Rainbow Cycle Forbidding Edge Colorings

by

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Abstract

It is well known that K_n can be edge colored using $n-1$ colors in order to avoid rainbow cycles; moreover, this is the maximum number of colors possible. We call such an edge coloring a JL-coloring. In previous work it has been shown that essentially different JL-colorings of K_n are in one-to-one correspondence with (can be encoded by) isomorphism classes of labeled full binary trees on n leafs. Later, this result was shown to be true for complete bipartite graphs as well, with a slight modification to the encoding. In this dissertation, we first show this correspondence extends to all complete multipartite graphs in chapter 2. In chapter 3 we show that if a connected graph G on n vertices is edge colored with $n-1$ colors and this coloring avoids rainbow cycles, then G has a monochromatic edge cut. It follows that the results from chapter 2 extend to all connected graphs. We also state some results on the sharpness of this result: specifically, what can we say about the number of colors used in an edge coloring that forbids rainbow cycles and monochromatic cuts, and what is the structure of such colorings.

In the last chapter we discuss proper edge colorings which avoid rainbow cycles. In particular, we give several results related to the classification of such edge colorings, including: $G \,\Box\, H$ is properly JL-colorable when both G and H are JL-colorable.

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Chapter 1

Introduction

If G is an edge colored graph, we say H is a **rainbow** subgraph of G if no two edges of H bear the same color. We say that rainbow subgraphs from a certain class of graphs are **forbidden** if every subgraph of G isomorphic to some graph in this class is not rainbow. In particular, an edge coloring of a graph G is said to be **rainbow-cycle-forbidding** (or **RCF** for short) if every cycle in G is not rainbow. If G is a multigraph with an RCF edge coloring, then necessarily G is loopless and for all $u, v \in V(G)$, the edges incident to both u and v must bear the same color. Therefore, we may as well consider only RCF colorings of simple graphs.

It is well known, but also highlighted in [8], that the maximum number of colors that can be used for an n-vertex RCF edge colored connected graph is $n-1$: for any edge coloring using n colors, consider the subgraph induced by n edges picked so that each edge bears a different color. This subgraph must contain a cycle and that cycle is necessarily rainbow.

Therefore, an RCF edge colored graph with c components contains at most $n - c$ colors. For a graph with n vertices and c components, it is always possible to construct RCF edge colorings on the graph using $n - c$ colors. The construction for such a coloring on complete graphs and complete bipartite graphs is given in section 1.1, on complete multipartite graphs in section 2.2, and for general connected graphs in section 3.1. We call an RCF coloring using this maximum possible number of colors a **JL-coloring** (we use **JL** in this nomenclature in honor of Hungarian mathematician Jenő Lehel).

A Ramsey problem in graph theory is about coloring the edges of K_n with as few colors as necessary so that no subgraph of a specified class is monochromatic. Similary, an anti-Ramsey problem is one where we maximize the number of colors used in an edge coloring of K_n so that no member of a specified class of subgraphs is rainbow. In the first paper on this problem, [5], the authors classified the construction of all JL-colorings of K_n and showed a correspondence between full binary trees and these JL-colorings. This classification of JL-colorings is in fact a particular anti-Ramsey result. Reference [4] is an excellent survey on anti-Ramsey results which includes a section on *Gallai* colorings. Gallai colorings are edge colorings that avoid rainbow K_3's. In [5] it was proven that an edge coloring of any chordal graph forbids rainbow K_3's if and only if the coloring forbids all rainbow cycles. The result from [5] on JL-colorings was therefore a result on Gallai colorings. This result followed easily from results on Gallai colorings mentioned in [4]; however the correspondence between JL-colorings of complete graphs and full binary trees in [5] does not, so as far as I know, appear elsewhere.

Recently, due to the work of Vitaly Voloshin on mixed hypergraphs which began in [10], the literature on mixed-Ramsey problems has grown. A mixed-Ramsey problem is one in which we avoid both monochromatic and rainbow subgraphs. In [5], it was shown that any JL-coloring of K_n also forbids monochromatic K_3's; this was a mixed-Ramsey result. In chapter 4 we will discuss proper RCF edge colorings; this is another mixed-Ramsey topic in which we color edges to avoid monochromatic $K_{1,2}$'s and rainbow cycles.

The results from [5] were generalized to complete bipartite graphs in [9]. My work began with the generalization to complete multipartite graphs in [8]. We were able to solve the classification problem of JL-colorings for all graphs in [7]. This dissertation will consist of the generalization to complete multipartite graphs in chapter 2, the complete solution for general graphs in chapter 3, and an introduction to proper RCF edge colorings in chapter 4.

Before beginning, we note the correlation between this work and a rather famous theorem by Ronald L. Graham and Henry O. Pollak. In [6], Graham and Pollak proved that at least $n-1$ complete bipartite graphs are required to decompose the edges of K_n. The theorems presented in [5, 7, 8, 9] show that JL-colorings provide a decomposition of the edges of a graph into (not necessarily complete) bipartite graphs; this is easily seen by the interpretation of the edge coloring via full binary trees explained in sections 1.1, 1.2, 2.2 and 3.1. In particular, the main result of [5] implies that JL-colorings of K_n are equivalent to certain decompositions of the edge set of K_n into $n-1$ complete bipartite graphs. Thus, the JL-colorings of K_n form some (but not all) of the decompositions discussed in [6]. Similar questions in random graphs were studied by the authors of [1, 2, 3]. The results on JL-colorings (including the main result of chapter 3), complement these results on decompositions into bipartite graphs.

To begin our work, we give the exact statement of the main theorem in [5].

Theorem 1.1. *Suppose $n \geq 2$. Every JL-coloring of $G = K_n$ is obtainable by the following: choose positive integers r and s such that $r + s = n$; partition $V(K_n)$ into sets R and S where $|R| = r$ and $|S| = s$. Color all $R - S$ edges in K_n with one color–say green. Color $G[R]$ and $G[S]$ each with some JL-coloring using disjoint sets of colors on the two cliques, and with green not appearing in either of these two sets. Further, any edge coloring arrived at by following the directions above is a JL-coloring.*

The main theorems on complete bipartite graphs in [9] and complete multipartite graphs in [8] are similar. The work for complete multipartite graphs is given in chapter 2.

We note that the above theorem implies there is a monochromatic cut in every JL-coloring of K_n. This idea plays an important role in the proof for general graphs in chapter 3.

1.1 Relation of JL-colorings of K_n and $K_{m,n}$ to Full Binary Trees

In [5] it is shown that the JL-colorings of K_n, $n > 1$ are in $1 - 1$ correspondence with (isomorphism classes of) full binary trees with n leafs. We first note what a full binary tree is and then we will describe this correspondence.

A **full binary tree** is a tree with exactly one vertex of degree two and all other vertices of degrees 1 or 3. The vertex of degree 2 is the **root** of the tree, and the vertices of degree 1 are **leafs**. Furthermore, every non-leaf (a vertex of degree 3 or the root) has exactly two **children**, which we call **siblings**. The non-leaf vertex is called the **parent** of the two children. The children of a vertex of degree 3 are its two neighbors other than the neighbor which is on the path joining it to the root.

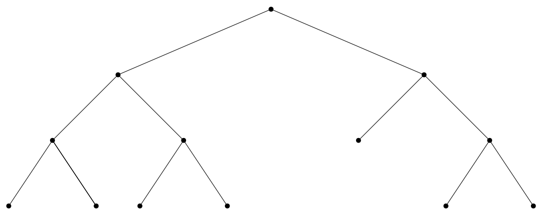

Figure 1.1: A full binary tree with 7 leafs.

To encode a JL-coloring on K_n we first draw a full binary tree on n leafs. We then label the vertices of these trees with a Huffman labeling. In a Huffman labeling of a full binary tree, each leaf is given a label from a commutative semigroup. Specifically, in the Huffman labelings of full binary tress representing JL-colorings of K_n, we use positive integers to label the vertices. Later we use m-tuples for m-partite graphs.

The label of each parent is the sum of the labels of its children (so the operation of the semigroup was addition) and each leaf is given weight 1. Afterwards, we assign a color to each sibling pair in the full binary tree. In these examples that color is represented by the lines between each sibling pair.

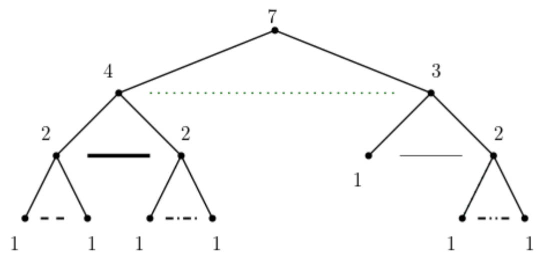

Figure 1.2: An encoded full binary tree with 7 leafs to represent a JL-coloring of K_7 using a Huffman labeling.

Let each leaf in the full binary tree represent a vertex in the graph we are JL-coloring; in this example, $G = K_7$. Alternatively, we can label each leaf as the vertex that it represents in the JL-colored graph and each parent vertex receives as its label the set union of its children's label.

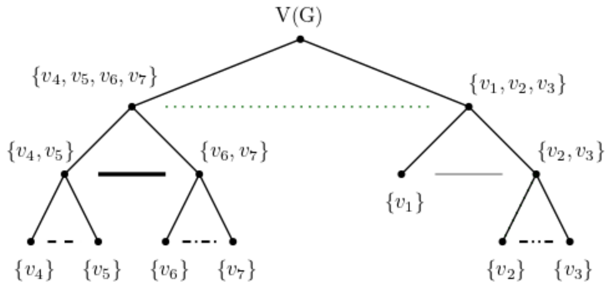

Figure 1.3: An encoded full binary tree with 7 leafs to represent a JL-coloring of K_7 using set labeling.

