5-cycle systems

by

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Abstract

A k-cycle system of a multigraph G is an ordered pair (V, C) where V is the vertex set of G and C is a set of k-cycles, the edges of which partition the edges of G. A k-cycle system of λK_v is known as a λ-fold k-cycle system of order v. A k-cycle system (V, C) of λK_v is said to be enclosed (embedded) in a k-cycle system $(V \cup U, P)$ of $(\lambda + m)K_{v+u}$ if $C \subset P$ and $u, m \geq 1$ ($m = 0$ and $u \geq 1$). We settle the enclosing problem for λ-fold 5-cycle systems when $u = 1$ or 2. We settle the embedding problem for λ-fold 5-cycle systems except possibly in two cases. Other analogues of this are considered and consequently settled.

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Chapter 1
Introduction

The goal of this dissertation is to construct 5-cycle systems of a specific family of graphs using tools from design theory. With that said, we begin with an overview of the terminology in Section 1.1 and history in Section 1.2 that will set the foundation for the rest of the dissertation. At the end of Section 1.1 there will be a brief outline of the entire dissertation.

1.1 Definitions

A *graph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a set of *vertices* and $E(G)$ is a set of unordered pairs of vertices called *edges*. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. To simplify notation, V and E will be used instead of $V(G)$ and $E(G)$ respectively when the underlying graph is not important. Two vertices u and v are *adjacent* if there is an edge $e = \{u, v\} \in E(G)$; e is said to join these vertices. A graph with no loops or multiple edges is called a *simple graph*. On the other hand, a graph that does have loops or multiple edges is called a *multigraph*. When a graph on v vertices has exactly one edge between every pair of vertices, it is called a *complete graph* and is denoted K_v. The λ-*fold complete graph* on v vertices, denoted by λK_v, is the multigraph in which every edge of K_v is repeated λ times (see Figure 1.1). The graph $(\lambda+m)K_{v+u} - \lambda K_v$ is the graph formed from $(\lambda+m)K_{v+u}$ on the vertex set $V \cup U$ by removing the edges of the subgraph λK_v on the vertex set V where $|V| = v$ and $|U| = u$ (see Figure 1.2).

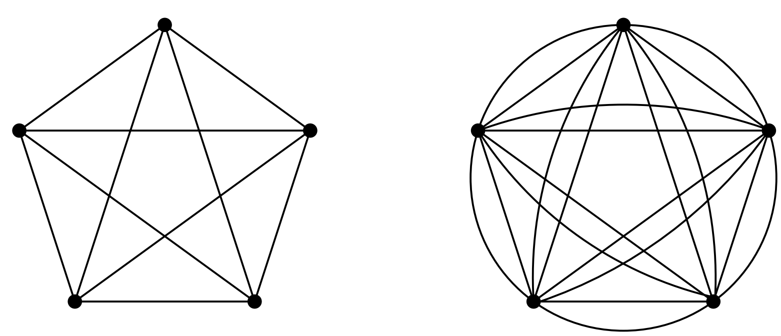

Figure 1.1: K_5 (left) and $2K_5$ (right)

It is easier to understand the structure of a graph by describing the properties of the edges and vertices of a graph. An edge $\{v_i, v_j\}$ is said to be *incident* to the vertices v_i and v_j. Two edges are incident if they share a common vertex. The

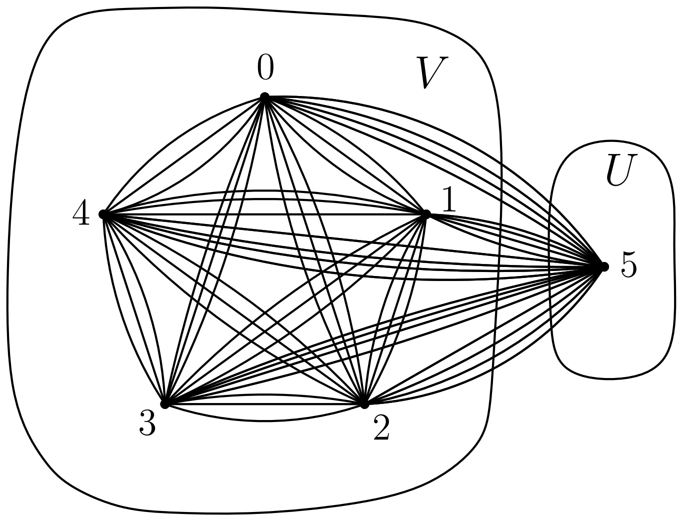

Figure 1.2: A $4K_6 - K_5$

degree of a vertex v_0 is the number of edges that are incident with v_0 and is denoted $\deg_G(v_0)$ or $\deg(v_0)$ when the underlying graph is not important. A sequence of vertices $(v_0, v_1, \ldots, v_n)$ such that $\{v_i, v_{i+1}\}$ is an edge for each $0 \leq i \leq n-1$ in the graph G is called a *trail*. A trail where $v_0 = v_n$ is called a *closed trail*. A trail where each vertex in the sequence is unique is called a *path*. Two vertices v_1 and v_2 in a graph are *connected* if there is a path between v_1 and v_2. If every pair of vertices is connected then the graph is said to be *connected*.

At times it will be convenient to talk about the set of integers from 0 to $k-1$. This notation is given as $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. A graph is *k-regular* if the degree of each vertex is k. A 2-regular connected graph is called a *cycle*. A cycle that contains every vertex in the graph G is called a *Hamilton cycle*. We will encounter situations where it will be very useful to talk about the length of a cycle. So a *k-cycle* $(v_0, v_1, \ldots, v_{k-1})$ is the graph on k distinct vertices $v_0, v_1, \ldots, v_{k-1}$ and with edge set $E = \{\{v_i, v_{i+1}\} \mid i \in \mathbb{Z}_k\}$, reducing sums modulo k. The graph in Figure 1.3 is a 7-cycle and a Hamilton cycle.

A *decomposition* of a graph G is a set of subgraphs that partition the edges of G. Typically, a graph is decomposed into copies of the same subgraph. This dissertation will only focus on the decomposition of graphs into cycles. A *k-cycle system*, k–CS, of a multigraph G is an ordered pair (V, C) where V is the vertex set of G and C is a set of k-cycles, the edges of which partition the edges of G. A k-cycle system of (a subgraph of) λK_v, k–CS(v, λ), is known as a (partial) *λ-fold k-cycle system of order v*. The graph in Figure 1.4 is a 5–CS of K_5 (i.e. a 1-fold 5-cycle system). When $k = 3$ and $\lambda = 1$, this structure is also called a *Steiner triple system* of order v, or an STS(v). A Steiner triple system is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a v-set of points and $\mathcal{B}$ is a collection of 3-subsets, called *blocks*, such that each pair of points of $\mathcal{V}$ is contained in exactly one block. The graph in Figure 1.5 is a decomposition of K_7 into 3-cycles

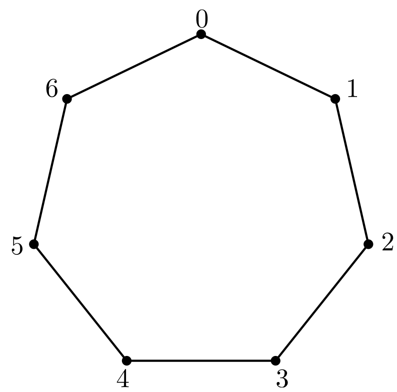

Figure 1.3: A 7-cycle (Hamilton cycle) with vertex set $\{0, 1, \ldots, 6\}$

(i.e. a Steiner triple system of order 7). Kirkman showed in [21] that there exists a Steiner triple system of order v if and only if $v \equiv 1$ or $3 \pmod 6$.

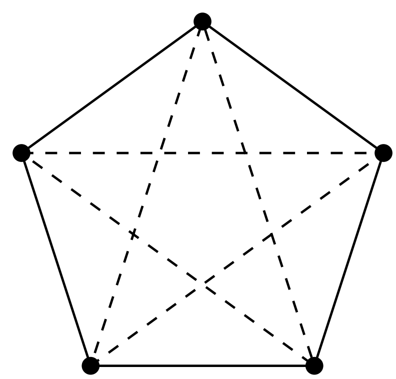

Figure 1.4: A 5–CS of K_5

A problem that has attracted much interest over the years is to find conditions which guarantee that a given (partial) k–CS of λK_v (V, C) is contained in a k–CS of $(\lambda + m)K_{v+u}$ $(V \cup U, P)$ with $C \subseteq P$. If $u, m = 0$ then (V, C) is said to be *completed* in $(V \cup U, P) = (V, P)$, if $m = 0$ then (V, C) is said to be *embedded*, if $u = 0$ then (V, C) is said to be *immersed*, and if $u, m \geq 1$ then (V, C) is said to be *enclosed* in $(V \cup U, P)$. This dissertation will focus exclusively on enclosings and embeddings. In particular, the focus will be placed on both the existence of enclosings of 5–CSs and the existence of 5–CSs of $(\lambda + m)K_{v+u} - \lambda K_v$. When finding such cycle systems, since the edges of λK_v have already been placed in the given k-cycles in C, it remains to find a k–CS of $(\lambda + m)K_{v+u} - \lambda K_v$. Figure 1.6 shows that a 5–CS of K_5 can be enclosed in a 5–CS of $4K_6$.

In order to attack these 5-cycle system problems, we need to construct 5-cycles in a flexible manner. To this end, Skolem-type sequences are used in a creative way. In 1957, while working on the construction of Steiner triple systems, Skolem published

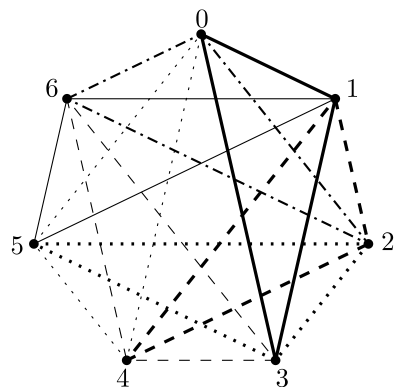

Figure 1.5: A Steiner triple system on 7 vertices (STS(7))

in [30] a construction that used sequences of integers with special properties. Later, an equivalent formulation in terms of sequences was made with an improvement on their construction in [1, 5, 26, 29]. Formally we define a *Skolem sequence* of order n as a sequence of integers $S = (d_1, \ldots, d_{2n})$ such that the following two conditions are met:

1. For every $\beta \in \{1, \ldots, n\}$ there exist exactly two elements $d_x, d_y \in S$ such that $d_x = d_y = \beta$, and
2. if $d_x = d_y = \beta$ with $x < y$, then $y - x = \beta$.

For example, $S = (1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$ is a Skolem sequence of order 5. Note that S can also be written as the collection of ordered pairs $\{(x_1, y_1) = (1, 2), (x_2, y_2) = (7, 9), (x_3, y_3) = (3, 6), (x_4, y_4) = (4, 8), (x_5, y_5) = (5, 10)\}$, the pairs being produced by the positions in S that contain the same integer; we refer to them as *Skolem pairs*.

An *extended Skolem sequence* $S = (d_1, \ldots, d_{2n+1})$ is defined in the same way as a Skolem sequence, with the added condition

3. there is exactly one $d_i \in S$ such that $d_i = 0$ (d_i is commonly called the *hook*).

A *hooked Skolem sequence* is an extended Skolem sequence where $d_{2n} = 0$.

The following existence results below for Skolem sequences and hooked Skolem sequences will assist us in our construction of 5-cycles.

Theorem 1.1.1. *[1] A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$.*

Theorem 1.1.2. *[26] A hooked Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$.*

One way to form cycle systems is to use differences. To explain how this is done, first let the set of vertices of a graph K_v be denoted as $\mathbb{Z}_v = \{0, 1, \ldots, v-1\}$. Let $d_v(i, j) = \min\{|i-j|, v - |i-j|\}$ where $\{i, j\}$ is an edge in K_v. We say that the

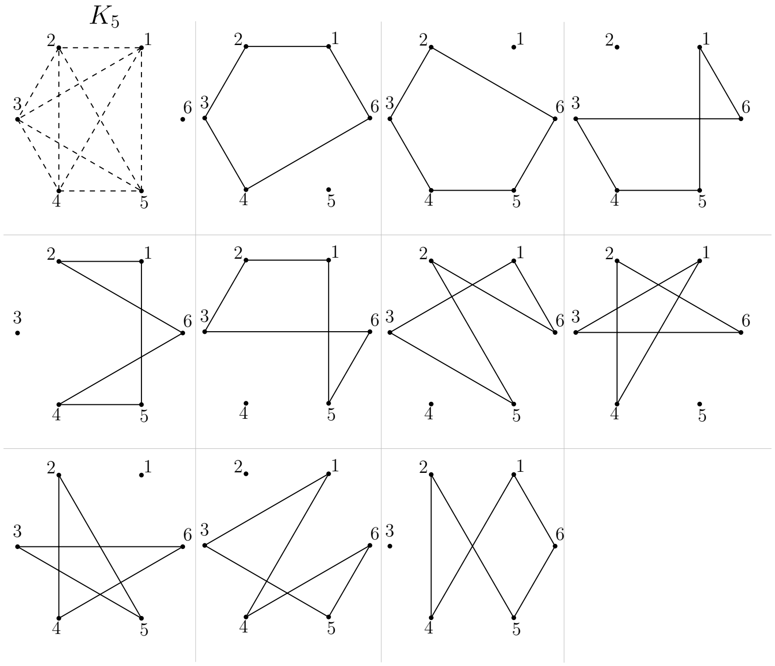

Figure 1.6: A 5–CS of K_5 is enclosed in a 5–CS of $4K_6$

edge $\{i, j\}$ has difference $d_v(i, j)$. So define $D(v) = \{1, 2, \ldots, \lfloor v/2 \rfloor\}$ to be the set of all differences of edges in K_v. If v is odd then there are v edges of each difference in $D(v)$. If v is even then there are v edges for each difference except the difference $v/2$. There are $v/2$ edges of difference $v/2$. Hence the difference $v/2$ is commonly referred to as the "half difference."

Consider the graph K_7 in Figure 1.5. The 3-cycle $c = (c_0, c_1, c_2) = (0, 1, 3)$ contains one edge of each difference 1, 2, and 3. We can then add 1 to each vertex in c to get another 3-cycle that does not contain any of the edges in c. This is no coincidence. The same is true if we add 2, 3, 4, 5, or 6 to c as long as the sums are reduced modulo $v = 7$. So $\{c + j \mid j \in \mathbb{Z}_7\}$ is a set of 3-cycles that is a decomposition of K_7, i.e. $\{(c_0 + j, c_1 + j, c_2 + j) \mid j \in \mathbb{Z}_7\}$ is a 3–CS of K_7 (see Figure 1.5 for a visual representation). This approach works well in general for cycles depending on the differences of the edges in the cycle. Since this method is a basic tool in the dissertation, the following notation will be useful. For any cycle $c' = (c_1, c_2, \ldots, c_x)$ let $c' + j = (c_1 + j, c_2 + j, \ldots, c_x + j)$, reducing the sums modulo v, and let $C + j = \{c' + j \mid c' \in C\}$.

One use of Skolem sequences is to build cyclic Steiner triple systems. A cyclic Steiner triple system is a Steiner triple system where for each block $b \in \mathcal{B}$, $b + 1 \in \mathcal{B}$. The graph in Figure 1.5 is an example of a cycle Steiner Triple System where $\{0, 1, 3\}$ would be the base block. The general construction of cyclic Steiner triple systems using Skolem sequences is split into two cases depending on the value of v.

Remember that there exists a Steiner triple system if $v \equiv 1$ or $3 \pmod 6$. Let

$G = K_{6n+1}$. So $\{1, 2, \ldots, 3n\}$ is the set of differences in G that represents the edges in G. We want to partition this set of differences into triples $\{a, b, c\}$ such that $a+b=c$ or $a+b+c \equiv 0 \pmod{6n+1}$. This is called the *first Heffter difference problem*. Form a Skolem or hooked Skolem sequence of order n such that (a_r, b_r) is the r^{th} Skolem pair. Then $T = \{(r, a_r+n, b_r+n) \mid 1 \leq r \leq n\}$ is a partition of $\{1, 2, \ldots, 3n\}$ and a solution to the first Heffter difference problem. Also, these differences can be used to form a 3–CS of K_{6n+1}; $\{T + j \mid j \in \mathbb{Z}_{6n+1}\}$.

Let $G = K_{6n+3}$. Then we want to partition $\{1, 2, \ldots, 3n+1\} \setminus \{2n+1\}$ into triples $\{a, b, c\}$ such that $a+b=c$ or $a+b+c \equiv 0 \pmod{6n+3}$. This is called the *second Heffter difference problem*. Notice that the difference $2n+1$ is removed so that the number of differences remaining is divisible by 3. Also, the different $2n+1$ can be used to form a 3-set by itself; $(0, 2n+1, 4n+2)$. Form an extended Skolem sequence of order n such that (a_r, b_r) is the r^{th} Skolem pair and the hook is in the n^{th} position. Then $T = \{(r, a_r+n, b_r+n) \mid 1 \leq r \leq n\}$ is a solution to the second Heffter difference problem. So $\{T + j \mid j \in \mathbb{Z}_{6n+3}\} \cup \{(0, 2n+1, 4n+1) + j \mid j \in \mathbb{Z}_{2n+1}\}$ is a 3–CS of K_{6n+3}.

The remainder of Chapter 1 will discuss the history of the decomposition problems related to the material presented in Chapters 2–5. In Chapter 2, the necessary conditions for the 5–CS enclosing problem will be established. Chapter 2 will also show that the necessary conditions for the 5–CS enclosing problem are sufficient when $u = 1$. Chapter 3 will show that the necessary conditions for the 5–CS enclosing problem are sufficient when $u = 2$. Chapters 4 and 5 will tackle the more general problem by establishing that the necessary conditions for the existence of a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$ are sufficient when $u = 1$ and $m = 0$ respectively except possibly in two cases in the latter case. Finally, Chapter 6 will look at open questions related to the work in Chapters 2-5 and possible future work related to finding 5–CS of $(\lambda+m)K_{v+u} - K_v$ and enclosings of 5–CSs. All other definitions not mentioned can be found in either "Design Theory" by Lindner and Rodger [22] or in "Introduction to graph theory" by West [31].

1.2 History

In 1981, Alspach conjectured that there exists a decomposition of K_v into t cycles of specified lengths $m_1, m_2, \ldots, m_t$ as long as v is odd, $3 \leq m_i \leq v$, and $m_1 + m_2 + \cdots + m_t = v(v-1)/2$ (see [2]). He also conjectured that there exists a decomposition of K_v into a 1-regular graph, say I, and t cycles of specified lengths $m_1, m_2, \ldots, m_t$ as long as v is even, $3 \leq m_i \leq v$, and $m_1 + m_2 + \cdots + m_t = v(v-1)/2 - v/2$. These conjectures are commonly referred to as the *Alspach Conjecture*. We shall focus our discussion on the former conjecture. Lately, there have been many papers published that aimed to solve this conjecture; most notably by Bryant and Horsley in the following result.

Theorem 1.2.1. [9] *There is an integer N such that for each integer $v \geq N$, there exists a decomposition of K_v into t cycles of lengths $m_1, m_2, \ldots, m_t$ if and only if*

1. v *is odd,*
2. $3 \leq m_1, m_2, \ldots, m_t \leq v$, *and*
3. $m_1 + m_2 + \cdots + m_t = v(v-1)/2$.

For the remainder of this dissertation we will focus on decompositions with cycles of the same length. Hoffman, Lindner, and Rodger were able to show that a K_v can be decomposed into m-cycles when $m \leq v \leq 3m$ and v is odd. By using this result and other methods, Alspach and Gavlas [3] were able to decompose $K_v - I$ or K_v into m-cycles for $3 \leq m \leq v$ when m and v are both even or both odd respectively. Later, Šajna [28] was able to decompose $K_v - I$ or K_v into m-cycles for $3 \leq m \leq v$ when m is odd and v even or m is even and v is odd respectively. This shows that there exists an m–CS of K_v for all values of m and v that satisfy the obvious necessary and sufficient conditions.

The next direction for this problem is to establish there exists a k–CS of λK_v for $\lambda > 1$. For a k–CS of λK_v, define v to be (k, λ)-*admissible* if the degree of each vertex is even, the number of edges in λK_v is divisible by k, and $v = 1$ or $v \geq k$. Through the efforts of Bermond, Hanani, Huang, Rosa, and Sotteau in [7, 8, 17, 18, 27], it was shown that for $3 \leq k \leq 14$ and $k = 16$, there exists a k–CS of λK_v if and only if v is (λ, k)-admissible. It was not until recently that Bryant, Horsley, Maenhaut, and Smith in [10] were able to show there exists a k–CS of λK_v if and only if v is (λ, k)-admissible. Of particular interest to this dissertation is the case when $k = 5$.

Theorem 1.2.2. [10] *A* 5–CS *of* λK_v *exists if and only if*

(a) $\lambda v(v-1) \equiv 0 \pmod 5$,
(b) $\lambda(v-1) \equiv 0 \pmod 2$, *and*
(c) either $v = 1$ *or* $v \geq 5$.

Since the immersing problem is exactly the same as showing there exists a k–CS of λK_v, this problem has already been completely solved by Bryant, Horsley, Maenhaut, and Smith in [10]. Since this paper focuses solely on building 5–CS, it will cause no confusion to refer to a $(\lambda, 5)$-admissible integer as simply being λ-admissible. Also λ is said to be v-admissible if the conditions in Theorem 1.2.2 are satisfied by λK_v. Some consequences of Theorem 1.2.2 are summarized in Table 1.1.

λ	Possible values of v
$0 \pmod{10}$	$v \notin \{2, 3, 4\}$
$1, 3, 7, 9 \pmod{10}$	$v \equiv 1, 5 \pmod{10}$
$2, 4, 6, 8 \pmod{10}$	$v \equiv 0, 1 \pmod 5$
$5 \pmod{10}$	$v \equiv 1 \pmod 2, v \neq 3$

Table 1.1: Necessary and sufficient conditions for the existence of a 5–CS(v, λ)

Doyen and Wilson were able to prove in 1973 the following result for embeddings on Steiner triple systems (i.e. embedding a 3–CS in another 3–CS); commonly called the *Doyen-Wilson theorem.*

Theorem 1.2.3. [16] *A Steiner triple system of order* u *can be embedded in a Steiner triple system of order* $v > u$ *if and only if* $v \geq 2u + 1$ *and* $v \equiv 1$ *or* $3 \pmod 6$*.*

Rodger and Bryant were able to extend the Doyen-Wilson theorem to hold for all odd k-cycle lengths when $k \leq 9$.

Theorem 1.2.4. [12, 13] *A* k–CS *of* K_v *can be embedded in a* k–CS *of* K_{v+u} *if and only if the following conditions hold:*

- *If* $k = 5$ *then* $v + u \equiv 1$ *or* $5 \pmod{10}$ *and* $u \geq v/2 + 1$*.*
- *If* $k = 7$ *then* $v + u \equiv 1$ *or* $7 \pmod{14}$ *and* $u \geq v/3 + 1$*.*
- *If* $k = 9$ *then* $v + u \equiv 1$ *or* $9 \pmod{18}$ *and* $u \geq v/4 + 1$*.*

Later, Bryant, Rodger, and Spicer were able to extend these results to embedding k-cycle systems when $k \leq 14$ in [14].

Some work has been seen in generalizing these results to establish the existence of a k–CS of $K_{v+u} - K_v$. Mendelsohn and Rosa were able to show the necessary conditions were sufficient for the existence of a 3-cycle system of $K_{v+u} - K_v$. Bryant, Hoffman and Rodger were able to establish the necessary and sufficient conditions for the existence of a 5-cycle system of $K_{v+u} - K_v$.

Theorem 1.2.5. [23] *There exists a* 3*-cycle system of* $K_{v+u} - K_v$ *if and only if* $u \geq v + 1$ *and*

1. $v + u, v \equiv 1$ *or* $3 \pmod 6$*, or*
2. $v + u \equiv v \equiv 5 \pmod 6$*.*

Theorem 1.2.6. [11] *There exists a* 5*-cycle system of* $K_{v+u} - K_v$ *if and only if* $u \geq v/2 + 1$ *and*

1. $v + u, v \equiv 1$ *or* $5 \pmod{10}$*, or*
2. $v + u \equiv v \equiv 3 \pmod{10}$*, or*
3. $v + u, v \equiv 7$ *or* $9 \pmod{10}$*.*

Again in [14], Bryant, Rodger, and Spicer were able to show there exists a k-cycle system of $K_{v+u} - K_v$ when $k = 4, 6, 7, 8, 10, 12,$ and 14.

Many papers have investigated the λ-fold 3–CS enclosing problem (see for example [15, 19, 20, 25]). The difficulty of this problem is clear since as yet there is no complete solution to this problem. Newman and Rodger [24] completely solved the problem of enclosing a λ-fold 4–CS of order v into a $(\lambda + m)$-fold 4–CS of order $v + u$ for all $u, m \geq 1$. Typically, problems involving odd cycles have many more difficulties than related even cycle problems, and that is certainly the case here. The techniques used in the subsequent chapters for the 5–CS enclosing problem are very different to those used in [24].

Chapter 2
Enclosings when $u = 1$

2.1 Introduction

In this chapter, we study the λ-fold 5–CS enclosing problem, settling the following conjecture in the case where $u = 1$ (see Theorem 2.3.2). The joint work in this chapter has been published in [4]. As in Lemma 2.1.2, all 5 conditions in this conjecture are necessary, i.e. the difficulty when studying Conjecture 2.1.1 lies in proving sufficiency.

Conjecture 2.1.1. *Let $u \geq 1$. Every 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda+m)K_{v+u}$ if and only if*

- *(a)* $(\lambda+m)u + m(v-1) \equiv 0 \pmod 2$,
- *(b)* $\binom{u}{2}(\lambda+m) + m\binom{v}{2} + vu(\lambda+m) \equiv 0 \pmod 5$,
- *(c) if $u = 1$, then $m(v-1) \geq 3(\lambda+m)$, and*
- *(d) if $u = 2$, then $m\binom{v}{2} - 2(\lambda+m) - (v-1)(\lambda+m)/2 \geq 0$.*
- *(e) if $u \geq 3$, then $\lceil vu(\lambda+m)/4 \rceil + 2\epsilon \leq mv(v-1)/2 + (\lambda+m)u(u-1)/2$ where $\epsilon = 0$ or 1 if $vu(\lambda+m) \equiv 0$ or $2 \pmod 4$ respectively.*

Lemma 2.1.2 establishes the necessary conditions in Conjecture 2.1.1.

Lemma 2.1.2. *Suppose there exists a 5–CS of λK_v. If this can be enclosed in a 5–CS of $(\lambda+m)K_{v+u}$, then:*

- *(a)* $(\lambda+m)u + m(v-1) \equiv 0 \pmod 2$,
- *(b)* $\binom{u}{2}(\lambda+m) + m\binom{v}{2} + vu(\lambda+m) \equiv 0 \pmod 5$,
- *(c) if $u = 1$, then $m(v-1) \geq 3(\lambda+m)$,*
- *(d) if $u = 2$, then $m\binom{v}{2} - 2(\lambda+m) - (v-1)(\lambda+m)/2 \geq 0$, and*
- *(e) if $u \geq 3$, then $\lceil vu(\lambda+m)/4 \rceil + 2\epsilon \leq mv(v-1)/2 + (\lambda+m)u(u-1)/2$ where $\epsilon = 0$ or 1 if $vu(\lambda+m) \equiv 0$ or $2 \pmod 4$ respectively.*

Proof. Suppose there exists a 5–CS (V, C_1) of λK_v that can be enclosed in a 5–CS $(V \cup U, C_2)$ of $(\lambda+m)K_{v+u}$. It is clear that each vertex in each of $(\lambda+m)K_{v+u}$ and λK_v has even degree in order for the 5–CSs of $(\lambda+m)K_{v+u}$ and λK_v to exist. So $(\lambda+m)(v+u-1)$ and $\lambda(v-1)$ are both even, proving (a). Since there exists a 5–CS of $(\lambda+m)K_{v+u}$ and a 5–CS of λK_v, the number of edges of each graph must be divisible by 5. Thus $(\lambda+m)u(u-1)/2 + (\lambda+m)v(v-1)/2 + (\lambda+m)uv$ and $\lambda v(v-1)/2$ are divisible by 5. So

$$(\lambda+m)\binom{u}{2} + (\lambda+m)\binom{v}{2} + (\lambda+m)uv - \lambda\binom{v}{2} = (\lambda+m)\binom{u}{2} + m\binom{v}{2} + vu(\lambda+m)$$
$$\equiv 0 \pmod 5.$$

Therefore condition (b) holds.

Suppose $u = 1$ and let $U = \{\infty\}$. There must be $\lambda + m$ edges connecting each vertex in V to ∞, so the degree of ∞ is $(\lambda+m)v$. Each 5-cycle in C_2 containing ∞ uses two of these edges, so the number of 5-cycles containing ∞ is $v(\lambda + m)/2$. Because mK_v has $m\binom{v}{2}$ edges, and since any 5-cycle containing ∞ uses 3 edges from this graph, it follows that the number of such 5-cycles in C_2 must be at most $mv(v-1)/6$, so $v(\lambda + m)/2 \leq mv(v-1)/6$. Thus condition (c) holds.

Suppose $u = 2$ and let $U = \{\infty_1, \infty_2\}$. An edge is *pure* if it joins two vertices in V or if it joins two vertices in U; otherwise it is said to be *mixed*. Each of the $\lambda + m$ edges joining ∞_1 to ∞_2 must be in a 5-cycle in C_2 that contains exactly 2 pure edges in mK_v and exactly 2 mixed edges. There are $2v(\lambda+m) - 2(\lambda+m) = 2(v-1)(\lambda+m)$ remaining mixed edges. Furthermore, the other 5-cycles in C_2 which contain at least one vertex in U must exhaust these remaining edges. Each 5-cycle in C_2 can use at most 4 of the mixed edges, and so each 5-cycle of this type uses at least one edge in mK_v. Thus it must be that

$$0 \leq m\binom{v}{2} - 2(\lambda + m) - \frac{2(v-1)(\lambda + m)}{4} = m\binom{v}{2} - 2(\lambda + m) - \frac{1}{2}(v-1)(\lambda + m).$$

Therefore condition (d) holds.

Suppose $u \geq 3$. Every 5-cycle in C_2 that contains a mixed edge must contain at least one pure edge, and at most 4 mixed edges. So there are at least $\lceil vu(\lambda + m)/4 \rceil$ 5-cycles in C_2 containing mixed edges. Furthermore, if $vu(\lambda + m) \equiv 2 \pmod 4$ then at least one such 5-cycle must contain 2 mixed and 3 pure edges. (Note that since each 5-cycle contains 0, 2, or 4 mixed edges it follows that $vu(\lambda + m)$ is even.) Thus, there must be at least $\lceil vu(\lambda+m)/4 \rceil + 2\epsilon$ pure edges in 5-cycles containing at least one vertex in U where $\epsilon \in \{0, 1\}$. So $\lceil vu(\lambda+m)/4 \rceil + 2\epsilon \leq mv(v-1)/2 + (\lambda+m)u(u-1)/2$ and condition (e) holds. □

Notice that if $u = 1$ and $\lambda = 0$ then the enclosing problem is equivalent to the existence of a 5–CS of mK_{v+1}, and Conditions Lemma 2.1.2(a-c) imply Conditions Theorem 1.2.2(a-c) with λ replaced by m and v by $v + 1$. So this case is already solved.

Section 2.2 addresses the cases when $v = 10k + \ell$ for $\ell \in \mathbb{Z}_{10}$ and m is small. We take the results from Section 2.2 and use them to prove our main result of this chapter, Theorem 2.3.2, in Section 2.3.

2.2 The small cases when $u = 1$

We now create the tools necessary to solve the problem of enclosing a λ-fold 5–CS of order v into a $(\lambda + m)$-fold 5–CS of order $v + u$ for all $m > 0$ and $u = 1$. For the remainder of this chapter, let $u = 1$ with $U = \{\infty\}$ being the vertex set of $V(K_{v+u}) \setminus V(K_v)$.

Recall that for any cycle $c = (c_1, \ldots, c_x)$ with entries in $\mathbb{Z}_v$, we define $c + j = (c_1 + j, \ldots, c_x + j)$, reducing the sums modulo v, and define $C + j = \{c + j \mid c \in C\}$. For any $D \subseteq \{1, \ldots, \lfloor v/2 \rfloor\}$, let $G_v(D)$ be the graph with vertex set $\mathbb{Z}_v$ and edge set $\{\{i, j\} \mid d_v(i, j) \in D\}$. Let $D(v) = \{1, \ldots, \lfloor v/2 \rfloor\}$.

Since we consider multigraphs in this dissertation, we need to introduce the multiset D_ℓ defined to contain ℓ copies of each element in D; then $G_v(D(v)_\ell)$ contains ℓ copies of each edge with difference in $D(v)$. When v and ℓ are both even, it is useful to let $D(v)^*_\ell$ be formed from $D(v)_\ell$ by removing $\ell/2$ copies of the difference $v/2$. This is very important notation because then the edge set of $\ell K_v = G_v(D(v)^*_\ell)$ can usefully be defined as

$$E(\ell K_v) = \{\{0, d\} + j \mid d \in D(v)^*_\ell, j \in \mathbb{Z}_v\}$$

since each occurrence of $v/2 \in D(v)^*_\ell$ gives rise to *two* copies of the edge $\{x, x + v/2\}$ in $E(G_v(D(v)_\ell))$ (namely when $j = x$ and when $d = x + v/2$). Another multiset convention we use is that for any positive integer t, tD is the multiset containing t copies of each element of D.

The following is a critical result, being used hand in hand with the observation that there exists a 5–CS of $G_v(\{v/5\})$ and of $G_v(\{2v/5\})$ when $v \equiv 0 \pmod 5$, since each component is a 5-cycle. So throughout this dissertation, when $v \equiv 0 \pmod 5$ let $(\mathbb{Z}_v, \Gamma_1)$, $(\mathbb{Z}_v, \Gamma_2)$ and $(\mathbb{Z}_v, \Gamma_3)$ be a 5–CS of $G_v(\{v/5\}), G_v(\{2v/5\})$, and $G_v(\{1, 2, 3\})$ respectively.

Lemma 2.2.1. *If 5 divides v, then there exists a 5–CS of $G_v(\{1, 2, 3\})$.*

Proof. $(\mathbb{Z}_v, \{(0, 1, 4, 5, 2) + 5j, (3, 4, 6, 8, 5) + 5j, (2, 3, 6, 7, 4) + 5j \mid j \in \mathbb{Z}_{v/5}\})$ is a 5–CS of $G_v(\{1, 2, 3\})$ since for each $d \in \{1, 2, 3\}$, the edges of differences d in the cycles $(0, 1, 4, 5, 2)$, $(3, 4, 6, 8, 5)$, and $(2, 3, 6, 7, 4)$ are of the form $\{x, x + d\}$ where each x is distinct modulo 5. □

The next lemma points to our main approach to proving Theorem 2.3.2, showing that the number of 5-cycles to be defined on the original v vertices during the enclosing is divisible by v. The parameter α will play a critical role throughout the rest of the dissertation.

Lemma 2.2.2. *Suppose there exists a 5–CS $(V \cup U, B)$ of $(\lambda + m)K_{v+1} - \lambda K_v$ where $U = \{\infty\}$. Then the number of edges in 5-cycles avoiding the vertex ∞ is $\alpha v = (m(v - 1)/2 - 3(\lambda + m)/2)v$.*

Proof. There are $mv(v - 1)/2$ edges in mK_v, $3v(\lambda + m)/2$ of which occur in 5-cycles in B, that contain the vertex ∞. So

$$m\frac{v(v-1)}{2} - \frac{3v(\lambda + m)}{2} = \left(\frac{m(v-1)}{2} - \frac{3(\lambda + m)}{2}\right) v = \alpha v \qquad (2.1)$$

is the number of edges in 5-cycles in B with all vertices in V. □

An integer x is said to be $\alpha(v,m)$-admissible if there exists an integer λ for which v is λ-admissible and for which Lemma 2.1.2(c) is satisfied by m, v and λ such that $\alpha = m(v-1)/2 - 3(\lambda+m)/2 = x$. (During the enclosing process, we usually fix v and m to consider all possible values of λ. Each such value of λ has an associated value of α, and the set of all such α values are the $\alpha(v,m)$-admissible integers.) The next lemma will be used when constructing the 5-cycles in the enclosing that contain the vertex ∞; each such 5-cycle uses exactly 3 edges joining vertices in mK_v.

Lemma 2.2.3. *Suppose that the Conditions Lemma* 2.1.2*(a-c) are satisfied and that there exists a* 5–CS *of* λK_v. *Then if* v *is odd, or if* v *is even and* m *is even,* 3 *divides* $m(v-1)/2 - \alpha$. *If* v *is even and* m *is odd then* 3 *divides* $m(v-1)/2 - \alpha - 3/2$.

Proof. If v is odd then by Lemma 2.1.2(a), $\lambda + m$ is even. If v is even then λ is even by Theorem 1.2.2(b) (since there exists a 5–CS of λK_v). Thus if v is odd or if v is even and m is even, then 3 divides $m(v-1)/2 - \alpha$. Now suppose v is even and m is odd. Then $m(v-1)/2 - \alpha - 3/2 = 3(\lambda + m - 1)/2$ is clearly divisible by 3, and otherwise $m(v-1)/2 - \alpha = 3(\lambda+m)/2$ is divisible by 3. □

The following result will be useful to refer to throughout the subsequent constructions. Note that it implies that all $\alpha(v,m)$-admissible integers are non-negative.

Lemma 2.2.4. *Suppose* $u = 1$. *Condition (c) Lemma* 2.1.2 *is equivalent to* $\alpha \geq 0$.

Proof. Clearly $m(v-1) \geq 3(\lambda+m)$ if and only if $m(v-1)/2 - 3(\lambda+m)/2 \geq 0$. □

The following lemma is constantly used in subsequent constructions.

Lemma 2.2.5. *If* $0 \leq d_1 \leq d_2 \leq d_3 \leq v/2$, $d_1 < d_3$, *and* $d_2 < v/2$ *then* $c = (d_1, 0, d_3, d_2 + d_3, \infty_i)$ *is a 5-cycle in* K_{v+u} *with vertex set* $\mathbb{Z}_v \cup \{\infty_0, \infty_1, \ldots, \infty_{u-1}\}$.

Proof. The 5 vertices in c are clearly distinct. □

For all positive integers v and m, let $\lambda_{\max}(v,m)$ and $\lambda_{\min}(v,m)$ be the largest and smallest integers for λ respectively satisfying Conditions (a-c) in Lemma 2.1.2 with $u = 1$.

It is probably no surprise that many larger enclosings can be found by combining two smaller enclosings. This observation is the basis of the inductive proof of Theorem 2.3.2. So now we focus on finding enclosings that cannot be found using this approach. This involves the cases where $m = 1$, and where $\lambda \in \{\lambda_{\max}(v,m), \lambda_{\min}(v,m)\}$ for some small values of m.

Lemma 2.2.6. *Let* $v = 10k$ *and assume conditions (a-c) of Lemma* 2.1.2 *are satisfied. If* $m = 1$, *or if* $m \in \{2,3\}$ *and* $\lambda = \lambda_{\max}(v,m)$, *then any* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda+m)K_{v+1}$.

Proof. Let $(V = \mathbb{Z}_{10k}, A)$ be a 5–CS of λK_v and let $U = \{\infty\}$. Since (V, A) exists, λ is even by Lemma 2.1.2(a).

First, note that there are $mv(v-1)/2$ edges in mK_v, $3v(\lambda+m)/2$ of which must be placed in 5-cycles that contain the vertex ∞. So we define αv as in Equation (2.1) of Lemma 2.2.2 to be the number of edges we must place in 5-cycles contained completely in mK_v. Since this is divisible by v, our plan is to create 5-cycles in mK_v using all the edges of α specially chosen differences. Note that $\alpha \geq 0$ by Lemma 2.2.4.

Suppose $m = 1$. Then $mK_v = K_v$ contains edges of each difference in $D(v) = \{1, \ldots, 5k\}$. To explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen), we make use of Skolem and hooked Skolem sequences.

Let $P' = \{(x_i', y_i') \mid 1 \leq i \leq k-1, y_i' > x_i'\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ from the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$ if $k-1 \equiv 0$ or $1 \pmod 4$ or $k-1 \equiv 2$ or $3 \pmod 4$ respectively (see Theorems 1.1.1 and 1.1.2). So each element in $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\} \setminus \{2k-2\}$ occurs in exactly one element of P'. If $k-1 \equiv 0$ or $1 \pmod 4$ then name the Skolem pairs so that (x_{k-1}', y_{k-1}') contains $2k-3$. If $k-1 \equiv 2$ or $3 \pmod 4$ then name the hooked Skolem pairs so that $x_{k-1}' = 1$. Now define $P = \{(x_i = x_i'+3, y_i = y_i'+3) \mid (x_i', y_i') \in P'\}$ when using Skolem sequences and $P = \{(x_i = x_i' + 2, y_i = y_i' + 2) \mid (x_i', y_i') \in P'\}$ when using hooked Skolem sequences; notice that $2k \in \{x_{k-1}, y_{k-1}\}$ and $x_{k-1} = 3$ respectively. Then form the set of 5-cycles

$$B = \left\{(0, x_i, x_i - y_i, 3k, 5k + 1 + i) + j \mid 1 \leq i \leq \left\lfloor \frac{\alpha}{5} \right\rfloor, j \in \mathbb{Z}_v, (x_i, y_i) \in P\right\} \cup E_\alpha$$

where computations are done modulo v, and where

$$E_\alpha = \begin{cases} \varnothing & \text{if } \alpha \equiv 0 \pmod 5, \\ \Gamma_2 & \text{if } \alpha \equiv 1 \pmod 5, \\ \Gamma_1 \cup \Gamma_2 & \text{if } \alpha \equiv 2 \pmod 5, \\ \Gamma_3 & \text{if } \alpha \equiv 3 \pmod 5, \\ \Gamma_2 \cup \Gamma_3 & \text{if } \alpha \equiv 4 \pmod 5. \end{cases} \tag{2.2}$$

Table 2.1 lists the differences of edges that occur in cycles in $B \setminus E_\alpha$ when α is as big as possible (from Lemma 2.2.2 we see this happens when λ is as small as possible and not 0, namely when $\lambda = 2$, and so when $\alpha = 5k - 5$).

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k+1$	$3, 4, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k\}$	$3k+1, 3k+2, \ldots, 4k-1$	$3k+1, 3k+2, \ldots, 4k-1$
$\{3k, 5k+1+i\}$	$2k+2, 2k+3, \ldots, 3k$	$2k+2, 2k+3, \ldots, 3k$
$\{5k+1+i, 0\}$	$4k, 4k+1, \ldots, 5k-2$	$4k, 4k+1, \ldots, 5k-2$

Table 2.1: Differences of edges in $B \setminus E_\alpha$ when α is as large as possible

Notice that when α is as large as possible:

(i) if $k-1 \equiv 0$ or $1 \pmod 4$, then the 5-cycle in $B \setminus E_\alpha$ defined when $i = \lfloor \alpha/5 \rfloor = k-1$ contains edges of differences $2k = v/5$ (since $2k \in \{x_{k-1}, y_{k-1}\}$) and $4k = 2v/5$, and

(ii) if $k-1 \equiv 2$ or $3 \pmod 4$, then the 5-cycle in $B \setminus E_\alpha$ defined when $i = \lfloor \alpha/5 \rfloor = k-1$ contains edges of differences 3 (since $x_{k-1} = 3$) and $4k = 2v/5$.

Fortunately, when α is as large as possible, $\alpha \equiv 0 \pmod 5$ since $\lambda = 2$, so $E_\alpha = \varnothing$. If α is smaller, then (i)-(ii) from above shows that the edges of differences $1, 2, 3, v/5$ and $2v/5$ do not occur in 5-cycles in $B \setminus E_\alpha$ (since then $i < k-1$), hence such edges are available to form the 5-cycles in E_α.

There are $(m(v-1)/2-\alpha)v = (m(v-1)/2-\alpha-3/2)v+3v/2$ edges remaining in mK_v which do not occur in a 5-cycle in B. These remaining edges can be partitioned by the corresponding remaining differences as follows. Using Lemma 2.2.3 we know that $m(v-1)/2-\alpha-3/2 \equiv 0 \pmod 3$, so we can form a partition P_0 of the remaining differences into one set $\{d, v/2\}$ of size 2 with $d \neq v/2$, and $(m(v-1)/2-\alpha-3/2)/3$ sets of size 3. (Note that there are $3v/2$ edges with differences in $\{d, v/2\}$). Define

$$\begin{aligned} C =& \{(d_1, 0, d_3, d_2+d_3, \infty) + j \mid \{d_1, d_2, d_3\} \in P_0 \text{ with } d_1 \leq d_2 \leq d_3, j \in \mathbb{Z}_v\} \\ &\cup \left\{ \left(d, 0, \frac{v}{2}, \frac{v}{2}+d, \infty\right) + j \mid \left\{\frac{v}{2}, d\right\} \in P_0, j \in \mathbb{Z}_{v/2} \right\}. \end{aligned} \tag{2.3}$$

Since $m = 1$ the differences are distinct, so by Lemma 2.2.5 the elements of C are indeed 5-cycles. Then $(\mathbb{Z}_{10k} \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+1)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$.

Now suppose $m = 2$ and $\lambda = \lambda_{\max}(v, m)$. By Lemma 2.1.2(c), $\lambda \leq m(v-1)/3 - m = m(v-4)/3$. Because λ must also be even, $\lambda_{\max}(v, 2) = 2\lfloor (v-4)/3 \rfloor$. So since $m = 2$ and $\lambda = \lambda_{\max}(v, 2)$, it follows that

$$\alpha = v - 4 - 3\lfloor (v-4)/3 \rfloor = 0, 1, \text{ or } 2 \text{ if } k \equiv 1, 2, \text{ or } 0 \pmod 3 \text{ respectively.}$$

So define $B = \varnothing$, Γ_1 or $\Gamma_1 \cup \Gamma_2$ if $k \equiv 1, 2$, or $0 \pmod 3$ respectively, and let $\beta = \varnothing, \{v/5\}$, or $\{v/5, 2v/5\}$ when $\alpha = 0, 1$, or 2 respectively. Using Lemma 2.2.3 we know that $|D(v)_2^* \setminus \beta| \equiv 0 \pmod 3$, so we can form a partition P_0 of the remaining differences of $D(v)_2^* \setminus \beta$ into sets of size 3. Then define

$$C = \{(d_1, 0, d_3, d_2+d_3, \infty) + j \mid \{d_1, d_2, d_3\} \in P_0 \text{ with } d_1 \leq d_2 \leq d_3, j \in \mathbb{Z}_v\}. \tag{2.4}$$

By Lemma 2.2.5 the elements of C are indeed 5-cycles. Therefore, $(\mathbb{Z}_{10k} \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+2)$ with $\lambda = \lambda_{\max}(v, 2)$.

Finally, suppose $m = 3$ and $\lambda = \lambda_{\max}(v, 3) = v - 4$. It follows that $\alpha = 0$ for all k. So we define $B = \varnothing$. Using Lemma 2.2.3 we know that $3(v-1)/2 - \alpha - 3/2 \equiv 0 \pmod 3$, so we can form a partition P_0 of $D(v)_2^* \cup D(v)$ into one set $\{d, v/2\}$ of size 2 with $d \neq v/2$, and $(3(v-1)/2 - \alpha - 3/2)/3$ sets (not multisets; this is easy to do) of size 3. Define C as in Equation (2.3). By Lemma 2.2.5 the elements of C are indeed 5-cycles. Therefore, $(\mathbb{Z}_{10k} \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+3)$ with $\lambda = \lambda_{\max}(v, 3)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$. □

The proof of the next three results follows that of Lemma 2.2.6 closely but are sufficiently different and important that they are described in detail.

Lemma 2.2.7. *Let* $v = 10k + 5$ *and assume conditions (a-c) of Lemma* 2.1.2 *are satisfied. If* $m = 1$ *or if* $m \in \{2, 3\}$ *and* $\lambda = \lambda_{\max}(v, m)$*, then any* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+1}$.

Proof. Let $(V = \mathbb{Z}_{10k+5}, A)$ be a 5–CS of λK_v and let $U = \{\infty\}$. Since (V, A) exists, $\lambda + m$ is even by Lemma 2.1.2(a).

By Lemma 2.2.2 the number of edges we must place in 5-cycles contained completely in mK_v is αv as defined by Equation (2.1). Since this is divisible by v, our plan is again to create 5-cycles in mK_v using all the edges of α specially chosen differences. By Lemma 2.2.4, $\alpha \geq 0$.

