Hamiltonian Decompositions of Complete Multipartite Graphs with Specified Leaves

by

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Abstract

For any 2-regular spanning subgraph G and H of the complete multipartite graph K with p parts each of size m, conditions are found which guarantee the existence of a 2-factorization of K or of $K - I$ (for some 1-factor I) in which

1. the first and second 2-factors are isomorphic to G and H respectively, and

2. each other 2-factor is a hamilton cycle.

These conditions are necessary and sufficient when m is odd, and solve the problem when m is even providing that m and p are each at least 6.

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Chapter 1

Introduction

1.1 History

One of the challenging problems over the past 30 years has been the Oberwolfach problem and its natural generalizations. The original problem requires one to find a 2-factorization of K_n in which all the cycles have the same length; this problem was solved over a decade ago [2, 8]. A much studied generalization of this problem is to simply require that each of the 2-factors be isomorphic to each other. To solve this would be an amazing feat, as so many possible 2-factors exist. Some progress has been made, including a complete solution when $n \leq 17$ [1], and in many cases where each 2-factor contains just two cycle lengths (see [5] for a survey of results).

Another direction in which research has developed is to allow a small number of the 2-factors to be anything, but then stipulate that the remaining 2-factors be hamilton cycles. Extending a result of Buchanan [6], in 2004 Bryant [4] found necessary and sufficient conditions for the existence of 2-factorizations of K_n and of $K_n - I$, where $K_n - I$ is the complete graph on n vertices with a 1-factor I removed, in which the cycle lengths in up to three of the 2-factors are freely specified, and all remaining 2-factors are hamilton cycles. Independently, Rodger [10] used a similar observation to settle the existence of 2-factorizations of all complete multipartite graphs, and of all complete multipartite graphs with a 1-factor removed, in which one 2-factor is freely specified and the rest of the 2-factors are hamilton cycles. One can think of this as the existence of a hamilton decomposition of the graph formed from $K(m, p)$ (the complete multipartite graph with m vertices in each of p parts) or from $K(m, p) - I$ by removing any 2-factor. Thought of in this way, the result has a relative in the world of matchings, where Plantholt [9] showed that the removal of any set of x edges from K_{2x+1} results in a graph whose edges can be partioned into $2x$ matchings ($2x+1$ matchings are needed if fewer edges are removed).

In this paper, we extend the result of Rodger, finding necessary and sufficient conditions for the existence of a hamilton decomposition of the graph $K(m,p)$ by removing the edges of any two 2-factors. More formally, for any two 2-regular graphs G and H of order mp, when m is odd we find necessary and sufficient conditions for the existence of a 2-factorization, $\{F_1, F_2, ..., F_{\lfloor m(p-1)\rfloor/2}\}$, of $K(m,p)$ such that $G \cong F_1, H \cong F_2$, and F_i is a hamilton cycle for $3 \leq i \leq \lfloor m(p-1)\rfloor/2$.

1.2 Preliminary Results

Before we can get to the results, some notation, lemmas, and theorems must be introduced. In this paper we use Z_n to denote the vertex set of a graph on n vertices. This allows us to define the *difference* of the edge $\{i,j\}$ to be $d(i,j) = \min\{j-i, n-(j-i)\}$ where $i < j$; thus $n/2 \geq d(i,j) > 0$. Let $\langle d_1, d_2, ..., d_x\rangle_n$ be the subgraph induced by the edges with differences in $\{d_1, d_2, ..., d_x\}$. Bermond et al [[3]] proved the following useful result that shows when the edges of two differences can be used to form two edge-disjoint hamilton cycles. If A is a set of positive integers, let $\gcd(A)$ denote the greatest common divisor of the elements of A. A hamilton cycle decomposition of the graph G is a 2-factorization of G, each 2-factor in which is a hamilton cycle.

Theorem 1.1. [3] Let s, t, n be positive integers with $s < t < n/2$. If $gcd(\{s,t,n\}) = 1$ then the graph $\langle s,t\rangle_n$ has a hamilton cycle decomposition.

The next lemma was proven separately by both Bryant and Rodger. It provides a key method used to prove our results.

Lemma 1.2. [4, 10] Let $n \geq 5$ and let F' be any 2-regular graph of order n. If $gcd(\{x,n\}) = 1$ then the subgraph $\langle x, 2x\rangle_n$ of K_n has a 2-factorization $\{F, H\}$ such that H is a hamilton cycle and $F' \cong F$. (See Figure 1.1)

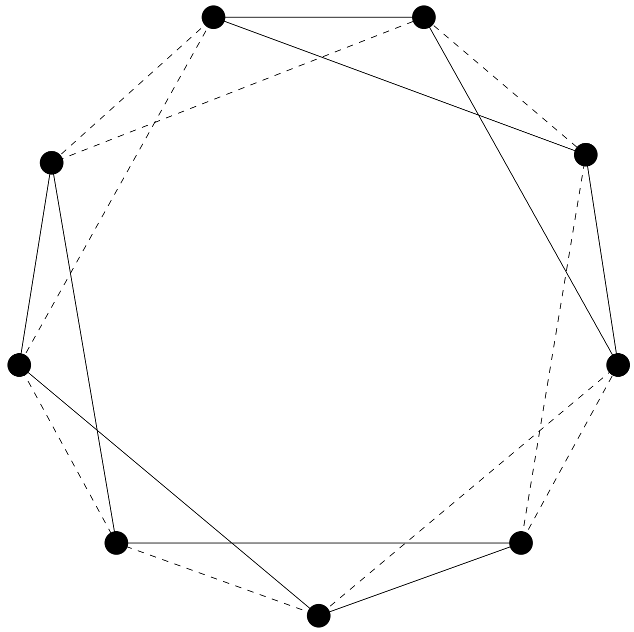

Figure 1.1: Referring to Lemma 1.2 with $F' = C_4 \cup C_5$ and $x = 1$

Now, we will introduce some specific results that will be used to clear up some of the cases we will encounter. Presented first is the result from Bryant's paper previously alluded to; one might also see the related results in [1, 7].