To edge color K_n from this encoding, if two vertices v_a, v_b from K_n appear in opposite sets of a sibling pair then color the edge v_av_b of K_n the same color that the sibling pair was given in the full binary tree representation.

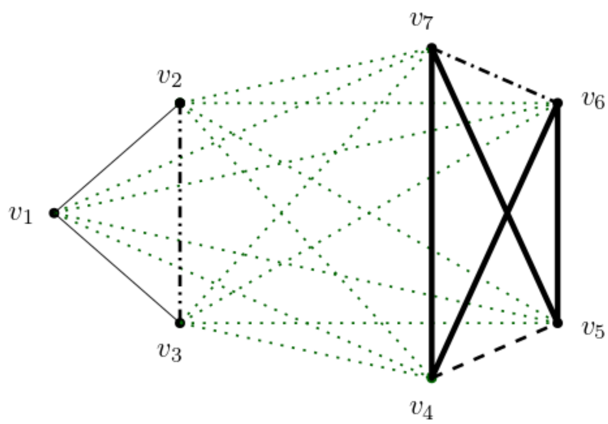

Figure 1.4: The JL-coloring on K_7 induced by Figure 1.2 or Figure 1.3.

It is easy to see that this in fact produces a JL-coloring of K_n. Let C be a cycle in K_n. C must traverse at least three vertices. For any subset of $V(K_n)$ with two or more elements, there will be a last vertex of the tree such that the set is entirely contained in the corresponding set of vertices: determine this vertex, v, for the set of vertices for this cycle. Then v is a parent vertex in the full binary tree representation. The cycle must have at least two edges colored with the same color in between the offspring of v. This guarantees that all cycles are not rainbow. Also, note that there are exactly $n-1$ colors used in this construction because in any binary tree with n leafs, there are exactly $n-1$ sibling pairs since there are exactly that number of non-leafs. This shows that there is always a JL-coloring of a K_n. Applying these same ideas to full binary tree representations of JL-colorings on complete bipartite graphs (done later in this section) and complete multipartite graphs (done in section 2.2), we show that all complete multipartite graphs can be JL-colored.

But moreover, the work in [5] states that any JL-coloring of K_n can be achieved by such a construction. Given a JL-coloring of K_n: we start constructing a full binary tree by letting the root represent all the vertices of K_n. The root has weight n. By Theorem 1.1 there are positive integers r and s, $r + s = n$, and sets R and S partitioning the vertex set of K_n, with $|R| = r$ and $|S| = s$, such that all $R - S$ edges are the same color (green), and the cliques induced by R and S are JL-colored with disjoint color sets where green does not appear in either set. Let the children of the root have weights r and s–alternatively, let them represent the sets R and S. Iterate this procedure (at the next stage each child of the root becomes a parent unless that child has weight 1).

The correspondence between JL-colorings of complete bipartite graphs $K_{m,n}$ and full binary trees with $m + n$ leafs is highlighted in [9]. Each JL-coloring of $K_{m,n}$ can be encoded by a certain labeling of a full binary tree, and conversely, every labeling of a full binary tree (with the labeling described next) produces a JL-coloring of a complete bipartite graph.

For this encoding, we must change the Huffman labeling. Each vertex in the full binary tree is labeled by an ordered pair of non-negative integers. The values in each coordinate represent the number of vertices, from each part of the bipartition of $K_{m,n}$, in the set of vertices represented by the particular vertex of the tree.

The other changes are as follows. Each leaf has weight one (there is a 1 in a single coordinate and a zero in the other coordinate). Sibling leafs must have the 1 in opposite coordinates (this ensures that an edge exists between the vertices represented by each leaf in the sibling pair). Each parent's label is the coordinate-wise sum of its children's labels.

Here we have an example of a JL-coloring on $K_{3,4}$. Figure 1.7 gives the JL-coloring of $K_{3,4}$ encoded by the full binary tree representations in Figure 1.5 and Figure 1.6 using a Huffman labeling and the set labeling, respectively.

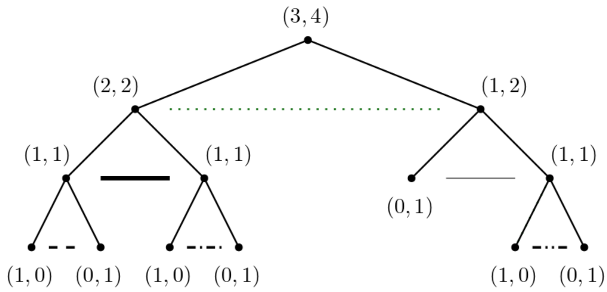

Figure 1.5: An encoded full binary tree with 7 leafs to represent the JL-coloring on $K_{3,4}$ with a Huffman labeling.

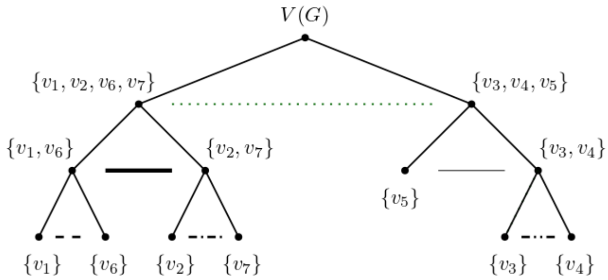

Figure 1.6: An encoded full binary tree with 7 leafs to represent the JL-coloring on $K_{3,4}$ with set labeling.

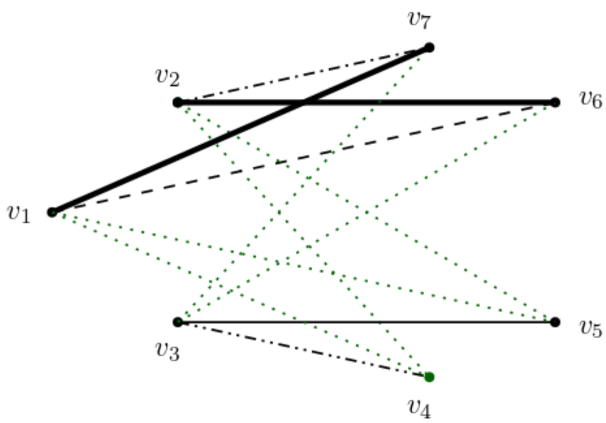

Figure 1.7: The JL-coloring on $K_{3,4}$ induced by Figure 1.5 or Figure 1.6.

We have one more example to consider before we end this chapter. In [5] it was noted that the (isomorphism classes of) full binary trees on n leafs were in $1-1$ correspondance with essentially different JL-colorings of K_n. For a complete bipartite graph, the same full binary tree may encode a different JL-coloring by changing the labeling. In fact, the underlying graph for the JL-coloring may in fact be different. In Figure 1.8 we give two different labelings on the same full binary tree that produces a JL-coloring on two different graphs, namely $K_{2,3}$ and $K_{1,4}$.

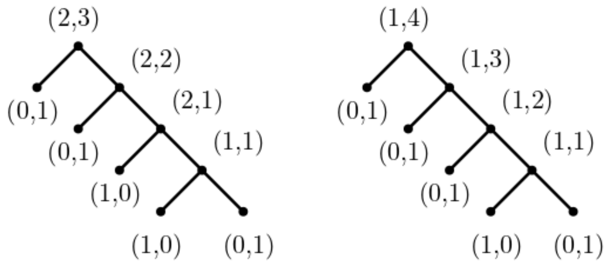

Figure 1.8: The same full binary tree with different labelings to produce JL-colorings of two different complete bipartite graphs.

Chapter 2

JL-colorings of Complete Multipartite Graphs

My work on this topic began with an attempt to generalize the theorems in [5] and [9] to complete multipartite graphs. While this was successful and the results were given in [8], the results are subsumed by later work in [7] that will be given in chapter 3.

2.1 Proof of the Main Theorem for Complete Multipartite Graphs

In this section we give the proof, in [8], of the main result for complete multipartite graphs, but first, two definitions and a comment on notation. In an edge colored graph, a color c is said to be **dedicated** to a vertex v if every edge colored c is incident to v. If a graph G is edge colored, and all edges incident to $v \in V(G)$ bear the same color, then v is said to be **unicolored**. Let $C[X]$ denote the set of colors on $G[X]$, the subgraph of G induced by X, for any $X \subseteq V(G)$. Let $C = C[V(G)]$ for short. We have two important lemmas before the main result.

Lemma 2.1. *If G is JL-colored, then every vertex in $V(G)$ has a color dedicated to it.*

Proof. Let $n = |V(G)|$. If $v \in V(G)$ has no color dedicated to it, then all $n-1$ colors appear on $G - v$, which has only $n-1$ vertices, and there are no rainbow cycles in $G - v$. But this is impossible since if we take the subgraph induced by picking $n-1$ different edges each bearing a different color, this induced subgraph must contain a cycle, which is necessarily rainbow. $\square$

Lemma 2.2. *If G is JL-colored, there are at least two vertices in G with exactly one dedicated color each.*

Proof. Let $n = |V(G)|$. We will count the number of ordered pairs in the set $\{(v,c) : v \in V(G),\ c \in C \text{ and } c \text{ is dedicated to } v\}$. For each $v \in V(G)$, let d_v be the number of colors dedicated to v. Note that $d_v \geq 1$ for all $v \in V(G)$ by Lemma 2.1. The number of such ordered pairs (v,c) is

$$\sum_{v \in V(G)} d_v \ \leq\ 2|C| = 2(n-1) = 2n-2. \tag{2.1}$$

The inequality holds since no color can be dedicated to more than two vertices. If each vertex had two or more colors dedicated to it, then $\sum_{v \in V(G)} d_v \geq 2n$. Similarly, $\sum_{v \in V(G)} d_v \geq 2n-1$ if exactly one vertex has one color dedicated to it. So we must have at least two vertices with exactly one color dedicated to each. $\square$

Now we can present the main result.

