Random and Vector Measures:
From "Toy" Measurable Systems to Quantum Probability

by

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Abstract

Vector measure theory and Bochner integration have been well-studied over the past century. This work is an introduction to both theories and explores various examples and applications in each. The theories and theorems are pre-existing, whereas the examples and discussions are mine. Our primary examples of vector measures are toy vector measures, which serve as a class of elementary yet nontrivial structures that enables us to grasp the spirit and essence of the advanced theory, both on the conceptual and technical level. We also discuss random measures as special cases of vector measures.

The theory of Bochner integration is introduced as a framework for the Radon-Nikodým Property, which comes from the failure of the Radon-Nikodým Theorem to hold when generalized to Banach spaces. The consequences of this failure as well as Rieffel's extension of the theorem are discussed in Chapter 2.

Finally, we conclude with a brief introduction to Hilbert quantum theory and quantum probability and introduce possible vector extensions of quantum probability theory.

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Chapter 1

Vector Measures

1.1 History and Motivation

Measure theory was developed primarily during the late 19th and early 20th century. By the 1930's investigations into extending real variable theory to functions taking values in Banach spaces had begun. Among the noteworthy results were Gel'fand's use of vector measure-theoretic techniques to prove that $L^1[0,1]$ is not the pre-dual of any Banach space (1938) and Pettis's contribution to the Orlicz-Pettis Theorem (weakly countably additive vector measures are norm countably additive)[2].

Naturally, representing linear operators as integrals of Banach-valued functions (i.e. Bochner integrals), which, requires some form of a Radon-Nikodým Theorem, has been of great interest. Hence, elements of vector measure theory, specifically elements concerning what is know as the Radon-Nikodým Property, are remarkably prolific in theories regarding classification of topological vector spaces, operators, and Banach spaces.

In 1933, Bochner introduced an integral of a Banach-valued function with respect to a scalar measure in *Integration von Funktionen deren Werte die Elemente eines Vectorraumes sind*, which is now called the Bochner integral. In 1936, J.A. Clarkson as well as N. Dunford and A.P. Morse established that absolutely continuous functions on a Euclidean space with values in a uniformly convex Banach space, or resp., a Banach space with a boundedly complete basis, are the integrals of their derivatives.[2] These were later recognized by Dunford as Radon-Nikodým theorems for the Bochner integral and hence, the first of the Radon-Nkiodým theorems for vector measures. [2] In 1943, Phillips extended the Dunford and Pettis' Radon-Nikodým Theorem.[11] In 1968 Rieffel published the strongest version of the Radon-Nikodým Theorem for the Bochner integral.

In the 1960's several results emerged in connection to Martingales, which lead to various purely geometrical characterizations of spaces with the Radon-Nikodým Property; one of the best known results is from Chatterji (1968)[1] pertaining to the relationship between the Radon-Nikodým Theorem and the martingale mean convergence theorem. In the 1970s, work in differentiating vector measures lead to geometric results involving the Radon-Nikodým Property in Banach spaces.[18] In 1983, R.D. Bourgin showed that the Radon-Nikodým Property is equivalent to several convergence properties for Bochner-valued martingales.[18]

In the first chapter, we will give a brief introduction to vector measures as well as an overview of several major results in vector measure theory. In the second chapter, we will progress to Bochner integration and the Radon-Nikodým Property with some attention paid to Pettis integration and the general Radon-Nikodým theorem for Bochner integrals. In the final chapter, we will introduce an axiomatic approach to quantum mechanics using so-called "Quantum Probability" and formulate a hypothesis on the existence of "quantum vector probabilites".

1.2 Vector Measures

Let $\mathcal{F}_0$ be a ring of subsets of a set Ω, $\mathcal{F}$ a σ-ring of subsets of Ω, Σ_0 an algebra on Ω, and Σ a σ-algebra on Ω,[1] Also, let X be a vector space over the field $\mathbb{K}(=\mathbb{C}$ or $\mathbb{R})$.

Definition 1.1. A vector-valued function $F : \mathcal{F}_0 \to X$ is called **additive** if, given disjoint $A, B \in \mathcal{F}_0$, then $F(A \cup B) = F(A) + F(B)$.[2]

This is well-defined since X is a vector space. If we assume that X is a topological vector space, then we can consider countable additivity as well.

Definition 1.2. Let X be a topological vector space $F : \mathcal{F} \to X$ is **countably additive** if, for all sequences (E_k) of pairwise disjoint members of $\mathcal{F}$ for which $\bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$,

[1]These terms are defined in Appendix A.

[2]Since $\emptyset \cap \emptyset = \emptyset$, this condition also implies that $\mu(\emptyset) = \mu(\emptyset \cup \emptyset) = \mu(\emptyset) + \mu(\emptyset) = 2\mu(\emptyset)$, and hence $\mu(\emptyset) = 0$.

$$F\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} F(E_k),$$

in the topology on X.

The first two immediate examples of vector measures are signed measures and complex measures from basic measure theory.

Example 1.3. Consider the vector space $\mathbb{R}$ and measurable space (Ω, Σ). Then a vector measure $F : \Sigma \to \mathbb{R}$ is called a signed measure on (Ω, Σ).

Example 1.4. Similarly, if the vector space is $\mathbb{C}$, then a vector measure $F : \Sigma \to \mathbb{C}$ is called a complex measure on (Ω, Σ).

The equation for countable additivity must hold in the topology of X in order for F to be well-defined. This means that, not only must the series converge, but, since the set union $\bigcup_{k=1}^{\infty} E_k$ is invariant under permutations of the E_k's, the series must converge unconditionally. To understand the strength of this requirement, consider a sequence (E_k) of pairwise disjoint members of $\mathcal{F}$ for which $\bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$. If F is well-defined, then $F\left(\bigcup_{k=1}^{\infty} E_k\right) = x_0$ for some $x_0 \in X$. Since F is finitely additive, for all n, $F\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} F(E_k)$. For each n, define $x_n \in X$ as $x_n := \sum_{k=1}^{n} F(E_k)$. Then, if F is well-defined, $x_n \to x_0$ in X. Hence, $F\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} F(E_k) = \sum_{k=1}^{\infty} F(E_k)$. Since the series converges conditionally, i.e. with any permutation of the $F(E_k)$'s, the series will also converge to x_0 in X with respect to the topology on X.

Therefore, a general topological space (or even a metric space) is not enough. Moreover, theorems about such spaces would be too few to be of interest of in depth study. Hence, for any reasonable purposes, all of the image spaces are F-spaces, which ensure luxury of addition, topological convergence, and a triangle inequality.

Definition 1.5. An additive vector-valued function from a σ-ring $\mathcal{F}$ into a topological vector space X is called a **vector measure** on $(\Omega, \mathcal{F})$, and a countably additive vector-valued function from a σ-ring $\mathcal{F}$ into a topological vector space X is called a **countably additive vector measure** on $(\Omega, \mathcal{F})$.

Example 1.6. Let $p \in [0, \infty)$, and define $F : ([0,1], \mathcal{B}, \lambda) \to L^p([0,1], \mathcal{B}, \lambda)$ by $F(E) := 1\!\!1_E$ for all $E \in \mathcal{B}$. Then, F is finitely additive and also countably additive by the countable additivity over domain of integration for the Lebesgue integrals [3].

Example 1.7. Consider the previous example but with $p = \infty$. F is still finitely additive; however, F is not countably additive: let $\{E_n\}_{n \in \mathbb{N}}$ be a pairwise disjoint collection of measurable sets with positive measure, and let $E := \bigcup_n E_n$. Then, $F(E) = 1\!\!1_E$, and $\sum_n F(E_n) = \sum_n 1\!\!1_{E_n}$. But for all $n \in \mathbb{N}$,

$$\|1\!\!1_E - \sum_{k=1}^{n} 1\!\!1_{E_k}\|_\infty = \|1\!\!1_E - 1\!\!1_{\bigcup_{k=1}^n E_k}\|_\infty = \|1\!\!1_{\bigcup_{k=n+1}^\infty E_k}\|_\infty = 1.$$

Therefore, $F(E) \neq \sum_n F(E_n)$.

Note that some texts use "vector-measure" to describe a countably additive vector-valued set function. Other texts reserve the name "vector-measure" for countably additive set functions that take values in a Banach space.[4] This is due, in large part, to the fact that Banach spaces admit a plethora of linear functionals and linear operators, which are crucial tools in characterizing vector measures. We will also make frequent use of the dual of our image space, X^*, which will be desirably rich if X is a Banach space (or at least locally convex).

1.3 "Toy" Vector Measures

The term toy vector measures comes from Paul-Andre Meyer's term toy Fock spaces in his book *Quantum Probability for Probabilists*,[12] in which the term was used to encompass a

[3] and because, for all $p \in [1, \infty)$ and all $E \in \mathcal{B}$, $|1\!\!1_E|^p = 1\!\!1_E$.

[4] However, many of the theorems we will see in this chapter to not require that our vector measures be countably additive.

class of elementary yet nontrivial structures to grasp the spirit and essence of the advanced theory, both on the conceptual and technical level. We have adopted this phrasing for a similar purpose.

Let X be an F-space[5] and $(\mathbb{N}, 2^{\mathbb{N}}, c)$ a measure space. Fix a sequence $(x_n)_{n=1}^{\infty} \in X^{\mathbb{N}}$, and define $F : 2^{\mathbb{N}} \to X$ by $F(E) := \sum_{n\in E} x_n$ for all $E \subseteq \mathbb{N}$.

Before we discuss whether or not F is a vector measure, we must first determine whether or not it is well-defined. (In fact, in doing so, we will have established both finite and countable additivity.) If $E \subseteq \mathbb{N}$ is finite, then $\sum_{n\in E} x_n \in X$. Suppose, then, that E is not finite. For the sake of simplicity, we will consider the case $E = \mathbb{N}$.

As mentioned before, since $\mathbb{N} = \bigcup_n \{n\}$, the well-defined claim reduces to the claim that the series $\sum_{n\in\mathbb{N}} x_n$ converges unconditionally. Since X is an F-space, Orlicz Theorem gives us useful characterizations of unconditional convergence:

Theorem 1.8. *In a complete metrizable topological vector space, the following are equivalent:*[6]

1. (x_k) *is summable, that is, for every* $\epsilon > 0$, *there is a finite set* $K \subseteq \mathbb{N}$ *such that, for every finite* $L \subseteq \mathbb{N}$ *that is disjoint with* K, $\left\| \sum_{n\in L} x_k \right\| < \epsilon$.

2. $\sum_{k\in\mathbb{N}} x_k$ *converges unconditionally in* X.

3. $\sum_{k\in\mathbb{N}} s_k x_k$ *converges for every sequence* $(s_k) \in \{-1, 1\}^{\mathbb{N}}$.

π. $\sum_{k\in\mathbb{N}} s_k x_k$ *converges for every sequence* $(s_k) \in \{0, 1\}^{\mathbb{N}}$.

4. *If* X *is also a Banach space, then* $(x_k) \in X^{\mathbb{N}}$ *is summable iff* $\sum_{k\in\mathbb{N}} s_k x_k$ *converges for any bounded sequence* (s_k) *of scalars.*

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[5] i.e. a complete metrizable topological vector space with a translation-invariant F-norm $\|x\| = d(x, 0)$.

[6] The origin of the particular formulations is hard to find as these claims have been well-integrated into the subject of functional analysis.

Furthermore, in even an F-space, absolute convergence with the triangle inequality immediately imply summability. However, only in finite dimensional Banach spaces are the two equivalent[5]. (The reverse implication comes from the fact that, in such spaces, convergence is equivalent to coordinatewise convergence.) An easy counterexample in the finite dimensional case is the series $\sum_{n=1}^{\infty} \frac{1}{n} e_n$ where (e_n) is the standard basis of the vector space of all sequences with finitely many non-zero terms. The proofs of the preceeding theorems may be found in the appendices.

Therefore, F is a countably additive vector measure if and only if the corresponding sequence (x_n) is summable.

1.4 Variation and Semivariation

An important concept in vector measures is variation, which acts similarly to a norm for our vector measures[7]

Definition 1.9. Let $F : \Sigma \to X$ be a vector measure and X an F-space. The **variation** of F is the extended nonnegative function $|F| : \Sigma \to [0, \infty]$ given for all $E \in \Sigma$ by

$$|F|(E) = \sup\{\sum_{k=1}^{n} ||F(E_k)||_X : \bigcup_{k=1}^{n} E_k \subseteq E \text{ and } E_k \in \Sigma \text{ are pairwise disjoint } 1 \leq k \leq n\}.$$

If $|F|(\Omega) < \infty$, then F is called a measure of **bounded variation**. If we require that F be σ-finite on (Ω, Σ), $|F| : \Sigma \to [0, \infty]$ on (Ω, Σ), will be contably additive, and hence a non-negative $\mathbb{R}$-valued measure.

Example 1.10. Supposing that $\mathbb{X}$ is a Banach space, consider bounded variation for our toy vector measure, $F : 2^{\mathbb{N}} \to \mathbb{X}$, given by $F(E) := \sum_{n \in E} x_n$ for all $E \subseteq \mathbb{N}$. (where (x_n) is summable). Let $E \subseteq \mathbb{N}$ and $\{E_k\}_{k=1}^{m}$ a pairwise disjoint collection subsets of $\mathbb{N}$ such that $\bigcup_{k=1}^{m} E_k \subseteq E$. Then,

$$\sum_{k=1}^{m} ||F(E_k)|| = \sum_{k=1}^{m} \left\| \sum_{j \in E_k} x_j \right\| \leq \sum_{k=1}^{m} \sum_{j \in E_k} \|x_j\| = \sum_{j \in \bigcup_{k=1}^{m} E_k} \|x_j\| \leq \sum_{j \in E} \|x_j\|.$$

[7] In fact, we can define a norm on vector measures on (Ω, Σ) by $\|F\| := |F|(\Omega)$.

Hence, $|F|(E) \leq \sum_{j\in E} \|x_j\|$ for all $E \in \Sigma$; that is, F is guaranteed to have bounded variation if (x_j) is absolutely convergent.[8]

Conversely, if F is of bounded variation, then (x_n) is absolutely convergent. To see this, let (x_n) be a summable sequence in $\mathbb{X}$ so that the F defined by (x_n) is of bounded variation. Then,

$$\begin{aligned}\infty > |F|(\mathbb{N}) &\geq \sup_{m\in\mathbb{N}} \left(\left(\sum_{n=1}^{m} \|F(\{n\})\| \right) + \|F(\{n \in \mathbb{N} | n > m\})\| \right) \\ &\geq \sup_{m\in\mathbb{N}} \left(\sum_{n=1}^{m} \|x_n\| \right) = \sum_{n=1}^{\infty} \|x_n\|.\end{aligned}$$

Example 1.11 (Vector Measure without Bounded Variation)**.** [2] Consider again the example where $F : ([0,1], \mathcal{B}, \lambda) \to L^\infty([0,1], \mathcal{B}, \lambda)$ is given by $F(E) := \mathbb{1}_E$ for all $E \in \mathcal{B}$. Let $\{E_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint collection of measurable sets with positive measure, and let $E := \bigcup_n E_n$. For each $n \in \mathbb{N}$, let $\pi_n = \{E_1, ..., E_{n-1}, \bigcup_{k=n}^\infty E_k\}$ be a partition of E into finite disjoint measurable sets each of finite measure. Then for each $n \in \mathbb{N}$,

$$\sum_{A\in\pi_n} \|F(A)\| = \sum_{k=1}^{n-1} \|\mathbb{1}_{E_k}\|_\infty + \|\mathbb{1}_{\bigcup_{k=n}^\infty E_k}\|_\infty = n.$$

Hence, $|F|(E) = \infty$ and F is not of bounded variation.

There are numerous interesting vector measures without bounded variation, but we will save those for the coming section on random measures.

Lemma 1.12. *: $|F| : \Sigma \to X$ is a monotone function on Σ.*

Proof. Let $E, A \in \Sigma$ and $A \subseteq E$. Then for any pairwise disjoint collection, $\{A_k\}_{k=1}^n \subseteq \Sigma$ such that $A_k \subseteq A$ for $1 \leq k \leq n$, $A_k \subseteq E$ for $1 \leq k \leq n$. Then,

$$\left\{ \sum_{k=1}^{n} ||F(A_k)||_X : \bigcup_{k=1}^{n} A_k \subseteq A \text{ and } A_k \in \Sigma \text{ are pairwise disjoint for } 1 \leq k \leq n \right\} \subseteq$$
$$\left\{ \sum_{k=1}^{n} ||F(A_k)||_X : \bigcup_{k=1}^{n} A_k \subseteq E \text{ and } A_k \in \Sigma \text{ are pairwise disjoint for } 1 \leq k \leq n \right\},$$

and hence $|F|(E) \geq |F|(A)$. $\square$

[8] In fact, $\|F\| = \sum_n \|x\|_n$.

Lemma 1.13. $|F| : \Sigma \to X$ *is finitely additive.*

Proof. It will suffice to show that for $E, A \in \Sigma$ with $E \cap A = \emptyset$ that $|F|(E \cup A) = |F|(E) + |F|(A)$. Let $\{B_i\}_{i=1}^l$ be a disjoint collection in Σ such that $B_i \subseteq E \cup A$ for $1 \leq i \leq l$. Let $E_i := B_i \cap E$ and $A_i := B_i \cap A$ for $1 \leq i \leq l$. Then $\{E_i\}_{i=1}^l \cup \{A_i\}_{i=1}^l$ is a pairwise disjoint collection in Σ for which $E_i \subseteq E$ and $A_i \subseteq A$ for $1 \leq i \leq l$. Then,

$$\begin{aligned}\sum_{i=1}^{l} ||F(B_i)|| &= \sum_{i=1}^{l} ||F(E_i \cup A_i)|| = \sum_{i=1}^{l} ||F(E_i) + F(A_i)|| \\ &\leq \sum_{i=1}^{l} ||F(E_i)|| + ||F(A_i)|| = \sum_{i=1}^{l} ||F(E_i)|| + \sum_{i=1}^{l} ||F(A_i)|| \\ &\leq |F|(E) + |F|(A).\end{aligned}$$

Now, let $\{E_k\}_{k=1}^n$ be a disjoint collection in Σ such that $E_k \subseteq E$ for $1 \leq k \leq n$, and let $\{A_j\}_{j=1}^m$ be a disjoint collection in Σ such that $A_j \subseteq A$ for $1 \leq j \leq m$. Let $B_i = E_i$ for $1 \leq i \leq n$ and $B_{n+i} = A_i$ for $1 \leq i \leq m$. Then, since $A \cap E = \emptyset$, $\{B_i\}_{i=1}^{n+m}$ is a disjoint collection in Σ such that $B_i \subseteq E \cup A$ for $1 \leq i \leq n+m$. Then,

$$|F|(E \cup A) \geq \sum_{i=1}^{n+m} ||F(B_i)|| = \sum_{k=1}^{n} ||F(E_k)|| + \sum_{j=1}^{m} ||F(A_j)||.$$

Taking the supermum over $\{E_k\}$ and $\{A_j\}$, $|F|(E \cup A) \geq |F|(E) + |F|(A)$. $\square$

Theorem 1.14. *If $F : \Sigma \to X$ is a a countably additive vector measure of bounded variation, then $|F|$ is countably additive.*

In other words, $|F|$ is a "true" or classic measure.

Proof. Let $\{E_k\}_{k \in \mathbb{N}}$ be a countable collection of disjoint measurable sets. If either $|F|(\bigcup_k E_k) = \infty$ or $\sum_k |F|(E_k) = \infty$, then the proof is immediate. Suppose, then, that F is of bounded variation.

1. Countable subadditivity:

Let $\{A_j\}_{j=1}^n$ be a disjoint collection of measurable subsets of $\bigcup_k E_k$. Then for each $k \in \mathbb{N}$, $\{A_j \cap E_k\}_{j=1}^n$ is a disjoint collection of measurable subsets of E_k and so $|F|(E_k) \geq \sum_{j=1}^{n} ||F(A_j \cap E_k)||$. Then,

$$\begin{aligned}\sum_k |F|(E_k) &\geq \sum_k \sum_{j=1}^{n} ||F(A_j \cap E_k)|| \\ &= \sum_{j=1}^{n} \sum_k ||F(A_j \cap E_k)|| \geq \sum_{j=1}^{n} \left\| \sum_k F(A_j \cap E_k) \right\| \\ &= \sum_{j=1}^{n} \left\| F\left(\bigcup_k (A_j \cap E_k) \right) \right\| = \sum_{j=1}^{n} ||F(A_j)||.\end{aligned}$$

Hence, $\sum_k |F|(E_k) \geq |F| \left(\bigcup_k E_k \right)$.

2. Countable superadditivity:

   For each $n \in \mathbb{N}$, $\bigcup_{k=1}^{n} E_k \subseteq \bigcup_k E_k$. Then, by monotonicity and finite additivity of $|F|$, for all $n \in \mathbb{N}$,

$$|F| \left(\bigcup_k E_k \right) \geq |F| \left(\bigcup_{k=1}^{n} E_k \right) = \sum_{k=1}^{n} |F|(E_k).$$

   Hence, $|F| \left(\bigcup_k E_k \right) \geq \sum_{k=1}^{\infty} |F|(E_k)$.

$\square$

Since, $|F|(\emptyset) = 0$, we have that $|F|$ is a scalar measure on $\mathbb{K}$ (and a non-negative scalar measure in case $\mathbb{K} = \mathbb{R}$). If F is of bounded variation, then for any collection $\{E_k\}_{k \in \mathbb{N}} \in \Sigma$, $\infty > |F|(\bigcup_k E_k) = \Sigma_k |F|(E_k)$. However, if F is not of bounded variation, then we will have divergent series $\sum_k |F|(E_k)$.

Another important concept for vector measures is weak-variation.

Definition 1.15. Let $F : \Sigma \to \mathbb{X}$ be a vector measure and $\mathbb{X}$ a Banach space. The **weak-variation** of F is the extended nonnegative function $|F|^* : \Sigma \to [0, \infty]$ given by

$$|F|^*(E) = \sup\{|x^*F|(E) : x^* \in \mathbb{X}^*, ||x^*|| \leq 1\},$$

where $|x^*F|$ is the variation of the $\mathbb{K}$-valued measure x^*F [9], for all $E \in \Sigma$.

If $|F|^*(\Omega) < \infty$, then F is said to be of **bounded weak-variation**.

Note that for any $x^* \in \mathbb{X}^*$ with $||x^*|| \leq 1$, $E \in \Sigma$, and pairwise disjoint collection $\{E_k\}_{k=1}^n \subseteq \Sigma$ such that $\bigcup_{k=1}^n E_k \subseteq E$,

$$\sum_{k=1}^n |x^*F(E_k)| \leq \sum_{k=1}^n ||x^*||\, ||F(E_k)|| \leq \sum_{k=1}^n ||F(E_k)||.$$

Therefore, $|F|^*(E) \leq |F|(E)$ for all $E \in \Sigma$.

For an instance of strict inequality, consider the Brownian Motion $X = \{X_t : t \geq 0\}$, and F be given by $F(A) = \int_A f\, dX$, for a bounded measurable $f : \mathbb{R}^+ \to \mathbb{R}$. Then $|F|(A) = \infty$ for all non-degenerate intervals A, but $|F|^*(A) < \infty$.[15]

Proposition 1.16. *[2] If $F : \Sigma \to \mathbb{X}$ is a vector measure of bounded variation, then a non-negative $\mathbb{R}$-valued measure $\mu : \Sigma \to [0, \infty]$ is the variation $|F|$ of F iff*

1. *$|x^*F|(E) \leq \mu(E)$ for all $E \in \Sigma$ and all $x^* \in \mathbb{X}^*$ with $||x^*|| \leq 1$, and*

2. *If $\nu : \Sigma \to [0, \infty]$ is a measure satisfying $|x^*F|(E) \leq \nu(E)$ for all $E \in \Sigma$ and all $x^* \in \mathbb{X}^*$ with $||x^*|| \leq 1$, then $\mu(E) \leq \nu(E)$ for all $E \in \Sigma$.*

In other words, $|F|$ is the least upper bound (if it exists) of $\{|x^*F| : x^* \in \mathbb{X}^*$ and $||x^*|| \leq 1\}$.

Proof. $\Rightarrow$ First, we will show that the two properties hold for $\mu = |F|$.

Let $x^* \in \mathbb{X}^*$ and $||x^*|| \leq 1$.

1. $|x^*F|(E) \leq |F|(E)$ comes from $|F|^*(E) \leq |F|(E)$ for all $E \in \Sigma$.

2. Let ν be a measure satisfying the assumptions of the implications in 2. Let $\{E_i\}_{i=1}^n$ be a disjoint collection of measurable subsets of E. Then,

[9] $|x^*F|(E) = \sup\{\sum_{k=1}^n |x^*F(E_k)| : \{E_k\}_{k=1}^n \subseteq \Sigma, \bigcup_{k=1}^n E_k \subseteq E\}$ is well-defined since x^*F is well-defined.

$$\nu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \nu(E_i) \geq \sum_{i=1}^{n} |x^*F|(E_i)$$

for all $x^* \in X^*$ with $||x^*|| \leq 1$. Since $||F(E)|| = \sup\{|x^*F|(E) : x^* \in X^*, ||x^*|| \leq 1\}$ for all $E \in \Sigma$, we have that $\sum_{i=1}^{n} \nu(E_i) \geq \sum_{i=1}^{n} ||F(E_i)||$. Therefore, $\nu(E) \geq |F|(E)$.