Theorem 1.3. [4] Let $n \geq 7$ be odd and let F_1', F_2', and F_3' be any three 2-regular graphs of order n. Then there exists a 2-factorization $\{F_1, F_2, ..., F_{(n-1)/2}\}$ of K_n in which $F_1 \cong F_1'$, $F_2 \cong F_2'$, $F_3 \cong F_3'$, and F_i is a hamilton cycle for $4 \leq i \leq (n-1)/2$, except that when $(n, F_1', F_2', F_3') \in \{(7, C_3 \cup C_4, C_3 \cup C_4, C_7), (9, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3), (9, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3, C_3 \cup C_6), (9, C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3, C_4 \cup C_5)\}$ no such two factorization exists.

Next we present Rodger's result.

Theorem 1.4. [10] Let $p \geq 3$ and $m \geq 1$. Let H be any 2-factor in $K(m, p)$. There exists a partition of the edge set of $K(m, p)$, one set in which induces a graph isomorphic to H, if $m(p-1)$ is odd then one set induces a 1-factor, and each other set induces a hamilton cycle.

The rest of the dissertation is organized as follows:

Chapter 2

The Case when mp is Odd

In this chapter we settle the existence of the specified 2-factorization when mp is odd. The proof relies heavily on Theorem 1.1, but in the case where $(m,p) = (5,3)$ several small cases must be considered in another way; this is accomplished by using a neat switching method.

Theorem 2.1. Let m be odd. Let G and H be any two 2-regular graphs of order mp. There exists a 2-factorization $\{F_1, F_2, \ldots, F_{\lfloor m(p-1)\rfloor/2}\}$ of $K(m,p)$ such that $F_1 \cong G$, $F_2 \cong H$, F_i is a hamilton cycle for $3 \le i \le \lfloor m(p-1)\rfloor/2$, if and only if

1. p is odd, and
2. $(m,p,G,H) \notin \{(1,7,C_3 \cup C_4, C_3 \cup C_4), (3,3,C_3 \cup C_3 \cup C_3, C_3 \cup C_3 \cup C_3), (3,3,C_3 \cup C_3 \cup C_3, C_3 \cup C_6), (3,3,C_3 \cup C_3 \cup C_3, C_4 \cup C_5)\}$.

Proof. If $K(m,p)$ is to have a 2-factorization, all vertices must have even degree, so $m(p-1)$ must be even, so the first condition is necessary since we are assuming that m is odd. Once one observes that the edges removed from K_9 to form $K(3,3)$ can be thought of as the edges in $C_3 \cup C_3 \cup C_3$, Theorem 1.3 clearly proves the four cases described in the second condition cannot be obtained. So we now turn to a proof of the sufficiency.

Since $K(m,p)$ is an $m(p-1)$-regular graph, and since it is assumed to contain at least two 2-factors, we know that $m(p-1) \ge 4$. So, since we also know that p is odd, clearly $p \ge 3$.

Notice that if we let the j^{th} part of $K(m,p)$ be $\{ip+j \mid i \in Z_m\}$ for $j \in Z_p$ then the edges of $K(m,p)$ are the same as the edges of the complete graph K_{mp} with edges of difference ip, $1 \le i \le \lfloor m/2 \rfloor$ removed. Therefore we will partition the edges of $K(m,p)$ by their differences, namely by the differences in the difference set $D = \{1, 2, \ldots, \lfloor (mp)/2 \rfloor\} \backslash \{ip \mid 1 \leq i \leq \lfloor m/2 \rfloor\}$. We now consider several cases in turn.

Case 1: Suppose $mp \geq 21$. Then $\{1, 2, 4, 8\} \subset D$. By Lemma 1.2, $\langle 1, 2 \rangle_{mp}$ and $\langle 4, 8 \rangle_{mp}$ each have a 2-factorization consisting of any 2-factor and a hamilton cycle; so we can choose the two 2-factors to be isomorphic to G and H respectively. It remains to partition the remaining edges into sets that induce hamilton cycles. We consider 4 subcases in turn.

Case 1(a): Suppose that $p \geq 9$. By pairing all except possibly the last of the differences in $D \backslash \{1, 2, 4, 8\} = D'$ in increasing order (that is, form pairs $\{3, 5\}, \{6, 7\}, \ldots$) we produce pairs of the form either $\{d, d+1\}$ or $\{d, d+2\}$, for some $d \in D'$.

Since $gcd(\{mp, (d+1)-d\}) = gcd(\{mp, 1\}) = 1$, it follows that $gcd(\{d, d+1, mp\}) = 1$. Also, since mp is odd, $gcd(\{mp, (d+2)-d\}) = gcd(\{mp, 2\}) = 1$ means that $gcd(\{mp, d+2, d\}) = 1$. Also, if $|D|$ is odd, then the last difference, $(mp-1)/2$, is not paired, but since $gcd(\{mp, (mp-1)/2\}) = 1$, the edges with difference $(mp-1)/2$ form a hamilton cycle. Therefore, by Theorem 1.1, there exists a hamilton cycle decomposition of the subgraph induced by the remaining edges.