Theorem 2.3. *Let $G = K_{n_1,\ldots,n_m}$ be a complete multipartite graph with parts of size $n_1, \ldots, n_m$, with $m \geq 3$. An edge coloring of G is a JL-coloring if and only if there is a partition of $V(G)$ into non-empty subsets R and S which satisfy the following:*

1. *All $R-S$ edges in G have the same color (let us call it green).*
2. *The sets of colors on the complete multipartite subgraphs $G[R]$ and $G[S]$ induced by R and S, respectively, are disjoint, and neither set contains green.*
3. *The induced colorings of $G[R]$ and $G[S]$ are JL-colorings.*

The main result of [9] is precisely Theorem 2.3 in the case $m = 2$.

Proof. Let $|V(G)| = n$. Suppose that $E(G)$ is colored, and that $V(G)$ is partitioned into R and S satisfying the stipulated requirements. Let $|R| = r$ and $|S| = s$.

We verify that the coloring of G is a JL-coloring. Since the colorings of $G[R]$ and $G[S]$ are JL-colorings, these colorings use $r-1$ and $s-1$ colors, respectively. Also, the set of colors on $G[R]$, $G[S]$ are disjoint and neither contains green, so we see that G is colored with $(r-1)+(s-1)+1 = r+s-1 = n-1$ colors appearing. Let C be any cycle in G. If C is contained in either $G[R]$ or $G[S]$, then C is not rainbow since both subgraphs are JL-colored. If C has vertices in both R and S, then, because C is a cycle, C must contain at least two $R-S$ edges. Then two of C's edges are green, so C is not rainbow. Thus, G is JL-colored.

Notice that the "if" claim of the theorem holds for any connected graph G if the requirement that $G[R]$ and $G[S]$ be connected is added. The forward implication is more difficult. From here on assume that G is JL-colored.

We note that if G has a JL-coloring and $v \in V(G)$ is unicolored in this coloring, then the partition $R = \{v\}$ and $S = V(G) \setminus \{v\}$ satisfies the three conditions in the main theorem.

To see this, let green be the color incident to v. Then 1, all $R-S$ edges are green.

For 2, note that $G[R] = v = K_1$ has no edges. So certainly the sets of colors on $G[R]$ and $G[S]$ are disjoint. Since v is unicolored by the color green, green must be dedicated to v and thus green cannot appear in the graph $G[S]$.

Finally, for 3 we know that $G[R]$ has a single vertex, so $r = 1$ and the number of colors used is $r-1 = 1-1 = 0$. Also, $G[S]$ has $n-1$ vertices so $s = n-1$. The JL-coloring of G has $n-1$ colors and the color green is not used in $G[S]$, so $G[S]$ is colored with $n-2 = (n-1)-1 = s-1$ colors appearing. Since G is JL-colored, we see that neither $G[R]$ nor $G[S]$ can have a rainbow cycle.

From here, the proof proceeds by induction on $n = n_1 + \cdots + n_m$. At some points in the proof we may be applying the induction hypothesis to a complete bipartite subgraph of G. The induction hypothesis holds in such cases by the main result of [9].

We start with $n = |V(G)| = 3$. Since we assume the number of maximal independent sets is m with $m \geq 3$, we have $m = 3$ in our base case and there is one vertex in each set. Thus K_3 is the graph in the base case. For a JL-coloring of K_3, we must use two colors. Let green be the color that appears on two edges and let v be the vertex incident to both of these green edges. Then v is unicolored and we are done.

Now we can assume $|V(G)| \geq 4$. By Lemma 2.2, we can find a vertex v with exactly one color dedicated to it. Let red be the color dedicated to v in G. So $G - v$ is colored with

$n-2=(n-1)-1=|V(G-v)|-1$ colors appearing. Since G has no rainbow cycles, $G-v$ has no rainbow cycles, and thus $G-v$ is JL-colored. Then by our induction hypothesis we have a partition $R_0 \neq \emptyset$ and $S_0 \neq \emptyset$ of $V(G) \setminus \{v\}$ that satisfies conditions 1, 2, and 3 of the main theorem.

First, we take care of the special case where one of R_0, S_0 is a singleton. Without loss of generality, let $S_0 = \{u\}$. Then all edges to u in $G-v$ are green. We may assume u is not in v's part; otherwise there is no uv edge and thus u is unicolored in G. For the same reason, we may assume the edge uv is not green. Let $c \neq$ green be the color on uv.

Case 1: Suppose c is dedicated to u in G.

Since c is dedicated to u in G, c does not appear in $C[V(G) \setminus \{u\}]$. In particular, this implies c is not in $C[R_0]$. The only colors in C not in $C[R_0]$ are green and red. Thus c must be red. Since red is dedicated in G to both v and u, then red appears only on the edge uv.

Consider an edge vw where the vertex w is not in u's part. Since G is a complete multipartite graph, there is a cycle in G with edges uv, uw, and vw. We know uv is red and uw is green, so vw is either red or green (since G has no rainbow cycles). This implies vw must be green since red appears only on uv. Thus all edges from v to parts other than u's part are green.

If all edges incident to v except uv are green, then the partition $R = R_0$ and $S = \{u, v\}$ satisfies the desired conditions.

Now, we may assume for some $w \in R_0$ in u's part, the edge vw is not green. Let vw be blue. If all edges incident to w are blue, then w is unicolored and we are done. Suppose there is some edge wx not colored blue. Note that $x \in R_0$. Then wx cannot be green since $w, x \in R_0$. Also, it is not red since red only appears on uv. Let wx be yellow. This creates a rainbow 4-cycle where uv is red, ux is green, wx is yellow and vw is blue. This impossibility finishes Case 1 under the supposition that $\min(|S_0|, |R_0|) = 1$.

Case 2: Suppose c is not dedicated to u in G.

Since green is the only other color incident to u, it follows that green is dedicated to u in G by Theorem 2.1. Therefore no $v - R_0$ edge is green.

Subcase 1: Suppose c is red. Let's consider the edge vw for any $w \in R_0$ not in u's part. Note that $G[\{u, v, w\}]$ is a three cycle. Also, uv is red and uw is green. This implies vw is red since green is dedicated to u. So every vw is red for every $w \in R_0$ where w is not in u's part.

Again, if v is unicolored we are done, so we can assume there is some $x \in R_0$ in u's part where vx is not colored red. Let's say vx is blue. As above, if x is unicolored, we are done. So for some $y \in R_0$ where y is not in u's part, $xy \notin \{blue, green, red\}$. Then $\{uv, uy, xy, vx\}$ is a rainbow four cycle which contradicts that G is JL-colored.

Subcase 2: Suppose c is not red. So $c \in C[R_0]$. Let's say c is blue. As in subcase 1, since green is dedicated in G to u we know that all edges vw, where $w \in R_0$ and w is not in u's part, are blue.

Red is dedicated to v, so there must be some x in u's part where vx is red. Pick any vertex $y \neq v$ not in u's part. Then $\{uv, uy, xy, vx\}$ is a four cycle where uv is blue, uy is green and vx is red. Since green is dedicated to u and red is dedicated to v, xy must be blue. We know blue is not dedicated to x in G (since it appears on uv). Since y was arbitrary, it follows that red is the only color other than blue incident to x and thus red is dedicated to x, in G. This means that red appears only on the edge vx.

Let's take $S = \{v, x\}$ and $R = V(G) - \{v, x\}$. It will suffice to show this choice satisfies conditions 1, 2, and 3 of our main theorem to dispose of the special case min$(|R_0|, |S_0|) = 1$.

We begin by showing that blue $\notin C[R]$. First we note that all edges incident to x in $G - v$ are blue; this was part of the proof that vx is red. Recall that $R_0 = V(G - v) - \{u\}$. By assumption $G[R_0]$ was JL-colored, and thus x has a color dedicated to it in $G[R_0]$. That color must be blue. Let $ab \in E(G[R])$. If either a or b is the vertex u, then the edge ab is green. If neither a nor b is u, then $ab \in E(G[R_0])$. Since blue is dedicated to x in $G[R_0]$, the edge ab cannot be blue.

Now we will show that either all $R - S$ edges are blue, or there is a unicolored vertex in G. We have already shown all edges from x to R are blue. We have also shown that all edges from v to $w \in R$ where w is not in u's part are blue. If all edges from v to vertices in u's part are blue (other than the red vx edge), then we have shown what we needed to show.

So consider $z \in R$ where z is in u's part and vz is not blue. We note that vz is not green, because green is dedicated to u in G.

Let yellow $\notin$ {blue, red, green} be the color on vz. For any $w \in V(G) \setminus \{v\}$ which is in neither v's part nor in z's (u's) part (which implies that $w \in R_0$), wz must be colored yellow; this can be seen since vwz is not rainbow, vw is blue, vz is yellow, and wz is not blue since $w, z \in R$ and blue $\notin C(R)$. If z is unicolored by yellow, then we are done. If z is not unicolored, let zz_0 be an edge not colored yellow. This color is in $C[R_0]$ since $z, z_0 \in R_0$. In particular, the color is neither blue nor green. If z_0 is not in v's part, then we have a rainbow three cycle $\{vz, vz_0, zz_0\}$ where vz is yellow, vz_0 is blue, and zz_0 is neither blue nor yellow. Alternatively, if z_0 is in v's part then we have a rainbow four cycle $\{uz_0, uv, vz, zz_0\}$ where uz_0 is green, uv is blue, vz is yellow, and zz_0 is not yellow, blue nor green. Thus we either have a unicolored vertex z, or condition 1 is satisfied and all $R - S$ edges are blue.