$\Leftarrow$ Suppose μ is a measure on Σ that satisfies the two conditions.

Then, by the second condition (with $\nu = |x^*F|$ for some fixed $x^* \in X^*$), we know $\mu(E) \geq |F|(E)$ for all $E \in \Sigma$, and by the fact that $|F|^*(E) \leq |F|(E)$ for all $E \in \Sigma$, we have the reverse inequality. $\square$

Proposition 1.17. *[2] Let $F : \Sigma \to X$ be a vector measure, then for all $E \in \Sigma$,*

1. $|F|^*(E) = \sup\left\{\left|\sum_{k=1}^{n} a_k F(E_k)\right| : a_k \in \mathbb{K}, |a_k| \leq 1 for 1 \leq k \leq n, and \{E_k\}_{k=1}^{n} \subseteq \Sigma is\ a\ finite\ partition\ of$
2. $\sup_{E \supseteq A \in \Sigma} \|F(A)\| \leq |F|^*(E) \leq 4 \sup_{E \supseteq A \in \Sigma} \|F(A)\|.$

Proof. 1. Let $E \in \Sigma$ and $\{E_k\}_{k=1}^{n}$ a partition of T into pairwise disjoint sets in Σ and $a_k \in \overline{B}^{\mathbb{K}}$[10] for $1 \leq k \leq n$. Then,

$$\left\|\sum_{k=1}^{n} a_k F(E_k)\right\| = \sup_{x^* \in \overline{B}^{\mathbb{X}^*}} \left|x^* \sum_{k=1}^{n} a_k F(E_k)\right| \leq \sup_{x^* \in \overline{B}^{\mathbb{X}^*}} \sum_{k=1}^{n} |x^*F(E_k)| \leq |F|^*(E).$$

For the reverse inequality, let $x^* \in X^*$ with $\|x^*\| \leq 1$, $E \in \Sigma$, and $\{E_k\}_{k=1}^{n}$ a partition of T into pairwise disjoint sets in Σ. Then,

$$\begin{aligned}
\sum_{k=1}^{n} |x^*F(E_k)| &= \sum_{k=1}^{n} \operatorname{sgn}(x^*F(E_k))x^*F(E_k) \\
&= \left|x^*\left(\sum_{k=1}^{n} \operatorname{sgn}(x^*F(E_k))F(E_k)\right)\right| \\
&\leq \left\|\sum_{k=1}^{n} \operatorname{sgn}(x^*F(E_k))F(E_k))\right\|.
\end{aligned}$$

2. For the first inequality, let $E \in \Sigma$. Then,

[10] $\overline{B}^{X}$ denotes the closed unit ball in the normed vector space X.

$$\sup_{E \supseteq A \in \Sigma} \|F(A)\| = \sup_{x^* \in \overline{B}^{X^*}} \sup_{E \supseteq A \in \Sigma} |x^* F(A)| \leq \|F(E)\|.$$

For the second inequality, we first assume $\mathbb{X}$ is a Banach space over $\mathbb{R}$. Let $E \in \Sigma$, $\{E_k\}_{k=1}^n$ a partition of T into pairwise disjoint sets in Σ, and $x^* \in \overline{B}^{X^*}$. Define $N^+ := \{k : 1 \leq k \leq n, x^* F(E_k) \geq 0\}$ and $N^- := \{k : 1 \leq k \leq n, x^* F(E_k) < 0\}$. Then,

$$\begin{aligned}\sum_{k=1}^n |x^* F(E_k)| &= \sum_{k \in N^+} x^* F(E_k) - \sum_{k \in N^-} x^* F(E_k) \\ &\leq \left| x^* \left(\sum_{k \in N^+} x^* F(E_k) \right) \right| + \left| x^* \left(\sum_{k \in N^-} F(E_k) \right) \right| \\ &\leq 2 \sup_{E \supseteq A \in \Sigma} \|F(A)\|\end{aligned}$$

If the scalar field is $\mathbb{C}$, the following argument will work for $x^* F$ split into real and imaginary parts, yielding the $4 \sup_{E \supseteq A \in \Sigma} \|F(A)\|$ in the proposition.

□

Note that the second claim implies that a vector measure is of bounded weak-variation iff its range is bounded in $\mathbb{X}$; hence, a vector measure of bounded weak-variation is called a bounded vector measure.

1.5 Linear Operators, Integrals with respect to Vector Measures, and descriptive theorems

As mentioned before, elements of vector measure theory can be very useful in describing the Banach spaces into which vector measures map. The following is a slight digression to illustrate such descriptions.

With weak-variation in hand, we are ready to construct a rudimentary integral of a bounded measurable function with respect to a bounded vector measure. We follow Diestel and Uhl's construction from Chapter 1 of *Vector Measures*:

Let Σ be an σ-algebra on Ω and $F : \Sigma \to \mathbb{X}$ a bounded vector measure. Let $B((\Omega, \Sigma), \mathbb{K})$ denote the space of all bounded Σ-measurable $\mathbb{K}$-valued function on Ω (with supremum norm $\|\cdot\|_\infty$), and let $B_0((\Omega, \Sigma), \mathbb{K}) = B_0$ denote the subspace of all simple scalar functions on Ω. Then, define $T_F : B_0 \to \mathbb{X}$ by

$$T_F f := \sum_{k=1}^{n} a_k F(E_k)$$

for each $f \in B_0$ (where f is given by $f := \sum_{k=1}^{n} a_k \mathbb{1}_{E_k}$ for some $n \in \mathbb{N}$ with $a_k \in \mathbb{K}$ and $E_k \in \Sigma$ for $1 \leq k \leq n$ and $\{E_k\}_{k=1}^n$ a partition of E). Then T_F is a linear map and, for all $f \in B_0$,

$$||T_F(f)|| = \left\| \sum_{k=1}^{n} a_k F(E_k) \right\| = ||f||_\infty \left\| \sum_{k=1}^{n} \frac{a_k}{||f||_\infty} F(E_k) \right\| \leq |F|^*(\Omega) ||f||_\infty.$$

Then T_F has a unique continuous linear extension to the space of all $\mathbb{K}$-valued functions on Ω that are uniform limits of simple functions Σ-measurable $\mathbb{K}$-valued functions on Ω, $B(\Sigma_0)$. Now, we are ready to define an integral with respect to our vector measure.

Definition 1.18. Let (Ω, Σ) be a measurable space and $F : \Sigma \to \mathbb{X}$ be a bounded vector measure, the for each $f \in B(\Sigma)$, we define $\int f \, dF$ by

$$\int f \, dF = T_F(f).$$

This is only a crude introduction to integration with respect to vector measures, but, as it is not in the scope of this paper, a crude introduction will have to do. This integral is, in fact, linear, and, as indicated above, satisfies

$$\left\| \int f \, dF \right\| \leq ||f||_\infty |F|^*(\Omega).$$

Our final remark on this integral is the following theorem:[2]

Theorem 1.19. *Let (Ω, Σ) be a measurable space and $\mathbb{X}$ a Banach space, and let $\mu : \Sigma \to [0, \infty]$ be a non-negative $\mathbb{R}$-valued measure on (Ω, Σ). Then there is a one-to-one linear correspondence ϕ between $\mathcal{L}(L^\infty(\mu); \mathbb{X}))$ and the space of all bounded vector measures $F : \Sigma \to \mathbb{X}$ that vanish on μ-null sets give by $\phi(T_F(f)) = \int f \, dF$ for all $f \in L^\infty(\mu)$.*

We will close this section with a few more definitions and important theorems, the discussions of which we will omit.

Definition 1.20. Let Σ_0 be an algebra on Ω, $F : \Sigma_0 \to X$ a vector measure, and $\mu : \mathcal{F} \to \mathbb{R}$ a finite measure on $\mathcal{F}$. Then F is called **μ-continuous**, $F << \mu$ if $\lim_{\mu(E)\to 0} F(E) = 0$.

Definition 1.21. Let Σ_0 be an algebra on Ω, and $F : \Sigma_0 \to X$ a vector measure. F is **strongly additive** whenever, given a pairwise disjoint sequence (E_k) in Σ_0, the series $\sum_{k=1}^{\infty} F(E_k)$ converges in the norm.

If Σ is a σ-algebra, this is weaker than countable additivity; note that, without an equation, convergence need not be unconditional.

Definition 1.22. A family $\{F_\tau : \Sigma_0 \to X : \tau \in T\}$ of strongly additive vector measures is **uniformly strongly additive** whenever, for any pairwise disjoint sequence $(E_k) \subseteq \Sigma_0$, $\lim_{k\to\infty} \left\| \sum_{m=k}^{\infty} F_\tau(E_m) \right\| = 0$ uniformly for $\tau \in T$.

Again, if $\{F_\tau : \Sigma_0 \to X : \tau \in T\}$ is a family of countably additive vector measures, then uniform strong additivity is just uniform countable additivity.

As previously mentioned, many of our theorems will hold for finitely additive vector measures; however, several require the slighly stronger requirement of strong additivity, such as the two following theorems.

Theorem 1.23 (Nikodým Boundedness Theorem)**.** *[2] Let (Ω, Σ) be a measurable space and $\{F_\tau : \tau \in T\}$ a family of X-valued vector measures defined on Σ. If $\sup_{\tau\in T} ||F_\tau(E)|| < \infty$ for each $E \in \Sigma$, then $\{F_\tau : \tau \in T\}$ is uniformly bounded, i.e. $\sup_{\tau\in T} |F_\tau|^*(\Omega) < \infty$.*

Theorem 1.24 (Vitali-Hahn-Saks-Nikodým Theorem)**.** *[2] Let Σ be a σ-field of subsets of Ω and (F_n) a sequence of strongly additive X-valued measures on Σ. If $\lim_n F_n(E)$ exists in X-norm for each $E \in \Sigma$, then the sequence (F_n) is uniformly strongly additive.*

Theorem 1.25 (Orlicz-Pettis). [11] *[2] Let $\sum_n x_n$ be a series in X such that every subseries of $\sum_n x_n$ is weakly convergent. Then $\sum_n x_n$ is unconditionally convergent in norm. Consequently, a weakly*[12] *countably addtitive vector measure on a σ-algebra is (norm) countably additive.*

Finally, we close this section with one of the most powerful theorems for vector (and scalar)-valued set functions.

Theorem 1.26. *[3] Let Σ be a σ-algebra of subsets of a set Ω, and let μ be a complex or signed measure on Σ. Suppose $\{F_n\}$ is a sequence of μ-continuous vector or scalar valued additive set functions on Σ such that $\lim_n F_n(E)$ exists for each $E \in \Sigma$. Then,*

$$\lim_{|\mu|(E)\to 0} F_n(E) = 0$$

uniformly for $n = 1, 2, ...$[13]

1.6 Vector Measures and Random Measures

One of the most natural examples[14] of a vector measure is a random measure. In this section we will introduce random measures and discuss where they fit in vector measure theory. First, note that there are at least two senses of the term "random measure". The first is a wider sense, which we will just mention:

In a wide sense, a random measure is simply a vector measure whose range is $L^0(S, \mathcal{S}, P)$, where $(S, \mathcal{S}, P)$ is a probability space.[15] Since members of $L^0(S, \mathcal{S}, P)$ are called random variables, the vector measure is called a random measure. A primary example of a random measure in the wide sense is "White Noise":

[11] This is often presented with the Orlicz Theorem given with the Toy Vector Measures example.

[12] that is, the series converges weakly in X

[13] It certainly bears mentioning that this theorem is very much akin to a corollary of the Uniform Boundedness Principle, and, in fact, the proofs of the Vitali-Hahn-Saks Theorem (whether they use the gliding-hump method or Bare Category Theorem) greatly resemble those for the Uniform Boundedness Principle

[14] "Example" should be used loosely here, for, as we will see, a random measure may not be additive.

[15] Note that, although $L^0(S, \mathcal{S}, P)$ is not a Banach space, we can define a complete metric on $L^0(S, \mathcal{S}, P)$, e.g. the metric given by $d(X, Y) = E\max\{1, |X - Y|\}$, and hence we consider $L^0(S, \mathcal{S}, P)$ an F-space.

Example 1.27. Say we have a Brownian motion X_t on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{B}[0,1] = \mathcal{B}$ represent the σ-algebra of sets on $[0,1]$. Then $\mathcal{B}$ is generated by the field, $\mathcal{B}_0$ of finite unions of intervals of the form $(a,b] \subseteq (0,1]$.

Define $X : \{(a,b] : (a,b] \subseteq (0,1]\} \to L^2(\Omega, \mathcal{F}, P)$ by $X(a,b] = X_b - X_a$, i.e. the increments of X_t.

For $A \in \mathcal{B}_0$ where A is the finite union of pairwise disjoint intervals $(a_k, b_k]$, we can establish finite additivity of X: Let $A = \bigcup_k (a_k, b_k]$ where $(a_k, b_k] \subseteq (0,1]$ are pairwise disjoint. Then,

$$XA = \sum_k X(a_k, b_k].$$

Since X_t is Brownian motion, X_t has orthogonal increments,

$$\|XA\|^2 = \sum_k \|X_{b_k} - X_{a_k}\|^2.$$

Furthermore, since our image space is $L^2(\Omega, \mathcal{F}, P)$,

$$\|X_b - X_a\|^2 = E|X_b - X_a|^2 = E|X_b|^2 - E|X_a|^2.$$

Then $F(t) := E|X_t|^2$ is a bounded nondecreasing function on $[0,1]$, which then generates a bounded Borel measure μ on $\mathcal{B}_0$. And so, we have for all $A \in \mathcal{B}_0$ where A is the finite union of pairwise disjoint intervals $(a_k, b_k]$

$$\|XA\|^2 = \sum_k \|X_{b_k} - X_{a_k}\|^2 = \sum_k \mu(a_k, b_k] = \mu(A).$$

Now, let $A \in \mathcal{B}$, and choose a collection $A_n \in \mathcal{B}_0$ such that $\mu(A \Delta A_n) \to 0$. Then, XA_n is Cauchy in $L^2(\Omega, \mathcal{F}, P)$ and hence converges. Define $X : \mathcal{B} \to L^2(\Omega, \mathcal{F}, P)$ by $XA := \lim_{n\to\infty} XA_n$ where $A_n \in \mathcal{B}_0$ and $\mu(A \Delta A_n) \to 0$. It follows that if $A_n \in \mathcal{B}$ are pairwise disjoint, then $X(\bigcup_n A_n) = \sum_n X(A_n)$ a.e., and hence X is a countably additive vector measure. This vector measure (or wide-sense random measure) generated by Brownian motion is called **white noise**. From real analysis, we know that if a function is of bounded variation, then its derivative exists almost everywhere. Therefore, since Brownian motion is (almost surely) nowhere differentiable, X is not of bounded variation.

The other sense of "random measure" is much narrower. Consider a countably-additive vector-valued function F from a measurable space $(\Omega, \mathcal{F})$ to an F-space, $X \subseteq L^0(S, \mathcal{S}, P)$, where $(S, \mathcal{S}, P)$ is a probability space. This F is then a vector measure and a random measure in the wide sense, but the narrow sense requires more.

Definition 1.28. A **random measure** is a kernel from a probability space $(S, \mathcal{S}, P)$ to a measurable space $(\Omega, \mathcal{F})$, that is, a function $\xi : S \times \mathcal{F} \to \overline{\mathbb{R}^+}$ such that $\xi(\cdot, E)$ is a random variable on $(S, \mathcal{S}, P)$[16] and $\xi(s, \cdot)$ is a measure on $\mathcal{F}$ P-a.s..

A random measure is often denoted by $\xi_s(E)$ for all $s \in S$ and $E \in \mathcal{F}$, and the subscript is often dropped. By using ξ to refer specifically to the mapping from $\mathcal{F}$ into $L^0(S, \mathcal{S}, P)$ given by $\xi(E) = \xi(\cdot, E)$, we can refer to ξ as a vector measure (provided, of course, that it is additive). Then we would say a random measure is a function $\xi : (\Omega, \mathcal{F}) \to L^0(S, \mathcal{S}, P)$ such that $\xi(E) \in L^0(S, \mathcal{S}, P)$ for all $E \in \mathcal{F}$ and ξ_s is a scalar measure on $\mathcal{F}$ for each $s \in S$. This notation also allows us to refer to ξ as the family of random variables $\{\xi E : E \in \mathcal{F}\}$.

Notice that a random measure in the narrow sense is not, by definition, additive. However, if, for any two disjoint measurable sets E_0 and E_1 in $\mathcal{B}$, $\xi(E_0 \cup E_1, \omega) = \xi(E_0, \omega) + \xi(E_1, \omega)$ a.e. for any $\omega \in \Omega$, then ξ is a vector measure. If, for any collection $(E_k)_{k \in \mathbb{N}}$ of pairwise disjoint open sets in $\mathcal{F}$, $\xi(\bigcup_k E_k, \omega) = \sum_k \xi(E_k, \omega)$ a.e. for any $\omega \in \Omega$, we say that ξ is a countably additive vector measure. Because ξ is originally defined as taking values in $\overline{\mathbb{R}}^+$, we can see that there is a direct correspondence in whether $\xi(\cdot, E)$ is (σ-)finite and whether $\xi(\omega, E)$ is (σ-)finite.

For the sake of an example, we will progress towards a Poisson process via random measures and point processes, a move which will be facilitated by the following theorem from T.E. Harris [9]:[17]

Theorem 1.29 (Harris)**.** *Let S be a (complete) metric space, $\mathcal{B}$ the Borel σ-field of S, and $X = \{X_B : B \in \mathcal{B}\}$ a random process on $\mathcal{B}$ such that*

[16] That is, an element of $L^0(S, \mathcal{S}, P)$.

[17] This is a version of the theorem from O. Kallenberg. The original statement of the theorem can be found in Appendix F.

1. $X_B \geq 0$ *for all* $B \in \mathcal{B}$,

2. $X_{B_1 \cup B_2} = X_{B_1} + X_{B_2}$ *a.s. for all pairs* B_1 *and* B_2 *of disjoint sets in* $\mathcal{B}$, *and*

3. $X_{B_n} \xrightarrow{P} 0$ *as* $B_n \to \emptyset$.

Then there exists a random measure ξ *on* S *such that* $\xi B = X_B$ *a.s. for all* $B \in \mathcal{B}$.

Definition 1.30. A random measure ξ on a measurable space has **independent increments** if for any disjoint sets $E_1, ..., E_n \in \mathcal{F}$, the random variables $\xi E_1, ..., \xi E_n$ are independent.

Definition 1.31. A **point process** ξ on a space S is a locally finite random measure from $(S, \mathcal{S}, P)$ to $(\mathbb{R}^d, \mathcal{B}^d)$ that takes values in $\mathbb{Z}$ for all bounded $B \in \mathcal{B}^d$.

A point process can be thought of as a count of a collection of random points.

Example 1.32. A classic example of a point process is a "counting process". Take a random sequence (X_n) with values in $\mathbb{R}^d$ and let ξE be a count of X_n's in E, i.e. $\xi E = \sum_n \mathbb{1}_E(X_n)$.

With these two definitions in mind, we can see that a standard Poisson process is an example of a random measure.

Definition 1.33. A **Poisson process** on a measurable space $(\Omega, \mathcal{F})$ with intensity measure μ, is a point process ξ on Ω with independent "increments" (i.e. disjoint sets) such that ξE has Poisson distribution with mean $\mu(E)$ for all $E \in \mathcal{F}$ such that $\mu(E) < \infty$.

If we have the luxury of having our point process on $(\mathbb{R}^+, \mathcal{B}, \lambda)$, our points will have a linear ordering, allowing us to discuss increments.

Definition 1.34. A **Poisson process** with rate λ is a family of random variables N_t, $t \geq 0$ such that

1. if $0 = t_0 < t_1 < ... < t_n$, then for each $1 \leq k \leq n$, $N_{t_k} - N_{t_{k-1}}$ are independent, and

2. $N_t - N_s$ is Poisson with mean $(\lambda(t-s))$ for all $t > s > 0$.

As mentioned before, ξ may be considered as the family of random variables $\{\xi E : E \in \mathcal{F}\}$. Hence, this is a specific case where $\Omega = [0, \infty)$ and $\mathcal{F} = \mathcal{B}$. (Specifically, $N(t_k) - N(t_{k-1}) = \xi(t_{k-1}, t_k])$. Furthermore, here μ is given by $\mu(s,t] = \lambda(t-s)$.

As defined, a Poisson process must be a random measure. However, even without this progression of definitions, it is not very suprising that a "counting" process would be a random measure:

Let $(S, \mathcal{S}, \mu)$ be a σ-finite separable measure space and $(\Omega, \mathcal{F}, P)$ a probability space. Let $X_n : \Omega \to S$ be random elements (i.e. measurable functions), and let a full event (i.e. an event with full measure) Ω_0 be the common domain of the sequence X_n. Define the counting measure as

$$(NS)(\omega) = \sum_n 1\!\!1_S(X_n(\omega)), \text{ for a given } S \in \mathcal{S} \text{ and all } \omega \in \Omega_0.$$

Then, for all $\omega \in \Omega_0$, it is the counting measure of the sequence $(x_n) = (X_n(\omega))$. However, we also have defined a vector measure with values in $L^2(\Omega, \mathcal{F}, P)$:

$$S \mapsto N(S)(\cdot) = \sum_n 1\!\!1_S(X_n(\cdot)) \text{ for all } S \in \mathcal{S}.$$

There do, however, exist vector measures of the form $F : (\Omega, \mathcal{F}) \to L^0(S, \mathcal{S}, P)$ such that $F(E, \cdot) \in L^0(S, \mathcal{S}, P)$ for all $E \in \mathcal{F}$ and $F(\cdot, \omega)$ is **not** a scalar measure on $\mathcal{F}$ for each $\omega \in \Omega$.

Example 1.35. Let (r_n) be a sequence of Rademacher functions[18] defined on $([0,1], \mathcal{B}, P)$ and $(a_n) \in \ell^2(\mathbb{R}) \backslash \ell^1(\mathbb{R})$. Define the Rademacher measure on $(\mathbb{N}, 2^{\mathbb{N}}, c)$ as the vector measure with values in $L^2([0,1], \mathcal{B}, P)$ by

$$F(K) = \sum_{n \in K} a_n r_n, \text{ for all } K \in 2^{\mathbb{N}}.$$

[18]defined in Appendix B

If $c(K) = \infty$, then the series converges in the metric space L^2. Hence, we immediately have unconditional convergence required for F to be a vector measure.

Then, for each infinite $K \subseteq \mathbb{N}$, there is a full event Ω_K such that for all $\omega \in \Omega_K$,

$$\sum_{n \in K} a_n r_n(\omega) = \lim_{N \to \infty} \sum_{n=1}^{N} \mathbb{I}_{n \in K} a_n r_n(\omega).$$

Now, to have a scalar measure a.s. on $2^{\mathbb{N}}$ when some $\omega \in \Omega_{\mathbb{N}}$ is fixed, we need, in particular, that

$$F(\mathbb{N})(\omega) = \sum_{n \in \mathbb{N}} F(\{n\})(\omega) = \sum_{n \in \mathbb{N}} a_n r_n(\omega) < \infty.$$

Since the collection $(\{n\})_n \in \mathbb{N}$ is invariant under permutations, $\sum_{n \in \mathbb{N}} a_n r_n(\omega)$ must be unconditionally convergent. In $\mathbb{R}$, this is equivalent to absolute convergence. However, $\sum_{n \in \mathbb{N}} |a_n r_n(\omega)| = \sum_{n \in \mathbb{N}} |a_n|$ where $(a_n) \notin \ell^1$.

Therefore, F is a vector measure, but not a random measure in the narrow sense.

An analogous argument shows that Brownian Motion is not a random measure in the narrow sense. (As previously established, it is a countable vector measure.) The claim that BM is a random measure amounts to the claim that almost all paths of a given BM have bounded variation on $[0, 1]$.

1.7 Conclusions

Now, we are equipped with a basic introduction to vector measures. In the next chapter, we will explore expressing vector measures as integrals of Banach-valued functions (and vice versa), which, in cases of a general domain space, requires some form of a Radon-Nikodým Theorem.

Chapter 2

Bochner Integration and the RNT

This section will develop a vector-valued integral of a vector-valued function with respect to a scalar measure, called the Bochner integral. The Bochner integral will allow us to expand the class of Lebesgue integrable functions to include vector-valued functions and to integrate this new class of functions, which we will call Bochner integration. By defining this vector-valued integral and extending the class of Lebesgue integrable functions, we will be able to formulate a Radon Nikodým Theorem in the more general context of Banach spaces. Since the theorem fails in some Banach spaces, the Radon Nikodým property emerges.

For this and subsequent sections, we will consider a probability space (Ω, Σ, μ) and Banach space $\mathbb{X}$ with underlying field $\mathbb{R}$. Again, all vector measures are understood to be countably additive. There are analogous results for $\mathbb{C}$, and many of these results extend to other scalar fields and σ-finite spaces, but since the aim is to understand concepts applying to functions between probability spaces and Banach spaces, we will not obscure these concepts with other scalar fields.

