Hamilton Cycles in Multipartite Graphs With Two Associate Classes

by

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Abstract

This dissertation focuses on graph decompositions of $K\left(n^{r};\lambda_1,\lambda_2\right)$, the r-partite multigraph in which each part has size n, where two vertices in the same part or different parts are joined by exactly λ_1 edges or λ_2 edges respectively. Assuming one condition, necessary and sufficient conditions are found to embed a k-edge-coloring of $K\left(n^{r};\lambda_1,\lambda_2\right)$ into a Hamilton decomposition of $K\left(n^{r+2};\lambda_1,\lambda_2\right)$. In the tightest case, this assumption is in fact proved to be a new necessary condition. In addition, it is also proved that there exists a maximal set of t edge-disjoint Hamilton cycles in $K\left(n^{r};\lambda_1,\lambda_2\right)$ for $\lambda_2 n\lfloor\frac{r+3}{4}\rfloor \leqslant t \leqslant min\{\lfloor\frac{\lambda_2 n^2(r-1)}{2}\rfloor, \lfloor\frac{\lambda_1(n-1)+\lambda_2 n^(r-1)}{2}\rfloor\}$, the upper bound being best possible. The results proved in both chapters make use of the method of amalgamations.

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Chapter 1

Introduction

1.1 Basic Definitions

A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is called the vertex set and $E(G)$ is called the edge set. An edge in G is a 2-set $\{u, v\}$ of vertices. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of the edges incident with the vertex u in G. An edge $\{u, v\}$ is called a loop if $u = v$ (so multisets are notatioanally useful here). A loop contributes two to the degree of its incident vertex. Let $\ell_G(u)$ denote the number of the loops incident with u in G. The multiplicity of two vertices u and v in a graph G is the number of edges joining u to v, which is denoted by $m_G(u, v)$. Let $\omega(G)$ denote the number of components of G. A null graph is a graph whose edge set is empty.

A k-edge-coloring of a graph G is a function from the edge set $E(G)$ to the color set $\{1, 2, \ldots, k\}$ which assign the colors to the edges of G. For $1 \leqslant j \leqslant k$, $G(j)$ is the spanning subgraph of G induced by the edges colored with j. Let $\omega(G(j)) = \omega_j$ for $1 \leqslant j \leqslant k$. A k-edge-coloring of G is said to equitable if for all $v \in V(G)$ and for $1 \leqslant i < j \leqslant k$,

$$|d_{G(i)}(v) - d_{G(j)}(v)| \leqslant 1,$$

and is said to be evenly equitable if for all $v \in V(G)$ and for $1 \leqslant i < j \leqslant k$,

(i) $d_{G(i)}(v)$ is even, and

(ii) $|d_{G(i)}(v) - d_{G(j)}(v)| \in \{0, 2\}$.

A Hamilton path in G is a path which contains all the vertices of G. Similarly, a Hamilton cycle in G is a cycle which contains all the vertices of G. A graph G is said to be Hamiltonian if it contains a Hamilton cycle. A k-factor in G is a k-regular spanning subgraph. So a Hamilton cycle can be considered to be a connected 2-factor, which is an important fact used in the proof of Theorem 3.7 in Section 3.3.

An H-decomposition of a graph G is a set $\mathscr{H} = \{H_i : i \in I\}$ of edge-disjoint subgraphs of G such that for each $i \in I$, H_i is isomorphic to H and $E(G) = \bigcup_{i\in I} E(H_i)$. It is useful to regard to a k-edge-coloring of G as an H-decomposition of G in the case where each color class $G(j)$ is an isomorphic copy of H. If H_i is a k-factor for each $i \in I$ then $\mathscr{H}$ is called a k-factorization of G, and if H_i is a Hamilton cycle for each $i \in I$ then $\mathscr{H}$ is called a Hamilton decomposition of G.

The union of two graphs G and H is a graph, denoted by $G \cup H$, whose vertex and edge sets are $V(G) \cup V(H)$ and $E(G) \cup E(H)$, respectively. The join of G and H is the graph, denoted by $G \vee H$, whose vertex set is $V(G) \cup V(H)$ and edge set is $E(G \vee H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G) \text{ and } v \in V(H)\}$. The complement of G is the graph $\overline{G}$ with the vertex set $V(\overline{G}) = V(G)$ such that $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$.

If S is a set of edge-disjoint Hamilton cycles in G, then let $E(S)$ denote the set of edges in the Hamilton cycles in S. If $G - E(S)$ is not Hamiltonian, then the set S is called a maximal set of edge-disjoint Hamilton cycles in G. The spectrum for maximal sets of edge-disjoint Hamilton cycles in G is the set $Sp(G)$ which consists of the sizes of maximal sets of edge-disjoint Hamilton cycles in G.

Let K_r be the complete graph on the vertex set $V = \{v_i \mid i = 0, 1, \ldots, r-1\}$. The edge-difference of an edge $e = \{v_i, v_j\}$ in K_r ($0 \leqslant i \neq j \leqslant r-1$) is defined by $D(e) = D(\{v_i, v_j\}) = min\{|i-j|, r-|i-j|\}$. Notice that $0 \leqslant D(e) \leqslant \lfloor \frac{r}{2} \rfloor$ for all $e \in E(K_r)$. Intuitively, if r is even, the edge-difference $r/2$ is called the half-difference. Notice that set of edges of half-difference in K_r (r is even) induce a 1-factor of K_r. This fact is used later in the proof of Lemma 3.2.

An amalgamation of a graph G to a graph H is a surjective function $f : V(G) \to V(H)$ associated with a bijective function $g : E(G) \to E(H)$ such that *i*. For each $e = \{u, v\} \in E(G)$ with $u \neq v$, $g(e) = \{f(u), f(v)\}$ is an edge (a loop if $f(u) = f(v)$) in H, and

ii. For each loop $\ell = \{u, u\} \in E(G)$, $g(\ell) = \{f(u), f(u)\}$ is a loop on $f(u)$ in H.

The graph H is called the f-amalgamation of G, and G is said to be a detachment of H. Notice that the set $\mathcal{F} = \{f^{-1}(u) : u \in V(H)\}$ is a partition of $V(G)$. The vertices in $f^{-1}(u)$ are said to be disentangled from u. Intuitively the amalgamation function f identifies the vertices in each element of $\mathcal{F}$ with a single vertex of H while the bijection g turns the edges and loops in G into the edges (possibly loops) and loops in H, respectively. So notice that any edge incident with a vertex in G becomes incident with the corresponding new vertex in the amalgamated graph H, and any edge whose endpoints are identified becomes a loop on the new vertex in H.

For each $u \in V(H)$, let $\psi(u)$ be a positive integer. Then a ψ-detachment of H is a graph obtained by detaching each vertex u in H into $\psi(u)$ new vertices. So if H is an f-amalgamation of G then the graph G is a ψ-detachment of H where $\psi(u) = |f^{-1}(u)|$. Clearly there is a one-to-one correspondence between the edges of the graphs G and H. Hence an edge-coloring of one of the graphs induces an edge-coloring on the other graph; so an amalgamation of an edge-colored graph is also an edge-colored graph.

1.2 Outline of the Dissertation

Through this dissertation, all graphs are finite, undirected and usually have loops and multiple edges. Especially the graph $K(n^r; \lambda_1, \lambda_2)$ is of particular interest, which denotes the graph with partition $\{P_1, \dots, P_r\}$ of the vertex set, each part of size n, in which for $1 \leqslant i < j \leqslant k$

$$m_G(u, v) = \begin{cases} \lambda_1 & \text{if } u, v \in P_i \\ \lambda_2 & \text{if } u \in P_i \text{ and } v \in P_j,\ i \neq j. \end{cases}$$

This dissertation contains two topics that are described in their own chapters.

In Chapter 2, assuming one condition, necessary and sufficient conditions are found to embed a k-edge-coloring of $K\left(n^r;\lambda_1,\lambda_2\right)$ into a Hamiltonian decomposition of $K\left(n^{r+2};\lambda_1,\lambda_2\right)$ (so each color class induces a Hamilton cycle). In the tightest case, this one assumption is in fact proved to be a new necessary condition. Unlike previous results, of particular interest here is a necessary condition involving the existence of certain components in a related bipartite graph.

In Chapter 3, the existence of a maximal set of t edge-disjoint Hamilton cycles in $K\left(n^r;\lambda_1,\lambda_2\right)$ is proved for any t in the range $\lambda_2 n\lfloor\frac{r+3}{4}\rfloor \leqslant t \leqslant min\{\lfloor\frac{\lambda_2 n^2(r-1)}{2}\rfloor, \lfloor\frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2}\rfloor\}$.

Chapter 2

Embedding

2.1 History

Decomposing a graph into edge-disjoint Hamilton cycles has been a subject undergoing intense study in graph theory for many years. The following are two important questions that are well-studied in the literature:

Problem 1 When does an H-decomposition of G exist?

Problem 2 When can a k-edge-coloring of G be embedded into an H-decomposition of G^* regarded as a k-edge-coloring (so G is a subgraph of G^*)?

Among the many results on this topic are the following that are particularly germane to this chapter. In the 1890s, Walecki proved that the complete graph K_n has a Hamiltonian decomposition if and only if n is odd [15]. In 1982, Hilton [10] found necessary and sufficient conditions to embed a k-edge-coloring of K_m into a k-edge-coloring of K_{m+n} whose color classes are Hamilton cycles. In 1976, Laskar and Auerbach [13] proved that complete r-partite graphs in which each part has n vertices is the union of $n(r-1)/2$ Hamilton cycles and a 1-factor which are mutually edge-disjoint if $n(r-1)/2$ is odd and $\geqslant 1$ or is the union of $n(r-1)/2$ edge-disjoint Hamilton cycles if $n(r-1)/2$ is even and $\geqslant 2$. Additionally, Hilton and Rodger [11] provided a procedure which constructs a Hamiltonian decomposition of the r-partite graph $K_{n,\dots,n}$ when $(r-1)n$ is even, and also conditions which are necessary and sufficient to embed a k-edge-coloring of the complete t-partite graph $K_{a_1,\dots,a_t}$ into a Hamiltonian decomposition of the r-partite graph $K_{n,\dots,n}$ for $2t \leqslant r$, $1 \leqslant a_1, \leqslant \cdots \leqslant a_t \leqslant n$.

Bahmanian and Rodger [3] found necessary and sufficient conditions to settle the existence of a Hamiltonian decomposition of the graph $G = K(n^r; \lambda_1, \lambda_2)$. They partially succeeded in embedding a k-edge-coloring of $G = K(n^r; \lambda_1, \lambda_2)$ into a Hamiltonian decomposition of $G^* = K(n^{r+t}; \lambda_1, \lambda_2)$ in Theorem 2 of [4] proving the following result under the assumption $\sum_{j=1}^{k} s_j \geqslant kt - \lambda_2 n^2 \binom{t}{2}$ where $s_j \equiv \omega_j \ (mod\ r)$ with $1 \leqslant s_j \leqslant r$ for $1 \leqslant j \leqslant k$ that: a k-edge-coloring of G can be embedded into a Hamiltonina decomposition of G^* if and only if: (i) $2k = \lambda_1(n-1) + \lambda_2 n(r+t) - 1$, (ii) $\lambda_1 \leqslant \lambda_2 n(r+t-1)$, (iii) every component of $G(j)$ is a path (of possibly length of 0) for $1 \leqslant j \leqslant k$, and (iv) $\omega_j \leqslant nt$ for $1 \leqslant j \leqslant k$. In particular, a corollary of this result is solved in the case where $G = K(n^r; \lambda_1, \lambda_2)$, $G^* = K(n^{r+t}; \lambda_1, \lambda_2)$, and H is a Hamilton cycle of G^*, in the cases $t = 1$ and $t \geqslant \frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{\lambda_2 n(n-1)}$ (see [4, Theorem 3 and Corollary 1]).