Case 1(b): Suppose that $p = 7$. If $m = 3$ then the result follows from Theorem 1.3, since we can choose each component in F_3' to be a 3-cycle, then remove these edges to form the independent vertices in the parts of $K(3, 7)$. In all other cases (so $mp > 21$), first form the pairs $\{3, 5\}, \{6, 10\}$, and $\{9, 11\}$ in turn (these exist since $mp > 21$). Notice that: $\gcd(3, 5, mp)$ divides $\gcd(5-3, mp) = 1$ since mp is odd; the $\gcd(6, 10, mp)$ divides $\gcd(10-6, mp) = 1$ since mp is odd; and, similarly, $\gcd(9, 11, mp) = 1$. All other pairs are of the form $\{d, d+1\}$ or $\{d, d+2\}$. Therefore we can apply Theorem 1.1 to each pair in turn to form sets of edges that induce hamilton cycles.

Case 1(c): Suppose that $p = 5$. If $mp \geq 35$ then apply Theorem 1.1 to each of the pairs $\{3, 7\}, \{6, 14\}, \{12, 13\}$, and $\{9, 11\}$ in turn. Pair the remaining differences in order and proceed as in Case 1a.

If $mp < 35$ then $mp = 25$. Apply Theorem 1.1 to each of the pairs $\{3,6\}, \{7,9\}$, and $\{11,12\}$ in turn.

Case 1(d): Suppose that $p = 3$. Pair the remaining differences in order and proceed as in Case 1a.

Case 2: Suppose $mp \leq 20$ and $(m,p) \neq (5,3)$. If $m = 1$ then $K(1,p)$ is just the complete graph K_p, so the result follows from Theorem 1.3. If $m = 3$ then $p \in \{3,5\}$ so the result also follows from Theorem 1.3, since when $m = 3$, the edges one removes from K_{mp} to form $K(m,p)$ induce the 2-factor consisting of p 3-cycles; consider this to be the third specified 2-factor.

Case 3: Suppose $(m,p) = (5,3)$. This case takes substantial effort to settle. It is too small to be able to apply Lemma 1.2 twice and be left with a difference that induces a hamilton cycle. The set of available differences is $\{1,2,4,5,7\}$, and Lemma 1.2 could be applied to the graphs $\langle 1,2\rangle_{15}$ and $\langle 4,8\rangle_{15}$ (since difference 7 is the same as difference 8), but that leaves difference 5 that induces five 3-cycles. So we do apply Lemma 1.2 to $\langle 4,8\rangle_{15}$ to obtain F_1, then obtain F_2 from $\langle 1,2,5\rangle_{15}$ in such a way that the edges left over form two hamilton cycles. We consider the various possible cycle lengths, $c_1, c_2, ..., c_x$ of the x components of F_2 in turn, written as $l = (c_1, c_2, ..., c_x)$.

We begin with the cases in which all the cycle lengths in F_2 are divisible by 3. To construct the required cycles, we always include the hamilton cycle $\langle 2\rangle_{15}$, then swap edges in $\langle 1\rangle_{15}$ with edges in $\langle 5\rangle_{15}$ to fuse components in $\langle 5\rangle_{15}$. In each case, we begin with l, then describe how to form F_2.

$(3,3,3,3,3)$: $\langle 1\rangle_{15}$ and $\langle 2\rangle_{15}$ are hamilton cycles, and difference 5 induces F_2.

$(3,3,3,6)$: Swap edges $\{0,1\}$ and $\{5,6\}$ in $\langle 1\rangle_{15}$ with edges $\{0,5\}$ and $\{1,6\}$ in $\langle 5\rangle_{15}$ to produce the hamilton cycle $(0,5,4,3,2,1,6,7,\ldots,14)$ and the graph consisting of the cycles $(0,1,11,6,5,10), (2,7,12), (3,8,13)$, and $(4,9,14)$ respectively. The next few cases proceed similarly, so we simply present the edges to be swapped. (Refer to Figure 2.1.)

$(3,3,9)$: Swap edges $\{0,1\},\{5,6\},\{6,7\}$, and $\{11,12\}$ in $\langle 1\rangle_{15}$ with edges $\{0,5\},\{1,6\},\{6,11\}$, and $\{7,12\}$ in $\langle 5\rangle_{15}$ (so just switch two more edges from the $(3,3,3,6)$ case).

$(3,12)$: Swap edges $\{0,1\},\{2,3\},\{5,6\},\{6,7\},\{7,8\}$, and $\{11,12\}$ in $\langle 1\rangle_{15}$ with edges $\{0,5\},\{1,6\},\{2,7\},\{3,8\}$, $\{6,11\}$, and $\{7,12\}$ in $\langle 5\rangle_{15}$ (so just switch two more edges from the $(3,3,9)$ case).

$(3,6,6)$: Swap edges $\{0,1\},\{5,6\},\{7,8\}$, and $\{12,13\}$ in $\langle 1\rangle_{15}$ with edges $\{0,5\},\{1,6\},\{7,12\}$, and $\{8,13\}$ in $\langle 5\rangle_{15}$ (so just switch two more edges from the $(3,3,3,6)$ case).