2.1 Bochner Integral for μ-Simple Functions

Definition 2.1. A function $f : \Omega \to \mathbb{X}$ is **μ-simple** if $f(\omega) = \sum_{i=1}^{n} \mathbb{I}_{E_i} x_i$ where $E_1, ..., E_n$ are disjoint[1] members of Σ, $\mu(E_i) < \infty$ for all i, and $x_1, ..., x_n \in \mathbb{X}$. [2]

We need not require that $\mu(E_i) < \infty$ for all i, but this requirement will allow our results to extend to σ-finite measure spaces, since it will force the integrals of our simple functions to be well-defined. Now, we will define the Bochner integral for μ-simple functions.

[1] They need not be disjoint yet, but WLOG, it's convenient to make this assumption at the outset.

[2] Of course, it is implicit that $f : (\Omega - \bigcup_{i=1}^{n} E_i) \to \{0\} \subseteq \mathbb{X}$.

Definition 2.2. Let $f : \Omega \to \mathbb{X}$ is μ-simple; then for any $E \in \Sigma$, its **Bochner integral** is defined as

$$\int_E f \; d\mu = \sum_{i=1}^{n} \mu(E \cap E_i) x_i$$

where $f(\omega) = \sum_{i=1}^{n} \mathbb{1}_{E_i} x_i$ with $E_1, ..., E_n$ disjoint members of Σ and $x_1, ..., x_n \in \mathbb{X}$.

Proposition 2.3. *The Bochner integral of a μ-simple function f is independent of the representation of f.*

Proof. Let $f(\omega) = \sum_{i=1}^{n} \mathbb{1}_{E_i} x_i = \sum_{j=1}^{m} \mathbb{1}_{A_j} y_j$ on a measurable set E where $E_1, ..., E_n$ and $A_1, ..., A_m$ are collections of disjoint measurable sets, $\bigcup_{i=1}^{n} E_i = \bigcup_{k=1}^{m} A_j = E$, and $x_1, ..., x_n, y_1, ..., y_m \in \mathbb{X}$. Then consider the refinement of both partitions, $\{F_{i,j}\}$ where $F_{i,j} = E_i \cap A_j$ for all i, j. Then, $f(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}_{F_{i,j}} x_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}_{F_{i,j}} y_j$. Furthermore,

$$\sum_{i=1}^{n} \mu(E \cap E_i) x_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E \cap F_{i,j}) x_i \text{ and } \sum_{j=1}^{m} \mu(E \cap A_j) y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E \cap F_{i,j}) y_j$$

Thus, it remains to show that $\sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E \cap F_{i,j}) x_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E \cap F_{i,j}) y_j$, which follows from the observation that, for each pair i, j, given $F_{i,j} \neq \emptyset$, we have that $x_i = y_j$. $\square$

Before we continue, we will establish linearity, additivity, and contractivity for Bochner integrals of μ-simple functions.

Proposition 2.4 (Linearity)**.** *Let $f, g : \Omega \to \mathbb{X}$ be μ-simple functions and $\alpha \in \mathbb{R}$. Then, for all $E \in \Sigma$,*

1. $\int_E \alpha f \; d\mu = \alpha \int_E f \; d\mu$
2. $\int_E f + g \; d\mu = \int_E f \; d\mu + \int_E g \; d\mu$

Proof. Since the claim for $E \subseteq \Omega$ will be analogous, we will show the claim for $E = \Omega$. Also, say $f = \sum_{i=1}^{n} \mathbb{1}_{B_i} x_i$ and $g = \sum_{j=1}^{m} \mathbb{1}_{A_j} y_j$ where $x_1, ..., x_n, y_1, ..., y_n \in \mathbb{X}$ and (B_i) and (A_i) are finite collections of disjoint measurable sets.

1. By the properties of vector spaces, $\int f \, d\mu = \sum_{i=1}^{n} \mu(B_i) \alpha x_i = \alpha \sum_{i=1}^{n} \mu(B_i) x_i = \alpha \int f \, d\mu.$
2. Let $E_{i,j} = B_i \cap A_j$ for all i, j. Then $(E_{i,j})$ is a refinement of both (B_i) and (A_j), and we can define the simple function $f + g$ as $\sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}_{E_{i,j}}(x_i + y_j)$. Then,

$$\begin{aligned}\int f \, d\mu + \int g \, d\mu &= \sum_{i=1}^{n} \mu(B_i) x_i + \sum_{j-1}^{m} \mu(A_j) y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(B_i \cap A_j)(x_i + y_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(E_{i,j})(x_i + y_j) = \int f + g \, d\mu\end{aligned}$$

□

Corollary 2.5 (Additivity). *If $(E_i)_{i=1}^{n}$ are disjoint measurable sets whose union is $E \in \Sigma$ and $f : \Omega \to \mathbb{X}$ is a μ-simple function then* $\sum_{i=1}^{n} \int_{E_i} f \, d\mu = \int_{E} f \, d\mu.$

If we let $f_i = f \cdot \mathbb{1}_{E_i}$, the proof follows from linearity.

Proposition 2.6 (Contractivity). *Let $f : \Omega \to \mathbb{X}$ be a μ-simple function. Then*

$$\left\| \int_E f \, d\mu \right\| \leq \int_E \|f\| \, d\mu.$$

Proof. WLOG, we will show the claim for $E = \Omega$. Let $f = \sum_{i=1}^{n} \mathbb{1}_{E_i} x_i$. Then,

$$\left\| \int f \, d\mu \right\| = \left\| \sum_{i=1}^{n} \mu(E_i) x_i \right\| \leq \sum_{i=1}^{n} \mu(E_i) \|x_i\| = \sum_{i=1}^{n} \int_{E_i} \|x_i\| \, d\mu = \int \|f\| \, d\mu$$ □

Example 2.7. For $n \in \mathbb{N}$ and $1 \leq i \leq n$, let $g_{i,n} : [0,1] \to \mathbb{R}$ be the constant function $g_{i,n}(x) = \dfrac{i}{n}$. Then, we can define the λ-simple functions $f_n : ([0,1], \mathcal{B}, \lambda) \to L^\infty[0,1]$ by $f_n(x) = \left(\sum_{i=1}^{n} \mathbb{I}_{\left(\frac{i-1}{n}, \frac{i}{n}\right]} g_{i,n}(x) \right)$. [3] Then for each $n \in \mathbb{N}$,

$$\int f_n \, d\lambda = \left(\sum_{i=1}^{n} \lambda \left(\frac{i-1}{n}, \frac{i}{n} \right] g_{i,n}(x) \right) = \sum_{i=1}^{n} \frac{1}{n} g_{i,n}(x).$$

Since $g_{i,n}(x)$ is the constant function $g_{i,n}(x) = \dfrac{i}{n}$, for each i, n, $\dfrac{1}{n} g_{i,n}(x)$ is the constant function $g'_{i,n}(x) = \dfrac{i}{n^2}$. Then $\sum_{i=1}^{n} \dfrac{1}{n} g_{i,n}(x) = \sum_{i=1}^{n} g'_{i,n}(x) = G_n(x)$ where $G_n(x)$ is the constant function $G_n(x) = \dfrac{n+1}{2n}$.

2.2 Bochner integral defined

Now, we will define the Bochner integral for more general functions, but first, we must define what it means for a function to be (strongly) μ-measurable.

Definition 2.8. $f : \Omega \to \mathbb{X}$ is **strongly μ-measurable** if there exists a sequence of μ-simple functions (f_n) where $\lim_{n\to\infty} \|f_n - f\| = 0$ μ-a.e..

Henceforth, we shall just call such a strongly μ-measurable function μ-measurable. (We will leave the discussion of weakly measurable function to section 2.4.)

Next, we define Bochner integrability for a μ-measurable function.

Definition 2.9. Let $f : \Omega \to \mathbb{X}$ be strongly μ-measurable; f is **Bochner integrable** if there exists a sequence of μ-simple functions (f_n) that converge to f μ-a.e. and

$$\lim_{n\to\infty} \int \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0.$$

Observe that, once we compose the norm of $\mathbb{X}$ with $f_n - f$, we have a real-valued function, and once we have a real-valued (Lebesgue integrable) function, the Bochner and Lebesgue

[3] Where $f_n(0) = 0 \cdot \mathbb{I}_{\{0\}}$.

integrals are the same. Hence, Bochner integration can be thought of as generalizing the class of Lebesgue integrable functions.

There is one exception to their coincidence, however. The Lebesgue integral (of a non-integrable function) can be infinity, whereas a infinity would have to be formally defined (and $\mathbb{R}$ extended to $\overline{\mathbb{R}}$) for a Bochner integral of a real-valued function to yield infinity. (In other words, the Lebesgue integral can be defined for a function that is not Lebesgue integrable, whereas the same makes no sense for the Bochner integral.)

Definition 2.10. Given that a function $f : \Omega \to \mathbb{X}$ is Bochner integrable, the **Bochner integral** of f is defined for each $E \in \Sigma$ as

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu$$

where (f_n) is a sequence of μ-simple functions such that $\lim\limits_{n\to\infty} \int \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0$.

Proposition 2.11. *The Bochner integral is well-defined, i.e.*

1. *Consistency: The Bochner integral of f is independent of the sequence of simple functions (f_n) such that $\lim\limits_{n\to\infty} \int \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0$.*
2. *Uniqueness: If $f : \Omega \to \mathbb{X}$ is Bochner integrable and (f_n) is a sequence of μ-simple functions such that $\lim\limits_{n\to\infty} \int \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0$, then $\lim\limits_{n\to\infty} \int_E f_n \, d\mu$ is unique.*

Proof. 1. Consistency: Suppose there exist two sequences of simple functions (f_n) and (g_n), each of which converge to f a.e. and

$$\lim_{n\to\infty} \int \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = \lim_{n\to\infty} \int \|g_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0.$$

WLOG, we will show the claim for $E = \Omega$. Let $\epsilon > 0$. Then there is a $N \geq 0$ such that for all $n \geq N$, $\int \|f_n - f\| \, d\mu < \epsilon/4$, $\int \|g_n - f\| \, d\mu < \epsilon/4$, and $\left\| \int g_n \, d\mu - \lim\limits_{n\to\infty} \int g_n \, d\mu \right\| < \epsilon/2$. Then for all $n \geq N$,

$$\int \|g_n - f_n\| \, d\mu \leq \int \|g_n - f\| \, d\mu + \int \|f - f_n\| \, d\mu < \epsilon/2.$$

Hence,

$$\epsilon/2 > \int \|g_n - f_n\| \, d\mu \geq \left\| \int g_n - f_n \, d\mu \right\| = \left\| \int g_n \, d\mu - \int f_n \, d\mu \right\|.$$

Then for all $n \geq N$,

$$\left\| \int f_n \, d\mu - \lim_{n\to\infty} \int g_n \, d\mu \right\| \leq \left\| \int f_n \, d\mu - \int g_n \, d\mu \right\| + \left\| \int g_n \, d\mu - \lim_{n\to\infty} \int g_n \, d\mu \right\| < \epsilon.$$

2. Uniqueness: WLOG, we will show the claim for $E = \Omega$. Let (f_n) a sequence of μ-simple functions from Ω to $\mathbb{X}$ that converges to $f : \Omega \to \mathbb{X}$ μ-a.e. such that $\lim_{n\to\infty} \int \|f_n - f\| \, d\mu = 0$. Suppose (g_m) is also a sequence of μ-simple functions from Ω to $\mathbb{X}$ that converges to $f : \Omega \to \mathbb{X}$ μ-a.e. such that $\lim_{n\to\infty} \int \|g_m - f\| \, d\mu = 0$. Let $\epsilon > 0$. Then, there exists an $N_0 \geq 0$ for which

$$\begin{aligned} \left\| \int g_m \, d\mu - \int f_n \, d\mu \right\| = \left\| \int g_m - f_n \, d\mu \right\| &\leq \int \|g_m - f_n\| \, d\mu \\ &\leq \int \|g_m - f\| \, d\mu + \int \|f - f_n\| \, d\mu < \epsilon \end{aligned}$$

Similarly, there is an $N_1 \geq 0$ for which for all $n, m \geq N$,

$$\left\| \int g_n \, d\mu - \int g_m \, d\mu \right\| < \epsilon \text{ and } \left\| \int f_n \, d\mu - \int f_m \, d\mu \right\| < \epsilon.$$

Hence, if we define the sequence (h_k) in $\mathbb{X}$ such that $h_{2i-1} = \int g_i \, d\mu$ and $h_{2i} = \int f_i \, d\mu$. Then there exists an $N \geq 0$ ($N = \max\{N_0, N_1\}$) such that for all $n, m \geq N$, $\|h_n - h_m\| < \epsilon$. Hence (h_k) is Cauchy and thus converges since $\mathbb{X}$ is complete as a Banach space. Since $\left(\int g_m \, d\mu\right)$ and $\left(\int f_n \, d\mu\right)$ are Cauchy subsequences of (h_k), they converge to the same limit.

$\square$

Next, we establish contractivity for Bochner integrable functions.

Proposition 2.12 (Contractivity). *If f is Bochner integrable, then* $\left\| \int f \, d\mu \right\| \leq \int \|f\| \, d\mu$.

Proof. Since f is Bochner integrable.

$$\left\|\int f\ d\mu\right\| = \left\|\lim_{n\to\infty}\int f_n\ d\mu\right\| = \lim_{n\to\infty}\left\|\int f_n\ d\mu\right\| \leq \lim_{n\to\infty}\int \|f_n\|\ d\mu$$

Let $\epsilon > 0$. Then there exists an $N > 0$ such that for all $n \geq N$,

$$\epsilon > \left|\int \|f - f_n\|\ d\mu\right| \geq \left|\int \|f\|\ d\mu - \int \|f_n\|\ d\mu\right|.$$

Hence $\lim_{n\to\infty}\int \|f_n\|\ d\mu = \int \|f\|\ d\mu$, and, therefore, $\left\|\int f\ d\mu\right\| \leq \int \|f\|\ d\mu$. □

Furthermore, by a classic diagonalization argument, we can now go beyond simple functions converging to f:

Corollary 2.13. *Suppose $f : \Omega \to \mathbb{X}$ is μ-measurable and (f_n) is a sequence of Bochner integrable functions that converge to f μ-a.e. such that $\lim_{n\to\infty}\int \|f_n - f\|\ d\mu = 0$. Then $\int f\ d\mu = \lim_{n\to\infty}\int f_n\ d\mu$.*

2.3 Bochner's Characterization

In this section, we will give an essential characterization of Bochner integrable functions. The result is attributed to Bochner.

Theorem 2.14 (Bochner's Characterization)**.** *Let $f : \Omega \to \mathbb{X}$ be a μ-measurable function, then f is Bochner integrable if and only if $\|f\|$ is Lebesgue integrable.*

Proof. $\Rightarrow$ Suppose f is Bochner integrable and (f_n) is a sequence of μ-simple functions such that $\lim_{n\to\infty}\int \|f_n - f\|\ d\mu = 0$. Then for any $\epsilon > 0$, there exists an $N > 0$ such that for all $n, m \geq N$,

$$\|f_n - f_m\|_{L^1(\mathbb{X})} = \int \|f_n - f_m\|\ d\mu \leq \int \|f_n - f\|\ d\mu + \int \|f - f_m\|\ d\mu < \epsilon.$$

In other words, (f_n) is Cauchy in $L^1(\mathbb{X})$, which is a metric space. Therefore, (f_n) is bounded in $L^1(\mathbb{X})$. Furthermore, since for any $\epsilon > 0$ there exists an $N > 0$ such that for all $n \geq N$, $\epsilon > \left|\int \|f_n - f\|\ d\mu\right| \geq \left|\int \|f_n\|\ d\mu - \int \|f\|\ d\mu\right|$, we know that $\lim_{n\to\infty}\|f_n\|_{L^1(\mathbb{X})} = \|f\|_{L^1(\mathbb{X})}$.

Therefore, since (f_n) is bounded in $L^1(\mathbb{X})$ and converges to f in $L^1(\mathbb{X})$, $\|f\|_{L^1} < \infty$, i.e. $\|f\| \in L^1(\mathbb{X})$.

$\Leftarrow$ Suppose $\displaystyle\int \|f\| \; d\mu < \infty$.

Let (g_n) be a sequence of μ-simple functions that converge to f μ-a.e.. (Then, by the triangle inequality, $\|g_n\| \to \|f\|$ μ-a.e..) Define $f_n := 1\!\!1_{\|g_n\| \leq 2\|f\|} g_n$.[4] Then (f_n) is a sequence of μ-simple functions that converge to f μ-a.e.. Furthermore, since $\|f_n\| \leq 2\|f\|$ for all n and $\displaystyle\int 2\|f\| \; d\mu < \infty$, by the Dominated Convergence Theorem, $\displaystyle\lim_{n\to\infty} \int \|f_n - f\| \; d\mu = 0$. □

A natural question is whether or not Bochner's characterization will still hold if (Ω, Σ, μ) is a σ-finite measure space. This is where our requirement that $\mu(E_i) < \infty$ (where the E_i are the measurable sets used to define the μ-simple functions in Definition 2.1) comes in handy. In fact, the same proof holds for σ-finite spaces. [5]

With Bochner's characterization, we will be able to use theorems already available to Lebesgue integration to prove corresponding theorems for Bochner integration.

Proposition 2.15. *Suppose f, g Bochner integrable and $\alpha \in \mathbb{R}$, then*

1. $\displaystyle\int \alpha f \; d\mu = \alpha \int d \; d\mu$ *and*
2. $\displaystyle\int f + g \; d\mu = \int f \; d\mu + \int g \; d\mu$

Proof. Let (f_n) and (g_n) be sequences of μ-simple functions that converge μ-a.e. to f and g respectively such that $\displaystyle\int f \; d\mu = \lim_{n\to\infty} \int f_n \; d\mu$ and $\displaystyle\int g \; d\mu = \lim_{n\to\infty} \int g_n \; d\mu$.

1. By Bochner's characterization, αf is Bochner integrable. Then, since (αf_n) is a sequence of μ-simple functions that converges μ-a.e. to αf and $\displaystyle\lim_{n\to\infty} \int \|\alpha f_n - \alpha f\| \; d\mu = 0$, we have that

[4] The choice of the f_n comes from [13].

[5] Without the stipulation that $\mu(E_i) < \infty$ for all i in the definition of μ-simple functions, a simple counterexample would be a $\mathbb{R}$-valued simple function $f = 1\!\!1_{[0,\infty)}$ where $\displaystyle\int f \; d\mu = \infty$.

$$\int \alpha f \, d\mu = \lim_{n\to\infty} \int \alpha f_n \, d\mu = \alpha \lim_{n\to\infty} \int f_n \, d\mu = \alpha \int f \, d\mu.$$

2. Since f and g are Bochner integrable, $\int \|f+g\| \, d\mu \leq \int \|f\| \, d\mu + \int \|g\| \, d\mu < \infty$, and therefore, $f+g$ is Bochner integrable by Bochner's characterization. Also, since $f_n \to f$ and $g_n \to g$ μ-a.e., $f_n + g_n \to f + g$ μ-a.e.

   Similarly, $\lim_{n\to\infty} \int \|(f_n + g_n) - (f+g)\| \, d\mu = 0$; thus $\int f + g \, d\mu = \lim_{n\to\infty} \int f_n + g_n \, d\mu$.

   Hence, it will suffice to show that $\int f \, d\mu + \int g \, d\mu = \lim_{n\to\infty} \int f_n + g_n \, d\mu$.

   Let $\epsilon > 0$. Then there exists an $N \geq 0$ such that for all $n \geq N$,

$$\begin{aligned} \left\| \int f_n + g_n \, d\mu - \left(\int f \, d\mu + \int g \, d\mu \right) \right\| &= \left\| \int f_n \, d\mu + \int g_n \, d\mu - \int f \, d\mu - \int g \, d\mu \right\| \\ &\leq \left\| \int f_n \, d\mu - \int f \, d\mu \right\| + \left\| \int g_n \, d\mu - \int g \, d\mu \right\| < \epsilon. \end{aligned}$$

□

Example 2.16. Consider the Banach space $C[0,1]$ (i.e. the space of all $\mathbb{R}$-valued continuous functions on the compact set $[0,1]$) and the measure space $(\mathbb{N}, 2^{\mathbb{N}}, c)$ where c is the counting measure, and we will construct not just one, but a sequence of simple functions $B_N : (\mathbb{N}, 2^{\mathbb{N}}, c) \to C[0,1]$. [6]

[6] Yes, $(\mathbb{N}, 2^{\mathbb{N}}, c)$ is not only not a probability space, but is not even finite. However, since Bochner's characterization holds in σ-finite spaces, we will avail ourselves of this fact in order to provide a particularly interesting example.

First, let T be the tent map on $[0,1]$ with height 1.

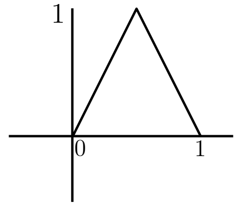


Figure 2.1: $T : [0,1] \to \mathbb{R}$

Next, extend T periodically to $\mathbb{R}$ (with period 1), and call this new map $b(x)$.

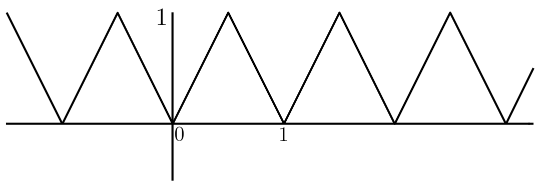


Figure 2.2: $b : \mathbb{R} \to \mathbb{R}$

Then, for all $n \in \mathbb{N}$, define $b_n(x) = \left.\dfrac{b(2^n x)}{2^n}\right|_{[0,1]}$

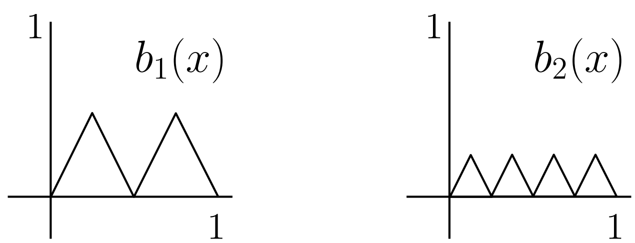


Figure 2.3: $b_1 : [0,1] \to \mathbb{R}$ and $b_2 : [0,1] \to \mathbb{R}$

Then $b_n \in C[0,1]$ and $\|b_n\| = \sup_{x \in [0,1]} b_n(x) = \dfrac{1}{2^n}$ for all n.

Define a sequence of simple functions $F_m : (\mathbb{N}, 2^{\mathbb{N}}, c) \to C[0,1]$ by $F_m = \displaystyle\sum_{n=1}^{m} b_n \mathbb{1}_{\{n\}}$. Then $\displaystyle\int \|F_m\| \, dc = \sum_{n=1}^{m} 2^{-n} < \infty$. Let $F : (\mathbb{N}, 2^{\mathbb{N}}, c) \to C[0,1]$ be given by $F = \displaystyle\sum_{n} b_n \mathbb{1}_{\{n\}}$. Then $F_m \to F$ a.e., and $\|F\| = \displaystyle\sum_{n} \|b_n\| \mathbb{1}_{\{n\}} = \sum_{n} \frac{1}{2^n} \mathbb{1}_{\{n\}}$. Hence,

$$\int_{\mathbb{N}} \|F\| \, dc = \int_{\mathbb{N}} \sum_n \frac{1}{2^n} \mathbb{1}_{\{n\}} \, dc = \sum_n \frac{1}{2^n} c(\{n\}) = \sum_n \frac{1}{2^n} < \infty$$

Therefore, F is Bochner integrable, and furthermore,

$$\int_{\mathbb{N}} F \, dc = \lim_{k\to\infty} \int_{\mathbb{N}} \sum_{n=1}^{k} b_n \mathbb{1}_{\{n\}} \, dc = \lim_{k\to\infty} \sum_{n=1}^{k} b_n c(\{n\}) = \lim_{k\to\infty} \sum_{n=1}^{k} b_n = \sum_{n=1}^{\infty} b_n$$

In fact, $\int_{\mathbb{N}} F \, dc$ is a continuous nowhere differentiable function called a Bolzano function.

2.4 Pettis Integral

Before we continue, we will digress briefly to at least mention the more general Pettis integral. First, we need to define weak μ-measurability.

Definition 2.17. A function a $f : \Omega \to \mathbb{X}$ is **weakly (or scalarly) μ-measurable** if $x^* f$ is μ-measurable for each $x^* \in \mathbb{X}^*$.

Proposition 2.18. *A (strongly) μ-measurable function is weakly μ-measurable.*

Proof. Suppose f is strongly μ-measurable, and (f_n) is a sequence of μ-simple functions such that $\lim_{n\to\infty} \|f_n - f\| = 0$, μ-a.e. Let $x^* \in \mathbb{X}^*$. Then for each f_n,

$$x^*(f_n) = x^* \left(\sum_{i=1}^{k_n} x_{i_n} \mathbb{1}_{E_{i_n}} \right) = \sum_{i=1}^{k_n} x^*(x_{i_n}) \mathbb{1}_{E_{i_n}}$$

is μ-simple. Furthermore,

$$\lim_{n\to\infty} \|x^*(f_n) - x^*(f)\| = \lim_{n\to\infty} \|x^*(f_n - f)\| \leq \lim_{n\to\infty} \|x^*\|_* \|f_n - f\| = 0$$

□

Bochner integral theory does not apply directly to functions that are weakly μ-measurable or whose norm is not Lebesgue integrable. Herein lies one of the merits of the Pettis integral. But first, a lemma.