In Theorem 2.4 of this chapter, the work in [4] is extended to the case where $t = 2$, providing necessary and sufficient conditions for the embedding of a k-edge-coloring of $K(n^r; \lambda_1, \lambda_2)$ into Hamiltonian decomposition of $K(n^{r+2}; \lambda_1, \lambda_2)$ under a general condition (see $(\star)$ of Theorem 2.4). We conjecture that this general condition is also necessary, proving in Corollary 2.6 that it is indeed necessary in a tightest case (i.e, when $|C_2|$, defined in Section 3, satisfies equality in Condition (v) of Theorem 2.4). At first sight, considering just one more of the unsolved values of t may not seem to be a lot of progress. But as will be clear, the nature of the necessary conditions changes dramatically when $t = 2$. Until now, the embedding has been completely determined by reasonably clear numerical necessary conditions. But now, when $t = 2$, the structure of a graph related to the given edge-coloring can determine whether or not the embedding is possible. This drastic change is reminiscent of the long-standing unsolved embedding problem for partial idempotent latin squares of order n into idempotent latin squares of order $n + t$ when t is small: when $t \geqslant n$ numerical conditions do prove to be sufficient, but for smaller values of t the existence of certain components in a closely related graph can prevent such an embedding (see [1, 2]). For this reason, being able to make the jump in Theorem 2.4 to the case where $t = 2$ is in fact substantial progress over the existing state of knowledge.

2.2 An Edge-coloring Lemma

Hilton proved that there exists an evenly equitable k-edge-coloring of every finite even graph (all vertices have even degree) for all $k \geqslant 1$ in [9, Theorem 8]. In the following lemma, his proof is manipulated to settle an interesting generalized notion, which produces a 2-edge-coloring of a bipartite graph that is evenly equitable on specified vertices and equitable on all the others.

Lemma 2.1. *Let B be a finite even bipartite graph with bipartition $\{V, C\}$ of its vertex set. For any subset $X \subseteq C$, there exists a 2-edge-coloring $\sigma : E(B) \to \{1, 2\}$ such that*

(i) $d_{B(1)}(v) = d_{B(2)}(v)$ *for all* $v \in V$,

(ii) $d_{B(1)}(c) = d_{B(2)}(c)$ *for all* $c \in X$,

(iii) $|d_{B(1)}(c) - d_{B(2)}(c)| = 2$ *for all* $c \in C \backslash X$

if and only if

(iv) $|V(D) \cap (C \backslash X)|$ *is even for each component D of B.*

Proof. To prove the necessity, let $\sigma : E(B) \to \{1, 2\}$ be a 2-edge-coloring of the bipartite graph B satisfying conditions $(i) - (iii)$ for a given subset $X \subseteq C$. Notice that the number of the edges with color 1 equals to the number of the edges with color 2 in each component D of B by (i). This, together with (ii) and (iii), implies that the number of the vertices $c \in V(D) \cap (C \backslash X)$ with $d_{B(1)}(c) = d_{B(2)}(c) + 2$ is the same as the number of the vertices $c \in V(D) \cap (C \backslash X)$ with $d_{B(2)}(c) = d_{B(1)}(c) + 2$. Therefore $|V(D) \cap (C \backslash X)|$ is even for each component D of B.

To prove the sufficiency, assume that $X \subseteq C$ and that $|V(D) \cap (C \backslash X)|$ is even for each component D of B. Form a new graph B' from the bipartite graph B by adding exactly one loop on c (i.e., a single edge contributing two to the degree of c) for each $c \in C \backslash X$. For all $v \in V$, $d_B(v)$ is even by (i). So each component D of B has an even number of edges (including loops). Also notice that the degree of each vertex in B' is even (a loop contributes 2 to the degree of its incident vertex). Therefore each component of B' has an Euler tour of an even length. Alternately color the edges of each of these Euler tours with colors 1 and 2. Since the length of each Eulerian tour is even, the color on its first edge is different from the color on its last edge. Therefore this results in a 2-edge coloring of B' in which: for all $v \in V$ $d_{B'(1)}(v) = d_{B'(2)}(v)$; and for all $c \in C$, $d_{B'(1)}(c) = d_{B'(2)}(c)$. The restriction of this edge-coloring onto the edges of the bipartite graph B is the required 2-edge coloring $\sigma : E(B) \to \{1, 2\}$ satisfying the conditions $(i) - (iii)$ (condition (iii) follows since each vertex in $C \backslash X$ loses exactly one loop in forming the restriction). $\square$

2.3 Main Results

In this section the main result, Theorem 2.4, is proved by using the method of amalgamations. This relies on the following result of Bahmanian and Rodger, a more generalized version of which is proved in [3, Theorem 3.1]. In Theorem 2.2, ψ is the detachment function producing H from G.

Theorem 2.2 ([3]). *Let G be a k-edge-colored graph and let ψ be a function from $V(G)$ into $\mathbb{N}$ such that for each $u \in V(G)$,*

(1) $\psi(u) = 1$ *implies* $\ell_G(u) = 0$,

(2) $d_{G(j)}(u)/\psi(u)$ *is an even integer for* $1 \leqslant j \leqslant k$,

(3) $\binom{\psi(u)}{2}$ *divides* $\ell_G(u)$,

(4) $\psi(u)\psi(v)$ *divides* $m_G(u, v)$ *for each* $v \in V(G) \backslash \{u\}$, *and*

(5) $G(j)$ *is connected for* $1 \leqslant j \leqslant k$.

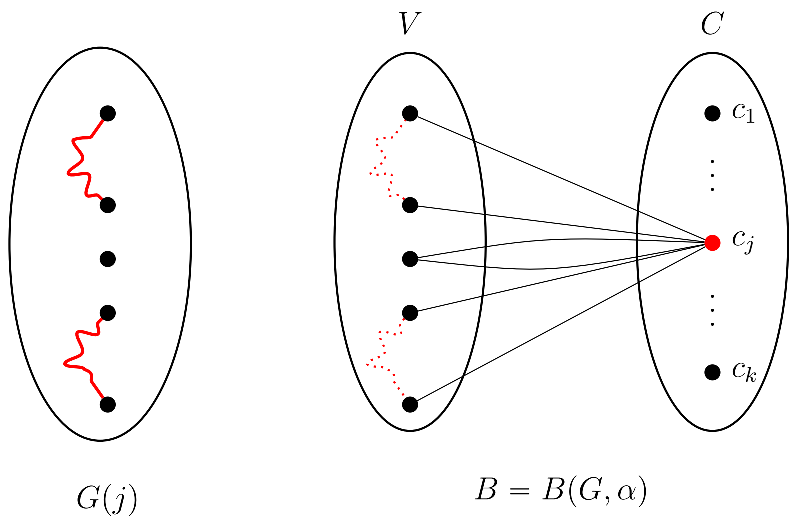
Then there exists a ψ-detachment H of G in which each $u \in V(G)$ is disentangled into vertices $u_1, ..., u_{\psi(u)}$, such that for all $u \in G$:

(i) $m_H(u_i, u_{i'}) = \ell_G(u)/\binom{\psi(u)}{2}$ *for all* $1 \leqslant i < i' \leqslant \psi(u)$ *if* $\psi(u) \geqslant 2$,

(ii) $m_H(u_i, v_{i'}) = m_G(u, v)/\psi(u)\psi(v)$ *for* $v \in V(G)\backslash\{u\}$, $1 \leqslant i \leqslant \psi(u)$, *and* $1 \leqslant i' \leqslant \psi(v)$,

(iii) $d_{H(j)}(u_i) = d_{H(j)}(u)/\psi(u)$ *for* $1 \leqslant i \leqslant \psi(u)$ *and* $1 \leqslant j \leqslant k$, *and*

(iv) *Color class* $H(j)$ *is connected for* $1 \leqslant j \leqslant k$.



The dotted path on V in B is solely used to indicate the components of $G(j)$.

Figure 2.1: The Bipartite Graph $B(G, \alpha)$

Define two bipartite graphs $B = B(G, \alpha)$ and $B^* = B(\pi(\alpha))$ associated with a special edge-coloring, α, of $G = K(n^r; \lambda_1, \lambda_2)$ as follows; these are critically important in the statement and proof of Theorem 3.2. Recall that $\{P_1, \ldots, P_r\}$ is the partition of $V(G)$. For a given k-edge-coloring $\alpha : E(G) \to \{1, \ldots, k\}$ of G in which

$$\text{each component in each color class is a path (possibly of length 0),} \tag{2.1}$$

define the bipartite graph $B = B(G,\alpha)$ with bipartition $\{V,C\}$ of its vertex set where $V = \bigcup_{i=1}^{r} P_i$, $C = \{c_1,\ldots,c_k\}$ is the set of color vertices, and for all $v \in V$ and $c_j \in C$,

$$m_B(v,c_j) = \begin{cases} 0 & \text{if } d_{G(j)}(v) = 2 \\ 1 & \text{if } d_{G(j)}(v) = 1 \\ 2 & \text{if } d_{G(j)}(v) = 0, \end{cases} \tag{2.2}$$

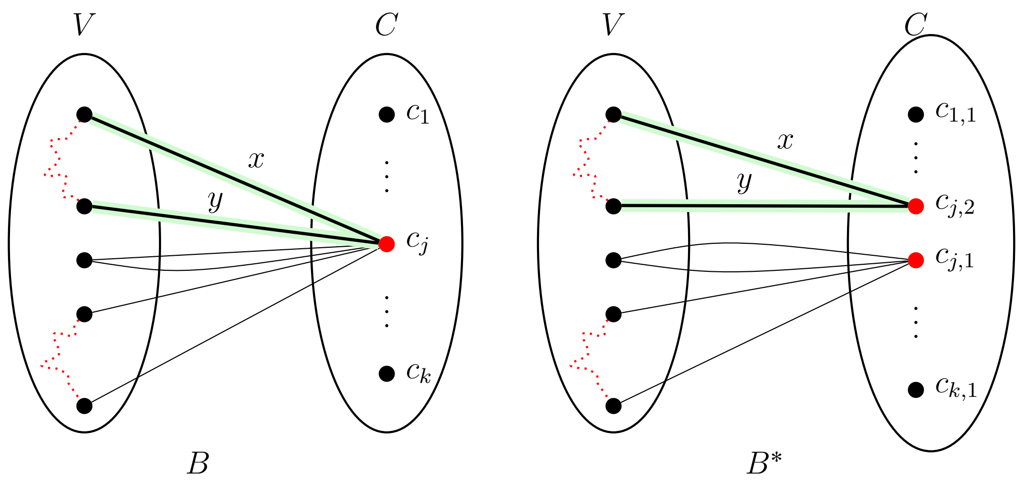
(see Figure 2.1). Notice that $d_B(c_j) \geqslant 2$ and is even for $1 \leqslant j \leqslant k$ by (2.1). Under the assumption that $k = (\lambda_1(n-1) + \lambda_2 n(r-1))/2$, it will also be important later to establish properties (2.3) and (2.4) below.

$$\begin{aligned} d_B(c_j) &= \sum_{v\in V(G)} m_B(v,c_j) \\ &= \sum_{v\in V(G)} (2 - d_{G(j)}(v)) \\ &= 2nr - \sum_{v\in V(G)} d_{G(j)}(v), \text{ so} \\ \sum_{j=1}^{k} d_B(c_j) &= 2nrk - \sum_{j=1}^{k}\sum_{v\in V(G)} d_{G(j)}(v) \\ &= 2nrk - 2|E(G)| \\ &= 2nrk - nr\left(\lambda_1(n-1) + \lambda_2 n(r-1)\right), \text{ so} \end{aligned}$$

$$\sum_{j=1}^{k} d_B(c_j) = 2\lambda_2 n^2 r. \tag{2.3}$$

Also by (2.2)

$$d_B(v) = \sum_{j=1}^{k} m_B(v,c_j) = 2k - d_G(v). \tag{2.4}$$



The edges indicated with x and y are selected to be joined with $c_{j,1}$ because they are in the same component of $G(j)$.

Figure 2.2: The Detached Graph $B^* = B^*(G, \pi(\alpha))$.