For 2, we have red appearing as the only color in $C[S]$ and since red is only on vx, red is not in $C[R]$. Since blue is dedicated to x in $G[R_0]$, and all edges incident to u are green except uv, blue $\notin C[R]$. So $C[R]$ and $C[S]$ are disjoint and neither set contains blue.

For 3, since G has no rainbow cycles, neither $G[R]$ nor $G[S]$ can have rainbow cycles. Since $G[S]$ has only one edge and that edge is red, $G[S]$ has the appropriate number of colors. We need $G[R]$ to have $n - 3$ colors. We note that G has $n - 1$ colors appearing. We have already shown red and blue are not in $C[R]$. All other $n - 3$ colors must appear somewhere and since the one $G[S]$ edge is red and all $R - S$ edges are blue (the only other edges are in $G[R]$), they must appear on $G[R]$.

We may now assume that $|V(R_0)|, |V(S_0)| \geq 2$. If R_0 is a subset of only one part of G, then there are no edges in $G[R_0]$ and since $|V(R_0)| \geq 2$, it follows that $G[R_0]$ is not JL-colored. Therefore, it follows that both R_0 and S_0 have representatives in at least two different parts. Also, we note that green is not dedicated to any vertex in $G - v$ (and thus not in G) since $|V(R_0)|, |V(S_0)| \geq 2$.

Without loss of generality, let $x \in R_0$ be such that vx is red. Consider any edge vw where $w \in S_0$ and w is not in x's part. By assumption vx is red. We know wx is green since $w \in S_0$ and $x \in R_0$. Since there are no rainbow cycles, this implies vw must be red or green.

Now consider an edge vw where $w \in S_0$ and w is in x's part. Pick a vertex $y \in R_0$ and $z \in S_0$ where neither y nor z are in x's part. In the four cycle $vxyw$ we have vx is red, xy is yellow where yellow is in $C[R_0]$, and yw is green. This implies vw is red, green, or yellow. In the four cycle $vxzw$ we have vx is red, xz is green, and zw is blue where blue is in $C[S_0]$. This implies vw is red, green, or blue. Thus vw must be red or green.

We now know that each $v - S_0$ edge is red or green. We have two final cases to consider to complete the proof. Either all $v - S_0$ edges are green, or at least one $v - S_0$ edge is red.

Assume all $v - S_0$ edges are green. We aim to show each $v - R_0$ edge is either red or a color in $C[R_0]$.

Consider the edge vw where $w \in R_0$ and w is not in x's part. Then we have a three cycle with the edges $\{vx, wx, vw\}$ where vx is red and wx is a color in $C[R_0]$ since $w, x \in R_0$. This implies vw is either red or a color in $C[R_0]$ since G has no rainbow cycles.

Now we consider the edge vw where $w \in R_0$, w is in x's part and $w \neq x$. There is some color dedicated to w in G. Red is not dedicated to w because red is on vx, green is not dedicated to w, because green is not dedicated to anything in G, and clearly no color dedicated to w could be a color in $C[S_0]$. Let yellow be a color dedicated to w where yellow is in $C[R_0]$ and say yellow appears on an edge uw where $u \in R_0$. Then we have a four cycle $\{vx, ux, uw, vw\}$ that cannot be rainbow. We note that ux is an edge in $G[R_0]$ and is thus colored by a color in $C[R_0]$. Moreover, it is a color in $C[R_0]$ other than yellow, since yellow is dedicated to w. This implies vw is either red or some color in $C[R_0]$.

We take $R = \{v\} \cup R_0$ and $S = S_0$. Since we have assumed all $v - S_0$ edges are green we now have that all $R - S$ edges are green. We have just shown $C[R]$ and $C[S]$ are disjoint and neither contains green. Again, neither $G[R]$ nor $G[S]$ has a rainbow cycle since G has no rainbow cycles. Finally, $|S| - 1$ colors appear on $G[S] = G[S_0]$, since the latter was JL-colored by the induction hypothesis. Also, the colors $C[R_0] \cup \{red\}$ appear on $G[R]$, so $(|R_0| - 1) + 1 = |R_0| = |R| - 1$ colors appear on $G[R]$. Thus, both $G[R]$ and $G[S]$ are JL-colored.

Now we assume there is some red $v - S_0$ edge. By the same argument as above, that showed that all $v - S_0$ edges are either red or green, this implies that all $v - R_0$ edges are red or green. We now show that this implies that all edges incident to v are red. Thus v is unicolored and we are done.

To see that all edges incident to v are red, assume that there is some edge vy that is green. There is a $v - S_0$ red edge by assumption and there is also a $v - R_0$ red edge (namely vx). So, without loss of generality, let $y \in R_0$.

If x and y are not in the same part, then we note that $\{vx, xy, vy\}$ is a rainbow three cycle since xy must be colored by some color in $C[R_0]$. Let us assume x and y are in the same part. Let blue be the color dedicated to x in the JL-coloring of $G - v$. Since green is not dedicated to any vertex in $G - v$, blue must be in $C[R_0]$. Let blue be on an edge ux where $u \in R_0$. Then the four cycle $\{vx, ux, uy, vy\}$ is rainbow since vx is red, ux is blue, uy is a color in $C[R_0]$ that is not blue since blue is dedicated to x, and uy is green. Thus, the green edge vy cannot exist and so all edges incident to v must be red. $\square$

2.2 The Encoding of a JL-coloring of Complete Multipartite Graphs by Full Binary Trees

Theorem 2.3 implies that full binary trees with $n_1 + \ldots + n_r$ leafs, given certain labelings on the tree's vertices with r-tuples of non-negative integers, encode JL-colorings of $K_{n_1,\ldots,n_r}$. As in the complete bipartite case, the Huffman labeling must change. Each leaf has weight 1 (a 1 appears in one coordinate and zeros appear elsewhere), sibling leafs are orthogonal, and for each $j \in \{1, \ldots, r\}$, at least one leaf has the label with a 1 in position j of the r-tuple. As before, the label of a parent vertex is the coordinate-wise sum of the labels of its children. Next, we provide an example of such an encoding.

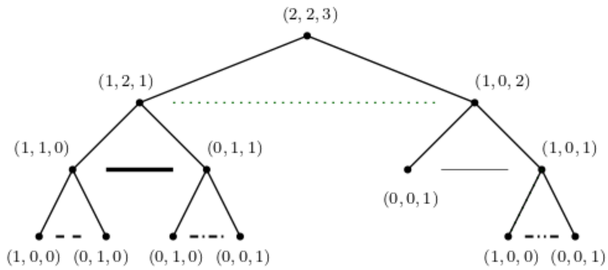

Figure 2.1: An encoded full binary tree with 7 leafs to represent the JL-coloring on $K_{2,2,3}$ illustrated below.

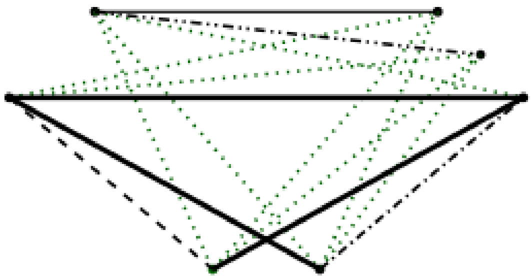

Figure 2.2: The JL-coloring of $K_{2,2,3}$ induced by Figure 2.1.

As in section 1.1, we noted that the same full binary tree with a different labeling can produce a JL-coloring on an entirely different graph. Figure 2.3 is a different labeling of the same full binary tree in Figure 2.1 that produces a JL-coloring of $K_{2,2,3}$ that is different than the JL-coloring in Figure 2.2. We see that the JL-coloring is different by noting that there are 9 dotted lines in Figure 2.2 and there would be 10 dotted lines in the JL-coloring induced by Figure 2.3.

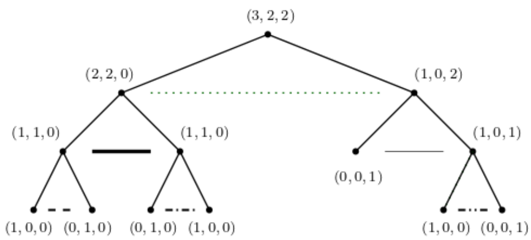

Figure 2.3: An encoded full binary tree with 7 leafs to represent a JL-coloring on $K_{2,2,3}$ that is different than the JL-coloring in Figure 2.2.

Chapter 3

JL-colorings of General Graphs

In this chapter we show that the structure theorems for JL-colorings in [5, 8, 9] hold for any finite connected graph. We first define a 'standard construction' for JL-colorings in section 3.1 and then in section 3.2 we show that all JL-colorings are, in fact, produced by this standard construction.

The proof relies heavily on showing that there is always a monochromatic cut in every JL-coloring of a connected graph. The contrapositive statement (an $n-1$ edge colored connected graph on n vertices without a monochromatic cut always contains a rainbow cycle) leads to an important and natural question regarding any possible stengthening of our result. So we ask, do similar results hold when we relax the number of colors appearing on edges. As we will see, the same conclusion is not reached even when using $n-2$ colors. However, in the case of K_n, we prove that all $n-2$ edge colorings of K_n without a monochromatic cut that are rainbow cycle free are created via a 'cloning construction.' This characterization of $n-2$ edge colorings with no monochromatic cut and no rainbow cycle does not remain true for general connected graphs.