Suppose $m = 1$. Then $mK_v = K_v$ contains edges of each difference in $D(v) = \{1, \ldots, 5k+2\}$. As in Lemma 2.2.6, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

We begin with the case where $k \geq 3$. Let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k-2, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k - 2$ from the set $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\}$ if $k - 2 \equiv 0$ or $1 \pmod 4$ or $k - 2 \equiv 2$ or $3 \pmod 4$ respectively (this exists by Theorems 1.1.1 and 1.1.2 since $k \geq 3$). So each element in $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\} \setminus \{2k-4\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 3, y_i = y'_i + 3) \mid (x'_i, y'_i) \in P'\}$. Then with

$$c_i = \begin{cases} (0, x_i, x_i - y_i, 4k+2, 7k+3-i) & \text{if } 1 \leq i \leq \lfloor \alpha/5 \rfloor - 1, \\ (0, 2k, 6k+1, 8k+3, 5k+2) & \text{if } i = \lfloor \alpha/5 \rfloor \text{ and } k-2 \equiv 0, 1 \pmod 4\text{, and} \\ (0, 2k-1, 6k, 8k+2, 5k+1) & \text{if } i = \lfloor \alpha/5 \rfloor \text{ and } k-2 \equiv 2, 3 \pmod 4, \end{cases}$$

we form the set of 5-cycles $B = \{c_i + j \mid 1 \leq i \leq \lfloor \alpha/5 \rfloor, j \in \mathbb{Z}_v\} \cup E_\alpha$ where the computations are done modulo v, $k \geq 3$, and where E_α is defined as in Equation (2.2). Table 2.2 lists the differences of edges that occur in cycles in $B \setminus E_\alpha$ if α is as big as possible (from Lemma 2.2.2 this happens when λ is as small as possible, namely when $\lambda = 1$ and so $\alpha = 5k - 1$). Fortunately, the edges of differences $1, 2, 3, v/5$ and

Edges	Differences when $k-2 \equiv 0, 1 \pmod 4$	Differences when $k-2 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k-1$	$4, 5, \ldots, 2k-2, 2k$
$\{x_i - y_i, 4k+2\}$	$4k+3, 4k+4, \ldots, 5k$	$4k+3, 4k+4, \ldots, 5k$
$\{4k+2, 7k+3-i\}$	$2k+3, 2k+4, \ldots, 3k$	$2k+3, 2k+4, \ldots, 3k$
$\{7k+3-i, 0\}$	$3k+3, 3k+4, \ldots, 4k$	$3k+3, 3k+4, \ldots, 4k$
c_i if $i = \lfloor \alpha/5 \rfloor$	$2k, 2k+2, 3k+1, 4k+1, 5k+2$	$2k-1, 2k+2, 3k+1, 4k+1, 5k+1$

Table 2.2: Differences of edges in $B \setminus E_\alpha$ when α is as large as possible

$2v/5$ do not occur in 5-cycles in $B \setminus E_\alpha$, hence are available to form the 5-cycles in

E_α.

By Lemma 2.2.3 we know that $m(v-1)/2-\alpha \equiv 0 \pmod 3$, and so the edges not occurring in a 5-cycle in B can be partitioned by the $m(v-1)/2-\alpha$ remaining differences. Form a partition P_0 of these differences into $((v-1)/2-\alpha)/3$ sets of size 3. Define C as in Equation (2.4). Notice that by Lemma 2.2.5 the edges in $\{\{d_1,0\},\{0,d_3\},\{d_3,d_3+d_2\}\}$ induce a 3-path in K_v. So the elements of C are all 5-cycles. Then, $(\mathbb{Z}_{10k+5}\cup\{\infty\}, A\cup B\cup C)$ is a 5–CS$(v+1,\lambda+1)$ for $v=10k+5$.

It is left to provide the required sets of 5-cycles when $m=1$ and $k=0,1$ and 2. Notice that if $k=0$, then $\lambda\le 0$, so this case need not be considered. Suppose $k=1$, so $v=15$. Then by Lemma 2.1.2(c) and since $\lambda+m=\lambda+1$ is even, it follows that $\lambda\in\{1,3\}$. So $(\lambda,\alpha)\in\{(1,4),(3,1)\}$. Define

$$C=\begin{cases}\{(d_1,0,d_3,d_2+d_3,\infty)+j \mid (d_1,d_2,d_3)=(4,5,7), j\in\mathbb{Z}_v\} & \text{if } \lambda=1, \text{ and}\\ \{(d_1,0,d_3,d_2+d_3,\infty)+j \mid (d_1,d_2,d_3)\in\{(1,2,3),(4,5,7)\}, j\in\mathbb{Z}_v\} & \text{if } \lambda=3,\end{cases}$$

and $B=E_\alpha$. Then $(\mathbb{Z}_{15}\cup\{\infty\}, A\cup B\cup C)$ is a 5–CS$(16,\lambda+1)$. If $k=2$, so $v=25$, then $\lambda\le 7$; so $\lambda=1,3,5$, or 7 and $\alpha=9,6,3$, or 0 respectively. Define

$$C=\begin{cases}\{(d_1,0,d_3,d_2+d_3,\infty)+j \mid (d_1,d_2,d_3)=(5,7,9), j\in\mathbb{Z}_v\} & \text{if } \lambda=1,\\ \{(d_1,0,d_3,d_2+d_3,\infty)+j \mid (d_1,d_2,d_3)\in\{(1,2,3),(5,7,9)\}, j\in\mathbb{Z}_v\} & \text{if } \lambda=3,\\ \{(d_1,0,d_3,d_2+d_3,\infty)+j \mid (d_1,d_2,d_3)\in\{(4,5,6),(7,8,9),(10,11,12)\}, j\in\mathbb{Z}_v\} & \text{if } \lambda=5, \text{ and}\\ \{(d_1,0,d_3,d_2+d_3,\infty)+j \mid (d_1,d_2,d_3)\in\{(x,x+1,x+2)\mid x\in\{1,4,7,10\}\}, j\in\mathbb{Z}_v\} & \text{if } \lambda=7,\end{cases}$$

and

$$B=\begin{cases}\{(0,4,10,22,8)+j \mid j\in\mathbb{Z}_{25}\}\cup E_\alpha & \text{if } \lambda\in\{1,3\},\\ E_\alpha & \text{if } \lambda\in\{5,7\}.\end{cases}$$

Then $(\mathbb{Z}_{25}\cup\{\infty\}, A\cup B\cup C)$ is a 5–CS$(26,\lambda+1)$.

Now suppose $m=2$ and $\lambda=\lambda_{\max}(v,m)$. By Lemma 2.1.2(c), $\lambda\le m(v-4)/3$, and by Lemma 2.1.2(a), λ is even. So $\lambda_{\max}(v,2)=2\lfloor (v-4)/3\rfloor$, and

$$\alpha=v-4-3\lfloor (v-4)/3\rfloor=0,1, \text{ or } 2 \text{ if } k\equiv 2,0, \text{ or } 1 \pmod 3 \text{ respectively.}$$

So define $B=\varnothing$, Γ_1, or $\Gamma_1\cup\Gamma_2$ if $k\equiv 2$, 0, or 1 (mod 3) respectively, and let $\beta=\varnothing,\{v/5\}$, or $\{v/5,2v/5\}$ when $\alpha=0,5$, or 10 respectively. Using Lemma 2.2.3 we know that $|D(v)_2\setminus\beta|\equiv 0 \pmod 3$, we can form a partition P_0 of $D(v)_2\setminus\beta$ into sets of size 3. Then define C as in Equation (2.4), again noting that the elements of C are 5-cycles by Lemma 2.2.5. Therefore, $(\mathbb{Z}_v\cup\{\infty\}, A\cup B\cup C)$ is a 5–CS$(v+1,\lambda+2)$ with $\lambda=\lambda_{\max}(v,2)$.

Finally, suppose $m=3$ and $\lambda=\lambda_{\max}(v,3)=v-4$. It follows that $\alpha=0$ for all k. So we define $B=\varnothing$. Since $3(v-1)/2-\alpha\equiv 0 \pmod 3$ by Lemma 2.2.3, form a partition P_0 of $D(v)_3$ into $(3(v-1)/2-\alpha)/3$ sets of size 3 (again it is easy to ensure that no element of P_0 contains a difference three times). Define C as in Equation (2.4). Then elements of C are all 5-cycles by Lemma 2.2.5. Then, $(\mathbb{Z}_v\cup\{\infty\}, A\cup B\cup C)$ is a 5–CS$(v+1,\lambda+3)$ with $\lambda=\lambda_{\max}(v,3)$. □

Lemma 2.2.8. *Let $v \equiv 2$ or $3 \pmod 5$ and assume conditions (a-c) of Lemma 2.1.2 are satisfied. If $m = 5$ or if $m \in \{10, 15\}$ and $\lambda = \lambda_{\max}(v, m)$, then any 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+1}$.*

Proof. Let $(V = \mathbb{Z}_v, A)$ be a 5–CS of λK_v where $v = 10k + \ell$ for some $\ell \in \{2, 3, 7, 8\}$ and let $U = \{\infty\}$. By Theorem 1.2.2(c), $v \geq 7$. In each case it is shown that (V, A) can be enclosed in a 5–CS $(V \cup \{\infty\}, A \cup B \cup C)$ of $(\lambda + m)K_{v+1}$ by defining B and C.

First, note that by Lemma 2.2.2, the number of edges we must place in 5-cycles contained completely in mK_v is αv as defined in Equation (2.1). Since this is divisible by v, our plan is to create 5-cycles in mK_v using the edges of α specially chosen differences. Notice that $\alpha \geq 0$ by Lemma 2.2.4.

Suppose $m = 5$. Then $\lambda \equiv 0$ or $5 \pmod{10}$ if $v \equiv 2, 8$ or $3, 7 \pmod{10}$ respectively. Note that when λ is as small as possible α is as large as possible: that is, when $\lambda = 10$, $\alpha = 25k - 20$ for $\ell = 2$ and $\alpha = 25k - 5$ for $\ell = 8$; and when $\lambda = 5$, $\alpha = 25k - 10$ for $\ell = 3$ and $\alpha = 25k$ for $\ell = 7$. The edges of $5K_v$ are considered to be $E(G_v(D(v)^*_4)) \cup E(G_v(D(v)))$ when v is even and $E(G_v(D(v)_5))$ when v is odd. For v even, care must be taken with edges of difference $v/2$ in $E(G_v(D(v)))$ since there are only $v/2$ of them. As in Lemma 2.2.6, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen). To form the set of cycles B, we consider the cases $\ell \in \{2, 3\}$ and $\ell \in \{7, 8\}$ in turn.

Suppose $\ell \in \{2, 3\}$. Let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k - 1, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k - 1$ for the set $\{1, 2, \ldots, 2k - 2\}$ or $\{1, 2, \ldots, 2k - 1\}$, if $k - 1 \equiv 0$ or $1 \pmod 4$ or $k - 1 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k - 2\}$ or $\{1, 2, \ldots, 2k - 1\} \setminus \{2k - 2\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 2, y_i = y'_i + 2) \mid (x'_i, y'_i) \in P'\}$. Then we form the set of 5-cycles

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k + 1, 5k + \ell + i) \mid 1 \leq i \leq k - 1, (x_i, y_i) \in P\}$$

where computations are done modulo v. We also need to define E_α when α is as large as possible. Let

$$E'_\alpha = \begin{cases} (0, 2k+1, 2k, 2k-1, 5k) & \text{if } k-1 \equiv 0, 1 \pmod 4 \text{ and } \ell = 2, \\ (0, 2k, 5k+1, 5k-1, 5k) & \text{if } k-1 \equiv 2, 3 \pmod 4 \text{ and } \ell = 2, \\ \bigcup_{i=1}^{3} \{(0, 2, 2k+3, 1, 5k+1)\} & \text{if } k-1 \equiv 0, 1 \pmod 4 \text{ and } \ell = 3, \text{ and} \\ \bigcup_{i=1}^{3} \{(0, 1, 2k+1, 7k+2, 5k)\} & \text{if } k-1 \equiv 2, 3 \pmod 4 \text{ and } \ell = 3. \end{cases}$$

Notice that when $\ell = 3$, E'_α is a multiset that contains 3 copies of the 5-cycle. Table 2.3 lists the differences of the edges that occur in cycles in B' when α is as large as possible and in E'_α. Let $B'' = 5B'$ be the multiset consisting of 5 copies of each 5-cycle in B'. When α is as large as possible, $|B'' \cup E'_\alpha| = 5k - 4$ or $5k - 2$ if $\ell = 2$ or 3 respectively, so $|B'' \cup E'_\alpha| = \alpha/5$. If α is not as big as possible then $\alpha \leq 25k - 25$

in all cases (since as λ increases by 10, α decreases by 15) so $|B''| = 5k - 5 \geq \alpha/5$. Therefore we define

$$B = \begin{cases} \bigcup_{j \in \mathbb{Z}_v} (B'' + j) \cup \{c + j \mid c \in E'_\alpha, j \in \mathbb{Z}_v\} & \text{if } \alpha \text{ is as large as possible, and} \\ \{c_i + j \mid i \in \mathbb{Z}_{\alpha/5}, c_{i \pmod{k-1}} \in B'', j \in \mathbb{Z}_v\} & \text{otherwise.} \end{cases}$$

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$3, 4, \ldots, 2k$	$3, 4, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k+1\}$	$3k+2, 3k+3, \ldots, 4k$	$3k+2, 3k+3, \ldots, 4k$
$\{3k+1, 5k+\ell+i\}$	$2k+\ell, 2k+1+\ell, \ldots, 3k-2+\ell$	$2k+\ell, 2k+1+\ell, \ldots, 3k-2+\ell$
$\{5k+\ell+i, 0\}$	$5k-1, 5k-2, \ldots, 4k+1$	$5k-1, 5k-2, \ldots, 4k+1$
E_α if $\alpha = 25k-20, \ell = 2$	$1, 1, 2k+1, 3k+1, 5k$	$1, 2, 2k, 3k+1, 5k$
E'_α if $\alpha = 25k-10, \ell = 3$	$2, 2, 2, 2k+1, 2k+1, 2k+1, 2k+2, 2k+2,$ $2k+2, 5k, 5k, 5k, 5k+1, 5k+1, 5k+1$	$1, 1, 1, 2k, 2k, 2k, 2k+2, 2k+2, 2k+2,$ $5k, 5k, 5k, 5k+1, 5k+1, 5k+1$

Table 2.3: Differences of edges in B' and E_α when α is as large as possible for $\ell \in \{2, 3\}$ and $m = 5$

Now suppose $\ell \in \{7, 8\}$. Let $P = \{(x_i, y_i) \mid 1 \leq i \leq k, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order k for the set $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\}$, if $k \equiv 0$ or $1 \pmod 4$ or $k \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\} \setminus \{2k\}$ occurs in exactly one element of P. Form the set of 5-cycles $B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k + 2, 5k + 4 + i) \mid 1 \leq i \leq k, (x_i, y_i) \in P\}$ where computations are done modulo v. Let $B'' = 5B'$ be the multiset consisting of 5 copies of each 5-cycle in B'. Define $B = \{c_i + j \mid i \in \mathbb{Z}_{\alpha/5}, c_{i \pmod k} \in B'', j \in \mathbb{Z}_v\}$. Table 2.4 lists the differences of edges that occur in cycles in B'.

Edges	Differences when $k \equiv 0, 1 \pmod 4$	Differences when $k \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k$	$1, 2, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k+2\}$	$3k+3, 3k+4, \ldots, 4k+2$	$3k+3, 3k+4, \ldots, 4k+2$
$\{3k+2, 5k+4+i\}$	$2k+3, 2k+4, \ldots, 3k+2$	$2k+3, 2k+4, \ldots, 3k+2$
$\{5k+4+i, 0\}$	$5k-5+\ell, 5k-6+\ell, \ldots, 4k-4+\ell$	$5k-5+\ell, 5k-6+\ell, \ldots, 4k-4+\ell$

Table 2.4: Differences of edges in B' when α is as large as possible for $\ell \in \{7, 8\}$ and $m = 5$

To define C, suppose $\ell \in \{2, 3, 7, 8\}$. Notice that by Lemma 2.2.3 we know that 3 divides $(5(v-1)/2 - \alpha)$ when v is odd and 3 divides $(5(v-1)/2 - \alpha - 3/2)$ when v is even. Form a partition P_0 of the differences not used to form B into $(5(v-1)/2 - \alpha)/3$ sets of size 3 when v is odd, and when v is even into $(5(v-1)/2 - \alpha - 3/2)/3$ sets of size 3 (not multisets) and one set $\{d, v/2\}$ of size 2 with $d \neq v/2$, when v is even; it is easy to ensure that $d_1 < d_3$ and $d_2 < v/2$. Form the set C of 5-cycles as in Equation (2.4) when v is odd and as in Equation (2.3) when v is even. By Lemma 2.2.5 the elements of C are 5-cycles. Thus $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+5)$ with $v \equiv 2$ or

3 (mod 5).

Now suppose $m = 10$ and $\lambda = \lambda_{\max}(v, m)$. By Table 1.1 since $v+1$ is $(\lambda+m)$-admissible $\lambda + m \equiv 0 \pmod{10}$ and so $\lambda \equiv 0 \pmod{10}$. It follows that $\alpha = 5(v-1) - 3(\lambda+m)/2 \equiv 0 \pmod 5$. It also follows that for any fixed values of v and m, v-admissible values of λ differ by multiples of 10, so $\alpha(v, m)$-admissible integers differ by multiples of 15 (by Lemma 2.2.2). So when $\lambda = \lambda_{\max}(v, m)$ it follows from Lemma 2.2.4 that $\alpha \in \{0, 5, 10\}$. (There is no need to be specific for this proof, but the reader may be interested to know that when $m = 10$ and $\lambda = \lambda_{\max}(v, 10)$: $\alpha = 0, 5$, or 10 if $k \equiv 2, 0$, or 1 (mod 3) respectively when $v \equiv 2$ or 8 (mod 10) ; $\alpha = 0, 5$, or 10 if $k \equiv 1, 2$, or 0 (mod 3) respectively when $v \equiv 3 \pmod{10}$; and $\alpha = 0, 5, 10$ if $k \equiv 0, 1$, or 2 (mod 3) respectively when $v \equiv 7 \pmod{10}$.) Define

$$B = \begin{cases} \varnothing & \text{if } \alpha = 0, \\ \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} & \text{if } \alpha = 5, \text{ and} \\ \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} \cup \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} & \text{if } \alpha = 10, \end{cases}$$

and let $\beta = \varnothing, \{1,2,3,4,6\}$, or $\{1,1,2,2,3,3,4,4,6,6\}$ when $\alpha = 0, 5$, or 10 respectively. By Lemma 2.2.3, $|D(v)_{10} \setminus \beta| \equiv 0 \pmod 3$ when v is odd and $|(D(v)^*_{10}) \setminus \beta| \equiv 0 \pmod 3$ when v is even. Form a partition P_0 of $D(v)_{10} \setminus \beta$ when v is odd and $D(v)^*_{10} \setminus \beta$ when v is even into $(m(v-1)/2 - \alpha)/3$ sets of size 3. Then define the 5-cycles in C as in Equation (2.4) (it is easy to ensure that $d_1 < d_3$ and $d_2 < v/2$, and so by Lemma 2.2.5 the elements of C are all 5-cycles). Thus $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+10)$ for $v \equiv 2$ or 3 (mod 5) and $\lambda = \lambda_{\max}(v, m)$.

Let $m = 15$ and $\lambda = \lambda_{\max}(v, 15) = 5(v-4)$. It follows that $\alpha = 0$ for all k. So define $B = \varnothing$. Using Lemma 2.2.3 we know that $15(v-1)/2 - \alpha \equiv 0 \pmod 3$ when v is odd and $15(v-1)/2 - \alpha - 3/2 \equiv 0 \pmod 3$ when v is even, so we can form a partition P_0 of $D(v)_{15}$ into $(15(v-1)/2 - \alpha)/3$ sets of size 3 when v is odd and when v is even form a partition P_0 of $D(v)^*_{14} \cup D(v)$ into $(15(v-1)/2 - \alpha - 3/2)/3$ sets of size 3 and one set $\{d, v/2\}$ of size 2 with $d \neq v/2$. Define C as in Equation (2.4) when v is odd and as in Equation (2.3) when v is even (it is easy to ensure $d_1 < d_3$ and $d_2 < v/2$ and so by Lemma 2.2.5 the elements of C are all 5-cycles). Therefore $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+15)$ for $v \equiv 2$ or 3 (mod 5) with $\lambda = \lambda_{\max}(v, 15)$. □

Lemma 2.2.9. *Let* $v \equiv 1$ *or* 4 (mod 5) *and suppose conditions (a-c) of Lemma* 2.1.2 *are satisfied. If* $m = 1$, *if* $m \in \{2, 3\}$ *and* $\lambda = \lambda_{\max}(v, m)$, *if* $m = 3$, $\lambda = 2$, *and* $v \equiv 6 \pmod{10}$, *or if* $m \in \{2, 3, \ldots, 9\}$, $\lambda + m = 10$ *and* $v \equiv 1 \pmod{10}$, *then any* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+1}$.

Proof. Let $(V = \mathbb{Z}_v, A)$ be a 5–CS of λK_v where $v = 10k + \ell \geq 6$ for some $\ell \in \{1, 4, 6, 9\}$ and let $U = \{\infty\}$. First, note that by Lemma 2.2.2, the number of edges we must place in 5-cycles contained completely in mK_v is αv as defined in Equation (2.1) Since this is divisible by v and $\alpha \geq 0$ by Lemma 2.2.4, our plan is to create 5-cycles in mK_v using the edges of α specially chosen differences.

Suppose $m = 1$. Then $mK_v = K_v$ contains edges of each difference in $D(v)$.

As in Lemma 2.2.6, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

For $\ell \in \{4, 6, 9\}$, let $P = \{(x_i, y_i) \mid 1 \leq i \leq k, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order k for the set $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\}$, if $k \equiv 0$ or $1 \pmod 4$ or $k \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\} \setminus \{2k\}$ occurs in exactly one element of P. For $\ell = 1$, let $P = \{(x_i, y_i) \mid 1 \leq i \leq k-1, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ for the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$, if $k-1 \equiv 0$ or $1 \pmod 4$ or $k-1 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\} \setminus \{2k-2\}$ occurs in exactly one element of P for $\ell = 1$. Then with

$$c_i = \begin{cases} (0, x_i, x_i - y_i, 3k, 5k+1+i) & \text{if } \ell = 1, \\ (0, x_i, x_i - y_i, 3k-1+\ell/2, 5k+\ell/2+i) & \text{if } \ell \in \{4, 6\}, \text{ and} \\ (0, x_i, x_i - y_i, 3k+3, 5k+5+i) & \text{if } \ell = 9, \end{cases}$$

form the set of 5-cycles $B = \{c_i + j \mid 1 \leq i \leq \lfloor \alpha/5 \rfloor, j \in \mathbb{Z}_v, (x_i, y_i) \in P\}$ where computations are done modulo v. Tables 2.5, 2.6, and 2.7 list the differences of the edges that occur in cycles in B when α is as big as possible (that is, when λ is as small as possible).

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k-2$	$1, 2, \ldots, 2k-3, 2k-1$
$\{x_i - y_i, 3k\}$	$3k+1, 3k+2, \ldots, 4k-1$	$3k+1, 3k+2, \ldots, 4k-1$
$\{3k, 5k+1+i\}$	$2k+2, 2k+3, \ldots, 3k$	$2k+2, 2k+3, \ldots, 3k$
$\{5k+1+i, 0\}$	$5k-1, 5k-2, \ldots, 4k+1$	$5k-1, 5k-2, \ldots, 4k+1$

Table 2.5: Differences of edges in B when α is as large as possible for $\ell = 1$ and $m = 1$

Edges	Differences when $k \equiv 0, 1 \pmod 4$	Differences when $k \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k$	$1, 2, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k-1+\ell/2\}$	$3k+\ell/2, 3k+1+\ell/2, \ldots, 4k-1+\ell/2$	$3k+\ell/2, 3k+1+\ell/2, \ldots, 4k-1+\ell/2$
$\{3k-1+\ell/2, 5k+\ell/2+i\}$	$2k+2, 2k+3, \ldots, 3k+1$	$2k+2, 2k+3, \ldots, 3k+1$
$\{5k+\ell/2+i, 0\}$	$5k-1+\ell/2, 5k-2+\ell/2, \ldots, 4k+\ell/2$	$5k-1+\ell/2, 5k-2+\ell/2, \ldots, 4k+\ell/2$

Table 2.6: Differences of edges in B when α is as large as possible for $\ell \in \{4, 6\}$ and $m = 1$

Notice that by Lemma 2.2.3 we know that 3 divides $((v-1)/2 - \alpha)$ when v is odd and 3 divides $((v-1)/2 - \alpha - 3/2)$ when v is even. Form a partition P_0 of the differences not used to form B into $((v-1)/2 - \alpha)/3$ sets (not multisets) of size 3 when v is odd and when v is even $((v-1)/2 - \alpha - 3/2)/3$ sets (not multisets) of size 3 and one set $\{d, v/2\}$ of size 2 with $d \neq v/2$. We form the set C of 5-cycles as in

Edges	Differences when $k \equiv 0, 1 \pmod 4$	Differences when $k \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k$	$1, 2, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k+3\}$	$3k+4, 3k+5, \ldots, 4k+3$	$3k+4, 3k+5, \ldots, 4k+3$
$\{3k+3, 5k+5+i\}$	$2k+3, 2k+4, \ldots, 3k+2$	$2k+3, 2k+4, \ldots, 3k+2$
$\{5k+5+i, 0\}$	$5k+3, 5k+2, \ldots, 4k+4$	$5k+3, 5k+2, \ldots, 4k+4$

Table 2.7: Differences of edges in B when α is as large as possible for $\ell = 9$ and $m = 1$

Equation (2.4) when v is odd and as in Equation (2.3) when v is even. Notice that by Lemma 2.2.5 the elements of C are all 5-cycles. Thus $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+1)$ for $v \equiv 1$ or $4 \pmod 5$.

Now suppose $m = 2$ and $\lambda = \lambda_{\max}(v, 2)$. By Table 1.1: if $v \equiv 1 \pmod 5$ then since $v+1$ is $(\lambda+m)$-admissible $\lambda + m \equiv 0 \pmod{10}$ and so $\lambda \equiv 8 \pmod{10}$; and if $v \equiv 4 \pmod 5$ then v is λ-admissible so $\lambda \equiv 0 \pmod{10}$ (and so $\lambda + m \equiv 2 \pmod{10}$). It follows that $\alpha = v - 1 - 3(\lambda+2)/2 \equiv 0 \pmod 5$. It also follows that for any fixed values of v and m, v-admissible values of λ differ by multiples of 10, so $\alpha(v, m)$-admissible integers differ by multiples of 15 (by Lemma 2.2.2). So when $\lambda = \lambda_{\max}(v, m)$ it follows from Lemma 2.2.4 that $\alpha \in \{0, 5, 10\}$. (There is no need to be more specific for this proof, but the reader may be interested to know that when $m = 2$ and $\lambda = \lambda_{\max}(v, 2)$: $\alpha = 0, 5$, or 10 if $k \equiv 0, 2$, or $1 \pmod 3$ respectively when $v \equiv 1$ or $4 \pmod{10}$; and $\alpha = 0, 5, 10$ if $k \equiv 1, 0$, or $2 \pmod 3$ respectively when $v \equiv 6$ or $9 \pmod{10}$.) Define

$$B = \begin{cases} \varnothing & \text{if } \alpha = 0, \\ \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} & \text{if } \alpha = 5, \text{ and} \\ \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} \cup \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} & \text{if } \alpha = 10, \end{cases}$$

and let $\beta = \varnothing, \{1,2,3,4,6\}$, or $\{1,1,2,2,3,3,4,4,6,6\}$ when $\alpha = 0, 5$, or 10 respectively. Using Lemma 2.2.3 we know that 3 divides $v - 1 - \alpha = |D(v)_2 \setminus \beta|$ if v is odd and $|D(v)_2^* \setminus \beta|$ if v is even. Form a partition P_0 of $D(v)_2 \setminus \beta$ when v is odd and $D(v)_2^* \setminus \beta$ if v is even into $(m(v-1)/2 - \alpha)/3$ sets of size 3. Form the set C of 5-cycles as in Equation (2.4). By Lemma 2.2.5 the elements of C are all 5-cycles. Thus $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+2)$ with $v \equiv 1$ or $4 \pmod 5$ and $\lambda = \lambda_{\max}(v, 2)$.

Now suppose $m = 3$ and $\lambda = \lambda_{\max}(v, 3) = (v - 4)$. It follows that $\alpha = 0$ for all k. So define $B = \varnothing$. Using Lemma 2.2.3 we know that $3(v-1)/2 - \alpha \equiv 0 \pmod 3$ when v is odd and $3(v-1)/2 - \alpha - 3/2 \equiv 0 \pmod 3$ when v is even, so we can form a partition P_0 of the differences in $D(v)_3$ into $(3(v-1)/2 - \alpha)/3$ sets of size 3 when v is odd and when v is even form a partition P_0 of the differences in $D(v)_2^* \cup D(v)$ into $(3(v-1)/2 - \alpha - 3/2)/3$ sets of size 3 and one set $\{d, v/2\}$ of size 2 with $d \neq v/2$; it is easy to ensure that each element of P_0 does not contain the same element 3 times. Define C as in Equation (2.4) when v is odd and as in Equation (2.3) when v is even.

By Lemma 2.2.5 the elements of C are cycles. Therefore $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+3)$ for $v \equiv 1$ or $4 \pmod 5$, $\lambda = \lambda_{\max}(v, 3)$.

Let $v \equiv 1 \pmod{10}$, $m \in \{2, 3, \ldots, 9\}$, and $\lambda = 10 - m$. Let $P = \{(x_i, y_i) \mid 1 \leq i \leq k-1, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ for the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$, if $k-1 \equiv 0$ or $1 \pmod 4$ or $k-1 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\} \setminus \{2k-2\}$ occurs in exactly one element of P. Form the set of 5-cycles

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k, 5k+1+i) \mid 1 \leq i \leq k-1, (x_i, y_i) \in P\}$$

where computations are done modulo v. Let $B'' = mB'$. Then B'' contains $m(k-1)$ 5-cycles, but we need $\lfloor \alpha/5 \rfloor = mk - 3$ of them. Hence for $m \geq 4$ we need $m-3$ more 5-cycles. Let $c'_1 = (0, 2k+1, 6k+1, k+1, 5k+1)$ and $c'_2 = (0, 2k, 4k, 6k, 8k)$. Also let E_α contain 1, 2, 3, 3, 4, and 4 copies of c'_1 if $m =$ 4, 5, 6, 7 8, and 9 respectively and contain 1, 1, and 2 copies of c'_2 if $m =$ 7, 8, and 9 respectively. Notice that the cycles in E_α contain at most m copies of edges of differences not used in cycles in B'', namely the differences $2k, 2k+1, 4k$ and $5k$ (B'' is also missing the differences $2k-1$ or $2k-2$). Then

$$B = \left\{c_i + j \mid 1 \leq i \leq \lfloor \alpha/5 \rfloor, c_{i \pmod{k-1}} \in B'', j \in \mathbb{Z}_v\right\} \cup \{c'_i + j \mid j \in \mathbb{Z}_v, c'_i \in E_\alpha\}.$$

Notice that regardless of our choice of m, $\lambda + m = 10$. So $(m(v-1)/2 - \alpha) = 15$ which is divisible by 3. So we use the elements not used to form B to form a partition P_0 of the differences in $D(v)_m$ into five sets of size 3. Define C as in Equation (2.4) (it is easy to ensure $d_1 < d_3$ and $d_2 < v/2$ so that by Lemma 2.2.5 the elements of C are all 5-cycles). Therefore $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+m)$ with $m \in \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $\lambda + m = 10$.

Let $v \equiv 6 \pmod{10}$, $m = 3$ and $\lambda = \lambda_{\min}(v, 3) = 2$. Let $P = \{(x_i, y_i) \mid 1 \leq i \leq k, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order k for the set $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\}$, if $k \equiv 0$ or $1 \pmod 4$ or $k \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\} \setminus \{2k\}$ occurs in exactly one element of P. Then we form the set of 5-cycles

$$B' = \{(0, x_i, x_i - y_i, 3k+2, 5k+3+i) \mid 1 \leq i \leq k, (x_i, y_i) \in P\}$$

where computations are done modulo v. Let $B'' = mB'$. Notice that $\lfloor \alpha/5 \rfloor = 3k = |B''|$. So define $B = \{c + j \mid c \in B'', j \in \mathbb{Z}_v\}$.

By Lemma 2.2.3 we know that $m(v-1)/2 - \alpha - 3/2 \equiv 0 \pmod 3$. So we use the elements not used to form B to form a partition P_0 into one set $\{d, v/2\}$ of size 2 with $d \neq v/2$ and two sets of size 3. Define C as in Equation (2.4) (it is easy to ensure that $d_1 < d_3$ and $d_2 < v/2$ so that by Lemma 2.2.5 the elements of C are all 5-cycles). Therefore $(\mathbb{Z}_v \cup \{\infty\}, A \cup B \cup C)$ is a 5–CS$(v+1, \lambda+m)$ with $v \equiv 6 \pmod{10}$, $m = 3$, and $\lambda = 2$. $\square$

Corollary 2.2.10. *Assume conditions (a-c) of Lemma* 2.1.2 *are satisfied. Let* $v \equiv 2$ *or* $3 \pmod 5$*. If*

- $m = 5$*, or*
- $m \in \{10, 15\}$ *and* $\lambda = \lambda_{\max}(v, m)$,

then any 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+1}$.

Let $v \not\equiv 2$ *or* $3 \pmod 5$*. If*

- $m = 1$,
- $m \in \{2, 3\}$ *and* $\lambda = \lambda_{\max}(v, m)$,
- $m = 3$, $\lambda = \lambda_{\min}(v, 3) = 2$, *and* $v \equiv 6 \pmod{10}$*, or*
- $m \in \{2, 3, \ldots, 9\}$, $\lambda + m = 10$ *and* $v \equiv 1 \pmod{10}$,

then any 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+1}$.

Proof. By Lemmas 2.2.6, 2.2.7, 2.2.8, and 2.2.9 the results follow. □

2.3 The Main Theorem

The following lemma will be used to guarantee an important point made in the main result.

Lemma 2.3.1. *If* $m \equiv 0 \pmod 3$ *then* $\lambda_{\max}(v, m) = m(v - 4)/3$.

Proof. Recall that $\lambda_{\max}(v, m)$ is the largest integer satisfying conditions (a-c) of Lemma 2.1.2. By Lemma 2.1.2(c), $\lambda_{\max}(v, m) \leq \lfloor m(v - 4)/3 \rfloor = m(v - 4)/3$. So the result follows since clearly

$$m(v - 1)/3 + m(v - 1) = 4m(v - 1)/3$$

is even and

$$mv(v - 1)/2 + mv(v - 1)/3 = 5mv(v - 1)/6$$

is divisible by 5. □

In the following theorem, Corollary 2.2.10 is used as a basis for an inductive argument to prove our main result, settling the enclosing problem when $u = 1$.

Theorem 2.3.2. *Let* $m \geq 1$*. A* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+1}$ *if and only if*

(a) $\lambda + mv \equiv 0 \pmod 2$,
(b) $m\binom{v}{2} + v(\lambda + m) \equiv 0 \pmod 5$*, and*
(c) $m(v - 1) \geq 3(\lambda + m)$.

Proof. The necessity follows from Lemma 2.1.2, so we now prove the sufficiency. Since we are assuming that a 5–CS of λK_v exists, it follows by Theorem 1.2.2 that $v \notin \{2, 3, 4\}$, and by condition (c) $v \neq 1$, so we know $v \geq 5$. If $\lambda = 0$ then by condition (a) $mv \equiv 0 \pmod 2$ and by condition (b) $mv(v+1) \equiv 0 \pmod 5$, so by Theorem 1.2.2 there exists a 5–CS of mK_{v+1}. So we can assume that $\lambda \geq 1$.

Let (V, A) be a 5–CS of λK_v. If $v \equiv 2$ or $3 \pmod 5$ then from Table 1.1 it follows that $m \equiv 0 \pmod 5$; in any other case m can potentially take on any positive integral value. To reflect this situation, let $t(v) = 1$ if $v \not\equiv 2$ or $3 \pmod 5$ and let $t(v) = 5$ if $v \equiv 2$ or $3 \pmod 5$. We often simply write t instead of $t(v)$ when the value of v is clear. Let $\lambda^-_{\text{max}}(v, m)$ be the second largest integer for λ satisfying conditions (a-c), and let $\lambda^+_{\text{min}}(v, m)$ be the second smallest integer for λ satisfying conditions (a-c).

We will first establish that for any given v, the difference between consecutive values of λ that satisfy conditions (a) and (b) is a constant; call this difference $\lambda_{\text{diff}}(v)$. Secondly, for all $m \geq 1$ and $v \geq 5$, we settle the enclosing problem for both the smallest and the largest values of λ satisfying conditions (a-c). Finally for all $m \geq 1$ and $v \geq 5$ it will be shown that for all λ satisfying conditions (a-c) there exist non-negative integers λ_1, λ_2 with $\lambda_1 + \lambda_2 = \lambda$ and positive integers m_1, m_2 with $m_1 + m_2 = m$ such that for $1 \leq i \leq 2$ there exists a 5–CS of $\lambda_i K_v$ that can be enclosed in a 5–CS of $(\lambda_i + m_i)K_{v+1}$. So the union of these two enclosings completes the proof.

We turn to the first step of the proof. For any given v and m, the difference between consecutive values of λ that satisfy conditions (a) and (b) is a constant; namely $\lambda_{\text{diff}}(v, m)$. Also by conditions (a) and (b), $\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 2$ and $v\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 5$. Therefore $v\lambda_{\text{diff}}(v, m) \equiv 0 \pmod{10}$. Since $\lambda_{\text{diff}}(v, m)$ must be the smallest such value, if $v \equiv 0, 5 \pmod{10}$ then $\lambda_{\text{diff}}(v, m) = 2$, and if $v \not\equiv 0, 5 \pmod{10}$ then $\lambda_{\text{diff}}(v, m) = 10$. Notice that if $m_1 \neq m_2$ then $\lambda_{\text{diff}}(v, m_1) = \lambda_{\text{diff}}(v, m_2)$. Therefore we define $\lambda_{\text{diff}}(v) = \lambda_{\text{diff}}(v, m)$.

We now turn to the second step in the proof; establishing the existence of the enclosing for the smallest and largest values of λ given v and m. Beginning with the smallest value, Tables 2.8 and 2.9 list the values of $\lambda_{\text{min}}(v, m)$, and were formed using Theorem 1.2.2 and Corollary 2.2.10. Three cases are considered in turn based on the value of v.

	m									
$v \pmod{10}$	0	1	2	3	4	5	6	7	8	9
1	0	9	8	7	6	5	4	3	2	1
6	0	4	8	2	6	0	4	8	2	6

Table 2.8: Checking $\lambda_{\text{min}}(v, m)$ when $v \equiv 1$ or $6 \pmod{10}$: each cell contains $\lambda_{\text{min}}(v, m)$

	$v \pmod{10}$							
m	0	2	3	4	5	7	8	9
odd	0	0	5	0	1	5	0	5
even	0	0	0	0	0	0	0	0

Table 2.9: Checking $\lambda_{\min}(v,m)$ when $v \not\equiv 1$ or $6 \pmod{10}$: each cell contains $\lambda_{\min}(v,m)$

Let $v \equiv 1 \pmod{10}$. The proof is by induction on m. By Corollary 2.2.10 if $m \leq 10$ then any 5–CS of $\lambda_{\min}(v,m)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}$. Suppose that for some $m \geq 11$ and for all s with $1 \leq s < m$, every 5–CS of $\lambda_{\min}(v,s)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v,s)+s)K_{v+1}$. Then, since $m \geq 11$, by induction each 5–CS (V,B_1) of $\lambda_{\min}(v,m-10)K_v$ can be enclosed in a 5–CS $(V \cup \{\infty\}, B_1')$ of $(\lambda_{\min}(v,m-10)+(m-10))K_{v+1}$ and there exists a 5–CS $(V \cup \{\infty\}, B_2)$ of $10K_{v+1}$ (by Theorem 1.2.2). Since Table 2.8 shows that $\lambda_{\min}(v,m-10) = \lambda_{\min}(v,m)$, (V,B_1) is a 5–CS of $\lambda_{\min}(v,m-10)K_v = \lambda_{\min}(v,m)K_v$ which is enclosed in a 5–CS $(V \cup \{\infty\}, B_1' \cup B_2)$ of

$$((\lambda_{\min}(v,m-10)+(m-10))+10)K_{v+1} = (\lambda_{\min}(v,m)+m)K_{v+1}.$$

Let $v \equiv 6 \pmod{10}$. The proof is by induction on m. We first settle the base cases where $m \leq 5$. By Corollary 2.2.10, if $m \in \{1,3\}$ then any 5–CS of $\lambda_{\min}(v,m)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}$. Hence a 5–CS of $\lambda_{\min}(v,1)K_v$ can be enclosed in a 5–CS of $(\lambda_{min}(v,1)+1)K_{v+1}$. Since Table 2.8 shows $\lambda_{\min}(v,1)+\lambda_{\min}(v,1) = 4+4 = \lambda_{\min}(v,2)$, the union of two copies of this enclosing proves that any 5–CS of $\lambda_{\min}(v,2)K_v$ can be enclosed in a 5–CS of $((\lambda_{\min}(v,1)+1)+(\lambda_{\min}(v,1)+1))K_{v+1} = (\lambda_{\min}(v,2)+2)K_{v+1}$. Similarly, since Table 2.8 shows that $\lambda_{\min}(v,1)+\lambda_{\min}(v,3) = 4+2 = \lambda_{\min}(v,4)$, any 5–CS of $\lambda_{\min}(v,4)K_v$ can be enclosed in a 5–CS of $((\lambda_{\min}(v,1)+1)+(\lambda_{\min}(v,3)+3))K_{v+1} = (\lambda_{\min}(v,4)+4)K_{v+1}$. If $m = 5$, by Table 2.8 $\lambda_{\min}(v,5) = 0$, so this enclosing has been established. Now suppose that for some $m \geq 6$ and for all s with $1 \leq s < m$, every 5–CS of $\lambda_{\min}(v,s)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v,s)+s)K_{v+1}$. Then, since $m \geq 6$, by induction each 5–CS (V,B_1) of $\lambda_{\min}(v,m-5)K_v$ can be enclosed in a 5–CS $(V \cup \{\infty\}, B_1')$ of $(\lambda_{\min}(v,m-5)+(m-5))K_{v+1}$, and by Theorem 1.2.2 there exists a 5–CS $(V \cup \{\infty\}, B_2)$ of $5K_{v+1}$. Since Table 2.8 shows that $\lambda_{\min}(v,m-5) = \lambda_{\min}(v,m)$, (V,B_1) is a 5–CS of $\lambda_{\min}(v,m-5)K_v = \lambda_{\min}(v,m)K_v$ which is enclosed in a 5–CS $(V \cup \{\infty\}, B_1' \cup B_2)$ of

$$((\lambda_{\min}(v,m-5)+(m-5))+5)K_{v+1} = (\lambda_{\min}(v,m)+m)K_{v+1}.$$

Let $v \not\equiv 1$ or $6 \pmod{10}$. The proof is by induction on m. If $m = t$, m is even (including the important small value $m = 2t$), or v is even, then by Table 2.9 $\lambda_{\min}(v,m) = 0$, so this enclosing has been established. It is left to show we can form an enclosing of a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}$ if v is odd and m is odd. Suppose that for some $m \geq 3t$ (recall that $m \equiv 0 \pmod{t}$) and for all s with $1 \leq s < m$, every 5–CS of $\lambda_{\min}(v,s)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v,s)+s)K_{v+1}$. Then, since $m \geq 2t$, each 5–CS (V,B_1) of $\lambda_{\min}(v,m-2t)K_v$ can be enclosed in a 5–CS (V,B_1') of $(\lambda_{\min}(v,m-2t)+(m-2t))K_{v+1}$ and by Theorem 1.2.2 there exists a 5–CS $(V\cup\{\infty\},B_2)$ of $2tK_{v+1}$. Since Table 2.9 shows that $\lambda_{\min}(v,m-t)=\lambda_{\min}(v,m)$, (V,B_1) is a 5–CS of $(\lambda_{\min}(v,m-2t))K_v=\lambda_{\min}(v,m)K_v$ which is enclosed in a 5–CS $(V\cup\{\infty\},B_1'\cup B_2)$ of

$$((\lambda_{\min}(v,m-2t)+(m-2t))+2t)K_{v+1}=(\lambda_{\min}(v,m)+m)K_{v+1}.$$

Now turning to the largest value of λ, we will enclose a 5–CS of $\lambda_{\max}(v,m)K_v$ in a 5–CS of $(\lambda_{\max}(v,m)+m)K_{v+1}$. The proof is by induction on the non-negative integer i, where $m=3ti+\epsilon$ and $\epsilon\in\{t,2t,3t\}$. In view of Corollary 2.2.10, the largest value of λ (i.e. $\lambda=\lambda_{\max}(v,m)$) when $i=0$ has already been solved. Suppose that for some $i\geq 1$ and for all s with $0\leq s<i$ we can enclose a 5–CS of $\lambda_{\max}(v,3st+\epsilon)K_v$ in a 5–CS of $(\lambda_{\max}(v,3st+\epsilon)+(3st+\epsilon))K_{v+1}$. By Lemma 2.3.1 it follows that $\lambda_{\max}(v,m)=m(v-4)/3$ when $m\equiv 0 \pmod 3$, and so $\lambda_{\max}(v,3(i-1)t+3t)+\lambda_{\max}(v,\epsilon)=\lambda_{\max}(v,m)$. In particular every 5–CS of $\lambda_{\max}(v,3t(i-1)+3t)K_v$ can be enclosed in a 5–CS of $((\lambda_{\max}(v,3t(i-1)+3t)+(3t(i-1)+3t))K_{v+1}$ and every 5–CS of $\lambda_{\max}(v,\epsilon)K_v$ can be enclosed in a 5–CS of $(\lambda_{\max}(v,\epsilon)+\epsilon)K_{v+1}$. The union of the previous two enclosings proves that any 5–CS of $(\lambda_{\max}(v,3t(i-1)+3t)+\lambda_{\max}(v,\epsilon))K_v=\lambda_{\max}(v,m)K_v$ can be enclosed in a 5–CS of

$$((\lambda_{\max}(v,3t(i-1)+3t)+(3t(i-1)+3t))+(\lambda_{\max}(v,\epsilon)+\epsilon))K_{v+1}=(\lambda_{\max}(v,m)+m)K_{v+1}.$$

Since the cases where $m=t$ and where $v\geq 5$, $m\geq t$ and $\lambda\in\{\lambda_{\min}(v,m),\lambda_{\max}(v,m)\}$ have now been settled, it is left to show that for $m\geq t$, $v\geq 5$, and for all λ satisfying conditions (a-c) with $\lambda_{\min}(v,m)<\lambda<\lambda_{\max}(v,m)$ there is an enclosing of a 5–CS of λK_v in a 5–CS of $(\lambda+m)K_{v+1}$. The proof is by induction on m. Tables 2.10, 2.11 and 2.12 were formed using Theorem 1.2.2 and Corollary 2.2.10, and will help in establishing each value of λ that the enclosing can be produced using two smaller enclosings. Suppose for some $m\geq 2t$ and for all s with

	$v \pmod{10}$									
m	0	1	2	3	4	5	6	7	8	9
$2t(v)$	0, 0, 2	0, 0, 10	10, 0, 0	0, 0, 10	0, 0, 10	2, 0, 0	10, 0, 0	0, 10, 0	10, 0, 0	10, 0, 0
$3t(v)$	2, 0, 2	0, 10, 10	10, 10, 0	10, 0, 10	0, 10, 10	2, 2, 0	10, 0, 10	0, 10, 10	10, 10, 0	10, 0, 10

Table 2.10: Checking the largest values of λ: each cell contains $\lambda_{\max}(v,m)-\lambda_{\max}(v,t(v))-\lambda_{\max}(v,m-t(v))$ for $k\equiv 0,1,$ and $2 \pmod 3$ in turn.