In this context, it is convenient to partition C into two sets C_0 and C_2: for each $i \in \{0, 2\}$ define $C_i = \{c_j \in C | d_B(c_j) \equiv i \ (mod\ 4)\}$. For each $c_j \in C$, choose a set $\mathcal{C}(c_j) = \{\{c_j, v_{j,1}\}, \{c_j, v_{j,2}\}\}$ of 2 edges incident with c_j in B such that $v_{j,1}$ and $v_{j,2}$ are in the same component of $G(j)$ (i.e., are the two ends of a path). Let $\pi(\alpha)$ be the set of all k such 2-element sets $\mathcal{C}(c_j)$. Now define the detached graph $B^* = B(\pi(\alpha))$ (see Figure 2.2) from B by detaching each color vertex $c_j \in C$, $1 \leqslant j \leqslant k$, into two new vertices $c_{j,1}$ and $c_{j,2}$ such that the edges $\{c_j, v_{j,1}\}$ and $\{c_j, v_{j,2}\}$ become incident with $c_{j,2}$ and all the other edges incident with c_j in B become incident with $c_{j,1}$. Note that $d_B(v) = d_{B*}(v)$ for all $v \in V$ and $d_B(c_j) = d_{B*}(c_{j,1}) + d_{B*}(c_{j,2})$ for all $c_j \in C$. Since $d_{B*}(c_{j,2}) = 2$ for $1 \leqslant j \leqslant k$, if $d_B(c_j) = 2$, then $d_{B*}(c_{j,1}) = 0$. It turns out to be notationally useful to have such vertices of degree 0, since by themselves they form components of B^* containing an odd number of vertices of degree divisible by 4, namely one. The number of such vertices is critical in the statement of Theorem 2.4.

In the following, edges in $K(n^r; \lambda_1, \lambda_2)$ joining vertices in the same part (or different parts) are said to be pure (or mixed, respectively). Note that it is assumed that $\lambda_1 \neq \lambda_2$ since otherwise $K(n^r, \lambda_1, \lambda_2) = \lambda_1 K_{nr}$, in which case the part structure of the graph is irrelevant.

Proposition 2.3. *Let $n > 1$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 1$ and $\lambda_1 \neq \lambda_2$. Let α be a k-edge-coloring of $G = K(n^r; \lambda_1, \lambda_2)$. If the k-edge-coloring α can be embedded into a Hamiltonian decomposition of $G^* = K(n^{r+2}; \lambda_1, \lambda_2)$, then*

(i) $k = \frac{1}{2}(\lambda_1(n-1) + \lambda_2 n(r+1))$,

(ii) $\lambda_1 \leqslant \lambda_2 n(r+1)$,

(iii) Each component of $G(j)$ is a path (possibly of length 0) for $1 \leqslant j \leqslant k$,

(iv) $\omega_j \leqslant 2n$ *for* $1 \leqslant j \leqslant k$, *and*

(v) $|C_2| \leqslant 2\lambda_1\binom{n}{2} + \lambda_2 n^2$.

Proof. Under different conditions, the necessity of conditions $(i) - (iv)$ are proved in [4, Theorem 2]. Nevertheless the following proofs of $(i) - (iv)$ are essentially the same. Suppose that the k-edge-coloring α of G is embedded into a k-edge-coloring α^* of G^* in which each color class is a Hamilton cycle. Obviously $d_{G^*}(v) = 2k$ for all $v \in V(G^*)$. Also, there are exactly $\lambda_1(n-1)$ pure edges and $\lambda_2 n(r+1)$ mixed edges incident with v in G^*. Hence $2k = d_{G^*}(v) = \lambda_1(n-1) + \lambda_2 n(r+1)$. This proves the necessity of (i).

As each color class $G^*(j)$ is a Hamilton cycle in G^*, it has at most $n-1$ pure edges in each part of G^*. Since $n > 1$ and each part of G^* has $\lambda_1\binom{n}{2}$ pure edges, as was shown in [4] the necessity of (ii) follows from the following inequalities:

$$\begin{aligned}
\lambda_1\binom{n}{2} &\leqslant (n-1)k \\
&= \frac{(n-1)}{2}(\lambda_1(n-1) + \lambda_2 n(r+1)), \text{ so} \\
\lambda_1 n &\leqslant \lambda_1(n-1) + \lambda_2 n(r+1), \text{ thus} \\
\lambda_1 &\leqslant \lambda_2 n(r+1).
\end{aligned}$$

For $1 \leqslant j \leqslant k$, $G(j)$ is a subgraph of $G^*(j)$ which is a Hamilton cycle in G^*. So the components of $G(j)$ can be only paths (possibly of length 0). This proves the necessity of (iii).

For $1 \leqslant j \leqslant k$, the components of $G(j)$ are paths (possibly of length 0), so the Hamilton cycle $G^*(j)$ has exactly two mixed edges joining vertices in $V(D)$ to vertices in $P_{r+1} \cup P_{r+2}$ for each component D of $G(j)$. Hence $\omega_j \leqslant 2n$ since for $1 \leqslant j \leqslant k$

$$\sum_{v \in P_{r+1} \cup P_{r+2}} d_{G^*(j)}(v) = 4n.$$

This proves the necessity of (iv).

Suppose $c_j \in C_2$ and $G^*(j)$ contains no mixed edge in $G^*[P_{r+1} \cup P_{r+2}]$. Let $\mathcal{D}$ be the set of all the components of $G(j)$. For each $D \in \mathcal{D}$, let $T_j(D)$ be the set of the two edges in the path in $G^*(j)$, each of which joins a vertex in $V(D)$ to a vertex in $P_{r+1} \cup P_{r+2}$. For each component D of $G(j)$, if both edges in $T_j(D)$ are incident with vertices in P_{r+1} or P_{r+2}, then place them in $\tau_{j,1}$ or $\tau_{j,2}$ respectively, and otherwise place them in $\tau_{j,3}$. So $\{\tau_{j,1}, \tau_{j,2}, \tau_{j,3}\}$ is partition of $\cup_{D \in \mathcal{D}} T_j(D)$. Notice that $G(j)$ contains $\sum_{i=1}^{3} |\tau_{j,i}|/2$ components. Clearly $|\tau_{j,1}|$ and $|\tau_{j,2}|$ are even. So, since $2|P_{r+1}| = \sum_{v \in P_{r+1}} d_{G^*(j)}(v) = 2|E(G^*[P_{r+1}](j))| + |\tau_{j,1}| + |\tau_{j,3}|/2$, it follows that $|\tau_{j,3}|/2$ is even; that is, there are an even number of the components of $G(j)$ which have vertices joined to vertices in the different parts of $G^*[P_{r+1} \cup P_{r+2}]$. As $c_j \in C_2$, $G(j)$ contains an odd number of components. Then $|\tau_{j,1}|/2$ and $|\tau_{j,2}|/2$ have different parity since $|\tau_{j,3}|/2$ is even. In particular, we can assume that $|\tau_{j,1}|/2 \geqslant |\tau_{j,2}|/2 + 1$. Thus, since

$$2n = \sum_{u \in P_{r+1}} d_{G^*(j)}(u) = 2|\tau_{j,1}| + |\tau_{j,3}| + 2|E\big(G^*[P_{r+1}](j)\big)|, \text{ and}$$
$$2n = \sum_{u \in P_{r+2}} d_{G^*(j)}(u) = 2|\tau_{j,2}| + |\tau_{j,3}| + 2|E\big(G^*[P_{r+2}](j)\big)|,$$

it follows that $|E(G^*[P_{r+2}](j))| > |E(G^*[P_{r+1}](j))|$. Therefore, if $c_j \in C_2$ then either there is at least one mixed edge or there is at least one pure edge in $G^*[P_{r+1} \cup P_{r+2}](j)$. This proves the necessity of (v). $\square$

Theorem 2.4. *Let $n > 1$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 1$ and $\lambda_1 \neq \lambda_2$. Let α be a k-edge-coloring of $G = K(n^r; \lambda_1, \lambda_2)$. Suppose that*

($\star$) *$\pi(\alpha)$ can be chosen such that in the detached graph $B^* = B(\pi(\alpha))$, the number of the components having an odd number of color vertices of degree divisible by 4 is at most $\lambda_2 n^2$.*

Then the k-edge-coloring α can be embedded into a Hamiltonian decomposition of $G^* = K(n^{r+2}; \lambda_1, \lambda_2)$ if and only if:

(i) $k = \frac{1}{2}(\lambda_1(n-1) + \lambda_2 n(r+1))$,

(ii) $\lambda_1 \leqslant \lambda_2 n(r+1)$,

(iii) *Each component of $G(j)$ is a path (possibly of length 0) for $1 \leqslant j \leqslant k$,*

(iv) *$\omega_j \leqslant 2n$ for $1 \leqslant j \leqslant k$, and*

(v) $|C_2| \leqslant 2\lambda_1\binom{n}{2} + \lambda_2 n^2$.

Proof. The necessity of conditions $(i) - (v)$ follows from Proposition 2.3.

To prove the sufficiency, consider a k-edge-coloring $\alpha : E(G) \to \{1, ..., k\}$ of the graph $G = K(n^r; \lambda_1, \lambda_2)$. By necessary condition (iii), the property (2.1) is satisfied by α, so form the bipartite graph $B = B(G, \alpha)$ and the detached graph $B^* = B(\pi(\alpha))$, as described in the preamble to Theorem 2.4, choosing $\pi(\alpha)$ so that Condition ($\star$) is satisfied.

Recall that the detached graph B^* is a bipartite graph with the parts $\{V, C^*\}$ where $V = \cup_{i=1}^{r} P_i$ and $C^* = \{c_{j,1}, c_{j,2} : 1 \leqslant j \leqslant k\}$. Notice that for all $v \in V$ and $1 \leqslant j \leqslant k$, $d_{B^*}(c_{j,2}) = 2$ and $d_{B^*}(v)$ and $d_{B^*}(c_{j,1})$ are even. Also by the necessary condition (iv):

$$d_{B^*}(c_{j,1}) \leqslant 4n - 2 \text{ with equality iff } \omega_j = 2n \text{ (in which case } c_j \in C_0) \tag{2.5}$$

For the sake of convenience, define a partition of C^* into two sets: $C_2^* = \{c_{j,1} : d_{B^*}(c_{j,1}) \equiv 2 \ (mod\ 4) \text{ and } 1 \leqslant j \leqslant k\} \cup \{c_{j,2} : 1 \leqslant j \leqslant k\}$ and $C_0^* = \{c_{j,1} : d_{B^*}(c_{j,1}) \equiv 0 \ (mod\ 4) \text{ and } 1 \leqslant j \leqslant k\}$ (recall that some vertices in C_0^* may have degree 0). Clearly

$$|C_0^*| = |C_2|. \tag{2.6}$$

Let $\mathcal{D}^*$ be the set of the components of B^*. For each $D \in \mathcal{D}^*$, choose a set $S(D)$ of $n(D)$ vertices in $V(D) \cap C_0^*$ such that $\sum\limits_{D \in \mathcal{D}^*} n(D)$ is as large as possible subject to the two conditions:

(1′) $n(D) \equiv |V(D) \cap C_0^*| \ (mod\ 2)$ for all $D \in \mathcal{D}^*$, and

(2′) $\sum\limits_{D \in \mathcal{D}^*} n(D) \leqslant \lambda_2 n^2$.

Clearly such a set $S(D)$ exists providing that there is at least one set S of $n(D)$ vertices satisfying (1′) and (2′). Construct such a set S^* as follows. If $|V(D) \cap C_0^*|$ is even, then let S^* contains no vertices from $V(D) \cap C_0^*$, and if $|V(D) \cap C_0^*|$ is odd, then let S^* contain any single element in $V(D) \cap C_0^*$; then S^* satisfies condition (1′), and Condition ($\star$) guarantees that $|S^*| \leqslant \lambda_2 n^2$, so S^* satisfies condition (2′).

Now consider the subset $N^* = \cup_{D \in \mathcal{D}^*} S(D) \subseteq C_0^*$ of size $\sum_{D \in \mathcal{D}^*} n(D)$. We now show that $|N^*| = \min\{\lambda_2 n^2, |C_0^*|\}$. If $|C_0^*| \leqslant \lambda_2 n^2$, then clearly $N^* = C_0^*$ since the maximality of $\sum n(D)$ forces $S(D)$ to be $V(D) \cap C_0^*$ for each $D \in \mathcal{D}^*$. If $|C_0^*| > \lambda_2 n^2$, then, since (1′) implies that $|N^*| \equiv |C_0^*| \ (mod\ 2)$, it follows by (2′) and the maximality of $\sum n(D)$ that $|N^*| \in \{\lambda_2 n^2, |C_0^*|\}$; the following argument shows that $|C_0^*| \equiv \lambda_2 n \ (mod\ 2)$, so this parity forces $|N^*| = \lambda_2 n^2$.