$(6,9)$: Swap edges $\{0,1\},\{3,4\},\{5,6\},\{6,7\},\{8,9\}$, and $\{11,12\}$ in $\langle 1\rangle_{15}$ with edges $\{0,5\},\{1,6\},\{3,8\},\{4,9\},\{6,11\}$, and $\{7,12\}$ in $\langle 5\rangle_{15}$ (so just switch two more edges from the $(3,3,9)$ case).

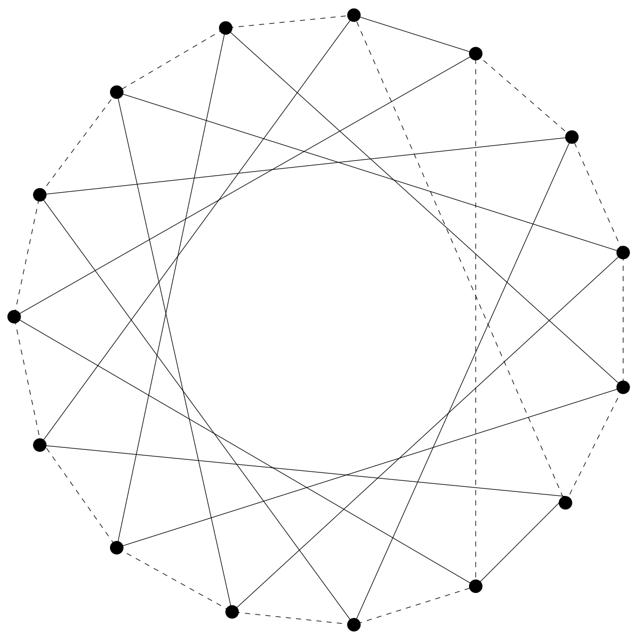

Figure 2.1: $K(m,p)$, using differences of 1 and 5 to produce a $C_3 \cup C_3 \cup C_3 \cup C_6$ and a hamilton cycle

All but one of the remaining cases are obtained by producing F_2 using Lemma 1.2, then switching edges between the resulting hamilton cycle and $\langle 5\rangle_{15}$ to obtain 2 hamilton cycles.

Since it is more complicated to describe, we simply provide the resulting decompositions of $\langle 1, 2, 5\rangle_{15}$.

$(3, 4, 4, 4)$: $(0, 1, 14, 13), (2, 3, 5, 4), (6, 7, 8), (9, 10, 12, 11)$,

$(0, 5, 10, 8, 3, 13, 12, 7, 2, 1, 11, 6, 4, 9, 14)$,

$(0, 2, 12, 14, 4, 3, 1, 6, 5, 7, 9, 8, 13, 11, 10)$.

$(3, 3, 4, 5)$: $(4, 5, 6), (11, 12, 13), (7, 8, 10, 9), (0, 2, 3, 1, 14)$,

$(0, 10, 12, 2, 7, 5, 3, 8, 6, 1, 11, 9, 4, 14, 13)$,

$(0, 5, 10, 11, 6, 7, 12, 14, 9, 8, 13, 3, 4, 2, 1)$.

$(5, 5, 5)$: $(0, 2, 3, 1, 14), (4, 6, 8, 7, 5), (9, 10, 12, 13, 11)$,

$(0, 5, 10, 11, 6, 1, 2, 12, 7, 9, 14, 4, 3, 8, 13)$,

$(0, 10, 8, 9, 4, 2, 7, 6, 5, 3, 13, 14, 12, 11, 1)$.

$(4, 5, 6)$: $(10, 11, 13, 12), (5, 6, 8, 9, 7), (0, 2, 4, 3, 1, 14)$,

$(0, 1, 11, 9, 4, 6, 7, 2, 12, 14, 13, 3, 8, 10, 5)$,

$(0, 10, 9, 14, 4, 5, 3, 2, 1, 6, 11, 12, 7, 8, 13)$.

$(4, 4, 7)$: $(6, 8, 9, 7), (10, 12, 13, 11), (0, 2, 4, 5, 3, 1, 14)$

$(0, 5, 7, 2, 12, 14, 4, 6, 1, 11, 9, 10, 8, 3, 13)$,

$(0, 10, 5, 6, 11, 12, 7, 8, 13, 14, 9, 4, 3, 2, 1)$.

$(3, 5, 7)$: $(11, 12, 13), (6, 7, 9, 10, 8), (0, 2, 4, 5, 3, 1, 14)$,

$(0, 5, 7, 2, 1, 6, 11, 10, 12, 14, 4, 9, 8, 3, 13)$,

$(0, 10, 5, 6, 4, 3, 2, 12, 7, 8, 13, 14, 9, 11, 1)$.

$(3, 4, 8)$: $(0, 5, 10), (1, 2, 7, 6), (3, 4, 14, 9, 11, 12, 13, 8)$,

$(0, 1, 3, 2, 4, 5, 6, 11, 10, 9, 8, 7, 12, 14, 13)$,

$(0, 2, 12, 10, 8, 6, 4, 9, 7, 5, 3, 13, 11, 1, 14)$.

$(5, 10)$: $(0, 2, 4, 6, 8, 7, 5, 3, 1, 14), (9, 11, 13, 12, 10)$,

$(0, 5, 4, 14, 9, 8, 13, 3, 2, 12, 7, 6, 1, 11, 10)$,

$(0, 1, 2, 7, 9, 4, 3, 8, 10, 5, 6, 11, 12, 14, 13)$.