$1\leq s<m$, every 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda+s)K_{v+1}$. From Table 2.10, $\lambda_{\max}(v,m)-\lambda_{\max}(v,t)-\lambda_{\max}(v,m-t)\in\{0,\lambda_{\text{diff}}(v)\}$, so $\lambda^-_{\max}(v,m)\leq\lambda_{\max}(v,t)+\lambda_{\max}(v,m-t)$. Similarly from Tables 2.11 and 2.12,

$$\lambda_{\min}(v,m-t)+\lambda_{\min}(v,t)-\lambda_{\min}(v,m)\in\{0,\lambda_{\text{diff}}(v)\},$$

	$v \pmod{10}$							
m	0	2	3	4	5	7	8	9
odd	0	0	0	0	0	0	0	0
even	0	0	10	0	2	10	0	10

Table 2.11: Checking the smallest values of λ for $v \not\equiv 1$ or $6 \pmod{10}$: each cell contains $\lambda_{\text{min}}(v, m - t(v)) + \lambda_{\text{min}}(v, t(v)) - \lambda_{\text{min}}(v, m)$.

	m									
$v \pmod{10}$	0	1	2	3	4	5	6	7	8	9
1	10	0	10	10	10	10	10	10	10	10
6	10	0	0	10	0	10	0	0	10	0

Table 2.12: Checking the smallest values of λ for $v \equiv 1$ or $6 \pmod{10}$: each cell contains $\lambda_{\text{min}}(v, m - t(v)) + \lambda_{\text{min}}(v, t(v)) - \lambda_{\text{min}}(v, m)$.

so $\lambda_{\text{min}}(v, m - t) + \lambda_{\text{min}}(v, t) \leq \lambda^{+}_{\text{min}}(v, m)$. Thus

$$\lambda_{\text{min}}(v, m - t) + \lambda_{\text{min}}(v, t) \leq \lambda \leq \lambda_{\text{max}}(v, m - t) + \lambda_{\text{max}}(v, t).$$

Since for each v we know that $\lambda_{\text{diff}}(v) = \lambda_{\text{diff}}(v, m)$ is constant for all m, it follows that there exist non-negative integers λ_1, λ_2 for which $\lambda_{\text{min}}(v, t) \leq \lambda_1 \leq \lambda_{\text{max}}(v, t)$, $\lambda_{\text{min}}(v, m - t) \leq \lambda_2 \leq \lambda_{\text{max}}(v, m - t)$, and $\lambda_1 + \lambda_2 = \lambda$, and there exist positive integers m_1, m_2 with $m_1 + m_2 = m$ such that for $1 \leq i \leq 2$ there exists a 5–CS of $\lambda_i K_v$ that can be enclosed in a 5–CS of $(\lambda_i + m_i)K_{v+1}$. So the union of these enclosings proves that any 5–CS of $(\lambda_1 + \lambda_2)K_v = \lambda K_v$ can be enclosed in a 5–CS of $((\lambda_1 + m_1) + (\lambda_2 + m_2))K_{v+1} = (\lambda + m)K_{v+1}$. □

Chapter 3
Enclosings when $u = 2$

3.1 Introduction

In this chapter, we settle Conjecture 2.1.1 in the case when $u = 2$; see Theorem 3.5.2. Yet again, new approaches are used in the proofs that follow.

In Theorem 3.5.2 we establish that the necessary conditions in Lemma 2.1.2 are sufficient when $u = 2$. Section 3.2 establishes two results: the first is very useful when finding 5-cycles in mK_v; the second shows that a judicious choice of parameters suggests a good approach to placing the edges incident with the 2 new vertices into 5-cycles. In Section 3.3, some general conditions on a set D are given that guarantee the multigraph induced by all the edges of differences in D have a closed trail in which each set of 3 consecutive edges induces a path. We will need to introduce Hamilton cycles in Section 3.3 in order to construct the closed trail. The proof of Theorem 3.5.2 builds on the results in Section 3.4 when small values of m are considered, and uses the results from Sections 3.2 and 3.3 to build the 5-cycles with mixed edges.

Recall that an integer v is said to be λ-admissible and λ is said to be v-admissible if the conditions in Theorem 1.2.2 are satisfied by λK_v.

The following decomposition result will be used to simplify some of the proofs given in Section 3.2.

Theorem 3.1.1. [6] *Let $1 \leq d_1, d_2 < v/2$. There exist 2 edge-disjoint Hamilton cycles in each component of $G_v(\{d_1, d_2\})$.*

3.2 Necessary tools

We now create some tools necessary to solve the problem of enclosing a λ-fold 5–CS of order v in a $(\lambda+m)$-fold 5–CS of order $v+u$ for all $m > 0$ and $u = 2$. For the remainder of this chapter, let $U = \{\infty_1, \infty_2\}$ be the vertex set of $V(K_{v+u}) \setminus V(K_v)$. Also for each pair of positive integers v and m, let $\lambda_{\max}(v, m)$ and $\lambda_{\min}(v, m)$ be the largest and smallest values of λ respectively for which conditions (a),(b) and (d) in Lemma 2.1.2 are satisfied when $u = 2$.

One such tool that we use is the concatenation of trails $T_1, T_2, \ldots, T_x$ in $G_v(D)$ which we denote by $\prod_{i=1}^{x} T_i = T_1 T_2 \cdots T_x$. (For example we use the trail $H_1 H_2$ formed by concatenating two Hamilton cycles in a component of $G_v(\{d_1, d_2\})$ obtained using Theorem 3.1.1.)

Again, Lemma 2.2.1 is a critical result, being used hand in hand with the observation that there exists a 5–CS of $G_v(\{v/5\})$ and of $G_v(\{2v/5\})$ when $v \equiv 0 \pmod 5$, since each component is a 5-cycle. So when $v \equiv 0 \pmod 5$ let $(\mathbb{Z}_v, \Gamma_1)$, $(\mathbb{Z}_v, \Gamma_2)$ and $(\mathbb{Z}_v, \Gamma_3)$ be a 5–CS of $G_v(\{v/5\}), G_v(\{2v/5\})$, and $G_v(\{1, 2, 3\})$ respectively.

For the remainder of this chapter, let

$$\begin{aligned}
a &= \lambda + m,\\
b &= \frac{2vd - (v+3)(\lambda+m)}{5},\\
c &= \frac{(3v-1)(\lambda+m) - vd}{5}, \text{ and}\\
d &= \begin{cases} \lambda + m & \text{if } \lambda + m < m(v-1)/2 \text{ and } v \not\equiv 4 \pmod 5,\\ \lambda + m - \epsilon & \text{if } \lambda + m < m(v-1)/2,\ v \equiv 4 \pmod 5, \text{ and}\\ & \lambda + m \equiv 3\epsilon \pmod 5 \text{ with } \epsilon \in \mathbb{Z}_5, \text{ and}\\ m(v-1)/2 & \text{if } \lambda + m \geq m(v-1)/2. \end{cases}
\end{aligned}$$

The following lemma guarantees that these parameters are all non-negative integers, and are specially chosen to satisfy even more properties, the reasons for requiring Conditions (*i*-*v*) being described after the lemma.

Lemma 3.2.1. *Let $u = 2$ and $v \geq 5$. Suppose there exists a 5–CS of λK_v. Let λ and m be positive integers that satisfy the conditions of Lemma 2.1.2. Then*

(i) a, b, c, *and* d *are all non-negative integers,*
(ii) b *is even,*
(iii) $2a + 3b + c = vd$,
(iv) $2a + 2b + 4c = 2v(\lambda + m)$,
(v) $mv(v-1)/2 - vd \geq 0$,
(vi) $c - a - 2b \geq 0$,
(vii) v *divides* $c - a - 2b$,
(viii) $m(v-1)/2 - d$ *is an integer, and*
(ix) $m(v-1)/2 - d \equiv 0 \pmod 5$ *when* $v \not\equiv 0 \pmod 5$.

Proof. First consider Condition (*i*). Clearly a is a non-negative integer since λ and m are non-negative integers. To show that b and c are non-negative integers, consider three cases that depend on the three values of d in turn.

Suppose $\lambda + m < m(v-1)/2$ and $v \not\equiv 4 \pmod 5$. So $d = \lambda + m$, $b = (\lambda+m)(v-3)/5$ and $c = (\lambda+m)(2v-1)/5$. It is clear that b and c are non-negative integers when $v \equiv 3 \pmod 5$. If $v \equiv 0$ or $1 \pmod 5$ then by Lemma 2.1.2(b) it follows $\lambda + m \equiv 0 \pmod 5$ and so b and c are non-negative integers. If $v \equiv 2 \pmod 5$ then $m \equiv 0 \pmod 5$ by Lemma 2.1.2(b) and $\lambda \equiv 0 \pmod 5$ by Theorem 1.2.2, so $\lambda + m \equiv 0 \pmod 5$ and thus b and c are non-negative integers.

Suppose $\lambda + m < m(v-1)/2$ and $v \equiv 4 \pmod 5$. Then $d = \lambda + m - \epsilon$ and

$\lambda + m \equiv 3\epsilon \pmod 5$ where $\epsilon \in \mathbb{Z}_5$. Since

$$\begin{aligned} 2vd - (v+3)(\lambda+m) &\equiv (v-3)(\lambda+m) - 2v\epsilon \\ &\equiv 3\epsilon(v-3) - 2v\epsilon \\ &\equiv \epsilon(v-9) \\ &\equiv 0 \pmod 5 \end{aligned}$$

and

$$\begin{aligned} (3v-1)(\lambda+m) - vd &\equiv (2v-1)(\lambda+m) + v\epsilon \\ &\equiv \epsilon(7v-3) \\ &\equiv 0 \pmod 5, \end{aligned}$$

it follows that b and c respectively are non-negative integers.

Finally suppose that $\lambda+m \geq m(v-1)/2$, so $d = m(v-1)/2$, $b = (mv(v-1)-(v+3)(\lambda+m))/5$ and $c = ((3v-1)(\lambda+m) - mv(v-1)/2)/5 = (2\lambda(3v-1) - m(v^2-7v+2))/10$. If $v \equiv 0$ or $1 \pmod 5$ then by Lemma 2.1.2(b), $\lambda + m \equiv 0 \pmod 5$, and so $mv(v-1)-(v+3)(\lambda+m) \equiv mv(v-1) \pmod 5$ and $(3v-1)(\lambda+m)-mv(v-1)/2 \equiv mv(v-1)/2 \pmod 5$; thus b and c are integers. If $v \equiv 2 \pmod{10}$ then $m \equiv 0 \pmod{10}$ by Lemma 2.1.2(a-b) and $\lambda \equiv 0 \pmod 5$ by Theorem 1.2.2, and so b and c are integers. If $v \equiv 7 \pmod{10}$ then $m \equiv 0 \pmod 5$ by Lemma 2.1.2(a) and so b and c are integers. If $v \equiv 3 \pmod 5$ then $\lambda \equiv 0 \pmod 5$ by Theorem 1.2.2 and we write $v = 5k+3$. Then

$$\begin{aligned} mv(v-1) - (v+3)(\lambda+m) &\equiv m(5k+3)(5k+2) - (5k+6)(\lambda+m) \\ &\equiv -6\lambda \\ &\equiv 0 \pmod 5 \end{aligned}$$

and

$$\begin{aligned} 2\lambda(3v-1) - m(v^2-7v+3) &\equiv 2\lambda(15k+8) - m((5k+3)^2 - 7(5k+3)+2) \\ &\equiv 2\lambda(15k+8) - m(25k^2+30k+9-35k-21+2) \\ &\equiv -m(25k^2-5k-10) \\ &\equiv -5m(5k^2-k-2) \\ &\equiv 0 \pmod{10}, \end{aligned}$$

since $5k^2-k-2$ is always even, and so b and c are integers. Let $v = 5k+4$. Then $\lambda \equiv 0 \pmod 5$, so b and c are integers since the numerator of b satisfies

$$\begin{aligned} mv(v-1) - (v+3)(\lambda+m) &\equiv m(5k+4)(5k+3) - (5k+7)(\lambda+m) \\ &\equiv 5m - \lambda(5k+7) \\ &\equiv 0 \pmod 5 \end{aligned}$$

and the numerator in the definition of c satisfies

$$\begin{aligned}2\lambda(3v-1)-m(v^2-7v+2) &\equiv 2\lambda(15k+11)-m((5k+4)^2-7(5k+4)+2)\\ &\equiv -m((25k^2+40k+16-35k-28+2)\\ &\equiv -m(25k^2+5k-10)\\ &\equiv -5m(5k^2+k-2)\\ &\equiv 0 \pmod{10}\end{aligned}$$

since $5k^2+k-2$ is always even.

Lastly, we will show d is a non-negative integer. By Lemma 2.1.2(a) and using Theorem 1.2.2(a) applied to the postulated existence of 5–CSs of λK_v and 5–CSs of $(\lambda+m)K_{v+u}$, $m(v-1)$ is even so d is an integer. It is clear that $d \geq 0$ if $\lambda+m < m(v-1)/2$ and $v \not\equiv 4 \pmod 5$ or $\lambda+m \geq m(v-1)/2$ so we let $\lambda+m < m(v-1)/2$ and $v \equiv 4 \pmod 5$ so that $d = \lambda+m-\epsilon$ where $\lambda+m \equiv 3\epsilon \pmod 5$ with $\epsilon \in \mathbb{Z}_5$. Since $v \equiv 4 \pmod 5$ and by Lemma 1.2.2(b), $\lambda \equiv 0 \pmod 5$ and so $\lambda \geq 5$ which ensures $d = \lambda+m-\epsilon \geq 0$ since $\epsilon \in \mathbb{Z}_5$.

It is left to show that Conditions (*ii-ix*) hold. If $\lambda+m < m(v-1)/2$, $v \not\equiv 4 \pmod 5$, and v is odd then $d = \lambda+m$ and so $b = (\lambda+m)(v-3)/5$ is even. If $\lambda+m < m(v-1)/2$, $v \not\equiv 4 \pmod 5$, and v is even then $\lambda+m$ is even by Lemma 2.1.2(a) and so b is even. Let $\lambda+m < m(v-1)/2$ and $v \equiv 4 \pmod 5$ so that $d = \lambda+m-\epsilon$ and $b = (m(v-3)-3\lambda+v(\lambda-2\epsilon))/5$. If $v \equiv 4 \pmod{10}$ then $\lambda \equiv 0 \pmod{10}$ and $m \equiv 0 \pmod 2$ by Theorem 1.2.2, so b is even. If $v \equiv 9 \pmod{10}$ then $b = (2v(m+\lambda-\epsilon)-(v+3)(\lambda+m))/5$ is even. Let $\lambda+m \geq m(v-1)/2$ so that $d = m(v-1)/2$ and $b = (mv(v-1)-(v+3)(\lambda+m))/5$. If v is odd, it is easy to see b is even. If v is even, then $\lambda+m$ is also even by Lemma 1.2.2(a) and so b is even. Thus Condition (*ii*) holds.

By using our definitions of a, b, c, and d, it follows that

$$\begin{aligned}2a+3b+c &= 2(\lambda+m)+3\left(\frac{2vd-(v+3)(\lambda+m)}{5}\right)+\frac{(3v-1)(\lambda+m)-vd}{5}\\ &= \frac{10\lambda+10m+6vd-3v\lambda-3vm-9\lambda-9m+3v\lambda+3vm-\lambda-m-vd}{5}\\ &= vd,\\ 2a+2b+4c &= 2(\lambda+m)+2\left(\frac{2vd-(v+3)(\lambda+m)}{5}\right)+4\left(\frac{(3v-1)(\lambda+m)-vd}{5}\right)\\ &= \frac{10(\lambda+m)+4vd-(2v+6)(\lambda+m)+(12v-4)(\lambda+m)-4vd}{5}\\ &= 2v(\lambda+m),\end{aligned}$$

$mv(v-1)/2 - vd \geq 0$, and

$$\begin{aligned} c-a-2b &= \frac{(3v-1)(\lambda+m)-vd}{5} - (\lambda+m) - 2\left(\frac{2vd-(v+3)(\lambda+m)}{5}\right) \\ &= \frac{(3v-1)(\lambda+m)-vd-5\lambda-5m-4vd+(2v+6)(\lambda+m)}{5} \\ &= v(\lambda+m-d). \end{aligned}$$

Thus Conditions (*iii-viii*) hold.

It remains to prove $m(v-1)/2 - d \equiv 0 \pmod 5$ when $v \not\equiv 0 \pmod 5$. Suppose $d = \lambda + m$ so that $m(v-1)/2 - d = (m(v-3) - 2\lambda)/2$. Since $\lambda \equiv 0 \pmod 5$ when $v \equiv 3 \pmod 5$, it follows that $m(v-3) - 2\lambda \equiv 0 \pmod 5$ and so $m(v-1)/2 - d \equiv 0 \pmod 5$. If $v \equiv 1 \pmod 5$ then $\lambda + m \equiv 0 \pmod 5$ and so $m(v-1)/2 - d \equiv 0 \pmod 5$. If $v \equiv 0$ or $2 \pmod 5$ then $\lambda \equiv m \equiv 0 \pmod 5$, and so $m(v-1)/2 - d \equiv 0 \pmod 5$.

Now suppose $d = \lambda + m - \epsilon$. Then $v \equiv 4 \pmod 5$ and $\lambda + m \equiv 3\epsilon \pmod 5$ and consequently $\lambda \equiv 0 \pmod 5$, so let $v = 5k + 4$. Since $m(v-1)/2 - d = (m(v-1) - 2d)/2$, it follows that

$$\begin{aligned} m(v-1) - 2d &\equiv 5km + m + 2\epsilon - 2\lambda \\ &\equiv (\lambda + m) + 2\epsilon - 3\lambda \\ &\equiv 3\epsilon + 2\epsilon - 3\lambda \\ &\equiv 5\epsilon - 3\lambda \\ &\equiv 0 \pmod 5. \end{aligned}$$

Finally suppose $d = m(v-1)/2$. Then clearly, $m(v-1)/2 - d = 0$ and so Condition (*ix*) is satisfied. □

At this point it is worth motivating the definitions of a, b, c, and d. There are 4 different types of 5-cycles in $(\lambda+m)K_{v+u}$ $(V \cup U, \mathcal{E})$ based on the number of pure and mixed edges in the 5-cycle and the location of the pure edges:

- 2 pure edges in V, 2 mixed edges, and 1 pure edge in U,
- 3 pure edges in V and 2 mixed edges,
- 1 pure edge in V and 4 mixed edges, and
- 5 pure edges in V.

These 5-cycles will be called Type $\mathcal{A}$, Type $\mathcal{B}$, Type $\mathcal{C}$, and pure cycles respectively. (See Figure 3.1 for a visual representation of these 5-cycles.)

When constructing a 5-cycle system of $(\lambda+m)K_{v+u} - \lambda K_v$ we endeavor to use a 5-cycles of Type $\mathcal{A}$, b of type $\mathcal{B}$, and c of Type $\mathcal{C}$. Lemma 3.2.1(i) shows this choice of a, b and c is well-defined. Lemma 3.2.1(iv) shows that this choice would use all $2v(\lambda+m)$ mixed edges. Lemma 3.2.1(ii) ensures that we can separate the 5-cycles of Type $\mathcal{B}$ so that $b/2$ of these 5-cycles are incident to ∞_1 and $b/2$ of these 5-cycles are incident

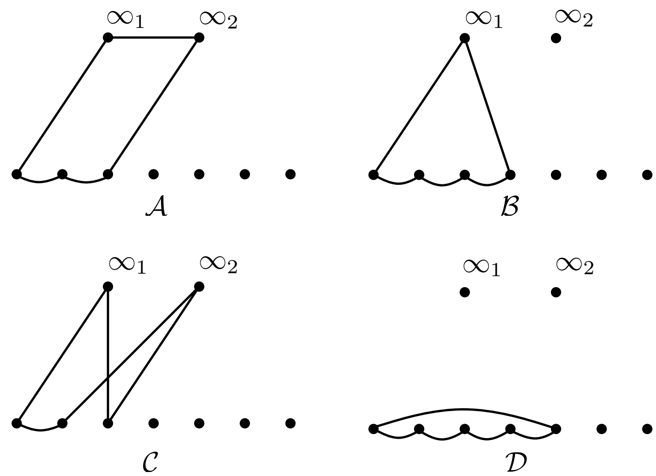

Figure 3.1: Possible 5-cycle Types

with ∞_2. The definition of a ensures that all $\lambda+m$ edges joining the two vertices in U would be in 5-cycles. By Lemma 3.2.1(*iii*) the choice of d ensures that the number of remaining edges joining vertices in V is $mv(v-1)/2-(2a+3b+c) = v(m(v-1)/2-d)$, which is a non-negative (by (v)) integral multiple of v or $v/2$ if v is odd or even respectively. This ensures that using difference methods is a viable technique to employ. In view of Lemma 3.2.1(*iii*) define α by

$$\frac{mv(v-1)}{2} - (2a+3b+c) = \left(\frac{m(v-1)}{2} - d\right) v = \alpha v. \tag{3.1}$$

So α is a measure of the number of differences we should use when defining the pure 5-cycles. Notice that if α is not an integer then v is even and α is divisible by $v/2$, so the "half" difference $v/2$ will play a vital role. Note that $\alpha \geq 0$ by Lemma 3.2.1(v).

3.3 Creating closed trail systems of $G_v(D)$

The following lemmas construct closed trail systems in components of $G_v(D)$ for various choices of D, each closed trail satisfying the important property that each set of 3 consecutive edges induces a path. These closed trails are eventually connected into a single closed trail in the proof of Lemma 3.3.7. Close attention is paid to the first and last edges in each closed trail so that when two such closed trails are concatenated the property that each set of 3 consecutive edges induces a path is maintained. In Lemma 3.3.8 this single closed trail is used to construct 5-cycles of Types $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

Within Lemmas 3.3.1-3.3.5, at times it will cause no confusion to simply use T_i instead of $T_i(D)$ to denote a trail of $G_v(D)$ that begins on vertex i.

Lemma 3.3.1. *Let D be a multiset consisting of x copies of each of $v/2$ and $v/3$. For some integer w there exists a closed trail system $\{T_i = T_i(x\{v/2, v/3\}) \mid i \in \mathbb{Z}_w\}$ of $G_v(D)$ such that for each $i \in \mathbb{Z}_w$:*

(i) *each set of* 3 *consecutive edges in* T_i *induces a path,*
(ii) *the first edge of* T_i *is an edge of difference* $v/3$*, the second edge of* T_i *is an edge of difference* $v/2$*, the last edge of* T_i *is an edge of difference* $v/2$*, and the second to last edge of* T_i *is an edge of difference* $v/3$*, and*
(iii) T_i *starts on vertex* i*.*

Proof. Since 2 and 3 divide v, let $v = 6w$. Each component of $G_v(\{v/3, v/2\})$ is isomorphic to $G_6(\{2,3\})$. Since $T = (0,4,1,5,2,0,3,1,4,2,5,3,0)$ is a closed trail that satisfies conditions (*i-ii*) for $G_v(\{2,3\})$, each closed trail T can be concatenated with another copy of T such that Conditions (*i-ii*) are satisfied. For each $i \in \mathbb{Z}_w$, form T_i from $\prod_{i=1}^{x} T$ by multiplying each vertex in the trail by w and adding i. Clearly conditions (*i-iii*) are satisfied by each closed trail T_i. So $\{T_i = T_i(x\{v/2, v/3\}) \mid i \in \mathbb{Z}_w\}$ is a closed trail system of $G_v(D)$ that satisfies Conditions (*i-iii*). □

Lemma 3.3.2. *Let* D *be a multiset consisting of* x *copies of each of* $v/3$ *and* $v/6$*. For some integer* w *there exists a closed trail system* $\{T_i = T_i(x\{v/3, v/6\}) \mid i \in \mathbb{Z}_w\}$ *of* $G_v(D)$ *such that for each* $i \in \mathbb{Z}_w$*:*

(i) *each set of* 3 *consecutive edges in* T_i *induces a path,*
(ii) T_i *can be chosen so that the first two edges are either* $\{i, i+v/6\}$ *and* $\{i + v/6, i+v/6-v/3\}$ *or* $\{i, i-v/6\}$ *and* $\{i-v/6, i-v/6+v/3\}$*,*
(iii) *the last two edges of* T_i *are both edges of difference* $v/3$*, and*
(iv) T_i *starts on vertex* i*.*

Proof. Each component of $G_v(\{v/3, v/6\})$ is isomorphic to $G_6(\{1,2\})$, so the required closed trail T_i can be formed from the concatenation of x copies of $T = (0,1,5,4,3,2,0,5,3,1,2,4,0)$ or of $T' = (0,5,1,2,3,4,0,1,3,5,4,2,0)$ (the choice is determined by the choice for the first and second edges of T_i in Condition (*ii*)) by multiplying each vertex in the trail by $v/6$ and adding i. Clearly Conditions (*i-iv*) are satisfied in each closed trail T_i. So $\{T_i = T_i(x\{v/3, v/6\}) \mid i \in \mathbb{Z}_v\}$ is the required closed trail system. □

Lemma 3.3.3. *Let* D *be a multiset consisting of* x *copies of* $v/3$ *and* x *elements in* $x\{1,2,\ldots,\lfloor v/2 \rfloor\} \setminus x\{v/2, v/3, v/6\}$*. Let* $j, k \in \mathbb{Z}_2$*. For some integer* w *there exists a closed trail system* $\{T_i = T_i(D, v/3) \mid i \in \mathbb{Z}_w\}$ *of* $G_v(D)$ *such that for each* $i \in \mathbb{Z}_w$*:*

(i) *each set of* 3 *consecutive edges in* T_i *induces a path,*
(ii) *the first edge in* T_i *is an edge of a difference in* $D \setminus x\{v/3\}$*,*
(iii) *the last two edges of* T_i *are edges of difference* $v/3$*,*
(iv) T_i *starts on vertex* i*, and*
(v) *if* $d \in D$ *and* $\gcd(\{d, v/3, v\}) = 1$ *then* $E(G_v(\{d, v/3\})) \subseteq E(T_0(D, v/3))$*.*

Proof. Let D be a multiset consisting of x copies of $v/3$ and x elements in $x\{1,2,\ldots,\lfloor v/2 \rfloor\} \setminus x\{v/2, v/3, v/6\}$. For each $d \in D \setminus x\{v/3\}$, name the components of $G_v(\{d, v/3\})$ with $G(d,i)$ for $0 \le i < \gcd(\{d, v/3, v\})$, where $G(d,i)$ contains vertex i. It is convenient to define $G(d,i)$ to be empty if $i \geq \gcd(\{d, v/3, v\})$. Notice that this implies that if $\gcd(\{d, v/3, v\}) = 1$ then $G(d,0) = G_v(\{d, v/3\})$, so $E(G_v(\{d, v/3\})) \subseteq E(T_0(D, v/3))$ as required by (v). We now form the required trail system as follows in which for each $i \in \mathbb{Z}_v$ $T_i(D, v/3)$ is a closed trail of $\bigcup_{d \in D \setminus x\{v/3\}} G(d,i)$.

By Theorem 3.1.1, for each $i \in \mathbb{Z}_v$ and each $d \in D \setminus x\{v/3\}$, if $G(d,i)$ is not empty then it can be decomposed into 2 Hamilton cycles $H_1(d,i)$ and $H_2(d,i)$. It is clear that in $G_v(\{v/3, d\})$ there are no 2-cycles, and since $v/6 \notin D$ the only 3-cycles are components of $G_v(\{v/3\})$. Clearly the edges between the vertices in $c(i) = \{i, i+v/3, i+2v/3\}$ cannot all appear in the same Hamilton cycle, so we can assume that:

(1) $H_1(d,i)$ and $H_2(d,i)$ contain one and two edges in $c(i)$ respectively, and
(2) vertex i is incident with edges of difference d and $v/3$ in each of $H_1(d,i)$ and $H_2(d,i)$.

Let $H_1(d,i)$ start on the edge of difference d, so by (2) it will end on the edge of difference $v/3$. Let $H_2(d,i)$ start on the edge of difference d, so by (1) it must end on two consecutive edges of difference $v/3$. Then clearly $H_1(d,i)H_2(d,i)$ has no set of 3 consecutive edges that induces a cycle. Since $v/6 \notin D$ and each $H_2(d,i)$ ends with two edges of difference $v/3$, $T_i(D, v/3) = \prod_{d \in D \setminus x\{v/3\}} H_1(d,i)H_2(d,i)$ also satisfies condition (i), so $T_i(D, v/3)$ is a closed trail of $\bigcup_{d \in D \setminus x\{v/3\}} G(d,i)$ satisfying Conditions (*i-iv*). Therefore $\{T_i(D, v/3) \mid i \in \mathbb{Z}_v\}$ is the required closed trail system. □

Lemma 3.3.4. *Let D be a multiset consisting of x copies of $v/2$ and x elements in $x\{1, 2, \ldots, \lfloor v/2 \rfloor\} \setminus x\{v/2, v/3, v/4\}$. For some integer w there exists a closed trail system $\{T_i = T_i(D, v/2) \mid i \in \mathbb{Z}_w\}$ of $G_v(D)$ such that for each $i \in \mathbb{Z}_w$:*

(i) each set of 3 consecutive edges in T_i induces a path,
(ii) T_i can be chosen so that if the first and second edges of T_i are $e_{i,1}$ and $e_{i,2}$ respectively then $(e_{i,1}, e_{i,2})$ is any of the 8 elements of $\{(\{i, i+(-1)^j y\}, \{i+(-1)^j y, i+(-1)^j y+(-1)^k z\}) \mid \{y, z\} = \{v/2, d\}, j, k \in \mathbb{Z}_2\}$ for some $d \in D \setminus (x\{v/2\})$,
(iii) exactly one of the last two edges of T_i has difference $v/2$,
(iv) T_i starts on vertex i,
(v) if $d \in D$ and $\gcd(\{d, v/2, v\}) = 1$ then $E(G_v(\{d, v/2\})) \subseteq E(T_0(D, v/2))$,
(vi) for any $d_i^* \in D \setminus x\{v/2\}$ if T_i contains an edge of difference d_i^* then the last edge of T_i has difference d_i^* or $v/2$, and
(vii) if $2 \in D \setminus (x\{v/2\})$ then $E(G_v(\{2, v/2\})) \subseteq E(\bigcup_{i=0}^{\gcd(\{v,v/2,2\})} T_i(D, v/2))$.

Proof. Let D be a multiset consisting of x copies of $v/2$ and x elements in $x\{1, 2, \ldots, \lfloor v/2 \rfloor\} \setminus x\{v/2, v/3, v/4\}$. For each $d \in D \setminus (x\{v/2\})$, let $v'(d) = v/\gcd(v, v/2, d)$ and $d' = d/\gcd(v, v/2, d)$. Then each component of $G_v(\{v/2, d\})$ is isomorphic to $G_{v'(d)}(\{v'(d)/2, d'\})$. Note that one of $v'(d)/2$ and d' must be odd. If at least one of them is even then a closed trail of $G_{v'(d)}(\{v'(d)/2, d'\})$ can be formed by alternately adding $v'(d)/2$ and d' (or choosing to always add $-d'$ instead of d') to reach the next vertex in the closed trail, starting with an edge of either difference. If both $v'(d)/2$ and d' are odd then a closed trail of $G_{v'(d)}(\{v'(d)/2, d'\})$ can be formed by alternately adding $v'(d)/2$ and d' (or $-d'$) for the first half of the closed trail, again starting with an edge of either difference, then alternately adding $v'(d)/2$ and $-d'$ (or d' respectively) for the second half of the closed trail. In this manner we can find closed trails in each component of $G_v(\{v/2, d\})$ which provide the required four choices for the first two edges, always ensuring that exactly one of the last two edges has difference $v/2$ which satisfies Conditions (ii) and (iii).

We now concatenate these closed trails to form the required closed trail system. Care must be taken to ensure that no 2-cycle and no 3-cycle is formed when two such trails (constructed using two elements of $D \setminus x\{v/2\}$) are concatenated. This is always ensured by carefully choosing the first two edges of the second path, based on the last two edges of the first path. If the first path ends on an edge of difference $v/2$ then since $d \neq v/4$ the choice of starting the second path with an edge of difference d or $-d$ allows 3-cycles to be avoided. If the first path ends on an edge of difference d' then starting the second path on an edge of difference $v/2$ and appropriately choosing the second edge to have difference d or $-d$ again avoids any 3-cycle.

For each $d \in D \setminus x\{v/2\}$ in turn, if $i < \gcd(\{v/2, d, v\})$ then let $P_i(\{v/2, d\})$ be the closed trail starting on vertex i in $G_v(\{v/2, d\})$ chosen as just described (so this is dependent on the order in which the trails are concatenated), and if $i \geq \gcd(\{v/2, d, v\})$ then let $P_i(\{v/2, d\}) = \varnothing$. Notice that if $\gcd(\{d, v, v/2\}) = 1$, then $G_v(\{v/2, d\})$ is a single component, so $E(G_v(\{d, v/2\}) \subseteq E(T_0(D, v/2))$ as required by (v). Also, notice that if $2 \in D \setminus (x\{v/2\})$ then $\gcd(\{2, v/2, v\}) = 1$ or 2 and so $P_i(\{v/2, 2\}) = \varnothing$ for $i > 0$ or $i > 1$ respectively as required by (vii). So for each $i \in \mathbb{Z}_v$, $T_i(D, v/2) = \prod_{d \in D \setminus (x\{v/2\})} P_i(\{v/2, d\})$ where the last trail in this concatenation is $P_i(\{v/2, d_i^*\})$ (so (vi) is satisfied) is a closed trail satisfying Conditions $(i$-$vii)$ and so $\{T_i(D, v/2) \mid i \in \mathbb{Z}_v\}$ is a closed trail system of $G_v(D)$ satisfying all of the listed conditions. □

In subsequent proofs, edges of difference $v/5$ can be problematic when trying to avoid 3-cycles in the concatenation process, so the following lemma uses a difference d^* at the end of the trail to separate edges of difference $v/5$ from edges in the next trail. To this end, d^* cannot be $2v/5$.

Lemma 3.3.5. *Let D be a multiset in which each element is $v/5$ except for a single element*
$d^* \notin \{v/2, v/3, v/5, 2v/5\}$. For some integer w there exists a closed trail system $\{T_i = T_i(D, v/5) \mid i \in \mathbb{Z}_w\}$ of $G_v(D)$ such that for each $i \in \mathbb{Z}_w$:

(i) each set of 3 consecutive edges in T_i induces a path,
(ii) the first and second edges of T_i can be chosen to be either $\{i, i + v/5\}$ and $\{i + v/5, i + 2v/5\}$, or $\{i, i - v/5\}$ and $\{i - v/5, i - 2v/5\}$,

(iii) T_i *ends on an edge of difference* d^**,*
(iv) T_i *starts on vertex* i*, and*
(v) if $\gcd(\{d^*, v, v/5\}) = 1$ *then* $E(G_v(\{d^*, v/5\})) \subseteq E(T_0(D, v/5))$*.*

Proof. Let D be a multiset in which each element is $v/5$ except for a single element $d^* \notin \{v/2, v/3, v/5, 2v/5\}$ and let $k = (v/5)/\gcd(\{v/5, d^*\})$. Let $P_i(\{v/5\})$ be the Euler tour of one component of $G_v((|D|-1)\{v/5\})$ chosen to meet condition (ii) (use edges of difference $(-1)^j v/5$ for the appropriate $j \in \mathbb{Z}_2$ to form the trail). The edges of difference d^* can be used to connect these tours as follows. If $i < \gcd(\{v/5, d^*\})$ then let

$$T_i = \left(\prod_{j\in\mathbb{Z}_k} P_{jd^*+i}(\{v/5\})(jd^*+i, (j+1)d^*+i)\right)(kd^*+i, (k+1)d^*+i, \ldots, -d^*+i, i).$$

Then $\{T_i = T_i(D, v/5) \mid i \in \mathbb{Z}_w\}$ is a closed trail system of $G_v(D)$ satisfying Conditions (*i-iv*). Notice that this implies that if $\gcd(\{d^*, v/5\}) = 1$ then $G_v(\{d^*, v/5\})$ is a single component, so $E(G_v(\{d, v/5\})) \subseteq E(T_0(D, v/5)))$. □

Each of the Lemmas 3.3.1 to 3.3.5 extend the definition of T_i so that if $i \in \mathbb{Z}_v \setminus \mathbb{Z}_w$ then define $T_i = \varnothing$. This will be important because trails formed using different lemmas will be concatenated, but the value of w may change from lemma to lemma.

Lemma 3.3.6. *Let the following trails be of those described in Lemmas* 3.3.1*,* 3.3.2*,* 3.3.3*,* 3.3.4*, and* 3.3.5 *with multisets* D_1, D_2, D_3, D_4 *and* D_5 *respectively are the multisets satisfying the respective conditions. Then in each case by judicious use of the choice available (as described in each lemma) for the first two edges of the second component trail, the following trails can be constructed so that each set of* 3 *consecutive edges induces a path:*

(i) $T_i((|D_1|/2)\{v/2, v/3\})T_i((|D_2|/2)\{v/3, v/6\})$,
(ii) $T_i((|D_1|/2)\{v/2, v/3\})T_i(D_3, v/3)$,
(iii) $T_i((|D_1|/2)\{v/2, v/3\})T_i(D_4, v/2)$,
(iv) $T_i((|D_1|/2)\{v/2, v/3\})T_i(D_5, v/5)$,
(v) $T_i((|D_1|/2)\{v/2, v/3\})T_i(\{d\})$ *where* $d \notin \{v/2, v/3, v/6\}$,
(vi) $T_i((|D_2|/2)\{v/3, v/6\})T_i(D_3, v/3)$,
(vii) $T_i((|D_2|/2)\{v/3, v/6\})T_i(D_4, v/2)$,
(viii) $T_i((|D_2|/2)\{v/3, v/6\})T_i(D_5, v/5)$,
(ix) $T_i((|D_2|/2)\{v/3, v/6\})T_i(\{d\})$ *where* $d \notin \{v/3, v/2\}$,
(x) $T_i(D_3, v/3)T_i(D_4, v/2)$,
(xi) $T_i(D_3, v/3)T_i(D_5, v/5)$,
(xii) $T_i(D_3, v/3)T_i(\{d\})$ *where* $d \notin \{v/2, v/3, v/6\}$,
(xiii) $T_i(D_4, v/2)T_i(D_5, v/5)$,
(xiv) $T_i(D_4, v/2)T_i(\{d\})$ *where* $d \notin \{v/2, v/3\}$,
(xv) $T_i(D_5, v/5)T_i(\{d\})$ *where* $d \notin \{v/2, v/3\}$*, and*

(xvi) $T_i(\{d\})T_i(\{d'\})$ *where* $d, d' \notin \{v/2, v/3, v/5\}$.

Proof. Let D_1, D_2, D_3, D_4, and D_5 be multisets defined as D is in Lemmas 3.3.1, 3.3.2, 3.3.3, 3.3.4, and 3.3.5 respectively. Then by Lemmas 3.3.1, 3.3.2, 3.3.3, 3.3.4, and 3.3.5, let $T_i(|D_1|/2\{v/2, v/3\})$, $T_i(|D_2|/2\{v/3, v/6\})$, $T_i(D_3, v/3)$, $T_i(D_4, v/2)$, and $T_i(D_5, v/5)$ respectively be the closed trails that satisfy the conditions outlined in the lemmas. Since in each case $d, d' \notin \{v/3, v/2\}$, each set of 3 consecutive edges in $T_i(\{d\})$ and in $T_i(\{d'\})$ induces a path. Therefore, in each case it remains to show that the choices permitted in how the trails start and end described in the lemmas can be exploited to ensure that as two trails are concatenated, the last 1 or 2 of the edges from the first closed trail followed by the first 2 or 1 edges from the second closed trail respectively induce a path.

Suppose the first of the two closed trails to be concatenated is $T_i(|D_1|/2\{v/2, v/3\})$. By Lemma 3.3.1, the last edge of $T_i(|D_1|/2\{v/2, v/3\})$ is an edge of difference $v/2$, and the second to last edge of $T_i(|D_1|/2\{v/2, v/3\})$ is an edge of difference $v/3$. Choose the first and second edge of the following trail as indicated:

- choose the first and second edge of $T_i(|D_2|/2\{v/3, v/6\})$ to be $\{i, i + v/6\}$ and $\{i + v/6, i + v/6 - v/3\}$ or $\{i, i - v/6\}$ and $\{i - v/6, i - v/6 + v/3\}$ respectively;
- choose the first and second edge of $T_i(D_3, v/3)$ to be $\{i, i - v/3\}$ and $\{i - v/3, i - v/3 - d_3\}$ or $\{i, i + v/3\}$ and $\{i + v/3, i + v/3 + d_3\}$ respectively where $d_3 \in D_3$;
- choose the first and second edge of $T_i(D_4, v/2)$ to be $\{i, i + v/2\}$ and $\{i + v/2, i + v/2 + d_4\}$ or $\{i, i + v/2\}$ and $\{i + v/2, i + v/2 - d_4\}$ respectively where $d_4 \in D_4$;
- choose the first and second edge of $T_i(D_5, v/5)$ to be $\{i, i + v/5\}$ and $\{i + v/5, i + 2v/5\}$ or $\{i, i - v/5\}$ and $\{i - v/5, i - 2v/5\}$ respectively; and
- choose the first and second edge of $T_i(\{d\})$ to be $\{i, i + d\}$ and $\{i + d, i + 2d\}$ respectively (since $d \notin \{v/2, v/3, v/6\}$).

This settles Conditions (*i*-*v*).

Now suppose the first closed trail is $T_i(|D_2|/2\{v/3, v/6\})$. By Lemma 3.3.2, the last two edges of $T_i(|D_2|/2\{v/3, v/6\})$ are edges of difference $v/3$. The first vertex in the second trail is of course vertex i, so it remains to ensure that the second and third vertices in the second trail avoid $v/3$ and $2v/3$ respectively, or avoid $2v/3$ and $v/3$ respectively, depending on whether the last edge of $T_i(|D_1|/2\{v/2, v/3\})$ is $\{i, i + v/3\}$ or $\{i, i + 2v/3\}$. In Case (*vi*) this is accomplished by Lemma 3.3.3(*ii*), noting that $v/6 \notin D_3$. This settles conditions (*vi*-*ix*).

If $T_i(D_3, v/3)$ is the first closed trail then since the last two edges of $T_i(D_3, v/3)$ are both edges of difference $v/3$, it follows that each set of 3 consecutive edges in $T_i(D_3, v/3)T_i(D_4, v/2)$, $T_i(D_3, v/3)T_i(D_5, v/5)$, and $T_i(D_3, v/3)T_i(\{d\})$ (since $d \notin \{v/2, v/3, v/6\}$) induces a path.

Suppose the first closed trail is $T_i(D_4, v/2)$. Since one of the last two edges of the closed trail $T_i(D_4, v/2)$ is an edge of difference $v/2$ and we have a choice of the first two edges of the closed trail $T_i(D_5, v/5)$, it follows that each set of 3 consecutive edges in the closed trail $T_i(D_4, v/2)T_i(D_5, v/5)$ induces a path. Since $d \notin \{v/2, v/3\}$, we can choose the first two edges of $T(\{d\})$ to be either $\{i, i+d\}$ and $\{i, i+2d\}$ or $\{i, i-d\}$ and $\{i, i-2d\}$ respectively so that each set of 3 consecutive edges in either $T_i(D_4, v/2)T_i((d))$ induces a path.

If the first closed trail is $T_i(D_5, v/5)$ then since $d^* \in D_5$ with $d^* \notin \{v/2, v/3, v/5, 2v/5\}$ and $d \notin \{v/2, v/3, v/5\}$, it follows that we can choose the first two edges of $T_i(\{d\})$ so that each set of 3 consecutive edges in the closed trail $T_i(D_5, v/5)T_i(\{d\})$ induces a path.

Finally, we can choose the first two edges of $T_i(\{d'\})$ so that each set of 3 consecutive edges in the closed trail $T_i(\{d\})T_i(\{d'\})$ induces a path. □

For the multiset D, let $m_D(d)$ be the number of times d appears in D. Note that we use the convention $G_v(\{d\}) = G_v(d)$ to simplify notation.

Lemma 3.3.7. *Let $v > 3$ be a positive integer. Let D be a multiset, each element of which is in $\{1, 2, \ldots, \lfloor v/2 \rfloor\}$. Suppose that the following conditions hold:*

(A) D contains a difference

$$d_1 = \begin{cases} 1 & \text{if } v \not\equiv 5 \pmod{10} \text{ and} \\ 4 & \text{otherwise} \end{cases}$$

and another difference $d_2 \notin \{v/2, v/3, v/4, v/5, 2v/5\}$ (possibly d_2 is a second copy of d_1 in D);

(B) $2m_D(v/2) + m_D(v/4) + \epsilon \leq |D \setminus \{d_1\}|$, where $\epsilon = 1$ if $\{v/2, v/3, v/6\} \subseteq D \setminus \{d_1, d_2\}$, $m_D(v/2) \geq m_D(v/3)$, and $m_D(v/6) > m_D(v/2) - \min(\{m_D(v/2), m_D(v/3)\})$ and $\epsilon = 0$ otherwise; and

(C) $2m_D(v/3) + m_D(2v/5) \leq |D \setminus \{d_1\}|$.

Then there exists a closed trail of $G_v(D)$ in which each set of 3 consecutive edges induces a path.

Proof. To form the required trail decomposition τ of $G_v(D)$, we begin by partitioning D into the 7 sets $D_1, D_2, \ldots, D_6$, and $\{d_1\}$ defined as follows. D will actually be partitioned in two different ways, depending on the occurrences of the differences $v/2$, $v/3$, and $v/6$ in D (as will be seen, this actually relates to whether or not a single copy of the difference $v/3$ needs to be set aside so as to be paired with a single copy of the difference $v/6$).

First, suppose that

$$\begin{gathered} \{v/2, v/3, v/6\} \subseteq D \setminus \{d_1, d_2\},\ m_D(v/2) \geq m_D(v/3), \\ \text{and } m_D(v/6) > m_D(v/2) - \min(\{m_D(v/2), m_D(v/3)\}); \end{gathered} \tag{3.2}$$

so in Condition (B), $\epsilon = 1$. Let D_1 consist of $z_1 = m_D(v/3) - 1$ copies of both $v/2$ and $v/3$. Let D_4 contain $m_D(v/2) - z_1$ copies of $v/2$ and $m_D(v/2) - z_1$ differences in $\delta_4 = D \setminus (D_1 \cup (m_D(v/2) - z_1)\{v/2\} \cup m_D(v/4)\{v/4\} \cup \{d_1, v/3\})$; choose these differences in D_4 so that $d_2 \in D_4$ only if $D_1 \cup D_4$ contains all of the differences in D besides $m_D(v/4)\{v/4\} \cup \{d_1, v/3\}$ (i.e. only if $|D_1|+|D_4| = |D \setminus (m_D(v/4)\{v/4\} \cup \{d_1, v/3\})|$). Notice that $D_1 \cup D_4$ contains $z_1 + m_D(v/2) - z_1 = m_D(v/2)$ copies of $v/2$ (that is, all the copies of $v/2$ in D). Also notice that by Condition (B) it follows that

$$\begin{aligned}|\delta_4| &= |D \setminus \{d_1\}| - m_D(v/2) - z_1 - m_D(v/4) - 1\\ &\geq 2m_D(v/2) + m_D(v/4) + 1 - m_D(v/2) - z_1 - m_D(v/4) - 1\\ &= m_D(v/2) - z_1 \geq 0\end{aligned}$$

so δ_4 is large enough for D_4 to be well defined. Let D_2 contain 1 copy of both $v/6$ and $v/3$. Notice that $D_1 \cup D_2$ contains all copies of $v/3$ in D. If $2v/5 \in D \setminus (D_1 \cup D_2 \cup D_4)$ then let D_5 contain $m_{D\setminus(D_1\cup D_2\cup D_4)}(v/5)$ copies of $v/5$ and a single copy of d_2 and otherwise, let $D_5 = \varnothing$. Let D_6 contain all of the remaining differences in D (i.e. $D_6 = D \setminus (D_1 \cup D_2 \cup D_4 \cup D_5 \cup \{d_1\})$).

Second, suppose that

$$\begin{aligned}\{v/2, v/3, v/6\} \nsubseteq D \setminus \{d_1, d_2\},\ m_D(v/2) < m_D(v/3),\\ \text{or } m_D(v/6) \leq m_D(v/2) - \min(\{m_D(v/2), m_D(v/3)\});\end{aligned} \tag{3.3}$$

so, in Condition (B), $\epsilon = 0$. Let D_1 consist of $z_2 = \min(\{m_D(v/3), m_D(v/2)\})$ copies of both $v/2$ and $v/3$. Let D_4 contain $m_D(v/2) - z_2$ copies of $v/2$ and $m_D(v/2) - z_2$ differences in $\delta_4 = D \setminus (D_1 \cup (m_D(v/2) - z_2)\{v/2\} \cup m_D(v/4)\{v/4\})$; choose these differences in D_4 so that D_4 contains as many copies of $v/6$ as possible and so that $d_2 \in D_4$ only if $D_1 \cup D_4$ contains all of the differences in $D \setminus (\{d_1\} \cup m_D(v/4)\{v/4\})$ (i.e. only if $|D_1| + |D_4| = |D \setminus (\{d_1\} \cup m_D(v/4)\{v/4\})|$). As before, $D_1 \cup D_4$ contains $z_2 + m_D(v/2) - z_2 = m_D(v/2)$ copies of $v/2$. Also notice that by Condition (B)

$$\begin{aligned}|\delta_4| &= |D \setminus \{d_1\}| - m_D(v/2) - z_2 - m_D(v/4)\\ &\geq 2m_D(v/2) + m_D(v/4) - m_D(v/2) - z_2 - m_D(v/4)\\ &= m_D(v/2) - z_2 \geq 0,\end{aligned}$$

so δ_4 is large enough for D_4 to be well defined. It will be important to note that under the set of assumptions in (3.3) one of the two following statements (i.e. (3.4) or (3.5)) must hold.

Case A: Suppose $\{v/2, v/3, v/6\} \subseteq D$ and either $m_D(v/2) < m_D(v/3)$ or $m_D(v/6) \leq m_D(v/2) - \min(\{m_D(v/2), m_D(v/3)\})$.