Note that for all $v \in V$ and $1 \leqslant j \leqslant k$,

$$\begin{aligned} m_B(v, c_j) + d_{G(j)}(v) &= 2, \text{ so} \\ d_B(v) + d_G(v) &= 2k \\ d_B(v) &= 2k - \lambda_1(n-1) + \lambda_2 n(r-1) \\ &= 2\lambda_2 n \text{ by } (i). \end{aligned}$$

Therefore

$$\sum_{v \in V} d_B(v) \equiv \begin{cases} 0 \ (mod\ 4) & \text{if } \lambda_2 n \text{ is even} \\ 2 \ (mod\ 4) & \text{if } \lambda_2 n \text{ is odd.} \end{cases} \tag{2.7}$$

If $\lambda_2 n$ is even, then the size of C_0^* is even since

$$\begin{aligned} 2|C_0^*| &= 2|C_2| \text{ by } (2.6) \\ &\equiv \sum_{c_j \in C_2} d_B(c_j) \ (mod\ 4) \\ &= \sum_{v \in V} d_B(v) - \sum_{c_j \in C_0} d_B(c_j) \\ &\equiv 0 \ (mod\ 4) \text{ by } (2.7). \end{aligned}$$

If $\lambda_2 n$ is odd, then r is odd by (i) and hence $|C_0^*|$ is odd since

$$\begin{aligned} 2|C_0^*| &= 2|C_2| \text{ by } (2.6) \\ &\equiv \sum_{c_j \in C_2} d_B(c_j) \ (mod\ 4) \\ &= \sum_{v \in V} d_B(v) - \sum_{c_j \in C_0} d_B(c_j) \text{ (since } |V| = nr \text{ is odd)} \\ &\equiv 2 \ (mod\ 4) \text{ by } (2.7). \end{aligned}$$

Therefore in both cases $|C_0^*| \equiv \lambda_2 n^2 \ (mod\ 2)$ as required.

Apply Lemma 2.1 to the detached graph B^* with the subset $X = C_2^* \cup N^* \subseteq C^*$ to obtain a 2-edge-coloring $\beta^* : E(B^*) \to \{1, 2\}$ such that

(a_1) $d_{B^*(1)}(v) = d_{B^*(2)}(v) = \lambda_2 n$ for all $v \in V$,

(a_2) $d_{B^*(1)}(c_{j,2}) = d_{B^*(2)}(c_{j,2}) = 1$ for all $c_{j,2} \in C^*$,

(a_3) $d_{B^*(1)}(c_{j,1}) = d_{B^*(2)}(c_{j,1}) \equiv 1 \ (mod\ 2)$ for all $c_{j,1} \in C_2^*$,

(a_4) $d_{B^*(1)}(c_{j,1}) = d_{B^*(2)}(c_{j,1}) \equiv 0 \ (mod\ 2)$ for all $c_{j,1} \in N^*$,

(a_5) $|d_{B^*(1)}(c_{j,1}) - d_{B^*(2)}(c_{j,1})| = 2$ for all $c_{j,1} \in C_0^* \backslash N^*$, and

(a_6) $d_{B^*(1)}(c_{j,1}) \equiv d_{B^*(2)}(c_{j,1}) \equiv 1 \ (mod\ 2)$ for all $c_{j,1} \in C_0^* \backslash N^*$.

Let $N = \{c_j : c_{j,1} \in N^*\} \subseteq C_2$. By way of the the natural one-to-one correspondence between $E(B)$ and $E(B^*)$, β^* can be used to define the 2-edge-coloring $\beta : E(B) \to \{1, 2\}$ with the following properties:

(b_1) $d_{B(1)}(v) = d_{B(2)}(v) = \lambda_2 n$ for all $v \in V$,

(b_2) $d_{B(1)}(c_j) = d_{B(2)}(c_j) \equiv 0 \ (mod\ 2)$ for all $c_j \in C_0$,

(b_3) $d_{B(1)}(c_j) = d_{B(2)}(c_j) \equiv 1 \ (mod\ 2)$ for all $c_j \in N$,

(b_4) $|d_{B(1)}(c_j) - d_{B(2)}(c_j)| = 2$ for all $c_j \in C_2 \backslash N$, and

(b_5) $d_{B(1)}(c_j) \equiv d_{B(2)}(c_j) \equiv 0 \ (mod\ 2)$ for all $c_j \in C_2 \backslash N$ (by (a_2) and (a_6)).

Let u_1 and u_2 be two distinct vertices not in $V = V(G)$. Define a new k-edge-colored graph G_1 by adding u_1 and u_2 to the given k-edge-colored graph G together with the edges of B as follows: for each edge $\{v, c_j\}$ colored $i \in \{1, 2\}$ in B (so $v \in V$ and $c_j \in C$), add the edge $\{v, u_i\}$ colored j to G. So for each $v \in V$ and $1 \leqslant i \leqslant 2$,

$$m_{G_1}(v, u_i) = d_{B_{(i)}}(v) = \lambda_2 n \text{ by } (b_1). \tag{2.8}$$

Define a new graph G_2 from G_1 by adding $\lambda_2 n^2$ mixed edges joining u_1 and u_2, and $\lambda_1\binom{n}{2}$ loops on each of u_1 and u_2. We now extend the k-edge-coloring of G_1 to a k-edge-coloring of G_2 in the following two steps.

Step 1: To make the degrees of the vertices u_1 and u_2 in each color class of G_2 both even and the same as each other,

- (A_1) Color exactly one mixed edge joining u_1 to u_2 with color j for each $c_j \in N$, and
- (A_2) For each $c_j \in C_2 \backslash N$, color exactly one loop with j at one of u_1 and u_2 , whichever has a smaller degree in $G_1(j)$.

By $(2')$, $|N| \leqslant \lambda_2 n^2$. So there are enough mixed edges to complete step (A_1). To see that there are enough loops to carry out step (A_2), consider the following. For $1 \leqslant i_1 \leqslant 2$ and $i_2 \in \{1,2\}\backslash\{i_1\}$, let κ_{i_1} be the number of the colors $j \in \{1,\ldots,k\}$ for which $d_{G_1(j)}(u_{i_1}) + 2 = d_{G_1(j)}(u_{i_2})$; so $\kappa_1 + \kappa_2 = |C_2\backslash N|$ loops are colored in (A_2). By (b_1), $|E(B(1))| = |E(B(2))|$. So $\kappa_1 = \kappa_2$ by (b_2), (b_3), and (b_4). Therefore there are enough loops to complete (A_2) since if $|C_2| \geqslant \lambda_2 n^2$ (otherwise (A_2) does nothing), then

$$\begin{aligned}
\kappa_1 = \kappa_2 = \frac{1}{2}|C_2\backslash N| &= \frac{|C_2| - |N|}{2} \\
&= \frac{|C_2| - \lambda_2 n^2}{2} \\
&\leqslant \lambda_1\binom{n}{2} \text{ by necessary condition } (iii) \\
&= \ell_{G_2}(u_i) \text{ for each } i \in \{1,2\}.
\end{aligned}$$

Let $\eta : \{1,\ldots k\} \to \mathbb{N}$ be the function defined by

$$2\eta(j) = \begin{cases} d_B(c_j) & \text{if } c_j \in C_0 \\ d_B(c_j) + 2 & \text{if } c_j \in C_2. \end{cases}$$

So after Step 1, considering only the edges in G_2 colored so far, for $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant 2$,

$$d_{G_2(j)}(u_i) = \eta(j). \tag{2.9}$$

Notice that $\eta(j) \leqslant 4n$ for $1 \leqslant j \leqslant k$ by (2.5).

Step 2: Greedily assign a color $j \in \{1, \ldots, k\}$ to each of the loops and mixed edges in G_2 which are left uncolored in Step 1 as follows:

Let $\mathcal{E}$ be a set of all the loops and mixed edges of G_2 left uncolored after Step 1. Let $\Delta(\mathcal{E})$ be a partition of $\mathcal{E}$ into subsets of size 2 each of which consists of either two mixed edges joining u_1 to u_2 or a loop on u_1 and another loop on u_2. Now partition $\Delta(\mathcal{E})$ into sets $E_1, \ldots, E_k$ such that $|E_j| = \frac{4n-2\eta(j)}{4}$ for each $1 \leqslant j \leqslant k$. Then color all the edges in E_j with color j for all $1 \leqslant j \leqslant k$.

Notice that the sum $\sum_{j=1}^{k} |E_j|$ counts the uncolored edges (including loops) in G_2 after Step 1 since

$$\begin{aligned}
\sum_{j=1}^{k} |E_j| &= \sum_{j=1}^{k} \frac{4n - 2\eta(j)}{4} \\
&= \sum_{j=1}^{k} n - \frac{1}{4} \sum_{j=1}^{k} 2\eta(j) \\
&= kn - \frac{1}{4} \sum_{c_j \in C_0} d_B(c_j) - \frac{1}{4} \sum_{c_j \in C_2} (d_B(c_j) + 2) \\
&= kn - \frac{1}{4} \sum_{j=1}^{k} d_B(c_j) - \frac{1}{4} \sum_{c_j \in C_2} 2 \\
&= kn - \frac{1}{4} \sum_{j=1}^{k} d_B(c_j) - \frac{1}{2} |C_2| \\
&= \frac{1}{2}(\lambda_1(n-1) + \lambda_2 n(r+1))n - \frac{1}{4}(2\lambda_2 n^2 r) - \frac{1}{2}|C_2| \text{ by } (i) \text{ and } (2.3) \\
&= \frac{1}{2}\left(2\lambda_1 \binom{n}{2} + \lambda_2 n^2 - |C_2|\right).
\end{aligned}$$

Therefore all the edges and loops in G_2 have been colored.

Let ψ be the function on $V(G_2)$ defined by $\psi(v) = 1$ for each $v \in V(G)$ and $\psi(u_i) = n$ for $1 \leqslant i \leqslant 2$. We now show that G_2 satisfies conditions $(1-5)$ of Theorem 2.2. Since $n \geqslant 2$ and v is loopless for each $v \in V(G)$, condition (1) is clearly satisfied. For all $v \in V(G)$, $d_{G_2(j)}(v) = 2$ by (2.2). For $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant 2$

$$\begin{aligned} d_{G_2(j)}(u_i) &= \eta(j) + 2|E_j| \text{ by } (2.9) \\ &= \eta(j) + 2\frac{4n - 2\eta(j)}{4} \\ &= 2n. \end{aligned}$$

So condition (2) is met. By the construction of G_2, $\ell_{G_2}(u_i) = \lambda_1\binom{n}{2}$ for $1 \leqslant i \leqslant 2$, so condition (3) is satisfied since $\psi(u_i) = n$. Notice that for all $v \in V(G)$ and $1 \leqslant i \leqslant 2$, $m_{G_2}(v, u_i) = \lambda_2 n$ by (2.3) and $m_{G_2}(u_1, u_2) = \lambda_2 n^2$ by the definition of G_2. Then condition (4) follows since for all $v_1, v_2 \in V(G)$ and for $1 \leqslant i \leqslant 2$, $\psi(v_1)\psi(v_2) = 1$, $\psi(v_1)\psi(u_i) = n$ and $\psi(u_1)\psi(u_2) = n^2$. By considering the k pairs of the edges in $\pi(\alpha)$, it is clear that for $1 \leqslant j \leqslant k$ the color class $G_2(j)$ is connected, since in the 2-edge-coloring β of the bipartite graph B, c_j is joined by two edges with different colors to vertices that correspond to the endpoints of a path forming a component in $G(j)$. So G_2 satisfies (5).

By Theorem 2.2, there exists a ψ-detachment G_3 of G_2,

$$\text{all of whose color classes are connected,} \tag{2.10}$$

in which the vertex u_i is detached into n new vertices $u_{i,1}, ..., u_{i,k}$ for $1 \leqslant i \leqslant 2$ such that:

(B_1) $m_{G_3}(u_{i,j}, u_{i,j'}) = \lambda_1\binom{n}{2}/\binom{n}{2} = \lambda_1$ for all $i \in \{1, 2\}$ and $j, j' \in \{1, \ldots, k\}$;

(B_2) $m_{G_3}(u_{1,j}, u_{2,j'}) = \lambda_2 n^2/n^2 = \lambda_2$ for all $j, j' \in \{1, \ldots, k\}$;

(B_3) $m_{G_3}(v, u_{i,j}) = \lambda_2 n/n = \lambda_2$ for all $v \in V(G)$, $i \in \{1, 2\}$, and $j \in \{1, \ldots, k\}$;

(B_4) $d_{G_3(j)}(u_{i,j'}) = 2n/n = 2$ for all $i \in \{1,2\}$ and $j, j' \in \{1,\dots,k\}$.

