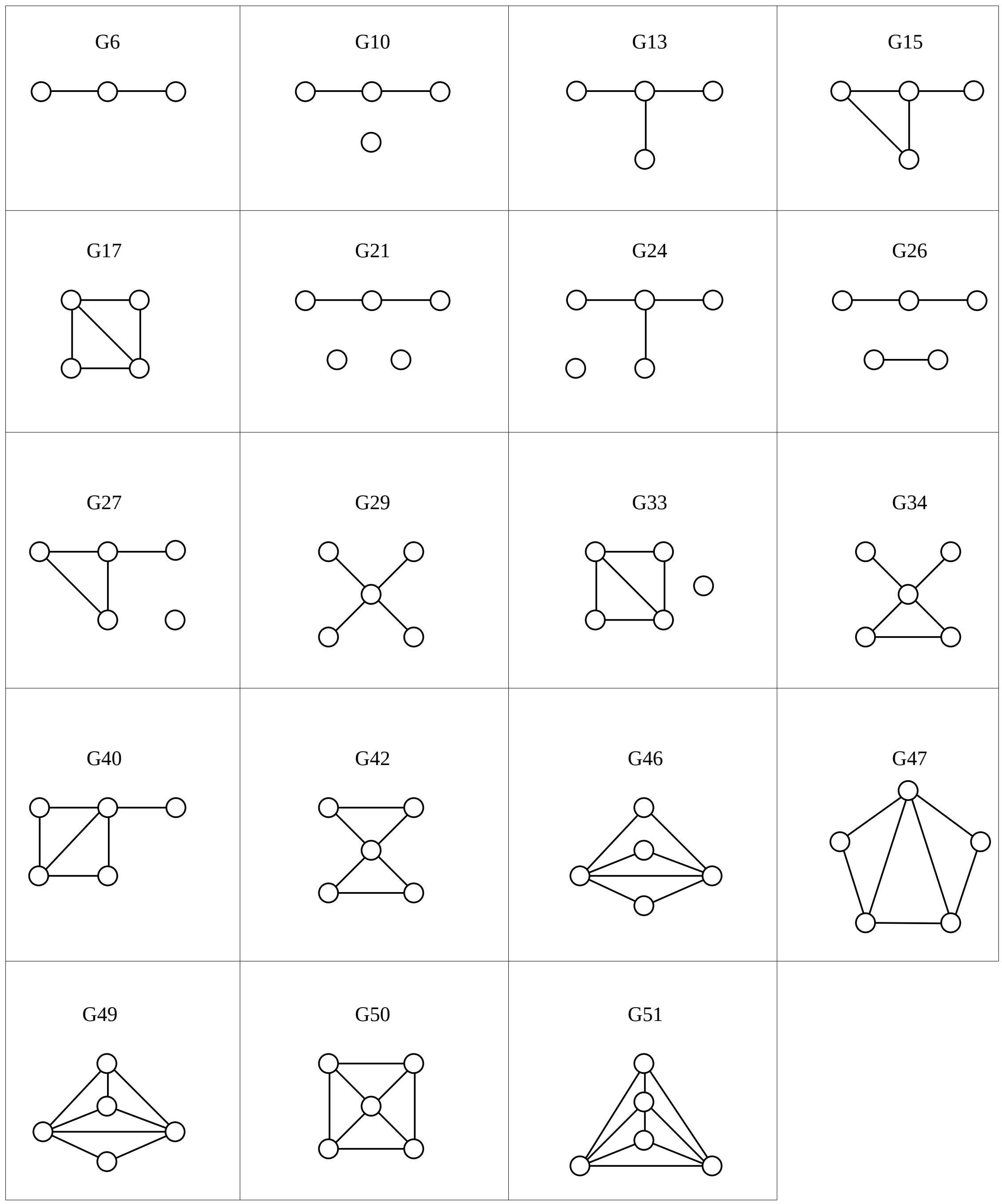

Figure 22: Class $\mathcal{P}$ graphs on 5 or fewer vertices.