Notice that in this case if $m_D(v/2) < m_D(v/3)$ then $v/3 \in D \setminus (D_1 \cup D_4)$, and if $m_D(v/6) \leq m_D(v/2) - \min(\{m_D(v/2), m_D(v/3)\})$ then since D_4 contains as many copies of $v/6$ as possible, $v/6 \notin D \setminus (D_1 \cup D_4)$. Therefore the following is true:

$$\text{if } v/6 \in D \setminus (D_1 \cup D_4) \text{ then } v/3 \in D \setminus (D_1 \cup D_4). \tag{3.4}$$

Case B: Suppose $\{v/2, v/3, v/6\} \nsubseteq D$. Then clearly the following is true:

$$\text{if } v/6 \in D \text{ then } \{v/2, v/3\} \nsubseteq D. \tag{3.5}$$

Returning to the partitioning of D under the set of assumptions in (3.3), D_2 contains $\min(\{m_D(v/3) - z_2, m_{D\setminus D_4}(v/6)\})$ copies of both $v/6$ and $v/3$. Note that by (3.4) in Case A,

$$\text{if } v/6 \in D \setminus (D_1 \cup D_4) \text{ then } D_2 \neq \varnothing. \tag{3.6}$$

Let D_3 contain $y = m_D(v/3) - z_2 - \min(\{m_D(v/3) - z_2, m_{D\setminus D_4}(v/6)\})$ copies of $v/3$ together with y differences in

$$\delta_3 = D \setminus (m_D(v/2)\{v/2\} \cup m_D(v/3)\{v/3\} \cup m_D(v/6)\{v/6\} \cup m_D(2v/5)\{2v/5\} \cup \{d_1\});$$

choose these differences in D_3 so that $d_2 \in D_3$ only if $D_1 \cup D_2 \cup D_3 \cup D_4$ contains every difference in D except d_1 (i.e. only if $|D_1| + |D_2| + |D_3| + |D_4| = |D \setminus \{d_1\}|$). Notice that since $D_3 \neq \varnothing$ implies $m_D(v/2) < m_D(v/3)$ and $m_D(v/3) - z_2 > m_{d\setminus D_4}(v/6)$, and by Condition (C)

$$\begin{aligned}
|\delta_3| &= \begin{cases} |D \setminus \{d_1\}| - m_D(v/2) - m_D(v/3) - m_D(v/6) - m_D(2v/5) & \text{if } D_3 \neq \varnothing, \\ 0 & \text{if } D_3 = \varnothing \end{cases} \\
&\geq \begin{cases} 2m_D(v/3) + m_D(2v/5) - m_D(v/2) - m_D(v/3) - m_D(v/6) - m_D(2v/5) & \text{if } D_3 \neq \varnothing, \\ 0 & \text{if } D_3 = \varnothing \end{cases} \\
&= \begin{cases} m_D(v/3) - m_D(v/2) - m_D(v/6) & \text{if } D_3 \neq \varnothing, \\ 0 & \text{if } D_3 = \varnothing \end{cases} \\
&= \begin{cases} m_D(v/3) - \min(\{m_D(v/2), m_D(v/3)\}) - \min(\{m_D(v/3) - z_2, m_D(v/6) & \text{if } D_3 \neq \varnothing, \\ 0 & \text{if } D_3 = \varnothing \end{cases} \\
&= \begin{cases} m_D(v/3) - z_2 - \min(\{m_D(v/3) - z_2, m_D(v/6) & \text{if } D_3 \neq \varnothing, \\ 0 & \text{if } D_3 = \varnothing. \end{cases}
\end{aligned}$$

Then there are enough differences in $D \setminus (D_1 \cup D_2 \cup D_4)$ to construct D_3 as described. If $2v/5 \in D \setminus (D_1 \cup D_2 \cup D_3 \cup D_4)$ then let D_5 contain $m_{D\setminus(D_1\cup D_2\cup D_3\cup D_4)}(v/5)$ copies of $v/5$ and a single copy of d_2 and otherwise, let $D_5 = \varnothing$. Let D_6 contain all of the remaining differences, i.e. $D_6 = D \setminus (D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup \{d_1\})$.

Form a sequence $S = (s_1, s_2, \ldots, s_{|D_6|+5})$, the elements of which form a partition of $D \setminus \{d_1\}$ where $s_k = D_k$ for $k \in \{1, 2, 3, 4, 5\}$ and $s_6, s_7, \ldots, s_{|D_6|+5}$ are sets of size 1 such that $\bigcup_{k=6}^{|D_6|+5} s_k = D_6$; by choosing the copies of the difference 2 in D_6 to appear as early as possible in this ordering, we can assume that

$$\text{if } s_{|D_6|+5} = \{2\} \text{ then } D_6 = |D_6|\{2\}. \tag{3.7}$$

Then by Lemmas 3.3.1-3.3.5 for each $i \in \mathbb{Z}_v$ we can form closed trails $T_i(|D_1|/2\{v/2, v/3\})$, $T_i(|D_2|/2\{v/3, v/6\})$, $T_i(D_3, v/3)$, $T_i(D_4, v/2)$, and $T_i(D_5, v/5)$ respectively in each of which every set of 3 consecutive edges induces a path. Let d_3, d_4, and d_5 be $v/3$, $v/2$, and $v/5$ respectively. So for each $i \in \mathbb{Z}_v$ define the concatenation

$$\tau_i = T_i(|D_1|/2\{v/2, v/3\}) T_i(|D_2|/2\{v/3, v/6\}) \left(\prod_{k=3}^{5} T_i(D_k, d_k) \right) \left(\prod_{k=6}^{|D_6|+5} T_i(s_k) \right)$$

of these trails. Note that some of the trails being concatenated may be empty.

It will be important later to note that

$$\text{if } d \in D \setminus \{d_1\} \text{ and } \gcd(\{d, v\}) = 1 \text{ then } E(G_v(\{d\})) \subseteq \tau_0; \tag{3.8}$$

this follows by applying Condition (v) of Lemmas 3.3.3–3.3.5 to $T_i(D_k, d_k)$ for $k = 3$, 4, and 5 respectively, and the fact that $G_v(\{d\})$ is connected (so $T_i(\{d\})$ is empty if $i > 0$).

To check that τ_i has the property that no 3 consecutive edges form a cycle of length 2 or 3 consider the following. So the concatenation of each pair of trails making up τ_i must be considered, but this is done in Lemma 3.3.6 in all cases except one. The one case not considered is when $T_i(|D_1|/2\{v/2, v/3\})$ is followed in the concatenation by $T_i(\{v/6\})$ (this could happen if $D_k = \varnothing$ for $2 \leq k \leq 5$). If this case did occur, a 3-cycle would necessarily be formed by 3 consecutive edges (such as $(i+v/3, i, i+v/6, i+v/3)$), but by (3.4), (3.5), and (3.8) the concatenation $T_i(|D_1|/2\{v/2, v/3\})T_i(\{v/6\})$ will never occur in τ_i. If we are in Case B then this is clearly so since by (3.5) one of the trails $T_i(|D_1|/2\{v/2, v/2\})$ or $T_i(\{v/6\})$ is empty. Otherwise, since the value of w is the same in Lemmas 3.3.1 and 3.3.2, if $T_i(|D_1|/2\{v/2, v/3\}) \neq \varnothing$ then $T_i(|D_2|/2\{v/3, v/6\}) \neq \varnothing$. If $v/6 \in D \setminus (D_1 \cup D_4)$ then since $D_2 \neq \varnothing$ (by (3.6) if we are in Case A), the concatenation of $T_i(|D_1|/2\{v/2, v/3\})T_i(\{v/6\})$ will never occur in τ_i under the set of assumptions in (3.2), nor under the set of assumptions in (3.3). In other words, in such a situation τ_i is formed by first concatenating $T_i(|D_1|/2\{v/2, v/3\})$ with $T_i(|D_2|/2\{v/3, v/6\})$ which is then concatenated with $T_i(\{v/6\})$. Thus by Lemma 3.3.6 every set of 3 consecutive edges in τ_i induces a path.

It is critical to note that for all $i \geq 1$ we still have the choice available (as described in Lemmas 3.3.4 to 3.3.5) for the first two edges in the first component trail in τ_i.

The final step in forming the required closed trail τ is to use the edges in $G_v(\{d_1\})$ to connect the trails τ_i, $i \in \mathbb{Z}_v$. In so doing, we need to make sure that we maintain the property that every set of 3 consecutive edges induces a path. To do so, it turns out that even more flexibility is required since τ_0 and τ_{d_1} may need to be replaced with the reverse trails τ_0^- and $\tau_{d_1}^-$ respectively defined formally in the next paragraph. This choice is only needed for τ_0 and τ_{d_1} and not for τ_i if $i \notin \{0, d_1\}$; τ_0 is unusual because from (3.8), if there is an edge of difference d where $\gcd(\{d, v\}) = 1$, then this edge could only be found in τ_0.

Let $\{i, i + f_{i,1}\}$ and $\{i + f_{i,1}, i + f_{i,1} + f_{i,2}\}$ be the first and second edges of τ_i respectively. Similarly, let $\{i - f_{i,-1} - f_{i,-2}, i - f_{i,-1}\}$ and $\{i - f_{i,-1}, i\}$ be the second-to-last and last edges of τ_i respectively. Let τ_i^- denote the closed trail formed by mapping each edge $\{j_1, j_2\}$ in the closed trail τ_i to the edge $\{-j_1 + 2i, -j_2 + 2i\}$ in τ_i^-. (Informally, τ_i^- is formed from τ_i by also starting at vertex i but proceeding in the reverse direction around the v vertices.) So clearly τ_i^- has the property that each set of 3 consecutive edges induces a path.

We are now ready to define the required trail τ.

$$\text{If } d_1 \in \{-f_{0,-1}, -(f_{0,-1} + f_{0,-2})\}, \text{ or} \qquad (\text{P}i)$$
$$\text{if both } -f_{0,-1} = 2d_1 \text{ and } \tau_{d_1} = \varnothing \qquad (\text{P}ii)$$

(i.e. d_1 is one of the second-to-last or third-to-last vertices in τ_0 or $2d_1$ is the second-to-last vertex in τ_0), then begin τ with $\tau_0^* = \tau_0^-$ and

$$\text{otherwise begin } \tau \text{ with } \tau_0^* = \tau_0. \qquad (\text{P}iii)$$

Extend the trail by concatenating τ_0^* with the trail $(0, d_1)$ of length 1. Recursively define τ_i^* for all $i \geq 1$ as follows:

$$\text{For all } i + d_1 \geq 0 \text{ if } \tau_{i+d_1} \neq \varnothing,\ \tau_i^* \neq \varnothing, \text{ and } d_1 + f_{i+d_1,1} \in \{0, -f_{i,-1}^*\} \qquad (\text{P}iv)$$
$$\text{where } \{i, i - f_{i,-1}^*\} \text{ is the last edge of } \tau_i^*$$

then $\tau_{i+d_1}^* = \tau_{i+d_1}^-$ and otherwise $\tau_{i+d_1}^* = \tau_{i+d_1}$. For all $i \in \mathbb{Z}_v$ let $\{i, i + f_{i,1}^*\}$, $\{i + f_{i,1}^*, i + f_{i,1}^* + f_{i,2}^*\}$, $\{i - f_{i,-1}^* - f_{i,-2}^*, i - f_{i,-1}^*\}$, and $\{i - f_{i,-1}^*, i\}$ be the first, second, second-to-last, and last edges of τ_i^*. Then define

$$\tau = \prod_{i=0}^{v-1} \left(\tau_{id_1}^*(id_1, (i+1)d_1)\right).$$

It now requires some effort to show that all sets of 3 consecutive edges that contain the edge $\{id_1, (i+1)d_1\}$ induce a path, especially in the case when $i = 0$.

If $\tau_0(0, d_1)$ contains 3 consecutive edges that form a cycle then d_1 must be one of the last 3 vertices of τ_0, so $d_1 \in \{0, -f_{0,-1}, -(f_{0,-1} + f_{0,-2})\}$, in which case $\tau_0^* = \tau_0^-$ since clearly $d_1 \neq 0$. To see that not both $\tau_0(0, d_1)$ and $\tau_0^-(0, d_1)$ can contain 3 consecutive edges that form a cycle, suppose that $d_1 \in \{-f_{0,-1}, -(f_{0,-1} + f_{0,-2})\} \cap \{f_{0,-1}, f_{0,-1} + f_{0,-2}\}$. First note that if $d_1 = -(f_{0,-1} + f_{0,-2})$ then $d_1 \neq f_{0,-1} + f_{0,-2}$ and so $d_1 = f_{0,-1}$. Similarly if $d_1 = -f_{0,-1}$ then $d_1 = f_{0,-1} + f_{0,-2}$. Therefore in either case the last two edges of τ_0 and τ_0^- have difference d_1 and $2d_1$. But by Lemmas 3.3.1–3.3.5 the last two edges of τ_0 are either both edges of the same difference or one of the two edges has difference $v/5$ or $v/2$. Since $d_1 \notin \{v/2, v/5\}$, it is impossible for both τ_0 and τ_0^- to have the vertex d_1 in the last three vertices of either trail. So τ_0^* has been chosen such that every 3 consecutive edges in $\tau_0^*(0, d_1)$ induces a path.

To see that $\tau_0^*(0, d_1)\tau_{d_1}^*$ when $\tau_{d_1}^* \neq \varnothing$ does not contain 3 consecutive edges that form a cycle, consider the following. First note that since $\gcd(\{d_1, v\}) = 1$ by Condition (A), if $d = d_1$ and $d \in D \setminus \{d_1\}$ (that is, d is another copy of d_1 in D) then $E(G_v(\{d\})) \subseteq E(\tau_0^*)$, so neither the second vertex of τ_{d_1} nor the second vertex of $\tau_{d_1}^-$ equal 0 (the last vertex of τ_0^*). Second, note that if $T_{d_1}(D_4, v/2)$ is the first trail in $\tau_{d_1}^*$ then by Lemma 3.3.4(ii), the preamble in Lemma 3.3.6, and since $m_D(v/2)\{v/2\} \subseteq D_1 \cup D_4$, we can choose the first edge of $T_{d_1}(D_4, v/2)$ to not have difference $v/2$. Therefore if $d_1 + f_{d_1,1} = -f_{0,-1}^*$ (so $\tau_{d_1}^* = \tau_{d_1}^-$) then $d_1 - f_{d_1,1} \neq -f_{0,-1}^*$

(for otherwise subtracting this equality from the first implies that $2f_{d_1,1} = 0$, so $f_{d_1,1} = v/2$), so the second vertex of $\tau^*_{d_1}$ is not $-f^*_{0,-1}$.

To finish showing that each set of 3 consecutive edges in $\tau^*_0(0, d_1)\tau^*_{d_1}$ induces a path, we must show that the third vertex of τ_{d_1}, namely $d_1 + f^*_{d_1,1} + f^*_{d_1,2}$, is not v. Since $d_1 > 0$, we need only show that the third vertex of $\tau^*_{d_1}$, namely $d_1 + f^*_{d_1,1} + f^*_{d_1,2}$, is not 0. This requires considering several cases, most of which rely on the claim that $\gcd(\{f^*_{d_1,1}, f^*_{d_1,2}, v\}) \neq 1$ (later we will show that if $f^*_{d_1,1} \neq f^*_{d_1,2}$ then $\gcd(\{f^*_{d_1,1}, f^*_{d_1,2}\}) \neq 1$). If $f^*_{d_1,1} = f^*_{d_1,2}$ (that is, the first two edges of $\tau^*_{d_1}$ have the same difference) then $d_1 + f^*_{d_1,1} + f^*_{d_1,2} \neq 0$, for otherwise $d_1 = v - f^*_{d_1,1} - f^*_{d_1,2} = v - 2f^*_{d_1,1}$ contradicting the facts that $\gcd(\{v, f^*_{d_1}\}) \neq 1$ (by (3.8)), and $\gcd(\{v, d_1\}) = 1$ (by Condition (A)). So we can assume that the first two edges of $\tau^*_{d_1}$ have different differences: $f^*_{d_1,1} \neq f^*_{d_1,2}$. Then the first trail in $\tau^*_{d_1}$ cannot be $T_{d_1}(\{d\})$ for any difference $d \in D_6$ and cannot be $T_{d_1}(D_5, v/5)$ by Lemma 3.3.5 (since in such cases the first two edges have the same differences). So the first trail in $\tau^*_{d_1}$ must be one of the following: $T_{d_1}(|D_1|/2\{v/2, v/3\})$, $T_{d_1}(|D_2|/2\{v/3, v/6\})$, $T_{d_1}(D_3, v/3)$, or $T_{d_1}(D_4, v/2)$. If the first trail of $\tau^*_{d_1}$ is $T_{d_1}(D_3, v/3)$ or $T_{d_1}(D_4, v/2)$ then $d_1 + f^*_{d_1,1} + f^*_{d_1,2} \neq 0$ since otherwise $\gcd(\{v, d_1\}) = 1$ would mean that $\gcd(\{f^*_{d_1,1}, f^*_{d_1,2}\}) = 1$ which contradicts Lemma 3.3.3(v) or 3.3.4(v) respectively (since then $\gcd(\{d, d', v\}) = 1$ where $d' = v/3$ or $v/2$ respectively and $\{f^*_{d_1,1}, f^*_{d_1,2}\} = \{d, d'\}$, so these edges would be in τ^*_0 instead of $\tau^*_{d_1}$). If the first trail of $\tau^*_{d_1}$ is $T_{d_1}(|D_1|/2\{v/2, v/3\})$ or $T_{d_1}(|D_2|/2\{v/3, v/6\})$ then $d_1 + f^*_{d_1,1} + f^*_{d_1,2} \neq 0$ since otherwise by the descriptions of the first two edges of the trails constructed in Lemmas 3.3.1 and 3.3.2, $d_1 = v/6$ so by Condition (A), $v = 6$, and so these edges would be in τ^*_0 instead of in $\tau^*_{d_1}$. So each set of 3 consecutive edges in $\tau^*_0(0, d_1)\tau^*_{d_1}$ induce a path.

It remains to show that in each of the following trails T every set of 3 consecutive edges induces a path:

- if $i \geq 0$ then $T = (i, i + d_1, i + 2d_1, i + 3d_1)$;
- if $i > 0$, $\tau^*_{i+d_1} \neq \varnothing$, and $\tau^*_i \neq \varnothing$ then $T = \tau^*_i(i, i + d_1)\tau^*_{i+d_1}$;
- if $i \geq 0$, $\tau^*_{i+d_1} = \varnothing$, and $\tau^*_{i+2d_1} \neq \varnothing$ then $T = (i, i + d_1, i + 2d_1)\tau^*_{i+2d_1}$ (by (A) and the definition of τ this only happens if $d_1 = 4$, so $v \equiv 5 \pmod{10}$); and
- if $i \geq 0$, $\tau^*_i \neq \varnothing$, $\tau^*_{i+d_1} = \varnothing$ then $T = \tau^*_i(i, i + d_1, i + 2d_1)$.

We consider each of the four remaining cases in turn.

First since $v > 3$, it is clear that $(i, i + d_1, i + 2d_1, i + 3d_1)$ is a path.

Second suppose that $i > 0$, $\tau^*_{i+d_1} \neq \varnothing$, and $\tau^*_i \neq \varnothing$. We only need to consider the sets of 3 consecutive edges in $\tau^*_i(i, i + d_1)\tau^*_{i+d_1}$ which contain $\{i, i + d_1\}$. So we need to show that: (a) $i + d_1 \neq i - f^*_{i,-1}$, (b) $i \neq i + d_1 + f^*_{i+d_1,1}$, (c) $i + d_1 \neq i - f^*_{i,-1} - f^*_{i,-2}$, (d) $i - f^*_{i,-1} \neq i + d_1 + f^*_{i+d_1,1}$, and (e) $i \neq i + d_1 + f^*_{i+d_1,1} + f^*_{i+d_1,2}$. By (3.8) and since $i > 0$, $i + d_1 \neq i - f^*_{i,-1}$ and $i \neq i + d_1 + f^*_{i+d_1,1}$, so (a) and (b) are true. To see (d), first we use the flexibility in the component trails (see preamble of Lemma 3.3.6) to ensure that $\tau^*_{i+d_1}$ does not begin with an edge of difference $v/2$. This follows by condition (ii) in Lemmas 3.3.4 and 3.3.5, opting to choose the first edge in $T_{i+d_1}(D_4, v/2)$ to not have difference $v/2$ if it is the first trail in $\tau^*_{i+d_1}$. Therefore by (Piv), if $i - f^*_{i,-1} = i + d_1 + f_{i+d_1,1}$ then $\tau^*_{i+d_1} = \tau^-_{i+d_1}$ so $i - f^*_{i,-1} \neq i + d_1 - f_{i+d_1,1}$ (for otherwise subtracting this equality from the first implies that $2f_{i+d_1,1} = 0$, so $f_{i+d_1,1} = v/2$). Similarly by (Piv), if $i - f^*_{i,-1} = i + d_1 - f_{i+d_1,1}$ then $\tau^*_{i+d_1} = \tau_{i+d_1}$ so $i - f^*_{i,-1} \neq i + d_1 + f_{i+d_1,1}$. So (d) is true regardless of which of the two values, $\pm f_{i+d_1,1}$, $f^*_{i+d_1,1}$ takes on. We now turn to showing (c) is true. If $f^*_{i,-1} = f^*_{i,-2}$ then $i + d_1 \neq i - f^*_{i,-1} - f^*_{i,-2}$ for otherwise $d_1 = -f^*_{i,-1} - f^*_{i,-2}$ contradicting that $\gcd(\{v, f^*_{i,-1}\}) \neq 1$ (by (3.8)) and $\gcd(\{v, d_1\}) = 1$ (by (A)). So we can assume that $f^*_{i,-1} \neq f^*_{i,-2}$. Notice that the last trail in τ^*_i cannot be $T_i(\{d\})$ for any difference $d \in D_6$ and cannot be $T_i(|D_2|/2\{v/3, v/6\})$ or $T_i(D_3, v/3)$ by Lemmas 3.3.2 and 3.3.3 (since in such cases the last two edges have the same difference). So the last trail in τ^*_i must be one of the following: $T_i(|D_1|/2\{v/2, v/3\})$, $T_i(D_4, v/2)$, or $T_i(D_5, v/5)$. If the last trail of τ^*_i is $T_i(D_4, v/2)$ or $T_i(D_5, v/5)$ then $i + d_1 \neq i - f^*_{i,-1} - f^*_{i,-2}$ since otherwise $\gcd(\{v, d_1\}) = 1$ would mean that $\gcd(\{v, f^*_{i,-1}, f^*_{i,-2}\}) = 1$ which contradicts Lemma 3.3.4(v) or Lemma 3.3.5(v) respectively since $i > 0$. If the last trail of τ^*_i is $T_i(|D_1|/2\{v/2, v/3\})$ then $i + d_1 \neq i - f^*_{i,-1} - f^*_{i,-2}$ since otherwise $d_1 = v/6$ since $f^*_{i,-1} + f^*_{i,-2} \in \{v/2 \pm v/3\}$ which implies $v = 6$ by Condition (A) and so these edges would be in τ^*_0 instead of τ^*_i. This shows that (c) is true. Lastly, we focus on showing (e) is true. Since $i + d_1 > 0$, we need only show that the third vertex of $\tau^*_{i+d_1}$, namely $i + d_1 + f^*_{i+d_1,1} + f^*_{i+d_1,2}$, is not i. This requires considering several cases, most of which rely on the claim that $\gcd(\{f^*_{i+d_1,1}, f^*_{i+d_1,2}, v\}) \neq 1$ (later we will show that if $f^*_{i+d_1,1} \neq f^*_{i+d_1,2}$ then $\gcd(\{f^*_{i+d_1,1}, f^*_{i+d_1,2}\}) \neq 1$). If $f^*_{i+d_1,1} = f^*_{i+d_1,2}$ (that is, the first two edges of $\tau^*_{i+d_1}$ have the same difference) then $i + d_1 + f^*_{i+d_1,1} + f^*_{i+d_1,2} \neq i$, for otherwise $i + d_1 = i + v - f^*_{i+d_1,1} - f^*_{i+d_1,2} = i + v - 2f^*_{i+d_1,1}$ contradicting the facts that $\gcd(\{v, f^*_{i+d_1}\}) \neq 1$ (by (3.8)), and $\gcd(\{v, d_1\}) = 1$ (by Condition (A)). So we can assume that the first two edges of $\tau^*_{i+d_1}$ have different differences: $f^*_{i+d_1,1} \neq f^*_{i+d_1,2}$. Then the first trail in $\tau^*_{i+d_1}$ cannot be $T_{i+d_1}(\{d\})$ for any difference $d \in D_6$ and cannot be $T_{i+d_1}(D_5, v/5)$ by Lemma 3.3.5 (since in such cases the first two edges have the same differences). So the first trail in $\tau^*_{i+d_1}$ must be one of the following: $T_{i+d_1}(|D_1|/2\{v/2, v/3\})$, $T_{i+d_1}(|D_2|/2\{v/3, v/6\})$, $T_{i+d_1}(D_3, v/3)$, or $T_{i+d_1}(D_4, v/2)$. If the first trail of $\tau^*_{i+d_1}$ is $T_{i+d_1}(D_3, v/3)$ or $T_{i+d_1}(D_4, v/2)$ then $i + d_1 + f^*_{i+d_1,1} + f^*_{i+d_1,2} \neq i$ since otherwise $\gcd(\{v, d_1\}) = 1$ would mean that $\gcd(\{f^*_{i+d_1,1}, f^*_{i+d_1,2}\}) = 1$ which contradicts Lemma 3.3.3(v) or 3.3.4(v) respectively (since then $\gcd(\{d, d', v\}) = 1$ where $d' = v/3$ or $v/2$ respectively and $\{f^*_{i+d_1,1}, f^*_{i+d_1,2}\} = \{d, d'\}$, so these edges would be in τ^*_0 instead of $\tau^*_{i+d_1}$). If the first trail of $\tau^*_{i+d_1}$ is $T_{i+d_1}(|D_1|/2\{v/2, v/3\})$ or $T_{i+d_1}(|D_2|/2\{v/3, v/6\})$ then $i + d_1 + f^*_{i+d_1,1} + f^*_{i+d_1,2} \neq i$ since otherwise by the descriptions of the first two edges of the trails constructed in Lemmas 3.3.1 and 3.3.2, $d_1 = v/6$ so by Condition (A), $v = 6$, and so these edges would be in τ^*_0 instead of in $\tau^*_{i+d_1}$. So each set of 3 consecutive edges in $(i, i + d_1)\tau^*_{i+d_1}$ and $\tau^*_0(0, d_1)\tau^*_{d_1}$ induce a path. So every set of 3 consecutive edges in $\tau^*_i(i, i + d_1)\tau^*_{i+d_1}$ induces a path when $i > 0$.

Third, suppose that $i \geq 0$, $\tau^*_{i+d_1} = \varnothing$, and $\tau^*_{i+2d_1} \neq \varnothing$, so $v \equiv 5 \pmod{10}$. We will show that $(i, i + d_1, i + 2d_1, i + 2d_1 + f^*_{i+2d_1,1})$ is a path; that is we show that $i \neq i + 2d_1 + f^*_{i+2d_1,1}$. This follows since $d_1 = 4$ and $v \equiv 5 \pmod{10}$, so

$\gcd(\{2d_1, v\}) = 1$ and by (3.8), $\gcd(\{f_{i+2d_1,1}, v\}) \neq 1$ since $i + 2d_1 > 0$.

Finally, suppose that $i \geq 0$, $\tau_i^* \neq \varnothing$, and $\tau_{i+d_1}^* = \varnothing$. We consider the sets of 3 consecutive edges in $\tau_i^*(i, i+d_1, i+2d_1)$ that contain the edge $\{i, i+d_1\}$. We will break this into 2 cases: $v \equiv 5 \pmod{10}$ and $v \not\equiv 5 \pmod{10}$.

Case 1: Suppose $v \equiv 5 \pmod{10}$; then $d_1 = 4$. We need to show that $i + 4 \neq i - (f_{i,-1}^* + f_{i,-2}^*)$, $i - f_{i,-1}^* \neq i + 4$, and $i - f_{i,-1}^* \neq i + 8$. By (3.8) and since v is odd, if $-f_{i,-1}^* = 4$ or 8 then $i = 0$. By (Pi), if $-f_{0,-1} = 4$ then $\tau_0^* = \tau_0^-$ so $-f_{0,-1}^* = -4$, and if $-f_{0,-1} = -4$ then $\tau_0^* = \tau_0$ so $-f_{0,-1}^* = -4$; so in any case $-f_{0,-1}^* \neq 4$ (since $v \neq 8$). Similarly, by (Pii), if $-f_{0,-1} = 8$ then $\tau_0^* = \tau_0^-$ so $-f_{0,-1}^* = -8$, and if $-f_{0,-1} = -8$ then $\tau_0^* = \tau_0$ so $-f_{0,-1}^* = -8$; so in any case $-f_{0,-1}^* \neq 8$ (since $v \neq 16$). If $-f_{i,-1}^* - f_{i,-2}^* = 4$ and the last two edges of τ_i^* have the same difference, then since v is odd this difference is 2, so by (3.8), $i = 0$. So by (Pi), if $-f_{0,-1} - f_{0,-2} = 4$ and the last two edges of τ_i^* have the same difference then $\tau_0^* = \tau_0^-$ so $-f_{0,-1}^* - f_{0,-2}^* = -4$, and if $-f_{0,-1} - f_{0,-2} = -4$ and the last two edges of τ_i^* have the same difference then $\tau_0^* = \tau_0$ so $-f_{0,-1}^* - f_{0,-2}^* = -4$. Therefore in any case $-f_{0,-1}^* - f_{0,-2}^* \neq 4$. If $-f_{i,-1}^* - f_{i,-2}^* = 4$ and the last two edges of τ_i^* have different differences then since v is odd, the last trail in τ_i^* must be $T_i(D_5, v/5)$ (see Lemmas 3.3.1-3.3.5). But then by the definition of D_5, since $D_5 \neq \varnothing$, it follows that $2v/5 \in D_6$, which contradicts $T_i(D_5, v/5)$ being the last trail in τ_i^* (since the number of components in $G_v(D_5)$ is at most the number of components in $G_v(\{2v/5\})$, so $T_i(D_5, v/5) \neq \varnothing$ implies $T_i(D_6) \neq \varnothing$). Thus each set of 3 consecutive edges in $\tau_i^*(i, i+4, i+8)$ induces a path.

Case 2: Suppose $v \not\equiv 5 \pmod{10}$; then $d_1 = 1$. We need to show that $i - f_{i,-1}^* - f_{i,-2}^* \neq i + 1$, $i - f_{i,-1}^* \neq i + 1$, and $i - f_{i,-1}^* \neq i + 2$; this is done by contradiction. First suppose that either $-f_{i,-1}^* = 1$ or $-f_{i,-1}^* = 2$ and v is odd; so by (3.8) it follows that $i = 0$. Then by (Pi), if $-f_{0,-1} = 1$ then $\tau_0^* = \tau_0^-$ so $-f_{0,-1}^* = -1$, and by (Piii) if $-f_{0,-1} = -1$ then $\tau_0^* = \tau_0$ so $-f_{0,-1}^* = -1$; so in any case $-f_{0,-1}^* \neq 1$. Similarly, by (Pii), if $-f_{0,-1} = 2$ then $\tau_0^* = \tau_0^-$ so $-f_{0,-1}^* = -2$, and by (Piii) if $-f_{0,-1} = -2$ then $\tau_0^* = \tau_0$ so $-f_{0,-1}^* = -2$; so in any case $-f_{0,-1}^* \neq 2$. Next suppose that $-f_{i,-1}^* - f_{i,-2}^* = 1$ and the last two edges of τ_i^* have the same difference. Then v is odd and this difference is $(v-1)/2$, so by (3.8) and since $\gcd(\{(v-1)/2, v\}) = 1$, $i = 0$. By (Pi), if $-f_{0,-1} - f_{0,-2} = 1$ and the last two edges of τ_i^* have the same difference then $\tau_0^* = \tau_0^-$ so $-f_{0,-1}^* - f_{0,-2}^* = -1$, and by (P$iii$) if $-f_{0,-1} - f_{0,-2} = -1$ and the last two edges of τ_i^* have the same difference then $\tau_0^* = \tau_0$ so $-f_{0,-1}^* - f_{0,-2}^* = -1$; so in any case $-f_{0,-1}^* - f_{0,-2}^* \neq 1$. Now suppose that $-f_{i,-1}^* - f_{i,-2}^* = 1$ and the last two edges of τ_i^* have a different difference. Then by Lemmas 3.3.1-3.3.5, the last trail in τ_i^* must be one of the following trails: $T_i(|D_1|/2\{v/2, v/3\})$, $T_i(D_4, v/2)$, or $T_i(D_5, v/5)$. As described at the end of Case 1, $T_i(D_5, v/5)$ cannot be the last trail in τ_i^* (since $2v/5 \notin D_6$, so $D_5 = \varnothing$, eliminating this case). If $T_i(|D_1|/2\{v/2, v/3\})$ is the last trail in τ_i^* then $-f_{i,-1}^* - f_{i,-2}^* \neq 1$ since otherwise $\{f_{i,-1}^*, f_{i,-2}^*\} = \{v/2, \pm v/3\}$ and so $v = 6$ and $i = 0$, and thus by (Pi) we reach a contradiction (since if $-f_{0,-1} - f_{0,-2} = 1$ or -1 then $\tau_0^* = \tau_0^-$ or τ_0 respectively and so in any case $-f_{0,-1}^* - f_{0,-2}^* = -1$). If $T_i(D_4, v/2)$ is the last trail in τ_i^*, then $-f_{i,-1}^* - f_{i,-2}^* \neq 1$ since otherwise $\{f_{i,-1}^*, f_{i,-2}^*\} = \{v/2, d\}$ where $d \in D_4 \setminus m_{D_4}(v/2)\{v/2\}$ and $\gcd(\{d, v/2, v\}) = 1$, so $i = 0$ and thus by (Pi) we reach a contradiction (since if $-f^*_{0,-1} - f^*_{0,-2} = 1$ or -1 then $\tau^*_0 = \tau^-_0$ or τ_0 respectively and so $-f^*_{0,-1} - f^*_{0,-2} = -1$).

Finally suppose that $-f^*_{i,-1} = 2$ and v is even. The last trail in τ^*_i must be one of the following: $T_i(|D_1|/2\{v/2, v/3\})$, $T_i(|D_2|/2\{v/3, v/6\})$, $T_i(D_3, v/3)$, $T_i(D_4, v/2)$, $T_i(D_5, v/5)$, and $T_i(\{2\})$. If $T_i(|D_2|/2\{v/3, v/6\})$ or $T_i(D_3, v/3)$ is the last trail in τ^*_i then by Lemma 3.3.2 or 3.3.3 respectively, the last edge has difference $\pm v/3 = 2$, so $v = 6$ and $i = 0$, and thus by (P*ii*) and (P*iii*) we reach a contradiction (since if $-f_{0,-1} = 2$ or -2 then $\tau^*_0 = \tau^-_0$ or τ_0 respectively and so in any case $-f^*_{0,-1} \neq 2$). Similarly, if $T_i(|D_1|/2\{v/2, v/3\})$ is the last trail in τ^*_i then by Lemma 3.3.1(*ii*), $-f^*_{i,-1} = \pm v/2 = 2$ and so $v = 4$, thus $v/3 \notin D$, a contradiction. As previously described, if $T_i(D_5, v/5)$ is the last trail in τ^*_i then $2v/5 \notin D_6$, so $D_5 = \varnothing$, eliminating this case.

We now address when the last trail of τ^*_i is $T_i(\{2\})$ or $T_i(D_4, v/2)$, still assuming that $-f^*_{i,-1} = 2$ (so the last edge of τ^*_i has difference 2) and v is even. By (3.8) and Lemma 3.3.4(*vii*) respectively, it follows that $i \leq 1$ (to apply Lemma 3.3.4(*vii*) note that $\gcd(\{v, v/2, 2\}) \in \{1, 2\}$). Suppose $T_i(D_4, v/2)$ is the last trail in τ^*_i and $i = 0$. Then by (P*ii*), if $-f_{0,-1} = 2$ then $\tau^*_0 = \tau^-_0$ so $-f^*_{0,-1} = -2$ and by (P*iii*) if $-f_{0,-1} = -2$ then $\tau^*_0 = \tau_0$ so $-f^*_{0,-1} = -2$; in any case $-f^*_{0,-1} \neq 2$. If $T_i(\{2\})$ is the last trail in τ^*_i then $i = 1$ (to see that $i \neq 0$, note that if $i = 0$ then since v is even τ^*_{i+1} would contain edges of difference 2 thus contradicting $\tau^*_{i+1} = \varnothing$). So in all cases we can now assume that $i = 1$. If $T_1(\{2\})$ is the last trail in τ^*_1, then by (3.7), $m_{D_6}(2)\{2\} = D_6$. Furthermore, if $T_1(D_4, v/2)$ is the last trail in τ^*_1 and if D_4 contains a difference d^* not in $\{2, v/2\}$ then by Lemma 3.3.4(*vi*) we avoid this case (since then the last edge in τ^*_i would have difference $d^* \neq 2$); so in this situation we can assume that $D_4 = m_{D_4}(2)\{2\} \cup m_{D_4}(v/2)\{v/2\}$.

We are now in the case where $\tau^*_j = \varnothing$ for all $j \geq 2$, and the last trail in τ^*_1 is either $T_1(D_4, v/2)$ or $T_1(\{2\})$ where $D_4 = m_{D_4}(2)\{2\} \cup m_{D_4}(v/2)\{v/2\}$ or $D_6 = m_{D_6}(2)\{2\}$ respectively. This situation is most easily handled by altering the definitions of τ^*_1 and τ^*_{v-1} by changing τ_1 and τ_{v-1}. The only change is that one of three things is done to avoid the new τ^*_1 ending on an edge of difference 2:

$$T_1(D_4, v/2) \prod_{k=6}^{|D_6|+5} T_1(\{2\}) \text{ or } T_1(D_4, v/2) \text{ or } \prod_{k=6}^{|D_6|+5} T_1(\{2\}) \text{ is detached} \tag{3.9}$$

from the end of τ^*_1 and moved to τ^*_{v-1} as described below.

This will ensure that τ^*_1 (as newly defined) does not end on an edge of difference 2. We will then need to check that the altering of these trails does not create a 2-cycle or 3-cycle.

For any trail T, let $\pi(T)$ be formed by mapping each vertex $\ell \in \mathbb{Z}_v$ in the trail to the vertex $\ell + v - 2$ modulo v. Let

$$T'_{v-1}(D_4, v/2) = \begin{cases} \pi(T_1(D_4, v/2)) & \text{if } D_4 = m_{D_4}(2)\{2\} \cup m_{D_4}(v/2)\{v/2\}, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

So if $T'_{v-1}(D_4, v/2) \neq \varnothing$ then it is a closed trail that begins on vertex $v-1$ and satisfies $E(T'_{v-1}(D_4, v/2)) = E(T_1(D_4, v/2))$. To see this equality, notice that $E(T_1(D_4, v/2))$ is the set of edges in the component of $G_v((|D_4|/2)\{2, v/2\})$ that contains vertex 1; since v is even, this component also contains the vertex $v-1$. Similarly define

$$T'_{v-1}(\{2\}) = \begin{cases} \pi(T_1(\{2\})) & \text{if } 2 \in D_6, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

Clearly $E(T_1(\{2\})) = E(T'_{v-1}(\{2\}))$ and $T'_{v-1}(\{2\})$ starts and ends on vertex $v-1$. Since π preserves the distance between consecutive vertices in the trails $T_1(D_4, v/2)$ and $T_1(\{2\})$, each set of 3 consecutive edges in $T'_{v-1}(D_4, v/2)$ and $T'_{v-1}(\{2\})$ induces a path. Also, π preserves the flexibility to choose the first 2 edges in $T'_{v-1}(D_4, v/2)$ as stated in Lemma 3.3.4(*ii*) and the flexibility in our choice of the direction of the first edge in $T'_{v-1}(\{2\})$. Let

$$T'_1(D_4, v/2) = \begin{cases} \varnothing & \text{if } D_4 = m_{D_4}(2)\{2\} \cup m_{D_4}(v/2)\{v/2\}, \text{ and} \\ T_1(D_4, v/2) & \text{otherwise.} \end{cases}$$

We are now ready to redefine

$$\tau_1 = T_1(|D_1|/2\{v/2, v/3\})T_1(|D_2|/2\{v/3, v/6\})T_1(D_3, v/3)T'_1(D_4, v/2)$$

(notice that $T_1(D_5, v/5) = \varnothing$ since $2v/5 \notin D_6$) and redefine

$$\tau_{v-1} = T'_{v-1}(D_4, v/2)\left(\prod_{k=6}^{|D_6|+5} T'_{v-1}(\{2\})\right).$$

As described in (P*iv*), for all $j \geq 0$ if $\tau_{j+1} \neq \varnothing$, $\tau^*_j \neq \varnothing$, and $1 + f_{j+1,1} \in \{0, f^*_{j,-1}\}$ then we still define $\tau^*_{j+1} = \tau^-_{j+1}$ and otherwise $\tau^*_{j+1} = \tau_{j+1}$. Since $\tau_j = \varnothing$ for $2 \leq j \leq v-2$ and since $v > 3$ in this lemma, $\tau^*_j = \tau_j$ for $2 \leq j \leq v-1$. Then since $d_1 = 1$ we define

$$\tau = \prod_{j=0}^{v-1}\left(\tau^*_j(j, j+1)\right) = \tau^*_0(0,1)\tau^*_1(1, 2, 3, \ldots, v-1)\tau^*_{v-1}(v-1, 0).$$

In view of (3.9) it remains to check there are no 2-cycles or 3-cycles in the following trails: $\tau^*_1(1,2,3)$ and $(v-3, v-2, v-1)\tau^*_{v-1}(v-1, 0)$. Note that since $v > 3$ in this lemma we know that there are at least 2 edges of difference 1 between τ^*_1 and τ^*_{v-1}. Since τ^*_1 cannot end on an edge of difference 2 by (3.9) (recall that τ^*_1 does not contain the trail $T_1(\{2\})$ and if $T'_1(D_4, v/2) \neq \varnothing$ then D_4 contains a difference not in $\{2, v/2\}$ and by Lemma 3.3.4(*vi*) we let it be d^* so that we avoid this case), following the original argument shows that each set of 3 consecutive edges in $\tau^*_1(1,2,3)$ induces a path. By exercising the choice we have in the first two edges of $T'_{v-1}(D_4, v/2)$ and $T'_{v-1}(\{2\})$, we can ensure that the first edge of τ^*_{v-1} is $\{v-1, 1\}$ (an edge of difference 2); so $f^*_{v-1,1} = 2$. It remains to show that: $v - 3 \neq v - 1 + f^*_{v-1,1} = 1$; $v - 2 \neq v - 1 + f^*_{v-1,1} = 1$ (this situation cannot occur since $v > 3$); $v - 2 \neq v-1+f^*_{v-1,1}+f^*_{v-1,2} = 1+f^*_{v-1,2}$; $v-1-f^*_{v-1,-1} \neq 0$; and $v-1-f^*_{v-1,-2}-f^*_{v-1,-1} \neq 0$. If $v = 4$ then by Lemma 3.3.4(v) all edges would occur in τ^*_0, contradicting $i = 1$ so the first situation cannot occur. By Lemma 3.3.4 and the definition of $T'_{v-1}(\{2\})$ the second edge of τ^*_{v-1} is either $\{1, 3\}$ or $\{1, v/2 + 1\}$. Then $v - 2 \neq 1 + f^*_{v-1,2}$ since otherwise $v = 5$ (a contradiction since $v \not\equiv 5 \pmod{10}$) or $v = 6$ (a contradiction since $2 = v/3$ and $v/3 \notin D_4$) respectively, so the third situation is resolved. Since each edge in τ^*_{v-1} is an edge of difference 2 or $v/2$, it follows that $v - 1 - f^*_{v-1,-1} \neq 0$. Also since each edge in τ^*_{v-1} is an of difference 2 or $v/2$, it follows that $v-1-f^*_{v-1,-2}-f^*_{v-1,-1} \neq 0$ since otherwise $v = 6$, and so $2 = v/3$, a contradiction since $v/3 \notin D_4$.

Thus τ is a closed trail such that every set of 3 consecutive edges in τ induce a path. □

The following lemma makes use of the trail decomposition formed in Lemma 3.3.7 to construct a 5–CS using a specified number of 5-cycles of Type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. This lemma will be useful in using up all of the differences needed to cover the $2v(\lambda + m)$ mixed edges in $(\lambda + m)K_{v+u} - \lambda K_v$. Define $\deg_H(v)$ to be the degree of v in the multigraph of H.

For any two vertex disjoint graphs G_1 and G_2, define $G_1 \vee_t G_2$ to be the multigraph with vertex set $V(G_1 \vee_t G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee_t G_2) = E(G_1) \cup E(G_2) \cup (t\{\{g_1, g_2\} \mid g_1 \in V(G_1), g_2 \in V(G_2)\})$.

Lemma 3.3.8. *Suppose that:*

(a) the conditions of Lemma 3.2.1 are satisfied;
(b) there exist sets Δ_1 and Δ_2 of differences such that $\Delta_1 \cup \Delta_2 \subseteq D(v)_m$;
(c) $D = \Delta_1$ satisfies the conditions of Lemma 3.3.7;
(d) $|\Delta_1| + |\Delta_2| = d$; and
(e) $|\Delta_2| = (c - a - 2b)/v$.

Then there exists a 5–CS *of $G_v(\Delta_1 \cup \Delta_2) \vee_{\lambda+m} (\lambda + m)K_2$.*

Remark. Since v divides $c - a - 2b$ by Lemma 3.2.1(vii) and $c - a - 2b \geq 0$ by Lemma 3.2.1(vi), $|\Delta_2|$ is a non-negative integer.

Proof. Define the required 5–CS $(\mathbb{Z}_v \cup \{\infty_1, \infty_2\}, B)$ by following the approach described at the end of Section 3.2. The set B will contain a, b, and c 5-cycles of Types $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ respectively, thereby exhausting all the pure edges (as will be seen).

By Lemma 3.2.1(iii), $2a + 3b + c = vd = v(|\Delta_1| + |\Delta_2|)$. Edges in $G_v(\Delta_1)$ will be used to construct b paths of length 3 and $2b$ paths of length 1 (that are combined into $b/2$ trails of length 10), and a paths of length 2 and a paths of length 1 (that are combined into a paths of length 3); the remaining $c-a-2b$ (≥ 0 by Lemma 3.2.1(vi)) paths of length 1 will occur in $G_v(\Delta_2)$. So $|E(G_v(\Delta_1))| = 3b + 2a + 2b + a = 3a + 5b$ and since $G_v(\Delta_1)$ is regular, each vertex has degree $2(3a + 5b)/v$. Notice that by

Lemma 3.2.1(*iii*,*vii*), $3a + 5b$ is divisible by v.

By Lemma 3.3.7 there exists a closed trail $T = (g_0, g_1, \ldots, g_{v|\Delta_1|} = g_0)$ of $G_v(\Delta_1)$ such that every 3 consecutive edges induces a path. The number of edges in T is $vd - (c - a - 2b) = 5b + 3a$ (by Lemma 3.2.1(*iii*)). Arbitrarily break T into a set T_1 of $b/2$ trails that are each of length 10 and a set T_2 of a paths that are each of length 3. Form a set of 5-cycles $B_1 = \bigcup_{t \in T_1 \cup T_2} B_1(t)$ where

$$\begin{aligned}
B_1(g_i, g_{i+1}, g_{i+2}, g_{i+3}) &= \{(g_i, g_{i+1}, g_{i+2}, \infty_2, \infty_1), (g_{i+2}, g_{i+3}, \infty_2, g_{i+1}, \infty_1)\} \quad \text{and} \\
B_1(g_i, g_{i+1}, \ldots, g_{i+10}) &= \{(g_i, g_{i+1}, \infty_2, g_{i+2}, \infty_1), (g_{i+1}, g_{i+2}, g_{i+3}, g_{i+4}, \infty_1), \\
&\quad (g_{i+4}, g_{i+5}, \infty_1, g_{i+3}, \infty_2), (g_{i+5}, g_{i+6}, \infty_1, g_{i+7}, \infty_2), \\
&\quad (g_{i+6}, g_{i+7}, g_{i+8}, g_{i+9}, \infty_2), (g_{i+9}, g_{i+10}, \infty_2, g_{i+8}, \infty_1)\}
\end{aligned}$$

Notice that for each $t \in T_1 \cup T_2$, each vertex g_j in t is joined to ∞_1 and ∞_2 in the 5-cycles in $B_1(t)$ with $\deg_t(g_j)/2$ edges except for the first vertex and the last vertex in t if they are different, in which case they have odd degree in t, so the first is joined with one less edge to ∞_2 than to ∞_1 and the last is joined with one less edge to ∞_1 than to ∞_2. Therefore, since T is a closed trail, each vertex z in $\mathbb{Z}_v$ is joined to each of ∞_1 and ∞_2 with $\deg_T(z)/2 = (3a + 5b)/v$ edges in 5-cycles in B_1.

Finally, let $B_2 = \{(z, z + d, \infty_2, z + d + 1, \infty_1) \mid d \in \Delta_2, z \in \mathbb{Z}_v\}$. Then each vertex in $\mathbb{Z}_v$ is joined to each of ∞_1 and ∞_2 with $2(c - a - 2b)/v$ edges in 5-cycles in B_2.