$(4,11)$: $(0,2,4,6,8,9,7,5,3,1,14),(10,12,13,11)$,
$(0,5,4,14,12,2,7,6,1,11,9,10,8,3,13)$,
$(0,1,2,3,4,9,14,13,8,7,12,11,6,5,10)$. □

Chapter 3

The Case when p is Even

3.1 A Number Theoretic Result

We begin this chapter with a general number theoretic result that will be used extensively in Section 3.2. The rest of this chapter deals with the case where p is even.

Lemma 3.1. Let $m, p \in Z^+$ with $p \neq 1$. Then there exists an $f \in Z$ such that $\gcd(f, mp) = 1$, $f \equiv -1 (\text{mod } p)$, and $0 < f < mp$.

Proof. Define $Q = \{q \mid q \text{ prime}, q \text{ divides } m, q \text{ does not divide } p\}$. For each $q \in Q$, choose $1 \leq a_q \leq q - 1$. By the Chinese Remainder Theorem, there exists a unique $f \in Z$ satisfying

1. $f \equiv -1 (\text{mod } p)$ and

2. $f \equiv a_q (\text{mod } q)$ for each $q \in Q$

with $0 \leq f < pD$, where D is the product of all the elements of Q. Obviously, $f \neq 0$ since $p \geq 2$. Also, $mp \geq pD$ since D is a product of primes dividing m, so D divides m. Since there are $q - 1$ for each a_q, there are $\phi(D)$ such f in each of the ranges $tpD < f < (t+1)pD$ for each $0 \leq t < \frac{m}{p}$. □

Corollary 3.2. In Lemma 3.1, there are $\frac{\phi(D)m}{D}$ such f's, where D is the product of all primes which divide m but do not divide p, and ϕ is the Euler ϕ-function.

Proof. Referring to the proof of Lemma 3.1, each a_q can be chosen in $q - 1$ ways, so the family of a_q's can be chosen in a total of $\Pi_{q \in Q}(q - 1) = \phi(D)$ ways. This gives $\phi(D)$ f's (mod pD), and the interval from 0 to mp contains $\frac{m}{D}$ copies of the integers (mod pD). □

Corollary 3.3. For p even, $p \geq 6, m \geq 5$, $\frac{\phi(D)m}{D} \geq 4$.

Proof. Let us consider the possible value of $\frac{\phi(D)m}{D}$ being 1, 2, or 3 in turn. First notice that, by definition, $2 \notin Q$ since p is even. Also, notice that $\phi(D) = 1$ if and only if $Q = \{2\}$, by definition of Q.

1. $\frac{\phi(D)m}{D} = 1$ if and only if both $\phi(D)$ and $\frac{m}{D}$ are 1. But we just showed that $\phi(D) \neq 1$.

2. $\frac{\phi(D)m}{D} = 2$ if and only if $\phi(D) = 2$ and $\frac{m}{D} = 1$ or $\phi(D) = 1$ and $\frac{m}{D} = 2$. The second option is not possible since $\phi(D) \neq 1$. If $\phi(D) = 2$ then either $Q = \{2, 3\}$ or $Q = \{3\}$. Since $2 \notin Q$, $Q = \{3\}$. Therefore $D = 3$. This implies that m is also 3 since $\frac{m}{D} = 1$. This contradicts the assumption that $m \geq 6$.

3. $\frac{\phi(D)m}{D} = 3$ if and only if $\phi(D) = 3$ and $\frac{m}{D} = 1$ or $\phi(D) = 1$ and $\frac{m}{D} = 3$. The second option is not possible since $\phi(D) \neq 1$. Also, since $\phi(D) = \Pi_{q \in Q}(q - 1)$, where each q is strictly prime, $\phi(D) \neq 3$.

Thus, $\frac{\phi(D)m}{D} \geq 4$. □

3.2 The Case when p is Even

We now use Lemma 3.1 and Theorem 1.1 to settle the case when p is even, and m is odd or even. When m is odd we will have the half-difference (1-factor), I.

Theorem 3.4. Let p be even, $p \geq 6, m \geq 5$, and suppose that G and H are any two 2-factors of $K(m, p)$. Then there exists a 2-factorization $S = \{F_1, F_2, \ldots, F_{\lfloor m(p-1)/2 \rfloor}\}$ of $K(m, p)$ when m is even and a 2-factorization of $K(m, p) - I$ when m is odd, such that $F_1 \cong G, F_2 \cong H$, and F_i is a hamilton cycle for $3 \leq i \leq \lfloor m(p - 1)/2 \rfloor$.

Proof. Notice that if we let the j^{th} part of $K(m, p)$ be the vertex set $\{ip + j \mid i \in Z_m\}$ for $j \in \mathbb{Z}_p$ then $K(m, p)$ is isomorphic to the subgraph of K_{mp} formed by removing the edges of difference ip, $1 \leq i \leq \lfloor m/2 \rfloor$. Therefore we will partition the edges of $K(m, p)$ by their differences in K_{mp}, namely by the differences in the difference set $D = \{1, 2, \ldots, mp/2\} \backslash \{ip \mid 1 \leq i \leq \lfloor m/2 \rfloor\}$. (Refer to Figure 3.1.)