The statements about JL-colorings hold for multigraphs as well and some of the proofs in this chapter are simpler on multigraphs. So in this chapter 'graph' will mean 'multigraph'; loops and multiple edges are allowed. However, as noted in the first paragraph of chapter 1, that if a multigraph has an edge coloring which admits no rainbow cycles, then the multigraph has no loops and for any distinct vertices u, v, all edges incident to both u and v bear the same color so the graph may as well be simple.

Now, we provide a bit of notation and a well known definition. If A, B are non-empty sets partitioning $V(G)$, then the set of edges of G with one end in A and one end in B, denoted $[A, B]$, is an **edge cut** in G.

3.1 The Standard Construction

In section 1.1 we saw that JL-colorings can be achieved on any complete or complete biparite graph. In section 2.2 we saw the construction to produce JL-colorings on complete multipartite graphs. In this section we will define a 'standard construction' for a JL-coloring of any connected graph. Before we can define the standard construction, we will show that every graph is JL-colorable and we will use that proof as a starting place for this standard construction.

Lemma 3.1. *Suppose the edges of G are colored, $[A, B]$ is a monochromatic edge cut in G with this coloring, and the coloring restricted to $G[A]$ and $G[B]$, the subgraphs of G induced by A and B, respectively, is RCF. Then the coloring on G is RCF.*

Proof. There are no rainbow cycles in $G[A] \cup G[B]$, and any cycle with at least one vertex in A and at least one vertex in B must have at least two different edges in $[A, B]$. Therefore there are no rainbow cycles in G. □

Lemma 3.2. *If G is a connected simple graph on n vertices, then there is an RCF edge coloring of G with $n - 1$ colors appearing.*

Proof. Let T be a spanning tree in G, and color the $n-1$ edges of T with $n-1$ different colors. These will be the colors appearing in the final coloring.

Choose an edge $uv \in E(T)$. Suppose that green is the color of uv. The vertices in G are naturally partitioned into vertex sets A and B (which we will call 'shores') by the choice of uv; that is, A is the set of vertices of G connected to u by a path in $T - uv$, and B is the set of vertices of G connected to v by a path in $T - uv$. Let every edge in the edge cut $[A, B]$ be colored green.

Since $T[A], T[B]$ are spanning trees in $G[A], G[B]$, respectively, both $G[A]$ and $G[B]$ are connected loopless graphs, and each already has a rainbow spanning tree installed.

If $T[A]$ has at least one edge (i.e. if $|A| > 1$), iterate the procedure just described with G, T replaced by $G[A], T[A]$, and proceed similarly with B. Continue until it is impossible to continue. Every edge will be colored since T is a spanning tree.

By induction on n we can conclude that the resulting edge colorings of $G[A], G[B]$ with $|A| - 1, |B| - 1$ colors, respectively, are RCF, and therefore the resulting edge coloring of G with $n - 1$ colors is RCF by Lemma 3.1. $\square$

Since, for any two vertices $u, v \in V(G)$, all edges between u and v bear the same color, the next corollary follows immediately.

Corollary 3.3. *If G is a loopless connected graph on n vertices, then there is an RCF edge coloring of G with $n - 1$ colors appearing.*

In the above proof we use a spanning tree to construct a JL-coloring. While this made it clear that $n - 1$ colors were used and that $G[A]$ and $G[B]$ were connected, the role of the spanning tree is not necessary. Every JL-coloring obtained by the method used in the proof is also achievable by the following procedure, which does not mention spanning trees. Conversely, every JL-coloring created by the following procedure can be achieved using a spanning tree as in Lemma 3.2.

The **Standard Construction** of RCF edge colorings, with $n - 1$ colors appearing, of a connected loopless multigraph G on n vertices is defined as follows:

1. If $n > 1$, find an edge cut $[A, B]$ in G such that $G[A]$ and $G[B]$ are connected. Color the edges of $[A, B]$ with a color that will not be used again.

2. If $|A| = 1$ there are no edges to color in $G[A]$. If $|A| > 1$, iterate step 1 on $G[A]$, and the same for $G[B]$ if $|B| > 1$. At each step, pick colors such that the color set to appear on $G[A]$ is disjoint from that on $G[B]$, and neither can contain the color on $[A, B]$. Continue until all edges are colored.

As before, every JL-coloring can be encoded by a full binary tree where vertices are labeled by subsets of $V(G)$. Each leaf will be a singleton labeled as the vertex the leaf represents in the graph G. Each parent's label is the set union of its children's labels. In this way, the root of the tree will be labeled as $V(G)$. We note that for a general graph (unlike for K_n) the full binary tree in this encoding is not necessarily unique, either as a labeled or unlabeled object. For instance, for the path P_n the unique JL-coloring can be encoded by *every* full binary tree, properly labeled. This can be seen by taking an ordered full binary tree with n leaves, labeling the leaves left-to-right with elements of the set $\{v_1, \ldots, v_n\}$, and taking the label of a parent to be the union of the labels of its children. On the other hand, for the star, $K_{1,n}$, while there is a unique isomorphism class of tree arising in such an encoding, there are $n!$ different labelings of this tree that encode the (again, unique) JL-coloring. In general, the labeled full binary tree representation will not be unique, as there may be more than one monochromatic edge cut.

We end this section with an example of such an encoding of a JL-coloring.

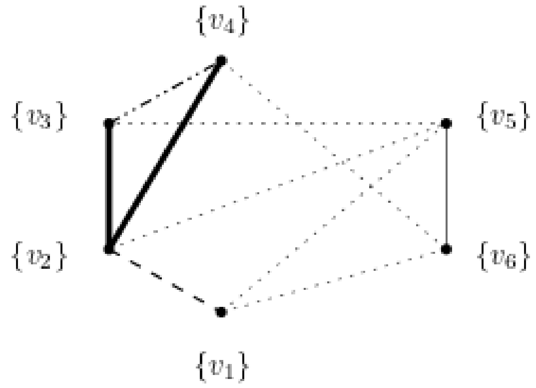

Figure 3.1: The JL-coloring of a connected graph on 6 vertices.

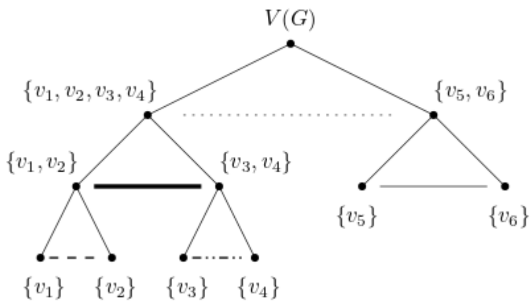

Figure 3.2: An encoded full binary tree with 6 leafs to represent the JL-coloring of the connected graph in Figure 3.1 using set labeling.

3.2 Proof of the Main Theorem for General Graphs

In this section we want to prove the following theorem.

Theorem 3.4. *All JL-colorings of finite connected graphs are achievable by the standard construction.*

Suppose that G is a graph on n vertices and c components, and $f : E(G) \to \{1, \dots, k\}$ is an edge coloring of G such that each color $1, \dots, k$ appears (i.e., f is surjective). The **slack** of f is $s(f) = n - c - k$. When discussing an RCF edge-coloring, the slack of f is exactly the number of colors we need to add to the edge coloring to make it a JL-coloring (how far are we from a JL-coloring?).

If $S \subseteq E(G)$ and $j \in \{1, \dots, k\}$, we will say that the color j is **dedicated to** S if and only if $f^{-1}(j) \subseteq S$. When S is the set of edges incident to a vertex $v \in V(G)$, we will say that a color dedicated to S is **dedicated to** v. This definition generalizes the definition of 'dedicated' given in chapter 2.

Lemma 3.5. *Let G and f be as above.*

i If f is a RCF edge-coloring, then $s(f) \geq 0$, with equality if and only if f is a JL-coloring.

ii A color $j \in \{1, \dots, k\}$ can be dedicated to at most two different vertices; further, j is dedicated to two different vertices if and only if j appears only on the edges between the two.

Proof. Claim (1) follows from our observation that G must be colored with at most $n - c$ colors in order to avoid rainbow cycles, and claim (2) is straightforward. $\square$

Lemma 3.6. *Suppose that G is a graph on n vertices with c components , f is an edge coloring of G, $S \subseteq E(G)$, $G' = G - S$ has $c' = c + x$ components, and S has d colors dedicated to it. If f restricted to G' is RCF, then $x \leq s(f) + d$.*

Proof. Let k be the number of colors appearing on the edges of G. Then $k' = k - d$ colors appear on the edges of G'. Let s' denote the slack of the restriction of f to G'. Then by Lemma

$3.5(i)$, if f' restricted to G' is RCF, we have $0 \leq s' = n - c' - k' = n - (c + x) - (k - d) = s(f) + d - x$. $\square$

Corollary 3.7. *Suppose that G is a connected graph with a JL-coloring, and $[A, B]$ is an edge cut in G. Then there is at least one color dedicated to $[A, B]$.*

Proof. Letting $n = |V(G)|$, f the JL-coloring of $E(G)$, and $S = [A, B]$, then the terms x and $s(f)$ in Lemma 3.6 satisfy $s(f) = n - (n - 1) - 1 = 0$ and $x \geq 1$, because S is an edge cut in a connected graph. Therefore, by Lemma 3.6, $d \geq x - s(f) \geq 1$. $\square$

Corollary 3.8. *In every JL-coloring of a loopless connected graph G on $n > 1$ vertices, for each $v \in V(G)$ some color is dedicated to v.*

Proof. Since $n > 1$, for each $v \in V(G)$ the set of edges incident to v is an edge cut, $[\{v\}, V(G) \setminus \{v\}]$. $\square$

Corollary 3.9. *If G is a JL-colored connected simple graph on $n > 1$ vertices, then at least one color appears exactly once in G.*

Proof. Each of the n vertices of G has a color dedicated to it and there are only $n - 1$ colors appearing. Therefore some color must be dedicated to two different vertices. The conclusion follows from Lemma 3.5 and the assumption that G is simple. $\square$

Theorem 3.10. *Suppose that G is a finite connected graph with a JL-coloring f which admits a monochromatic edge cut $[A, B]$. Then $G[A], G[B]$ are connected, the restrictions of f to each of $G[A], G[B]$ are JL-colorings, the color sets on $G[A], G[B]$ are disjoint, and neither contains the single color on $[A, B]$.*

Proof. Let $S = [A, B], n = |V(G)|$, and $G' = G - S$. Let c be the number of components of G, $c' = c+x$ be the number of components of G', and d be the number of colors dedicated to S. Then $c = 1$, because G is connected, and $c' = c + x \geq 2$ because S is an edge cut. Therefore, $x \geq 1$.