Then each vertex $z \in \mathbb{Z}_v$ is joined to each of ∞_1 and ∞_2 with $\deg_T(z)/2 = (3a + 5b)/v$ edges in cycles in B_1 and $2(c-a-2b)/v$ edges in cycles in B_2. So each $z \in \mathbb{Z}_v$ is joined to each of ∞_1 and ∞_2 with $((3a+5b)+2(c-a-2b))/v = (a+b+2c)/v = \lambda+m$ by Lemma 3.2.1(*iv*). Also, each of the a trails of length 3 give rise to a edges joining ∞_1 to ∞_2 in B_1, so ∞_1 and ∞_2 are joined by $a = \lambda + m$ edges as required. Since clearly the edges of $G_v(\Delta_1 \cup \Delta_2)$ occur in T, $(\mathbb{Z}_v \cup \{\infty_1, \infty_2\}, B = B_1 \cup B_2)$ is a 5–CS of $G_v(\Delta_1 \cup \Delta_2) \vee_{\lambda+m} (\lambda + m)K_2$. □

3.4 The small cases for m

As in Chapter 2, we embark on a proof of the main result by beginning with a section which will prove the necessary conditions in Lemma 2.1.2 are sufficient when $u = 2$ and m is small.

Lemma 3.4.1. *Let $v = 10k$, λ and m be integers satisfying Conditions (a,b,d) of Lemma 2.1.2 with $u = 2$. If $m = 2$ or if $m \in \{4, 6, 8\}$ and $\lambda + m = 10$, then any 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+2}$.*

Proof. Let $(V = \mathbb{Z}_{10k}, A)$ be a 5–CS of λK_v and let $V \cup U$ be the vertex set of $(\lambda + m)K_{v+2}$ where $U = \{\infty_1, \infty_2\}$. Since (V, A) exists, it follows by Theorem 1.2.2 that λ is even. So m is even by Lemma 2.1.2(a). By Lemma 2.1.2(b) $\lambda + m \equiv 0 \pmod 5$. Therefore $\lambda + m \equiv 0 \pmod{10}$. Thinking of $v/2$ as half a difference, we begin by choosing αv edges to place in cycles containing only pure edges (edges between vertices only in V or only in U) using α differences in $D_v(m)$ where α is defined in Equation (3.1).

Suppose $m = 2$. Then $mK_v = 2K_v$ contains edges of each difference in $D(v)_2$. Care must be taken with edges of difference $v/2$ in $E(K_v)$ since there are only $v/2$ of them. To explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen), we make use of Skolem and hooked Skolem sequences.

By Theorem 1.1.1 or 1.1.2, let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k-2, y'_i > x'_i\}$ correspond to a Skolem sequence or hooked Skolem sequence of order $k-2$ with $k-2 \equiv 0$ or $1 \pmod 4$ or $k-2 \equiv 2$ or $3 \pmod 4$ respectively; so each element in $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\} \setminus \{2k-4\}$ respectively occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 3, y_i = y'_i + 3) \mid (x'_i, y'_i) \in P'\}$ if $k-2 \equiv 0$ or $1 \pmod 4$ and $P = \{(x_i = x'_i + 4, y_i = y'_i + 4) \mid (x'_i, y'_i) \in P'\}$ if $k-2 \equiv 2$ or $3 \pmod 4$. Then with

$$c_i = \begin{cases} (0, x_i, x_i - y_i, 3k, 5k+1+i) & \text{if } 1 \leq i \leq k-2, \\ (0, 3, 4k+2, k+2, 5k+1) & \text{if } i = k-1, \\ (0, 2k+1, 4k+1, 2k, 5k) & \text{if } i = k \text{ and } k-2 \equiv 0, 1 \pmod 4, \text{ and} \\ (0, 4, 2k+4, 5k+4, 5k) & \text{if } i = k \text{ and } k-2 \equiv 2, 3 \pmod 4, \end{cases} \tag{3.10}$$

(where arithmetic is done modulo v) form the set of 5-cycles $B' = \left\{c''_{i-1} = c_i \mid 1 \leq i \leq k\right\}$ where computations are done modulo v. Table 3.1 lists the differences of edges that occur in cycles in B' when α is as big as possible (from Lemma 3.2.1 this happens when λ is as small as possible and not 0, namely when $\lambda = 8$, and so when $\alpha = 10k - 11$; notice that when $\lambda = 0$ a 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+u}$ by Theorem 1.2.2).

Edges	Differences when $k-2 \equiv 0, 1 \pmod 4$	Differences when $k-2 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k-1$	$5, 6, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k\}$	$3k+1, 3k+2, \ldots, 4k-2$	$3k+1, 3k+2, \ldots, 4k-2$
$\{3k, 5k+1+i\}$	$2k+2, 2k+3, \ldots, 3k-1$	$2k+2, 2k+3, \ldots, 3k-1$
$\{5k+1+i, 0\}$	$5k-2, 5k-3, \ldots, 4k+1$	$5k-2, 5k-3, \ldots, 4k+1$
c_i if $i = k-1$	$3, 3k, 4k-1, 4k-1, 5k-1$	$3, 3k, 4k-1, 4k-1, 5k-1$
c_i if $i = k$	$2k, 2k+1, 2k+1, 3k, 5k$	$4, 4, 2k, 3k, 5k$

Table 3.1: Edges of differences in B' when α is as large as possible, $v \equiv 0 \pmod{10}$, and $m = 2$

Let $B'' = 2B'$ be the multiset consisting of 2 copies of each 5-cycle in B'. Define $B = \{c''_i + j \mid i \in \mathbb{Z}_{\alpha/5}, c''_{i \pmod k} \in B'', j \in \mathbb{Z}_v\} \cup E_\alpha$ where

$$E_\alpha = \begin{cases} \varnothing & \text{if } \alpha \equiv 0 \pmod 5, \\ \Gamma_2 & \text{if } \alpha \equiv 1 \pmod 5, \\ \Gamma_1 \cup \Gamma_2 & \text{if } \alpha \equiv 2 \pmod 5, \\ \Gamma_3 & \text{if } \alpha \equiv 3 \pmod 5, \text{ and} \\ \Gamma_2 \cup \Gamma_3 & \text{if } \alpha \equiv 4 \pmod 5. \end{cases}$$

Notice that c_{k-1} and c_k will only be chosen once in B. Fortunately when λ is as small as possible (α is as large as possible) $\alpha = 10k - 11$ and so $E_\alpha = \varnothing$ allowing the differences 3 and $2k$ to be used in c_{k-1}.

Let D be the set of differences not used in forming B. Notice that because $m = 2$, $\{1, 2\} \subseteq D$. Partition D into two sets Δ_1 and Δ_2 so that $\{1, 2\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/2$, $v/3$, $v/4$, $v/5$, or $2v/5$ as possible. When λ is as small as possible, it is clear that Conditions (A-D) of Lemma 3.3.7 are satisfied. Whenever λ increases α will decrease by 10 and so 10 more differences (at most 2 copies of the same difference) will be contained in D. So there are still enough different differences for any α value ensuring Δ_1 still satisfies Conditions (A-D) of Lemma 3.3.7. So by Lemma 3.3.8, there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda + m)K_2$ which covers each mixed edge $2v(\lambda + m)$ times. Thus, $(\mathbb{Z}_{10k} \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v + 2, \lambda + 2)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$.

Let $m \in \{4, 6, 8\}$ and $\lambda + m = 10$, so $d = \lambda + m$. Now let $P' = \{(x_i, y_i) \mid 1 \le i \le k - 2, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k - 2$ for the set $\{1, 2, \ldots, 2k - 4\}$ or $\{1, 2, \ldots, 2k - 3\}$ if $k - 2 \equiv 0$ or $1 \pmod 4$ or $k - 2 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k - 4\}$ or $\{1, 2, \ldots, 2k - 3\} \setminus \{2k - 4\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 3, y_i = y'_i + 3) \mid (x'_i, y'_i) \in P'\}$ when using Skolem sequences and $P = \{(x_i = x'_i + 4, y_i = y'_i + 4) \mid (x'_i, y'_i) \in P'\}$. Then with c_i as defined in Equation (3.10) form the set of 5-cycles $B' = 2\{c_i + j \mid j \in \mathbb{Z}_v, 1 \le i \le k - 2, (x_i, y_i) \in P\} \cup \{c_i + j \mid j \in \mathbb{Z}_v, i \in \{k - 1, k\}\}$ where computations are done modulo v. Let $P^* = \{(x_i, y_i) \mid 1 \le i \le k - 1, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k - 1$ for the set $\{1, 2, \ldots, 2k - 2\}$ or $\{1, 2, \ldots, 2k - 3\}$ if $k - 1 \equiv 0$ or $1 \pmod 4$ or $k - 1 \equiv 2$ or $3 \pmod 4$ respectively. Then with

$$c'_i = \begin{cases} (0, x_i, x_i - y_i, 3k, 5k + i) & \text{if } 1 \le i \le k - 1, \text{ and} \\ (0, 2k, 6k, 9k, 5k) & \text{if } i = k, \end{cases}$$

for the set of 5-cycles $B'' = 2\{c'_i + j \mid j \in \mathbb{Z}_v, 1 \le i \le k - 1, (x_i, y_i) \in P^*\} \cup \{c'_k + j \mid j \in \mathbb{Z}_v\}$. Table 3.2 lists the differences of edges that occur in cycles in B'' when α is as large as possible (from Lemma 3.2.1 this happens when λ is as small as possible and not 0; notice that when $\lambda = 0$ a 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+u}$ by Theorem 1.2.2).

Edges	Differences when $k - 1 \equiv 0, 1 \pmod 4$	Differences when $k - 1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k - 2$	$1, 2, \ldots, 2k - 3, 2k - 1$
$\{x_i - y_i, 3k\}$	$3k + 1, 3k + 2, \ldots, 4k - 1$	$3k + 1, 3k + 2, \ldots, 4k - 1$
$\{3k, 5k + i\}$	$2k + 1, 2k + 3, \ldots, 3k - 1$	$2k + 1, 2k + 3, \ldots, 3k - 1$
$\{5k + i, 0\}$	$5k - 1, 5k - 2, \ldots, 4k + 1$	$5k - 1, 5k - 2, \ldots, 4k + 1$
c'_i if $i = k$	$2k, 3k, 4k, 4k, 5k$	$2k, 3k, 4k, 4k, 5k$

Table 3.2: Edges of differences in B'' when α is as large as possible, $v \equiv 0 \pmod{10}$, and $m \in \{4, 6, 8\}$

Then $((m-2)/2)B'' \cup B'$ contains $(m-2)(2k-1)/2 + 2k - 2 = mk - m/2 - 1$ 5-cycles, but we need $\lfloor \alpha/5 \rfloor = mk-3$ of them. Hence we need 1 or 2 more 5-cycles when $m = 6$ or $m = 8$ respectively. Let $c_1'' = (0, 2k, 4k, 6k, 3k)$ and $c_2'' = (0, 2k, 4k, 7k, 3k)$. Then let E'_α contain 1 copy of c_1' if $m = 6$ or 8 and 1 copy of c_2' if $m = 8$. Notice that the 5-cycles in E'_α contain at most m copies of differences of edges not used in cycles in $((m-2)/2)B'' \cup B'$. Form the set of 5-cycles

$$B = ((m-2)/2)B'' \cup B' \cup \{c_i'' + j \mid j \in \mathbb{Z}_v, c_i'' \in E'_\alpha\} \cup E_\alpha.$$

Since $\lambda + m = 10$, it follows that $(m(v-1)/2 - \alpha) = \lambda + m = 10$. Let D be the set of differences not used in B. Notice that D contains at least $(m-2)/2$ copies of $\{2k, 3k, 2k-1\}$ when $k-1 \equiv 0$ or $1 \pmod 4$ or $(m-2)/2$ copies of $\{2k, 3k, 2k-2\}$ when $k-1 \equiv 2$ or $3 \pmod 4$. Partition D into two sets Δ_1 and Δ_2 so that $\{1, 2\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/2$, $v/3$, $v/4$, $v/5$, or $2v/5$ as possible. In this way, Conditions (A-D) of Lemma 3.3.7 are satisfied for Δ_1. So by Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Therefore $(\mathbb{Z}_v \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+m)$ with $m \in \{4, 6, 8\}$ and $\lambda + m = 10$. □

The proof of the next several results follows that of Lemma 3.4.1 closely, but are sufficiently different and important that they are described in detail.

Lemma 3.4.2. *Let $v = 10k+5$ and assume Conditions (a,b,d) of Lemma* 2.1.2 *are satisfied. If $m = 1$ or if $m \in \{2, 3, 4\}$ and $\lambda + m = 5$, then any* 5–CS *of λK_v can be enclosed in a* 5–CS *of $(\lambda+m)K_{v+2}$.*

Proof. Let $(V = \mathbb{Z}_{10k+5}, A)$ be a 5–CS of λK_v and let $U = \{\infty_1, \infty_2\}$. Since (V, A) exists, $\lambda + m \equiv 0 \pmod 5$ by Lemma 2.1.2(a). As in Lemma 3.2.1, the number of edges we must place in 5-cycles contained completely in mK_v is

$$\left(\frac{m(v-1)}{2} - \frac{2a+3b+c}{v}\right) v = \alpha v.$$

Since this is divisible by v, our plan is again to create 5-cycles in mK_v using all the edges of α specially chosen differences. By Lemma 3.3.8, α is a non-negative integer.

Suppose $m = 1$. Then $mK_v = K_v$ contains edges of each difference in $D(v) = \{1, \ldots, 5k+2\}$. As in Lemma 3.4.1, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

We begin with the case where $k \geq 2$. Let $P' = \{(x_i', y_i') \mid 0 \leq i \leq k-3, y_i' > x_i'\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ from the set $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\}$ if $k-2 \equiv 0$ or $1 \pmod 4$ or $k-2 \equiv 2$ or $3 \pmod 4$ respectively (this exists by Theorems 1.1.1 and 1.1.2 since $k \geq 3$). So each element in $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\} \setminus \{2k-4\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x_i' + 4, y_i = y_i' + 4) \mid (x_i', y_i') \in P'\}$

when using Skolem sequences and $P = \{(x_i = x'_i + 5, y_i = y'_i + 5) \mid (x'_i, y'_i) \in P'\}$ when using hooked Skolem sequences. Notice that because $\alpha = 5k+2-d = 5k+2-(\lambda+m)$ or 0 (depending on d) and $\lambda + m \equiv 0 \pmod 5$, it follows that $\alpha \equiv 0$ or $2 \pmod 5$. Then with

$$c_i = \begin{cases} (0, x_i, x_i - y_i, 4k+2, 7k+3-i) & \text{if } 0 \leq i \leq k-3 \text{ and,} \\ (0, 3k+1, 6k+3, k+2, 5k+3) & \text{if } i = k-2, \end{cases} \tag{3.11}$$

we form the set of 5-cycles $B = \left\{c_{i \pmod{k-1}} + j \mid i \in \mathbb{Z}_{\lfloor \alpha/5 \rfloor}, j \in \mathbb{Z}_v\right\} \cup E_\alpha$ where the computations are done modulo v, $k \geq 2$, and where $E_\alpha = \varnothing$ or $\Gamma_1 \cup \Gamma_2$ if $\alpha \equiv 0$ or $2 \pmod 5$ respectively. Table 3.3 lists the differences of edges that occur in cycles in $B \setminus E_\alpha$ if α is as big as possible (from Lemma 3.2.1 this happens when λ is as small as possible, namely when $\lambda = 4$ and so $\alpha = 5k-3$). By design, the edges of differences

Edges	Differences when $k-2 \equiv 0,1 \pmod 4$	Differences when $k-2 \equiv 2,3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$5, 6, \ldots, 2k$	$6, 7, \ldots, 2k, 2k+2$
$\{x_i - y_i, 4k+2\}$	$4k+3, 4k+4, \ldots, 5k$	$4k+3, 4k+4, \ldots, 5k$
$\{4k+2, 7k+3-i\}$	$3k, 3k-1, \ldots, 2k+3$	$3k, 3k-1, \ldots, 2k+3$
$\{7k+3-i, 0\}$	$3k+3, 3k+4, \ldots, 4k$	$3k+3, 3k+4, \ldots, 4k$
c_i if $i = k-2$	$3k+1, 3k+2, 4k+1, 5k+1, 5k+2$	$3k+1, 3k+2, 4k+1, 5k+1, 5k+2$

Table 3.3: Edges of differences in $B \setminus E_\alpha$ when α is as large as possible and $v \equiv 5 \pmod{10}$

$v/5$ and $2v/5$ do not occur in 5-cycles in $B \setminus E_\alpha$, hence are available to form the 5-cycles in E_α. If $k = 0$ then $v = 5$ and it is easy to see how to form the set 5-cycles B using differences so that B contains only 5-cycles with 5 pure edges. If $k = 1$ then $\alpha \leq 6 - \lambda$, so form the set of 5-cycles $B = \lfloor \alpha/5 \rfloor \{(0, 1, 3, 6, 10) + j \mid j \in \mathbb{Z}_v\} \cup E_\alpha$.

Let D be the set of differences not used in B. From Table 3.3 we know $\{2, 4\} \in D$. Partition D into two sets Δ_1 and Δ_2 so that $\{2, 4\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/3$, $v/5$, or $2v/5$ as possible. If λ is as small as possible (α is as large as possible) then $\alpha = 5k-3$ and so $1, 3 \in D$. From this, it is clear that Conditions (A-D) in Lemma 3.3.7 are satisfied for Δ_1 when λ is as small as possible. As λ increases (α decreases) the differences used to construct c_i are added to D. In this manner, it is clear that Δ_1 still satisfies Conditions (A-D) of Lemma 3.3.7. So by Lemma 3.3.8 we can form a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Thus, $(\mathbb{Z}_{10k+5} \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+1)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$.

Let $m \in \{2, 3, 4\}$ and $\lambda + m = 5$. Now let $P' = \{(x_i, y_i) \mid 0 \leq i \leq k-3, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ for the set $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\}$ if $k-2 \equiv 0$ or $1 \pmod 4$ or $k-2 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\} \setminus \{2k-4\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 4, y_i = y'_i + 4) \mid (x'_i, y'_i) \in P'\}$ when using Skolem sequences and

$P = \{(x_i = x'_i + 5, y_i = y'_i + 5) \mid (x'_i, y'_i) \in P'\}$ when using hooked Skolem sequences. Define the set $B' = \{c_i \mid i \in \mathbb{Z}_{k-1}, (x_i, y_i) \in P\}$. Then we form the set of 5-cycles

$$B = \{c_{i \pmod{k-1}} \mid i \in \mathbb{Z}_{\lfloor \alpha/5 \rfloor}\} \cup (m-1)G_v(\Gamma_1) \cup (m-1)G_v(\Gamma_2) \cup (m-1)G_v(\Gamma_3)$$

where computations are done modulo v and c_i is defined in Equation (3.11). Since $\lfloor \alpha/5 \rfloor = mk - 1$ and B' contains $mk - m$ 5-cycles, the extra $m-1$ copies of $G_v(\Gamma_1) \cup G_v(\Gamma_2) \cup G_v(\Gamma_3)$ ensures we have enough pure 5-cycles in B.

Notice that regardless of our choice of m, $\lambda + m = 5$ and so $(m(v-1)/2 - \alpha) = \lambda + m = 5$. Let D be the set of differences not used in B. There are m copies of both 4 and $2k+2$ in D when $k - 2 \equiv 0$ or $1 \pmod{10}$ and m copies of both 4 and 5 when $k-2 \equiv 2$ or $3 \pmod 4$. Partition D into two sets Δ_1 and Δ_2 so that $\{4, 2k+2\} \subseteq \Delta_1$ or $\{4, 5\} \subseteq \Delta_1$ if $v \equiv 0$ or $1 \pmod 4$ or $v \equiv 2$ or $3 \pmod 4$ respectively and Δ_1 contains as many differences that are not $v/3$, $v/5$, or $2v/5$ as possible. It is clear that Δ_1 satisfies Conditions (A-D) of Lemma 3.3.7. By Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Therefore $(\mathbb{Z}_v \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+m)$ with $m \in \{2,3,4\}$ and $\lambda + m = 5$. □

Lemma 3.4.3. *Let* $v \equiv 1, 3,$ *or* $9 \pmod{10}$ *and assume Conditions (a,b,d) of Lemma* 2.1.2 *are satisfied. If* $m = 1$ *or if* $m \in \{2,3,4\}$, $\lambda + m = 5$ *and* $v \equiv 1 \pmod{10}$, *then any* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda+m)K_{v+2}$.

Proof. Let $(V = \mathbb{Z}_{10k+\ell}, A)$ be a 5–CS of λK_v and let $U = \{\infty_1, \infty_2\}$. Since (V, A) exists, $\lambda + m \equiv 0 \pmod 5$ when $v \equiv 1 \pmod{10}$ and $\lambda \equiv 0 \pmod 5$ when $v \equiv 3, 9 \pmod{10}$ by Lemma 2.1.2(a-b). As in Lemma 3.2.1, the number of edges we must place in 5-cycles contained completely in mK_v is

$$\left(\frac{m(v-1)}{2} - \frac{2a+3b+c}{v}\right) v = \alpha v.$$

Since this is divisible by v, our plan is again to create 5-cycles in mK_v using all the edges of α specially chosen differences.

Suppose $m = 1$. Then $mK_v = K_v$ contains edges of each difference in $D(v) = \{1, \ldots, 5k + (\ell-1)/2\}$. As in Lemma 3.4.1, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

If $\ell = 1$, let $P' = \{(x'_i, y'_i) \mid 1 \le i \le k-2, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ from the set $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\}$ if $k - 2 \equiv 0$ or $1 \pmod 4$ or $k - 2 \equiv 2$ or $3 \pmod 4$ respectively. If $\ell \in \{3, 9\}$ let $P' = \{(x'_i, y'_i) \mid 1 \le i \le k-1, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ from the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$ if $k - 1 \equiv 0$ or $1 \pmod 4$ or $k - 1 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\} \setminus \{2k-4\}$ occurs in exactly one element of P' when $\ell = 1$. When $\ell \in \{3, 9\}$, each element in $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\} \setminus \{2k-4\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x_i' + 4, y_i = y_i' + 4) \mid (x_i', y_i') \in P'\}$ if $\ell = 1$ or $P = \{(x_i = x_i' + 3, y_i = y_i' + 3) \mid (x_i', y_i') \in P'\}$ if $\ell \in \{3, 9\}$. Then with

$$c_i = \begin{cases} (0, x_i, x_i - y_i, 3k, 5k + 2 + i) & \text{if } 1 \le i \le \lfloor \alpha/5 \rfloor - 1 \text{ and } \ell = 1, \\ (0, x_i, x_i - y_i, 3k + 1 + \ell/3, 8k - 2 + \ell - i) & \text{if } 1 \le i \le \lfloor \alpha/5 \rfloor - 1 \text{ and } \ell \in \{3, 9\}, \\ (0, 2k + 1, 6k, 10k, 5k + 1) & \text{if } i = \lfloor \alpha/5 \rfloor,\ k - 2 \equiv 0, 1 \pmod 4 \text{ and } \ell = 1, \\ (0, 4, 2k + 4, 2, 5k + 1) & \text{if } i = \lfloor \alpha/5 \rfloor,\ k - 2 \equiv 2, 3 \pmod 4\text{, and } \ell = 1, \\ (0, x_i, x_i - y_i, 3k + 1 + \ell/3, 8k - 2 + \ell - i) & \text{if } i = \lfloor \alpha/5 \rfloor \text{ and } \ell = 3, \\ (0, 2k + 2, 5k + 4, k, 4k + 3) & \text{if } i = \lfloor \alpha/5 \rfloor,\ k - 1 \equiv 0, 1 \pmod 4\text{, and } \ell = 9\text{, and} \\ (0, 2k + 1, 5k + 4, 2k + 2, 6k + 5) & \text{if } i = \lfloor \alpha/5 \rfloor,\ k - 1 \equiv 2, 3 \pmod 4\text{, and } \ell = 9, \end{cases} \tag{3.12}$$

we form the set of 5-cycles $B = \{c_i + j \mid 1 \le i \le \lfloor \alpha/5 \rfloor, j \in \mathbb{Z}_v, (x_i, y_i) \in P\}$ where the computations are done modulo v. By Lemma 3.2.1, $\alpha \equiv 0 \pmod 5$. Tables 3.4 and 3.5 list the differences of edges that occur in cycles in B if α is as big as possible (from Lemma 3.2.1 this happens when λ is as small as possible, namely when $\lambda = 4$ if $\ell = 1$ and $\lambda = 5$ if $\ell \in \{3, 9\}$ and so $\alpha = 5k - 5$ if $\ell \in \{1, 3\}$ and $\alpha = 5k$ if $\ell = 9$).

Edges	Differences when $k - 2 \equiv 0, 1 \pmod 4$	Differences when $k - 2 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$5, 6, \ldots, 2k$	$5, 6, \ldots, 2k - 1, 2k + 1$
$\{x_i - y_i, 3k\}$	$3k + 1, 3k + 2, \ldots, 4k - 2$	$3k + 1, 3k + 2, \ldots, 4k - 2$
$\{3k, 5k + 2 + i\}$	$2k + 3, 2k + 4, \ldots, 3k$	$2k + 3, 2k + 4, \ldots, 3k$
$\{5k + 2 + i, 0\}$	$5k - 2, 5k - 3, \ldots, 4k + 1$	$5k - 2, 5k - 3, \ldots, 4k + 1$
c_i if $i = \lfloor \alpha/5 \rfloor$	$2k + 1, 4k - 1, 4k, 5k - 1, 5k$	$4, 2k, 2k + 2, 5k - 1, 5k$

Table 3.4: Edges of differences in B when α is as large as possible for $\ell = 1$

Edges	Differences when $k - 1 \equiv 0, 1 \pmod 4$	Differences when $k - 1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k + 1$	$4, 5, \ldots, 2k, 2k + 2$
$\{x_i - y_i, 3k + \ell/3\}$	$3k + 1 + \ell/3, 3k + 2 + \ell/3, \ldots, 4k - 1 + \ell/3$	$3k + 1 + \ell/3, 3k + 2 + \ell/3, \ldots, 4k - 1 + \ell/3$
$\{3k + \ell/3, 8k - 2 + \ell - i\}$	$5k - 2 + 2\ell/3, 5k - 3 + 2\ell/3, \ldots, 4k + 2\ell/3$	$5k - 2 + 2\ell/3, 5k - 3 + 2\ell/3, \ldots, 4k + 2\ell/3$
$\{8k - 2 + \ell - i, 0\}$	$2k + 3, 2k + 4, \ldots, 3k + 1$	$2k + 3, 2k + 4, \ldots, 3k + 1$
c_i if $i = \lfloor \alpha/5 \rfloor$ and $\ell = 9$	$2k + 2, 3k + 2, 3k + 3, 4k + 3, 4k + 4$	$2k + 1, 3k + 2, 3k + 3, 4k + 3, 4k + 4$

Table 3.5: Edges of differences in B when α is as large as possible for $\ell \in \{3, 9\}$

Let D be the set of differences not used in B. When λ is as large as possible (α is as small as possible)

$$D = \begin{cases} \{1, 2, 3, 2k + 2\} & \text{if } v \equiv 1 \pmod{10} \text{ and } k \equiv 0, 1 \pmod 4, \\ \{1, 2, 3, 4k - 1, 4k\} & \text{if } v \equiv 1 \pmod{10} \text{ and } k \equiv 2, 3 \pmod 4, \\ \{1, 2, 3, 2k + 2, 5k + 1\} & \text{if } v \equiv 3 \pmod{10}, \\ \{1, 2, 3, 4k + 5\} & \text{if } v \equiv 9 \pmod{10} \text{ and } k \equiv 0, 1 \pmod 4\text{, and} \\ \{1, 2, 3, 4k + 5\} & \text{if } v \equiv 9 \pmod{10} \text{ and } k \equiv 2, 3 \pmod 4. \end{cases}$$

Partition D into two sets Δ_1 and Δ_2 so that $\{1,2,3\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/3$ as possible. It is clear that Δ_1 satisfies Conditions (A-D) of Lemma 3.3.7 when λ is as small as possible. As λ increases, the differences represented in c_i are added to D. So it is clear that Conditions (A-D) of Lemma 3.3.7 will still be satisfied and so by Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Thus, $(\mathbb{Z}_{10k+\ell} \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+1)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$.

Let $v \equiv 1 \pmod{10}$, $m \in \{2,3,4\}$, and $\lambda + m = 5$. As before, let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k-2, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ from the set $\{1,2,\ldots,2k-4\}$ or $\{1,2,\ldots,2k-3\}$ if $k-2 \equiv 0$ or $1 \pmod 4$ or $k-2 \equiv 2$ or $3 \pmod 4$ respectively. Now define $P = \{(x_i = x'_i + 4, y_i = y'_i + 4) \mid (x'_i, y'_i) \in P'\}$. Then with c_i defined as in Equation (3.12) we form the set of 5-cycles $B' = \{c_i \mid 1 \leq i \leq k-1, (x_i, y_i) \in P\}$ where computations are done modulo v. Let $B'' = mB'$. Then B'' contains $m(k-1)$ 5-cycles, but we need $\lfloor \alpha/5 \rfloor = mk - 1$ of them. Hence we need $m-1$ more 5-cycles. Let

$$c'_i = \begin{cases} (0,1,2k+3,2k+5,3) & \text{if } i = 1 \text{ and } k \equiv 0,1 \pmod 4, \\ (0,1,5,7,3) & \text{if } i = 2 \text{ and } k \equiv 0,1 \pmod 4, \\ (0,1,4k,4k+2,3) & \text{if } i = 1 \text{ and } k \equiv 2,3 \pmod 4, \text{ and} \\ (0,1,4k+1,4k+3,3) & \text{if } i = 2 \text{ and } k \equiv 2,3 \pmod 4. \end{cases}$$

Then let E'_α contain 1, 1, or 2 copies of c'_1 if $m = 2$, 3, or 4 respectively, and contain 1 or 2 copies of c'_2 if $m = 3$ or 4 respectively. Notice that the cycles in E'_α contain at most m copies of edges not used in cycles in B''. Then we form the set of 5-cycles

$$B = \{c_i + j \mid 1 \leq i \leq \lfloor \alpha/5 \rfloor, c_{i \pmod{k-1}} \in B'', j \in \mathbb{Z}_v\} \cup \{c'_i + j \mid j \in \mathbb{Z}_v, c'_i \in E'_\alpha\}.$$

Notice that regardless of our choice of m, $\lambda + m = 5$. So $(m(v-1)/2 - \alpha) = \lambda + m = 5$. Let D be the set of differences not used in B. It is clear that D contains $1, 2$, and 3. Partition D into two sets Δ_1 and Δ_2 so that $\{1,2,3\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/3$ as possible. Since $v/5, 2v/5, v/2 \notin D$, it is clear that Δ_1 satisfies Conditions (A-D) of Lemma 3.3.7 and by Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Therefore $(\mathbb{Z}_v \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+m)$ with $m \in \{2,3,4\}$ and $\lambda + m = 5$. □

Lemma 3.4.4. *Let* $v \equiv 4, 6,$ *or* $8 \pmod{10}$ *and assume Conditions (a,b,d) of Lemma* 2.1.2 *are satisfied. If* $m = 2$ *or if* $m \in \{4,6,8\}$, $\lambda + m = 10$, *and* $v \equiv 6 \pmod{10}$, *then any* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda+m)K_{v+2}$.

Proof. Let $(V = \mathbb{Z}_v, A)$ be a 5–CS of λK_v and let $U = \{\infty_1, \infty_2\}$. Since (V, A) exists, $\lambda \equiv m \equiv 0 \pmod 2$ by Lemma 2.1.2(a). As in Lemma 3.2.1, the number of edges we must place in 5-cycles contained completely in mK_v is

$$\left(\frac{m(v-1)}{2} - \frac{2a+3b+c}{v}\right) v = \alpha v.$$

Since this is divisible by v, our plan is again to create 5-cycles in mK_v using all the edges of α specially chosen differences.

Suppose $m = 2$. Then $mK_v = 2K_v$ contains edges of each difference in $D(v)_2$. Care must be taken with edges of difference $v/2$ in $E(K_v)$ since there are only $v/2$ of them. As in Lemma 3.4.1, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

Let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k-1, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ from the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$ if $k-1 \equiv 0$ or $1 \pmod 4$ or $k-1 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\} \setminus \{2k-2\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 3, y_i = y'_i + 3) \mid (x'_i, y'_i) \in P'\}$. Then with

$$c_i = \begin{cases} (0, x_i, x_i - y_i, 3k+1, 5k+3+i) & \text{if } 1 \leq i \leq k-1 \text{ and } \ell \in \{4, 6\}, \\ (0, x_i, x_i - y_i, 3k+3, 5k+5+i) & \text{if } 1 \leq i \leq k-1 \text{ and } \ell = 8, \end{cases} \tag{3.13}$$

we form the set of 5-cycles $B' = \left\{c''_{i-1} = c_i \mid 1 \leq i \leq k-1\right\}$ where the computations are done modulo v. Notice that based on our choice of d, $\alpha \equiv 0 \pmod 5$. Tables 3.6 and 3.7 list the differences of edges that occur in cycles in B if α is as big as possible (from Lemma 3.2.1 this happens when λ is as small as possible, namely when $\lambda = 10$ if $v \equiv 4, 8 \pmod{10}$ or $\lambda = 8$ if $v \equiv 6 \pmod{10}$).

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k+1$	$4, 5, \ldots, 2k, 2k+2$
$\{x_i - y_i, 3k+1\}$	$3k+2, 3k+3, \ldots, 4k$	$3k+2, 3k+3, \ldots, 4k$
$\{3k+1, 5k+3+i\}$	$2k+3, 2k+4, \ldots, 3k+1$	$2k+3, 2k+4, \ldots, 3k+1$
$\{5k+3+i, 0\}$	$5k-4+\ell, 5k-5+\ell, \ldots, 4k-2+\ell$	$5k-4+\ell, 5k-5+\ell, \ldots, 4k-2+\ell$

Table 3.6: Edges of differences in B' when α is as large as possible for $\ell \in \{4, 6\}$

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k+1$	$4, 5, \ldots, 2k, 2k+2$
$\{x_i - y_i, 3k+3\}$	$3k+4, 3k+5, \ldots, 4k+2$	$3k+4, 3k+5, \ldots, 4k+2$
$\{3k+3, 5k+5+i\}$	$2k+3, 2k+4, \ldots, 3k$	$2k+3, 2k+4, \ldots, 3k$
$\{5k+5+i, 0\}$	$5k+2, 5k+1, \ldots, 4k+4$	$5k+2, 5k+1, \ldots, 4k+4$

Table 3.7: Edges of differences in B when α is as large as possible for $\ell = 8$

Let $c''_{k-1} = (0, 1, 3, 4k+4, 4k+1)$ or $(0, 1, 3, 3k+4, 3k+1)$ if $\ell \in \{4, 6\}$ or $\ell = 8$ respectively. Now let $B'' = 2B' \cup \{c''_{k-1}\}$ so that we form the set of 5-cycles $B = \{c''_i + j \mid i \in \mathbb{Z}_{\alpha/5}, c''_{i \pmod k} \in B'', j \in \mathbb{Z}_v\}$.

Let D be the set of differences not used in B. It is clear that $1, 2$, and 3 are contained in D. Partition D into two sets Δ_1 and Δ_2 so that $\{1, 2, 3\} \subseteq \Delta_1$ and Δ_1

contains as many differences that are not $v/2$, $v/3$, or $v/4$ as possible. Since $v/5$ and $2v/5$ are not in D, it follows that Δ_1 satisfies Conditions (A-D) of Lemma 3.3.7 and so by Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Thus, $(\mathbb{Z}_{10k+\ell} \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+2)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$.

Let $\ell = 6$, $m \in \{4,6,8\}$, and $\lambda + m = 10$. Let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k-1, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ from the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$ if $k - 1 \equiv 0$ or $1 \pmod 4$ or $k - 1 \equiv 2$ or $3 \pmod 4$ respectively. Now define $P = \{(x_i = x'_i + 3, y_i = y'_i + 3) \mid (x'_i, y'_i) \in P'\}$. Then with c_i defined as in Equation (3.13) we form the set of 5-cycles $B' = \{c''_{i-1} = c_i \mid 1 \leq i \leq k-1, (x_i, y_i) \in P\}$ where computations are done modulo v. Let $B'' = mB'$ be the multiset consisting of m copies each 5-cycle in B'. Then B'' contains $m(k-1)$ 5-cycles, but we need $\lfloor \alpha/5 \rfloor = mk - 2 + m/2$ of them. Hence we need $3m/2 - 2$ more 5-cycles. Let $c'_1 = (0, 1, 3, 4k+5, 4k+2)$, $c'_2 = (0, 2k+1, 4k+2, 8k+5, 4k+3)$, $c'_3 = (0, 2k+2, 4k+4, 8k+6, 4k+3)$, $c'_4 = (0, 2k+1, 6k+3, k, 5k+3)$, and $c'_5 = (0, 1, 3, 4k+4, 4k+1)$. If $k-1 \equiv 0$ or $1 \pmod 4$, then let E'_α:

- contain 1, 2, or 2 copies of c'_1 if $m = 4$, 6, or 8 respectively,
- contain 1, 2, or 4 copies of c'_3 if $m = 4$, 6, or 8 respectively,
- and contain $m/2$ copies of c'_5.

If $k - 1 \equiv 2$ or $3 \pmod 4$, then let E'_α:

- contain 0, 1, or 2 copies of c'_1 if $m = 4$, 6, or 8 respectively,
- contain 1, 2, or 2 copies of c'_2 if $m = 4$, 6, or 8 respectively,
- contain 1, 1, and 2 copies of c'_4 if $m = 4$, 6, or 8 respectively,
- and contain $m/2$ copies of c'_5.

Notice that the cycles in E'_α contain at most m copies of edges not used in cycles in B''. Then

$$B = \{c_i + j \mid 1 \leq i \leq k-1, c_{i \pmod k} \in B'', j \in \mathbb{Z}_v\} \cup \{c'_i + j \mid j \in \mathbb{Z}_v, c'_i \in E'_\alpha\}.$$

Notice that regardless of our choice of m, $\lambda + m = 10$. So $(m(v-1)/2 - \alpha) = \lambda + m = 10$. Let D be the set of differences not used in B. By examining the remaining differences we know differences 1 and 2 appear in D and when $k \equiv 0$ or $1 \pmod 4$, D has at most 1 copy of $v/2$. Partition D into two sets Δ_1 and Δ_2 so that $\{1,2\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/2$, $v/3$, or $v/4$ as possible. Based on the remaining differences it is clear that $v/3$, $v/6$, $2v/5$, and $v/5$ are not in D. Thus Δ_1 satisfies Conditions (A-D) of Lemma 3.3.7 and by Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Therefore $(\mathbb{Z}_v \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+m)$ with $m \in \{4,6,8\}$ and $\lambda + m = 10$. □

Lemma 3.4.5. *Let $v \equiv 2 \pmod{10}$ and assume Conditions (a,b,d) of Lemma* 2.1.2 *are satisfied. If $m = 10$ then any* 5–CS *of λK_v can be enclosed in a* 5–CS *of $(\lambda + m)K_{v+2}$.*

Proof. Let $(V = \mathbb{Z}_{10k+2}, A)$ be a 5–CS of λK_v and let $U = \{\infty_1, \infty_2\}$. Since (V, A) exists, $\lambda \equiv m \equiv 0 \pmod{10}$. As in Lemma 3.2.1, the number of edges we must place in 5-cycles contained completely in mK_v is

$$\left(\frac{m(v-1)}{2} - \frac{2a + 3b + c}{v}\right) v = \alpha v.$$

Since this is divisible by v, our plan is again to create 5-cycles in mK_v using all the edges of α specially chosen differences.

Suppose $m = 10$. Then $mK_v = K_v$ contains edges of each difference in $D(v)_{10}$. Care must be taken with edges of difference $v/2$ in $E(K_v)$ since there are only $v/2$ of them. As in Lemma 3.4.1, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

Let $P' = \{(x'_i, y'_i) \mid 1 \le i \le k-2, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ from the set $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\}$ if $k - 2 \equiv 0$ or $1 \pmod 4$ or $k - 2 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\} \setminus \{2k-4\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 4, y_i = y'_i + 4) \mid (x'_i, y'_i) \in P'\}$. Then with

$$c'_i = \begin{cases} (0, 4, 4k+3, 8k+3, 4k+2) & \text{if } i = 1, \\ (0, 1, 3, 3k+3, 3k) & \text{if } i = 2, \\ (0, 1, 3, 2k+4, 2k+1) & \text{if } i = 3 \text{ and } k-2 \equiv 0, 1 \pmod 4, \text{ and} \\ (0, 1, 3, 2k+3, 2k) & \text{if } i = 3 \text{ and } k-2 \equiv 2, 3 \pmod 4, \end{cases}$$

we form the set of 5-cycles

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k, 5k+1+i) \mid 1 \le i \le k-2, (x_i, y_i) \in P\}$$

and let E'_α contain 10 copies of c'_1, 5 copies of c'_2, and 2 copies of c'_3. Notice that based on our choice of d, $\alpha \equiv 0 \pmod 5$. Table 3.8 lists the differences of edges that occur in cycles in $B' \cup E'_\alpha$ if α is as big as possible (from Lemma 3.2.1 this happens when λ is as small as possible and not 0, namely when $\lambda = 10$ and so $\alpha = 50k - 15$). Then form the set of 5-cycles $B = \{c_i + j \mid i \in \mathbb{Z}_{z_1}, c_{i \pmod{k-1}} \in B', j \in \mathbb{Z}_v\} \cup z_2\{c + j \mid j \in \mathbb{Z}_v, c \in E'_\alpha\}$ where $z_1 = \min(\{\alpha/5, m(k-2)\})$, $z_2 = \max(\{0, \alpha/5 - m(k-2)\})$, and the computations are done modulo v.

Let D be the set of differences not used in B. There are 3 copies of 1, 2, and 3 in D when λ is as small as possible (α is as large as possible. Partition D into two sets Δ_1 and Δ_2 so that $\{1, 2, 3\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not

Edges	Differences when $k-2 \equiv 0,1 \pmod 4$	Differences when $k-2 \equiv 2,3 \pmod 4$
$\{0,x_i\},\{x_i,x_i-y_i\}$	$5,6,\ldots,2k$	$5,6,\ldots,2k-1,2k+1$
$\{x_i-y_i,3k\}$	$3k+1,3k+2,\ldots,4k-2$	$3k+1,3k+2,\ldots,4k-2$
$\{3k,5k+1+i\}$	$2k+2,2k+3,\ldots,3k-1$	$2k+2,2k+3,\ldots,3k-1$
$\{5k+1+i,0\}$	$5k,5k-1,\ldots,4k+3$	$5k,5k-1,\ldots,4k+3$
c_i' if $i=1$	$4,4k-1,4k,4k+1,4k+2$	$4,4k-1,4k,4k+1,4k+2$
c_i' if $i=2$	$1,2,3,3k,3k$	$1,2,3,3k,3k$
c_i' if $i=3$	$1,2,3,2k+1,2k+1$	$1,2,3,2k,2k$

Table 3.8: Edges of differences in $B' \cup E'_\alpha$ when α is as large as possible for $\ell = 2$

$v/2$, $v/3$, or $v/4$ as possible. As λ increases, the differences from c_i are added to D. In this manner, it is clear that Δ_1 will satisfy Conditions (A-D) of Lemma 3.3.7 and by Lemma 3.3.8 there exists a 5–CS $(V \cup U, C)$ of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Thus, $(\mathbb{Z}_{10k+2} \cup \{\infty_1,\infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2,\lambda+10)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$. □

Lemma 3.4.6. *Let $v \equiv 7 \pmod{10}$ and assume Conditions (a,b,d) of Lemma* 2.1.2 *are satisfied. If $m = 5$, then any* 5–CS *of λK_v can be enclosed in a* 5–CS *of $(\lambda + m)K_{v+2}$.*

Proof. Let $(V = \mathbb{Z}_{10k+7}, A)$ be a 5–CS of λK_v and let $U = \{\infty_1,\infty_2\}$. Since (V,A) exists, $\lambda + m \equiv 0 \pmod 5$. As in Lemma 3.2.1 in Lemma 3.4.1, the number of edges we must place in 5-cycles contained completely in mK_v is

$$\left(\frac{m(v-1)}{2} - \frac{2a+3b+c}{v}\right) v = \alpha v.$$

Since this is divisible by v, our plan is again to create 5-cycles in mK_v using all the edges of α specially chosen differences.

Suppose $m = 5$. Then $mK_v = K_v$ contains edges of each difference in $D(v)_5$. As in Lemma 3.4.1, we will use Skolem and hooked Skolem sequences to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen).

Let $P' = \{(x_i', y_i') \mid 1 \le i \le k-1, y_i' > x_i'\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ from the set $\{1,2,\ldots,2k-2\}$ or $\{1,2,\ldots,2k-1\}$ if $k-1 \equiv 0$ or $1 \pmod 4$ or $k-1 \equiv 2$ or $3 \pmod 4$ respectively. So each element in $\{1,2,\ldots,2k-2\}$ or $\{1,2,\ldots,2k-1\} \setminus \{2k-2\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x_i'+3, y_i = y_i'+3) \mid (x_i', y_i') \in P'\}$. Then with

$$c_i' = \begin{cases} (0,2k+2,5k+4,k+1,5k+3) & \text{if } i=1 \text{ and } k-1 \equiv 0,1 \pmod 4,\\ (0,1,2,5,3) & \text{if } i=2 \text{ and } k-1 \equiv 0,1 \pmod 4,\\ (0,1,2k+2,7k+5,4k+3) & \text{if } i=1 \text{ and } k-1 \equiv 2,3 \pmod 4,\\ (0,2,4,1,4k+3) & \text{if } i=2 \text{ and } k-1 \equiv 2,3 \pmod 4, \text{ and}\\ (0,2,2k+3,2k+1,5k+3) & \text{if } i=3 \text{ and } k-1 \equiv 2,3 \pmod 4, \end{cases}$$

we form the set of 5-cycles

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k+2, 5k+4+i) \mid 1 \le i \le k-1, (x_i, y_i) \in P\}$$

and let E'_α contain 5 copies of c'_1 and 1 copy of c'_2 if $k-1 \equiv 0$ or $1 \pmod 4$, and contain 4 copies of c'_1, 1 copy of c'_2 and 1 copy of c'_3 if $k-1 \equiv 2$ or $3 \pmod 4$. Notice that based on our choice of d, $\alpha \equiv 0 \pmod 5$. Table 3.9 lists the differences of edges that occur in cycles in $B' \cup E'_\alpha$ if α is as big as possible (from Lemma 3.2.1 this happens when λ is as small as possible, namely when $\lambda = 5$ and so $\alpha = 25k+5$).

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$4, 5, \ldots, 2k+1$	$4, 5, \ldots, 2k, 2k+2$
$\{x_i - y_i, 3k+2\}$	$3k+3, 3k+4, \ldots, 4k+1$	$3k+3, 3k+4, \ldots, 4k+1$
$\{3k+2, 5k+4+i\}$	$2k+3, 2k+4, \ldots, 3k+1$	$2k+3, 2k+4, \ldots, 3k+1$
$\{5k+4+i, 0\}$	$5k+2, 5k+1, \ldots, 4k+4$	$5k+2, 5k+1, \ldots, 4k+4$
c'_i if $i=1$	$2k+2, 3k+2, 4k+2, 4k+3, 5k+3$	$1, 2k+1, 3k+2, 4k+3, 5k+3$
c'_i if $i=2$	$1, 1, 2, 3, 3$	$2, 2, 3, 4k+2, 4k+3$
c'_i if $i=3$	$-$	$2, 2, 2k+1, 3k+2, 5k+3$

Table 3.9: Edges of differences in $B' \setminus E'_\alpha$ when α is as large as possible for $\ell = 7$

Define $B = \{c_i + j \mid i \in \mathbb{Z}_{z_1}, c_{i \pmod{k-1}} \in B', j \in \mathbb{Z}_v\} \cup z_2\{c + j \mid j \in \mathbb{Z}_v, c \in E'_\alpha\}$ where $z_1 = \min(\{\alpha/5, m(k-1)\})$, $z_2 = \max(\{0, \alpha/5 - m(k-1)\})$, and the computations are done modulo v.

Let D be the set of differences not used in B. By examining Table 3.9, we see that 1, 2, and 3 are contained in D and $v/2, v/4, v/5, 2v/5, v/6 \notin D$. Partition D into two sets Δ_1 and Δ_2 so that $\{1,2,3\} \subseteq \Delta_1$ and Δ_1 contains as many differences that are not $v/3$ as possible. Thus Δ_1 satisfies Conditions (A-D) of Lemma 3.3.7 and by Lemma 3.3.8 there exists a 5–CS of $G_v(D) \vee_{\lambda+m} (\lambda+m)K_2$ ensuring we cover each mixed edge $2v(\lambda+m)$ times. Thus, $(\mathbb{Z}_{10k+7} \cup \{\infty_1, \infty_2\}, A \cup B \cup C)$ is a 5–CS$(v+2, \lambda+5)$ that encloses the given 5–CS $(\mathbb{Z}_v, A)$. □

We can gather results from Lemmas 3.4.1, 3.4.2, 3.4.3, 3.4.4, 3.4.5, and 3.4.6 into the following corollary.

