Hamilton Decompositions of Graphs with Primitive Complements

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Hamilton Decompositions of Graphs with Primitive Complements

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Vita

Sibel Özkan, daughter of Salih and Rükiye Özkan, was born on October 4, 1978 in Alanya, Turkey. She has one older brother Önder. She graduated from Anatolian Teacher Training High School, Edirne, Turkey in 1996. She then attended Boğaziçi University in Istanbul as Vehbi Koç Foundation Scholar, and received her Bachelor of Science in Mathematics in January, 2003. She enrolled in Graduate School at Auburn University, in August 2003.

Dissertation Abstract

Hamilton Decompositions of Graphs with Primitive Complements

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A k-factor of a graph G is a k-regular spanning subgraph of G. A Hamilton cycle is a connected 2-factor. A graph G is said to be primitive if it contains no proper factors. A Hamilton decomposition of a graph G is a partition of the edges of G into sets, each of which induces a Hamilton cycle. In this dissertation, by using a graph homomorphism technique called amalgamation, we find necessary and sufficient conditions for the existence of a $2x$-regular graph G on n vertices which:

1. has a Hamilton decomposition, and
2. has a complement in K_n that is primitive.

This extends the conditions studied by Hoffman, Rodger and Rosa [7] who considered maximal sets of Hamilton cycles and 2-factors. It also sheds light on construction approaches to the Hamilton-Waterloo problem.

We also give sufficient conditions, by using amalgamation technique, for the existence of $2x$-regular graph G on mp vertices which:

1. has a Hamilton decomposition, and

2. has a complement in K_m^p that is primitive.

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First, I would like to thank Dr. Chris A. Rodger for being a great mentor to follow. I would also like to thank my committee members Dr. Curt Lindner and Dr. Peter Johnson and the rest of the faculty for making my Auburn years worthwhile and memorable. Your ideas, advice, and support were the main ingredients of my Ph.D. study. Next, I would like to thank my friends for making me feel home in Auburn.

I wish to thank Dr. Haluk Oral for his encouragement and support to start a graduate study. Finally, I would like to thank my family -Rükiye, Salih, and Önder Özkan- for their love and support, not only during my overseas education but also throughout my life.

Style manual or journal used Journal of Approximation Theory (together with the style known as "auphd"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `auphd.sty`.

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Chapter 1

Introduction

1.1 Definitions

Let's start with giving some definitions. A *complete graph* on n vertices, denoted by K_n, is a simple graph in which there is an edge between every pair of its vertices. A *complete multipartite graph* K_m^p is the graph with p parts, each of size m, in which there is an edge between any pair of its vertices if and only if they are in different parts.

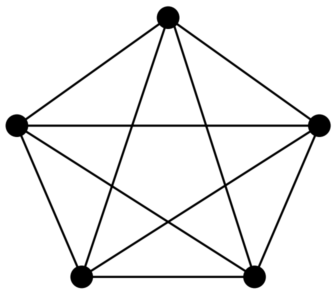

Figure 1.1: K_5

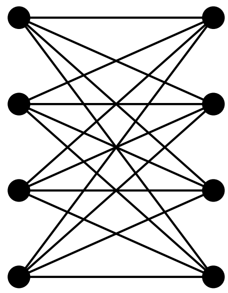

Figure 1.2: K_4^2

A *k-factor* of a graph G is a spanning k-regular subgraph of G. In other words, it is a subgraph of G which uses all the vertices of G, and the degree of each of its vertices is k. G is said to be *primitive* if it contains no k-factors with $1 \leq k < \Delta$ (Δ is the maximum degree of G). A *Hamilton cycle* in a graph G is a spanning cycle in G. So, we can consider Hamilton cycles as connected 2-factors.

A graph which has a Hamilton cycle is called *Hamiltonian*, and a *Hamilton decomposition* of a graph G is a partition of the edges of G into sets, each of which induces a Hamilton cycle. A set S of Hamilton cycles in G is said to be *maximal* if $G - E(S)$ contains

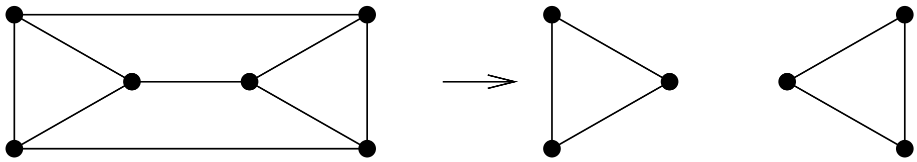

Figure 1.3: A 2-factor in a given graph G

no Hamilton cycles. Similarly, a set S of edge-disjoint k-factors in G is said to be *maximal* if $G - E(S)$ contains no k-factors. In either case, the *spectrum* is the set that contains the integer s if and only if such a maximal set of size s exists.