Since f is a JL-coloring, $s(f) = 0$. Therefore, by Lemma 3.6, $1 \leq x \leq s(f) + d = d$. By the assumption that $[A, B]$ is monochromatic, $d = 1$. By the inequality above, it follows that $x = 1$, so $c' = c + x = 1 + 1 = 2$. From this we conclude that $G[A]$ and $G[B]$ are connected.

Because the one color on S is dedicated to S, it does not appear in $G[A] \cup G[B]$. Let the sets of colors appearing in the restrictions of f to $G[A]$ and $G[B]$ be $C(A)$ and $C(B)$, respectively. Then $|C(A) \cup C(B)| = n-2$, and $|C(A)| \leq |A|-1$, $|C(B)| \leq |B|-1$, because the colorings on $G[A]$ and $G[B]$ are RCF. Since $|A| + |B| = n$, from these facts we conclude that $|C(A)| = |A| - 1$, $|C(B)| = |B| - 1$, and $C(A) \cap C(B) = \emptyset$. Therefore, $G[A]$ and $G[B]$ are JL-colored, with disjoint color sets. □

Theorem 3.10 implies that if G is a connected JL-colored graph, with $|V(G)| > 1$, and it has a monochromatic edge cut, then G is obtainable by the standard construction, and is therefore representable by vertex-labeled binary trees, as described in section 3.1. Next we prove that every connected JL-colored graph on more than 1 vertex admits a monochromatic edge cut.

Theorem 3.11. *If G is a finite connected graph on $n > 1$ vertices with a JL-coloring, then there is a monochromatic edge cut in G.*

Proof. Let G be a counterexample to the claim of the theorem with minimum $n+|E(G)|$. So G has a JL-coloring which admits no monochromatic edge cut. Then G must be loopless, because the coloring is RCF, and must have no multiple edges between vertices—otherwise, the simple graph obtained by collapsing multiple edges into simple edges with the color borne by those multiple edges would be a counterexample with the same number of vertices and fewer edges.

Since G is connected and simple on $n > 1$ vertices, and JL-colored, by Corollary 3.9 there is an edge $e = uv \in E(G)$ bearing a color—let us call it red—which appears only on e. Let $G^* = G/e$, the result of contracting e. (The edge e disappears and the vertices u and v merge into a new vertex w which is incident in G^* to any edge of G, except e, which was incident to either u or v.) Let each edge of G^* bear the color that it bore in G. The number of colors on G^* is one less than $n-1$, the number of colors on G, because the color red was dedicated to $\{e\}$. Also, $|V(G^*)| = n-1$. Since we can suppose that $n > 2$, we have $n-1 > 1$. Clearly G^* is connected. If G^* is RCF, then G^* is JL-colored, connected on more than one vertex, and $|V(G^*)| + |E(G^*)| < |V(G)| + |E(G)|$. It would then follow that there is a monochromatic edge cut $[A^*, B^*]$ in G^*. But then there is a monochromatic cut $[A, B]$ in G: if, without loss of generality, $w \in A^*$, take $A = (A^* \setminus \{w\}) \cup \{u, v\}$ and $B = B^*$.

The proof will be over if we show that there are no rainbow cycles in G^*. Since there are no rainbow cycles in G, the only cycles in G^* that might be rainbow must contain the vertex w. Let C^* be a rainbow cycle in G^* containing w; let x, y be the neighbors of w on the cycle. The possibilities are:

(i) x=y and C^* is a double edge arising from the edges xu, xv in G. But then $G[\{u, v, x\}]$ is a rainbow C_3 in G.

(ii) $x \neq y$ and the edges xw, yw on C^* arise from edges xu, yv (or xv, yu) in G. But then we have a rainbow cycle in G with all the edges of C^*, letting xu replace xw and yv replace yw, together with e.

(iii) $x \neq y$ and the edges xw, yw arise from edges xu, yu (or xv, yv) in G; then the edges of C^* define a rainbow cycle in G.

□

Proof of Theorem 3.4. This follows immediately by combining Theorem 3.10 and 3.11. □

3.3 Sharpness Results

We begin this section with the statement of the contrapositive to Theorem 3.11, listed next as Corollary 3.12.

Corollary 3.12. *Suppose G is an $n-1$-edge colored graph on n vertices without a monochromatic cut. Then G contains a rainbow cycle.*

As stated at the beginning of this chapter, Corollary 3.12 is a good place to start when considering any strengthening of this result. I.e., if we use less than $|V(G)|-1$ colors for some RCF edge-coloring, do we have a monochromatic cut? If not, under what conditions can this occur? For easier readability, we first have a definition.

If G is edge colored we say that G is **robustly colored** if $V(G)$ cannot be partitioned into two non-empty parts $X, \bar{X}$ so that $[X, \bar{X}]$ is monochromatic.

The following example shows that K_4 can be robustly 2-colored to avoid rainbow cycles, so we cannot relax the number of colors used without additional restrictions.

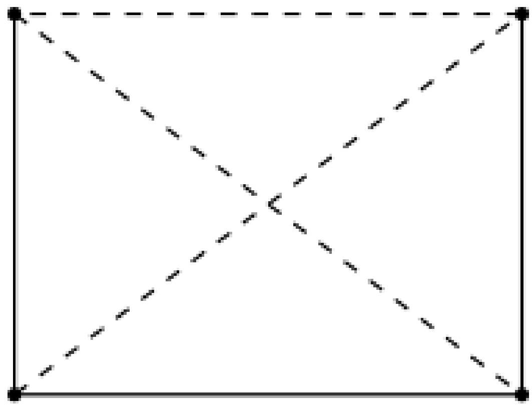

Figure 3.3: An RCF robust edge coloring of K_4 with $n-2=2$ colors.

Naturally we want to ask if we can classify all such edge colorings, as in, do we know the structure of all RCF robust edge colorings using $n-2$ colors. We have such an answer for K_n for all $n \geq 4$. We first show that via the **cloning construction** that we define next, we can find an RCF robust edge coloring of K_n for all $n \geq 4$. Moreover, we later show that all such edge colorings must be derived via this cloning construction.

Suppose that $n \geq 4$ and K_n is robustly colored with exactly $n-2$ colors appearing so that there are no rainbow cycles in K_n. Let a coloring of K_{n+1} be formed as follows: pick any arbitrary vertex v and clone it—if we call the cloned vertex w, then for all vertices $x \in V(K_4)$ with $x \neq v$, give the edge wx the same color that appeared on vx. Give the edge vw a color not appearing in K_n.

Exactly $n-2+1=(n+1)-2$ colors appear on K_{n+1}. The single edge bearing the new color cannot be an edge cut in K_{n+1}, because $n \geq 4$; therefore, if there is a monochromatic edge cut $[A, B]$ in K_{n+1}, v and w must be on the same side of the cut. Without loss of generality, let $v, w \in A$. But then $[A \setminus \{w\}, B]$ is a monochromatic edge cut in K_n. Therefore the edge coloring of K_{n+1} is robust.

Suppose C is a rainbow cycle in K_{n+1}. It would necessarily have to contain w. If $v \notin V(C)$, then $(V(C) \setminus \{w\}) \cup \{v\}$ is the vertex set of a rainbow cycle in K_n. So both v and w appear on C. Let $x \in V(K_n) \setminus \{v\}$ be a neighbor of w on C. If $vx \in E(C)$ then C is not rainbow. Therefore vx is a chord of C, of the same color as wx. It follows that one of the cycles in $C \cup vx$ is a rainbow cycle in K_n. Thus, no such C exists.

The following theorem and corollary show that every RCF robust edge coloring of K_n for $n > 4$ is created via this cloning construction.

Theorem 3.13. *If K_n is edge colored with $n-2$ colors avoiding monochromatic cuts with $n > 4$ and each color appears at least twice, then there is a rainbow cycle.*

Proof. Let $G = K_n$ be edge colored with $n-2$ colors, where each color appears at least twice so that there are no monochromatic edge cuts. Suppose also that there are no rainbow cycles.

We claim that there is no isolated edge in G of any color. That is, each edge is adjacent to another edge of the same color. Suppose, to the contrary, that $ww_1 \in E(G)$ is colored red, and neither w nor w_1 is incident to a red edge other than ww_1. Contract ww_1 —let w_2 be the new vertex obtained by merging w and w_1— to obtain $G' = G/ww_1$, a robustly edge colored graph on $n-1$ vertices, with $n-2$ colors appearing. If the coloring of G' forbids rainbow cycles then G' is JL-colored, which implies that the coloring of G' is not robust after all.