Lemma 2.19 (Dunford)**.** *[2] Suppose $f : \Omega \to \mathbb{X}$ is a weakly μ-measurable function and $x^* f \in L^1(\mu)$ for each $x^* \in \mathbb{X}^*$. Then for each $E \in \Sigma$, there exsits $\varphi_E \in \mathbb{X}^{**}$ such that, for all $x^* \in \mathbb{X}^*$,*

$$\varphi_E(x^*) = \int_E x^*(f)\ d\mu$$

Definition 2.20. [2] If f is a weakly μ-measurable $\mathbb{X}$-valued function on Ω such that $x^*f \in L^1(\mu)$ for all $x^* \in \mathbb{X}^*$, then f is called **Dunford integrable**. The **Dunford integral** of f is the functional $\varphi \in \mathbb{X}^{**}$ such that, for all $x^* \in \mathbb{X}^*$,

$$\varphi(x^*) = \int x^*(f)\ d\mu.$$

Notice that $x^*(f)$ takes values in the underlying scalar field. Hence, the integral, $\int x^*(f)\ d\mu$, is the Lebesgue integral, and so, for any $E \in \Sigma$, we can define

$$\int_E x^*(f)\ d\mu := \int x^*(f)|_E\ d\mu = \int x^*(f|_E)\ d\mu.$$

For each $E \in \Sigma$, we call the functional $\varphi_E \in \mathbb{X}^{**}$ given by $\varphi_E(x^*) = \int_E x^*(f)\ d\mu$ for all $x^* \in \mathbb{X}^*$ the **Dunford integral of f over E.**

If $\varphi_E \in \mathbb{X}$[7] for each $E \in \Sigma$, then f is called **Pettis integrable**, and φ_E is the **Pettis integral of f over E**. If $\mathbb{X}$ is a reflexive, then Dunford integrability is Pettis integrability.

Now, consider a function $f : \Omega \to \mathbb{X}$ that is μ-measurable (and hence weakly μ-measurable) and Bochner integrable. By Bochner's characterization, that means $\int \|f\|\ d\mu < \infty$. Let $x^* \in \mathbb{X}^*$. Then,

$$\int |x^*(f)|\ d\mu \leq \|x^*\| \int \|f\|\ d\mu < \infty.$$

Hence, Bochner integrability implies Dunford integrability.

Example 2.21. Let $\mathbb{X}$ be a Banach space, and $(x_n) \in \mathbb{X}^{\mathbb{N}}$ be summable. Define $f : \mathbb{N} \to \mathbb{X}$ by $f(n) = x_n$ for each $n \in \mathbb{N}$. Then, f is μ measurable (and hence μ-weakly measurable).

To see that f is Dunford integrable, let $x^* \in \mathbb{X}^*$. Then,

$$\begin{aligned}\int |x^*(f)|\ dc &= \int \left| x^*\left(\sum_{n\in\mathbb{N}} x_n \mathbb{1}_{\{n\}}\right)\right|\ dc = \int \left|\sum_{n\in\mathbb{N}} x^*(x_n)\mathbb{1}_{\{n\}}\right|\ dc \\ &\leq \int \sum_{n\in\mathbb{N}} |x^*(x_n)|\mathbb{1}_{\{n\}}\ dc = \sum_{n\in\mathbb{N}} |x^*(x_n)|.\end{aligned}$$

[7] Of course, by $\phi_E \in \mathbb{X}$, we mean $\phi_E \in J(\mathbb{X}) \subseteq X^{**}$ where $J : \mathbb{X} \to \mathbb{X}^{**}$ is the natural embedding where for each $x \in \mathbb{X}$ and $x^* \in \mathbb{X}^*$,$Jx(x^*) = x^*(x)$.

Since x^* is continuous, $(x^*(x_n))$ is summable in $\mathbb{R}$, which means it is also absolutely convergent. Hence $\sum_{n\in\mathbb{N}} |x^*(x_n)| < \infty$, and f is Dunford integrable.

The Dunford integral for each $E \subseteq \mathbb{N}$ is the functional $\phi_E = \mathbb{X}^{**}$ given by

$$\phi_E(x^*) = \int_E x^* \left(\sum_{n\in\mathbb{N}} x_n \mathbb{1}_{\{n\}} \right) \, dc = \int \sum_{n\in\mathbb{N}} x^*(x_n) \mathbb{1}_{\{n\}} \, dc = \sum_{n\in E} x^*(x_n)$$

for all $x^* \in \mathbb{X}^{**}$. In bracket notation, $\phi_E = \sum_{n\in E} \langle \cdot, x_n \rangle = \langle \cdot, \sum_{n\in E} x_n \rangle$. In other words, f is Pettis integrable.

As for Bochner integrability, by Bochner's characterization, $f \in L^1(\mathbb{X}, c)$ iff $\|f\| \in L^1(c)$ iff $\int \|f\| \, dc = \sum_{n\in\mathbb{N}} \|x_n\| < \infty$. However, from Orlicz's Theorem (in Chapter 1), we know that summability and absolute convergence are the same iff $\mathbb{X}$ is a finite-dimensional Euclidean space. [8] Hence, if $\mathbb{X}$ is finite-dimensional, then f is Dunford integrable iff f is Bochner integrable, and if $\mathbb{X}$ is infinite-dimensional, then we have only that Bochner integrability of f implies Dunford integrability of f.

However, even in this discrete space there are Dunford integrable functions that are not Pettis integrable.

Example 2.22. Consider $\mathbb{X} = c_0$ and $f : \mathbb{N} \to c_0$ given by $f(n) = e_n$ where (e_n) is the standard basis for c_0. Let $x^* \in c_0^*$, $x = (x_n) \in \ell^1$ such that $x^* = \langle x, \cdot \rangle$, and $E \in 2^{\mathbb{N}}$. Then,

$$\int_E |x^*(f)| \, dc = \sum_{n\in E} |x_n| < \infty.$$

However, for any $x^* \in c_0^*$ with $x = (x_n) \in \ell^1$ such that $x^* = \langle x, \cdot \rangle$ and $E \in 2^{\mathbb{N}}$,

$$\int_E x^*(f) \, dc = \sum_{n\in E} x_n = \langle x, (1)_{n=1}^{\infty} \rangle.$$

But, $(1)_{n=1}^{\infty} \notin c_0$.

This example extends readily to one defined on a measurable space that is not discrete.

[8] The "only if" portion comes from [5].

Example 2.23. Consider c_0 with standard basis (e_n), and let our measure space be $([0,1], \mathcal{B}, \lambda)$[9] Define $f_n : [0,1] \to c_0$ by $f_n := n e_n \mathbb{1}_{(0,\frac{1}{n}]}$ for all $n \in \mathbb{N}$ where (e_n) is the standard basis for c_0, and define $f : [0,1] \to c_0$ by $f := \sum_{n \in \mathbb{N}} f_n$. To see that f is Dunford integrable, let $x^* \in c_0^*$ and $x = (x_n) \in \ell^1$ such that $x^* = \langle x, \cdot \rangle$. Then,

$$\begin{aligned}\int |x^*(f)| \, d\lambda &= \int |x^* \sum_{n \in \mathbb{N}} n e_n \mathbb{1}_{(0,\frac{1}{n}]}| \, d\lambda \leq \int \sum_{n \in \mathbb{N}} n |x^*(e_n)| \mathbb{1}_{(0,\frac{1}{n}]} \, d\lambda \\ &= \sum_{n \in \mathbb{N}} n |x_n| \frac{1}{n} = \sum_{n \in \mathbb{N}} |x_n| < \infty\end{aligned}$$

However, $\varphi \notin c_0$. To see this, let $x^* \in c_0^*$ and $x = (x_n) \in \ell^1$ such that $x^* = \langle x, \cdot \rangle$. Then,

$$\begin{aligned}\int x^*(f) \, d\lambda &= \int x^* \sum_{n \in \mathbb{N}} n e_n \mathbb{1}_{(0,\frac{1}{n}]} \, d\lambda = \int \sum_{n \in \mathbb{N}} n (x^*(e_n)) \mathbb{1}_{(0,\frac{1}{n}]} \, d\lambda \\ &= \sum_{n \in \mathbb{N}} n (x_n) \frac{1}{n} = \sum_{n \in \mathbb{N}} x_n\end{aligned}$$

However, $\sum_{n \in \mathbb{N}} x_n = \langle x, (1)_{n=1}^{\infty} \rangle$ where $(1)_{n=1}^{\infty} \notin c_0$.

2.5 Extending $\mathbb{R}$ results

Several important results still hold even after we leave $\mathbb{R}$ for more general spaces, aside from the ones already mentioned. In this section we will discuss a few that do extend and why others do not. First, we will note that not all Banach spaces have a partial ordering. Hence many theorems requiring or guaranteeing an inequality will no longer hold in all Banach spaces, e.g. Monotonicity, Fatou's Lemma, and the Monotone Convergence Theorem. Theorems with non-negative assumptions also may not extend. However, we do have an analogue to Lebesgue's Dominated Convergence Theorem.

Theorem 2.24 (Dominated Convergence Theorem). *[2] Let $f_n : \Omega \to \mathbb{X}$ be a sequence of Bochner integrable functions, $g : \Omega \to \mathbb{R}$, $g \in L^1(\mu)$, $f : \Omega \to \mathbb{X}$ is a μ-measurable function*

[9] where λ is the Lebesgue measure

such that (f_n) *converges to* f *in measure, and* $\|f_n\| \leq |g|$ μ*-a.e. for all* n*. Then* f *is Bochner integrable and* $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ *for all* $E \in \Sigma$*, and* $\lim_{n\to\infty} \int_\Omega \|f - f_n\| \, d\mu = 0$*.*

Notice that, according to proceeding prepositions, all that we need to show is that $\lim_{n\to\infty} \int \|f_n - f\| \, d\mu = 0$. Also, note that a sequence of functions converges in (finite) measure iff every subsequence contains an almost everywhere convergent sub-subsequence (all of which converge to the same limit). Since we are working in a metric space, this subsequence clause is just μ-a.e. convergence. Therefore, since we are working with functions from a finite measure space to a metric space, we can just assume that $(f_n) \to f$ μ-a.e.. (If we were working in a σ-finite space, a slightly weaker version of the DCT would hold, i.e. where $(f_n) \to f$ μ-a.e..) [10]

Proof. Since (f_n) converge to f μ-a.e., $\|f_n\|$ converges to $\|f\|$ μ-a.e., and $\lim_{n\to\infty} \|f_n - f\| = 0$. Furthermore, since $\|f_n\| \leq g$ for all n, $\|f\| \leq g$. Hence, $\|f_n - f\| \leq \|f_n\| + \|f\| \leq |2g|$ μ-a.e.. Then, by the classic Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int \|f_n - f\| \, d\mu = \int \lim_{n\to\infty} \|f_n - f\| \, d\mu = \int 0 \, d\mu = 0.$$

□

We also get a generalized case of Egoroff's Theorem:

Theorem 2.25. *For a* σ*-finite measure space* $(S, \mathcal{S}, \nu)$*, if* $E \in \mathcal{S}$*,* $\nu(E) < \infty$*, and* (f_n) *a sequence of measurable functions on* E*, each of which is finite a.e. in* E*, that converges a.e. to a finite* ν*-measurable* f*, then for all* $\epsilon > 0$*, there is an* $A_\epsilon \subseteq E$ *such that* $\nu(E - A_\epsilon) < \epsilon$ *and* (f_n) *converges to* f *uniformly on* A_ϵ*.*

The proof for this theorem is simply the standard proof for Egoroff's Theorem with a general norm in place of absolute value.

[10] Notice that Bochner's characterization is not required for the Dominated Convergence Theorem. In fact, given the relationship between Bochner and Lebesgue integration and the Dominated Convergence Theorem for Lebesgue integration, the Dominated Convergence Theorem for Bochner integrals is more of a corollary than a theorem.

Theorem 2.26. *[2] If f is Bochner integrable with respect to μ, then*

1. $\displaystyle\lim_{\mu(E)\to 0} \int_E f\ d\mu = 0,$

2. *(Countable Additivity) If (E_n) is a sequence of pairwise disjoint members of Σ and $E = \displaystyle\bigcup_{n=1}^{\infty} E_n$, then*

$$\int_E f\ d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\ d\mu,$$

where the sum is absolutely convergent, and

3. *If $F(E) = \displaystyle\int_E f\ d\mu$, then F is of bounded variation and $|F|(E) = \displaystyle\int_E \|f\|\ d\mu$ for all $E \in \Sigma$.*

Proof. 1. Since $\displaystyle\lim_{\mu(E)\to 0} \int_E \|f\|\ d\mu = 0$ for $f \in L^1(\mu)$, and f is Bochner integrable iff $\|f\| \in L^1(\mu)$,

$$0 = \lim_{\mu(E)\to 0} \int_E \|f\|\ d\mu \geq \lim_{\mu(E)\to 0} \left\| \int_E f\ d\mu \right\| = \left\| \lim_{\mu(E)\to 0} \int_E f\ d\mu \right\|.$$

2. Note first that $\displaystyle\sum_{n=1}^{\infty} \int_{E_n} f\ d\mu$ is dominated term by term by the nonnegative series $\displaystyle\sum_{n=1}^{\infty} \int_{E_n} \|f\|\ d\mu$ and that, since $\displaystyle\sum_{n=1}^{\infty} \int_{E_n} \|f\|\ d\mu \leq \int_{\Omega} \|f\|\ d\mu < \infty$, the series converges. Thus $\displaystyle\sum_{n=1}^{\infty} \int_{E_n} f\ d\mu$ is absolutely convergent.

Note also that (by the finiteness of Ω), $\displaystyle\lim_{m\to\infty} \mu\left(\bigcup_{n=m+1}^{\infty} E_n \right) = 0$, and thus by the first property, $\displaystyle\lim_{m\to\infty} \left\| \int_{\bigcup_{n=m+1}^{\infty} E_n} f\ d\mu \right\| = 0$. Then,

$$\left\| \int_E f\ d\mu - \sum_{n=1}^{m} \int_{E_n} f\ d\mu \right\| = \left\| \int_{\bigcup_{n=m+1}^{\infty} E_n} f\ d\mu \right\| = 0.$$

Hence, $\displaystyle\int_E f\ d\mu = \lim_{m\to\infty} \sum_{n=1}^{m} \int_{E_n} f\ d\mu.$

3. Let $E \in \Sigma$

$\leq$: Let π be a partition of E. Then

$$\sum_{A\in\pi} \|F(A)\| = \sum_{A\in\pi} \left\| \int_A f \; d\mu \right\| \leq \sum_{A\in\pi} \int_A \|f\| \; d\mu = \int_E \|f\| \; d\mu.$$

Therefore, $|F|(E) \leq \displaystyle\int_E \|f\| \; d\mu$.

Furthermore, since f is Bochner integrable, $|F|(E) \leq \displaystyle\int_E \|f\| \; d\mu < \infty$, so F is of bounded variation.

$\geq$: [2] Let $\epsilon > 0$ and choose $\{f_n\}$ simple such that $\displaystyle\lim_{n\to\infty} \int \|f - f_n\| \; d\mu = 0$. Choose m sufficiently large such that $\displaystyle\int \|f - f_m\| \; d\mu < \frac{\epsilon}{2}$ and a partition π of E such that $\displaystyle\sum_{A\in\pi} \left\| \int_A f_m \; d\mu \right\| = \int_E \|f_m\| \; d\mu$.[11] Choose a refinement π' of π such that $\displaystyle\left| |F|(E) - \sum_{B\in\pi'} \left\| \int_B f \; d\mu \right\| \right| < \frac{\epsilon}{2}$.[12] Then, we still have $\displaystyle\int_E \|f_m\| \; d\mu = \sum_{B\in\pi'} \left\| \int_B f_m \; d\mu \right\|$.
Furthermore,

$$\begin{aligned}
\left| \sum_{B\in\pi'} \left\| \int_B f \; d\mu \right\| - \left\| \int_B f_m \; d\mu \right\| \right| &\leq \sum_{B\in\pi'} \left| \left\| \int_B f \; d\mu \right\| - \left\| \int_B f_m \; d\mu \right\| \right| \\
&\leq \sum_{B\in\pi'} \left\| \int_B f \; d\mu - \int_B f_m \; d\mu \right\| \\
&\leq \sum_{B\in\pi'} \int_B \|f - f_m\| \; d\mu \\
&= \int_E \|f - f_m\| \; d\mu < \frac{\epsilon}{2}.
\end{aligned}$$

Therefore,

$$\left| |F|(E) - \int_E \|f_m\| \; d\mu \right| = \left| |F|(E) - \sum_{B\in\pi'} \left\| \int_B f_m \; d\mu \right\| \right| < \epsilon$$

[11] We can choose π such that $A = A_i$ for $\displaystyle\sum_{i=1}^{n} \mathbb{I}_{A_i} x_i = f_m$ on E.
Then, $\displaystyle\sum_{A\in\pi} \left\| \int_A f_m \; d\mu \right\| = \sum_{A\in\pi} \|\mu(A) x_A\| = \sum_{A\in\pi} \mu(A) \|x_A\| = \sum_{A\in\pi} \int_A \|f_m\| \; d\mu = \int_E \|f_m\| \; d\mu$.

[12] We can do so by the the definition of $|F|(E)$.

Since this holds for all sufficiently large m, $|F|(E) = \lim_{n\to\infty} \int_E \|f_n\|\, d\mu \geq \int_E \|f\|\, d\mu$.[13]

$\square$

Corollary 2.27. *[2] If f and g are Bochner integrable and $\int_E f\ d\mu = \int_E g\ d\mu$ for each $E \in \Sigma$, then $f = g$ a.e. with respect to μ.*

Proof. Let $F(E) = \int_E f - g\ d\mu = 0$. Then by (3), $0 = |F|(E) = \int_E \|f - g\|\ d\mu$. Hence $\|f - g\| = 0$, which means, $f = g$ a.e.$-\mu$. $\square$

However, the Bochner integral does have results with no nontrivial Lebesgue analogue, such as the following property:

Theorem 2.28. *[2] Let T be a bounded linear operator [14] If f is Bochner integrable with respect to μ, then so is Tf and for all $E \in \Sigma$*

$$T\left(\int_E f\ d\mu\right) = \int_E Tf\ d\mu.$$

Proof. Since T is bounded, $\int_E \|Tf\|\ d\mu \leq \int_E M\|f\|\ d\mu < \infty$. Hence, Tf is Bochner integrable. Also, the claim holds for f μ-simple by linearity.

Let (f_n) be a sequence of μ-simple functions such that $\lim_{n\to\infty} \int_E f_n\ d\mu = \int_E f\ d\mu$. Then by the continuity of T,

$$T\left(\int_E f\ d\mu\right) = T\left(\lim_{n\to\infty} \int_E f_n\ d\mu\right) = \lim_{n\to\infty} T\left(\int_E f_n\ d\mu\right) = \lim_{n\to\infty} \int_E Tf_n\, d\mu = \lim_{n\to\infty} \int_E Tf\, d\mu$$

$\square$

Finally, before we move forward, we should note a new class of spaces at our disposal:

Definition 2.29. For $1 \leq p < \infty$, define $L^p(\mu, \mathbb{X})$ to be the vector space of all (equivalence classes) of μ-measurable functions $f : \Omega \to \mathbb{X}$ for which, $\int \|f\|^p\ d\mu < \infty$.

[13] by Fatou's Lemma

[14] The theorem holds for closed linear operators as well, but we shall only require the result for bounded linear operators. The more general theorem is attributed to Hille.

Much like the L^p spaces defined with the Lebesgue integral, $L^p(\mu, \mathbb{X})$, with the norm $\|\cdot\|_{L^p(\mu,\mathbb{X})}$ given by $\|f\|_{L^p(\mu,\mathbb{X})} = \left(\int \|f\|^p \, d\mu\right)^{1/p}$ for all $f \in L^p(\mu, \mathbb{X})$, is also a Banach space. In particular, $L^1(\mu, \mathbb{X})$ is the space of all Bochner integrable functions from Ω to $\mathbb{X}$.

Definition 2.30. Define $L^\infty(\mu, \mathbb{X})$ to be the vector space of all (equivalence classes) essentially bounded μ-measurable functions $f : \Omega \to \mathbb{X}$

Again, $L^\infty(\mu, \mathbb{X})$, with the norm $\|\cdot\|_{L^\infty(\mu,\mathbb{X})}$ given by

$$\|f\|_{L^\infty(\mu,\mathbb{X})} = \inf\{M \in \mathbb{R}_+ : \|f\| \leq M \text{ a.e..}\},$$

is also a Banach space.

2.6 The Radon Nikodým Property

In our extended class of Lebesgue integrable functions, several results (such as the Dominated Convergence Theorem) that held for the class of real-valued Lebesgue integrable functions hold trivially (after a little reformulation) for the extended class. However, we shall see that the Radon Nikodým Theorem requires alteration before it holds in this larger class of functions. Our focus in this section will be not in altering the theorem but in understanding the significance of its failure to hold for measures taking values in certain Banach spaces.

First, let us put the classic Radon Nikodým Theorem in the context of Banach spaces and Bochner integrals:

Theorem 2.31 (Radon-Nikodým Theorem)**.** *Assume (Ω, Σ, μ) is a finite measure space*[15] *and $\mathbb{X}$ a Banach space. If $F : \Sigma \to \mathbb{X}$ is a μ-continuous vector measure of bounded variation, then there exists a Bochner integrable $g \in L^1(\mu, \mathbb{X})$ such that $F(E) = \int_E g \, d\mu$ for all $E \in \Sigma$. We call g the Radon-Nikodým Derivative (RND) of F.*

In the general context of Banach spaces and Bochner integrals, this theorem no longer holds. The following is a classic incidence of the failure of the Radon-Nikodým Theorem from Diestel and Uhl:

[15] That is, a finite, countably additive set function on Σ that takes values in $[0, \infty)$.

Proposition 2.32. *[2] There exists a countably additive c_o-valued vector measure of bounded variation with no Radon-Nikodým Derivative.*

Proof. Let λ be the Lebesgue measure on $([0,1],\mathcal{B})$. For $E \in \mathcal{B}$, write $\phi_n(E) = \displaystyle\int_E \sin(2^n\pi t)\, dt$ and let $F(E) = (\phi_n(E))_{n=1}^{\infty}$.[16] Then $F : \mathcal{B} \to c_0$ by the Riemann-Lebesgue Lemma[17], and for all $E \in \mathcal{B}$,

$$\|F(E)\|_{c_0} \le \sup_n \int_E |\sin(2^n\pi t)|\, dt \le \int_E 1\, dt = \lambda(E).$$

Then, F is countably additive, λ-continuous, and of bounded variation.

1. To see that F is countably additive, let $\{E_k\}$ be a countable disjoint collection of elements of $\mathcal{B}$, and let $E := \bigcup_k E_k$. Then,

   $$\|F(E)\|_{c_0} = \|(\phi_n(\textstyle\bigcup_k E_k))_n\|_{c_0} = \|(\textstyle\sum_k \phi_n(E_k))_n\|_{c_0} \le 1.$$

   Hence, $|\sum_k \phi_n(E_k)| \le 1$ for all $n \in \mathbb{N}$. If we let $(e_n)_{n=1}^{\infty}$ be the standard basis of c_0, then we have

   $$F(E) = (\sum_k \phi_n(E_k))_n = \sum_n \left(\sum_k \phi_n(E_k)\right) e_n = \sum_k \sum_n \phi_n(E_k) e_n = \sum_k F(E_k)$$

2. To see that F is λ-continuous, first note that, since F and λ are countably additive and $\mathcal{B}$ is a σ-algebra, it will suffice to show that $\lambda(E) = 0 \Rightarrow F(E) = 0$ for all $E \in \mathcal{B}$, which follows immediately from the inequality.

3. To see that F is of bounded variation, let $\{E_k\}_{k=1}^{n}$ be a pairwise disjoint collection in $\mathcal{B}$. Then, $\displaystyle\sum_{k=1}^{n} \|F(E_k)\|_{c_0} \le \sum_{k=1}^{n} \lambda(E_k) \le 1$. Hence, $|F|([0,1]) \le 1 < \infty$.

Now, suppose that F does have a RND, i.e. a Bochner integrable function $f : [0,1] \to c_0$ such that $F(E) = \displaystyle\int_E f\, d\lambda$ for all $E \in \mathcal{B}$. Then, $f = (f_n)_{n=1}^{\infty}$ where, for each n, f_n is the

[16] Note that $\sin(2^n\pi t)$ is merely the sine function with period $\frac{1}{2^{n-1}}$.

[17] Specifically, $\displaystyle\lim_{k\to\infty} \int_0^1 \sin(kt)\, dt = 0$, since $E \subseteq [0,1]$ and $2^n\pi \to \infty$, $\displaystyle\lim_{n\to\infty} \phi_n(E) = 0$

nth coordinate functional on c_0 for (e_n). Hence, each f_n is bounded (continuous) and thus measurable.

By Corollary 2.25, $|F|(E) = \int_E \|f\| \, d\lambda$ for all $E \in \mathcal{B}$. Hence,

$1 \geq \int_E \|f\| \, d\lambda = \int_E \sup_n |f_n| \, d\lambda$ for all $E \in \mathcal{B}$. It follows that $\sum_{n=1}^{m} f_n e_n$ is Bochner integrable for each $m \in \mathbb{N}$.[18] Therefore, by the general Dominated Convergence Theorem, for all $E \in \mathcal{B}$

$$F(E) = \int_E f \, d\lambda = \lim_{n\to\infty} \int_E \sum_{n=1}^{m} f_n e_n \, d\lambda = \lim_{n\to\infty} \sum_{n=1}^{m} \left(\int_E f_n \, d\lambda \right) e_n = \left(\int_E f_n \, d\lambda \right)_{n=1}^{\infty}$$

Therefore, by Corollary 2.26, $f_n(t) = \sin(2^n \pi t)$ for almost all $t \in [0,1]$.