So for $1 \leqslant j \leqslant k$, $G_3(j)$ is an Hamilton cycle in G_3 (by (2.10) and (B_4)), which clearly contains $G(j)$ as a subgraph. Furthermore, G_3 is isomorphic to $G^* = K(n^{r+2}; \lambda_1, \lambda_2)$ (by $B_{(1-3)}$). Therefore the k-edge-coloring α of $K(n^r; \lambda_1, \lambda_2)$ is embedded into a Hamiltonian decomposition of $K(n^{r+2}; \lambda_1, \lambda_2)$, as required. □

We think that Condition $(\star)$ of Theorem 2.4 is in fact a necessary condition, so we make the following conjecture.

Conjecture 2.5. *Let $n > 1$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 1$ and $\lambda_1 \neq \lambda_2$. Let α be a k-edge-coloring of $G = K(n^r; \lambda_1, \lambda_2)$. If the k-edge-coloring α can be embedded into a Hamiltonian decomposition of $G^* = K(n^{r+2}; \lambda_1, \lambda_2)$, then $\pi(\alpha)$ can be chosen such that in the detached graph $B^* = B(\pi(\alpha))$, the number of the components having an odd number of color vertices of degree divisible by 4 is at most $\lambda_2 n^2$.*

Three corollaries of Theorem 2.4 are now presented. The first corollary shows this conjecture is true in the tightest case, namely when $|C_2|$ meets equality condition (v) in Theorem 2.4, thus completely settling the embedding problem in this case. The other two corollaries describe interesting cases where Condition $(\star)$ is clearly satisfied.

Corollary 2.6. *Let $n > 1$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 1$ and $\lambda_1 \neq \lambda_2$. Let α be a k-edge-coloring of $G = K(n^r; \lambda_1, \lambda_2)$ with*

$$|C_2| = 2\lambda_1 \binom{n}{2} + \lambda_2 n^2. \tag{\#}$$

Then α can be embedded into a Hamilton decomposition of $G^* = K(n^{r+2}; \lambda_1, \lambda_2)$ if and only if:

(i) $k = \frac{1}{2}(\lambda_1(n-1) + \lambda_2 n(r+1))$,

(ii) $\lambda_1 \leqslant \lambda_2 n(r+1)$,

(iii) *Each component of $G(j)$ is a path (possibly of length 0) for $1 \leqslant j \leqslant k$,*

(iv) $\omega_j \leqslant 2n$ *for* $1 \leqslant j \leqslant k$, *and*

(v) $\pi(\alpha)$ *can be chosen such that in the detached graph $B^* = B(\pi(\alpha))$, the number of the components having an odd number of color vertices of degree divisible by 4 is at most $\lambda_2 n^2$.*

Proof. The necessity of conditions (i)-(iv) follow from Proposition 2.3. So it remains to prove that condition (v) is necessary. For $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant k$, let $E(i, j)$ be the set of the edges in $G^*(j)$ joining the vertices in $V = V(G)$ to vertices in P_{r+i}. Let $c_j \in C_2$. If $G^*(j)$ has no edges in $G^*[P_{r+1} \cup P_{r+2}]$, then $|E(1, j)| = |E(2, j)| = 2n$ which contradicts $c_j \in C_2$. Therefore, by Condition (#),

$$\begin{aligned} &|E\big(G^*[P_{r+1} \cup P_{r+2}](j)\big)| = 1 \text{ for each } c_j \in C_2, \text{ and} \\ &|E\big(G^*[P_{r+1} \cup P_{r+2}](j)\big)| = 0 \text{ for each } c_j \in C_0. \end{aligned} \tag{2.11}$$

That is, for each $c_j \in C_2$, since $G^*[P_{r+1} \cup P_{r+2}](j)$ must have exactly one mixed or exactly one pure edge, it follows that $|E(1, j)| = |E(2, j)|$ or $\big||E(1, j)| - |E(2, j)|\big| = 2$ respectively. It also follows that

$$2n = |E(1, j)| = |E(2, j)| \text{ for each } c_j \in C_0. \tag{2.12}$$

The embedding of α into a Hamilton decomposition of G^* yields the 2-edge-coloring β of the bipartite graph $B = B(G, \alpha)$ with colors 1 and 2 defined as follows: an edge $\{v, c_j\}$ in B is colored with color i for $1 \leqslant i \leqslant 2$ if and only if the vertex v is joined to a vertex in P_{r+i} by an edge colored j. Let X be the subset of C consisting of all the color vertices in C_0 together with all the color vertices in C_2 for which $|E(1, j)| = |E(2, j)|$. Notice that

$$|X| = |C_0| + \lambda_2 n^2 \text{ and } |X \cap C_2| = \lambda_2 n^2. \tag{2.13}$$

Also, β has the following properties:

(d_1) $d_{B(1)}(v) = d_{B(2)}(v) = \lambda_2 n$ for all $v \in V$,

(d_2) $d_{B(1)}(c_j) = d_{B(2)}(c_j)$ for all $c_j \in X$, and

(d_3) $|d_{B(1)}(c_j) - d_{B(2)}(c_j)| = 2$ for all $c_j \in C_2 \backslash X$.

The 2-edge-coloring β satisfies (d_1) since $m_{G^*}(v, u_1) = m_{G^*}(v, u_2) = \lambda_2$ for all $v \in V$ and $u_i \in P_{r+i}$ ($1 \leqslant i \leqslant 2$); and ($ii$) and ($iii$) are satisfied by (2.11), (2.12), and (2.13).

Suppose $c_j \in C_2 \cap X$. Then as noted above, (2.11) implies that there is exactly one edge in $G^*[P_{r+1} \cup P_{r+2}](j)$, and it is mixed, so $|E(1, j)| = |E(2, j)| = 2n - 1$, which is clearly odd. So, since each component in $G(j)$ is a path (see (iii)), there is at least one component in $G(j)$ with exactly two vertices, say $v_{j,1}$ and $v_{j,2}$ (which are the ends of the path), such that $v_{j,i}$ is joined to a vertex in P_{r+i} for $1 \leqslant i \leqslant 2$. So the edge $\{v_{j,i}, c_j\}$ in B is colored i for $1 \leqslant i \leqslant 2$.

Now suppose $c_j \in C_0 \cup (C_2 \backslash X)$. Then $G^*[P_{r+1} \cup P_{r+2}](j)$ has no mixed edges. So, since $G^*(j)$ is connected, there is at least one component in $G(j)$ (see (iii)) with exactly two vertices, say $v_{j,1}$ and $v_{j,2}$ (which are the ends of the path), such that for $1 \leqslant i \leqslant 2$ $v_{j,i}$ is joined to a vertex in P_{r+i}. So the edge $\{v_{j,i}, c_j\}$ in B is colored i for $1 \leqslant i \leqslant 2$.

Therefore for each $c_j \in C$, define the set $\mathcal{C}(c_j) = \{\{c_j, v_{j,1}\}, \{c_j, v_{j,2}\}\}$ to be the set of these two edges in B, and let $\pi(\alpha) = \{\mathcal{C}(c_j) : 1 \leqslant j \leqslant k\}$. This can be used to define the 2-edge-colored detached graph $B^* = B(\pi(\alpha))$ derived from the 2-edge-colored bipartite graph B. So it remains to check that B^* satisfies condition (v). (Recall that since $d_{B^*}(c_{j,1}) = d_B(c_j) - 2$, $c_{j,1} \in C_0^*$ or C_2^* if and only if $c_j \in C_2$ or C_0 respectively.) Let $X^* = \{c_{j,1} : c_j \in X\}$. Then

(f_1) $d_{B^*(1)}(v) = d_{B^*(2)}(v) = \lambda_2 n$ for all $v \in V$,

(f_2) $d_{B^*(1)}(c_{j,2}) = d_{B^*(2)}(c_{j,2}) = 1$ for all $c_{j,2} \in C^*$,

(f_3) $d_{B^*(1)}(c_{j,1}) = d_{B^*(2)}(c_{j,1}) \equiv 1 \ (mod\ 2)$ for all $c_{j,1} \in C_2^*$,

(f_4) $d_{B*(1)}(c_{j,1}) = d_{B*(2)}(c_{j,1}) \equiv 0$ $(mod\ 2)$ for all $c_{j,1} \in X^* \cap C_0^*$,

(f_5) $d_{B*(1)}(c_{j,1}) \equiv d_{B*(2)}(c_{j,1}) \equiv 0$ $(mod\ 2)$ for all $c_{j,1} \in C_0^* \backslash X^*$, and

(f_6) $|d_{B*(1)}(c_{j,1}) - d_{B*(2)}(c_{j,1})| = 2$ for all $c_{j,1} \in C_0^* \backslash X^*$.

By (f_2), (f_3) and (f_4), each of the vertices $c_{j,2} \in C^*$ and $c_{j,1} \in C_2^* \cup (X^* \cap C_0^*)$ is incident with the same number of the edges colored 1 as the number of the edges colored 2 in each component of B^*. So each component of B^* must have an even number of vertices in $C_0^* \backslash X^*$ by (f_5) and (f_6) since it is guaranteed by (f_1) that each component of B^* has the same number of the edges colored 1 as the number of the edges colored 2. (That is, for each component D of B^*, half the vertices in $V(D) \cap \left(C_0^* \backslash X^*\right)$ are incident with 2 more edges colored 1 than 2; and the other half of these vertices are incident with 2 more edges colored 2 than 1.) So, for each component D of B^*, the number of color vertices of degree divisible by 4 in D is $|V(D) \cap (C_0^* \backslash X^*)| + |V(D) \cap (C_0^* \cap X^*)| \equiv |V(D) \cap \left(C_0^* \cap X^*\right)|$ $(mod\ 2)$. Therefore, condition (v) is necessary since $|C_0^* \cap X^*| = \lambda_2 n^2$ by (2.13).

The proof sufficency is identical with the proof of Theorem 2.4 except for that here condition (v) is used instead of condition $(\star)$.

□

The second corollary replaces the assumption $(\star)$ in Theorem 2.4 with a condition more clearly related to the given edge-coloring.

Corollary 2.7. *Let $G = K\left(n^r; \lambda_1, \lambda_2\right)$ with $n > 1$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 1$ and $\lambda_1 \neq \lambda_2$ and let α be a k-edge-coloring of G such that the number of the color classes $G(j)$ with an odd number of components is at most $\lambda_2 n^2$. Then α can be embedded into a Hamiltonian decomposition of $K\left(n^{r+2}; \lambda_1, \lambda_2\right)$ if and only if*

(i) $k = \frac{1}{2}\left(\lambda_1 (n-1) + \lambda_2 n (r+1)\right)$,

(ii) $\lambda_1 \leqslant \lambda_2 n (r+1)$,

(iii) *Each component of* $G(j)$ *is a path (possibly of length* 0*) for* $1 \leqslant j \leqslant k$*, and*

(iv) $\omega_j \leqslant 2n$ *for* $1 \leqslant j \leqslant k$*.*

Proof. In Theorem 2.4, since $d_B(c_j) \equiv 2\ (mod\ 4)$ if and only if $G(j)$ has an odd number of components, it is clear in this case that $|C_2|$ is postulated to be at most $\lambda_2 n^2$; so condition (v) of Theorem 2.4 is satisfied. This also implies that Condition $(\star)$ in Theorem 2.4 is always satisfied, regardless of the choice of $\mathcal{C}(c_j)$ for $1 \leqslant j \leqslant k$, as the following shows. Each vertex $c_{j,1}$ in B^* has degree divisible by 4 if and only if c_j has degree $2\ (mod\ 4)$ in B. So clearly the number of the components containing an odd number of vertices $c_{j,1}$ of degree divisible by 4 is at most $|C_2|$, which is itself at most $\lambda_2 n^2$ by assumption. This completes the proof. □

A case that may be of particular interest is where each color class is a Hamiltonian path. The following settles this problem in a more general setting.