Suppose G' contains a rainbow cycle C'. If C' either contains no red edge, or does not pass through w_2, then there is a rainbow cycle in G. Therefore, C' contains w_2 and does contain a red edge, say xy, $x, y \in V(G) \setminus \{w, w_1\}$, by the assumption that ww_1 is isolated from other red edges. This implies that the cycle C in G obtained by 'opening' w_2 into ww_1 is of length at least 4 (note that if ww_1 was not an edge in C, then the edges of C' are all edges of C and thus there is a rainbow cycle in G). Further, at least one of the edges wx, wy, w_1x, w_1y is a chord of C which creates, with C, two cycles, at least one of which is rainbow. This establishes that the edge ww_1 which is not adjacent to another edge of G bearing its color cannot exist.

We also note there exists an induced subgraph $H = K_{n-1}$ on $n-1$ vertices that has all $n-2$ colors appearing on the edges. If not, then for every vertex $y \in V(G)$ we have that there is at least one color dedicated to y. If any color is dedicated to both y_1 and y_2 for some

$y_1, y_2 \in V(G)$ with $y_1 \neq y_2$, then that color appears on only one edge. Since we assume every colors appears at least twice, then all of these dedicated colors must be distinct. This is impossible since we have n vertices and $n-2$ colors.

Let v be the vertex in G missing from H. If H is not JL-colored, then there is a rainbow cycle in H and thus in G. So, we may assume H is JL-colored. By Corollary 3.9, we know that some color appears exactly once. Call this color green and say that it appears on the edge xy.

Then, without loss of generality, xv is green since there is no isolated edge. For every $x' \in V(G) \setminus \{v, x, y\}$ the edge from v to x' is either the same color as xx' or it is green; otherwise we have a rainbow K_3.

Moreover, there is a monochromatic cut $[X, Y]$ in $V(G) \setminus \{v\}$ with x and y in the same part. If x and y were in different parts then the monochromatic cut would be green, but $V(G) \setminus \{v\} \geq 4$, and green supposedly appears only on xy, among edges of $G - v$. Say $x, y \in X$ and the edges in $[X, Y]$ are colored red. The edges from v to Y must be red or green (otherwise we have a rainbow K_3). Also, at least one edge must be green or $[X \cup \{v\}, Y]$ would be a monochromatic cut. Since no edge in the subgraph induced by Y is green or red, this implies all edges from v to Y are green. This also implies all v to X edges are red or green. The proof is finished after the following observations.

If $|Y| = 1$, then $|X| \geq 3$. Then for $z \in X$ with $z \neq x, y$ we have vz is colored green. (If it were red, then vzx would be a rainbow K_3). This forces vy to be green; but then $[\{v\}, X \cup Y]$ is a monochromatic cut.

If $|Y| \geq 2$ then the subgraph induced by Y is JL-colored and therefore admits a color that appears only on one edge in $G - v$; this color is neither red nor green. This color must also appear on an edge incident to v, because there is no isolated edge in G of any color. But, this color cannot appear on an edge incident to v since all such edges are red or green.

$\square$

Corollary 3.14. *For $n > 4$, every robust edge coloring of K_n with $n - 2$ colors that avoids rainbow cycles is created by the cloning construction.*

Proof. We now know that in all robust edge colorings of K_n with $n > 4$ using $n - 2$ colors in which we avoid rainbow cycles, at least one color appears exactly once. Pick some edge where its assigned color appears only on that edge; call that edge uv. For all $x \in V(K_n)$ with $x \neq u, v$, if the color on xu is different than the color on xv we have a rainbow K_3. So we can regard the coloring as arising from an RCF robust edge coloring of $K_n - u$ with $n - 3$ colors by the cloning construction—here u is a 'clone' of v. □

We have now shown that all RCF robust edge colorings of K_n with $n - 2$ colors ($n > 4$) are derived by the cloning construction from the RCF robust edge coloring of K_4. The idea of the cloning construction may be extended to general graphs; start with a RCF robust edge coloring of a graph G on n vertices using $n - 2$ colors, and let $x \in V(G)$. We may extend to a graph on $n + 1$ vertices by partially cloning x: introduce a new vertex v and add the edge zv for some, but not necessarily all, vertices $z \in N_G(x)$. Color each added edge zv with the color on zx, and let the new edge vx bear a new color. For the same reasons that the original cloning construction gave us the desired results, this method creates a robustly colored graph on $n + 1$ vertices, uses $n - 1$ colors and avoids rainbow cycles.

Now we may ask if all RCF robust edge colorings of a general graph are created via this method. The following example shows this is false.

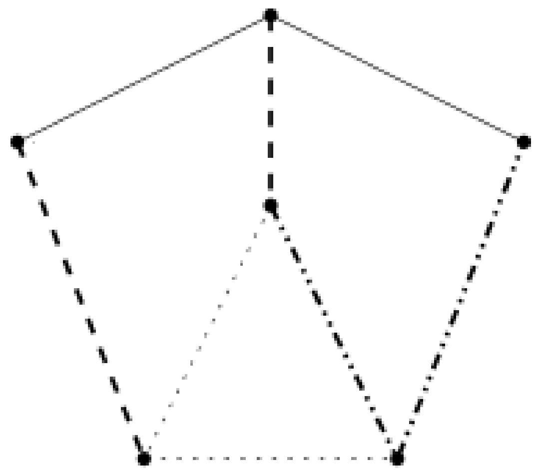

Figure 3.4: An RCF robust edge coloring of a graph on 6 vertices using $|V(G)| - 2 = 4$ colors

Chapter 4

Proper RCF edge colorings

In a proper edge coloring of a graph G, no two adjacent edges — i.e., edges both incident to some vertex — bear the same color. The smallest number of colors for which such a coloring exists is the *edge chromatic number*, or *chromatic index*, of G, and is denoted $\chi'(G)$. Clearly G will have a proper coloring with exactly t colors appearing for any $t \in \{\chi'(G), \ldots, |E(G)|\}$, and only for such t.

Can a graph be properly edge colored so that there are no rainbow cycles? For some graphs the answer is yes, for others, no. As with many ”mixed hypergraph coloring” phenomena, when there is such a coloring the question arises: For which positive integers t can this graph be so colored with exactly t colors appearing?

For a graph G let $SPRCF(G) = \{\ t \in \mathbb{N} \mid$ there is a proper rainbow-cycle-forbidding edge coloring of G with exactly t colors appearing$\}$. The ”S” in SPRCF is for ”spectrum.” We know that the maximum number of colors appearing on a connected JL-colored graph G is $|V(G)| - 1$, thus $SPRCF(G) \subseteq \{\chi'(G), \ldots, |V(G)| - 1\}$.

If $SPRCF(G) = \emptyset$, we will say that G is $PRCF$**-bad**; otherwise, G is $PRCF$**-good**. When G is connected and $SPRCF(G) = \{\chi'(G), \ldots, |V(G)| - 1\} \neq \emptyset$, we will say that G is $PRCF$**-excellent**. (If we omitted the condition that $SPRCF(G) \neq \emptyset$ in this definition, then complete graphs of odd order would be simultaneously $PRCF$-bad and $PRCF$-excellent.) If G is connected and $|V(G)| - 1 \in SPRCF(G)$, we will say that G is **properly JL-colorable**.

If G can be properly edge colored so that no cycle in G is rainbow, then each subgraph of G can be so colored. Therefore each subgraph of a $PRCF$-good graph is $PRCF$-good. Therefore the class of $PRCF$-good graphs has a forbidden subgraph characterization.

A graph G is $PRCF$**-critically bad** if it is $PRCF$-bad, but every proper subgraph of G is $PRCF$-good. The collection of $PRCF$-critically bad graphs is the smallest (with respect to inclusion) forbidden subgraph collection that characterizes the $PRCF$-good graphs. We will see some examples of these graphs at the end of this chapter.

4.1 Known Theorems for the Spectrum of Proper RCF edge colorings

In this section we shall state (and prove when necessary) what we know of proper JL-colorings. We omit the straightforward proofs of the claims in the following lemma.

Lemma 4.1. *1. G is $PRCF$-good if and only if every component of G is $PRCF$-good.*

2. K_1 and K_2 are $PRCF$-excellent

3. If G is critically $PRCF$-bad, then G is connected.

4. G is $PRCF$-good if and only if G has no $PRCF$-critically bad subgraph.

5. G is critically $PRCF$-bad if and only if G is $PRCF$-bad, G has no isolated vertices, and $G - e$ is $PRCF$-good for every $e \in E(G)$.

Regarding proper JL-colorability, the proof of Proposition 3.2 suggested that it is a rare property. If the method of this proof is used to JL-color a graph G, then, at the very first stage, after an edge is chosen and the shore sets A and B are thereby determined, if any two $A - B$ edges happen to be adjacent, the coloring will be improper; this danger exists as the algorithm rolls on, coloring $G[A]$ and $G[B]$. Since the process in the proof is essentially that of the Standard Construction for the JL-colorings, our proof that every JL-coloring arises from an instance of the Standard Construction further indicates that proper JL-colorability is seldom encountered. However, we do have the following.

Proposition 4.2. *Cycles of order ≥ 4 and trees are $PRCF$-excellent.*

Proof. Every coloring of a tree trivially forbids rainbow cycles. Therefore, $SPRCF = \{\chi'(G), \ldots, |E(T)|\}$, the set of numbers of colors in proper colorings of T. Since $|E(T)| = |V(T)| - 1$, T is $PRCF$-excellent.

Suppose $n \geq 4$. Then $\chi'(C_n) \leq n-1$. Since C_n has n edges, no edge coloring of C_n with fewer than n colors can make C_n rainbow. Therefore, $SPRCF = \{\chi'(C_n), \ldots, n-1\}$. $\square$

Let the degree of a vertex $v \in V(G)$ be denoted $d_G(v)$.