Now, let $E_n = \{t \in [0,1] : f_n(t) \geq \frac{1}{\sqrt{2}}\}$. Then $\lambda(E_n) = \frac{1}{4}$ for all n,[19] and for all $t \in \limsup_n E_n$, $f(t) \notin c_0$ since $f_n(t) \geq \frac{1}{\sqrt{2}}$ for all n. Also,

$$\lambda(\limsup_n(E_n)) = \lambda\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \right) \geq \inf_{k\geq 1} \left(\lambda\left(\bigcup_{n=k}^{\infty} E_n \right) \right) \geq \inf_{k\geq 1} (\sup_{n\geq k} \lambda(E_n)) \geq \frac{1}{4}$$

Therefore, $\lambda\{t \in [0,1] : f(t) \in c_0\} \leq \frac{3}{4} < 1$, contradicting $f : [0,1] \to c_0$. □

Because this theorem may or may not hold in a given Banach space, we can think the success or failure of the theorem to hold in the space as a property of the space itself, that is, we have the following property:

Definition 2.33. A Banach space $\mathbb{X}$ has the **Radon Nikodým Property (RNP)** with respect to (Ω, Σ, μ) if for each μ-continuous vector measure, $F : \Sigma \to \mathbb{X}$ of bounded variation, there exisits a Bochner integrable $f : \Omega \to \mathbb{X}$ such that $F(E) = \int_E f \, d\mu$ for all $E \in \Sigma$.

Definition 2.34. A Banach space $\mathbb{X}$ has the **Radon Nikodým Property** if it has the Radon Nikodým Property with respect to every finite measure space. [20]

[18] Using Bochner's Characterization, $\int \left\| \sum_{n=1}^{m} f_n e_n \right\| \, d\lambda \leq \int \sum_{n=1}^{m} |f_n| \, d\lambda < \infty$

[19] Again, $f_n(t)$ is merely the sine function with period $\frac{1}{2^{n-1}}$. Since $\sin(t) > \frac{1}{\sqrt{2}}$ for $\frac{1}{4}$ of each period, it follows that $\lambda(E_n) = \frac{1}{4}$.

[20] In fact, it is enough that $\mathbb{X}$ has the RNP with respect to $([0,1], \mathcal{B}, \lambda)$.

Hence, from proposition 2.32, c_0 does not have the Radon Nikodým Property.

2.6.1 Significant Theorems

As we have noted, the RNP gives significant insight into analytic properties and the geometry of a Banach space. To illustrate this, we shall briefly enumerate several significant results pertaining to the RNP. The following theorems and proofs can be found in Diestel and Uhl's *Vector Measures*, [2].

Theorem 2.35 (Dunford-Pettis)**.** *Separable dual spaces have the Radon-Nikodým Property.*

Theorem 2.36 (Phillips)**.** *Reflexive Banach spaces have the Radon-Nikodým Property.*

Theorem 2.37 (Uhl)**.** *If every separable closed linear subspace of X is isomorphic to a subspace of a separable dual space, then X has the Radon-Nikodým Property.*

Theorem 2.38 (Von Neumann)**.** *Hilbert spaces have the Radon-Nikodým Property.*

Theorem 2.39. *Let (Ω, Σ, μ) be a finite measure space and X a Banach space such that X and X^* have the RNP. A subset K of $L^1(\mu, X)$ is relatively weakly compact if*

1. *K is bounded,*

2. *K is uniformly integrable, and*

3. *for each $E \in \Sigma$, $\{\int_E f \; d\mu : f \in K\}$ is relatively weakly compact.*[21]

Theorem 2.40. *For a Banach space X, the following are equivalent:*

1. *X^* has the RNP.*

2. *X^* has the Kreĭn-Mil'man property*[22]*.*

3. *Every separable subspace of X has a separable dual.*

[21] In fact, without the RNP, we can guarantee a K that is not relatively weakly compact event if the three criteria hold.

[22] That is, if each closed bounded convex subset of X is the norm closed convex hull of its extreme points

4. Ever separable subspace of X^* is a subspace of a separable dual space.

Theorem 2.41. *If X is a weakly sequentially complete Banach space and X^* has the RNP, then X is reflexive.*

Theorem 2.42 (Davis-Phelps)**.** *A Banach space has the RNP iff each of its equivalent norms has a dentable*[23] *closed unit ball.*

Theorem 2.43 (Huff-Morris)**.** *For a Banach space X, the following are equivalent:*

1. *X has the RNP.*
2. *Every closed bounded subset of X contains an extreme point*[24] *of its closed convex hull.*
3. *Every closed bounded subset of X contains an extreme point of its convex hull.*
4. *For each closed bounded subset A of X, there is a nonzero $x^* \in \mathbb{X}^*$ and $x_0 \in A$ such that $x^*(x_0) = \sup x^*(A)$.*
5. *For each closed bounded subset A of X the collection of $x^* \in \mathbb{X}^*$ that attain their maxima on A is norm-dense in X^*.*

Theorem 2.44. *A Banach space lacks the RNP iff there is a bounded open convex set K in X and a norm closed subset A of K such that $\overline{co}(A) = \overline{K}$.*

2.7 Radon Nikodým Theorem for Bochner Integration

Thanks to the illuminary properties of the Radon Nikoým Property, the direct translation of the classic Radon Nikodým Theorem into the context of Banach spaces is of most interest. However, there exist several versions of the theorem that hold in Banach spaces, and we will not leave this chapter without a brief discussion. The first version is by Dunford

[23] A subspace D of a Banach space is dentable if for all $\epsilon > 0$, there exists an $x \in D$ such that $x \notin \overline{co}(D \backslash B_\epsilon(x))$.

[24] An extreme point of a set is a point that is not an interior point of any line segment lying entirely within the set– a vertex, if you will.

and Pettis in 1940. In 1943, Phillips proved an extension of their result, then Metivier proved that the converse of Phillips's theorem holds. Finally, in 1968, Rieffel proved an even more extensive version.[11]

We shall not prove all of these here. In fact, this section will closely follow the paper "Radon-Nikodým Theorems for the Bochner and Pettis Integrals" by S. Moedomo and J.J. Uhl, Jr.([11]). We will establish the necessary conditions for Rieffel's statement of the theorem (as it appears in Moedomo and Uhl's paper) and Phillips Theorem. Let (Ω, Σ, μ) be a probability space and $\mathbb{X}$ a Banach space[25].

Theorem 2.45 (Dunford-Pettis). *[11] Let $T : L^1(\Omega, \Sigma, \mu) \to \mathbb{X}$ be a weakly compact operator whose range is separable. Then there exists an essentially bounded μ-measurable $g : \Omega \to \mathbb{X}$ such that $T(f) = \int_\Omega fg \, d\mu$ for all $f \in L^1(\Omega, \Sigma, \mu)$.*

Theorem 2.46 (Phillips). *[11] A vector measure $F : \Sigma \to \mathbb{X}$ is of the form $F(E) = \int_E f \, d\mu$ for all $E \in \Sigma$ for some Bochner integrable $f : \Omega \to \mathbb{X}$ if*

1. *F is μ-continuous,*
2. *F is of bounded variation, and*
3. *for each $\epsilon > 0$ there exists an $E_\epsilon \in \Sigma$ with $\mu(\Omega - E_\epsilon) < \epsilon$ such that for some weakly compact $A \subseteq \mathbb{X}$, $\{F(E)/\mu(E) : E \subseteq E_\epsilon, \mu(E) > 0, E \in \Sigma\} \subseteq A$.*

Theorem 2.47 (Rieffel). *[11][26] A vector measure $F : \Sigma \to \mathbb{X}$ is of the form $F(E) = \int_E f \, d\mu$ for all $E \in \Sigma$ for some Bochner integrable $f : \Omega \to \mathbb{X}$ iff*

1. *F is μ-continuous,*
2. *F is of bounded variation, and*

[25] Uhl and Moedomo assume finite, whereas Rieffel assumes σ-finite, but we will stick with our probability space from before.

[26] In his paper, ([16]), Rieffel's statement of the theorem includes two equivalent statements of the third condition.

3. for each $\epsilon > 0$ there exists an $E_\epsilon \in \Sigma$ with $\mu(\Omega - E_\epsilon) < \epsilon$ such that for some compact $A \subseteq_\epsilon \mathbb{X}$ (with respect to the uniform operator topology on $\mathbb{X}$),

$\{F(E)/\mu(E) : E \subseteq E_\epsilon, \mu(E) > 0, E \in \Sigma\} \subseteq A_\epsilon$.

Moedomo and Uhl's proof of the necessary half of the claim is a simplified version of Rieffel's proof that has been tweaked to show the claim holds (save the second part) for Pettis integrable functions f. However, we shall stick to Bochner integrable functions. First, we will show the necessary implication for Rieffel's Theorem. The first two are satisfied by Theorem 2.26[27]. Hence, all that remains is to prove (3).

Proof. Let $F : \Sigma \to \mathbb{X}$ be of the form $F(E) = \displaystyle\int_E f \, d\mu$ for all $E \in \Sigma$ for some Bochner integrable $f : \Omega \to \mathbb{X}$, and let (f_n) be a sequence of μ-simple functions that converge a.e. to f. Let $\epsilon > 0$. Then, by Egoroff's Theorem, there is an $E_\epsilon \in \Sigma$ with $\mu(\Omega - E_\epsilon) < \epsilon$ such that (f_n) converges to f uniformly on E_ϵ. Hence, since each f_n is bounded, so is f on E_ϵ. Then, for all $g \in L^1(\mu)$, define $T, T_n : L^1(\mu) \to \mathbb{X}$ by $T(g) = \displaystyle\int_{E_\epsilon} gf \, d\mu$ and $T_n(g) = \displaystyle\int_{E_\epsilon} gf_n \, d\mu$ for all n. Since f is bounded and f_n are bounded for all n, $\|gf\| \in L^1(\mu)$ and $\|gf\| \in L^1(\mu)$ for all n. Hence, $T, T_n \in \mathcal{L}(L^1(\mu), \mathbb{X})$ for all n.

Claim 1: $\lim\limits_{n\to\infty} T_n = T$, i.e. $\|T - T_n\| = \sup\limits_{\|g\|_1 \le 1} \|(T - T_n)g\| \to 0$.

Let $g \in L^1(\mu)$ with $\|g\|_1 \le 1$. Then,

$$\begin{aligned}
\|(T - T_n)g\| &= \left\| \int_{E_\epsilon} (gf - gf_n) \, d\mu \right\| \le \int_{E_\epsilon} |g| \; \|f - f_n\| \, d\mu \\
&\le \int_{E_\epsilon} |g| \sup_{\omega \in E_\epsilon} \|f(\omega) - f_n(\omega)\| \, d\mu \\
&= \sup_{\omega \in E_\epsilon} \|f(\omega) - f_n(\omega)\| \int_{E_\epsilon} |g| \, d\mu \\
&\le \sup_{\omega \in E_\epsilon} \|f(\omega) - f_n(\omega)\|.
\end{aligned}$$

[27](1) by (1) and (2) by (3)

Since (f_n) converges to f uniformly on E_ϵ, (T_n) converges to T on the uniform operator topology.

Claim 2: For each n, $T_n(L^1(\mu))$ has finite dimension.

For each n, let $f_n = \sum_{i=1}^{k_n} x_i \mathbb{1}_{A_i}$ (where $\bigcup_{i=1}^{k_n} A_i = E_\epsilon$). Then,

$$T_n(L^1(\mu)) = \left\{ \int_{E_\epsilon} g f_n \, d\mu : g \in L^1(\mu) \right\} = \left\{ \int_{E_\epsilon} g \sum_{i=1}^{k_n} x_i \mathbb{1}_{A_i} \, d\mu : g \in L^1(\mu) \right\}$$
$$= \left\{ \sum_{i=1}^{k_n} \left(x_i \int_{A_i} g \right) \, d\mu : g \in L^1(\mu) \right\} = \left\{ \sum_{i=1}^{k_n} \left(\int_{A_i} g \, d\mu \right) x_i : g \in L^1(\mu) \right\}$$

Then for all $x \in T_n(L^1(\mu))$, $x = \sum_{i=1}^{k_n} \left(\int_{A_i} g \, d\mu \right) x_i$. Therefore, $\{x_1, ..., x_{k_n}\}$ spans $T_n(L^1(\mu))$, and hence $\dim(T_n(L^1(\mu))) \leq k_n$.

Therefore, each T_n is a compact operator. Since the collection of all compact operators from $L^1(\mu)$ into $\mathbb{X}$ is closed in $\mathcal{L}(L^1(\mu), \mathbb{X})$, and since (T_n) converges to T in the uniform operator topology, T is also a compact operator.

Now, consider the bounded set $S = \{\mathbb{1}_E/\mu(E) : E \in \Sigma, \mu(E) > 0\}$ in $L^1(\mu)$. For each measurable $E \subseteq E_\epsilon$, $T(\mathbb{1}_E) = \int_{E_\epsilon} \mathbb{1}_E f \, d\mu = \int_E f \, d\mu = F(E)$. Therefore, the closure of

$$T(S) = \{T(\mathbb{1}_E/\mu(E)) = \tfrac{1}{\mu(E)} \int_E f \, d\mu = \frac{F(E)}{\mu(E)} : E \in \Sigma, E \subseteq E_\epsilon\}$$

is compact with respect to the uniform operator topology in $\mathbb{X}$.

□

Now, we will use this result as well as the Dunford-Pettis Theorem to get Phillips Theorem (i.e. the sufficiency). But first, a lemma:

Lemma 2.48. *[11] A weakly compact operator $T : L^1(\Omega, \Sigma, \mu) \to \mathbb{X}$ has a separable range.*

Proof. Let $S = \{\mathbb{1}_E : E \in \Sigma\}$. Then the linear span of S is the collection of μ-simple functions on Ω, which is dense in $L^1(\Omega, \Sigma, \mu)$. Therefore, by the linearity and continuity of T, it will suffice to show that $T(S)$ is separable.

Let $\{\mathbb{1}_{E_n}\}_{n=1}^{\infty} \in S^{\mathbb{N}}$ and let Σ_0 be the σ-algebra generated by $\{\mathbb{1}_{E_n}\}$[28]. Then Σ_0 is countably generated and so $L^1(\Omega, \Sigma_0, \mu) = \{g \in L^1(\Omega, \Sigma, \mu) : g \in \Sigma_0\}$ is separable. Since T is continuous, $T : L^1(\Sigma_0) \to \mathbb{X}$ is a weakly compact operator whose range is separable. By Dunford-Pettis, there is an essentially bounded, Σ_0-measurable $f : \Omega \to \mathbb{X}$ for which $T(g) = \displaystyle\int_\Omega gf \; d\mu$ for all $g \in L^1(\Omega, \Sigma, \mu)$.

Now, let $\epsilon > 0$, and $\|f\|_\infty = M \in \mathbb{R}^+$. Then, by the necessary direction of Rieffel's Theorem, there is a set $E_\epsilon \in \Sigma_0$ such that $\mu(\Omega - E_\epsilon) < \frac{\epsilon}{M+1}$ and a (norm) compact set $A_\epsilon \subseteq \mathbb{X}$ such that $\{\dfrac{\int_E f \; d\mu}{\mu(E)} : E \in \Sigma_0, E \subseteq E_\epsilon\} \subseteq A_\epsilon$. Let $A'_\epsilon = \{\alpha x : 0 \le \alpha \le \mu(\Omega) = 1$ and $x \in A_\epsilon\}$. Since A_ϵ is compact, so is A'_ϵ. Now, note that

$$T(\mathbb{1}_{E_n}) = \int_{E_n} f \; d\mu = \int_{E_n \cap E_\epsilon} f \; d\mu + \int_{E_n - E_\epsilon} f \; d\mu.$$

Since $E_n \cap E_\epsilon \subseteq E_\epsilon$, we have that

$$\frac{1}{\mu(E_n \cap E_\epsilon)} \int_{E_n \cap E_\epsilon} f \; d\mu \in A_\epsilon, \text{ and hence, } \int_{E_n \cap E_\epsilon} f \; d\mu \in \mu(E_n \cap E_\epsilon) A_\epsilon \subseteq A'_\epsilon.$$

Moreover,

$$\left\| \int_{E_n - E_\epsilon} f \; d\mu \right\| \le \int_{E_n - E_\epsilon} \|f\| \; d\mu \le M\mu(E_n - E_\epsilon) \le M\mu(\Omega - E_\epsilon) < \epsilon.$$

Thus, $T(\mathbb{1}_{E_n})$ is within ϵ of a member of the compact set A'_ϵ. Then $\{T(\mathbb{1}_{E_n})\}$ is totally bounded and hence is (norm) relatively compact. Thus $\{T(\mathbb{1}_{E_n})\}$ has a sequence that converges in $\mathbb{X}$. Then, for any infinite subset $B \subseteq T(S)$, there is a $\{\mathbb{1}_{E_n}\} \subseteq T^{-1}(B)$ such that $\{T(\mathbb{1}_{E_n})\}$ has a convergent subsequence in $\mathbb{X}$; hence B has a limit point in $\mathbb{X}$. Then, $T(S)$ is relatively compact[29]. Since a compact subset of a metric space is separable, $\overline{T(S)}$ is separable and hence so is $T(S)$. □

Now, we are ready to prove the main theorem.

[28] That is the smallest σ-algebra containing $\{\mathbb{1}_{E_n}\}$.

[29] Because the previous property is equivalent to relative (or conditional) compactness

Theorem 2.49 (Phillips)**.** *[11] Let $F : \Sigma \to \mathbb{X}$ be a μ-continuous vector measure of bounded variation such that for each $\epsilon > 0$, there exists $E_\epsilon \in \Sigma$ with $\mu(\Omega - E_\epsilon) < \epsilon$ such that*

$$B_\epsilon = \{F(E)/\mu(E) : E \subseteq E_\epsilon, E \in \Sigma, \mu(E) > 0\}$$

is contained in a weakly compact subset of $\mathbb{X}$. Then, there exists a μ-measurable Bochner integrable function $f : \Omega \to \mathbb{X}$ such that for $E \in \Sigma$,

$$F(E) = \int_E f \; d\mu.$$ [30]

Proof. For each $n \in \mathbb{Z}_+$, choose $E_n \in \Sigma$ such that $\mu(\Omega - E_n) < \frac{1}{n}$ and

$$B_n := \{F(E)/\mu(E) : E \subseteq E_n, E \in \Sigma, \mu(E) > 0\}$$

is contained in a weakly compact $A_n \subseteq \mathbb{X}$. Let

$$S := \{g : \Omega \to \mathbb{R} : g = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i}, \alpha_i \in \mathbb{R}, \{E_i\} \subseteq \Sigma \text{ are pairwise disjoint}\} \subseteq L^1(\mu),$$

and define $t_n : S \to \mathbb{X}$ by[31]

$$t_n\left(\sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i}\right) = \sum_{i=1}^{n} \alpha_i F(E_i \cap E_n) = \sum_{i=1} \alpha_i \mu(E_i \cap E_n) \frac{F(E_i \cap E_n)}{\mu(E_i \cap E_n)}$$

The linearity of t_n comes directly from the form of the elements of S.

Now, let S' be the intersection of S and the unit ball in $L^1(\mu)$; hence S' is a dense subset of the unit ball of $L^1(\mu)$. Then for any $f \in S'$, $\|f\|_1 \leq 1$, and so

$$\sum_{i=1} |\alpha_i \mu(E_i \cap E_n)| \leq \sum_{i=1}^{n} |\alpha_i|(E_i) = \|f\|_1 \leq 1.$$

Therefore,

$$t_n(f) \in co(B_n - B_n) \subseteq \overline{co}(B_n - B_n) = \overline{co}(\overline{B_n - B_n}).$$

[30] In the paper, the authors also prove (without the requirement that F be of bounded variation) that F can be given as the integral of a Pettis integrable function. However, for the sake of remaining conscise, we will avoid the digression into Pettis integration.

[31] The last equality holds cleanly if we adopt the convention of $0/0 = 0$.

Since A_n is weakly compact, so is $A_n - A_n$; since $B_n \subseteq A_n$, we have that $\overline{B_n - B_n} \subseteq A_n - A_n$. Hence $\overline{B_n - B_n}$ is weakly compact in $\mathbb{X}$. Then, by the Krein-Smulian theorem[32], $\overline{co}(\overline{B_n - B_n})$ is also weakly compact. Therefore, $t_n(S')$ is contained in a weakly compact set. Since S' is dense in the unit ball of $L^1(\mu)$, there is an extension T_n of t_n on all of $L^1(\mu)$ that maps the closed unit ball into a weakly compact set in $\mathbb{X}$. Hence, T_n is weakly compact. By the preceeding Lemma, T_n has a separable range, and hence by the Dunford-Pettis theorem, there is a μ-measurable $f : \Omega \to \mathbb{X}$ with support E_n[33] such that $T_n(g) = \int_{E_n} g f_n \, d\mu$ for all $g \in L^1(\mu)$.

If we do this for each $n \in \mathbb{Z}_+$, we can produce an increasing sequence of measurable sets (E_n) such that $\mu(\Omega - E_n) \to 0$ and a sequence (f_n) of μ-measurable Bochner integrable functions such that

$$F(E \cap E_n) = T_n(\mathbb{1}_{E \cap E_n}) = \int_{E_n} f_n \, d\mu.$$

Then $f_n \mathbb{1}_{E_m} = f_m$ for $n \geq m$ since (E_n) is increasing. Define $f : \Omega \to \mathbb{X}$ by $f := \sum_{n \in \mathbb{N}} f_n \mathbb{1}_{E_n}$. Since $(E_n) \nearrow \Omega$, $f_n \to f$ uniformly, and hence f is μ-measurable, and $f_n = f \mathbb{1}_{E_n}$ for all $n \in \mathbb{Z}_+$.

Let $E \in \Sigma$. Since $F << \mu$, and $\lim_n \mu(\Omega - E_n) = 0$,

$$F(E) = \lim_n F(E \cap E_n) = \lim_n \int_{E \cap E_n} f_n \, d\mu = \lim_n \int_{E \cap E_n} f \mathbb{1}_{E_n} \, d\mu = \lim_n \int_{E \cap E_n} f \, d\mu$$

in the metric topology on $\mathbb{X}$. Since F is of bounded variation,

$$\infty > |F|(\Omega) \geq \int \|f_n\| \, d\mu = \int_{E_n} \|f\| \, d\mu \text{ for all } n \in \mathbb{Z}_+.$$

Since $\|f_n\| \nearrow \|f\|$, by the Monotone Convergence Theorem,

$$\infty > |F|(\Omega) \geq \lim_n \int_{E_n} \|f\| \, d\mu = \lim_n \int \|f_n\| \, d\mu = \int \|f\| \, d\mu.$$

Therefore, $\|f\|$ is integrable, and hence f is Bochner integrable.

[32] The closed convex hull of a weakly compact subset of a Banach space is weakly compact

[33] That is, $f_n = f_n \mathbb{1}_{E_n}$.

Therefore, $(f_n) \to f$ in measure, $\|f_n\| \leq \|f\|$ for all n where f is Bochner integrable; hence, by the Dominated Convergence Theorem, $\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$ for all $E \in \Sigma$. Therefore, we have that f is Bochner integrable and

$$F(E) = \lim_n \int_{E \cap E_n} f \, d\mu = \lim_n \int_E f_n \, d\mu = \int_E f \, d\mu.$$

$\square$

Chapter 3

Quantum Measures

Our final chapter will explore an avenue of vector measure theory that branches from what is called Quantum Probability. The operators discussed here are defined and briefly discussed in section D.4 of the Appendix.

3.1 Hilbert Space Quantum Mechanics

Among the contending axiomatic bases for quantum mechanics, the traditional approach uses the structure of the Hilbert space and its self-adjoint operators to describe states and observables within a given physical system. This approach was developed in the early 1930's by P. Dirac and J. von Neumann with the ideas of M. Born[8].

In both classical and quantum mechanics, a state represents a theoretically complete description of a given physical system, and observables correspond to measurable quantities in the physical system.[8] In the traditional axiomatic basis for quantum mechanics, the physical system is represented by a complex Hilbert space, and states and observables are described by entities in the Hilbert space. Hence the traditional axiomatic structure is called Hilbert quantum mechanics. Von Neumann gave the following axioms for Hilbert space quantum mechanics [8]:

(A1) The states of a quantum system are [described by] unit vectors in a complex Hilbert space $\mathbb{H}$.

(A2) The observables are [described by] self-adjoint operators on $\mathbb{H}$.

(A3) The probability that an observable T has a value in a Borel set $A \subseteq \mathbb{R}$ when the system is in the state ψ is $\langle P^T(A)\psi, \psi\rangle$ where $P^T(\cdot)$ is the resolution of identity (spectral measure) for T.

(A4) If the state at time $t = 0$ is ψ, then the state at time t is $\psi_t = e^{-itH/h}\psi$, where H is the energy observable and h is a constant (Planck's constant).

In this chapter, we will focus on explicating A1 and A2. After which, we will introduce a natural extension of those probabilities to vector measures on $\mathbb{H}$, in the spirit of the classical vector measure theory.