Corollary 2.8. *Let* $G = K(n^r; \lambda_1, \lambda_2)$ *with* $n > 1$*,* $\lambda_1 \geqslant 0$*,* $\lambda_2 \geqslant 1$ *and* $\lambda_1 \neq \lambda_2$ *and let* α *be a* k*-edge-coloring of* G *such that each color class* $G(j)$ *either has an even number of components or is a Hamiltonian path. Then* α *can be embedded into a Hamiltonian decomposition of* $G^* = K(n^{r+2}; \lambda_1, \lambda_2)$ *if and only if*

(i) $k = \frac{1}{2}\left(\lambda_1 (n-1) + \lambda_2 n (r+1)\right)$,

(ii) $\lambda_1 \leqslant \lambda_2 n (r+1)$,

(iii) *Each component of* $G(j)$ *is a path (possibly of length* 0*) for* $1 \leqslant j \leqslant k$*,*

(iv) $\omega_j \leqslant 2n$ *for* $1 \leqslant j \leqslant k$*, and*

(v) $|C_2| \leqslant \lambda_2 n^2$.

Proof. Assume α is embedded into a Hamiltonian decomposition of G^*. Conditions $(i)-(iv)$ are necessary by Proposition 2.3. Since $G(j)$ is a subgraph of the Hamilton cycle $G^*(j)$, $G^*[P_{r+1} \cup P_{r+2}](j)$ is a Hamiltonian path in $G^*[P_{r+1} \cup P_{r+2}]$ so there is at least one mixed edge colored j in $G^*[P_{r+1} \cup P_{r+2}]$. Since there are exactly $\lambda_2 n^2$ mixed edges in $G^*[P_{r+1} \cup P_{r+2}]$, the necessity of (v) follows.

By assumption the only color classes with an odd number of components are the Hamiltonian paths, so Condition $(\star)$ in Theorem 2.4 is satisfied by the same reasoning used in the proof of Corollary 2.7. $\square$

Chapter 3

Maximal Set of Hamilton Cycles

3.1 History

Determining whether a graph is Hamiltonian (contains a Hamilton cycle) or not is one of the earliest problems in the history of Graph Theory (see [8]). Another interesting problem releted to determining if there are Hamilton cycles in a graph is to find a maximal set of edge-disjoint Hamilton cycles. In 1989, Hoffman, Rodger, and Rosa [12] determined that the spectrum for the maximal sets of edge-disjoint Hamilton cycles in K_r is $Sp(K_r) = \{\lfloor \frac{r+3}{4} \rfloor, \lfloor \frac{r+3}{4} \rfloor + 1, \ldots, \lfloor \frac{r-1}{2} \rfloor\}$. In 2000, Bryant, El-zanati, and Rodger [5] proved that there exists a maximal set of x edge-disjoint Hamilton cycle in $K_{n,n}$ if and only if $\frac{n}{4} < x \leqslant \frac{n}{2}$.

In 2002, Daven, MacDougall, and Rodger [6] solved the existence problem of a maximal set of x edge-disjoint Hamilton cycles in the complete multipartite graph $K(n^r) = K(n^r; 0, 1)$ except for the smallest value of x in the case $n \equiv r \equiv 1$ $(mod\ 2)$ by showing in all other cases that there exists a maximal set of x edge-disjoint Hamilton cycles in $K(n^r)$ if and only if: $\lceil \frac{n(r-1)}{4} \rceil \leqslant x \leqslant \lfloor \frac{n(r-1)}{2} \rfloor$; and $x > \frac{n(r-1)}{4}$ if either n is odd and $r \equiv 1$ $(mod\ 4)$ or $p = 2$ and $n = 1$. In 2005, Logan and Rodger [14] solved the existence problem in the case where r is odd and $n = 3$ by proving that if $\lceil \frac{n(r-1)}{4} \rceil + 1 \leqslant x \leqslant \lfloor \frac{(n+1)(r-1)-2}{4} \rfloor$ when $r \equiv 3$ $(mod\ 4)$ or $\lceil \frac{n(r-1)}{4} \rceil + 1 < x \leqslant \lfloor \frac{(n+1)(r-1)-2}{4} \rfloor$ when $r \equiv 1$ $(mod\ 4)$, then there exists a maximal sets of x edge-disjoint Hamilton cycles in $K(n^r)$. In [16], Noble and Rodger proved that if $r \equiv 1$ $(mod\ 4)$ and $n = 3$, then there exists a maximal set of $\lceil \frac{3(r-1)}{4} \rceil$ edge-disjoint Hamilton cycles in $K(n^r)$.

In Theorem 3.7 of this chapter, these results are extended to multipartite graphs with 2 associate classes. Using the result in [12], here it is proved that if $\lambda_2 n \lfloor \frac{r+3}{4} \rfloor \leqslant t$, then there exists a maximal set of t edge-disjoint Hamilton cycles in $K(n^r; \lambda_1, \lambda_2)$ if and only if

$t \leqslant min\{\lfloor\frac{\lambda_2 n^2(r-1)}{2}\rfloor, \lfloor\frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2}\rfloor\}$ by using the method of amalgamtions. It is still an open problem whether or not $\lambda_2 n\lfloor\frac{r+3}{4}\rfloor$ is a lower bound for the spectrum $Sp(K(n^r;\lambda_1,\lambda_2))$ (see Problem 5).

3.2 A Useful Result

Theorem 3.1 ([15], Walecki Construction)**.** $\lambda K_r = K(1^r; 0, \lambda)$ *has a Hamilton decomposition (or a Hamilton decomposition with a* 1*-factor leave) if and only if* $\lambda(r-1)$ *is even (or odd, respectively).*

Let K_{2s} be the complete graph with the vertex set $V = \{v_j : j \in \mathbb{Z}_{2s}\}$, and let P be the Hamilton path $(e_1, e_2, \ldots, e_{2s-1})$ in the complete graph K_{2s} whose vertices are ordered as follows: $v_0, v_{2s-1}, v_1, v_{2s-2}, v_2, \ldots, v_{s-1}, v_s$. For $0 \leqslant i \leqslant s-1$, let P_i be the Hamilton path in K_{2s} obtained from P as follows:

$$e + i = \{v_{j+i}, v_{j'+i}\} \in E(P_i) \text{ if and only if } e = \{v_j, v_{j'}\} \in E(P)$$

(reducing the sum in the subscript modulo $2s$). Then $P_0, ..., P_{s-1}$ are edge-disjoint, and so form a Hamilton path decomposition of K_{2s} made using the Walecki Construction (a companion result to Theorem 3.1). Here, we can make effective use of this structure. For $1 \leqslant c \leqslant 2s$, define $q = \lfloor\frac{2s-c}{2}\rfloor$. For $0 \leqslant i \leqslant s-1$, let $P_{i,1}(q)$ and $P_{i,2}(q)$ be the two subpaths $(e_1 + i, e_2 + i \ldots, e_q + i)$ and $(e_{2s-q} + i, e_{2s-q+1} + i, \ldots, e_{2s-1} + i)$ of length q in P_i respectively. For $0 \leqslant i \leqslant s-1$, $P_{i,1}(q)$ and $P_{i,2}(q)$ are edge-disjoint simply because: they consist of the first and last q edges of P_i respectively; the length of P_i is $2s-1$; and $2s-1-2q \geqslant 0$. Notice that for $0 \leqslant i \leqslant s-1$, c is the number of the components in $P_{i,1}(q) \cup P_{i,2}(q)$ or $P_{i,1}(q) \cup P_{i,2}(q) \cup \{e_s + i\}$ if c is even or odd, respectively.

Let $Q^{(c)}(2s)$ be the graph with the vertex set V and edge set $E = E(Q^{(c)}(2s))$ defined as follows: if c is even and $2s - c = 2q \geqslant 2$,

$$E(Q^{(c)}(2s)) = \bigcup_{i=0}^{s-1} \big(E(P_{i,1}(q)) \cup E(P_{i,2}(q))\big),$$

and if c is odd and $2s - c = 2q + 1 > 1$, then

$$E(Q^{(c)}(2s)) = E(Q^{(c+1)}(2s)) \cup \{e_s + i : i = 0, 1, \ldots, s - 1\}.$$

For $1 < c < 2s$, define an s-edge-coloring of $Q^{(c)}(2s)$ as follows:

- if c is even, then color all the edges in $E(P_{i,1}(q)) \cup E(P_{i,2}(q))$ with color i for $0 \leqslant i \leqslant s - 1$, and
- if c is odd, then color the edge $e_s + i$ and all the edges in $E(P_{i,1}(q)) \cup E(P_{i,2}(q))$ in the graph $Q^{(c+1)}(2s)$ with color i for $0 \leqslant i \leqslant s - 1$.

Regardless of whether c is even or odd, $Q^{(c)}(2s)$ is regular of degree $2s - c$ and each color class of the edge-coloring has $2s - c$ edges. So for $1 \leqslant c \leqslant 2s$,

$$Q^{(c)}(2s) \text{ satisfies the conditions required to apply Lemma 3.3.} \tag{3.1}$$

Lemma 3.2. *If c is even, then the complement of $Q^{(c)}(2s)$ in K_{2s} contains a 1-factor.*

Proof. The result is clear in the case $c = 2s$, since the graph $Q^{(2s)}(2s)$ has no edges; in other words,

$$\overline{Q^{(2s)}(2s)} = K_{2s}. \tag{3.2}$$

Now let $c = 2m$ with $1 \leqslant m < s$. For each $e = \{v_j, v_{j'}\} \in E = E(Q^{(2m)}(2s))$, the edge-difference $D(e)$ is positive since $Q^{(2m)}(2s)$ is a loopless graph. It is also obvious that for all $e \in E$ and for $0 \leqslant i \leqslant s-1$,

$$D(e+i) = D(e) \tag{3.3}$$

since for all $e = \{v_j, v_{j'}\}$ in E by considering $j > j'$ without loss of generality,

$$\begin{aligned} D(e+i) =& D(\{v_{j+i}, v_{j'+i}\}) \\ =& min\{(j+i)-(j'+i), 2s-((j+i)-(j'+i))\} \\ =& min\{j-j', 2s-(j-j')\} \\ =& D(\{v_j, v_{j'}\}) \\ =& D(e). \end{aligned}$$

By (3.3), after now we consider only the case $i = 0$ for convenience.

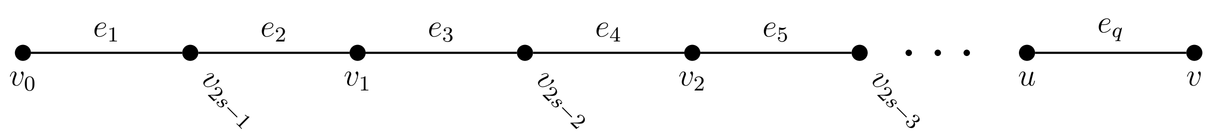


Figure 3.1: The Path $P_{0,1}(q)$ in $Q^{(2m)}(2s)$.

For $1 \leqslant j \leqslant q$, let e_j be an edge in $P_{0,1}(q)$ (see Figure 3.1), so

$$\begin{aligned} e_j &= \left\{v_{\frac{j-1}{2}}, v_{2s-\frac{j+1}{2}}\right\} && \text{if } j \text{ is odd, and} \\ e_j &= \left\{v_{2s-\frac{j}{2}}, v_{\frac{j}{2}}\right\} && \text{if } j \text{ is even.} \end{aligned}$$

Then for $1 \leqslant j \leqslant q$,

$$D(e_j) = min\{2s-j, j\} = j \tag{3.4}$$

since $q = s-m$ and $m < s$.

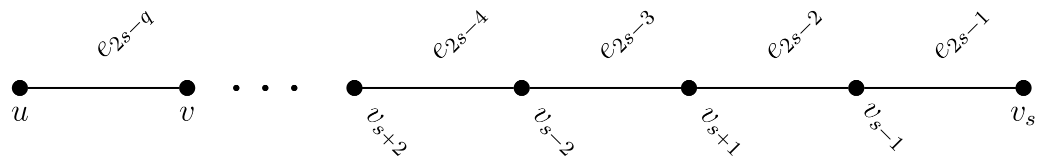


Figure 3.2: The Path $P_{0,2}(q)$ in $Q^{(2m)}(2s)$.