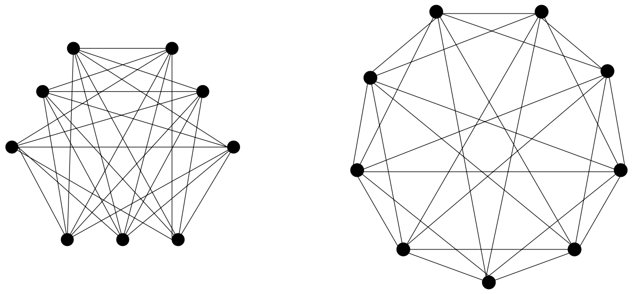

Figure 3.1: $K(3,3)$ represented as a K_9 with edges of difference 3 removed

We define

$$[f] = \begin{cases} f & \text{if } f < mp/2, \text{ and} \\ mp - f & \text{if } f > mp/2 \end{cases}$$

so $\langle f \rangle_{mp} = \langle [f] \rangle_{mp}$. This special difference f will be chosen from D such that

1. $f \equiv -1 (\text{mod } p)$,

2. $\gcd(f, mp) = 1$,

3. $0 < f < mp$, and

4. $f \notin \{mp/2 - 1, mp - 1\}$.

Since property 4 excludes two possible values of f described in Lemma 3.1, by Corollary 3.3 there are at least two choices for f. In most cases, just one value is used, but in Case 2, both will be needed.

By Lemma 1.2, $\langle 1, 2 \rangle_{mp}$ and $\langle f, 2f \rangle_{mp}$ each have a 2-factorization consisting of any 2-factor and a hamilton cycle; so we can choose the two 2-factors to be isomorphic to G and H respectively. It remains to partition the remaining edges (differences) into sets that will induce hamilton cycles by applying Theorem 1.1. If m is odd the half difference, $mp/2$, will induce the 1-factor, I. We now consider several cases in turn.

Case 1: Suppose that $p \equiv 0, 1$, or $3 \pmod 4$ or $m \equiv 0, 1$, or $2 \pmod 4$. Define $D' = D \backslash \{1, 2, f, 2f, mp/2\}$. So either i: $|D'|$ is even or

ii: $|D'|$ is odd and $mp/2-1$ is odd.

In the latter case, $\langle mp/2-1\rangle_{mp}$ induces a hamilton cycle which will be placed in S; so, in this case, further modify D' by removing the difference $mp/2-1$. So in both cases, $|D'|$ is even. If there are an odd number of differences in D' that are less than f or $[f]$, then modify D' to form D'' as follows.

Case 1(a): If 3 does not divide m or if 3 does not divide $f-1$ then remove the pair $\{d, d+3\}$ from D' where

i: $d = f-1$ if $f < mp/2$

ii: $d = [f]-2$ if $f > mp/2$.

Case 1(b): If 3 divides m and 3 divides $f-1$ then remove the pair $\{d, d+9\}$ from D' where

i: $d = f-3$ if $f < mp/2$

ii: $d = [f]-4$ if $f > mp/2$.

Consider this new set D'' (possibly $D' = D''$). Now pair the differences in D'' in increasing order. We now show that Theorem 1.1 can be applied to each of the defined pairs. That is, we will show that for each pair $\pi = \{z_1, z_2\}$, $\gcd(z_1, z_2, mp) = 1$.

We consider each of the possible pairs π of differences in turn.

1. Suppose $\pi = \langle d, d+1\rangle$ for some $d \in D''$. Then $\gcd(d, d+1, mp)$ divides $\gcd((d+1)-(d), mp) = \gcd(1, mp) = 1$.

2. Suppose $\pi = \langle d, d+2\rangle$ for some $d \in D''$. Notice that such a pair only occurs when $d+1$ is a multiple of p, is $2f$, is $2[f]$, or is $[2[f]]$. So, in each case, d and $d+2$ are both odd. Thus, $\gcd(d, d+2, mp)$ divides $\gcd((d+2)-(d), mp) = \gcd(2, mp) \in \{1, 2\}$. So, since d is odd, $\gcd(d, d+2, mp) = 1$.

3. Suppose $\pi = \langle d, d+3 \rangle$. Such pairs only occur in case 1(a). So, we consider the situations in turn.

   - 1(a)i: In this case $d = f - 1$, so we need to consider $\gcd(f-1, f+2, mp)$ which divides $\gcd((f+2)-(f-1), mp) = \gcd(3, mp) \in \{1, 3\}$. Recall that in this case 3 does not divide m or 3 does not divide $f-1$. If 3 does not divide $f-1$ then the $gcd(f-1, f+2, mp) = 1$. Otherwise, 3 does not divide m and 3 divides $f-1$; so 3 does not divide p since $f \equiv -1(\mod p)$. Thus, 3 does not divide mp. So, $\gcd(f-1, f+2, mp) = \gcd(3, mp) = 1$.

   - 1(a)ii: Here we have that $d = [f] - 2$. We must consider $\gcd([f]-2, [f]+1, mp)$ which divides $\gcd(([f]+1) - ([f]-2)), mp) = \gcd(3, mp) \in \{1, 3\}$. Since $[f] - 2 \equiv -1(\mod p)$, 3 cannot divide both $[f]-2$ and p, so the only way for $\gcd([f]-2, [f]+1, mp) = 3$ is if 3 divides both $[f]-2$ and m. But $[f] - 2 = mp - f - 2 \equiv mp - (f - 1 (\text{mod} 3)$ so then 3 would divide $f-1$, contradicting the assumption that either 3 does not divide m or 3 does not divide $f-1$. Thus, $\gcd([f]-2, [f]+1, mp) = \gcd(3h, mp) = 1$.