3.2 Basics of Quantum Probability

We begin the endeavor of defining a probability with respect to our physical system, represented by $\mathbb{H}$.

3.2.1 Events

To define a probability, we will first need to define events, which, in classic probability, are elements of a σ-algebra on which the probability measure is defined. We begin with the following axiom:[8]

The events of a quantum system can be represented by projections on a complex Hilbert space.

For the sake of restricting this discussion to merely one chapter, we will restrict ourselves to bounded orthogonal projections on $\mathbb{H}$. Because of our restriction, we now have a one-to-one correspondence between events and closed subspaces of $\mathbb{H}$, i.e.

Proposition 3.1. *For each closed subspace E of $\mathbb{H}$, there exists a unique orthogonal projection $P_E = proj_E$ that maps $\mathbb{H}$ onto E and is defined such that for each $x \in \mathbb{H}$, $P_E(x)$ is the unique element in $\mathbb{H}$ such that $(x - P_E(x)) \in E^\perp$.Furthermore, if P is a continuous projection, then $R(P)$ is closed and $R(P) \oplus R(P)^\perp = \mathbb{H}$.*

[1]

Hence, the events for our probability are identified with closed subspaces of $\mathbb{H}$. Furthermore, a subset E_0 of $\mathbb{H}$ may be viewed as an event through its closed span:

$$E_0 \to \overline{\text{span} E_0} \leftrightarrow P_{\overline{\text{span} E_0}}.$$

Example 3.2. To say a single vector $h = (h_j) \in \mathbb{H}$ is an event is to say that $\{\alpha h : \alpha \in \mathbb{C}\}$ or the projection $P_h : \mathbb{H} \to \mathbb{H}$ given by $P_h = hh^* = [h_j h_k]$.

We will denote the family of events (i.e. bounded orthogonal projections onto closed subspaces of $\mathbb{H}$) by $\mathscr{E}$ and the "unit" by $1 \leftrightarrow \mathbb{H} \leftrightarrow P_{\mathbb{H}}$.

3.2.2 Intersections and Unions

If we are to define a probability on elements of $\mathscr{E}$, we must first define unions and intersections of "events" that are identified with bounded orthogonal projections.

The role of **"intersection"** will be played by product, i.e. composition. Here arise some disanalogies between quantum and classic probabilities. First, the product is not commutative in general, which leaves us with left and right "intersections". Second, $\mathscr{E}$ is not necessary closed under intersection. [2]

There are two important exceptions to these two dis-analogies. First, if $E \subseteq F$, then $P_E P_F = P_E = P_F P_E$. The second exception is of greater interest to us as it introduces the notion of a disjoint union:

Proposition 3.3. *If E and F are closed subspaces of $\mathbb{H}$, then E and F are orthogonal iff $P_E P_F = \{0\} = P_F P_E$.*

Proof. First, an intermediate claim: $P_F E = \{0\}$ iff $E \subseteq F^{\perp}$.

Recall that every $h \in \mathbb{H}$ can be written uniquely as $h = x + y$ where $x \in F$ and $y \in F^{\perp}$, and $P_F h = x$. Then, $y \in F^{\perp} \iff y = 0 + y$ where $y \in F^{\perp}$ and $0 \in F$ is the unique

[1] As the proof of this proposition is standard and routine, we will omit it here.

[2] However, we need only the notion of disjoint events (which we will introduce momentarily) to talk about a probability.

representation of y in $\mathbb{H} = F \oplus F^{\perp} \iff P_F y = 0$. This proves our claim. Now,

$$\begin{aligned} P_E P_F = 0 &\iff P_E P_F x = 0 \ \forall x \in \mathbb{H} \iff P_E(F) = \{0\} \iff F \subseteq E^{\perp} \\ &\iff E \text{ and } F \text{ are orthogonal } \iff E \subseteq F^{\perp} \iff P_F(E) = \{0\} \\ &\iff P_F P_E x = 0 \ \forall x \in \mathbb{H} \iff P_F P_E x = 0 \end{aligned}$$

□

Definition 3.4. We say two events P_E and P_F are **disjoint** if $P_E P_F = \{0\} = P_F P_E$.[3]

Intuitively, we say two events are disjoint if one's occurance precludes the other's occurance and vice versa. Similarly, as observables correspond to physical phenomena that either do or do not occur within a physical system [8], "disjoint" events (which are simple observables) should be mutually exclusive physical phenomena.

The role of **"union"** will be played by sums of projections; however, just as with intersection, $\mathscr{E}$ is not necessarily closed under "unions":

Proposition 3.5. *If $P_E : \mathbb{H} \to \mathbb{H}$ and $P_F : \mathbb{H} \to \mathbb{H}$ are orthogonal projections onto closed subspaces E and F respectively, then TFAE:*

1. *$P_E + P_F$ is an orthogonal projection.*
2. *E and F are orthogonal.*
3. *$P_E P_F = 0 = P_F P_E$.*

Proof. Since we have already shown that $(ii) \iff (iii)$, we may use those as interchangeable. So, we will first assume (ii) and (iii).

To prove that $P_E + P_F$ is an orthogonal projection, it will suffice to prove that it is self-adjoint and idempotent. First, idempotent:

$$(P_E + P_F)^2 = P_E^2 + P_E P_F + P_F P_E + P_F^2 = P_E + 0 + 0 + P_F = P_E + P_F.$$

[3] In terms of closed subspaces of $\mathbb{H}$, E and F represent "disjoint" events if they are orthogonal.

Next, self-adjoint: Let $x, y \in \mathbb{H}$. Then,

$$\begin{aligned}\langle (P_E + P_F)x, y\rangle &= \langle P_E x + P_F x, y\rangle = \langle P_E x, y\rangle + \langle P_F x, y\rangle \\ &= \langle x, P_E y\rangle + \langle x, P_F y\rangle = \langle x, P_E y + P_F y\rangle \\ &= \langle x, (P_E + P_F)y\rangle.\end{aligned}$$

Now, assume (i), and we will show (ii). Now, by the claim in the preceeding proposition, it will suffice to show that $P_F(E) = \{0\}$. Then,

$$P_E + P_F = (P_E + P_F)^2 = P_E^2 + P_E P_F + P_F P_E + P_F^2.$$

So, $0 = P_E P_F + P_F P_E$, and hence $P_E P_F = -P_F P_E$. First, note that if $x \in E \cap F$, then

$$x = P_E x = P_E P_F x = -P_F P_E x = -Fx = -x.$$

Hence $x = 0$. Now, let $x \in E$. Then,

$$P_E P_F x = -P_F(P_E x) = -P_F(x).$$

Since E and F are linear, $-(-P_F x) \in E \cap F = \{0\}$. Thus, $P_F x = \{0\}$, and therefore, E and F are orthogonal.

$\square$

Hence, we have an analogy for "disjoint unions" (which extends readily to finite "disjoint unions"). Intuitively, the union of two disjoint events is the event that occurs iff only one of the two events occurs. In fact, provided that E and F are orthogonal, the range of $P_E + P_F$ is $E \oplus F$.

Therefore, for $P_E, P_F \in \mathscr{E}$, their "union", $P_E \vee P_F$ is in $\mathscr{E}$ iff E and F are orthogonal, and hence we have

Definition 3.6. For orthogonal $E, F \in \mathscr{E}$, $P_E \vee P_F := P_{E \oplus F}$.

Furthermore, we can easily define an arbitrary union for all $P_E, P_F \in \mathscr{E}$ as $P_E \vee P_F := P_{\overline{\mathrm{Span} E \cup F}} = P_{E \oplus F}$.

As for countable disjoint unions, nontrivial countable collection of pairwise disjoint orthogonal projections will not exist in a finite dimensional vector space. However, in the case of an infinite dimensional complex Hilbert space, we have the following proposition:

Proposition 3.7. *Let $\mathbb{H}$ be an infinite dimensional complex Hilbert space and $\{P_{E_k}\}$ a collection of mutually disjoint orthogonal projections. Then, $\sum_k P_{E_k}$ exists and is an orthogonal projection.*

Proof. Given the existence of $P = \sum_k P_{E_k}$, to show that it is an orthogonal projection, we must show that it is self-adjoint and indempotent. Note first that each partial sum $\sum_{k=1}^{n} P_{E_k}$ is an orthogonal projection. Then, the self-adjoint and indempotent properties come from the fact that the square function is continuous and the inner product is jointly continuous.

$\square$

Therefore, $\mathscr{E}$ is closed under countable "disjoint unions". Henceforth, events will be represented by both orthogonal projections and their ranges (primarily by their ranges unless there is a risk of ambiguity); $\mathbb{H}$ will be denoted with 1. We will call these **quantum events**.

Under a partial ordering $\leq$ on $\mathscr{E}$, where $P_E \leq P_F$ if $E \subseteq F$ for all $P_E, P_F \in \mathscr{E}$, then $\mathscr{E}$ is a lattice. In fact, given any subset A of $\mathscr{E}$, $\inf A = P_{\bigcap_{E \in A} E}$ and $\sup A = \bigvee_{E \in A} P_E$[4].

3.2.3 A Probability

Classically, a probability p is a countably additive mapping from a σ-algebra of subsets of a set Ω into the interval $[0, 1]$ such that $p(\Omega) = 1$. Hence, an analogous function defined on $\mathscr{E}$ should be a mapping $p : \mathscr{E} \to [0, 1]$ such that for every sequence of mutually disjoint events $\{E_j\}$,

$$p\left(\sum_j E_j\right) = \sum_j p(E_j) \text{ and } p(1) = 1.$$

[4]Recall that, if the P_E are not "disjoint", then the union is actually the projection onto the closed linear span of their union

Instead of "probability," this mapping defined on quantum events is often called a **state map** because "[it] gives a theoretically complete description of the system. Since quantum mechanics is a probabilistic theory, a complete description of a quantum system is given by a probability measure on its set of events."[8] In fact, probabilities are usually determined by states.

Example 3.8. Let P be a positive semi-definite matrix (operator) with unit trace.[5] In other words, P is a density operator or, more specifically, a state. For each $E \in \mathscr{E}$, define

$$p(E) := \mathrm{tr}(PE).$$

Proposition 3.9. *$p : \mathscr{E} \to [0, 1]$ satisfies the following:*

1. *$p(1) = 1$*

2. *Given a countable collection $\{E_k\}$ of pairwise disjoint events, $p\left(\sum_k E_k\right) = \sum p(E_k)$.*

Proof. Let p be defined as above.

1. First, $p(1) = \mathrm{tr}(P1) = \mathrm{tr}(PI) = \mathrm{tr}(P) = 1$.

2. Finite additivity comes from the linearity of the trace map, and countable additivity comes from the fact that the trace (or Schatten) class operators[6] form an ideal, $\mathscr{I}_1(\mathbb{H}) := \{T \in \mathscr{I}_\infty(\mathbb{H}) : \sum_n |s_n(T)| < \infty\}$, that is, the class of all operators on $\mathbb{H}$ with absolutely summable singular values (s_n) [7]. Therefore, for any collection $\{E_j\}$ of pairwise disjoint events, $tr\left(P \sum_{j=1}^{\infty} E_j\right) < \infty$.

□

In fact, in a separable Hilbert space of dimension at least 3, this is the only example of a probability on $\mathscr{E}$ by Gleason's theorem.

[5] Both properties are characterizable by eigenvalues: A positive semi-definate matrix has all nonnegative eigenvalues, and the trace of a matrix is the same as the sum of its eigenvalues.

[6] operators with finite trace

[7] A singular value of T is the square root of an eigenvalue of T^*T.

Theorem 3.10 ([7], Gleason)**.** *Let μ be a measure on the closed subspaces of a separable Hilbert space $\mathbb{H}$ of dimension at least three. There exists a positive semi-definite self-adjoint operator P of the trace class such that for all closed subspaces E of $\mathbb{H}$, $\mu(E) = tr(PP_E)$ where P_E is the orthogonal projection of $\mathbb{H}$ onto E.*

Restricting our positive semi-definite self-adjoint operators of the trace class to density operators, we get a probability measure and a special case of Gleason's Theorem:

Theorem 3.11 ([8], Gleason)**.** *If $dim(\mathbb{H}) \geq 3$ and p is a probability on $\mathscr{E}$, then there exists a unique density operator (state) P on $\mathbb{H}$ such that $p(E) = tr(PE)$ for all $E \in \mathscr{E}$.*

3.2.4 Observables as Random Variables

To define integration with respect to our probability, we must first find an analog of a random variable. The natural choice would be a bounded linear operator or matrix X. In which case, we would define the integral as

$$\int X \ dp = tr(XP).^{8}$$

If X is Hermitian, then it is an analog to a real random variable, and the integral is merely EX. In the complex case, we want guarantee at least diagonalizability. Hence, normal matrices will be our extension of complex random variables. These are our **observables**, and we will denote their class by $\mathscr{O}$. Note that $\mathscr{O} = \overline{\text{span}}(\mathscr{E})$[9].

Since an observable is not necessarily compact, we will distinguish the space of compact observables (i.e. with eigenvalues converging to 0) and denote the subclass by $\mathscr{O}_\infty := \mathscr{O} \cap \mathscr{I}_\infty(\mathbb{H})$, where $\mathscr{I}_\infty(\mathbb{H})$ denotes the ideal of compact operators.

By the same token, we can also define moments $\mathrm{E}X^k$, the Fourier transform $\mathrm{E}e^{itX}$, other transforms such as $\mathrm{E}(XY)$, variance $\mathrm{Var}(X) = \mathrm{E}X^2 - (\mathrm{E}X)^2$, covariance, etc..

[8] We can also denote the integral $\int X \ dP$ since the probability is uniquely determined by the density matrix P.

[9] The simplest finite-valued observable is a finite linear combination of bounded orthogonal projections. Hence all finite-valued observables are in $\text{span}(\mathscr{E})$. Therefore, $\overline{\text{span}}\mathscr{E} \subseteq \mathscr{O}$. Furthermore, any observable X can be written as the limit of a sequence of finite-valued observables, which is in $\overline{\text{span}}(\mathscr{E})$.

3.3 Quantum Measures

Now, we will begin extending our quantum "probability". First, we will extend it to a quantum "measure". To that end, we will begin with a countably-additive function $\mu : \mathscr{E} \to \mathbb{R}$ (or $\mathbb{C}$). This function can be extended to a linear mapping on $\overline{\text{span}}(\mathscr{O})$ in the strong operator topology[10]. Hence, μ would be a continuous functional on $\mathscr{I}_\infty(\mathbb{H})$ by the Hahn-Banach Theorem.

By the Schatten Theorem[11], μ must then be of the form

$$\mu(T) = \text{tr}QT \ , \, T \in \mathscr{I}_\infty(\mathbb{H})$$

for some unique $P \in \mathscr{I}_1(\mathbb{H})$.

3.4 Quantum Vector Measures

Now, we will find the natural extension of this notion to vector measures. To do so we will replace the range space with a Banach space $\mathbb{X}$:

$$F : \mathscr{E} \to \mathbb{X},$$

which, as before is extendable to $F : \mathscr{I}_\infty(\mathbb{H}) \to \mathbb{X}$ (by linearity and continuity).

By theorem 2.26, we have an immediate example of a vector measure on a measure space $(S, \mathcal{S}, \mu)$ is given by the Bochner integral of a Bochner integrable function $f : S \to \mathbb{X}$:

$$F(E) := \int_E f(s)\mu\ (ds), \text{ for all } E \in \mathcal{S},$$

If f is simple, i.e. $f := \sum_k x_k \mathbb{1}_{E_k}$ (where $x_k \in \mathbb{X}$ and $E_k \in \mathcal{S}$ for $1 \leq k \leq n$),

$$F(E) = \sum_k x_k\ \mu(E \cap E_k) \text{ for all } E \in \mathcal{S}.$$

We may, instead, simply use n arbitrary finite measures $\mu_k : \mathcal{S} \to \mathbb{R}$ (or $\mathbb{C}$):

$$F(E) = \sum_k x_k\ \mu_k(E) \text{ for all } E \in \mathcal{S}.$$

[10] Under the assumption of continuity

[11] $\mathscr{I}_1(\mathbb{H})$ is isometrically isomorphic to the dual of $\mathscr{I}_\infty(\mathbb{H})$.

Now, to make F a "quantum vector measure", we let our "measurable space" be $(\mathbb{H}, \mathscr{E})$ and let the μ_k be "quantum measures": $\mu_k : \mathscr{E} \to \mathbb{R}$ (or $\mathbb{C}$) where for each k, $\mu_k(E) = \operatorname{tr}(Q_k E)$ for all $E \in \mathscr{E}$ where $Q_k \in \mathscr{I}_1(\mathbb{H})$ is unique for each $1 \leq k \leq n$. Hence, our simple "quantum vector measure" is

$$F(E) = \sum_k x_k \operatorname{tr}(Q_k E) = \sum_k \int_E x_k E_k \, d(\operatorname{tr}(Q_k \cdot)), \text{ for all } E \in \mathscr{E}.^{12}$$

Now, to look at a case that is not discrete, consider a Bochner integrable function $f : [0,1] \to \mathbb{X}$, and the functions $Q : [0,1] \to \mathscr{I}_1$ and $T_E : [0,1] \to \mathbb{R}$ given by $T_E(s) = \operatorname{tr}(Q(s)E)$ for a given $E \in \mathscr{E}$. We would like to define a "quantum vector measure" as

$$M(E) = \int_0^1 f(s) T_E(s) \, ds \text{ for } E \in \mathscr{E}.$$

However, we must first consider under what conditions fT_E is Bochner integrable. It is immediate that if $|T_E(s)|$ were Lebesgue integrable either $\|f(s)\|$ or $|T_E(s)|$ were bounded, then fT_E would be Bochner integrable, for then

$$M(E) = \int_0^1 \|f(s) T_E(s)\| \, ds = \int_0^1 \|f(s)\| |T_E(s)| \, ds$$

would be the product of two Lebesgue integrable functions, one of which is bounded.

3.5 Questions

The preceeding section is merely the beginning. Of course, there is the immediate question of sufficient conditions for our quantum vector measure M being the Bochner integral of fT_E as above. Furthermore, under what conditions will Gleason's theorem extend to the quantum vector measure?

A next step would be in the direction of random quantum measures (in the wide sense from Chapter 1), in which case we would choose Banach subspaces of $L^0(\Omega, \mathcal{F}, P)$ or $C(\Omega)$ (where Ω is a "nice" topological space) for our Banach space. Furthermore, our density

[12] Recall that we denote our events (which are projections) by their ranges, i.e. P_E is identified with E. Therefore, we write E_k instead of $\mathbb{1}_{E_k}$ because, for our quantum measures, the projection itself plays the role of an indicator function for the closed subspace E_k of $\mathbb{H}$.

maxtirx Q would be a random matrix (operator), and, just as with quantum vector measures, we would want to know if there is a random version of Gleason's theorem.

Appendices

Appendix A

Measure Theory

First, we will define different types of collections of subsets: Let Ω be a nonempty set.

Definition A.1. A **ring** is a collection of subsets of Ω that is closed under finite unions and finite intersections.

Definition A.2. A **δ-ring** is a ring that is closed under countable intersection.

Definition A.3. A **σ-ring** is a ring that is closed under countable uions.

Definition A.4. An **algebra** is a collection of subsets of Ω that contains Ω and is closed under finite unions and relative complements.

To begin, let Ω be a set and let Σ be a σ-algebra of subsets of Ω, that is, Σ contains Ω and is closed under complements and countable unions[1]. Then we call (Ω, Σ) a **measurable space**, and call a subset $E \subseteq \Omega$ measurable (with respect to Σ) if $E \in \Sigma$.

A.1 Measures

A **measure** μ on (Ω, Σ) is a set function from Σ into $[0, \infty)$ for which $\mu(\emptyset) = 0$ and which is countably additive. A **measure space** (Ω, Σ, μ) is a measurable space together with a measure μ defined on Σ. Note that a measure is inherently nonnegative. We call measures that map to $\overline{\mathbb{R}}$ "signed measures." A **signed measure** on (Ω, Σ) is a set function $\nu : \Sigma \to [-\infty, \infty]$ which assumes at most one of $-\infty, \infty$, has $\nu(\emptyset) = 0$, and is countably additive. By the Jordan Decomposition Theorem, any signed measure ν can be written as the

[1] An algebra is a collection of subsets of Ω that contians Ω and is closed under complements and finite unions.

difference between two mutually singular measures, ν^+ and ν^-. By **mutually singular**, we mean there exist measurable sets A and B whose disjoint union is Ω and $\nu^+(A) = \nu^-(B) = 0$; this is denoted $\nu^+ \perp \nu^-$. Due to the existence of signed measures, nonnegative measures will sometimes be referred to as "true" measures, but we will usually refer to true measures as measures.

Now, let ν be a σ-finite signed measure on the measure space (Ω, Σ, μ). Then there is a Jordan Decomposition $\nu = \nu^+ - \nu^-$ on (Ω, Σ) where ν^+, ν^- are true measures on (Ω, Σ) and $\nu^+ \perp \nu^-$.[2] Then $|\nu| = \nu^+ + \nu^-$ is a "true" measure on (Ω, Σ) called the **variation** of ν.

We say that measure ν is **absolutely continuous** with respect to a measure μ, denoted $\nu << \mu$, if for all $E \in \Sigma$, if $\mu(E) = 0$, then $\nu(E) = 0$. Then, $\nu << \mu$ iff $|\nu| << \mu$ iff $\nu^+ << \mu$ and $\nu^- << \mu$.

An extremely important measure, for our purposes, is the Lebesgue measure, which we will denote with λ, on $(\mathbb{R}^n, \mathcal{L})$, where $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets.

A measure μ on (Ω, Σ) is considered **finite** if $\mu(\Omega) < \infty$; it is considered **σ-finite** if Ω is the union of a countable collection of measurable sets, each of which has finite measure. Many of our results will hold for σ-finite spaces. However, we will often assume that our space is finite, or, better yet, a probability space, which we will define in Appendix B.

A.2 Integration and Noteworthy Theorems

A measurable function f on Ω is called **integrable** if $\int |f| \, d\mu < \infty$. Note, however, that the integral is still defined when $\int |f| \, d\mu \not< \infty$.

The following are a few of many useful theorems from measure theory: Let (Ω, Σ, μ) be a measure space.

[2] That is, the two are mutually singular, which means there exists measurable sets A and B whose disjoint union is Ω and $\nu^+(A) = \nu^-(B) = 0$.

Theorem A.5 (Egoroff's Theorem). *Let* (f_n) *be a sequence of measurable functions on* Ω *that converge pointwise a.e.*[3] *on* Ω *to a function* f *that is finite a.e. on* Ω*, then for all* $\epsilon > 0$*, there exists a measurable set* $X_\epsilon \subseteq \Omega$ *such that* $\mu(\Omega - X_\epsilon) < \epsilon$ *and* (f_n) *converges uniformly to* f *on* X_ϵ*.*

Theorem A.6 (Fatou's Lemma). *If* (f_n) *is a sequence of nonnegative measurable functions on* Ω *that converge pointwise a.e. to a function* f *on* Ω*, then*

$$\int \liminf f_n \; d\mu \leq \liminf \int f_n \; d\mu.$$

Theorem A.7 (Lebesgue Dominated Convergence Theorem). *Let* (f_n) *be a sequence of measurable functions on* Ω *which converge pointwise a.e. to a measurable function* f *on* Ω*. If there is a nonnegative integrable function* g *on* Ω *that dominated the sequence* (f_n)*, then* f *is integrable, and*

$$\lim_{n\to\infty} \int f_n \; d\mu = \int f \; d\mu.$$

There are several more, but they have their own places in this introductory chapter.

[3] "a.e." or "almost everywhere" means everywhere except on a set of measure zero.

Appendix B

Probability Theory

B.1 Measure Theory and Probability Theory

A great deal of terms in probability theory have equivalent versions in measure theory. So, we will begin this section pointing out these terms. First, a **probability measure space**, or probability space, is a finite measure space (Ω, Σ, P) where $P(\Omega) = 1$.

The following excerpt from a table from Folland may prove useful, [6]:

Analysts' Term	Probabilists' Term
σ-algebra	σ-field
Measurable Set	Event
Measurable function f into $(\mathbb{R}, \mathcal{B})$	Random variable (r.v.) X
Measurable function f into $(\mathbb{R}^d, \mathcal{B}^d), d > 1$	Random vector X
Integral of f	Expectation of X
$\int f \, d\mu$	$E(X)$
Almost every(where) (a.e.)	Almost sure(ly) (a.s.)
Characteristic function χ	Indicator function $\mathbb{1}$

In this paper, there is a slight mixture of terms from each field. In this chapter and chapter 2, the language will be primarily in measure theoretic terms, with the primary exception being the use of $\mathbb{1}$ in place of χ. This is to prevent confusion between the measure theory characteristic function and the probability theory characteristic function, two very differnt functions[1].

[1] And because "indicator" seems to be a more appropriate name for the function.

B.2 Other Basics

The following are some other useful definitions from Durett, [4]:

Definition B.1. Let X be a random variable, then the probability measure on $\mathbb{R}$ (with the Borel σ-algebra) induced by X is called the **distribution** of X, and is given by

$$\mu(A) = P(X \in A).^2$$

Definition B.2. The distribution of a random variable is usually described by its **distribution function**, $F(x) = P(X \leq x)$.

Definition B.3. Two random variables, X and Y are **identically distributed (id)**, $X \stackrel{d}{=} Y$, if they induce the same distribution μ on $\mathbb{R}$ with the Borel σ-algebra.