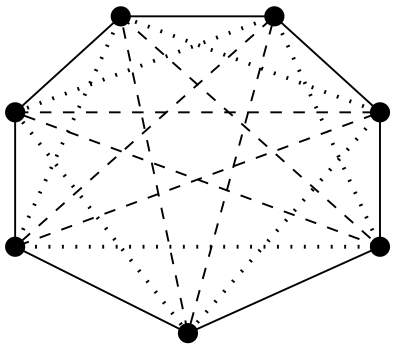

Figure 1.4: A Hamilton decomposition of K_7

In this dissertation, graphs may have multiple edges and loops, with each loop contributing 2 to the degree of the incident vertex. The number of edges between w and v in G is denoted by $m_G(w, v)$ or simply by $m(w, v)$. If G has an edge-coloring, then let $G(i)$ be the subgraph of G induced by the edges colored i, and let $\omega(G)$ be the number of components in G. An edge-coloring of G is said to be *equitable* if for each pair of colors i and j and for each $v \in V(G)$, $|d_{G(i)}(v) - d_{G(j)}(v)| \in \{0, 1\}$, and called *evenly-equitable* if for each pair of colors i and j and for each $v \in V(G)$, $d_{G(i)}(v)$ is even and $|d_{G(i)}(v) - d_{G(j)}(v)| \in \{0, 2\}$.

1.2 History

The popularity of Hamilton cycles rises from optimization problems like the Traveling Salesman Problem: Given a number of cities and the costs of traveling from any city to any other city, what is the cheapest round-trip route that visits each city exactly once and then returns to the starting city? Solution to this problem is equivalent to finding a Hamilton cycle with the least weight in a weighted complete graph. Although finding a Hamilton cycle in a graph is an NP-complete problem, the literature contains classic results which give necessary and sufficient conditions for a graph to be Hamiltonian; some also exist for the decomposition of certain graphs into Hamilton cycles.

In 1847, Kirkman [10] solved the existence and the spectrum problem for 3-cycle systems of K_n. Since then, cycle decomposition has been an interesting problem and many results have appeared on this subject. Hamilton decompositions, where each cycle in the system needs to be a spanning cycle, also dates back to at least 19th century. In 1894, Walecki [12] determined the spectrum for Hamilton decompositions of K_n, and in 1976, Laskar and Auerbach [11] determined this spectrum for complete multipartite graphs.

In 1982, A.J.W. Hilton [5] wrote a gem of a paper, giving necessary and sufficient conditions for an edge-colored copy of K_n to be embedded in an edge-colored copy of K_v in such a way that each color class induces a Hamilton cycle. In so doing, he demonstrated the power of using graph homomorphisms to construct graph decompositions, calling them *amalgamations*. The flexibility of his technique has been demonstrated over the next 25 years in a variety of settings, and this dissertation uses it too!

In 1993, Hoffman, Rodger, and Rosa [7] determined the spectrum for maximal sets of Hamilton cycles of K_n by using amalgamations. They also determined the complete spectrum for maximal sets of 2-factors of K_n by using Tutte's f-factor Theorem, proving the following result.

Theorem 1.1 ([7]) *There exists a maximal set S of x edge-disjoint 2-factors in K_n if and only if*

1. $x = \frac{n-1}{2}$ *if n is odd, and*

2. $\frac{n-\sqrt{n}}{2} \leq x \leq \frac{n-2}{2}$ *if n is even.*

This result stands at the other end to an avenue of research in the literature, setting limits on results that seek to extend sets of edge-disjoint 2-factors of one kind to a (complete) 2-factorization of K_n in which the added 2-factors have another property. For example, Buchanan [4], in his dissertation written under the supervision of A.J.W. Hilton, used amalgamations to show that $K_n - E(U)$ has a Hamilton decomposition for any odd n and for any 2-factor U of K_n in 1997. This result and extensions of it have now also been proved using difference methods [3, 17].

Another such example of research in the literature is the Hamilton-Waterloo problem: For which values of s, t, and z does there exist a 2-factorization of K_n (or of $K_n - F$, the spouse avoiding version of the problem where n is even and F is a 1-factor) in which z of the 2-factors consist entirely of s-cycles, and the rest consist of t-cycles? Horak, Nedela, and Rosa [8] recently addressed this problem, making progress in the case when $s = n$ (so these 2-factors are Hamilton cycles) and $t = 3$. Results also address the situation where both s and t are small (see [1] for example).