4. Suppose $\pi = \langle d, d+9 \rangle$. Such pairs only occur in case 1(b). So, we consider the cases in turn.

   - 1(b)i: In this case $d = f - 3$, so we need to consider $\gcd(f-3, f+6, mp)$ which divides $\gcd((f+6)-(f-3), mp) = \gcd(9, mp) \in \{1, 3, 9\}$. Recall that in this case 3 divides m, 3 divides $f-1$, and $f \equiv -1(\mod p)$. So 3 does not divide $f-3$. Thus, $\gcd(f-3, f+6, mp) = \gcd(9, mp) = 1$.

   - 1(b)ii: Here we have that $d = [f]-4$. So we must consider $\gcd([f]-4, [f]+5, mp)$ which divides $\gcd(([f]+5) - ([f]-4), mp) = \gcd(9, mp) \in \{1, 3, 9\}$. Notice that $[f] - 4 = mp - f - 4 = mp - (f-1) - 5$. In this case, 3 divides m and 3 divides $(f-1)$; so if 3 divides $([f]-4)$ then 3 divides -5, a contradiction. Thus 3 does not divide $[f]-4$, so the $\gcd([f]-4, [f]+5, mp) = 1$.

Case 2: If $p \equiv 2(\text{mod}4)$ and $m \equiv 3(\text{mod}4)$, then $mp/2 - 1$ is even. Thus $\langle mp/2 - 1\rangle_{mp}$ is not a hamilton cycle. Consider a difference, g, defined in the same way as f. That is, by Lemma 3.1 there exists a $g \in Z_{mp}$ with $g \equiv -1(\text{mod}p), \gcd(g, mp) = 1, g \neq mp - 1$, and $g \neq f$. Then $\langle g\rangle_{mp}$ is a hamilton cycle. Remove this difference as well from D; so in this case define our new difference set $D' = D\backslash\{1, 2, f, 2f, mp/2, g\}$. In the following situations we will modify D' to form D'' as follows.

- (a):
  - i: If $[f] = g + 2$, then remove $\{3, g + 3\}$ from D'.
  - ii: If $[g] = f + 2$, then remove the pair $\{3, f + 3\}$ from D'.
- (b): If $2f = g - 1$, then remove the pair $\{g - 2, g + 2\}$ from D'.

Consider this new set D''. Differences in D'' are now paired in increasing order and Theorem 1.1 applied to each pair. We now show that for each possible pair $\pi = \{z_1, z_2\}$, $\gcd(z_1, z_2, mp) = 1$.

1. Suppose $\pi = \langle d, d + 1\rangle$ for some $d \in D''$. Then $\gcd(d, d + 1, mp)$ divides $\gcd((d + 1) - (d), mp) = \gcd(1, mp) = 1$.

2. Suppose $\pi = \langle d, d + 2\rangle$ for some $d \in D''$. Notice that such a pair only occurs when $d + 1$ is a multiple of p, is $2f$, is $2[f]$, or is $[2[f]]$. In each situation, d and $d + 2$ are both odd. Thus, $\gcd(d, d+2, mp)$ divides $\gcd((d+2) - (d), mp) = \gcd(2, mp) \in \{1, 2\}$. So, since d is odd, $\gcd(d, d + 2, mp) = 1$.

3. Suppose $\pi = \langle 3, d + 3\rangle$. Such pairs occur in case $2(a)$. So, we consider the situations in turn.

   - (Case 2i): In this case $d = g$, so we need to consider $\gcd(3, g + 3, mp)$ which divides $\gcd((g + 3) - 3, mp) = \gcd(g, mp)$. By definition of g, $\gcd(g, mp) = 1$.

- (Case 2ii): In this case $d = f$, so we need to consider $\gcd(3, f + 3, mp)$ which divides $\gcd((f+3) - 3, mp) = \gcd(f, mp)$. By definition of f, $\gcd(f, mp) = 1$.

4. Suppose $\pi = \langle g - 2, g + 2 \rangle$. This pair occurs in case $2(b)$. We need to consider $\gcd(g-2, g+2, mp)$ which divides $\gcd((g+2) - (g-2), mp) = \gcd(4, mp) \in \{1, 2, 4\}$. Notice that in this case $g - 2$ is odd since g is odd, thus $\gcd(g-2, g+2, mp)$ is odd. So, $\gcd(g-2, g+2, mp) = 1$.

Thus every pair of differences induces two hamilton cycles to be placed into S. In some cases, the difference $mp/2-1$ or g is used alone to form a hamilton cycle. The sets $\langle 1, 2 \rangle_{mp}$, $\langle f, 2f \rangle_{mp}$ induce G and H and two hamilton cycles in S. If the half-difference, $mp/2$, is present in D, it induces the 1-factor, I. So the required 2-factorization has been constructed in all cases. □

Chapter 4

Another Number Theoretic Result

In this section a result is obtained that was, early in the research, thought to be pivotal. However, it was concluded that a simpler approach could be used. By adopting a different approach in the proof of Theorem 3.4, it turns out that Lemma 4.1 was not needed. Nevertheless it is a result that may be of some consequence in future endeavors. For example, it could be a useful tool in attacking results that would generalize Theorem 3.4.