Definition B.4. When the distribution function $F(x) = P(X \leq x)$ can be written as

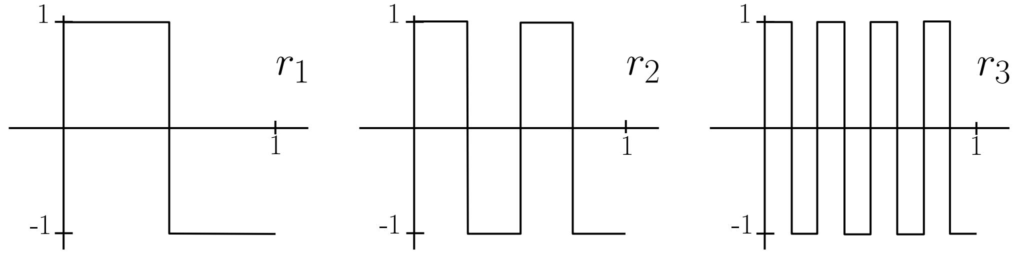
$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

for some integrable function f, then we say that P has **density function** f.

Finally, we should mention the Rademacher functions, which are useful examples in Chapter 1.

Definition B.5. The Rademacher functions are the functions $r_n : [0,1] \to \mathbb{R}$ given by $r_n(x) = \mathrm{sgn}(\sin(2^n \pi x)$ for all $x \in [0,1]$ for all $n \in \mathbb{Z}^+$.

However nice the definition is, the best understanding of the Rademacher functions comes from just a glance:



[2] That is $P(X^{-1}(A))$.

Appendix C
Radon-Nikodým Theorem

One of the most influential theorems from Measure theory and Probability theory is the Radon-Nikodým Theorem. As it is a primary focus for this paper, we will devote a slightly larger section to it. We will discuss some of the applications of the RNT (as it is usually abbreviated) in later sections, but first, the standard statement of the theorem.

Theorem C.1. *Let (Ω, Σ, μ) be a σ-finite measure space and ν a σ-finite measure on (Ω, Σ) that is absolutely continuous with respect to μ. Then there exists a function $f : \Omega \to [0, \infty)$ such that for all $A \in \Sigma$,*

$$\nu(A) = \int_A f d\mu.$$

Furthermore, f is unique modulo variations on μ-null sets.

C.1 Radon Nikodým Derivative

The non-negative function f is called the Radon Nikodým Derivative, and is usually denoted $\frac{d\nu}{d\mu}$ (which should be understood as the class of functions that are equal to f μ-a.e.). Hence, we say the RNT guarantees us a nonnegative function f such that $d\nu = f\ d\mu$. However, when we are working in a probability space, (Ω, Σ, P), the RND is referred to as a density function.

Proposition C.2. *Let P be a probability measure (usually a distribution of a random variable X) on $(\mathbb{R}, \mathcal{B})$. P is said to have density f iff $P << \lambda$ and P has RND $\frac{dP}{d\lambda} = f$ where λ is the Lesbegue measure on $(\mathbb{R}, \mathcal{B})$.*

Furthermore, f on $\mathbb{R}^k$ is called a **probability density** if $f \geq 0$, f is measurable with respect to λ^k, and $\displaystyle\int f\ d\lambda^k = 1$. And if a probability measure P on $(\mathbb{R}^k, \mathcal{B})$ is given by $P(A) = \displaystyle\int_A f\ d\lambda^k$ for all $A \in \mathcal{B}$, then $f \in \frac{dP}{d\lambda^k}$ is the density of P.

C.2 Other Forms

As the RNT is a significant theorem in numerous fields, it takes on different forms in different subjects. Some forms will be developed later in this work, but for the sake of illustration, the following some forms that are more immediate from the classic statement of the theorem.

Theorem C.3 (RNT for Signed Measures)**.** *Let μ be a σ-finite measure on (Ω, Σ) and ν a σ-finite signed measure on (Ω, Σ) such that $\nu << \mu$. Then there exists a μ-integrable function $f : \Omega \to \mathbb{R}$ such that $\nu(A) = \displaystyle\int_A f d\mu$ for all $A \in \Sigma$. Furthermore, f is unique modulo alterations on μ-null sets.*

The function f is simply $f_{\nu^+} - f_{\nu^-}$.

Theorem C.4. *Let (Ω, Σ, μ) be a finite measure space and ν a complex valued measure on Σ where $\nu << \mu$. Then there is a unique μ-integrable function f such that $\nu(A) = \displaystyle\int_A f d\mu$ for all $A \in \Sigma$.*

C.3 Some Significant Applications

C.3.1 Derivatives

The RNT introduces a notion of derivative of a (possibly signed) σ-finite measure ν with respect to a measure μ. Hence when $(\Omega, \Sigma, \mu) = (\mathbb{R}^n, \mathcal{B}, \lambda)$, we can define the derivative of a measure ν with respect to μ at the point $x \in \mathbb{R}^n$ as

$$F(x) = \lim_{r \to 0} \frac{\nu(B_r(x))}{\lambda(B_r(x))}$$

provided the limit exists. If $\nu << \lambda$, then for some λ-integrale f, $d\nu = f\ d\lambda$. Then $\dfrac{\nu(B_r(x))}{\lambda(B_r(x))}$ is the average value of f on $B_r(x)$. If we assume ν finite, then we are guaranteed $F = f$ a.e. with respect to λ. When $F = f$ a.e. with respect to λ, we get something like a generalization of the Fundamtal Theorem of Calculus: the derivative of the indefinite integral of f (namely ν) is f. [6]

C.3.2 Conditional Expectation

In the realm of Probability, the RND is not only the source of density functions, but also a primary instance of conditional expectation.

Definition C.5. Given a probability space (Ω, Σ, P), $\Sigma_0 \subseteq F$ a σ-field, and $X \in \Sigma$ a random variable with $E|X| < \infty$, the **conditional expectation**, $E(X|\Sigma_0)$ is the class of random variables Y such that $Y \in \Sigma_0$ and for all $A \in \Sigma_0$, $\displaystyle\int_A X\ dP = \int_A Y\ dP$.

If we let $X \geq 0$ be a r.v. on the probability space (Ω, Σ, P), and define $\nu(A) = \int_A X\ dP$ for all $A \in \Sigma$, then ν is a measure by the the countable additivity over domains of integration (which comes from the Monotone Convergence Theorem and linearity of integration), and $\nu << P$ by definition. Then $\nu(A) = \displaystyle\int_A \frac{d\nu}{dP}\ dP$. Taking $A = \Omega$, we get $\frac{d\nu}{dP} \geq 0$ integrable and $\frac{d\nu}{dP}$ is a version of $E(X|\Sigma)$.

If X is not nonnegative, then simply let $\nu(A) = \displaystyle\int_A X^+\ dP - \int_A X^-\ dP$ for all $A \in \Sigma$ is a signed measure, for which we have defined a RND.

Appendix D

Vector Spaces

Definition D.1. A **vector space** X is an abelian group with a field $\mathbb{K}$ [1] and an associated scalar product $m : \mathbb{K} \times X \to X$ given by $\cdot(\alpha, x) = \alpha \cdot x$ for which the following hold for all $\alpha, \beta \in \mathbb{K}$ and $u, v \in X$:

1. $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$
2. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$
3. $(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$
4. $1 \cdot u = u$ where 1 is the multiplicative identity in $\mathbb{K}$.

These spaces are also called linear spaces; however, we will adhere to the "vector" terminology to save from confusion when we begin to define vector measures. We will assume that the field corresponding to X is $\mathbb{K}$ unless otherwise stated.

Definition D.2. Let S be a nonempty subset of a vector space X. The **span** of S ($\text{span}(S)$) is the collection of all linear combinations of vectors in S.

Definition D.3. The **closed span** of S, $\overline{\text{span}}(S)$, is the smallest closed linear ubspace containing the span of S.

Definition D.4. A subset A of a vector space X is **convex** if, given $x, y \in A$ and $0 \leq a \leq 1$, $ax + (1 - a)y \in A$.

In other words, for any pair of points in the space, the "line" between them is also contained in the space. A classic example of a convex set is $S^n \subseteq \mathbb{R}^{n+1}$.

[1] For our purposes, $\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$.

Definition D.5. If A is a subset of a linear space X, then the **convex hull** of A, $\mathrm{co}(A)$, is the intersection of all convex sets containing A:

$$\mathrm{co}(A) = \{\sum_{i=1}^{n} a_i x_i | x_i \in A, 0 \leq a_i \leq 1, \sum_{i=1}^{n} a_i = 1\}.$$

Definition D.6. If A is a subset of a linear topological space X, then the **closed convex hull** of A, $\overline{\mathrm{co}}(A)$, is the intersection of all closed convex sets containing A.

D.1 Some Significant Spaces

The following are several significant types vector spaces. But before we introduce them, we must first introduce the concept of Cauchy and completeness of a metric.

Definition D.7. A sequence (x_n) in a metric space X is **Cauchy** if for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, y_n) < \epsilon$.

Definition D.8. A metric is **complete** if every Cauchy Sequence converges with respect to that metric.

Intuitively, it is best to understand a complete space as a space with no "holes" in it.

Definition D.9. An F**-space** is a metric topological vector space X over $\mathbb{K}$ such that,

1. The metric on X is translation invariant.
2. Scalar multiplication, $\cdot : \mathbb{K} \times X \to X$, is continuous with respect to the metric on $\mathbb{K} \times X$ and X.
3. Vector addition $+ : X \times X \to X$, is continuous with respect to the metric on $X \times X$ and X.
4. X is complete.

Save the completeness requirement, all normed vector spaces satisfy all of these criteria. Now, for a normed vector space that is complete.

Definition D.10. A **Banach space** is a normed vector space that is complete with respect to the norm metric.

Note that a Banach Space is an F-space where $||\alpha x|| = |\alpha|||x||$ for all $x \in X$ and $\alpha \in \mathbb{K}$.

Definition D.11. A **Hilbert space** H is a vector space over $\mathbb{C}$ together with a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ (known as an inner product for H) for which for all $x, y, z \in H$ and $\alpha \in \mathbb{C}$

1. $\langle x, x \rangle = 0$ iff $x = 0$
2. $\langle x, x \rangle \geq 0$
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
4. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
5. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
6. If $(x_n) \in H$ and $\lim\limits_{n,m \to \infty} \langle x_n - x_m, x_n - x_m \rangle = 0$, then there is an $x \in H$ with $\lim\limits_{n \to \infty} \langle x_n - x, x_n - x \rangle = 0$.

Given the inner product on a Hilbert Space, we can define a norm on the space as

$||x|| = \sqrt{< x, x >}$.

D.1.1 Example Spaces

Next, we will define several significant spaces which will be alluded to throughout this text; all of these spaces are Banach spaces.

Let $\mathbb{K}$ be a scalar field (usually $\mathbb{R}$ or $\mathbb{C}$).

1. Let $1 \leq p < \infty$ and $n \in \mathbb{Z}^+$. Then $l_n^p = \{x = (a_1, ..., a_n) : a_i \text{ in } \mathbb{K} \text{ for } 1 \leq i \leq n\}$.

   The norm on l_n^p is $||x|| = \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p}$.

   Sometimes it is written as $l_n^p(\mathbb{K})$ to make the associated scalar field explicit.

2. Let $1 \leq p < \infty$, then $\ell^p = \{x = (a_i) \text{ in } \mathbb{K}\colon \sum_{i=1}^{\infty} |a_i|^p < \infty\}$.

   The the norm on ℓ^p is $||x|| = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}$.

3. Let $n \in \mathbb{Z}^+$. Then $l_n^p = \{x = (a_1, ..., a_n) : a_i \text{ in } \mathbb{K} \text{ for } 1 \leq i \leq n\}$ with norm $||x|| = \sup_{1 \leq i \leq n} |a_i|$.

4. $\ell^\infty = \{x = (a_i) \text{ in } \mathbb{K}\colon (a_i) \text{ is bounded }\}$. The norm on ℓ^∞ is given by $||x|| = \sup_i |a_i|$.

5. $c = \{x = (a_i) \text{ in } \mathbb{K} : a_i \to a \text{ for some } a \in \mathbb{K}\}$. The norm on c is given by $||x|| = \sup_i |a_i|$.

6. $c_0 = \{x = (a_i) \text{ in } \mathbb{K} : a_i \to 0\}$. The norm on c_0 is given by $||x|| = \sup_i |a_i|$.

7. $c_s = \{x = (a_i) \text{ in } \mathbb{K} : \sum a_i <\text{converges}\}$. The norm on c_s is given by $||x|| = \sup_n \left|\sum_{i=1}^{n} a_i\right|$.

8. If A is a nonempty set and X a normed vector space, then $B(A, X) : \{f \in X^A : \sup_{x \in A} ||f(x)|| < \infty\}$ is a vector space with norm given by $||f||_\infty := \sup_{x \in A} ||f(x)||$.

9. If S is a compact topological space, then $C(S) = \{f : S \to \mathbb{K} : f \text{ is continuous}\}$. The norm on $C(S)$ is given by $||f|| = \sup_{s \in S} |f(s)|$.

10. L^p spaces we will save for later.

To get an idea of basic open sets in l_n^p, consider the unit circle in $l_2^2(\mathbb{R})$ (i.e. the set $\{x \in \mathbb{R}^2 : ||x|| = 1 \text{ where } ||\cdot|| \text{ is the Euclidean norm}\}$).

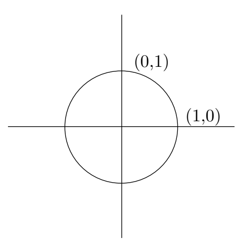


Figure D.1: Unit Circle for $l_2^2(\mathbb{R})$

And the unit circle for l_2^1:

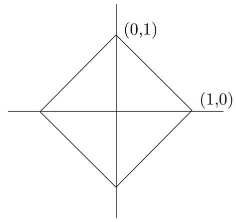


Figure D.2: Unit Circle for $l_2^1(\mathbb{R})$

And, finally, the unit circle for l_2^∞:

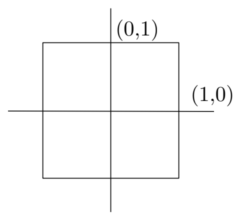


Figure D.3: Unit Circle for $l_2^\infty(\mathbb{R})$

Notice that all of these unit circles are convex.

D.2 L^p Spaces

We will now take a moment to consider L^p spaces in particular.

D.2.1 Definitions

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and (Ω, Σ, μ) be a measure space.

Consider the space $\mathcal{G}$ of μ-measurable $\mathbb{K}$-valued functions on (Ω, Σ, μ). We will (for the sake of norms soon to be defined) consider the equivalence relation $\simeq$ on $\mathcal{G}$ such that for $f, g \in \mathcal{G}$, $f \simeq g$ if $f = g$ a.e. on Ω. Then we partition $\mathcal{G}$ into a disjoint collection of equivalence classes, $\mathcal{G}/\simeq$. We will simply refer to the equivalence classes as functions and will denote them by f as opposed to $[f]$.

Then define $L^p(\Omega, \Sigma, \mu)$=$\{f \in \mathcal{G}/\simeq : \int |f|^p \, d\mu < \infty\}$.

Unless the particular measure space we are working over is relevant, we will just write L^p or $L^p(\mu)$.

The norm on L^p is given by $||f||_p = \left(\int |f|^p d\mu\right)^{1/p}$.

The first two requirements for a norm are proven readily (especially since the equivalance classes on $\mathcal{G}$ ensure that $||f-g||_p = 0$ iff $f-g=0$) from the linearity of integration and the absolute value norm on $\mathbb{K}$. The triangle inequality is satisfied by the Minkowski Inequality, which we will state later.

Next, we will introduce the L^∞ space. But first, a definition.

Definition D.12. If f is a measurable function on Ω, we say f is **essentially bounded** if there is a real $M \in \mathbb{R}_+$ such that $|f(x)| \leq M$ a.e.. In this case M is called an essential upper bound for f.

Then, define $L^\infty(\Omega, \Sigma, \mu)$=$\{f \in \mathcal{G}/\simeq : \ f$ is essentially bounded $\}$.
Unless the particular measure space we are working over is relevant, we will just write L^∞ or $L^\infty(\mu)$.

If f is a measurable function on Ω and $0 < p < \infty$, then $||f||_p = \left(\int |f|^p \, d\mu\right)^{1/p}$ (where $||f||_p$ can be infinite).

The norm on L^∞ is given by $||f||_\infty = \inf\{M \geq 0 : |f(x)| \leq M$ a.e. $\}$. Again, the first two requirements are met readily from the equivalence classes, absolute value norm, and linearity of integration; the triangle inequality falls from the fact that $||f||_\infty$ is an essential upper bound for f.

When Ω is countable and μ is the counting measure, then $L^p = \ell^p$ and $L^\infty = \ell^\infty$ where the $a_i \in \mathbb{K}$ are $f(x_i) \in \mathbb{K}$.

D.2.2 L^p for $0 \leq p < 1$

L^p and ℓ^p spaces are also defined for $0 < p < 1$. These spaces are defined in the same way as their $1 \leq p < \infty$ counterparts, i.e. $L^p(\Omega, \Sigma, \mu) = \{f \in \mathcal{G}/\simeq\, : \int |f|^p \, d\mu < \infty\}$. However, these spaces are not Banach (though they can be F-spaces). Whereas the basis for the L^p and ℓ^p spaces for $1 \leq p \leq \infty$ consists of convex sets, the basis for L^p and ℓ^p for $0 < p < 1$ consists of concave sets (i.e. non-convex sets).

The unit circle for $l_2^p(\mathbb{R})$ for $0 < p < 1$ looks like this:

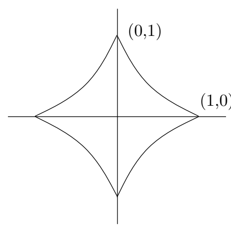


Figure D.4: Unit Circle for $l_2^p(\mathbb{R})$

Finally, we reach $L^0(\Omega, \Sigma, \mu)$, the collection of $\mathbb{K}$-valued, μ-measurable functions on Ω. This vector-space is extremely rich, but not normed. However, if we add the assumption that $\mu(\Omega) < \infty$, we can define a metric $d : L^0 \times L^0 \to \mathbb{R}$ on the space by $d(f, g) = \displaystyle\int |f - g| \wedge 1 d\mu$. This metric is translation invariat, but clearly does not have absolute homogeneity, and hence is not determined by a norm.

D.3 Linear Operators

Definition D.13. Let X and Y be vector spaces over the same field $\mathbb{K}$. An **operator** (or mapping or transformation) is a function from a linear subspace of X into Y.

Definition D.14. Let X and Y be vector spaces over $\mathbb{K}$. $T : X \to Y$ is a **linear operator** if for all $u, v \in X$ and $\alpha, \beta \in \mathbb{K}$, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$.

Definition D.15. Let X and Y be vector spaces. For linear operators $T, S : X \to Y$ and $\alpha, \beta \in \mathbb{K}$, define $\alpha T + \beta S : X \to Y$ pointwise by $(\alpha T + \beta S)(u) = \alpha T(u) + \beta S(u)$ for all $u \in X$.

Definition D.16. Let X and Y be normed vector spaces over $\mathbb{K}$. A linear operator $T : X \to Y$ is **bounded** if there exists an $M \in \mathbb{R}_+$ such that $||Tu|| \leq M||u||$ for all $u \in X$.

Let $\mathcal{L}\ (X, Y)$ denote the space of all bounded linear operators from X into Y.

Proposition D.17. *Let X and Y be normed vector spaces. Then $\mathcal{L}\ (X, Y)$ is a normed linear space with norm $|| \cdot ||_{\mathcal{L}} : \mathcal{L}(X, Y) \times \mathcal{L}(X, Y) \to [0, \infty)$ given by $||T||_{\mathcal{L}} = \inf\{M \in \mathbb{R}_+ : ||Tu|| \leq M||u|| \forall u \in X\}$.*

Proof. Since $\mathcal{L}(X, Y)$ is a linear subspace of Y^X, it remains to show that $|| \cdot ||_{\mathcal{L}(X,Y)}$ is a norm.

We will check the three criteria for a function to be a norm.

1. Clearly, $|| \cdot ||_{\mathcal{L}} \geq 0$. If $T \equiv 0$, then $\inf\{M \geq 0 : 0 \leq M||u|| \forall u \in X\} = 0$. If $\inf\{M \geq 0 : ||Tu|| \leq M||u|| \forall u \in X\} = 0$, then since $|| \cdot ||$ is a norm, $T \equiv 0$.

2. Let $\alpha \in \mathbb{F}$. Then,

   $||\alpha T||_{\mathcal{L}} = \inf\{M \geq 0 : ||\alpha Tu|| \leq M||u|| \forall u \in X\} = \inf\{M \geq 0 : |\alpha|||Tu|| \leq M||u|| \forall u \in X\} = \inf\{M \geq 0 : ||Tu|| \leq 1/|\alpha| M||u|| \forall u \in X\} = \inf\{|\alpha| M \geq 0 : ||Tu|| \leq M||u|| \forall u \in X\} = |\alpha|||T||_{\mathcal{L}}$.

3. Let $T, S \in \mathcal{L}\ (X, Y)$.

   Then for all $M \geq 0$ such that for all $u \in X$, $M||u|| \geq ||Tu|| + ||Su||$, $M||u|| \geq ||Tu|| + ||Su|| \geq ||Tu + Su|| \geq ||(T + S)u||$. Hence, $||T||_{\mathcal{L}} + ||S||_{\mathcal{L}} \geq ||T + S||_{\mathcal{L}}$.

□

Henceforth, if it is clear we are taking the norm of an element of $\mathcal{L}(X, Y)$, we will drop the $\mathcal{L}(X, Y)$ from the norm. Next, we give equivalent definitions of the norm on $\mathcal{L}\ (X, Y)$.

Proposition D.18. $||T|| = \inf\{M \in \mathbb{R}_+ : ||Tu|| \leq M||u||, \forall u \in X\}$

$$= \sup\{||Tu|| : u \in X, ||u|| \leq 1\}$$

$$= \sup\{||Tu|| : u \in X, ||u|| = 1\}$$

$$= \sup\{\tfrac{||Tu||}{||u||} : u \in X, u \neq 0\}$$

Proof.

1. $\sup\{||Tu|| : u \in X, ||u|| \leq 1\} \geq \sup\{||Tu|| : u \in X, ||u|| = 1\}$

   This comes from the fact that $\{||Tu|| : u \in X, ||u|| = 1\} \subseteq \{||Tu|| : u \in X, ||u|| \leq 1\}$.

2. $\sup\{||Tu|| : u \in X, ||u|| = 1\} \geq \inf\{M \geq 0 : ||Tu|| \leq< M||u||, \forall u \in X\}$.

   Comes from $\{M \geq 0 : ||Tu|| \leq< M||u||, \forall u \in X\} \supseteq \{M \geq 0 : ||Tu|| \leq< M, u \in X, ||u|| = 1\} = \{||Tu|| : u \in X, ||u|| = 1\}$.

3. $\sup\{||Tu|| : u \in X, ||u|| = 1\} \leq \sup\{\frac{||Tu||}{||u||} : u \in X, u \neq 0\}$

   Comes from $\{\frac{||Tu||}{||u||} : u \in X, u \neq 0\} \supseteq \{\frac{||Tu||}{||u||} : u \in X, ||u|| = 1\} = \{||Tu|| : u \in X, ||u|| = 1\}$.

4. $\inf\{M \geq 0 : ||Tu|| \leq< M||u||, \forall u \in X\} \geq \sup\{||Tu|| : u \in X, ||u|| = 1\}$

   Let $M \geq 0$ such that $||Tu|| \leq M||u||$ for all $u \in X$. Let $w \in X$ such that $||w|| = 1$. Then, $||Tw|| \leq M||w|| = M$. Thus, the claim holds.

5. $\sup\{\frac{||Tu||}{||u||} : u \in X, u \neq 0\} \leq \sup\{||Tu|| : u \in X, ||u|| = 1\}$

   Let $x \in X$ with $x \neq 0$ and $z = \frac{x}{||x||}$. Then $||z|| = 1$ and $\frac{||Tx||}{||x||} = ||Tz||$. Thus, the claim holds.

6. $\sup\{||Tu|| : u \in X, ||u|| \leq 1\} \leq \sup\{||Tu|| : u \in X, ||u|| = 1\}$

   Let $x \in X$ with $||x|| \leq 1$. Then by (5), $||Tx|| \leq \sup\{||Tu|| : u \in X, ||u|| = 1\} \cdot ||x|| \leq \sup\{||Tu|| : u \in X, ||u|| = 1\}$. Thus, the claim holds.

□

One of the more valuable characterizations of linear operators is given by the following theorem.

Theorem D.19. *Let X and Y be normed linear spaces and $T : X \to Y$ be a linear operator. Then, the following are equivalent:*

1. *T is bounded.*

2. *$diam(T(\{x \in X : ||x||_X = 1\})) < \infty$.*

3. *$diam(T(\{x \in X : ||x||_X \leq 1\})) < \infty$.*

4. *There exists a nonempty $A \subseteq X$ such that $diam T(A) < \infty$.*

5. *T is continuous at 0.*

6. *T is Lipschitz continuous.*

Proof. (1) $\Rightarrow$ (2)

Suppose T is a bounded linear operator. Then there is some $c \in \mathbb{R}_+$ such that $||Tx||_Y \leq c||x||_X$ for all $x \in X$. Let $S^X = \{x \in S : ||x||_X = 1\}$. Note that for all $x \in S^X$, $||Tx||_Y \leq c||x||_X = c$. Hence $T(S^X) \subseteq \overline{B}_c(0)$.