Proposition 4.3. *Suppose that* $V(G \cap H) = \{v\}$. *If* $a \in SPRCF(G)$ *and* $b \in SPRCF(H)$, *then* $SPRCF(G \cup H)$ *contains every integer* t *satisfying* $\max(a, b, d_G(v) + d_H(v)) \leq t \leq a+b$

Proof. We can obtain $PRCF$ edge colorings of $G \cup H$ from $PRCF$ edge colorings of G and H by naming the colors so that the set of colors on the edges incident to v in G is disjoint from the set of colors on the edges incident to v in H. Since the edge colorings of G and H separately are proper this will make $d_G(v) + d_H(v)$ colors appearing on edges incident to v in $G \cup H$.

Suppose G and H are $PRCF$ edge colored with a and b colors, respectively, and the colors have been named so that $d_G(v) + d_H(v)$ colors appear on the edges in $G \cup H$ incident to v. Let A be the set of colors appearing on G, B the set of colors appearing on H, C the set of $d_G(v)$ colors appearing on the edges incident to v in G, and C' the set of $d_H(v)$ colors appearing on the edges incident to v in H. We want to show that $|A \cup B|$ can be any integer from $M = \max(a, b, d_G(v) + d_H(v))$ to $a + b$. We can control $|A \cup B|$ by renaming colors in $A \setminus C$ and $B \setminus C'$ with colors not in those sets, respectively; clearly no such renaming will make the coloring of $G \cup H$ improper, or create a rainbow cycle, if the coloring before renaming was a $PRCF$ coloring.

Clearly $|A \cup B| \geq M$, no matter how the colors are renamed. If we can rename the colors in $A \setminus C$ and $B \setminus C'$ so that $|A \cup B| = M$, then we are done: for a color that is in both $A \setminus C$ and $B \setminus C'$— and therefore appears on the edges of both $g - v$ and $H - v$—we can change the name of that color on one of $G - v$ or $H - v$ to a color name that did not appear on $G \cup H$ before the renaming, and thus increasing the value of $|A \cup B|$ by 1. Continuing in this way we will arrive at $|A \cup B| = a + b$ when A and B are disjoint.

Without loss of generality, assume $a \leq b$. We will show that by renaming $A \setminus C$ and $B \setminus C'$ we can achieve $|A \cup B| = \max(b, d_G(v) + d_H(v)) = M$. For the rest of this proof, when we command "rename some of the colors in set Y with names of colors in X," any color in X that also appears in Y is not to be used to rename any other color in Y.

Case 1: $b \leq d_G(v) + d_H(v)$.

Then $|B \setminus C'| = b - d_H(v) \leq d_G(v) = |C|$ and $|A \setminus C| = a - d_G(v) \leq b - d_G(v) \leq d_H(v) = |C'|$. In this case rename all of the colors in $B \setminus C'$ with names of colors in C, and all of the colors in $A \setminus C$ with names of colors in C'. Then $|A \cup B| = |C \cup C'| = |C| + |C'| = d_G(v) + d_H(v) = M$.

Case 2: $a \leq d_G(v) + d_H(v) < b$.

Then $|A \setminus C| = a - d_G(v) \leq d_H(v) = |C'|$ and $|C| = d_G(v) < b - d_H(v) = |B \setminus C'|$. In this case, rename all of the colors in $A \setminus C$ with color names from C' and rename the colors in C with names from $B \setminus C'$. Now, every color on G appears on H, so with this renaming, $A \subseteq B$, so $|A \cup B| = |B| = b$.

Case 3: $d_G(v) + d_H(v) < a \leq b$.

Then $d_H(v) = |C'| < a - d_G(v) = |A \setminus C|$ and $d_G(v) = |C| < b - d_H(v) = |B \setminus C'|$. First, rename $d_H(v)$ of the colors in $A \setminus C$ with color names from C', and rename $d_G(v)$ of the colors in $B \setminus C'$ with color names from C. This leaves $|A \setminus C| - d_H(v) = a - (d_G(v) + d_H(v))$ colors on $G - v$ from the original $PRCF$ coloring of G with a colors, and $b - (d_G(v) + d_H(v))$ colors on $H - v$ from the original $PRCF$ coloring of H with b colors. Since $a \leq b$ we can rename all of the remaining $a - (d_G(v) + d_H(v))$ colors in $A \setminus C$ from the original colorig with remaining colors from $B \setminus C'$; again we have that the new A is a subset of the new B, so $|A \cup B| = |B| = b$. $\square$

In structure theorems, it is common (but often difficult) to determine the following question: suppose two graphs G and H have a given property P, then does their cartesian product have property P. Our final set of theorems answers this question for proper JL-colorability.

Theorem 4.4. *If G and H are properly colored with no rainbow cycles, then $G \,\square\, H$ can be properly colored with no rainbow cycles.*

Proof. We shall construct a coloring of $G \,\square\, H$ from the colorings on G and H. Without loss of generality, we can assume that $C[V(G)]$ is disjoint from $C[V(H)]$.

We can color each copy, $G \,\square\, \{w\}$, $w \in V(H)$, of G with the same color scheme as G, and the same for H. In particular, if there is an edge between (u, v) and (u', v') in $G \,\square\, H$ color this edge the same color of the edge uu' in G if $v = v'$ and color the edge the same as vv' if $u = u'$.

First, we show that $G \,\square\, H$ is proper. Let (u, v) be a vertex in$G \,\square\, H$ adjacent to (u_i, v_i), $i = 1, 2$, with $(u_1, v_1) \neq (u_2, v_2)$. Let e_1 be the edge from (u, v) to (u_1, v_1) and e_2 be the edge from (u, v) to (u_2, v_2).

Suppose $u = u_1$. If $u = u_2$, then both e_1 and e_2 received the colors of the edges vv_1 and vv_2 in H. Since vv_1 and vv_2 are incident in H, they must be different colors. This implies e_1 and e_2 are different colors. If $v = v_2$, then the color of e_1 is in $C[V(G)]$ and the color of e_2 is in $C[V(H)]$. Since these two sets were disjoint, e_1 and e_2 must have different colors. A similar argument works if we first suppose $v = v_1$.

Now, we show that this coloring of $G \,\square\, H$ forbids rainbow cycles. If a cycle lies entirely in one of the copies of G, then the cycle cannot be rainbow since G is colored so that no rainbow cycle exists.

Suppose a cycle in $G \,\square\, H$ has vertices in multiple copies of G. The collection of all edges in this cycle with vertices of the form (u, v) and (u, v') where $v \neq v'$ correspond to a closed walk in H, with edge colors from the coloring of H. If this closed walk contains a cycle, then there must be a repeated color, because the coloring of H forbids rainbow cycles. Otherwise, there must be a repeated edge on this closed walk (in fact, every edge on the walk is repeated), which implies that the original cycle in $G \,\square\, H$ is not rainbow. $\square$

Corollary 4.5. *If $a \in SPRCF(G)$, $b \in SPRCF(H)$ and $n = |V(H)|$ then $an + b \in SPRCF(G \,\square\, H)$.*

Proof. We follow the construction used in the proof of Theorem 4.4, except that we require that each copy of G uses a set of colors disjoint from these on any other copy of G and disjoint from these on H. Then $an + b$ colors appear in this construction. $\square$

The next corollary follows immediately with $a = |V(G)| - 1$ and $b = |V(H)| - 1$.

Corollary 4.6. *If G and H are properly JL-colored, then $G \,\square\, H$ can be properly JL-colored.*

We have reached the end of our current set of theorems for proper RCF edge colorings. Although, it is likely there is more "low-hanging fruit" in regards to this topic.

Proving to be a much more difficult task is the subject of the next (and last!) section of this dissertation: finding the set of forbidden subgraphs for a graph to be properly RCF edge colored.

4.2 Forbidden Subgraphs for RCF edge-colorings

As noted at the beginning of this chapter, proper RCF edge colorable graphs have a forbidden subgraph characterization. It is easy to see that both K_3 and $K_{2,4}$ cannot be properly edge colored in order to avoid rainbow cycles. Figure 4.1 shows another graph that cannot be properly colored in order to avoid rainbow cycles.

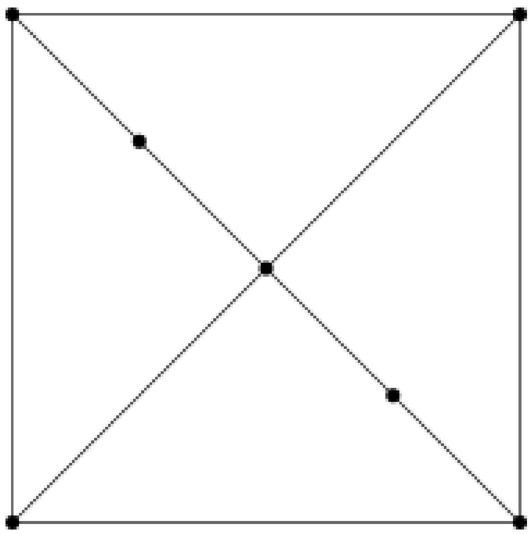

Figure 4.1: A bipartite PRCF-critically bad graph on 7 vertices.

We note that although these are the only graphs in the set of forbidden subgraphs that we will directly identify, there are many more. A colleague has found several more bipartite PRCF-critically bad graphs via a computer algorithm. We believe the set of PRCF-critically bad graphs to be infinite and one primary goal of future work will be to find a nice characterization of an infinite antichain of such graphs.

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