Corollary 3.4.7. *Let v, m and λ be integers satisfying Conditions (a,b,d) of Lemma 2.1.2 with $u = 2$. Let $v = 10k + \ell$ with $\ell \in \{0, 1, \ldots, 9\}$. If:*

- *$m = 1$ when $\ell \in \{1,3,5,9\}$, $m = 2$ when $\ell \in \{0,4,6,8\}$, $m = 5$ when $\ell = 7$, $m = 10$ when $\ell = 2$,*
- *$m \in \{2,3,4\}$, $\lambda + m = 5$, and $\ell \in \{1,5\}$, or*
- *$m \in \{4,6,8\}$, $\lambda + m = 10$, and $\ell \in \{0,6\}$,*

then any 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda+m)K_{v+2}$.

3.5 The Main Result

The following lemma will be useful in constructing the enclosings when λ is as large as possible.

Lemma 3.5.1. *Let v and m be integers satisfying conditions (a), (b), and (d) of Lemma 2.1.2 with $u = 2$ and let $\lambda = \lambda_{\max}(v, m)$. Then a 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+2}$.*

Proof. Let $(V = \mathbb{Z}_v, A)$ be a 5–CS of λK_v. Since $\lambda = \lambda_{\max}(v, m)$, $\lambda + m \geq m(v-1)/2$ and so it follows that $d = m(v-1)/2$ by definition and so $\alpha = 0$ from Equation (3.1). Partition $D = D(v)_m$ into two sets Δ_1 and Δ_2 so that $\{1, 2\} \subseteq \Delta_1$, Δ_1 contains as many differences that are not $v/2$, $v/3$, $v/4$, $v/5$, or $2v/5$ as possible. So Conditions (A-D) of Lemma 3.3.7 hold for Δ_1 and therefore by Lemma 3.3.8 a 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+2}$ when $\lambda = \lambda_{\max}(v, m)$. □

In the following theorem, Corollary 3.4.7 is used as a basis for an inductive argument to settle the enclosing problem when $u = 2$.

Theorem 3.5.2. *A 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + m)K_{v+2}$ if and only if*

(a) $(\lambda + m)u + m(v - 1) \equiv 0 \pmod 2$,
(b) $\binom{u}{2}(\lambda + m) + m\binom{v}{2} + vu(\lambda + m) \equiv 0 \pmod 5$,
(c) $m\binom{v}{2} - 2(\lambda + m) - \frac{1}{2}(v - 1)(\lambda + m) \geq 0$.

Proof. The necessity follows from Lemma 2.1.2(a,b,d), so we now prove the sufficiency. Since we are assuming that a 5–CS of λK_v exists, it follows by Theorem 1.2.2 that $v \notin \{2, 3, 4\}$, and by Condition (c) that $v \neq 1$, so we know that $v \geq 5$. If $\lambda = 0$ then by Condition (a) $m(v + 1) \equiv 0 \pmod 2$ and by Condition (b) $m(v + 2)(v + 1) \equiv 0 \pmod 5$, so by Theorem 1.2.2 there exists a 5–CS of mK_{v+2} as required; so we can assume that $\lambda \geq 1$.

Let (V, A) be a 5–CS of λK_v. If $v \equiv 0, 4, 6,$ or $8 \pmod{10}$, $v \equiv 2 \pmod{10}$, or $v \equiv 7 \pmod{10}$, then it follows from Theorem 1.2.2 that $\lambda \equiv m \equiv 0 \pmod 2$, $\lambda \equiv m \equiv 0 \pmod{10}$, and $\lambda + m \equiv 0 \pmod 5$ respectively; in any other case m can potentially take on any positive integral value. To reflect this situation, let

$$t(v) = \begin{cases} 1 & \text{if } v \equiv 1, 3, 5, 9 \pmod{10}, \\ 2 & \text{if } v \equiv 0, 4, 6, 8 \pmod{10}, \\ 5 & \text{if } v \equiv 7 \pmod{10}, \text{ and} \\ 10 & \text{if } v \equiv 2 \pmod{10}. \end{cases}$$

We often simply write t instead of $t(v)$ when the value of v is clear. For any choice of v and m, let $\lambda^-_{\max}(v, m)$ be the second largest integer satisfying Conditions (a-c), and let $\lambda^+_{\min}(v, m)$ be the second smallest integer satisfying Conditions (a-c).

We will first establish that for any given v, the difference between consecutive values of λ that satisfy Conditions (a) and (b) is a constant; call this difference $\lambda_{\text{diff}}(v)$. Secondly, for all $m \geq 1$ and $v \geq 5$, we settle the enclosing problem for both the smallest and the largest values of λ satisfying Conditions (a-c). Finally for all $m \geq 1$ and $v \geq 5$ it will be shown that for all λ satisfying Conditions (a-c) there exist non-negative integers λ_1, λ_2 with $\lambda_1 + \lambda_2 = \lambda$ and positive integers m_1, m_2 with $m_1 + m_2 = m$ such that for $1 \leq i \leq 2$ there exists a 5–CS of $\lambda_i K_v$ that can be enclosed in a 5–CS of $(\lambda_i + m_i)K_{v+2}$. So the union of these two enclosings completes the proof.

We turn to the first step of the proof. For any given v and m, the difference between consecutive values of λ that satisfy Conditions (a) and (b) is a constant; namely $\lambda_{\text{diff}}(v, m)$. Also by Conditions (a) and (b) and by Theorem 1.2.2, $\lambda \equiv 0 \pmod 2$ if v is even and $(2v + 1)\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 5$. Therefore $\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 5$ and $\lambda_{\text{diff}}(v, m) \equiv 0 \pmod{10}$ if v is odd and even respectively. Since $\lambda_{\text{diff}}(v, m)$ must be the smallest such value, if v is odd then $\lambda_{\text{diff}}(v, m) = 5$, and if v is even then $\lambda_{\text{diff}}(v, m) = 10$. Notice that if $m_1 \neq m_2$ then $\lambda_{\text{diff}}(v, m_1) = \lambda_{\text{diff}}(v, m_2)$. Therefore we define $\lambda_{\text{diff}}(v) = \lambda_{\text{diff}}(v, m)$.

We now turn to the second step in the proof; establishing the existence of the enclosing for the smallest and largest values of λ given v and m. Beginning with the smallest value, Tables 3.10 and 3.11 list the values of $\lambda_{\min}(v, m)$, and were formed using Theorem 1.2.2 and Corollary 3.4.7. Three cases are considered in turn based on the value of v.

	$m \pmod{5t}$				
$v \pmod{10}$	0	t	$2t$	$3t$	$4t$
0,6	0	8	6	4	2
1,5	0	4	3	2	1

Table 3.10: Checking $\lambda_{\min}(v, m)$ when $v \equiv 0, 1, 5$ or $6 \pmod{10}$: each cell contains $\lambda_{\min}(v, m)$

	$v \pmod{10}$					
m	2	3	4	7	8	9
any m	0	0	0	0	0	0

Table 3.11: Checking $\lambda_{\min}(v, m)$ when $v \not\equiv 0, 1, 5$ or $6 \pmod{10}$: each cell contains $\lambda_{\min}(v, m)$

Let $v \equiv 0, 1, 5,$ or $6 \pmod{10}$. The proof is by induction on m. By Corollary 3.4.7 if $m \leq 4t$ then any 5–CS of $\lambda_{\min}(v, m)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v, m) + m)K_{v+2}$. If $m = 5t$, by Table 2.8 $\lambda_{\min}(v, 10) = 0$, so this enclosing has been established by Theorem 1.2.2 and we say there exists a 5–CS $(V \cup \{\infty_1, \infty_2\}, B_2)$

of $(5t)K_{v+2}$. Suppose that for some $m \geq 6t$ and for all s with $1 \leq s < m$, every 5–CS of $\lambda_{\min}(v,s)K_v$ can be enclosed in a 5–CS of $(\lambda_{\min}(v,s)+s)K_{v+2}$. Then, since $m \geq 6t$, by induction each 5–CS (V, B_1) of $\lambda_{\min}(v, m-5t)K_v$ can be enclosed in a 5–CS $(V \cup \{\infty_1, \infty_2\}, B_1')$ of $(\lambda_{\min}(v, m-5t) + (m-5t))K_{v+2}$. Since Table 3.10 shows that $\lambda_{\min}(v, m-5t) = \lambda_{\min}(v, m)$, (V, B_1) is a 5–CS of $\lambda_{\min}(v, m-5t)K_v = \lambda_{\min}(v,m)K_v$ which is enclosed in a 5–CS $(V \cup \{\infty_1, \infty_2\}, B_1' \cup B_2)$ of

$$((\lambda_{\min}(v, m-5t) + (m-5t)) + 5t)K_{v+2} = (\lambda_{\min}(v,m) + m)K_{v+2}.$$

Now let $v \not\equiv 0, 1, 5$ or $6 \pmod{10}$. By Table 3.11 $\lambda_{\min}(v,m) = 0$, and so clearly a 5–CS of $\lambda_{\min}(v,m)$ can be enclosed in a 5–CS of $(\lambda + m)K_{v+u}$.

Now turning to the largest value of λ, by Lemma 3.5.1 a 5–CS of $\lambda_{\max}(v,m)K_v$ can be enclosed in a 5–CS of $(\lambda_{\max}(v,m) + m)K_{v+2}$.

Since the cases where $m = t$ (by Corollary 2.2.10) and where $v \geq 5$, $m \geq t$ and $\lambda \in \{\lambda_{\min}(v,m), \lambda_{\max}(v,m)\}$ have now been settled, it is left to show that for $m \geq t$, $v \geq 5$, and for all λ satisfying Conditions (a-c) with $\lambda_{\min}(v,m) < \lambda < \lambda_{\max}(v,m)$ there is an enclosing of a 5–CS of λK_v in a 5–CS of $(\lambda + m)K_{v+2}$. The proof is by induction on m. Tables 3.12, 3.13, 3.14 and 3.15 were formed using Theorem 1.2.2 and Corollary 3.4.7, and will help in establishing for each value of λ that the enclosing can be produced using two smaller enclosings.

	$v \pmod{10}$									
m	0	1	2	3	4	5	6	7	8	9
$2t(v)$	0	5	0	5	10	0	0	0	10	5
$3t(v)$	0	0	0	0	10	0	10	0	0	5

Table 3.12: Checking the largest values of λ: each cell contains $\lambda_{\max}(v,m) - \lambda_{\max}(v,t(v)) - \lambda_{\max}(v, m-t(v))$ except for some special values listed in Table 3.13.

	v												
m	10	12	13	15	16	17	18	22	23	27	28	32	33
$2t(v)$	0	10	5	0	10	5	10	0	5	0	10	0	5
$3t(v)$	10	10	5	5	0	0	10	10	5	5	10	10	5

Table 3.13: Checking the largest values of λ: each cell contains $\lambda_{\max}(v,m) - \lambda_{\max}(v,t(v)) - \lambda_{\max}(v, m-t(v))$.

Suppose for some integer $m \geq 2t$ and for all s with $1 \leq s < m$, every 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda + s)K_{v+2}$. From Tables 3.12 and 3.13, $\lambda_{\max}(v,m) - \lambda_{\max}(v,t) - \lambda_{\max}(v,m-t) \in \{0, \lambda_{\text{diff}}(v)\}$, so $\lambda_{\max}^-(v,m) \leq \lambda_{\max}(v,t) + \lambda_{\max}(v,m-t)$. Similarly from Tables 3.14 and 3.15, $\lambda_{\min}(v,m-t) + \lambda_{\min}(v,t) - \lambda_{\min}(v,m) \in \{0, \lambda_{\text{diff}}(v)\}$, so $\lambda_{\min}(v,m-t) + \lambda_{\min}(v,t) \leq \lambda_{\min}^+(v,m)$. Thus $\lambda_{\min}(v,m-t) + \lambda_{\min}(v,t) \leq \lambda \leq \lambda_{\max}(v,m-t) + \lambda_{\max}(v,t)$. Since for each v we know that

m	$v \pmod{10}$ 2	3	4	7	8	9
any m	0	0	0	0	0	0

Table 3.14: Checking the smallest values of λ for $v \not\equiv 0, 1, 5$ or $6 \pmod{10}$: each cell contains $\lambda_{\min}(v, m - t(v)) + \lambda_{\min}(v, t(v)) - \lambda_{\min}(v, m)$.

$v \pmod{10}$	m: 0	t	$2t$	$3t$	$4t$
0,6	10	0	10	10	10
1,5	5	0	5	5	5

Table 3.15: Checking the smallest values of λ for $v \equiv 1$ or $6 \pmod{10}$: each cell contains $\lambda_{\min}(v, m - t(v)) + \lambda_{\min}(v, t(v)) - \lambda_{\min}(v, m)$.

$\lambda_{\text{diff}}(v) = \lambda_{\text{diff}}(v, m)$ is constant for all m, it follows that there exist non-negative integers λ_1, λ_2 for which $\lambda_{\min}(v, t) \leq \lambda_1 \leq \lambda_{\max}(v, t)$, $\lambda_{\min}(v, m-t) \leq \lambda_2 \leq \lambda_{\max}(v, m-t)$, and $\lambda_1 + \lambda_2 = \lambda$, and there exist positive integers m_1, m_2 with $m_1 + m_2 = m$ such that for $1 \leq i \leq 2$ there exists a 5–CS of $\lambda_i K_v$ that can be enclosed in a 5–CS of $(\lambda_i + m_i)K_{v+2}$. So the union of these enclosings proves that any 5–CS of $(\lambda_1 + \lambda_2)K_v = \lambda K_v$ can be enclosed in a 5–CS of $((\lambda_1 + m_1) + (\lambda_2 + m_2))K_{v+2} = (\lambda + m)K_{v+2}$. $\square$

Chapter 4
5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$ when $u = 1$

4.1 Introduction

We now turn to generalizing the enclosing results from Chapter 2. As said in Chapter 1, showing that a 5–CS of λK_v can be enclosed in a 5–CS of $(\lambda+m)K_{v+u}$ is the same as showing there exists a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$ in the case where there exists a 5–CS of λK_v. We will solve some of the cases where the existence condition of a 5–CS of λK_v is removed.

Not surprisingly, necessary conditions for the existence of a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$ in Lemma 4.1.1 are very similar to the necessary conditions in Lemma 2.1.2. Lemma 2.1.2 will help establish some of the conditions in the following lemma since in Lemma 2.1.2 the existence of a 5–CS of λK_v was not used to establish every condition in this theorem.

Lemma 4.1.1. *Let v, m, and u be positive integers. If there exists a 5–CS of $(\lambda + m)K_{v+u} - \lambda K_v$, then:*

(a) $(\lambda+m)u + m(v-1) \equiv 0 \pmod 2$,
(b) $(v+u-1)(\lambda+m) \equiv 0 \pmod 2$,
(c) $\binom{u}{2}(\lambda+m) + m\binom{v}{2} + vu(\lambda+m) \equiv 0 \pmod 5$,
(d) if $u = 1$, *then* $m(v-1) \geq 3(\lambda+m)$,
(e) if $u = 2$, *then* $m\binom{v}{2} - 2(\lambda+m) - (v-1)(\lambda+m)/2 \geq 0$, *and*
(f) if $u \geq 3$, *then* $\lceil vu(\lambda+m)/4 \rceil + 2\epsilon \leq mv(v-1)/2 + (\lambda+m)u(u-1)/2$ *where* $\epsilon = 0$ *or* 1 *if* $vu(\lambda+m) \equiv 0$ *or* $2 \pmod 4$ *respectively.*

Proof. Suppose there exists a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$ $(V \cup U, C)$ where V contains the vertices in λK_v, U contains the other vertices in $(\lambda+m)K_{v+u}$ and C contains the 5-cycles that partition the edge set of $(\lambda+m)K_{v+u} - \lambda K_v$. Then each vertex in V and U must have even degree. So $(\lambda+m)(u-1) + (\lambda+m)v = (\lambda+m)(v+u-1)$ and $m(v-1) + (\lambda+m)u$ are both even, proving (a) and (b). Since the number of edges in $(\lambda+m)K_{v+u} - \lambda K_v$ must be divisible by 5, it follows that $(\lambda+m)\binom{u}{2} + m\binom{v}{2} + vu(\lambda+m) \equiv 0 \pmod 5$, thus Condition (c) is established.

Using the work in the proof of Lemma 2.1.2, Conditions (d-f) can be established since the existence of a 5–CS of λK_v is not used to prove Conditions (c-e) of Lemma 2.1.2. For completeness, we will reconstruct the argument here.

Suppose $u = 1$. Then every 5-cycle that contains at least one mixed edge must have three pure edges joining vertices in V and two mixed edges that join a vertex in V to a vertex in U. Since there are $mv(v-1)/2$ pure edges, $mv(v-1)/6 \geq v(\lambda+m)/2$, thus establishing Condition (d).

Suppose $u = 2$. Then there must be $\lambda + m$ 5-cycles that contain a pure edge joining vertices in U. These 5-cycles also contain $2(\lambda+m)$ pure edges joining vertices in V and $2(\lambda + m)$ mixed edges joining a vertex in V to a vertex in U. Since each remaining 5-cycle must contain at least one pure edge and at most four mixed edges and there are $2(v-1)(\lambda+m)$ remaining mixed edges, it follows that

$$0 \le m\binom{v}{2} - 2(\lambda+m) - \frac{2(v-1)(\lambda+m)}{4} = m\binom{v}{2} - 2(\lambda+m) - \frac{(v-1)(\lambda+m)}{2}.$$

Therefore Condition (e) holds.

Suppose $u \ge 3$. Then every 5-cycle must contain at least one pure edge and at most 4 mixed edges. So there are at least $\lceil vu(\lambda+m)/4 \rceil$ 5-cycles that contain mixed edges. But if $vu(\lambda+m) \equiv 2 \pmod 4$, one of these 5-cycles must contain 2 mixed edges and 3 pure edges. So $\lceil vu(\lambda+m)/4\rceil + 2\epsilon \le mv(v-1)/2 + (\lambda+m)u(u-1)/2$ which shows Condition (f) holds. □

For the remainder of the chapter let $U = \{\infty\}$. Sections 4.2 and 4.3 will show that the conditions in Lemma 4.1.1 are sufficient for $u = 1$. In particular, Section 4.2 addresses the cases when m is small. These cases will then be used to construct the main result of this chapter, Theorem 4.3.1, in Section 4.3.

4.2 5–CSs of $(\lambda+m)K_{v+1} - \lambda K_v$: The small cases

Notice that when $v \not\equiv 2, 3, 7$, or $8 \pmod{10}$ the λ-admissible values and corresponding m values that satisfy Theorem 2.3.2 are identical to the λ-admissible values and corresponding m values that satisfy Lemma 4.1.1(a-d). So all we need to do is prove that a 5–CS of $(\lambda+m)K_{v+1} - \lambda K_v$ exists when $v \equiv 2, 3, 7$ or $8 \pmod{10}$.

We will use almost the same definitions used in Chapter 2. The only change is that we will define $D(v)^*_\ell$ differently. When v and ℓ are both even, it is useful to let $D(v)^*_\ell$ be formed from $D(v)_\ell$ by removing $\ell/2$ copies of the difference $v/2$. Let $D(v)^*_\ell = D(v)_\ell$ if v and ℓ are not both even.

Lemma 2.2.2 shows that the number of 5-cycles defined on the vertices in V is divisible by v and also points to the main approach used in Lemma 4.2.3. So the parameter

$$\alpha = \frac{m(v-1) - 3(\lambda+m)}{2},$$

restated here, will play a critical role throughout the remainder of the paper.

The next lemma will be used when constructing the 5-cycles that contain the vertex ∞; each such 5-cycle uses exactly 3 pure edges in V.

Lemma 4.2.1. *Suppose that Conditions (a-d) in Lemma* 4.1.1 *are satisfied. Then if v is odd, or if both v and m are even,* 3 *divides $m(v-1)/2 - \alpha$. If v is even and m is odd then* 3 *divides $m(v-1)/2 - \alpha - 3/2$.*

Proof. If v is odd then by Lemma 4.1.1(a), $\lambda + m$ is even. If v is even then λ is even by Lemma 4.1.1(b). So if v is odd or if both v and m are even, then 3 divides $m(v-1)/2 - \alpha$. Now suppose v is even and m is odd. It is clear that 3 divides $m(v-1)/2 - \alpha - 3/2 = 3(\lambda + m - 1)/2$ and otherwise 3 divides $m(v-1)/2 - \alpha = 3(\lambda + m)/2$. □

The next lemma will be most useful in Theorem 4.3.1 but also plays a role in Lemma 4.2.3.

Lemma 4.2.2. *If Conditions (a-d) of Lemma* 4.1.1 *hold for any given* v *and* m *when* $u = 1$ *then the difference between consecutive* λ *values that are admissible is* 10 *if* $v \not\equiv 0 \pmod 5$ *and is* 2 *if* $v \equiv 0 \pmod 5$. *Furthermore, this difference is constant for all values of* m.

Proof. For any given v and m, the difference between consecutive values of λ that satisfy Conditions (a-d) of Lemma 4.1.1, say $\lambda_{\text{diff}}(v, m)$, is a constant. Also by Lemma 4.1.1(a-c), $\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 2$ and $v\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 5$, so $v\lambda_{\text{diff}}(v, m) \equiv 0 \pmod{10}$ if $v \not\equiv 0 \pmod 5$ and $v\lambda_{\text{diff}}(v, m) \equiv 0 \pmod 2$ if $v \equiv 0 \pmod 5$. Since $\lambda_{\text{diff}}(v, m)$ must be the smallest such value, $\lambda_{\text{diff}}(v, m) = 10$ or $\lambda_{\text{diff}}(v, m) = 2$ if $v \not\equiv 0 \pmod 5$ or $v \equiv 0 \pmod 5$ respectively. Notice that if $m_1 \neq m_2$ then $\lambda_{\text{diff}}(v, m_1) = \lambda_{\text{diff}}(v, m_2)$. □

For all positive integers v and m, let $\lambda_{\max}(v, m)$ and $\lambda_{\min}(v, m)$ be the largest and smallest integers for λ respectively satisfying Conditions (a-d) in Lemma 4.1.1 with $u = 1$. As was done in Chapters 2 and 3 we will use smaller 5–CSs to build larger 5–CSs. This is the basis behind the induction proof given in Theorem 4.3.1. So we now focus on finding a 5–CS of $(\lambda + m)K_{v+1} - \lambda K_v$ where $m = 1$ and where $\lambda \in \{\lambda_{\min}(v, m), \lambda_{\max}(v, m)\}$ for small values of m.

Lemma 4.2.3. *Let* $v \equiv 2, 3, 7$ *or* $8 \pmod{10}$ *and assume Conditions (a-d) of Lemma* 4.1.1 *are satisfied. Suppose at least one of the following statements is satisfied:*

- $m = 1$ *or both* $m \in \{2, 3\}$ *and* $\lambda = \lambda_{\max}(v, m)$*;*
- $m = 2$*,* $\lambda = 2$*, and* $v \equiv 2 \pmod{10}$*;*
- $(\lambda, m) \in \{(2, 4), (1, 7)\}$ *and* $v \equiv 3 \pmod{10}$*; or*
- $(\lambda, m) \in \{(2, 4), (6, 2), (4, 3)\}$ *and* $v \equiv 8 \pmod{10}$.

Then there exists a 5–CS *of* $(\lambda + m)K_{v+1} - \lambda K_v$.

Proof. Let $V = \mathbb{Z}_v$ be the vertex set of λK_v where $v = 10k + \ell$ for some $\ell \in \{2, 3, 7, 8\}$ and let $U = \{\infty\}$. If $\lambda = 0$ then there exists a 5–CS of mK_v by Theorem 1.2.2. Notice that by Lemma 4.1.1(d), $v \geq 7$ when $\lambda \geq 1$. By defining the sets of 5-cycles B and C, our aim is to show that there exists a 5–CS $(V \cup \{\infty\}, B \cup C)$ of $(\lambda + m)K_{v+1} - \lambda K_v$.

First, note that by Lemma 2.2.2, the number of edges, αv, from 5-cycles with 5 pure edges is given in Equation (3.1). Since this is divisible by v, our plan is to create these 5-cycles using the edges of α specially chosen differences. Notice that by Lemma 4.1.1(d), $\alpha \geq 0$ and by Lemma 4.1.1(c),

$$m\binom{v}{2} + v(\lambda + m) \equiv \ell(\lambda + m) + 3\ell(v-1)m \equiv 0 \pmod 5$$

so that $-3(\lambda + m) \equiv 4m(v-1) \pmod 5$. Then

$$\begin{aligned} m(v-1) - 3(\lambda + m) &\equiv m(v-1) + 4m(v-1) \\ &\equiv 5m(v-1) \\ &\equiv 0 \pmod 5, \end{aligned}$$

and so $\alpha \equiv 0 \pmod 5$. This shows we can distribute the α specially chosen differences into 5-cycles.

Suppose $m = 1$ or $(m, \lambda, \ell) \in \{(2,2,2), (4,2,3), (7,1,3), (2,6,8), (3,4,8)\}$. Skolem and hooked Skolem sequences will be used to explicitly describe the 5-cycles containing all the edges of α differences (yet to be chosen). To form the set of 5-cycles B we consider the cases $\ell \in \{2,3\}$ and $\ell \in \{7,8\}$ in turn.

Suppose $\ell \in \{2,3\}$. Let $P' = \{(x'_i, y'_i) \mid 1 \leq i \leq k-1, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ for the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$, if $k - 1 \equiv 0$ or $1 \pmod 4$ or $k - 1 \equiv 2$ or $3 \pmod 4$ respectively (see Theorems 1.1.1 and 1.1.2). So each element in $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\} \setminus \{2k-2\}$ occurs in exactly one element of P'. Now define $P = \{(x_i = x'_i + 2, y_i = y'_i + 2) \mid (x'_i, y'_i) \in P'\}$. Then let

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k, 5k+1+i) \mid 1 \leq i \leq k-1, (x_i, y_i) \in P\} \tag{4.1}$$

where computations are done modulo v. So for $m = 1$, form the set of 5-cycles $B = \{c_i + j \mid i \in \mathbb{Z}_{\alpha/5}, c_i \in B', j \in \mathbb{Z}_v\}$. Table 4.1 lists the differences of the edges that occur in 5-cycles in B' when α is as large as possible. When α is as large as possible (λ is as small as possible), $\alpha/5 = k-2$ or $k-1$ and $|B'| = k-2$ or $k-1$ if $\ell = 2$ or 3 respectively. So B' is large enough to construct the required 5-cycles.

Edges	Differences when $k-1 \equiv 0, 1 \pmod 4$	Differences when $k-1 \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$3, 4, \ldots, 2k$	$3, 4, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k\}$	$3k+1, 3k+2, \ldots, 4k-1$	$3k+1, 3k+2, \ldots, 4k-1$
$\{3k, 5k+1+i\}$	$2k+2, 2k+3, \ldots, 3k$	$2k+2, 2k+3, \ldots, 3k$
$\{5k+1+i, 0\}$	$5k-2+\ell, 5k-3+\ell, \ldots, 4k+\ell$	$5k-2+\ell, 5k-3+\ell, \ldots, 4k+\ell$

Table 4.1: Edges of differences in B' when α is as large as possible for $\ell \in \{2,3\}$ and $m = 1$

If $m = 2$, $\lambda = 2$, and $\ell = 2$, then $\alpha/5 = 2k-1$ and $|2B'| = 2k-2$, so we must create one more 5-cycle constructed from differences. To account for this, form the set of 5-cycles $B = 2\{c_i + j \mid i \in \mathbb{Z}_{k-1}, j \in \mathbb{Z}_v, c_i \in B'\} \cup \{c + j \mid j \in \mathbb{Z}_v\}$ where B' is defined in Equation (4.1) and

$$c = \begin{cases} (0, 1, 2, 2k+3, 2k+1) & \text{if } k-1 \equiv 0, 1 \pmod 4, \text{ and} \\ (0, 2k, 2k+1, 1, 2) & \text{if } k-1 \equiv 2, 3 \pmod 4. \end{cases}$$

The 5-cycle c uses the differences 1, 1, 2, $2k+1$, and $2k+1$ if $k-1 \equiv 0, 1 \pmod 4$ and the differences 1, 1, 2, $2k$, and $2k$ if $k-1 \equiv 2, 3 \pmod 4$.

If $m = 4$, $\lambda = 2$, and $\ell = 3$, then $\alpha/5 = 4k - 1$ when α is as large as possible and $|4B'| = 4k - 4$, so we must construct three more 5-cycles using differences. In this case form the set of 5-cycles $B = 4\{c_i + j \mid i \in \mathbb{Z}_{k-1}, j \in \mathbb{Z}_v, c_i \in B'\} \cup \{c'_{i \pmod 2} + j \mid j \in \mathbb{Z}_v, i \in \mathbb{Z}_3\}$ where B' is defined in Equation (4.1) and

$$c'_i = \begin{cases} (0, 1, 2k+2, 4k+3, 4k+1) & \text{if } i = 0 \text{ and } k-1 \equiv 0, 1 \pmod 4, \\ (0, 1, 4k+2, 2k+2, 2) & \text{if } i = 0 \text{ and } k-1 \equiv 2, 3 \pmod 4, \text{ and} \\ (0, 1, 2, 4k+3, 4k+1) & \text{if } i = 1. \end{cases} \quad (4.2)$$

Notice that c'_0 is repeated twice in B so that if $k-1 \equiv 0$ or $1 \pmod 4$ then $\{c'_{i \pmod 2} \mid i \in \mathbb{Z}_3\}$ uses three copies of the difference 2 and four copies of each of the differences $1, 2k+1$, and $4k+1$ and if $k-1 \equiv 2$ or $3 \pmod 4$ then $\{c'_{i \pmod 2} \mid i \in \mathbb{Z}_3\}$ uses three copies of the difference 2 and four copies of each of the differences $1, 2k$, and $4k+1$.

If $m = 7$, $\lambda = 1$, and $\ell = 3$, then $\alpha/5 = 7k - 1$, and $|7B'| = 7k - 7$, and so we need to construct six more 5-cycles using differences. Under these conditions form the set of 5-cycles

$$B = 7\{c_i + j \mid i \in \mathbb{Z}_{k-1}, j \in \mathbb{Z}_v, c_i \in B'\} \cup 3\{c'_0 + j \mid j \in \mathbb{Z}_v\} \cup \{c'_1 + j \mid j \in \mathbb{Z}_v\} \\ \cup 2\{c'_2 + j \mid j \in \mathbb{Z}_v\}$$

where B' is defined in Equation (4.1), c'_0 and c'_1 are defined as in Equation (4.2), and $c'_2 = (0, 1, 4k+3, 8k+4, 4k+2)$. The 5-cycles c'_0, c'_1 and c'_2 in B use four copies of difference 2, six copies of the difference $4k+2$, seven copies of both of the differences 1 and $4k+1$, and six copies of the difference $2k+1$ or the difference $2k$ if $k \equiv 0$ or $1 \pmod 4$ or $k \equiv 2$, or $3 \pmod 4$ respectively.

Now suppose $\ell \in \{7, 8\}$. Let $P = \{(x_i, y_i) \mid 1 \le i \le k, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order k for the set $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\}$, if $k \equiv 0$ or $1 \pmod 4$ or $k \equiv 2$ or $3 \pmod 4$ respectively. Define

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k+2, 5k+4+i) \mid 1 \le i \le k, (x_i, y_i) \in P\} \quad (4.3)$$

where computations are done modulo v. So for $m = 1$ or $(m, \lambda, \ell) \in \{(2, 6, 8), (3, 4, 8)\}$, form the set of 5-cycles $B = \{c_i + j \mid i \in \mathbb{Z}_{\alpha/5}, c_{i \pmod k} \in B', j \in \mathbb{Z}_v\}$. Table 4.2 lists the differences of edges that occur in cycles in B when α is as large as possible (λ

is as small as possible). Since $|mB'| = mk$ and $\alpha/5$ is k, $k-2$, $2k-1$, or $3k$ when (m, λ, ℓ) is $(1,1,7)$, $(1,8,8)$, $(2,6,8)$, or $(3,4,8)$ respectively (note that if $m = 1$ and α is as large as possible then $\lambda = 1$ or 8 when $\ell = 7$ or 8 respectively), there are enough 5-cycles in mB' to construct all of the 5-cycles with only pure edges when $m = 1$ or $(m, \lambda, \ell) \in \{(2,6,8),(3,4,8)\}$.

Edges	Differences when $k \equiv 0, 1 \pmod 4$	Differences when $k \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k$	$1, 2, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k+2\}$	$3k+3, 3k+4, \ldots, 4k+2$	$3k+3, 3k+4, \ldots, 4k+2$
$\{3k+2, 5k+4+i\}$	$2k+3, 2k+4, \ldots, 3k+2$	$2k+3, 2k+4, \ldots, 3k+2$
$\{5k+4+i, 0\}$	$5k-5+\ell, 5k-6+\ell, \ldots, 4k-4+\ell$	$5k-5+\ell, 5k-6+\ell, \ldots, 4k-4+\ell$

Table 4.2: Edges of differences in B' when α is as large as possible for $\ell \in \{7,8\}$ and $m = 1$

Now suppose $m = 4$, $\lambda = 2$, and $\ell = 8$. Since $\alpha/5 = 4k+1$ and $|4B'| = 4k$, we need to construct one more 5-cycle constructed from differences. So we form the set of 5-cycles

$$B = 4\{c_i + j \mid i \in \mathbb{Z}_k, j \in \mathbb{Z}_v, c_i \in B'\} \cup \{(0, 4k+3, 2k+1, 6k+4, 8k+6) + j \mid j \in \mathbb{Z}_v\}$$

where B' is defined in Equation (4.3) and computations are done modulo v and $(0, 4k+3, 2k+1, 6k+4, 8k+6)$ uses the differences $2k+2$, $2k+2$, $2k+2$, $4k+3$, and $4k+3$.

To define C, let $\ell \in \{2,3,7,8\}$. By Lemma 4.2.1, if v is odd or both v and m are even then 3 divides $(m(v-1)/2 - \alpha)$ and if v is even and m is odd then 3 divides $(m(v-1)/2 - \alpha - 3/2)$. So form a partition P_0 from the differences not used to form B into $(m(v-1)/2 - \alpha)/3$ sets of size 3 (not multisets) when either v is odd or both v and m are even, and form a partition P_0 from the differences not used to form B into $(m(v-1)/2 - \alpha - 3/2)/3$ sets of size 3 (not multisets) and one set $\{d, v/2\}$ of size 2 with $d \neq v/2$ when v is even and m is odd; this ensures that $d_1 < d_3$ and $d_2 < v/2$ when defining C. Form the set C of 5-cycles

$$C = \{(d_1, 0, d_3, d_2 + d_3, \infty) + j \mid \{d_1, d_2, d_3\} \in P_0 \text{ with } d_1 \leq d_2 \leq d_3, j \in \mathbb{Z}_v\}. \quad (4.4)$$

when either v is odd or both v and m are even and the set

$$\begin{aligned} C =& \{(d_1, 0, d_3, d_2 + d_3, \infty) + j \mid \{d_1, d_2, d_3\} \in P_0 \text{ with } d_1 \leq d_2 \leq d_3, j \in \mathbb{Z}_v\} \quad (4.5) \\ &\cup \left\{ \left(d, 0, \frac{v}{2}, \frac{v}{2} + d, \infty\right) + j \mid \left\{\frac{v}{2}, d\right\} \in P_0, j \in \mathbb{Z}_{v/2} \right\}. \end{aligned}$$

when v is even and m is odd. By Lemma 2.2.5 the elements of C are 5-cycles. Thus $(\mathbb{Z}_v \cup \{\infty\}, B \cup C)$ is a 5–CS of $(\lambda + m)K_{v+1} - \lambda K_v$ with $v \equiv 2, 3, 7$, or $8 \pmod{10}$ and $m = 1$ or with $(\lambda, m, \ell) \in \{(2,2,2),(2,4,3),(1,7,3),(2,6,8),(3,4,8),(4,2,8)\}$.

Now suppose $m = 2$, $\lambda = \lambda_{\max}(v, m)$, and $v \equiv 2, 3, 7$, or $8 \pmod{10}$. For any fixed values of v and m, v-admissible values of λ differ by multiples of 10 by Lemma 4.2.2, and so the corresponding α values differ by multiples of 15 (by Lemma 2.2.2). So when $\lambda = \lambda_{\max}(v, m)$ it follows that $\alpha \in \{0, 5, 10\}$. Then let

$$B = \begin{cases} \varnothing & \text{if } \alpha = 0, \\ \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} & \text{if } \alpha = 5, \text{ and} \\ \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} \cup \{(0,1,4,2,6) + j \mid j \in \mathbb{Z}_v\} & \text{if } \alpha = 10, \end{cases}$$

and let $\beta = \varnothing, \{1,2,3,4,6\}$, or $\{1,1,2,2,3,3,4,4,6,6\}$ when $\alpha = 0, 5$, or 10 respectively. By Lemma 4.2.1, $|D(v)^*_2 \setminus \beta| \equiv 0 \pmod 3$. Notice that if $v = 7$, $m = 2$, and $\lambda_{\max}(7,2) = 2$, then $\alpha = 0$ and if $v = 8$ and $m = 2$ then there are no 5–CSs of $(\lambda+m)K_{v+1} - \lambda K_v$ by Lemma 4.1.1(a-d), so these 5-cycles in B can be constructed. Form C using Equation (4.4) when either v is odd or both v and m are even and using Equation (4.5) when v is even and m is odd. Thus $(\mathbb{Z}_v \cup \{\infty\}, B \cup C)$ is a 5–CS of $(\lambda+2)K_{v+1} - \lambda K_v$ for $v \equiv 2, 3, 7$, or $8 \pmod{10}$ and $\lambda = \lambda_{\max}(v, m)$.

Let $m = 3$ and $\lambda = \lambda_{\max}(v, 3) = v - 4$. It follows that $\alpha = 0$ for all k, so define $B = \varnothing$. Form C using Equation (4.4) when either v is odd or both v and m are even and using Equation (4.5) when v is even and m is odd. Therefore $(\mathbb{Z}_v \cup \{\infty\}, B \cup C)$ is a 5–CS of $(\lambda+3)K_{v+1} - \lambda K_v$ for $v \equiv 2, 3, 7$, or $8 \pmod{10}$ with $\lambda = \lambda_{\max}(v, 3)$. □

4.3 5–CSs of $(\lambda+m)K_{v+1} - \lambda K_v$: The Main Result

In the following theorem, Lemma 4.2.3 is used as a basis for an inductive argument to settle the problem of constructing a 5–CS of $(\lambda+m)K_{v+1} - \lambda K_v$.

Theorem 4.3.1. *Let $m \geq 1$. There exists a* 5–CS *of $(\lambda+m)K_{v+1} - \lambda K_v$ if and only if*

(a) $(\lambda+m) + m(v-1) \equiv 0 \pmod 2$,
(b) $v(\lambda+m) \equiv 0 \pmod 2$,
(c) $m\binom{v}{2} + v(\lambda+m) \equiv 0 \pmod 5$, *and*
(d) $m(v-1) \geq 3(\lambda+m)$.

Proof. The necessity follows from Lemma 4.1.1, so we now prove the sufficiency. If $\lambda = 0$ then by Condition (a), $mv \equiv 0 \pmod 2$ and by Condition (c), $mv(v+1)/2 \equiv 0 \pmod 5$; so by Theorem 1.2.2 there exists a 5–CS of mK_{v+1}. So we can also assume that $\lambda \geq 1$. By Condition (d), $v \notin \{1,2,3,4\}$, so we know $v \geq 5$. By Theorem 2.3.2, there exists a 5–CS of $(\lambda+m)K_{v+1} - \lambda K_v$ for $v \not\equiv 2, 3, 7$ or $8 \pmod{10}$, and so we can assume $v \equiv 2, 3, 7$ or $8 \pmod{10}$. Let V be the vertex set of λK_v and U contains the other vertex in $(\lambda+m)K_{v+1}$.

First we establish that for any given v, the difference between consecutive values of λ that satisfy Conditions (a-c) is a constant; call this difference $\lambda_{\text{diff}}(v)$. Lemma 4.2.2 establishes this claim and shows that $\lambda_{\text{diff}}(v) = 10$. Secondly, for all $m \geq 1$ and $v \geq 5$, we will show there exists a 5–CS of $(\lambda+m)K_{v+1} - \lambda K_v$ for both the smallest and the largest values of λ satisfying all of the conditions. Finally for all $m \geq 1$ and $v \geq 5$ it will be shown that for all λ satisfying Conditions (a-d) there exist non-negative integers λ_1, λ_2 with $\lambda_1 + \lambda_2 = \lambda$ and positive integers m_1, m_2 with $m_1 + m_2 = m$ such that there exists a 5–CS of $(\lambda_i + m_i)K_{v+1} - \lambda_i K_v$ for $i \in \{1, 2\}$. So the union of these two 5–CSs completes the proof.

We turn to the second step in the proof; establishing the existence of the 5–CS for the smallest and largest values of λ given v and m. Beginning with the smallest value, Table 4.3 lists the values of $\lambda_{\min}(v, m)$ using Theorem 1.2.2 and Lemma 4.2.3. Four cases are considered in turn based on the value of v.

	$m \pmod{10}$									
$v \pmod{10}$	1	2	3	4	5	6	7	8	9	0
2	6	2	8	4	0	6	2	8	4	0
3	3	6	9	2	5	8	1	4	7	0
7	1	2	3	4	5	6	7	8	9	0
8	8	6	4	2	0	8	6	4	2	0

Table 4.3: List of $\lambda_{\min}(v, m)$ values

Suppose $v \equiv 2 \pmod{10}$. By Lemma 4.2.3 a 5–CS of $(\lambda_{\min}(v, m) + m)K_{v+1} - (\lambda_{\min}(v, m))K_v$ exists for $m = 1$ and 2. From examining Table 4.3 it is easy to see that the union of

$$(\lambda_{\min}(v, m') + m')K_{v+1} - (\lambda_{\min}(v, m'))K_v$$

and

$$(\lambda_{\min}(v, m'') + m'')K_{v+1} - (\lambda_{\min}(v, m''))K_v$$

where $m' = 1$ and $m'' = 2$ or $m' = 2$ and $m'' = 2$ form a 5–CS of $(\lambda_{\min}(v, m) + m)K_{v+1} - (\lambda_{\min}(v, m))K_v$ where $m = 3$ or 4 respectively. Now we can form a 5–CS of $(\lambda_{\min}(v, 5) + 5)K_{v+1} - (\lambda_{\min}(v, 5))K_v$ by taking the union of

$$(\lambda_{\min}(v, 2) + 2)K_{v+1} - (\lambda_{\min}(v, 2))K_v$$

and

$$(\lambda_{\min}(v, 3) + 3)K_{v+1} - (\lambda_{\min}(v, 3))K_v.$$

We will revisit what happens when $m > 5$ after we examine when $v \equiv 8 \pmod{10}$ and $m \leq 5$

Suppose $v \equiv 8 \pmod{10}$. It follows by Lemma 4.2.3 that a 5–CS of $(\lambda_{\min}(v, m) + m)K_{v+1} - (\lambda_{\min}(v, m))K_v$ exists for $m \leq 4$. By Theorem 2.3.2, there exists a 5–CS of $(\lambda_{\min}(v, 5) + 5)K_{v+1} - (\lambda_{\min}(v, 5))K_v$.

Now suppose $v \equiv 2$ or $8 \pmod{10}$. By using the 5–CS of $5K_{v+1}$ (which exists by Theorem 1.2.2) we will generate the remaining 5–CS for $m \geq 6$ when $\lambda = \lambda_{\min}(v, m)$ using induction on m. Suppose for some $m \geq 6$ and for all s with $1 \leq s < m$

there exists a 5–CS of $((\lambda_{\min}(v,s)+s)K_{v+1}-(\lambda_{\min}(v,s))K_v$. Since $\lambda_{\min}(v,m-5)=\lambda_{\min}(v,m)$ and there exists a 5–CS $(V\cup U,B_1)$ of

$$(\lambda_{\min}(v,m-5)+m-5)K_{v+1}-(\lambda_{\min}(v,m-5))K_v$$

and a 5–CS $(V\cup U,B_2)$ of $5K_{v+1}$, it follows that there exists a 5–CS $(V\cup U,B_1\cup B_2)$ of

$$\begin{aligned}((\lambda_{\min}(v,m-5)+m-5)+5)K_{v+1}-(\lambda_{\min}(v,m-5))K_v\\ =(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v.\end{aligned}$$

Suppose $v\equiv 3 \pmod{10}$. By Lemma 4.2.3 we have established the existence of a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v$ for $m\in\{1,4,7\}$. So by taking the union of particular copies of these 5–CSs, we can form a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v$ for $m\in\{2,5,8\}$. Since we can build a 5–CS of $(\lambda+m)K_{v+1}-\lambda K_v$ in the case when λ is as small as possible and $m\in\{1,2,4,5,7,8\}$, we can form a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v$ for $m\in\{3,6,9,10\}$ by taking the union of 2 particular copies of the 5–CSs we have already constructed. We will revisit what happens when $m>10$ after we examine when $m\leq 10$ and $v\equiv 7 \pmod{10}$.

Suppose $v\equiv 7 \pmod{10}$. Then by Lemma 4.2.3, we can form a 5–CS of $(\lambda_{\min}(v,1)+1)K_{v+1}-(\lambda_{\min}(v,1))K_v$. So by using successive copies of this 5–CS, we can form a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v$ for $2\leq m\leq 10$.

Now suppose $v\equiv 3$ or $7 \pmod{10}$. We will use induction on m in this proof to construct the remaining 5–CSs. As before, we use the existence of a 5–CS of $10K_{v+1}$ (which exists by Theorem 1.2.2) and a 5–CS of $(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v$ for $m\leq 10$ to assist constructing larger 5–CSs. So suppose for some $m\geq 11$ and for all s with $1\leq s<m$, there exists a 5–CS of $(\lambda_{\min}(v,s)+s)K_{v+1}-(\lambda_{\min}(v,s))K_v$. Since there exists a 5–CS $(V\cup U,B_1)$ of

$$(\lambda_{\min}(v,m-10)+m-10)K_{v+1}-(\lambda_{\min}(v,m-10))K_v,$$

since there exists a 5–CS $(V\cup U,B_2)$ of $10K_{v+1}$, and since $\lambda_{\min}(v,m-10)=\lambda_{\min}(v,m)$, it follows that there exists a 5–CS $(V\cup U,B_1\cup B_2)$ of $(\lambda_{\min}(v,m)+m)K_{v+1}-(\lambda_{\min}(v,m))K_v$.

We now turn to constructing a 5–CS of $(\lambda+m)K_{v+1}-\lambda K_v$ where $\lambda=\lambda_{\max}(v,m)$. The proof of this case will be done using induction on m. By Lemma 4.2.3, there exists a 5–CS of $(\lambda_{\max}(v,m)+m)K_{v+1}-(\lambda_{\max}(v,m))K_v$ for $m\in\{1,2,3\}$. Suppose for some $m\geq 4$ and for all s with $1\leq s<m$ there exists a 5–CS of $(\lambda_{\max}(v,s)+s)K_{v+1}-(\lambda_{\max}(v,s))K_v$. Since $\lambda_{\max}(v,m-3)+\lambda_{\max}(v,3)=\lambda_{\max}(v,m)$ by Lemma 2.3.1, there exists a 5–CS of

$$(\lambda_{\max}(v,m-3)+m-3)K_{v+1}-(\lambda_{\max}(v,m-3))K_v$$

and there exists a 5–CS of $(\lambda_{\max}(v,3)+3)K_{v+1}-(\lambda_{\max}(v,3))K_v$, it follows that the union of these two 5–CSs form a 5–CS of $(\lambda_{\max}(v,m)+m)K_{v+1}-(\lambda_{\max}(v,m))K_v$.