For $1 \leqslant j \leqslant q$, let e_{2s-j} be an edge in $P_{0,2}(q)$ (see Figure 3.2), so

$$\begin{aligned} e_{2s-j} &= \left\{v_{s+\frac{j-1}{2}}, v_{s-\frac{j+1}{2}}\right\} && \text{if } j \text{ is odd, and} \\ e_{2s-j} &= \left\{v_{s-\frac{j}{2}}, v_{s+\frac{j}{2}}\right\} && \text{if } j \text{ is even.} \end{aligned}$$

Then for $1 \leqslant j \leqslant q$,

$$D(e_{2s-j}) = min\{2s-j, j\} = j \tag{3.5}$$

since $q = s - m$ and $m < s$. Now, notice that the differences of the edges in $P_{0,1}(q)$ increase from 1 to q through the path $P_{0,1}(q)$ by (3.4) while the differences of the edges in $P_{0,2}(q)$ decrease from q to 1 through the path $P_{0,2}(q)$ by (3.5). So for $0 \leqslant i \leqslant s-1$ and $1 \leqslant j \leqslant 2s-1$ it is always the case that

$$1 \leqslant D(e_j + i) \leqslant s - m. \tag{3.6}$$

Therefore, by (3.2) and (3.6), all the edges of half-difference of K_{2s} are in the complement of the graph $Q^{(2m)}(2s)$ which induce a 1-factor. $\square$

As stated in (3.1), let $G = Q^{(c)}(2s)$ in the following lemma which was proved in [12].

Lemma 3.3 ([12]). *Suppose $1 \leqslant c \leqslant 2s$. Let G be a graph with $2s$ vertices that is regular of degree $2s-c$ for which there exists an s- edge-coloring in which each color class consists of $2s-c$ edges that induce a subgraph of G consisting of vertex disjoint paths. Then $\overline{K_c} \vee G$ has a Hamilton decomposition.*

Lemma 3.3 was then used in [12] to prove the following theorem.

Theorem 3.4 ([12]). *For $r \geqslant 3$, there exists a maximal set of x edge-disjoint Hamilton cycles in K_r if and only if*

$$x \in Sp(K_r) = \left\{ \left\lfloor \frac{r+3}{4} \right\rfloor, \left\lfloor \frac{r+3}{4} \right\rfloor + 1, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor \right\}.$$

In the following proposition, (a_1) is explicitly stated at the end of the proof of Lemma 3.4 in [12]. However, (a_2) is a new result we prove here.

Proposition 3.5. *For each $x \in Sp(K_r)$, there exists a maximal set M_x of x Hamilton cycles in K_r such that*

(a_1) *$K_r - E(M_x)$ is disconnected, and*

(a_2) *if r is even, then $K_r - E(M_x)$ contains a 1-factor.*

Proof. For each $x \in Sp(K_r)$, if $x = (r-1)/2$ (so r is odd), then define $H_x = K_r$; otherwise, let $H_x = \overline{K}_{r-2x} \vee Q^{(r-2x)}(2x)$. Notice that for each $x \in Sp(K_r)$, H_x is a spanning subgraph of K_r and it can be decomposed into x Hamilton cycles by Theorem 3.1 or Lemma 3.3 respectively. Then let M_x be the set of the Hamilton cycles in any Hamilton decomposition of H_x. Since $\overline{H_x} = K_r - E(M_x)$ is obviously disconnected, M_x is a maximal set of x edge-disjoint Hamilton cycles in K_r.

To prove (a_2), let r be even. As defined above, $H_x = \overline{K_{r-2x}} \vee Q^{(r-2x)}(2x)$ where K_{r-2x} and $Q^{(r-2x)}(2x)$ are vertex disjoint subgraphs of K_r. Then the complement of H_x in K_r is the graph $\overline{H_x} = K_{r-2x} \cup \overline{Q^{(r-2x)}(2x)}$. Notice that K_{r-2x} has a 1-factor simply because it has an even number of vertices. Also the complement $\overline{Q^{(r-2x)}(2x)}$ has a 1-factor by Lemma 3.2 since $r - 2x$ is even. So (a_2) follows. $\square$

The following theorem, which is also known as Petersen's Theorem in literature, is very important for the proof of the main result of this chapter, Theorem 3.7.

Theorem 3.6 ([17]). *Every regular graph of even degree has a 2-factorization.*

3.3 Main Result

If S is a set of Hamilton cycles in any graph G, then let $E(S) = \bigcup_{C \in S} E(C)$. Define the edge-cut $[V_1, \overline{V_1}]_G = \{e : e = \{v_1, v_2\} \in E(G), v_1 \in V_1, v_2 \in \overline{V_1}\}$ for a subset $V_1 \subseteq V(G)$. Now we are ready to prove the main result of this chapter.

Theorem 3.7. *Let $G = K(n^r, \lambda_1, \lambda_2)$ with $n > 1$, $r > 2$, $\lambda_1 \geqslant 1$, $\lambda_2 \geqslant 1$, and $\lambda_1 \neq \lambda_2$. Assume that $t \geqslant \lambda_2 n \lfloor \frac{r+3}{4} \rfloor$. Then there exists a maximal set of t edge-disjoint Hamilton cycles in G if and only if:*

$$t \leqslant min\{\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor, \lfloor \frac{\lambda_1(n-1) + \lambda_2 n(r-1)}{2} \rfloor\}. \tag{3.7}$$

Proof. To prove the necessity, let M be a maximal set of t edge disjoint Hamilton cycles in G. For all $v \in V$, $d_G(v) = \lambda_1(n-1) + \lambda_2 n(r-1)$. So $t \leqslant \lfloor \frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2} \rfloor$. Furthermore, there must be at least r mixed edges in a Hamilton cycles in the graph G simply because every Hamilton cycle is connected (see Figure 3.3).

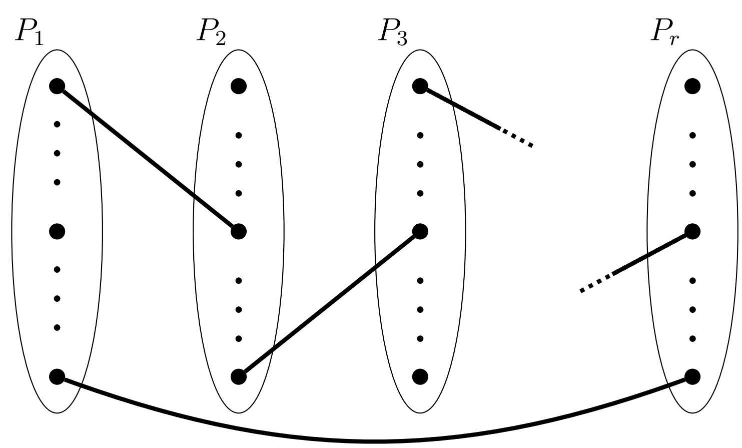


Figure 3.3: The Parts of $G = K(n^r, \lambda_1, \lambda_2)$.

Then

$$rt \leqslant \lambda_2 n^2 \binom{r}{2}, \text{ so}$$

$$t \leqslant \lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor.$$

So Condition (3.7) is necessary. In what follows, we give the proof of the sufficiency.

For each $x \in Sp(K_r)$, let M_x be a maximal set of x edge-disjoint Hamilton cycles in K_r on the vertex set V which is obtained as described in Proposition 3.5, and let $S(\lambda_2 n^2 x) = \{C_1, \ldots, C_{\lambda_2 n^2 x}\}$ be the set of Hamilton cycles in $\lambda_2 n^2 K_r$ formed by exactly $\lambda_2 n^2$ copies of each Hamilton cycle in M_x. Then $K_r - E(M_x)$ is disconnected by Proposition 3.5 (a_1), so K_r has an edge-cut $[V_1, V_2]_{K_r} \subseteq E(M_x)$. Clearly each $e \in [V_1, V_2]_{K_r}$ appears in exactly $\lambda_2 n^2$ Hamilton cycles in $S(\lambda_2 n^2 x)$, so $[V_1, V_2]_{\lambda_2 n^2 K_r}$ is an edge-cut of $\lambda_2 n^2 K_r$ which is a subset of $E(S(\lambda_2 n^2 x))$. So the set $S(\lambda_2 n^2 x)$ is a maximal set of Hamilton cycles in $\lambda_2 n^2 K_r$.

By Theorem 3.1, $\lambda_2 n^2 K_r$ can be decomposed into $\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor$ Hamilton cycles or $\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor$ Hamilton cycles and one 1-factor if $\lambda_2 n^2 (r-1)$ is even or odd, respectively. Notice that if r is odd, then $\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor = \lambda_2 n^2 \lfloor \frac{r-1}{2} \rfloor$; so let $S(\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor) = S(\lambda_2 n^2 \lfloor \frac{r-1}{2} \rfloor)$ as defined in the last paragraph. But if r is even then $\lambda_2 n^2 \lfloor \frac{r-1}{2} \rfloor < \lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor$; so in this case define an additional set $S(\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor)$ to be the set of all the edge-disjoint Hamilton cycles in $\lambda_2 n^2 K_r$ obtained by Theorem 3.1. So when r is even, the set $S(\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor)$ is a maximal set of Hamilton cycles $\lambda_2 n^2 K_r$ since $\lambda_2 n^2 K_r - E(S(\lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor))$ is either a null graph or a 1-factor if $\lambda_2 n^2 (r-1)$ is even or odd, respectively.

If t is in the range $\lambda_2 n \lfloor \frac{r+3}{4} \rfloor \leqslant t \leqslant \lambda_2 n \lfloor \frac{r-1}{2} \rfloor$, then there exists an element $x \in Sp(K_r)$ such that $\lambda_2 n x \leqslant t \leqslant \lambda_2 n^2 x$ since

$$\begin{aligned}
& 2 \leqslant && x \\
\Leftrightarrow\quad & \tfrac{n}{n-1} \leqslant && x \\
\Leftrightarrow\quad & n \leqslant && (n-1)x \\
\Leftrightarrow\quad & \lambda_2 n^2 \leqslant && \lambda_2 n(n-1)x \\
\Leftrightarrow\quad & \lambda_2 nx \leqslant && \lambda_2 n^2 x - \lambda_2 n^2 \\
\Leftrightarrow\quad & \lambda_2 nx \leqslant && \lambda_2 n^2(x-1);
\end{aligned}$$

so greedily form a partition $S(t) = \{S_1, \dots, S_t\}$ of the set $S(\lambda_2 n^2 x)$ in which $1 \leqslant |S_i| = s_i \leqslant n$ for $1 \leqslant i \leqslant t$. This is possible simply because $|S(\lambda_2 n^2 x)| = \lambda_2 n^2 x$, so at the extremes of t in the range $\lambda_2 nx \leqslant t \leqslant \lambda_2 n^2 x$, either all sets in $S(t)$ have size n or they have size 1.

Similarly if t is in the range $\lambda_2 n \lfloor \frac{r-1}{2} \rfloor < t \leqslant min\{\lfloor \frac{\lambda_2 n^2(r-1)}{2} \rfloor, \lfloor \frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2} \rfloor\}$, then greedily form a partition $S(t) = \{S_1, \dots, S_t\}$ of the set $S(\lfloor \frac{\lambda_2 n^2(r-1)}{2} \rfloor)$ in which $1 \leqslant |S_i| = s_i \leqslant n$ for $1 \leqslant i \leqslant t$. Notice that r must be even for this range of t existing. So such a partition $S(t)$ of the set $S(\lfloor \frac{\lambda_2 n^2(r-1)}{2} \rfloor)$ exists since if $\lambda_2 n$ is even, then

$$\begin{aligned}
& 2 \leqslant && n \\
\Leftrightarrow\quad & \tfrac{r-1}{r-2} \leqslant && n \\
\Leftrightarrow\quad & n \leqslant && (n-1)x \\
\Leftrightarrow\quad & \tfrac{\lambda_2 n^2(r-1)}{2} \leqslant && \lambda_2 n^3 \lfloor \tfrac{r-1}{2} \rfloor \\
\Leftrightarrow\quad & \frac{\lfloor \frac{\lambda_2 n^2(r-1)}{2} \rfloor}{n} \leqslant && \lambda_2 n^2 \lfloor \tfrac{r-1}{2} \rfloor,
\end{aligned}$$

and if $\lambda_2 n$ is odd, then

$$\begin{aligned}
& 2 \leqslant && n \\
\Leftrightarrow\quad & \tfrac{r-1}{r-2} \leqslant && n \\
\Leftrightarrow\quad & 0 \leqslant && n(r-2) - (r-1), \text{ so} \\
& 0 \leqslant && \lambda_2 n^2(n(r-2) - (r-1)) + 1 \\
\Leftrightarrow\quad & \tfrac{\lambda_2 n^2(r-1)-1}{2} \leqslant && \lambda_2 n^3 \tfrac{r-2}{2} \\
\Leftrightarrow\quad & \frac{\lfloor \frac{\lambda_2 n^2(r-1)}{2} \rfloor}{n} \leqslant && \lambda_2 n^2 \lfloor \tfrac{r-1}{2} \rfloor.
\end{aligned}$$

For $\lambda_2 n \lfloor \frac{r+3}{4} \rfloor \leqslant t \leqslant min\{\lfloor \frac{\lambda_2 n^2(r-1)}{2} \rfloor, \lfloor \frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2} \rfloor\}$, let $S^1(t) = \{S^1_1, \dots, S^1_t\}$ in which each S^1_i is the union of the Hamilton cycles in S_i; that is,

$$S_i^1 = \bigcup_{C \in S_i} C.$$

Notice that for $1 \leqslant i \leqslant t$, S_i^1 is both connected and regular of degree $2s_i \leqslant 2n$ since it is the union of s_i Hamilton cycles.