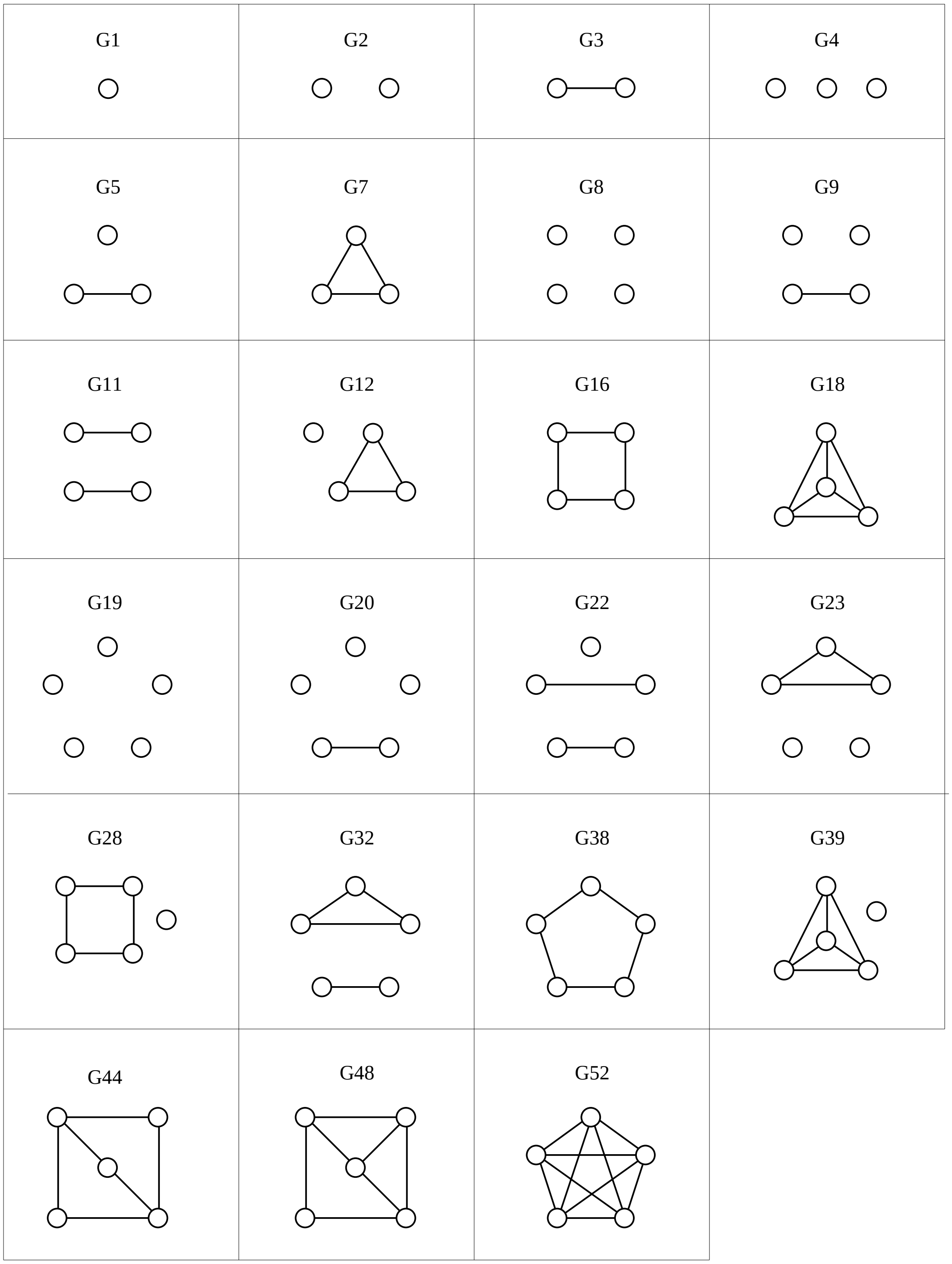

Figure 23: Class $\mathcal{E}$ graphs on 5 or fewer vertices.