	$v \pmod{10}$			
m	2	3	7	8
2	0,10,0	10,0,0	0,0,10	0,10,0
3	10,10,0	10,0,10	0,10,10	10,10,0

Table 4.4: Each cell contains $\lambda_{\max}(v,m) - \lambda_{\max}(v,m-1) - \lambda_{\max}(v,1)$ for $k \equiv 0,1,2 \pmod{3}$ in turn

	$m \pmod{10}$									
$v \pmod{10}$	1	2	3	4	5	6	7	8	9	0
2	0	10	0	10	10	0	10	0	10	10
3	0	0	0	10	0	0	10	0	0	10
7	0	0	0	0	0	0	0	0	0	10
8	0	10	10	10	10	0	10	10	10	10

Table 4.5: Each cell contains $\lambda_{\min}(v,m-1) + \lambda_{\min}(v,1) - \lambda_{\min}(v,m)$

Now we turn to the third and final step of the proof using induction on m. We have settled the cases where $m = 1$ and where $v \geq 5$, $m > 1$, and $\lambda \in \{\lambda_{\min}(v,m), \lambda_{\max}(v,m)\}$. It is left to show that there exists a 5–CS of $(\lambda+m)K_{v+1} - \lambda K_v$ for $m > 1$ and $\lambda_{\min}(v,m) < \lambda < \lambda_{\max}(v,m)$. Tables 4.4 and 4.5 were formed using Theorem 1.2.2 and Lemma 4.2.3. These tables will be instrumental in showing that we can use smaller 5–CSs to form larger 5–CSs. Suppose for some $m > 1$ and for all s with $1 \leq s < m$ there exists a 5–CS of $(\lambda+s)K_{v+1} - \lambda K_v$. By Lemma 4.2.2, $\lambda_{\text{diff}}(v)$ is constant for all m. From Table 4.4,

$$\lambda_{\max}(v,m) - \lambda_{\max}(v,m-1) - \lambda_{\max}(v,1) \in \{0, \lambda_{\text{diff}}(v)\}$$

and so

$$\lambda_{\max}(v,m) - \lambda_{\text{diff}}(v,m) = \lambda_{\max}(v,m) - 10 \leq \lambda_{\max}(v,m-1) + \lambda_{\max}(v,1).$$

Similarly, Table 4.5 shows that

$$\lambda_{\min}(v,m-1) + \lambda_{\min}(v,1) - \lambda_{\min}(v,m) \in \{0, \lambda_{\text{diff}}(v)\},$$

and so

$$\lambda_{\min}(v,m-1) + \lambda_{\min}(v,1) \leq \lambda_{\min}(v,m) + \lambda_{\text{diff}}(v,m) = \lambda_{\min}(v,m) + 10.$$

This shows that

$$\lambda_{\min}(v,m-1) + \lambda_{\min}(v,1) \leq \lambda \leq \lambda_{\max}(v,m-1) + \lambda_{\max}(v,1).$$

It follows that there exist non-negative integers λ_1, λ_2 for which $\lambda_{\min}(v,1) \leq \lambda_1 \leq \lambda_{\max}(v,1)$, $\lambda_{\min}(v,m-1) \leq \lambda_2 \leq \lambda_{\max}(v,m-1)$, and $\lambda_1 + \lambda_2 = \lambda$, and there exist positive integers m_1, m_2 with $m_1 + m_2 = m$ such that for $1 \leq i \leq 2$ there exists a 5–CS of $(\lambda_i + m_i)K_{v+1} - \lambda_i K_v$. So the union of these two 5–CSs proves there exists a 5–CS of $(\lambda + m)K_{v+1} - \lambda K_v$. $\square$

Chapter 5
5-cycle systems when $m = 0$

5.1 Introduction

In this chapter we will settle the existence of a 5–CS of $\lambda K_{v+u} - \lambda K_v$ except possibly in the case where $\lambda \equiv 5 \pmod{10}$, $v \equiv v + u \equiv 1 \pmod{2}$, and $\lambda < v \leq (u-1)/3$ and in the case where $\lambda \equiv 2, 4, 6$, or $8 \pmod{10}$, $u \equiv 0 \pmod{5}$, and $v/2 + 1 \leq u < v/2 + 5$. To do this, the methods in this chapter draw heavily from the proof methods established in Chapters 2-4. Notice that if a 5–CS of a λK_v exists then this is an embedding type problem rather than an enclosing.

In the literature there is one result which settles the existence problem in the special case where $\lambda = 1$ and $m = 0$ [11]:

Theorem 5.1.1. [11] *There exists a* 5*-cycle system of* $K_{v+u} - K_v$ *if and only if*

(a) $u \geq v/2 + 1$, *and*
(b) $v + u \equiv v \equiv 3 \pmod{10}$, *or* $v + u, v \equiv 1$ *or* $5 \pmod{10}$, *or* $v + u, v \equiv 7$ *or* $9 \pmod{10}$.

Section 5.2 will find some necessary conditions for the existence of a 5–CS of $\lambda K_{v+u} - \lambda K_v$ and establish some conditions that will be useful in subsequent lemmas. In Section 5.3, we will establish that these necessary conditions are sufficient in the case where $\lambda \equiv 5 \pmod{10}$ and $u < 3v + 1$. Section 5.4 will show the necessary conditions in Lemma 5.2.1 are sufficient except possibly in two cases.

5.2 5–CSs of $\lambda K_{v+u} - \lambda K_v$: Preliminary Results

The following necessary conditions are remarkably similar to those given in Theorem 5.1.1. Notice that Theorem 5.1.1 proves the conditions in Lemma 5.2.1(a) are necessary and sufficient by taking λ copies of a 5–CS of $K_{v+u} - K_v$. Recall that an edge is said to be pure if it joins two vertices in V or if it joins two vertices in U; otherwise it is said to be mixed.

Lemma 5.2.1. *If there exists a* 5–CS *of* $\lambda K_{v+u} - \lambda K_v$ *then* $u \geq v/2 + 1$ *and*

(a) If $\lambda \equiv 1, 3, 7,$ *or* $9 \pmod{10}$ *then*

$v + u, v \equiv 1$ *or* $5 \pmod{10}$;
$v + u, v \equiv 7$ *or* $9 \pmod{10}$; *and*
$v + u \equiv v \equiv 3 \pmod{10}$.

(b) If $\lambda \equiv 0 \pmod{10}$ *then any* $v \geq 1$ *and* $v + u \geq 5$.

(c) If $\lambda \equiv 2, 4, 6,$ *or* $8 \pmod{10}$ *then*

$$v + u, v \equiv 0, 1 \pmod 5;$$
$$v + u, v \equiv 2, 4 \pmod 5; \text{ and}$$
$$v + u \equiv v \equiv 3 \pmod 5.$$

(d) If $\lambda \equiv 5 \pmod{10}$ *then* v *and* $v + u$ *are both odd.*

Proof. Let $V \cup U$ be the vertex set of $\lambda K_{v+u} - \lambda K_v$ where V is the vertex set of λK_v and U contains the other vertices in λK_{v+u}. If there exists a 5–CS of $\lambda K_{v+u} - \lambda K_v$ then there is a limit to the number of mixed edges since every 5-cycle must contain at least one pure edge. Hence $4\lambda u(u-1)/2 \geq \lambda vu$ shows $u \geq v/2 + 1$.

As said before, Condition (a) holds by Theorem 5.1.1. If $\lambda \equiv 0 \pmod{10}$ then it is clear that $v + u \geq 5$ and so Condition (b) holds. Suppose that $\lambda \equiv 5 \pmod{10}$. Since the degree of each vertex must be even, $v - 1$ and $v + u - 1$ must be even and so v and $v + u$ must be odd showing that Condition (d) is satisfied. Suppose that $\lambda \equiv 2, 4, 6,$ or $8 \pmod{10}$. Since the number of edges must be divisible by 5, $\lambda vu + \lambda u(u-1)/2 \equiv 0 \pmod 5$ and so $vu + u(u-1)/2 \equiv u(2v + u - 1) \equiv 0 \pmod 5$ since $\lambda \not\equiv 0 \pmod 5$. Condition (c) follows from this observation. □

There are three types of 5-cycles in $\lambda K_{v+u} - \lambda K_v$ based on the number of mixed and pure edges in each 5-cycle:

- one pure edge and four mixed edges,
- three pure edges and two mixed edges, and
- five pure edges.

Define these 5-cycles as Type $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ respectively. A visual representation of these 5-cycles is given in Figure 5.1. For the remainder of the chapter, let a, b, and

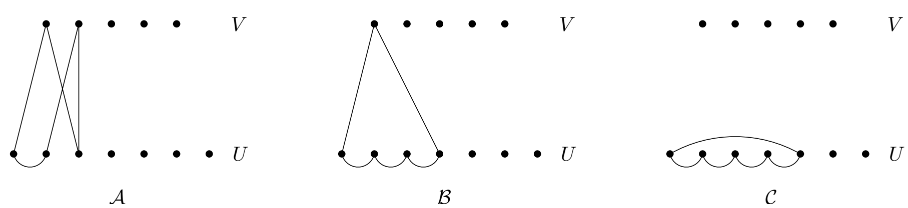

Figure 5.1: Possible Types of 5-cycle

c be the number of 5-cycles of Type $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ respectively. We may need to set aside certain differences in order for the conditions of Lemma 5.2.2 to hold when $u \equiv 0 \pmod 5$; let ϵ be the number of differences we choose to set aside. If $\lambda \equiv 5 \pmod{10}$ then ϵ is always chosen to be 0. Furthermore, if $u \not\equiv 0 \pmod 5$ then ϵ is always chosen to be 0 and if $u \equiv 0 \pmod 5$ then ϵ is always chosen to be in $\mathbb{Z}_5$. Since there are λvu mixed edges and $\lambda\binom{u}{2}$ pure edges in $\lambda K_{v+u} - \lambda K_v$, we can set up the following system of equations:

$$4a + 2b = \lambda v u \tag{5.1}$$
$$a + 3b + 5c = \lambda \binom{u}{2} - \epsilon u.$$

For the rest of the chapter, define

$$a = \begin{cases} 0 & \text{if } u \geq 3v+1 \text{ and} \\ u(\lambda(3v-u+1)+2\epsilon)/10 & \text{if } u < 3v+1, \end{cases}$$
$$b = \begin{cases} \lambda uv/2 & \text{if } u \geq 3v+1 \text{ and} \\ u(\lambda(2u-v-2)-4\epsilon)/10 & \text{if } u < 3v+1, \end{cases} \tag{5.2}$$
$$c = \begin{cases} u(\lambda(u-1-3v)-2\epsilon)/10 & \text{if } u \geq 3v+1 \text{ and} \\ 0 & \text{if } u < 3v+1. \end{cases}$$

The values for a, b, and c in (5.2) satisfy the system of equations in (5.1).

The following lemma will be useful for showing that the values for a, b, and c in (5.2) satisfy conditions that will become vital in Theorem 5.4.1.

Lemma 5.2.2. *Suppose that λ, v, and u satisfy Conditions (a-d) of Lemma 5.2.1 and $\lambda \not\equiv 1, 3, 7,$ or $9 \pmod{10}$. If $u \geq 3v+1$ then*

(i) *b and c are non-negative integers,*
(ii) *u divides c,*
(iii) *vu divides b if λ is even or $vu/2$ divides b if λ is odd,*
(iv) *$\lambda(u-1)/2 - \lambda(u-3v-1)/2 \equiv 0 \pmod{3}$ when λ is even, and*
(v) *$\lambda(u-1)/2 - \lambda(u-3v-1)/2 - 3v/2 \equiv 0 \pmod{3}$ when λ is odd,*

If $v/2 + 1 \leq u < 3v+1$ then

(vi) *a and b are non-negative integers if $\lambda \equiv 0 \pmod{5}$,*
(vii) *a and b are non-negative integers if $\lambda \equiv 2, 4, 6,$ or $8 \pmod{10}$ and $u \not\equiv 0 \pmod{5}$ or if $\lambda \equiv 2, 4, 6,$ or $8 \pmod{10}$, $u \equiv 0 \pmod{5}$, and $u \geq v/2 + 5$,*
(viii) *u divides a, and*
(ix) *u divides b if λ is even.*

Proof. Suppose that $u \geq 3v+1$. Then it is clear that b and c are non-negative. Conditions (*i*) and (*ii*) will be discussed in turn based on λ.

If $\lambda \equiv 0 \pmod{10}$ then c is an integer and u divides c. If $\lambda \equiv 5 \pmod{10}$ then v is odd, u is even, and $\epsilon = 0$ so that $u - 1 - 3v$ is even and consequently c is an integer and u divides c. Now suppose that $\lambda \equiv 2, 4, 6,$ or $8 \pmod{10}$. Then it is clear that $b = \lambda uv/2$ is an integer. Suppose $u \not\equiv 0 \pmod{5}$ so that $\epsilon = 0$. We will now show that $(u - 1 - 3v) \equiv 0 \pmod{5}$ to show c is an integer and u divides c

when $\lambda \equiv 2, 4, 6$, or $8 \pmod{10}$. Let $u = 5k_1 + e_1$ and $v = 5k_2 + e_2$ satisfy Condition (c) of Lemma 5.2.1 where $e_1 \in \mathbb{Z}_5 \setminus \{0\}$ (i.e. $u \not\equiv 0 \pmod 5$) and $e_2 \in \mathbb{Z}_5$. So the possible values of e_1 and e_2 are $(e_1, e_2) \in \{(1, 0), (4, 1), (2, 2), (3, 4)\}$. It follows that $(u - 1 - 3v) \equiv e_1 - 1 - 3e_2 \equiv 0 \pmod 5$ which shows c is an integer and consequently u divides c. Now let $e_1 = 0$ so that $u \equiv 0 \pmod 5$. Then

$$\lambda(u - 1 - 3v) - 2\epsilon \equiv -\lambda - 3\lambda e_2 - 2\epsilon \pmod{10}$$

and so for some $\epsilon \in \mathbb{Z}_5$, $-\lambda - 3\lambda e_2 - 2\epsilon \equiv 0 \pmod{10}$. This shows that c is an integer and u divides c and so Conditions (i) and (ii) are satisfied.

Since $b = \lambda uv/2$, it is clear that vu divides b or $vu/2$ divides b when λ is even or odd respectively, so Condition (iii) holds. Since $\lambda(u-1)/2 - \lambda(u-3v-1)/2 = 3v\lambda/2$ is divisible by 3 when λ is even and $\lambda(u-1)/2 - \lambda(u-3v-1)/2 - 3v/2 = 3v(\lambda-1)/2$ is divisible by 3 when λ is odd, it follows that Conditions (iv) and (v) are satisfied.

Finally, suppose that $u < 3v + 1$. Then $a \geq 0$ and $b \geq 0$ (since $u \geq v/2 + 5$ if $\lambda \equiv 2, 4, 6$, or $8 \pmod{10}$ in this lemma). To show a is divisible by u we go through the same argument as before. If $\lambda \equiv 0 \pmod{10}$ then a is an integer and u divides a. If $\lambda \equiv 5 \pmod{10}$ then v is odd, u is even, and $\epsilon = 0$ so that $3v - u + 1$ is even and consequently a is an integer and u divides a. Now suppose that $\lambda \equiv 2, 4, 6$, or $8 \pmod{10}$. Also, suppose $u \not\equiv 0 \pmod 5$ so that $\epsilon = 0$. We will now show that $(u - 1 - 3v) \equiv 0 \pmod 5$ to show a is an integer and u divides a when $\lambda \equiv 2, 4, 6$, or $8 \pmod{10}$. Let $u = 5k_1 + e_1$ and $v = 5k_2 + e_2$ satisfy Condition (c) of Lemma 5.2.1 where $e_1 \in \mathbb{Z}_5 \setminus \{0\}$ (i.e. $u \not\equiv 0 \pmod 5$) and $e_2 \in \mathbb{Z}_5$. So the possible values of e_1 and e_2 are $(e_1, e_2) \in \{(1, 0), (4, 1), (2, 2), (3, 4)\}$. It follows that $(3v - u + 1) \equiv 3e_2 - e_1 + 1 \equiv 0 \pmod 5$ which shows a is an integer and consequently u divides a. Now let $e_1 = 0$ so that $u \equiv 0 \pmod 5$. Then

$$\lambda(3v - u + 1) - 2\epsilon \equiv \lambda + 3\lambda e_2 - 2\epsilon \pmod{10}$$

and so for some $\epsilon \in \mathbb{Z}_5$, $\lambda + 3\lambda e_2 - 2\epsilon \equiv 0 \pmod{10}$. This shows that a is an integer and u divides a and so Conditions (i) and (ii) are satisfied. Conditions (vi), (vii), and (ix) will be discussed in turn based on λ.

When $\lambda \equiv 0 \pmod{10}$, Conditions (vi) and (ix) are satisfied. If $\lambda \equiv 5 \pmod{10}$ then u is even and so it is clear that Condition (vi) is satisfied. Let $\lambda \equiv 2, 4, 6$, or $8 \pmod{10}$. Suppose that $u \not\equiv 0 \pmod 5$ so that $\epsilon = 0$. Let $u = 5k_1 + e_1$ and $v = 5k_2 + e_2$ satisfy Condition (c) of Lemma 5.2.1 where $e_1 \in \mathbb{Z}_5 \setminus \{0\}$ and $e_2 \in \mathbb{Z}_5$, so the possible values of e_1 and e_2 are $(e_1, e_2) \in \{(1, 0), (4, 1), (2, 2), (3, 4)\}$. Hence it is clear that $\lambda(2u - v - 2) \equiv \lambda(2e_1 - e_2 - 2) \equiv 0 \pmod{10}$, so b is an integer and u divides b. Now suppose that $u \equiv 0 \pmod 5$. If λ is even then there exists some $\epsilon \in \mathbb{Z}_5$ such that $\lambda(2u - v - 2) - 8\epsilon \equiv \lambda(-e_2 - 2) - 8\epsilon \equiv 0 \pmod{10}$. Therefore Conditions (vii) and (ix) are satisfied. □

When $u \equiv 0 \pmod 5$ and $\lambda \not\equiv 5 \pmod{10}$, we need to use edges of 0, 1, 2, 3, or 4 differences (depending on the value of ϵ) to construct some 5-cycles with only pure edges so that the total number of remaining differences needed to build 5-cycles with only pure edges is divisible by 5. To this end, remember back to Section 2.2 where $G_u(D(u)^*_\ell)$ is defined as containing ℓ copies of each edge with difference in $D(u)^*$. We will use the observation that there exists a 5–CS of $G_u(\{u/5\})$ and a 5–CS of $G_u(\{2u/5\})$ when $u \equiv 0 \pmod 5$, since each component is a 5-cycle. Recall that $G_u(\{1,2,3\})$ is a 5–CS by Lemma 2.2.1 when $u \equiv 0 \pmod 5$. Let $(\mathbb{Z}_u, \Gamma_1)$, $(\mathbb{Z}_u, \Gamma_2)$ and $(\mathbb{Z}_u, \Gamma_3)$ be a 5–CS of $G_u(\{u/5\}), G_u(\{2u/5\})$, and $G_u(\{1,2,3\})$ respectively when $u \equiv 0 \pmod 5$.

5.3 5–CSs of $\lambda K_{v+u} - \lambda K_v$: $\lambda \equiv 5 \pmod{10}$ and $u < 3v + 1$

In this section we will focus solely on the case where $\lambda \equiv 5 \pmod{10}$ and $u < 3v+1$. This case needs to be dealt with separately from the other values of λ. Recall that if $\lambda \equiv 5 \pmod{10}$ then u is even, v is odd, and $\epsilon = 0$. Partition the vertices of U into two sets U_0 and U_1 of equal size such that $U = U_0 \cup U_1 = \mathbb{Z}_{u/2} \times \mathbb{Z}_2$ and $(x,i) \in U_i$ for $x \in \mathbb{Z}_{u/2}$ and $i \in \{0,1\}$. Let $G = \lambda K_{v+u} - \lambda K_v$. Then for each $i \in \mathbb{Z}_2$, $G[U_i] \cong \lambda K_{u/2} \cong G_{u/2}(D(u/2)^*_\lambda)$. For each $i \in \mathbb{Z}_2$ the edge joining (j_0, i) to (j_1, i) is said to have difference $d_{u/2}(j_0, j_1)$ and the difference is said to be Type i. So for each $i \in \mathbb{Z}_2$ the edges joining vertices in U_i are partitioned by the set $D'_i = D(u/2)^*_\lambda$ of Type i differences. Each edge joining a vertex $(j_0, 0)$ in U_0 to a vertex $(j_1, 1)$ is U_1 is said to have Type 2 difference $j_1 - j_0 \pmod{u/2}$ (so the edges of Type 2 difference j form a 1-factor in $G[U]$); so the edges joining a vertex in U_0 to a vertex in U_1 are partitioned by the set $D'_2 = \lambda\mathbb{Z}_{u/2}$ of Type 2 differences. For each $i \in \mathbb{Z}_2$ let $S_i = \{u/4\}$ if $u \equiv 0 \pmod 4$ and $S_i = \varnothing$ if $u \equiv 2 \pmod 4$. Let $S_2 = 2\{u/4\}$ if $u \equiv 0 \pmod 4$ and $S_2 = \varnothing$ if $u \equiv 2 \pmod 4$. For each $i \in \mathbb{Z}_3$ let $D_i = D'_i \setminus S_i$.

Since we are currently focused on the case when $u < 3v + 1$, by definition of c there are no Type $\mathcal{C}$ 5-cycles. Additionally, we can split the classification of Type $\mathcal{A}$ and Type $\mathcal{B}$ 5-cycles further based on how many edges these 5-cycles have in $E(U_0)$, $E(U_1)$, or between vertices in U_0 and U_1:

- 1 pure edge, and it either joins vertices in U_0 or it joins vertices in U_1,
- 1 pure edge, that joins a vertex in U_0 to a vertex in U_1,
- 3 pure edges, each joining a vertex in U_0 to a vertex in U_1,
- 3 pure edges, one edge between vertices in U_0, one edge between vertices in U_1, and one edge joining a vertex in U_0 to a vertex in U_1, and
- 3 pure edges, one edge of difference in S_i for some $i \in \mathbb{Z}_2$ joining two vertices in U_i and two edges of difference in S_2 joining a vertex in U_0 to a vertex in U_1.

Define these 5-cycles as Type $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ respectively; so $\{\mathcal{A}_1, \mathcal{A}_2\}$ is a partition of $\mathcal{A}$ and $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ is a partition of $\mathcal{B}$. A visual representation of these 5-cycles are given in Figure 5.2. For the remainder of the paper, let a_1, a_2, b_1, b_2, and b_3 represent the number of 5-cycles of Type $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ respectively. Recall that since $\lambda \equiv 5 \pmod{10}$, ϵ is chosen to be 0. Notice that $a_1 + a_2 = a$ and

$b_1 + b_2 + b_3 = b$. The following system of equations defines the relationship between these values and the number of edges in λK_u:

$$\begin{aligned} a_1 + a_2 &= a, \\ b_1 + b_2 + b_3 &= b, \\ a_2 + 3b_1 + b_2 + 2b_3 &= \lambda(u/2)^2, \text{ and} \\ a_1 + 2b_2 + b_3 &= \lambda u(u-2)/4. \end{aligned} \tag{5.3}$$

It is easy to check that one solution of this system is:

$$\begin{aligned} a_1 &= \begin{cases} \lambda u(3v - u + 1)/10 & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ \lambda u(u-2)/4 - b_3 & \text{if } 7u - 6(2+v) \leq 0, \end{cases} \\ a_2 &= \begin{cases} 0 & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ \lambda u(-7u + 6(2+v))/20 + b_3 & \text{if } 7u - 6(2+v) \leq 0, \end{cases} \\ b_1 &= \begin{cases} \lambda u(2v + u + 4)/40 - b_3/2 & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ \lambda u(2u - v - 2)/10 - b_3 & \text{if } 7u - 6(2+v) \leq 0, \end{cases} \\ b_2 &= \begin{cases} \lambda u(7u - 6(2+v))/40 - b_3/2 & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ 0 & \text{if } 7u - 6(2+v) \leq 0, \end{cases} \\ b_3 &= \begin{cases} u/2 & \text{if } u \equiv 0 \pmod 4 \text{ and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.4}$$

For the rest of the chapter, let a_1, a_2, b_1, b_2, and b_3 be defined as in (5.4). The following lemma will be useful for showing that a_1, a_2, b_1, and b_2 satisfy the conditions that will become vital in Lemma 5.3.2.

Lemma 5.3.1. *Suppose that* $\lambda \equiv 5 \pmod{10}$, *that* $v/2 + 1 \leq u < 3v + 1$, *and that* λ, v, *and* u *satisfy Condition (d) of Lemma* 5.2.1. *Then*

(A) a_1, a_2, b_1, *and* b_2 *are non-negative,*
(B) a_1, a_2, b_1, *and* b_2 *are each divisible by* $u/2$, *and*
(C) $a_1/(u/2)$ *and* $a_2/(u/2)$ *are even.*

Proof. Let $\lambda = 10k_3 + 5$ so then by Lemma 5.2.1(d) we can let $u = 2k_1$ and $v = 2k_2 + 1$.

First suppose that $7u - 6(2+v) > 0$. Then $a_1 \geq 0$ since $u < 3v + 1$ in this lemma, and $a_2 = 0$. By definition,

$$\begin{aligned} b_1 &= \frac{\lambda u(2v + u + 4)}{40} - \frac{b_3}{2} \\ &\geq \frac{\lambda u(2v + u + 4)}{40} - \frac{u}{4} \\ &= \frac{u(\lambda(2v + u + 4) - 10)}{40} \\ &\geq 0. \end{aligned}$$

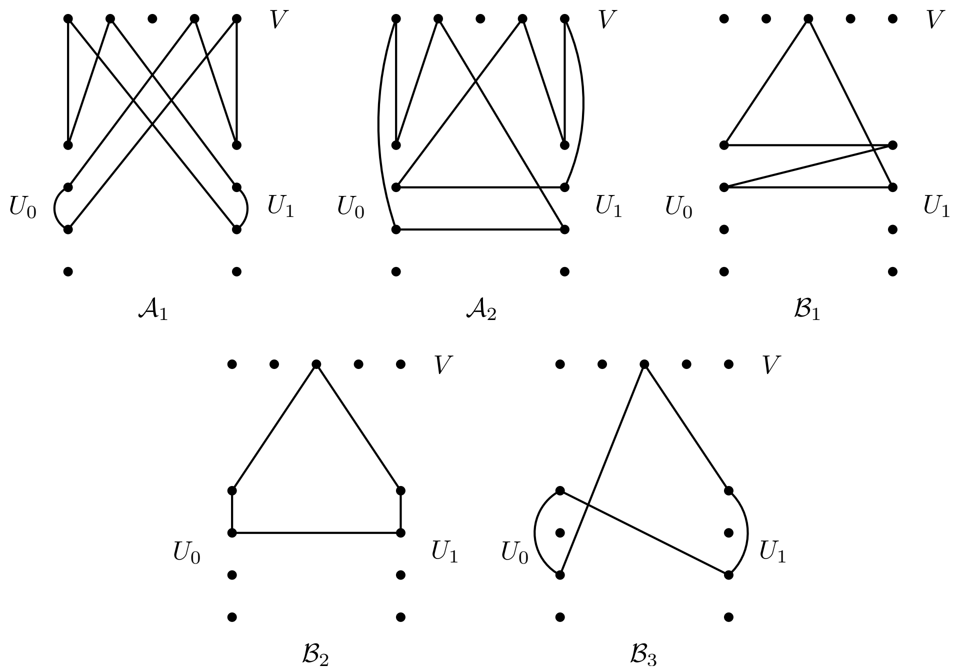

Figure 5.2: Possible 5-cycle Types when $\lambda \equiv 5 \pmod{10}$ and $u < 3v+1$

Since $u > (6v+12)/7$ and $(6v+13)/7$ is never an even integer, $u \geq (6v+14)/7$ and so

$$\begin{aligned} b_2 &\geq \frac{u(\lambda(7u - 6(2+v)) - 10)}{40} \\ &\geq \frac{(6v+14)(\lambda((6v+14) - 6(2+v)) - 10)}{280} \\ &= \frac{(3v+7)(\lambda - 5)}{70} \geq 0. \end{aligned}$$

So Condition (A) is satisfied. By definition

$$\begin{aligned} a_1 &= 2k_1(3k_2 - k_1 + 2)(2k_3 + 1), \\ b_1 &= k_1(2k_2 + k_1 + 3)(2k_3 + 1)/2 - b_3/2, \textit{and} \\ b_2 &= k_1(7k_1 - 6k_2 - 9)(2k_3 + 1)/2 - b_3/2. \end{aligned}$$

Clearly, a_1 and $a_2 = 0$ are divisible by $u/2 = k_1$ and both $a_1/(u/2) = 2a_1/u = a_1/k_1$ and $2a_2/u$ are even, so (C) is satisfied. If $b_3 = 0$ then k_1 is odd and so $2b_1/u = b_1/k_1$ and $2b_2/u = b_2/k_1$ are integers. If $b_3 = u/2$ then $u \equiv 0 \pmod 4$ and k_1 is even, so

$$\begin{aligned} 2b_1/u &= \frac{\lambda(2v + u + 4)}{20} - \frac{b_3}{u} \\ &= \frac{(2k_2 + k_1 + 3)(2k_3 + 1)}{2} - \frac{1}{2} \\ &= 1 + k_2 + 3k_3 + 2k_2k_3 + \frac{k_1(2k_3 + 1)}{2} \end{aligned}$$

and

$$\begin{aligned}
2b_2/u &= \frac{\lambda(7u-6(2+v))}{20} - \frac{b_3}{u} \\
&= \frac{(7k_1-6k_2-9)(2k_3+1)}{2} - \frac{1}{2} \\
&= \frac{k_1(14k_3+7)}{2} - 9k_3 - 3k_2(2k_3+1) - 5.
\end{aligned}$$

So it follows that $2b_1/u$ and $2b_2/u$ are integers and thus Condition (B) is satisfied.

Finally, suppose that $7u-6(2+v) \leq 0$. It is clear by (5.4) that a_1, a_2, and b_2 are non-negative. Since v is odd, $u \neq v/2+1$ and thus $u \geq v/2+3/2$. So if $b_3 = u/2$ then

$$b_1 \geq \frac{\lambda(v+3)(2(v/2+3/2)-v-2)}{20} - \frac{v+3}{4} = \frac{(\lambda-5)(v+3)}{20} \geq 0,$$

and so b_1 is non-negative. If $b_3 = 0$ then it is clear that b_1 is non-negative. Since $u/2 = k_1$ and since

$$\begin{aligned}
a_1 &= 5k_1(k_1-1)(2k_3+1) - b_3, \\
a_2 &= k_1(6k_2-7k_1+9)(2k_3+1) + b_3, \text{ and} \\
b_1 &= k_1(4k_1-2k_2-3)(2k_3+1) - b_3,
\end{aligned}$$

it follows that Condition (B) is satisfied. If $b_3 = u/2$ then k_1 is even and so $2a_1/u = a_1/k_1 = 5(k_1-1)(2k_3+1)-1$ is even. If $b_3 = 0$ then k_1 is odd and so $2a_1/u = 5(k_1-1)(2k_3+1)$ is even. If $b_3 = u/2$ then k_1 is even and so $2a_2/u = (6k_2+9-7k_1)(2k_3+1)+1$ is even. If $b_3 = 0$ then k_1 is odd and so $2a_2/u = (6k_2+9-7k_1)(2k_3+1)$ is even; thus Condition (C) is satisfied. □

The next lemma will cover one of the cases of Theorem 5.4.1 using a slightly different technique to construct the 5-cycles from the methods used so far.

Lemma 5.3.2. *Suppose that* $\lambda \equiv 5 \pmod{10}$, *that* $v/2+1 \leq u < 3v+1$, *and that* λ, v, *and* u *satisfy Condition (d) of Lemma* 5.2.1. *Then there exists a* 5–CS *of* $\lambda K_{v+u} - \lambda K_v$.

Proof. By Lemma 5.2.1(d), u is even and v is odd. Let $U = \mathbb{Z}_{u/2} \times \mathbb{Z}_2$ be the vertex set of λK_u. Let $V = \{\infty_i \mid i \in \mathbb{Z}_v\}$ be the vertex set of λK_v. For $i \in \mathbb{Z}_2$ let $D_i = D(u/2)^*_\lambda \setminus S_i$ where $S_i = \{u/4\}$ if $u \equiv 0 \pmod 4$ and $S_i = \varnothing$ if $u \equiv 2 \pmod 4$. Let $D_2 = \mathbb{Z}_{u/2} \setminus S_2$ where $S_2 = 2\{u/4\}$ if $u \equiv 0 \pmod 4$ and $S_2 = \varnothing$ if $u \equiv 2 \pmod 4$. Note that $D_0 \cup S_0$, $D_1 \cup S_1$, and $D_2 \cup S_2$ represent all of the Type 0, 1 and 2 differences respectively in λK_u and will be used to keep track of the edges in λK_u as we decompose the edges of $\lambda K_{v+u} - \lambda K_v$ into 5-cycles.

The aim of this construction will be to form a_1, a_2, b_1, b_2, and b_3 5-cycles of Type $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ respectively which will be shown to decompose $\lambda K_{v+u} - \lambda K_v$

into 5-cycles. By Lemma 5.3.1, a_1, a_2, b_1, b_2, and b_3 are non-negative integers. For each $i \in \mathbb{Z}_{2a_1/u+2a_2/u}$ define

$$s_i = \begin{cases} (i, i+1) & \text{if } i \text{ is even and } i < r, \\ (i+1, i) & \text{if } i \text{ is odd and } i < r, \text{ and} \\ (2i, 2i+1) & \text{if } i \geq r, \end{cases}$$

where calculations are done modulo v and

$$r = \min(\{v(\lambda - 1)/2, 2a_1/u + 2a_2/u\}).$$

The ordered pairs s_i will be used in the formation of the Type $\mathcal{A}$ 5-cycles to denote the subscripts of the vertices in V.

Before forming the 5-cycles, the sets of differences D_0, D_1, and D_2 need to be partitioned appropriately. By Lemma 5.3.1, $2b_1/u$, $2b_2/u$ and $2a_2/u$ are integers, and by the system of equations in (5.3), $3(2b_1/u) + (2b_2/u) + (2a_2/u) = |D_2|$. Form a partition $T' \cup T^* \cup T'''$ of D_2 where T' contains $2b_1/u$ sets of size 3 such that at least one difference in each set in T' is unique (for example, if $u \geq 6$ then the i^{th} set of T' could be $\{3i, 3i+1, 3i+2\}$ reducing the sum modulo $u/2$, and if $u = 4$ then each element in T' is $\{0, 1, 1\}$ or $\{0, 0, 1\}$), $T^* = \{g_0, g_1, \ldots, g_{2b_2/u-1}\}$ contains $2b_2/u$ differences, and $T''' = \{q_0, q_1, \ldots, q_{2a_2/u-1}\}$ contains the remaining $2a_2/u$ differences. By the system of equations in (5.3), $|D_i| = 2b_2/u + a_1/u$ for $i \in \mathbb{Z}_2$ and by Lemma 5.3.1, $2a_1/u$ is an even integer. Form a partition $F_1 \cup F_2$ of the set D_0 so that $F_1 = \{f_0, f_1, \ldots, f_{2b_2/u-1}\}$ contains $2b_2/u$ differences and $F_2 = \{q_{2a_2/u}, q_{2a_2/u+1}, \ldots, q_{2a_2/u+a_1/u-1}\}$ contains a_1/u differences. Form a partition $H_1 \cup H_2$ of the set D_1 such that $H_1 = \{h_0, h_1, \ldots, h_{2b_2/u-1}\}$ contains $2b_2/u$ differences and $H_2 = \{q_{2a_2/u+a_1/u}, q_{2a_2/u+a_1/u+1}, \ldots, q_{2a_2/u+2a_1/u-1}\}$ contains a_1/u differences. The differences in $F_2 \cup H_2$ will be used to build the Type $\mathcal{A}_1$ 5-cycles. Let $T'' = \{\{f_i, g_i, h_i\} \mid f_i \in F_1, g_i \in T^*, h_i \in H_1, i \in \mathbb{Z}_{2b_2/u}\}$. Later, the differences in T' will be used to build the Type $\mathcal{B}_1$ 5-cycles, the differences in T'' will be used to build the Type $\mathcal{B}_2$ 5-cycles, and the differences in T''' will be used to build the Type $\mathcal{A}_2$ 5-cycles.

Let $t = 0$ if $q_j \in F_2$ and let $t = 1$ if $q_j \in H_2$. Define a mapping

$$f : s_j \to \begin{cases} \{(0, j), (q_j, j+1)\} & \text{if } 0 \leq j \leq 2a_2/u - 1 \text{ and} \\ \{(0, t), (q_j, t)\} & \text{if } 2a_2/u \leq j \leq 2a_2/u + 2a_1/u - 1. \end{cases}$$

The function f will be used to indicate the pure edge in the Type $\mathcal{A}$ 5-cycle that is to be joined with the corresponding vertices in V.

For each $x \in \mathbb{Z}_v$ let $n(x)$ be the number of ordered pairs in $\{s_i \mid i \in \mathbb{Z}_{2a_1/u+2a_2/u}\}$ that contain x. Partition T into v sets $T_0, T_1, \ldots, T_{v-1}$ such that $|T_x| = \lambda - n(x)$ for $x < v - 1$ and $|T_{v-1}| = \lambda - n(v-1) - 2b_3/u$. We will now show: $\lambda - n(x) \geq 0$ for $x \in \mathbb{Z}_{v-1}$, $\lambda - n(v-1) - 2b_3/u \geq 0$, and $\sum_{j=0}^{v-1} |T_j| = |T|$.

First we aim to show $\sum_{j=0}^{v-1} |T_j| = |T|$. The other 2 inequalities are established in the next paragraph. Notice that $\sum_{j=0}^{v-1} n(j) = 2(2a_1/u + 2a_2/u)$. Then

$$\begin{aligned}
\sum_{j=0}^{v-1} |T_j| &= -2b_3/u + \sum_{j=0}^{v-1} (\lambda - n(j)) \\
&= \lambda v - 2b_3/u - 2(2a_1/u + 2a_2/u) \\
&= \begin{cases} \lambda v - 2b_3/u - 2\left(\frac{\lambda(3v-u+1)}{5} + \frac{2(0)}{u}\right) & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ \lambda v - 2b_2/u - 2\left(\frac{\lambda(u-2)}{2} - \frac{2b_3}{u} + \frac{\lambda(-7u+6(2+v))}{10} + \frac{2b_3}{u}\right) & \text{if } 7u - 6(2+v) \le 0, \end{cases} \\
&= \lambda(2u - v - 2)/5 - 2b_3/u \\
&= (2/u)(\lambda u(2u - v - 2)/10 - b_3) \\
&= \begin{cases} \frac{2}{u}\left(\frac{\lambda u((7u-6v-12)+(2v+u+4))}{40} - b_3\right) & \text{(use this expression if } 7u - 6(2+v) > 0) \\ \frac{2}{u}\left(\frac{\lambda u(2u-v-2)}{10} - b_3 + 0\right) & \text{(use this expression if if } 7u - 6(2+v) \le 0), \end{cases} \\
&= 2b_1/u + 2b_2/u = |T|.
\end{aligned}$$

If $r = 2a_1/u + 2a_2/u$ and $r < v(\lambda - 1)/2$, then clearly at most $\lambda - 1$ ordered pairs in $\{s_i \mid i \in \mathbb{Z}_{2a_1/u+2a_2/u}\}$ contain x for all $x \in \mathbb{Z}_v$. Suppose that $r = v(\lambda - 1)/2$ so that $r \le 2a_1/u + 2a_2/u$. If we can show that $v/2 > 2a_1/u + 2a_2/u - r$ then for all $x \in \mathbb{Z}_v$, $\lambda - 1$ ordered pairs in $\{s_i \mid i \in \mathbb{Z}_r\}$ contain x and at most 1 ordered pair in $\{s_i \mid i \in \mathbb{Z}_{2a_1/u+2a_2/u} \backslash \mathbb{Z}_r\}$ contain x (to be precise, $n(x) = \lambda$ if $x \le 2(2a_1/u+2a_2/u-r)$ and $n(x) = \lambda - 1$ otherwise); so it follows that for all $x \in \mathbb{Z}_{v-1}$, $\lambda - n(x) \ge 0$ and that $\lambda - n(v-1) - 2b_3/u \ge 0$. So if $r = v(\lambda - 1)/2$ then

$$\begin{aligned}
\frac{v}{2} + r - \frac{2a_1}{u} - \frac{2a_2}{u} &= \frac{\lambda v}{2} - \frac{2a_1}{u} - \frac{2a_2}{u} \\
&= \begin{cases} \frac{\lambda v}{2} - \left(\frac{\lambda(3v-u+1)}{5} + \frac{2(0)}{u}\right) & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ \frac{\lambda v}{2} - \left(\frac{\lambda(u-2)}{2} - \frac{2b_3}{u} + \frac{\lambda(6(v+2)-7u)}{10} + \frac{2b_3}{u}\right) & \text{if } 7u - 6(2+v) \le 0, \end{cases} \\
&= \begin{cases} \frac{5\lambda v - 2\lambda(3v-u+1)}{10} & \text{if } 7u - 6(2+v) > 0 \text{ and} \\ \frac{5\lambda v - 5\lambda(u-2) - \lambda(6v+12-7u)}{10} & \text{if } 7u - 6(2+v) \le 0, \end{cases} \\
&= \frac{\lambda(2u - v - 2)}{10} > 0,
\end{aligned}$$

The last inequality follows from $u \ge v/2 + 1$ and v is odd. Therefore $\lambda - n(x) \ge 0$ if $x \in \mathbb{Z}_{v-1}$ and $\lambda - n(v-1) - 2b_3/u \ge 0$.

We will now define the sets of 5-cycles that will form our desired 5–CS. Let $p_1 = p_3 = 1$ and $p_2 = 0$ if $T_i \subseteq T'$ and $p_1 = 0$ and $p_2 = p_3 = 1$ if $T_i \subseteq T''$. Then define

$$\begin{aligned}
C_0 = \{&(\infty_{t_1}, (0, i), (q_i, i+1), \infty_{t_2}, (q_i + 1, i+1)) + (j, j) \mid i \in \mathbb{Z}_{2a_2/u}, \\
&f(s_i) = \{(0, i), (q_i, i+1)\}, j \in \mathbb{Z}_{u/2}, s_i = (t_1, t_2)\}, \\
C_1 = \{&(\infty_{t_1}, (0, t), (q_i, t), \infty_{t_2}, (0, t+1)) + (j, j) \mid i \in \mathbb{Z}_r \setminus \mathbb{Z}_{2a_2/u}, \\
&f(s_i) = \{(0, t), (q_i, t)\}, j \in \mathbb{Z}_{u/2}, s_i = (t_1, t_2)\}, \\
C_2 = \{&(\infty_{2i}, (0, t), (q_i, t), \infty_{2i+1}, (0, t+1)) + (j, j) \mid i \in \mathbb{Z}_{2a_1/u+2a_2/u-r},
\end{aligned}$$

$f(s_i) = \{(0,t),(q_i,t)\}, j \in \mathbb{Z}_{u/2}\},$

$$C_3 = \{(\infty_i, (0,0), (d_1,p_1), (d_1 - d_2, p_2), (d_1 - d_2 + d_3, p_3)) + (j,j) \mid i \in \mathbb{Z}_v, \{d_1,d_2,d_3\} \in T_i, j \in \mathbb{Z}_{u/2}, \text{ if } T_i \subseteq T' \text{ then } d_2 \text{ is unique}\}, \text{ and}$$

$$C_4 = \begin{cases} \varnothing & \text{if } b_3 = 0, \text{ and} \\ \{((0,0),(v/4,0),(0,1),\infty_{v-1},(v/4,1)) + (j,j) \mid j \in \mathbb{Z}_{u/2}\} & \text{if } b_3 = u/2, \\ \cup\{((0,1),(v/4,1),(0,0),\infty_{v-1},(v/4,0)) + (j,j) \mid j \in \mathbb{Z}_{u/2}\} & \end{cases}$$

where calculations are done modulo v (including indicies). So C_0 is a set of Type $\mathcal{A}_2$ 5-cycles, $C_1 \cup C_2$ is a set of Type $\mathcal{A}_1$ 5-cycles, C_3 is a set of Type $\mathcal{B}_1$ and $\mathcal{B}_2$ 5-cycles (depending on whether $T_i \subseteq T'$ or $T_i \subseteq T''$ respectively), and C_4 is a set of Type $\mathcal{B}_3$ 5-cycles. Since every difference in $D_0 \cup S_0$, $D_1 \cup S_1$, and $D_2 \cup S_2$ represents every edge in λK_u, it follows that $C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4$ decomposes the edges in λK_u into 5-cycles. It remains to show that each vertex in U is joined to each vertex in V λ times.

By Lemma 5.3.1, $2a_2/u$ is even, so by definition of s_i, each ∞_i in V is adjacent to every vertex in U the same number of times, though ∞_i may not be joined to every vertex in U the same number of times as ∞_j. To be precise, ∞_i is adjacent to every vertex in U $n(i)$ times. Since $|T_i| = \lambda - n(i)$ for $i < v-1$ and $|T_{v-1}| = \lambda - n(v-1) - 2b_3/u$, it follows that in C_3, ∞_i is adjacent to each vertex in U $\lambda - n(i)$ times for $i < v-1$ and ∞_{v-1} is adjacent to each vertex in U $\lambda - n(i) - 2b_3/u$ times. Thus $(V \cup U, C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4)$ is a 5–CS of $\lambda K_{v+u} - \lambda K_v$ when $\lambda \equiv 5 \pmod{10}$ and $u < 3v + 1$. □

5.4 5–CSs of $\lambda K_{v+u} - \lambda K_v$: Main Result

The following theorem uses the same proof method used in Chapters 2-4 to prove the sufficient conditions for a 5–CS of $\lambda K_{v+u} - \lambda K_v$ except possibly in the case where $\lambda \equiv 5 \pmod{10}$ and $\lambda < v \leq (u-1)/3$ or in the case where $\lambda \equiv 2, 4, 6,$ or $8 \pmod{10}$, $u \equiv 0 \pmod 5$, and $v/2 + 1 \leq u < v/2 + 5$. When λ is odd, u is even, and $\lambda < v \leq (u-1)/3$, the proof falls apart since each set of differences used to construct the 5-cycles of Type $\mathcal{B}$ or $\mathcal{C}$ will join each vertex in U to a vertex in V twice unless the difference $u/2$ is used. As long as the number of copies of difference $u/2$ is more than the number of vertices in V (i.e. $\lambda \geq v$) the method described in Theorem 5.4.1 is viable. If $\lambda \equiv 2, 4, 6,$ or $8 \pmod{10}$, $u \equiv 0 \pmod 5$, and $v/2 + 1 \leq u < v/2 + 5$, then b is possibly negative since ϵ can be at most 4.

Theorem 5.4.1. *There exists a 5–CS of $\lambda K_{v+u} - \lambda K_v$ if and only if $u \geq v/2 + 1$ and*

(a) If $\lambda \equiv 1, 3, 7,$ or $9 \pmod{10}$ then

$v + u, v \equiv 1$ *or* $5 \pmod{10}$*;*

$v + u, v \equiv 7$ *or* $9 \pmod{10}$*; and*

$v + u \equiv v \equiv 3 \pmod{10}$,

(b) If $\lambda \equiv 0 \pmod{10}$ *then any* $v \geq 1$ *and* $v + u \geq 5$,

(c) If $\lambda \equiv 2, 4, 6,$ *or* $8 \pmod{10}$ *then*

$v + u, v \equiv 0, 1 \pmod 5$*;*
$v + u, v \equiv 2, 4 \pmod 5$*; and*
$v + u \equiv v \equiv 3 \pmod 5$,

(d) If $\lambda \equiv 5 \pmod{10}$ *then* v *and* $v + u$ *are both odd,*

except possibly in the case where $\lambda \equiv 5 \pmod{10}$*,* $v \equiv v + u \equiv 1 \pmod 2$*, and* $\lambda < v \leq (u-1)/3$ *or in the case where* $\lambda \equiv 2, 4, 6,$ *or* $8 \pmod{10}$*,* $u \equiv 0 \pmod 5$*, and* $v/2 + 1 \leq u < v/2 + 5$.

Proof. Let $V = \{\infty_0, \infty_1, \ldots, \infty_{v-1}\}$, $u = 10k + \ell \geq 2$ for $\ell \in \mathbb{Z}_{10}$, and $\mathbb{Z}_u$ is the vertex set of U. The necessary conditions were proved in Lemma 5.2.1 so it remains to show sufficiency. As said before, Condition (a) is satisfied by Theorem 5.1.1. So we can assume $\lambda \geq 2$. Notice that if $v = 0$ then $\lambda K_{v+u} - \lambda K_v = \lambda K_u$ in which case a 5–CS exists by Theorem 1.2.2 and if $v = 1$ then $\lambda K_{v+u} - \lambda K_v = \lambda K_{u+1}$ in which case a 5–CS exists again by Theorem 1.2.2; so $v \geq 2$

First suppose $u \geq 3v + 1$. By Lemma 5.2.2, b and c are non-negative integers, u divides c, and vu divides b. We will need $3b$ pure edges to construct the 5-cycles of Type $\mathcal{B}$ and so the number of pure edges that will be used to construct the Type $\mathcal{C}$ 5-cycles is

$$\lambda u(u-1)/2 - 3b = (\lambda(u - 3v - 1)/2)u = (5c/u + \epsilon)u = \alpha u$$

where α represents the number of differences used to construct the 5-cycles of Type $\mathcal{C}$. It is clear that when $\epsilon = 0$, $\alpha \equiv 0 \pmod 5$. Recall that the value of ϵ indicates the number of differences that will be used to construct Γ_i for $i \in \{1, 2, 3\}$ (possibly more than one of the 5–CSs of Γ_i will be constructed) so that the $\alpha - \epsilon$ remaining differences needed to construct Type $\mathcal{C}$ 5-cycles is divisible by 5. Our plan is to create Type $\mathcal{C}$ 5-cycles using α specially chosen differences. Skolem sequences will be used as in the proof to Lemma 4.2.3 to construct the Type $\mathcal{C}$ 5-cycles. First, we will construct the set B of Type $\mathcal{C}$ 5-cycles by addressing the values of ℓ in turn. Afterwards, we will construct the set C of Type $\mathcal{B}$ 5-cycles.