Here we continue the tradition begun by A.J.W. Hilton by finding necessary and sufficient conditions on (x, n) to be able to partition $E(K_n)$ into 2 sets, one of which induces a

$2x$-regular graph that has a Hamilton decomposition, the other of which induces a primitive graph. Not only is this an interesting graph decomposition, but it also has the appeal of setting limits on results like those addressed by Horak, Nedela, and Rosa and by Buchanan described above. The results proved here show that when n is even, one could select x edge-disjoint Hamilton cycles for any $x \geq \frac{n-\sqrt{n}}{2}$ and be left with no 2-factors of any type in the complement.

In Chapter 5, we extend the results for the graphs that have primitive complements in complete multipartite graphs K_m^p.

Chapter 2

Primitive graphs

If d is even, then Petersen's Theorem [16] precludes any non-trivial primitive d-regular graphs. It is known that there exist primitive regular graphs for every odd degree d [7]. In fact, there exists a primitive regular graph of order n and odd degree d if and only if $n \geq (d+1)^2$ and n is even. We now define a family of such graphs. For each $d \geq 3$ (d odd) and each $n \geq (d+1)^2$, define a set of d-regular graphs on n vertices $G(n,d)$ as follows:

$$G \in G(n,d) \text{ if and only if}$$

(a) G contains a cut-vertex v such that $G-v$ has a partition into d subgraphs $C_1, C_2, \ldots, C_d$, where $C_1, C_2, \ldots, C_{d-1}$ are components of $G-v$, and C_d is the union of the remaining components of $G-v$,

(b) Each of $C_1, C_2, \ldots, C_{d-1}$ contains exactly $d+2$ vertices, exactly one of which is adjacent to v,

(c) C_d has $n-(d-1)(d+2)-1$ vertices, exactly one of which is adjacent to v, and

(d) G is d-regular.

Note that (c) is implied by the other conditions.

In this dissertation it is easily shown that any $G \in G(n,d)$ is primitive, and then the amalgamation technique is used to show that $K_n - E(G)$ has a Hamilton decomposition for some $G \in G(n,d)$. This shows that the spectrum of edge-disjoint Hamilton cycles that have primitive complements is equal to the spectrum of maximal sets of 2-factors. Computing this first spectrum is the main result of Chapter 4.

Lemma 2.1 *For each odd $d \geq 3$ and even $n \geq (d+1)^2$, any $G \in G(n,d)$ is primitive.*

Proof Note that, by construction, each of $C_1, \ldots, C_d$ in $G - v$ has an odd number of vertices. Suppose there exists a proper d'-factor F. Since $d' < d$, there would be two of $C_1, \ldots, C_d$, say C_i and C_j, such that F contains the edge joining C_i to v but does not contain the edge joining C_j to v. But then in the components induced by $V(C_i)$ and $V(C_j)$ in $F - v$, the number of vertices of odd degree differs by 1. So, one of the components has an odd number of vertices of odd degree, which is a contradiction. ∎