Now let $G_1(t)$ be the union of the graphs in the set $S^1(t)$; that is,

$$G_1(t) = \bigcup_{i=1}^{t} S_i^1.$$

So $G_1(t)$ is both connected and regular of even degree. Then define

$$\begin{aligned}\overline{G_1(t)} =& \lambda_2 n^2 K_r - E(G_1(t)) \\ =& \lambda_2 n^2 K_r - E(S(t))\end{aligned}$$

to be the complement of $G_1(t)$ in the multigraph $\lambda_2 n^2 K_r$. So $\overline{G}_1(t)$ is disconnected and regular.

If $\overline{G_1(t)}$ is regular of even degree, then it has a 2-factorization $\mathcal{F}$ by Theorem 3.6. Now suppose that $\overline{G_1(t)}$ is regular of odd degree. Then $\lambda_2 n^2(r-1) \equiv 1$ $(mod\ 2)$. This is possible if and only if $\lambda_2 \equiv n \equiv r-1 \equiv 1$ $(mod\ 2)$. So r must be even. If $\lambda_2 n \lfloor \frac{r+3}{4} \rfloor \leqslant t \leqslant \lambda_2 n^2 \lfloor \frac{r-1}{2} \rfloor$, then there exists an element $x \in Sp(K_r)$ such that $\lambda_2 n x \leqslant t \leqslant \lambda_2 n^2 x$; so $\overline{G_1(t)}$ can be decomposed to a 2-factorization $\mathcal{F}$ and a 1-factor F by Theorem 3.6 and Proposition 3.5 (a_2) since $\overline{G_1(t)}$ is the union of $\lambda_2 n^2$ copies of $K_r - E(M_x)$. If $\lambda_2 n^2 \lfloor \frac{r-1}{2} \rfloor < t \leqslant \lfloor \frac{\lambda_2 n^2 (r-1)}{2} \rfloor$, then $\overline{G_1(t)}$ can be decomposed to a 2-factorization $\mathcal{F}$ and a 1-factor F due to the Walecki construction.

Let G' and $G_2(t)$ be the graphs obtained from $\lambda_2 n^2 K_r$ and $G_1(t)$, respectively by adding exactly $\lambda_1 \binom{n}{2}$ loops on each vertex. Notice that $G_2(t)$ is a spanning subgraph of G' simply because $G_1(t)$ is a spanning subgraph of $\lambda_2 n^2 K_r$.

Define a partition $\mathcal{L} = \{L_1, \ldots, L_{\lambda_1\binom{n}{2}}\}$ of the set of the $\lambda_1\binom{n}{2}r$ loops in G' such that for $1 \leqslant i \leqslant \lambda_1\binom{n}{2}$, each vertex in G' is incident with exactly one loop in L_i; so $|L_i| = r$.

Now let $\mathcal{E}(t) = \{E_1, \ldots, E_t\}$ be a partition of a subset of the union $\mathcal{F} \cup \mathcal{L}$ in which each set E_i has size $n - s_i$. For $\lambda_2 n\lfloor\frac{r+3}{4}\rfloor \leqslant t \leqslant min\{\lfloor\frac{\lambda_2 n^2(r-1)}{2}\rfloor, \lfloor\frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2}\rfloor\}$, such a partition $\mathcal{E}(t)$ exists since

$$\begin{aligned}
t \leqslant{}& \lfloor\frac{\lambda_1(n-1)+\lambda_2 n(r-1)}{2}\rfloor \text{ by Condition (3.7)}\\
\leqslant{}& \tfrac{1}{n}\lfloor\frac{\lambda_1 n(n-1)+\lambda_2 n^2(r-1)}{2}\rfloor\\
\Leftrightarrow \quad nt \leqslant{}& \lfloor\frac{\lambda_1 n(n-1)+\lambda_2 n^2(r-1)}{2}\rfloor\\
\Leftrightarrow \quad nt \leqslant{}& \lfloor\frac{\lambda_2 n^2(r-1)-2|E(S(t))|}{2}\rfloor + \frac{\lambda_1 n(n-1)}{2} + |E(S(t))|\\
\Leftrightarrow \quad nt - |E(S(t))| \leqslant{}& \lfloor\frac{\lambda_2 n^2(r-1)-2|E(S(t))|}{2}\rfloor + \frac{\lambda_1 n(n-1)}{2}\\
\Leftrightarrow \quad nt - \sum_{i=1}^{t} s_i \leqslant{}& \lfloor\frac{\lambda_2 n^2(r-1)-2|E(S(t))|}{2}\rfloor + \lambda_1\binom{n}{2}\\
\Leftrightarrow \quad \sum_{i=1}^{t} n - s_i \leqslant{}& \lfloor\frac{\lambda_2 n^2(r-1)-2|E(S(t))|}{2}\rfloor + \lambda_1\binom{n}{2}\\
\Leftrightarrow \quad \textstyle\sum_{i=1}^{t}|E_i| \leqslant{}& |\mathcal{F}| + |\mathcal{L}|.
\end{aligned}$$

Now define a $(t+1)$-edge-coloring of G' by coloring all the edges in S_i^1 and in E_i with color i, and color all the other edges in G' with color $t+1$. For $1 \leqslant i \leqslant t+1$, let S_i^2 denote the i^{th} color class in the $(t+1)$edge-colored graph G'. Notice that for $1 \leqslant i \leqslant t$, S_i^2 is regular of degree $2n$ since for all $v \in V$,

$$\begin{aligned}
d_{S_i^2}(v) ={}& d_{S_i^1}(v) + 2|E_i|\\
={}& 2s_i + 2(n - s_i)\\
={}& 2n
\end{aligned}$$

Also notice that

$$\text{the color class } S_i^2 \text{ is connected for } 1 \leqslant i \leqslant t \tag{3.8}$$

since it contains S_i^1 which is the union of s_i Hamilton cycles. However,

$$\text{the color class } S_{t+1}^2 \text{ is disconnected by Proposisition 3.5 } (a_1). \tag{3.9}$$

Let ψ be a function from the vertex set V into $\mathbb{N}$ defined by $\psi(v) = n$ for all $v \in V$. We now show that ψ satisfies the conditions $(1) - (5)$ of Theorem 2.2. Condition (1) is satisfied simply because $n \geqslant 2$. For all $v \in V$, $d_{G_2}(v) = 2nt$, so $d_{G_2}(v)/\psi(v) = 2t$. Then condition (2) is satisfied. For all $v \in V$, $\binom{\psi(v)}{2} = \binom{n}{2}$, so $\binom{\psi(v)}{2}$ divides $\ell_{G_2}(v) = \lambda_1\binom{n}{2}$. So condition (3) is satisfied. For all $v_j, v_{j'} \in V$, $\psi(v_j)\psi(v_{j'}) = n^2$ divides $m_{G_2}(v_j, v_{j'}) = \lambda_2 n^2$. So condition (4) is satisfied. By (3.8) and (3.9), condition (5) is satisfied for all the color classes $S_1^2, \ldots, S_t^2$ but for S_{t+1}^2. In Theorem 2.2, the condition (5) is needed only for (iv), so if a color class $G(j)$ is disconnected then definitely the color class $H(j)$ is also disconnected simply because $|E(G(j))| = |E(H(j))|$ while $|V(H)| \geqslant |V(G)|$. So by Theorem 2.2, there exists a ψ-detachement G'' of the graph G', in which the vertices $v_j, v_{j'} \in V$ are detached into n new vetices $v_{j,1}, \ldots, v_{j,n}$ and $v_{j',1}, \ldots, v_{j',n}$ such that:

(g_1) $d_{G''(i)}(v_{j,\tau}) = 2n/n = 2$ for $1 \leqslant i \leqslant t$ and $1 \leqslant \tau \leqslant n$,

(g_2) $m_{G''}(v_{j,\tau}, v_{j,\tau'}) = \lambda_1\binom{n}{2}/\binom{n}{2} = \lambda_1$ for $1 \leqslant \tau < \tau' \leqslant n$, and

(g_3) $m_{G''}(v_{j,\tau}, v_{j',\tau'}) = \lambda_2 n^2/n^2 = \lambda_2$ for $1 \leqslant \tau \leqslant \tau' \leqslant n$.

Now notice that G'' is isomorphic to the graph $G = K(n^r; \lambda_1, \lambda_2)$ by (g_2) and (g_3). So let S_i^3 be the i^{th} color class in the $(t+1)$-edge-colored graph G induced from the edges in the color class $S^2(t)$. Then for $1 \leqslant i \leqslant t$, the color class S_i^3 is a Hamilton cycle in G by (3.8) and (g_1). Let $S^3(t) = \{S_1^3, \ldots, S_t^3\}$. Then $S^3(t)$ is a maximal set of Hamilton cycles in G since $G - E(S^3(t)) = S_{t+1}^3$ is disconnected by (3.9). Therefore, the proof of the sufficiency follows.

□

Chapter 4

Future Directions

4.1 Some Open Problems

In this dissertation, Chapter 2 focusses on the embedding problem: when can a edge-coloring of $K(n^r;\lambda_1,\lambda_2)$ be embedded into a Hamiltonian decompostion of $K(n^{r+2};\lambda_1,\lambda_2)$? The embedding problem has not been settled completely for quite some time. So solutions of the following open questions are of great interest to those working on these types of problems.

Problem 1. *Prove Conjecture conjecture 2.5.*

For convenience, this conjecture is restated here.

Conjecture 2.5. *Let $n>1$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 1$ and $\lambda_1 \neq \lambda_2$. Let α be a k-edge-coloring of $G = K(n^r;\lambda_1,\lambda_2)$. If the k-edge-coloring α can be embedded into a Hamiltonian decomposition of $G^* = K(n^{r+2};\lambda_1,\lambda_2)$, then $\pi(\alpha)$ can be chosen such that in the detached graph $B^* = B(\pi(\alpha))$, the number of the components having an odd number of color vertices of degree divisible by 4 is at most $\lambda_2 n^2$.*

Perhaps this is currently intractible, but it would still be of interest to settle the conjecture with some additional restrictions, such as the following problem.

Problem 2. *Show that Conjecture conjecture 2.5 is true if $|C_2| > \gamma$ for some particular $\gamma < 2\lambda_1\binom{n}{2} + \lambda_2 n^2$.*

The following problem also appears to be very difficult.

Problem 3. *Solve the embedding problem in case $t = 3$.*

Problem 4. *What can be the largest value of t in the embedding problem in order for numerical conditions to be suffcient for the embedding (i.e., until the components are not an issue in the embedding problem)?*

In Chapter 3, maximal sets of t edge-disjoint Hamilton cycles in $K(n^r; \lambda_1, \lambda_2)$ is studied. The following problem is left open:

Problem 5. *Prove whether or not $\lambda_2 n \lfloor \frac{r+3}{4} \rfloor$ is a lower bound for the spectrum $Sp(K(n^r; \lambda_1, \lambda_2))$.*

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