Suppose $\ell \geq 4$ and $\ell \neq 5$. Then let $P = \{(x_i, y_i) \mid 1 \leq i \leq k, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order k for the set $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k+1\}$ if $k \equiv 0$ or $1 \pmod 4$ or $k \equiv 2$ or $3 \pmod 4$ respectively (see Theorems 1.1.1 and 1.1.2). If $\ell = 8$ then let $t_1 = 1$ and let $t_1 = 0$ otherwise. If $\ell = 9$ then let $t_2 = 1$ and let $t_2 = 0$ otherwise. Then let

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k-1+\lfloor \ell/2 \rfloor - t_1, 5k+i+\lfloor \ell/2 \rfloor + t_2) \mid 1 \leq i \leq k, (x_i, y_i) \in P\}.$$

Table 5.1 lists the differences of edges that occur in 5-cycles in B' when α is as large as possible (λ is as small as possible). Notice that $\lambda|B'| = \lambda k \geq \alpha/5 = \lambda(10k + \ell - 3v -$

$1)/10$ if $\ell \leq 7$ or $v \geq 3$. If $v = 2$ and $\ell \in \{8, 9\}$, then $\lambda/10$ and $\lambda/5$ more 5-cycles using differences need to be constructed when $\ell = 8$ and 9 respectively. To account for this, form the set of 5-cycles $B = \{c_i + j \mid j \in \mathbb{Z}_u, i \in \mathbb{Z}_{t'_1}, c_{i \pmod k} \in B'\} \cup t'_2\{c + j \mid j \in \mathbb{Z}_u\}$ where $t'_1 = \min(\{\lambda k, \alpha/5\})$, $t'_2 = \max(\{0, \alpha/5 - \lambda k\})$ and

$$c = \begin{cases} (0, 2k+1, 4k+2, 6k+4, 4k+3) & \text{if } \ell = 8 \text{ and } k \equiv 0, 1 \pmod 4, \\ (0, 2k, 6k+3, 4k+3, 6k+5) & \text{if } \ell = 8 \text{ and } k \equiv 2, 3 \pmod 4, \text{ and} \\ (0, 2k+2, 4k+4, 6k+6, 3k+3) & \text{if } \ell = 9. \end{cases}$$

Since $v \geq 2$ and

$$\frac{\alpha}{5} - \lambda k = \frac{\lambda(u - 3v - 1)}{10} - \lambda k = \lambda\left(k + \frac{\ell}{10}\right) - \lambda k - \frac{\lambda(3v+1)}{10} = \frac{\lambda(\ell - 3v - 1)}{10},$$

it follows that the number of times c is copied is no more than $\lambda/10$ or $\lambda/5$ times if $\ell = 8$ or 9 respectively. So we have enough copies of the differences to construct c.

Suppose $\ell = 0$. Let $P' = \{(x'_i, y'_i) \mid 0 \leq i \leq k-3, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ for the set $\{1, 2, \ldots, 2k-4\}$ or $\{1, 2, \ldots, 2k-3\}$ if $k - 2 \equiv 0$ or $1 \pmod 4$ or $k - 2 \equiv 2$ or $3 \pmod 4$ respectively. If $k - 2 \equiv 0$ or $1 \pmod 4$ then name the Skolem pairs so that (x'_{k-3}, y'_{k-3}) contains $2k-3$. Define $P = \{(x_i = x'_i + 3, y_i = y'_i + 3) \mid (x'_i, y'_i) \in P'\}$; notice that $2k \in \{x_{k-3}, y_{k-3}\}$ when $k - 2 \equiv 0$ or $1 \pmod 4$. To simplify how the 5-cycles are defined, let $z = \lfloor \lambda/2 \rfloor$. Then with $c'_i = (0, x_i, x_i - y_i, 3k, 5k + 2 + i)$ and

$$c_i = \begin{cases} (0, 1, 3, 2k+4, 2k+1) & \text{if } i = 1 \text{ and } k - 2 \equiv 0, 1 \pmod 4, \\ (0, 1, 3, 3k+3, 3k) & \text{if } i = 1 \text{ and } k - 2 \equiv 2, 3 \pmod 4, \\ (0, 3k, 6k, k+1, 5k+1) & \text{if } i = 2 \text{ and } k - 2 \equiv 0, 1 \pmod 4, \\ (0, 2k, 6k-1, k, 5k-1) & \text{if } i = 2 \text{ and } k - 2 \equiv 2, 3 \pmod 4, \\ (0, 1, 2, 5, 3) & \text{if } 3 \leq i \leq z+1, \\ (0, 2, 2k+3, 7k+2, 5k+1) & \text{if } z+2 \leq i \leq 2z \text{ and } k - 2 \equiv 0, 1 \pmod 4, \\ (0, 2, 2k+2, 7k+1, 5k+1) & \text{if } z+2 \leq i \leq 2z \text{ and } k - 2 \equiv 2, 3 \pmod 4, \\ (0, 3k, 6k, 2k+1, 6k+1) & \text{if } 2z+1 \leq i \leq 3z-1, \text{ and} \\ c'_{i-3z \pmod{k-2}} & \text{if } 3z \leq i \leq \lambda(k-2) + 3z - 1, \end{cases}$$

form the set of 5-cycles $B' = \{c''_{i-1} = c_i \mid 1 \leq i \leq \lambda(k-2) + 3z - 1, (x_i, y_i) \in P\}$. Notice that when $\lambda = 2$, c_i will never be any of the following 5-cycles: $(0, 1, 2, 5, 3)$, $(0, 2, 2k+3, 7k+2, 5k+1)$, $(0, 2, 2k+2, 7k+1, 5k+1)$, or $(0, 3k, 6k, 2k+1, 6k+1)$. So when $k > 2$ we form the set of 5-cycles $B = \{c''_i + j \mid i \in \mathbb{Z}_{\lfloor \alpha/5 \rfloor}, j \in \mathbb{Z}_u, c''_i \in B'\} \cup E_\alpha$

Edges	Differences when $k \equiv 0, 1 \pmod 4$	Differences when $k \equiv 2, 3 \pmod 4$
$\{0, x_i\}, \{x_i, x_i - y_i\}$	$1, 2, \ldots, 2k$	$1, 2, \ldots, 2k-1, 2k+1$
$\{x_i - y_i, 3k - 1 + \lfloor \ell/2 \rfloor - t_1\}$ if $\ell \neq 8$	$3k + \lfloor \ell/2 \rfloor, 3k + 1 + \lfloor \ell/2 \rfloor, \ldots, 4k - 1 + \lfloor \ell/2 \rfloor$	$3k + \lfloor \ell/2 \rfloor, 3k + 1 + \lfloor \ell/2 \rfloor, \ldots, 4k - 1 + \lfloor \ell/2 \rfloor$
$\{x_i - y_i, 3k - 1 + \lfloor \ell/2 \rfloor - t_1\}$ if $\ell = 8$	$3k+3, 3k+4, \ldots, 4k+2$	$3k+3, 3k+4, \ldots, 4k+2$
$\{3k - 1 + \lfloor \ell/2 \rfloor - t_1, 5k + i + \lfloor \ell/2 \rfloor + t_2\}$ if $\ell \leq 7$	$2k+2, 2k+3, \ldots, 3k+1$	$2k+2, 2k+3, \ldots, 3k+1$
$\{3k - 1 + \lfloor \ell/2 \rfloor - t_1, 5k + i + \lfloor \ell/2 \rfloor + t_2\}$ if $\ell \geq 8$	$2k+3, 2k+4, \ldots, 3k+2$	$2k+3, 2k+4, \ldots, 3k+2$
$\{5k + i + \lfloor \ell/2 \rfloor + t_2, 0\}$ if $\ell \neq 9$	$5k - 1 + \lceil \ell/2 \rceil, 5k - 2 + \lceil \ell/2 \rceil, \ldots, 4k + \lceil \ell/2 \rceil$	$5k - 1 + \lceil \ell/2 \rceil, 5k - 2 + \lceil \ell/2 \rceil, \ldots, 4k + \lceil \ell/2 \rceil$
$\{5k + i + \lfloor \ell/2 \rfloor + t_2, 0\}$ if $\ell = 9$	$5k+3, 5k+2, \ldots, 4k+4$	$5k+3, 5k+2, \ldots, 4k+4$

Table 5.1: Edges of differences in B' when α is as large as possible for $\ell \geq 4$ and $\ell \neq 5$

where computations are done modulo u and where

$$E_\alpha = \begin{cases} \varnothing & \text{if } \alpha \equiv 0 \pmod 5, \\ \Gamma_2 & \text{if } \alpha \equiv 1 \pmod 5, \\ \Gamma_1 \cup \Gamma_2 & \text{if } \alpha \equiv 2 \pmod 5, \\ \Gamma_3 & \text{if } \alpha \equiv 3 \pmod 5, \text{ and} \\ \Gamma_2 \cup \Gamma_3 & \text{if } \alpha \equiv 4 \pmod 5. \end{cases}$$

Since

$$\frac{\alpha - \epsilon}{5} = \frac{\lambda(10k - 3v - 1) - 2\epsilon}{10} = \lambda k - \frac{\lambda(3v+1) + 2\epsilon}{10} \leq \lambda k - \left\lceil \frac{7\lambda}{10} \right\rceil,$$

since $|B'| = 2k-2$ if $\lambda = 2$, and since $|B'| = \lambda(k-2)+3z \geq \lambda(k-2)+3(\lambda/2-1/2) = \lambda k - \lambda/2 - 3/2$ if $\lambda > 2$, it follows that $\lfloor (\alpha - \epsilon)/5 \rfloor \leq |B'|$, so we have enough 5-cycles to construct the Type $\mathcal{C}$ 5-cycles. Notice that if $v = 2$ and $\lambda < 8$, then $\alpha \not\equiv 2 \pmod 5$. Fortunately, when α is as large as possible then either $\alpha \not\equiv 2 \pmod 5$ or $\lceil \alpha/5 \rceil < |B'|$, so in either case $2k$ is an available difference and thus the appropriate differences are available to be used in E_α. Table 5.2 lists the differences of edges that occur in cycles in B' when α is as large as possible. Note that $j \in \mathbb{Z}_{k-2}$ in the table and that the first 4 rows of this table represent c_i when $i \geq 3z$. If $u = 10$ ($k = 1$), then since $\lfloor \alpha/5 \rfloor = \lfloor 3\lambda/10 \rfloor \leq \lambda/2$ when $v = 2$ and $\lceil \alpha/5 \rceil = 0$ when $v = 3$ ($v \leq 3$ since $u \geq 3v+1$ in this case), it follows that $B = \lfloor \alpha/5 \rfloor \{(0,1,2,5,3) + j \mid j \in \mathbb{Z}_u\} \cup E_\alpha$ is a set of 5-cycles with the required number of 5-cycles. If $u = 20$, let

$$S = \{(0,1,4,5,2), (3,4,6,8,5), (2,3,6,7,4), (0, u/5, 2u/5, 3u/5, 4u/5), (0, 2u/5, 4u/5, u/5, 3u/5)\}, \tag{5.5}$$

$f = \min(\{\lambda - 1, \lfloor \alpha/5 \rfloor\})$, and $g = \max(\{0, \lfloor \alpha/5 \rfloor - (\lambda - 1)\})$. So then form the set

Edges	Differences when $k-2 \equiv 0,1 \pmod 4$	Differences when $k-2 \equiv 2,3 \pmod 4$
$\{0, x_j\}, \{x_j, x_j - y_j\}$	$4, 5, \ldots, 2k$ (λ copies)	$4, 5, \ldots, 2k-1, 2k+1$ (λ copies)
$\{x_j - y_j, 3k\}$	$3k+1, 3k+2, \ldots, 4k-2$ (λ copies)	$3k+1, 3k+2, \ldots, 4k-2$ (λ copies)
$\{3k, 5k+j+2\}$	$2k+2, 2k+3, \ldots, 3k-1$ (λ copies)	$2k+2, 2k+3, \ldots, 3k-1$ (λ copies)
$\{5k+j+2, 0\}$	$5k-2, 5k-3, \ldots, 4k+1$ (λ copies)	$5k-2, 5k-3, \ldots, 4k+1$ (λ copies)
c_i if $i = 1$	$1, 2, 3, 2k+1, 2k+1$	$1, 2, 3, 3k, 3k$
c_i if $i = 2$	$3k, 3k, 4k, 5k-1, 5k-1$	$2k, 4k-1, 4k-1, 5k-1, 5k-1$
c_i if $3 \leq i \leq z+1$	$1, 1, 2, 3, 3$ ($z-1$ copies)	$1, 1, 2, 3, 3$ ($z-1$ copies)
c_i if $z+2 \leq i \leq 2z$	$2, 2k+1, 2k+1, 5k-1, 5k-1$ ($z-1$ copies)	$2, 2k, 2k, 5k-1, 5k-1$ ($z-1$ copies)
c_i if $2z+1 \leq i \leq 3z-1$	$3k, 3k, 4k-1, 4k-1, 4k$ ($z-1$ copies)	$3k, 3k, 4k-1, 4k-1, 4k$ ($z-1$ copies)

Table 5.2: Edges of differences in B' when α is as large as possible for $\ell = 0$

of 5-cycles $B = f\{s + j \mid j \in \mathbb{Z}_u, s \in S\} \cup g\{(0,5,11,6,13) + j \mid j \in \mathbb{Z}_u\} \cup E_\alpha$. Notice that S represents the set of 5-cycles formed from the differences 1, 2, 3, $u/5$, and $2u/5$ and f represents the number of copies of the 5-cycles constructed from the differences 1, 2, 3, $u/5$, and $2u/5$. Also notice that B uses at least one less than the number of copies of the differences 1, 2, 3, $u/5$, and $2u/5$ in $D(u)^*_\lambda$. Since $v \geq 2$ and

$$\lfloor \alpha/5 \rfloor - (\lambda - 1) = \left\lfloor \frac{\lambda(u - 3v - 1)}{10} \right\rfloor - (\lambda - 1) \leq \lambda + 1 - \frac{\lambda(3v+1)}{10},$$

it follows that there are enough copies of the differences to construct the cycle $(0, 5, 11, 6, 13)$ g times. So there are enough copies of each difference to construct the 5-cycles in B.

Suppose $\ell \in \{1, 3\}$. Let $P = \{(x_i, y_i) \mid 1 \leq i \leq k-1, y_i > x_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-1$ for the set $\{1, 2, \ldots, 2k-2\}$ or $\{1, 2, \ldots, 2k-1\}$ if $k - 1 \equiv 0$ or 1 (mod 4) or $k - 1 \equiv 2$ or 3 (mod 4) respectively. Then let

$$B' = \{c_{i-1} = (0, x_i, x_i - y_i, 3k, 5k + i) \mid 1 \leq i \leq k-1, (x_i, y_i) \in P\}.$$

Table 5.3 lists the differences of edges that occur in cycles in B' when α is as large as possible (λ is as small as possible). Since $\alpha/5 < \lambda k$ and $|B'| = \lambda(k-1)$ we need to form λ more 5-cycles constructed from differences. (Note that by Conditions (b-d), λ must be even since u is odd.) To account for these 5-cycles, form the set of 5-cycles

$$B = \{c_i + j \mid i \in \mathbb{Z}_{x_1}, j \in \mathbb{Z}_u, c_{i \pmod{k-1}} \in B'\} \cup x_2\{c'_0 + j \mid j \in \mathbb{Z}_u\} \cup x_3\{c'_1 + j \mid j \in \mathbb{Z}_u\}$$

where $x_1 = \min(\{\alpha/5, \lambda(k-1)\})$, $x_2 = \max(\{0, \lfloor(\alpha/5 - \lambda(k-1))/2\rfloor\})$, $x_3 = \max(\{0, \lceil(\alpha/5 - \lambda(k-1))/2\rceil\})$, and where

$$c'_i = \begin{cases} (0, 2k+1, 4k, 2k, 6k), & \text{if } \ell = 1,\ k \equiv 0, 1 \pmod 4, \text{ and } i \in \mathbb{Z}_2, \\ (0, 2k-2, 4k-2, 2k, 4k) & \text{if } \ell = 1,\ k \equiv 2, 3 \pmod 4, \text{ and } i = 0, \\ (0, 2k+1, 4k+2, 8k+2, 4k+1) & \text{if } \ell = 1,\ k \equiv 2, 3 \pmod 4, \text{ and } i = 1, \text{ and} \\ (0, 2k, 4k+1, 1, 4k+2) & \text{if } \ell = 3 \text{ and } i \in \mathbb{Z}_2. \end{cases}$$

Since $v \geq 2$ and

$$\begin{aligned} \frac{\alpha}{5} - \frac{\lambda(k-1)}{2} &= \frac{\lambda(u - 3v - 1)}{10} - \frac{\lambda(k-1)}{2} \\ &= \lambda k + \frac{\lambda(\ell - 3v - 1)}{10} - \frac{\lambda k}{2} - \frac{\lambda}{2} \\ &= \frac{\lambda k}{2} + \frac{\lambda(\ell - 3v - 6)}{10} \\ &< \frac{\lambda k}{2} \end{aligned}$$

there are enough copies of the differences to construct c'_i.

Suppose $\ell = 2$. Let $P' = \{(x'_i, y'_i) \mid 0 \leq i \leq k-2, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence of order $k-1$ for the set $\{1, 2, \ldots, 2k-2\}$ or

Edges	Differences when $k-1 \equiv 0,1 \pmod 4$	Differences when $k-1 \equiv 2,3 \pmod 4$
$\{0,x_i\},\{x_i,x_i-y_i\}$	$1,2,\ldots,2k-2$	$1,2,\ldots,2k-3,2k-1$
$\{x_i-y_i,3k\}$	$3k+1,3k+2,\ldots,4k-1$	$3k+1,3k+2,\ldots,4k-1$
$\{3k,5k+i\}$	$2k+2,2k+3,\ldots,3k$	$2k+2,2k+3,\ldots,3k$
$\{5k+i,0\}$	$5k-1+\lceil\ell/2\rceil,5k-2+\lceil\ell/2\rceil,\ldots,$ $4k+1+\lceil\ell/2\rceil$	$5k-1+\lceil\ell/2\rceil,5k-2+\lceil\ell/2\rceil,\ldots,$ $4k+1+\lceil\ell/2\rceil$

Table 5.3: Edges of differences in B' when α is as large as possible for $\ell \in \{1,3\}$

$\{1,2,\ldots,2k-1\}$ if $k-1 \equiv 0$ or $1 \pmod 4$ or $k-1 \equiv 2$ or $3 \pmod 4$ respectively. Define $P = \{(x_i = x'_i+1, y_i = y'_i+1) \mid (x'_i,y'_i) \in P'\}$. Then with

$$c_i = \begin{cases} (0,x_i,x_i-y_i,3k+1,5k+i+1) & \text{if } 0 \le i \le k-2,\\ (0,1,3k+1,7k+2,4k+1) & \text{if } i=k-1 \text{ and } k-1 \equiv 0,1 \pmod 4,\\ (0,1,2k+1,5k+1,2k) & \text{if } i=k \text{ and } k-1 \equiv 0,1 \pmod 4, \text{ and}\\ (0,1,2k,6k+1,3k+1) & \text{if } i=k-1 \text{ and } k-1 \equiv 2,3 \pmod 4, \end{cases}$$

let

$$\begin{aligned} B' = \{c'_i = c_{i \pmod{k-1}} \mid i \in \mathbb{Z}_{\lambda(k-1)}, (x_i,y_i) \in P\}\\ \cup \{c'_i = c_{k-1} \mid \lambda(k-1) \le i \le \lambda(k-1)+\lfloor\lambda/2\rfloor-1\}\\ \cup \{c'_i = c_k \mid \lambda(k-1)+\lfloor\lambda/2\rfloor \le i \le \lambda(k-1)+2\lfloor\lambda/2\rfloor-1\} \end{aligned}$$

if $k-1 \equiv 0$ or $1 \pmod 4$ or let

$$B' = \{c'_i = c_{i \pmod{k-1}} \mid i \in \mathbb{Z}_{\lambda(k-1)}, (x_i,y_i) \in P\} \cup \{c'_i = c_{k-1} \mid \lambda(k-1) \le i \le \lambda k\}$$

if $k-1 \equiv 2$ or $3 \pmod 4$. Since $|B'| \ge \lambda(k-1)+2\lfloor\lambda/2\rfloor-1 \ge \lambda k-2$ and $\alpha/5 \le \lambda k - \lambda/2$ we have enough 5-cycles to form the appropriate number of 5-cycles of Type $\mathcal{C}$. Table 5.4 lists the differences of edges that occur in 5-cycles in B' when α is as large as possible (λ is as small as possible). Then form the set of 5-cycles $B = \{c'_i + j \mid i \in \mathbb{Z}_{\alpha/5}, j \in \mathbb{Z}_u, c'_i \in B'\}$. If $k=0$ then it is impossible to construct a 5-cycle.

Edges	Differences when $k-1 \equiv 0,1 \pmod 4$	Differences when $k-1 \equiv 2,3 \pmod 4$
$\{0,x_i\},\{x_i,x_i-y_i\}$	$2,3,\ldots,2k-1$	$2,3,\ldots,2k-2,2k$
$\{x_i-y_i,3k+1\}$	$3k+2,3k+3,\ldots,4k$	$3k+2,3k+3,\ldots,4k$
$\{3k+1,5k+i+1\}$	$2k+1,2k+2,\ldots,3k-1$	$2k+1,2k+2,\ldots,3k-1$
$\{5k+i+1,0\}$	$5k,5k-1,\ldots,4k+2$	$5k,5k-1,\ldots,4k+2$
c_i if $i=k$	$1,3k,3k+1,4k+1,4k+1$	$1,2k-1,3k,3k+1,4k+1$
c_i if $i=k+1$	$1,2k,2k,3k,3k+1$	$-$

Table 5.4: Edges of differences in B' when α is as large as possible for $\ell=2$

Suppose $\ell=5$. Let $P' = \{(x'_i,y'_i) \mid 1 \le i \le k-2, y'_i > x'_i\}$ be the collection of pairs from a Skolem sequence or hooked Skolem sequence of order $k-2$ for the set $\{1,2,\ldots,2k-4\}$ or $\{1,2,\ldots,2k-3\}$ if $k-2\equiv 0$ or $1 \pmod 4$ or $k-2\equiv 2$ or $3 \pmod 4$ respectively. Define $P=\{(x_i=x_i'+3, y_i=y_i'+3) \mid (x_i',y_i')\in P'\}$. Let

$$c_i=\begin{cases}(0,x_i,x_i-y_i,4k+2,7k+3-i) & \text{if } 1\le i\le k-2, \text{ and}\\ (0,3k+1,6k+3,k+2,5k+3) & \text{if } i=k-1,\end{cases}$$

and $B'=\{c'_{i-1}=c_i \mid 1\le i\le k-1, (x_i,y_i)\in P\}$. Table 5.4 lists the differences of edges that occur in cycles in B' when α is as large as possible (λ is as small as possible). Let S be defined as in Equation (5.5) and let $w_1=\min(\{\lambda(k-1),\lfloor\alpha/5\rfloor\})$ and $w_2=\max(\{0,\lfloor\alpha/5\rfloor-\lambda(k-1)\})$. So when $k\ge 1$ we form the set of 5-cycles

$$B=\{c_i'+j \mid i\in\mathbb{Z}_{w_1}, j\in\mathbb{Z}_u, c'_{i \text{ (mod k)}}\in B'\}\cup w_2\{s+j \mid s\in S, j\in\mathbb{Z}_u\}\cup E_\alpha$$

where computations are done modulo u. Table 5.5 lists differences of edges in the 5-cycles in B' when α is as large as possible. Since $\lfloor\alpha/5\rfloor<\lambda k$ and since

$$\begin{aligned}\left\lfloor\frac{\alpha}{5}\right\rfloor-\lambda(k-1) &= \left\lfloor\frac{\lambda(u-3v-1)}{10}\right\rfloor-\lambda(k-1)\\ &= \left\lfloor\lambda k+\frac{4-3v}{10}\right\rfloor+\lambda-\lambda k\\ &= \lambda+\left\lfloor\frac{4-3v}{10}\right\rfloor\le\lambda-1,\end{aligned}$$

it follows that E_α can be constructed and there are enough copies of the differences 1, 2, 3, $u/5$, and $2u/5$ in S. If $k=0$ then let $c_i=(0,1,2,3,4)$ for $0\le i\le\lfloor\lambda/2\rfloor-1$ and let $c_i=(0,2,4,1,3)$ for $i>\lfloor\lambda/2\rfloor-1$, and so we can form the set of 5-cycles $B=\{c_i+j \mid i\in\mathbb{Z}_{\lfloor\alpha/5\rfloor}, j\in\mathbb{Z}_v$.

Edges	Differences when $k-2\equiv 0,1 \pmod 4$	Differences when $k-2\equiv 2,3 \pmod 4$
$\{0,x_i\},\{x_i,x_i-y_i\}$	$4,5,\ldots,2k-1$	$4,5,\ldots,2k-2,2k$
$\{x_i-y_i,4k+2\}$	$4k+3,4k+4,\ldots,5k$	$4k+3,4k+4,\ldots,5k$
$\{4k+2,7k+3-i\}$	$3k,3k-1,\ldots,2k+3$	$3k,3k-1,\ldots,2k+3$
$\{7k+3-i,0\}$	$3k+3,3k+4,\ldots,4k$	$3k+3,3k+4,\ldots,4k$
c_i if $i=k-1$	$3k+1,3k+2,4k+1,5k+1,5k+2$	$3k+1,3k+2,4k+1,5k+1,5k+2$

Table 5.5: Edges of differences in B' when α is as large as possible for $\ell=5$

We will now construct the Type $\mathcal{B}$ 5-cycles. By Lemma 5.2.2, $\lambda(u-1)/2-\alpha$ or $\lambda(u-1)/2-\alpha-3v/2$ is divisible by 3 when λ is even or odd respectively. We can form a partition P_0 of the remaining differences after constructing B into $\lambda(u-1)/2-\alpha$ sets of size 3 when λ is even or into v sets $\{d,u/2\}$ of size 2 with $d\neq u/2$ and $\lambda(u-1)/2-\alpha-3v/2$ sets of size 3 when λ is odd; make sure that at most one copy of $v/2$ is in each set of size 3 and that there is at least one unique element in each set of size 3. These last conditions are easily met based on the differences used to construct each B. Since $\lambda \geq v$ in this lemma, there are enough copies of the difference $u/2$ to construct the desired partition. Let $\{d_{1,i}, d_{2,i}, d_{3,i}\}$ and $\{d_i, u/2\}$ denote the i^{th} set in P_0 and let p be the number of sets of size 3 in P_0. If λ is even then define

$$C = \{(d_{1,i}, 0, d_{3,i}, d_{2,i} + d_{3,i}, \infty_{i \text{ (mod v)}}) + j \mid \{d_{1,i}, d_{2,i}, d_{3,i}\} \in P_0 \tag{5.6}$$
$$\text{with } d_{1,i} \leq d_{2,i} \leq d_{3,i}, j \in \mathbb{Z}_u\} \tag{5.7}$$

and define

$$\begin{aligned} C = \{&(d_{1,i}, 0, d_{3,i}, d_{2,i} + d_{3,i}, \infty_{i \text{ (mod v)}}) + j \mid \{d_{1,i}, d_{2,i}, d_{3,i}\} \in P_0 \\ &\text{with } d_{1,i} \leq d_{2,i} \leq d_{3,i}, j \in \mathbb{Z}_u\} \\ &\cup \{(d_i, 0, u/2, u/2 + d_i, \infty_{i+p \text{ (mod v)}}) + j \mid \{u/2, d_i\} \in P_0, j \in \mathbb{Z}_{u/2}\} \end{aligned}$$

when λ is odd. This ensures the elements of C are indeed 5-cycles. Then $(\mathbb{Z}_u \cup \{\infty_0, \ldots, \infty_{v-1}\}, B \cup C)$ is a 5–CS of $\lambda K_{v+u} - \lambda K_v$ when $u \geq 3v + 1$.

Finally suppose $u < 3v+1$. If λ is odd then by Lemmas 5.1.1 and 5.3.2 there is a 5–CS of $\lambda K_{v+u} - \lambda K_v$. So let λ be even. Remember that if $\lambda \equiv 2, 4, 6, \text{ or } 8 \pmod{10}$ and $u \equiv 0 \pmod 5$ then $u \geq v/2 + 5$. So $u \neq 5$ when $\lambda \equiv 2, 4, 6, \text{ or } 8 \pmod{10}$. By Lemma 5.2.2, a and b are non-negative integers, u divides a, and either u divides b when u is odd or $u/2$ divides b when u is even.

If $u = 1$ or 2, then it is impossible to form any 5-cycles. If $u = 3$ then $v = 2, 3,$ or 4. Since only Type $\mathcal{A}$ 5-cycles can be formed when $u = 3$, since there are 3λ pure edges in $\lambda K_{v+3} - \lambda K_v$, and since there are 6λ or 9λ mixed edges when $v = 2$ or 3 respectively, it follows that there are not enough mixed edges to form all of the required 5-cycles, so a 5–CS cannot be formed under these conditions. When $u = 3$ and $v = 4$, define

$$C = \{(0, \infty_0, 2, \infty_1, 1) + j \mid j \in \mathbb{Z}_3\} \quad \text{and} \quad C' = \{(0, \infty_2, 2, \infty_3, 1) + j \mid j \in \mathbb{Z}_3\}.$$

So then $(\mathbb{Z}_u \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}, \lambda C \cup \lambda C')$ is a 5–CS of $\lambda K_7 - \lambda K_4$ when $u < 3v+1$.

Suppose $u \in \{4, 5\}$. Then form a partition P_0 of $3b/u$ of the differences in $D(u)^*_\lambda$ into b/u sets of size 3 such that each set in P_0 is $\{1, 1, 2\}$. Let $\{d_{1,i}, d_{2,i}, d_{3,i}\}$ denote the i^{th} set of size 3 in P_0 and let $D = \{d_1, d_2, \ldots, d_s\}$ be the remaining differences after forming the partition P_0. Define C as in Equation (5.6) and

$$B = \{(0, d_i, \infty_{2i-2+|P_0| \text{ (mod } v)}, d_i + 1, \infty_{2i-1+|P_0| \text{ (mod } v)}) + j \mid d_i \in D, j \in \mathbb{Z}_u\}.$$

Then $(\mathbb{Z}_u \cup \{\infty_0, \infty_1, \ldots, \infty_v\}, B \cup C)$ is a 5–CS of $\lambda K_{v+4} - \lambda K_v$.

Suppose $u = 6$. Then form a partition $P_0 = P_1 \cup P_2$ of $3b/u$ of the differences in $D(u)^*_\lambda$ into b/u sets of size 3 such that P_0 does not contain $\{2, 2, 2\}$ or $\{3, 3, 3\}$ and P_2 only contains copies of $\{1, 2, 3\}$. It is easy to ensure that P_0 does not contain $\{2, 2, 2\}$ or $\{3, 3, 3\}$ since every difference in $D(u)^*_\lambda$ is available when P_0 is constructed. Let $\{d_{1,i}, d_{2,i}, d_{3,i}\}$ denote the i^{th} set of size 3 and $D = \{d_1, d_2, \ldots, d_s\}$ be the remaining differences after forming the partition P_0. Define

$$C_1 = \{(0, d_{2,i}, d_{1,i} + d_{2,i}, d_{1,i} + d_{2,i} + d_{3,i}, \infty_{i \text{ (mod } v)}) + j \mid \{d_{1,i}, d_{2,i}, d_{3,i}\} \in P_1, j \in \mathbb{Z}_u,$$
$$d_{1,i} \leq d_{2,i} \leq d_{3,i}\},$$
$$C_2 = \{(d_{1,i}, 0, d_{3,i}, d_{2,i} + d_{3,i}, \infty_{i+|P_1| \text{ (mod } v)}) + j \mid \{d_{1,i}, d_{2,i}, d_{3,i}\} \in P_2, j \in \mathbb{Z}_u, \}, \text{ and}$$
$$B = \{(0, d_i, \infty_{2i-2+|P_0| \text{ (mod } v)}, d_i + 1, \infty_{2i-1+|P_0| \text{ (mod } v)}) + j \mid d_i \in D, j \in \mathbb{Z}_u\}.$$

Then $C = C_1 \cup C_2$ and so $(\mathbb{Z}_u \cup \{\infty_0, \infty_1, \ldots, \infty_v\}, B \cup C_1 \cup C_2)$ is a 5–CS of $\lambda K_{v+6} - \lambda K_v$.

Let $u \geq 7$. Let P_α contain the ϵ differences used to construct E_α. Form a partition P_0 of $3b/u$ differences from $D(u)^*_\lambda \setminus P_\alpha$ into b/u sets of size 3 so that each set contains at most one copy of $v/2$ and at least one unique element. We can guarantee that such a partition can be constructed since $u \geq 7$ and every difference in $D(u)^*_\lambda \setminus P_\alpha$ is available at this point. Notice that $3b/u$ is no larger than $D(u)^*_\lambda \setminus P_\alpha$. Define

$$C = \{(d_{1,i}, 0, d_{3,i}, d_{2,i} + d_{3,i}, \infty_{i \text{ (mod v)}}) + j \mid \{d_{1,i}, d_{2,i}, d_{3,i}\} \in P_0$$
$$\text{with } d_{1,i} \leq d_{2,i} \leq d_{3,i}, j \in \mathbb{Z}_u\} \cup$$

where calculations are performed modulo u. Thus C is a set of 5-cycles. Let $D = \{d_1, d_2, \ldots, d_s\}$ be the remaining differences in $D(u)^*_\lambda$ after partitioning $3b/u + \epsilon$ of the differences into P_0. Then define

$$B = \{(0, d_i, \infty_{2i-2+|P_0| \text{ (mod } v)}, d_i + 1, \infty_{2i-1+|P_0| \text{ (mod } v)}) + j \mid d_i \in D, j \in \mathbb{Z}_u\} \cup E_\alpha.$$

Then $(\mathbb{Z}_u \cup \{\infty_0, \ldots, \infty_{v-1}\}, B \cup C)$ is a 5–CS of $\lambda K_{v+u} - \lambda K_v$ for $u < 3v + 1$. □

Chapter 6
Conclusion/Future Work

As a result of the work in this dissertation, we have come up with some notable conjectures that will be investigated in the future.

Conjecture 6.1.1. *Suppose there exists a* 5–CS *of* λK_v. *If the necessary conditions in Lemma 2.1.2 are satisfied then a* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+u}$ *when* $u \geq 3$.

Conjecture 6.1.2. *If the necessary conditions in Lemma 4.1.1 are satisfied then there exists a* 5–CS *of* $(\lambda + m)K_{v+u} - \lambda K_v$ *for* $m \geq 1$ *and* $u \geq 2$

Conjecture 6.1.3. *Suppose there exists a* 5–CS *of* $\lambda K_v - F$ *where* F *is a* 1*-factor. Then a* 5–CS *of* $\lambda K_v - F$ *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+u}$ *if and only if*

1. $(\lambda + m)u + m(v - 1) - 1 \equiv 0 \pmod 2$,
2. $\binom{u}{2}(\lambda + m) + m\binom{v}{2} + vu(\lambda + m) + v/2 \equiv 0 \pmod 5$,
3. *if* $u = 1$, *then* $(m(v - 1) + 1) \geq 3(\lambda + m)$,
4. *if* $u = 2$, *then* $m\binom{v}{2} + v/2 - 2(\lambda + m) - (v - 1)(\lambda + m)/2 \geq 0$, *and*
5. *if* $u \geq 3$, *then* $\lceil vu(\lambda + m)/4\rceil + 2\epsilon \leq v(m(v - 1) + 1)/2 + (\lambda + m)u(u - 1)/2$ *where* $\epsilon = 0$ *or* 1 *if* $vu(\lambda + m) \equiv 0$ *or* $2 \pmod 4$ *respectively.*

Conjecture 6.1.4. *Suppose there exists a* 5–CS *of* λK_v. *Then a* 5–CS *of* λK_v *can be enclosed in a* 5–CS *of* $(\lambda + m)K_{v+u} - F$ *where* F *is a* 1*-factor if and only if*

1. $(\lambda + m)u + m(v - 1) - 1 \equiv 0 \pmod 2$,
2. $\binom{u}{2}(\lambda + m) + m\binom{v}{2} + vu(\lambda + m) - (v + u)/2 \equiv 0 \pmod 5$,
3. *if* $u = 1$, *then* $(m(v - 1) - 1) \geq 3(\lambda + m)$,
4. *if* $u = 2$, *then* $m\binom{v}{2} - v/2 - 2(\lambda + m) - (v - 1)(\lambda + m)/2 \geq 0$, *and*
5. *if* $u \geq 3$, *then* $\lceil vu(\lambda + m)/4\rceil + 2\epsilon \leq v(m(v - 1) - 1)/2 + (\lambda + m)u(u - 1)/2$ *where* $\epsilon = 0$ *or* 1 *if* $vu(\lambda + m) \equiv 0$ *or* $2 \pmod 4$ *respectively.*

Conjecture 6.1.5. *There exists a* 5–CS *of* $(\lambda + m)K_{v+u} - \lambda K_v - F$ *where* F *is a* 1*-factor if and only if*

1. $(\lambda + m)u + m(v - 1) - 1 \equiv 0 \pmod 2$,
2. $(\lambda + m)(v + u - 1) - 1 \equiv 0 \pmod 2$,
3. $\binom{u}{2}(\lambda + m) + m\binom{v}{2} + vu(\lambda + m) - (v + u)/2 \equiv 0 \pmod 5$,
4. *if* $u = 1$, *then* $(m(v - 1) - 1) \geq 3(\lambda + m)$,
5. *if* $u = 2$, *then* $m\binom{v}{2} - v/2 - 2(\lambda + m) - (v - 1)(\lambda + m)/2 \geq 0$, *and*

6. *if* $u \geq 3$, *then* $\lceil vu(\lambda+m)/4 \rceil + 2\epsilon \leq v(m(v-1)-1)/2 + (\lambda+m)u(u-1)/2$ *where* $\epsilon = 0$ *or* 1 *if* $vu(\lambda+m) \equiv 0$ *or* $2 \pmod 4$ *respectively.*

Conjecture 6.1.6. *If the necessary conditions in Lemma 2.1.2 are satisfied then a* t–CS *of* λK_v *can be enclosed in a* t–CS *of* $(\lambda+m)K_{v+u}$ *when* $t > 5$ *is odd.*

Since it was shown in Chapter 4 when a 5–CS of $(\lambda+m)K_{v+1} - \lambda K_v$ exists by using the construction in Chapter 2, we believe that a proof of Conjecture 6.1.2 when $u = 2$ can be found by using the construction in Chapter 3. This will soon be investigated.

The difficulty in extending our work from Chapters 2 and 3 to $u \geq 3$ in Conjecture 6.1.1 can be seen from the extra work in Section 3.3 to prove Theorem 3.5.2. A trail with a very special property was constructed so as to form a 5–CS subgraph that contains all of the mixed edges of a given multiset of differences. Unless another method is developed to obtain such a trail we believe that it will be extremely difficult to use the methods presented in this dissertation to solve Conjecture 6.1.1 when $u = 3$ or 4. There may be some hope when $u \geq 5$ since Theorem 5.4.1 can be used to construct some 5–CSs of $(\lambda+m)K_{v+u} - \lambda K_v$. If there exists a 5–CS of $(\lambda+m)K_{v+u} - (\lambda+m)K_{v+u}$ and there exists a 5–CS of mK_v, then there exists a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$. By checking the conditions in Theorem 1.2.2 and Theorem 5.4.1, if $v \equiv 1 \pmod{10}$ or $\lambda + m \equiv 0 \pmod{10}$ then there exists a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$. Also, if $\lambda \equiv 1, 3, 7,$ or $9 \pmod{10}$ and one of the following is true if:

- $\lambda + m \equiv 0 \pmod 5$,
- $\lambda + m \equiv 2, 4, 6,$ or $8 \pmod{10}$ and $u \equiv 0$ or $1 \pmod 5$, or
- $\lambda + m \equiv 1, 3, 7,$ or $9 \pmod{10}$ and $u \equiv 6 \pmod{10}$,

then there exists a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$. Once $\lambda \not\equiv 0, 1, 3, 7,$ or $9 \pmod{10}$ then the cases settled using this approach become very sparse.

There may be other hope in finishing the $u \geq 5$ case in Conjecture 6.1.1 since the types of 5-cycles based on the number of pure edges and mixed edges are equal for every $u \geq 5$. A visual representation of these 5-cycles is given in Figure 6.1.

The same comments made for Conjecture 6.1.1 can be said for Conjecture 6.1.2 since the work on the enclosing problem in this dissertation was used to construct a 5–CS of $(\lambda+m)K_{v+u} - \lambda K_v$.

Following the two parts of the Alspach Conjecture stated in Section 1.2, there is a second version of the enclosing problem as described in Conjectures 6.1.3 and 6.1.4. Since having a 5–CS of $\lambda K_v - F$ or a 5–CS of $(\lambda+m)K_{v+u} - F$ are different problems, there are two conjectures that involve a 1-regular graph as opposed to the one conjecture that involves a 1-regular graph in the Alspach Conjecture. Both of these conjectures can be extended by removing the condition that a 5–CS of $\lambda K_v - F$ or a 5–CS of λK_v exists respectively. Removing this condition in Conjecture 6.1.4 will give us Conjecture 6.1.5 while removing this condition from Conjecture 6.1.3 will give us Conjecture 6.1.2.

Another avenue to take this research is to consider enclosing problems for t–CSs for $t > 7$ and t odd, as seen in Conjecture 6.1.6. Conjectures 6.1.3-6.1.5 can also be generalized to t–CSs. As long as a t-cycle can be constructed using differences, the arguments described in this dissertation could be transferred to create constructions of other t–CSs. The hitch in this idea is that we have not explored whether a t-cycle can be constructed using Skolem sequences. There may also be options other than Skolem sequences to build the t-cycles using differences. This too will be investigated soon.

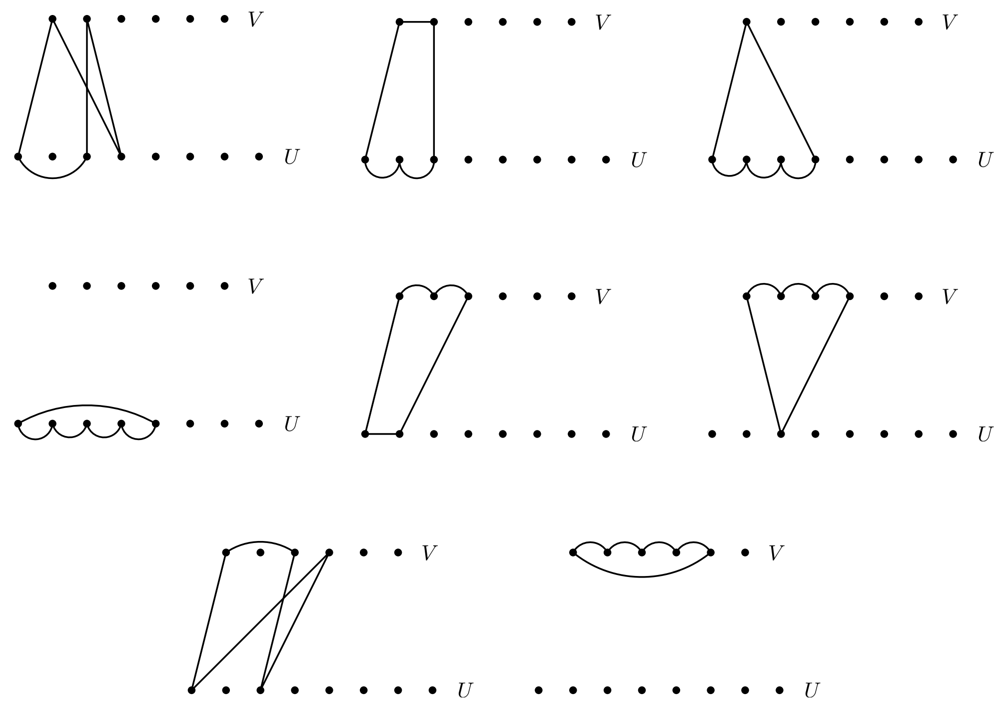

Figure 6.1: All types of 5-cycles based on the number of pure and mixed edges

Bibliography

[1] Jaromir Abrham and Anton Kotzig, *Skolem sequences and additive permutations*, Discrete Math. **37** (1981), no. 2-3, 143–146.

[2] B. Alspach, *Research problem* 3, Discrete Math. **36** (1981), 333.

[3] Brian Alspach and Heather Gavlas, *Cycle decompositions of* K_n *and* $K_n - I$, J. Combin. Theory Ser. B **81** (2001), no. 1, 77–99.

[4] J. Asplund, M. Keranen, and C. Rodger, *Enclosings of* λ*-fold* 5*-cycle systems for* $u = 1$, preprint.

[5] C. A. Baker, *Extended Skolem sequences*, J. Combin. Des. **3** (1995), no. 5, 363–379.

[6] J.-C. Bermond, O. Favaron, and M. Mahéo, *Hamiltonian decomposition of Cayley graphs of degree* 4, J. Combin. Theory Ser. B **46** (1989), no. 2, 142–153. MR 992988 (90g:05126)

[7] J.-C. Bermond, C. Huang, and D. Sotteau, *Balanced cycle and circuit designs: even cases*, Ars Combin. **5** (1978), 293–318.

[8] J.-C. Bermond and D. Sotteau, *Cycle and circuit designs odd case*, Contributions to graph theory and its applications (Internat. Colloq., Oberhof, 1977) (German), Tech. Hochschule Ilmenau, Ilmenau, 1977, pp. 11–32.

[9] Darryn Bryant and Daniel Horsley, *An asymptotic solution to the cycle decomposition problem for complete graphs*, J. Combin. Theory Ser. A **117** (2010), no. 8, 1258–1284.

[10] Darryn Bryant, Daniel Horsley, Barbara Maenhaut, and Benjamin R. Smith, *Cycle decompositions of complete multigraphs*, J. Combin. Des. **19** (2011), no. 1, 42–69.

[11] Darryn E. Bryant, D. G. Hoffman, and C. A. Rodger, 5*-cycle systems with holes*, Des. Codes Cryptogr. **8** (1996), no. 1-2, 103–108, Special issue dedicated to Hanfried Lenz.

[12] Darryn E. Bryant and C. A. Rodger, *The Doyen-Wilson theorem extended to* 5*-cycles*, J. Combin. Theory Ser. A **68** (1994), no. 1, 218–225.

[13] ———, *On the Doyen-Wilson theorem for m-cycle systems*, J. Combin. Des. **2** (1994), no. 4, 253–271.

[14] Darryn E. Bryant, C. A. Rodger, and Erin R. Spicer, *Embeddings of m-cycle systems and incomplete m-cycle systems: $m \leq 14$*, Discrete Math. **171** (1997), no. 1-3, 55–75.

[15] Charles J. Colbourn, Rose C. Hamm, and Alexander Rosa, *Embedding, immersing, and enclosing*, Proceedings of the sixteenth Southeastern international conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1985).

[16] Jean Doyen and Richard M. Wilson, *Embeddings of Steiner triple systems*, Discrete Math. **5** (1973), 229–239.

[17] Haim Hanani, *The existence and construction of balanced incomplete block designs*, Ann. Math. Statist. **32** (1961), 361–386.

[18] Charlotte Huang and Alexander Rosa, *On the existence of balanced bipartite designs*, Utilitas Math. **4** (1973), 55–75.

[19] Spencer P. Hurd, Patrick Munson, and Dinesh G. Sarvate, *Minimal enclosings of triple systems. I. Adding one point*, Ars Combin. **68** (2003), 145–159.

[20] Spencer P. Hurd and Dinesh G. Sarvate, *Minimal enclosings of triple systems. II. Increasing the index by 1*, Ars Combin. **68** (2003), 263–282.

[21] Thomas P Kirkman, *On a problem in combinations*, Cambridge and Dublin Math. J **2** (1847), 191–204.

[22] C. C. Lindner and C. A. Rodger, *Design theory*, second ed., Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2009.

[23] Eric Mendelsohn and Alexander Rosa, *Embedding maximal packings of triples*, Congr. Numer. **40** (1983), 235–247.

[24] N. A. Newman and C. A. Rodger, *Enclosings of λ-fold 4-cycle systems*, Des. Codes Cryptogr. **55** (2010), no. 2-3, 297–310.

[25] ———, *Enclosings of λ-fold triple systems*, Util. Math. **83** (2010), 149–158.

[26] Edward S. O'Keefe, *Verification of a conjecture of Th. Skolem*, Math. Scand. **9** (1961), 80–82.

[27] Alexander Rosa and Charlotte Huang, *Another class of balanced graph designs: balanced circuit designs*, Discrete Math. **12** (1975), 269–293.

[28] Mateja Šajna, *Cycle decompositions. III. Complete graphs and fixed length cycles*, J. Combin. Des. **10** (2002), no. 1, 27–78.

[29] James E. Simpson, *Langford sequences: perfect and hooked*, Discrete Math. **44** (1983), no. 1, 97–104.

[30] Th. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. **5** (1957), 57–68.

[31] Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

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