Lemma 4.1. Let $m \geq 5$. If $m, p \in \mathbb{Z}^+$ and p even, then there exists an $f \in Z$ such that $\gcd(f, mp) = 1$, $f \equiv p - 1 (\bmod 2p)$, and $0 \leq f < mp$.

Proof. Let m, $p \in Z^+$ and let p be even. Let $Q = \{q_i \mid q_i$ is prime, q_i divides m, and q_i does not divide $p\}$. Notice that the $\gcd(q_i, 2p) = 1$ for each $q_i \in Q$ since p is even. By the Chinese Remainder Theorem, there exists a unique solution modulo $2p \prod_{q_i \in Q} q_i$ to the system of congruences:

1. $f \equiv p - 1 (\bmod 2p)$
2. $f \equiv 1$ or $2 (\bmod q_i)$ for each $q_i \in Q$.

Notice that $0 \leq f < 2p \prod_{q_i \in Q} q_i$. (*) We first check that $\gcd(f, mp) = 1$. Let $r \in Z$ be a prime such that r divides f and r divides mp. We consider two cases in turn.

Case 1: r divides f and r divides p

By (1), $f = 2px + p - 1$ for some $x \in Z$. Then, r divides $2px$, r divides p, and r divides f, so r divides -1. This implies that $r = 1$, so $\gcd(f, mp) = 1$.

Case 2: r divides f, r divides m, but r does not divide p

We can assume that $r = q_i$ for some $q_i \in Q$. By (2), $f = ry + 1$ or $f = rz + 2$ for $y, z \in Z$.

If $f = ry + 1$, then r divides 1, so $\gcd(f, mp) = 1$. If $f = rz + 2$, then r divides 2, which implies that r divides 1 since $q_i \neq 2$, so $\gcd(f, mp) = 1$.

Now that we have an f such that $\gcd(f, mp) = 1$ and $f \equiv p - 1 (\text{mod } 2p)$, and clearly, by (1) and (2), $0 < f < 2mp$, it just remains to show that f can be chosen so that $f < mp$. We consider two cases in turn.

Case 1: Suppose m is not square free; say q_1^2 divides m. Then, by (*) $f < 2p \prod_{q_i \in Q} q_i \leq 2pm \div q_1 \leq mp$. So, $f < mp$ as required.

Case 2: If m is square free, $m \geq 5$, and $\gcd(m, p) = 1$, then $2p \prod q_i = 2mp$. An obvious problem since we need $f < mp$, not just $f < 2mp$.

We know that $f = 2px + p - 1$ for some $x \in Z$, and $\gcd(f, p) = 1$. We need $\gcd(f, m) = 1$. Since $0 \leq f < 2mp$, we have $0 \leq x \leq m - 1$. These m values for x form a complete residue class modulo m. Then, since $\gcd(m, 2p) = 1$, the resulting m values of f are a complete residue class modulo m. Notice, because m is odd, there are $m + 1/2$ possible values between 0 and mp, and $m - 1/2$ values between mp and $2mp$. Thus, we have slighly more candidates for f's in the desired range, $[0, mp)$. We need to show that one of these values satisfies $\gcd(f, mp) = 1$.

Define two functions, $g, h : Z \to Z^+$, where $g(x) = \gcd(x, m)$ and $h(x) = \gcd(2px + p - 1, m) = \gcd(f, m)$.

We will now show that for some fixed $y \in Z$, $h(x + y) = g(x)$ for all $x \in Z_m$.
Since $\gcd(2p, m) = 1$, we can find an integer t satisfying $2pt \equiv 1 (\text{mod } m)$. Let $y = t(1 - p)$.

Then:
$h(x + y) = \gcd(2p(x + y) + p - 1, m) = \gcd(2px + 2py + p - 1, m) = \gcd(2px + 2pt(1 - p) + p - 1, m) = \gcd(2px + 1 - p + p - 1, m) = \gcd(2px, m) = \gcd(x, m) = g(x)$.
Since $g(-1) = g(1) = g((m - 1)/2) = g((m + 1)/2) - 1$, it follows that if we evaluate h at each of the values, $-1 + y (\text{mod } m)$, $1 + y (\text{mod } m)$,

$(m-1)/2+y(\text{mod } m)$, and $(m+1)/2+y(\text{mod } m)$, we get 1 in each case; and clearly at least one of these four values, say x^k, must be at most $(m-1)/2$. So, let $f = 2px^k + p - 1 \leq 2p((m-1)/2) + p - 1 < mp$.

□

Chapter 5

Conclusion

This dissertation shows the existence of the cases where mp is odd and is settled in Theorem 2.1. However, there are many smaller cases where p is even that need to be considered, namely, when either $p \leq 5$ or $m \leq 4$. The method used when settling the existence problem when mp is small and odd may be able to be adapted for these unsettled cases. Because we are looking for two 2-factors, the degree of each vertex must be at least 4, thus $m(p-1) \geq 4$. So to obtain a complete solution to this problem it suffices to consider the following cases:

1. $m = 2$ and $p \geq 3$
2. $m = 3$, $p \geq 4$, and p is even
3. $m = 4$ and $p \geq 2$
4. $p = 2$ and $m \geq 5$
5. $p = 3$, $m \geq 6$, and m is even
6. $p = 4$ and $m \geq 5$
7. $p = 5$, $m \geq 6$, and m is even.

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