(2) $\Rightarrow$ (3)

Suppose $T(S^X) \subseteq \overline{B}_r^Y(0)$ for some $r \in \mathbb{R}_+$. Let $x \in \overline{B}^X$, and assume $x \neq 0$ (since $T0 = 0$). Then $\frac{x}{||x||_x} \in S^X$, and so $T(\frac{x}{||x||_X}) \in \overline{B}_r^Y(0)$. Then, $||Tx||_y + ||\ ||x||_X T(\frac{x}{||x||_X})||_Y = \|x\|_X \|T(\frac{x}{\|x\|_X})\|_Y \leq \|T(\frac{x}{\|x\|_X})\|_Y \leq r$. Hence $T(\overline{B}^X)$ is bounded.

(3) $\Rightarrow$ (4) Immediate

(4) $\Rightarrow$ (5)

Suppose G is a nonempty subset of X and $T(G)$ is bounded in Y. Then $T(G) \subseteq \overline{B}_r^Y(0)$ for some $r \in \mathbb{R}_+$. Let $x_0 \in G$ and $\rho \in \mathbb{R}_+$ such that $\mathring{B}_\rho^X(x_0) \subseteq G$. Then $T(\mathring{B}_\rho^X(x_0)) \subseteq \overline{B}_r^Y(0)$. So, for all $x \in \mathring{B}_\rho^X(x_0)$, $||Tx||_Y \leq r$. Now, let $\epsilon > 0$, $0 < \delta < \frac{\epsilon\rho}{2r}$, and let $x \in \mathring{B}_\delta^Y(0)$. Then $x_0 + \frac{\rho}{\delta}x \in \mathring{B}_\rho^X(x_0)$. Hence $\|T(x_0 + \frac{\rho}{\delta}x)\|_Y \leq r$. But then,

$\|Tx\|_Y = \|\frac{\delta}{\rho}T(\frac{\rho}{\delta}x)\|_Y = |\frac{\delta}{\rho}|\|T(\frac{\rho}{\delta}x + x_0) - Tx_0\|_Y \leq \frac{\delta}{\rho}\|T(\frac{\rho}{\delta}x + x_0)\|_Y + \|Tx_0\|_Y \leq 2\frac{\delta}{\rho}r < \epsilon$.

Since $T0 = 0$, T is continuous at 0.

(5) $\Rightarrow$ (6).

Suppose T is continuous at 0, then choose $\delta > 0$ such that $T(\mathring{B}_\delta^X(0)) \subseteq \mathring{B}_1^Y(0)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Then $\frac{x_1 - x_2}{\|x_1 - x_2\|_X}\frac{\delta}{2} \in \mathring{B}_\delta^X(0)$. Hence, $\|T(\frac{x_1 - x_2}{\|x_1 - x_2\|_X}\frac{\delta}{2})\|_Y \leq 1$. Then,

$\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y = \|\frac{2\|x_1 - x_2\|_X}{\delta} T(\frac{x_1 - x_2}{\|x_1 - x_2\|_X}\frac{\delta}{2})\|_Y$

$= \frac{2\|x_1 - x_2\|_X}{\delta}\|T(\frac{x_1 - x_2}{\|x_1 - x_2\|_X}\frac{\delta}{2})\|_Y \leq 2/\delta\|x_1 - x_2\|_Y$.

(6) $\Rightarrow$ (1)

Lipschitz continuity with $x_2 = 0$ gives boundedness. □

A slightly larger class of linear operators is the class of closed linear operators.

Definition D.20. Let X and Y be vector spaces over $\mathbb{K}$, and let $T : \mathrm{D}(T) \subseteq X \to Y$ be a linear operator. Then T is called a **closed operator** if its graph $\{(x, Tx) : x \in \mathrm{D}(T)\}$ is closed in $X \times Y$.

Proposition D.21. *Let T be a linear operator from its domain, $D(T) \subseteq X$ into Y. Then, T is closed iff given (x_n) in $D(T)$ such that x_n converges to $x \in X$ and Tx_n converges to $y \in Y$, then $x \in D(T)$ and $Tx = y$.*

Notice that a continuous linear operator defined on X is closed, but a closed linear operator need not be continuous. (By the Closed Graph Theorem, if X and Y are Banach spaces, then a linear operator $T : X \to Y$ is closed iff it is continuous.) Usually, we will require that T be densely defined, i.e. that $\mathrm{D}(T)$ is dense in X.

D.3.1 Dual Spaces

Definition D.22. Given a vector space X over $\mathbb{K}$, a **linear functional** is a linear operator $T : X \to \mathbb{K}$.

Definition D.23. The **dual** of a vector space X is the collection of all bounded linear functionals on X. The dual of X is denoted X^* and its elements are usually denoted x^*.

By the previous Proposition, since $\mathbb{K}$ is a field it is a vector space over itself; therefore X^* is a vector space with the operator norm. Furthemore, if $\mathbb{K}$ is a Banach Space, then so is

X^*. As for examples of Linear Operators, Linear Functionals, and Dual Spaces; an excellent example of each can be found in the construction of the duals of L^p spaces.

For an example, consider the duals of the L^p spaces, where $1 \leq p < \infty$.

We begin by defining a linear functional $T_f : L^p(\Omega, \mu) \to \mathbb{K}$ with respect to a function $f \in L^q(\Omega, \mu)$ (where q is the conjugate of p) by

$$T_f(g) = \int fg \, d\mu \text{ for all } g \in L^p(\Omega, \mu).$$

Hölder's Inequality grants that T_f is a bounded linear functional (and is hence a member of $(L^p)^*$) and that $||T_f|| = ||f||_q$. Hence, the linear operator $T : L^q \to (L^p)^*$, give by $f \mapsto T_f$ for all $f \in L^q$, is an isometry. The isomorphism is due to Riez, and so the following theorem is often attributed to Riez:[17]

Theorem D.24 (The Riesz Representation Theorem for the Dual of $L^p(\Omega, \mu)$)**.** *Let (Ω, Σ, μ) be a σ-finite measure space, $1 \leq p < \infty$, and q the conjugate of p. Then T is an isometric isomorphism of L^q onto $(L^p)^*$ (where T and T_f are defined as above for all $f \in L^q$).*

In other words, we consider $(L^p)^* = L^q$ for $1 \leq p < \infty$, and by the symmetry of the conjugate relationship, we also have $(L^q)^* = L^p$ (for $p \neq 1, \infty$). Hence, $(L^p)^{**} = L^p$! We would like for the same to hold true for $p = 1$ or $p = \infty$. But, so far, all we have is that $(L^1)^* = L^\infty$.

So, is L^∞ the pre-dual of L^1?

It turns out that the dual of L^∞ is isometric and isomorphic to the normed linear space of bounded finitely additive signed measures on Σ that are absolutely continuous with respect to μ. In fact, (a fact not easily verified) L^1 does not have a pre-dual!

D.3.2 More on Operators

The following definitions are from Linear Operators by Dunford and Schwartz

Definition D.25. The **uniform operator topology** in $\mathcal{L}(X, Y)$ is the metric topology of $\mathcal{L}(X, Y)$ induced by the operator norm.

Definition D.26. The **strong operator topology** on $\mathcal{L}(X,Y)$ is defined by the basis

$$\mathcal{B} = \{N(T;A,\epsilon) = \{R \in \mathcal{L}(X,Y) : \|(T-R)x\| < \epsilon, x \in A\} : A \subseteq X \text{ finite and } \epsilon > 0\}.$$

Definition D.27. The **weak operator topology** on $\mathcal{L}(X,Y)$ is defined by the basis

$\mathcal{B} = \{N(T;A,B,\epsilon) = \{R \in \mathcal{L}(X,Y) : |y^*(T-R)x| < \epsilon, y^* \in B, x \in A\} : A \subseteq X, B \subseteq Y^* \text{ finite and } \epsilon > 0\}$

As far as relationships between the three go, the weak operator topology is contained in the strong operator topology, which is contained in the uniform operator topology.

Now that we have defined the dual of a Banach space, we can define the weak topology on a Banach Space X.

Definition D.28. The **weak topology** of X is the inverse image topology generated by X^*.[2]

With our topologies defined, we may now name two particular types of operators that will be used in the course of this paper.

Definition D.29. Let $T \in \mathcal{L}(X,Y)$ and S the closed unit sphere in X. T is a **compact linear operator** if the strong closure of TS is compact in the norm topology of Y.

Equivalently, $T \in \mathcal{L}(X,Y)$ is compact iff the image of any bounded set in X is relatively compact in Y, i.e. its closure is compact in Y.

Definition D.30. Let $T \in \mathcal{L}(X,Y)$ and S the closed unit sphere in X. T is a **weakly compact linear operator** if the weak closure of TS is weakly compact, i.e. it is compact in the weak topology of Y.

Equivalently, $T \in \mathcal{L}(X,Y)$ is weakly compact iff the image of any bounded set in X is weakly sequentially compact, i.e. every sequence (y_n) in B contains a subsequence (y_n') which converges weakly to a point $y \in Y$, i.e. for all $y^* \in Y^*$, $\lim_n y^* y_n' = y^* y$.

[2] It is the smallest topology on X with respect to which all x^* in X^* are continuous.

D.4 Operators in a Hilbert Space

Some of the definitions and propositions in this section can be generalized. However, we present them in the form that will be relevant for this particular work. Let $\mathbb{H}$ be a Hilbert space.

Definition D.31. A **projection** $P : \mathbb{H} \to \mathbb{H}$ is a linear operator onto a subset of $\mathbb{H}$ such that $P^2 = P$

Proposition D.32. *A projection $P : \mathbb{H} \to \mathbb{H}$ maps $\mathbb{H}$ onto a closed linear subspace of $\mathbb{H}$ iff P is bounded.*

Definition D.33. Two elements x and y of $\mathbb{H}$ are **orthogonal** if $\langle x, y\rangle = 0$

Definition D.34. Given a closed linear subspace E of $\mathbb{H}$, the **orthogonal complement** of E, $E^\perp$ is the collection of all elements of $\mathbb{H}$ that are orthogonal to all members of E.

Definition D.35. Two subspaces E and F of $\mathbb{H}$ are **orthogonal** if $E \subseteq F^\perp$ (and, consequently, $F \subseteq E^\perp$).

Proposition D.36. *For each closed subspace E of $\mathbb{H}$, there exists a unique projection $P_E = proj_E$ that maps $\mathbb{H}$ onto E and is defined such that for each $x \in \mathbb{H}$, $P_E(x)$ is the unique element in $\mathbb{H}$ such that $(x - P_E(x)) \in E^\perp$. Furthermore, if P is a continuous projection, then $R(P)$ is closed and $R(P) \oplus R(P)^\perp = \mathbb{H}$.*

Definition D.37. An **orthogonal projection** is a projection such that $N(P)$ and $R(P)$ are orthogonal.

Definition D.38. Let $T : \mathbb{H} \to \mathbb{H}$ be a linear operator. The **adjoint** T^* of T is the unique operator on $\mathbb{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathbb{H}$.

Definition D.39. A linear operator $T : \mathbb{H} \to \mathbb{H}$ is **self-adjoint** if $T = T^*$.

Definition D.40. A bounded self-adjoint linear operator $T : \mathbb{H} \to \mathbb{H}$ is called **Hermitian**. A bounded linear operator that commutes with its adjoint (i.e. $TT^* = T^*T$) is called **normal**.

A Hermitian operator is characterized by having real eigenvalues, whereas normal operators may have complex eigenvalues. (In finite dimensional cases, where operators are matrices, normal matrices are guaranteed to be diagonalizable.)

Proposition D.41. *A bounded projection T is orthogonal iff it is self-adjoint*[3].

Proof. Let $T : \mathbb{H} \to \mathbb{H}$ be a bounded projection.

$\Rightarrow$ Suppose T is orthogonal, and let $x, y \in \mathbb{H}$. Then there exist $u, v \in R(T) \perp$ such that $x = u + Tx$ and $y = v + Ty$. Then,

$$\langle Tx, y\rangle = \langle Tx, v\rangle + \langle Tx, Ty\rangle = 0 + \langle Tx, Ty\rangle = \langle Tx, Ty\rangle$$

and

$$\langle x, Ty\rangle = \langle u, Ty\rangle + \langle Tx, Ty\rangle = 0 + \langle Tx, Ty\rangle = \langle Tx, Ty\rangle$$

Hence, $T = T^*$

$\Leftarrow$ Suppose T is self-adjoint. Then, $N(T) = N(T^*) = R(T)^\perp$.

$\square$

Definition D.42. A **positive semi-definite** matrix/operator is an operator T such that for any nonzero $x \in \mathbb{H}$, $x^*\mathbb{H}x$ is real and nonnegative.

Definition D.43. The **trace** of an $n \times n$ matrix $A = [a_{ij}]$ is given by $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$. A matrix A has unit trace if $\mathrm{tr}(A) = 1$. More generally, the trace of a bounded operator T is given by $\mathrm{tr}(T) := \sum_k \langle Te_k, e_k\rangle$ where $\{e_k\}$ is an orthonormal basis of a separable hilbert space $\mathbb{H}$.

[3] An operator T is self-adjoint if $T = T^*$ where T^* is the adjoint of T, i.e. the unique operator $T^* : \mathbb{H} \to \mathbb{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

If we consider $B(\mathbb{H})$ (the collection of bounded operators on $\mathbb{H}$) a ring, then some classes of operators may be considered ideals. For example, the class of compact operators is an ideal and is denoted $\mathscr{I}_\infty$.

The ideal of trace class operators, $\mathscr{I}_1 = \{T \in \mathscr{I}_\infty : \text{tr}(T)\text{is absolutely convergent}\}$, a.k.a. Schatten operators, is the ideal of compact operators with finite trace.

Appendix E

Summability

In Chapter 1, we gave several characterizations of unconditional convergence. In this section, we will give the proofs of those theorems.

First, we will define summability in an F-space:

Definition E.1. A sequence $(x_k) \in X^{\mathbb{N}}$ is **summable** if for every $\epsilon > 0$, there is a finite set $K \subseteq \mathbb{N}$ such that, for every finite $L \subseteq \mathbb{N}$ that is disjoint with K, $\left\| \sum_{n \in L} x_k \right\| < \epsilon$.

Theorem E.2 (Orlicz)**.** *In a complete metrizable topological vector space, the following are equivalent:*

1. *(x_k) is summable.*
2. *$\sum_{k \in \mathbb{N}} x_k$ converges unconditionally in X.*
3. *$\sum_{k \in \mathbb{N}} s_k x_k$ converges for every sequence $(s_k) \in \{-1, 1\}^{\mathbb{N}}$.*

π. *$\sum_{k \in \mathbb{N}} s_k x_k$ converges for every sequence $(s_k) \in \{0, 1\}^{\mathbb{N}}$.*

Proof. Let (x_k) be a sequence in a complete metrizable t.v.s. X.

1. ((3) $\Rightarrow$ (π)) Suppose $\sum_{k \in \mathbb{N}} s_k x_k$ converges for every sequence $(s_k) \in \{-1, 1\}^{\mathbb{N}}$, and let $(s_k) \in \{0, 1\}^{\mathbb{N}}$.

   Let $(s'_k) \in \{-1, 1\}^{\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $s'_k := \begin{cases} -1 & : s_k = 0 \\ 1 & : s_k = -1 \end{cases}$.

   Then $\sum_{k \in \mathbb{N}} x_k$ and $\sum_{k \in \mathbb{N}} s'_k x_k$ converge, and hence so does

$$\sum_{k\in\mathbb{N}} x_k + \sum_{k\in\mathbb{N}} s'_k x_k = \sum_{k\in\mathbb{N}} 2s_k x_k = 2\sum_{k\in\mathbb{N}} s_k x_k.$$

Therefore, $\sum_{k\in\mathbb{N}} s_k x_k$ converges.

2. $((\pi) \Rightarrow (3))$ Suppose $\sum_{k\in\mathbb{N}} s_k x_k$ converges for every sequence $(s_k) \in \{0,1\}^{\mathbb{N}}$, and let $(s_k) \in \{-1,1\}^{\mathbb{N}}$.

   Let (s_k^-) and $(s_k^+) \in \{0,1\}^{\mathbb{N}}$ such that for each $k \in \mathbb{N}$,

$$s_k^- := \begin{cases} 0 & : s_k = 1 \\ 1 & : s_k = -1 \end{cases} \quad \text{and} \quad s_k^+ := \begin{cases} 1 & : s_k = 1 \\ 0 & : s_k = -1 \end{cases}.$$

   Then $\sum_{k\in\mathbb{N}} -s_k^- x_k$ and $\sum_{k\in\mathbb{N}} s_k^+ x_k$ converge, and hence so does

$$\sum_{k\in\mathbb{N}} -s_k^- x_k + \sum_{k\in\mathbb{N}} s_k^+ x_k = \sum_{k\in\mathbb{N}} s_k x_k.$$

3. $((1) \Rightarrow (2))$ Suppose (x_k) is summable and $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation on $\mathbb{N}$. Let $\epsilon > 0$ and $K \subseteq \mathbb{N}$ finite such that for any finite subset L of $\mathbb{N}$ that is disjoint with K, $\left\|\sum_{k\in L} x_k\right\| < \epsilon$. Let $N \in \mathbb{N}$ such that $\sigma(k) \notin K$ for all $k \geq N$.

   Then, for all $m \geq n \geq N$, $\left\|\sum_{k=n}^{m} x_{\sigma(k)}\right\| < \epsilon$.

4. $((2) \Rightarrow (\pi))$ Suppose that there exists a sequence $(s_k) \in \{0,1\}^{\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} s_k x_k$ does not converge. Then there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n, m \geq N$ such that $\left\|\sum_{k=n^m} s_k x_k\right\| \geq \epsilon$.

   Let $n_1, m_1 \geq 1$ such that $\left\|\sum_{k=n_1}^{m_1} s_k x_k\right\| \geq \epsilon$.

   Let $N_2 = m_1 + 1$ and $n_2, m_2 \geq N_2$ such that $\left\|\sum_{k=n_2}^{m_2} s_k x_k\right\| \geq \epsilon$.

   Then, for all $j \geq 2$, define $N_j := m_{j-1}+1$ and let $m_j, n_j \geq N_j$ such that $\left\|\sum_{k=n_j}^{m_j} s_k x_k\right\| \geq \epsilon$.

Define $M_j := \{x_k : n_j \leq k \leq m_j \text{ and } s_k = 1\}$. (Note that the M_j's are disjoint.) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation that maps each M_j onto a block of consecutive numbers. Then $\sum_{k\in\mathbb{N}} x_{\sigma(k)}$ diverges, and hence (x_k) is not unconditionally convergennt.

5. $((\pi) \Rightarrow (1))$ Suppose there exists an $\epsilon > 0$ such that for all finite $K \subseteq \mathbb{N}$ there exists a finite $L \subseteq \mathbb{N}$ disjoint with K such that $\left\|\sum_{k\in L} x_k\right\| \geq \epsilon$.

   Let $K_0 = \{1\}$ and $K_0' \subseteq \mathbb{N}$ disjoint with K_0 such that $\left\|\sum_{k\in K_0'} x_k\right\| \geq \epsilon$.

   Let $K_1 = K_0 \cup K_0'$ and $K_1' \subseteq \mathbb{N}$ disjoint with K_1 such that $\left\|\sum_{k\in K_1'} x_k\right\| \geq \epsilon$.

   Let $K_n := K_{n-1}' \cup K_n$ and $K_n' \subseteq \mathbb{N}$ disjoint with K_n such that $\left\|\sum_{k\in K_n'} x_k\right\| \geq \epsilon$.

   Let $K := \bigcup_{k\in\mathbb{N}} K_n$, and define $(s_k) \in \{0,1\}^{\mathbb{N}}$ by $s_k := \begin{cases} 0 & : k \notin K \\ 1 & : k \in K \end{cases}$.

   Then, $\sum_{k\in\mathbb{N}} s_k x_k$ diverges.

□

Theorem E.3. *Let X be a Banach space. Then $(x_k) \in X^{\mathbb{N}}$ is summable iff $\sum_{k\in\mathbb{N}} s_k x_k$ where (s_k) is any bounded sequence of scalars.*

Proof. The $(\Leftarrow)$ implication is immediate from the preceding theorem since any sequence $(s_n) \in \{0,1\}^{\mathbb{N}}$ is bounded. For the other direction, let $(a_k) \in \ell^\infty$. Let $\epsilon > 0$, $K \subseteq \mathbb{N}$ finite such that for all finite $L \subseteq \mathbb{N}$ that are disjoint with K, $\left\|\sum_{k\in L} x_k\right\| < \epsilon$, and $N \in \mathbb{N}$ such that $k < N$ for all $k \in K$. Let $m \geq n \geq N$. Then,

$$\left\|\sum_{k=n}^{m} a_k x_k\right\| = \sup_{x^*\in \overline{B}^{X^*}} \left|x^*\left(\sum_{k=n}^{m} a_k x_k\right)\right| = \sup_{x^*\in \overline{B}^{X^*}} \left|\sum_{k=n}^{m} a_k x^*(x_k)\right| \leq \|(a_k)\|_\infty \sup_{x^*\in \overline{B}^{X^*}} \sum_{k=n}^{m} |x^*(x_k)|.$$

Let $x^* \in \overline{B}^{X^*}$. Since $|x^*(x)| \leq |\mathrm{Re}(x^*(x))| + |\mathrm{Im}(x^*(x))|$ for all $x \in X$, we may assume WLOG that $x^* : X \to \mathbb{R}$. Define $E^+ := \{k \in \mathbb{N} : n \leq k \leq m \text{ and } x^*(x_k) \geq 0\}$ and

$E^- := \{k \in \mathbb{N} : n \leq k \leq m \text{ and } x^*(x_k) < 0\}$. Then,

$$\begin{aligned}
\sum_{k=n}^{m} |x^*(x_k)| &= \left|\sum_{k\in E^+} x^*(x_k)\right| + \left|\sum_{k\in E^-} x^*(x_k)\right| \\
&= \left|x^*\left(\sum_{k\in E^+} x_k\right)\right| + \left|\left(x^*(\sum_{k\in E^-} x_k\right)\right| \\
&\leq \left\|\sum_{k\in E^+} x_k\right\| + \left\|\sum_{k\in E^-} x_k\right\| < 2\epsilon.
\end{aligned}$$

$\square$

Appendix F

Harris Theorem

In Section 2 of [9], Harris gives the theorem that was restated in the section on random measures in Chapter 1:[1]

Let X be a separable metric space and (V_n) a sequence of proper open subsets of X such that $V_1 \subset V_1^- \subset V_2 \subset V_2^- ... \uparrow X$. Let M be the class of Borel measures ξ in X such that $\xi(V_n) < \infty$, $n = 1, 2, ...,$ and let C_V be the class of functions $f \in C(X)$ such that f is supported b some V_n. We can make M into a Polish space with convergence $\xi_n \to \xi$ iff $\xi_n(f) \to \xi(f)$ for all $f \in C_V$. Let $\mathcal{M}$ be the Borel sets in M, and let $\mathcal{A}$ the class of Borel sets A in X such that $A \subseteq V_n$ for some $n \in \mathbb{N}$.

Theorem F.1 (2.3)**.** *For each $A_1, ..., A_k \in \mathcal{A}$, $k = 1, 2, ...,$ let $Q(A_1, .., A_k; \cdot)$ be a probability measure in $\mathbb{R}^k$. Suppose the Q determine a stochastic process $\{\xi(A), A \in \mathcal{A}\}$ with values in $[0, \infty)$ such that ξ is a finitely additive random set function on $\mathcal{A}$. Suppose further that if $A_n \downarrow \emptyset$, where $A_n \in \mathcal{A}$, then $Q(A_n; \cdot)$ converges weakly to the unit step at 0. Then there is a unique probability measure P on $\mathcal{M}$ such that the joint distribution of each set $\{\xi(A_1), ...\xi(A_n)\}$ under P is $Q(A_1, ..., A_k; \cdot)$.*

[1] For a proof of the theorem, Harris directs us to the proof for finite random measures in his paper, "Counting Measures, Monotone Random Set Functions" in the same journal, Vol 10, Issue 2, pp 102-119.

Appendix G

Conclusions

The theory of vector measures and Bochner integration has a rich history in functional analysis, probability theory, Banach space theory, and several other fields. Random measures, in particular, whether in the narrow or wide sense, appear in numerous forms in probability theory. Our particular contribution to this chapter is the study of “toy” vector measures and other examples.

Bochner integrals, by Bochner’s characterization, generalize the class of Lebesgue integrable functions. As in the first chapter, the examples here (for Bochner and Pettis integrals) are our particular contribution to the material. Among several theorems that are generalized by Bochner integrals, we have the Radon Nikodým theorem, which fails to hold for all Banach spaces; this, in turn, yields the Radon Nikodým property. Although, with revision, we can give a generalized version of the Radon Nikodým theorem, the Radon Nikodým property and the numerous resulting theorems on Banach spaces is the upshot of our second chapter.

Quantum probability is a newer subject that uses the structure of a Hilbert space and language of probability to describe states, events, and observables in quantum mechanics. There are ready generalizations of these probabilities to quantum vector measures; however, our preliminary characterizations are merely a peek through a keyhole. Future endeavors are promising.

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