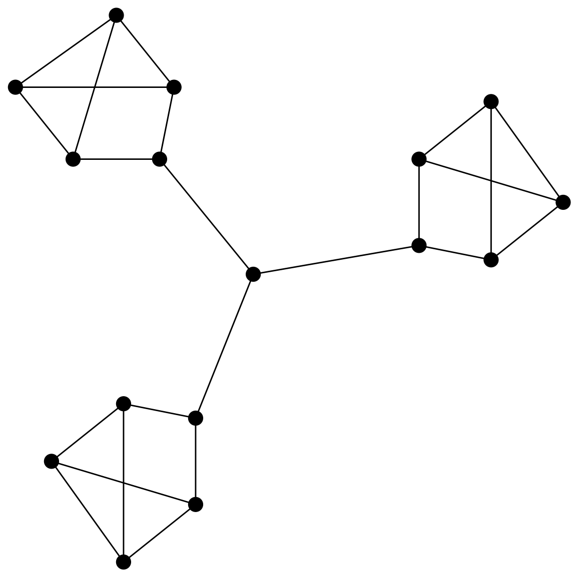

Figure 2.1: The only graph in $G(16,3)$

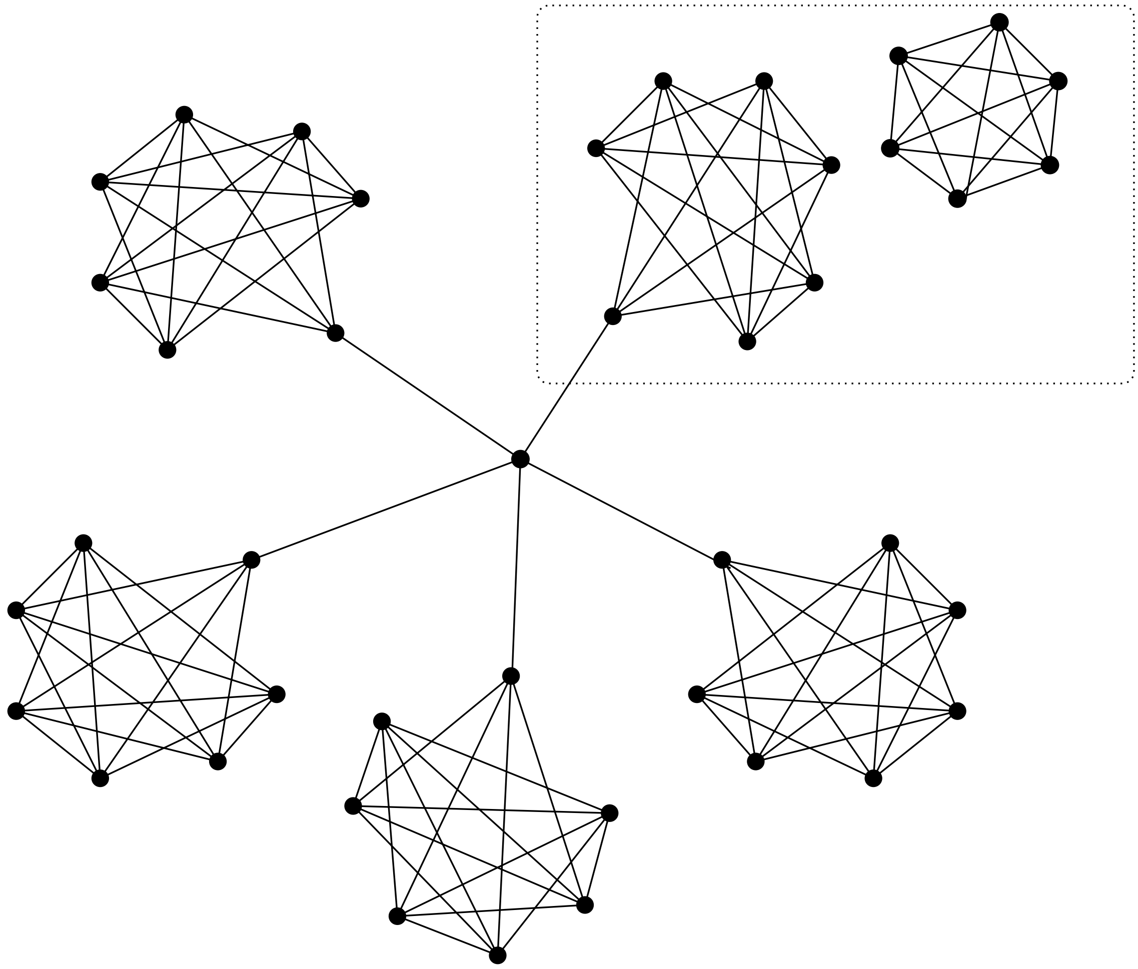

Figure 2.2: One graph in $G(42, 5)$: C_d does not need to be connected

Chapter 3

Amalgamations and preliminary results

Informally, an amalgamation of a graph H is a new graph A, formed by partitioning the vertices of H and representing each element p of the partition P with a single vertex in A, where edges incident with this single vertex are in one-to-one correspondence with the edges incident with original vertices of H in P; so edges in H joining two vertices in p correspond to loops in A. In other words, for each edge $\{u, v\}$ in H, if $u \in p_1$ and $v \in p_2$, then we add an edge $\{p_1, p_2\}$ in A (edges between two vertices in the same element of the partition correspond to loops in A).

Formally, an amalgamation A of a graph H is formed by a graph homomorphism $f : V(H) \to V(A)$, where each vertex v of A represents $\eta(v) = |f^{-1}(v)|$ vertices of H. $\eta(v)$ is called the *amalgamation number* of v, and f is called the *amalgamation function* of H. Notice that any edge coloring of H naturally induces an edge-coloring of A under the homomorphism f. In an edge coloring of A, $A(k)$ represents the subgraph of A induced by the edges colored k.

So, how do we use amalgamations? Given a graph A with amalgamation numbers, one could try to find graphs which have A as an amalgamation. Conceptually, this could be achieved by taking each vertex v with $\eta(v) > 1$ and "peeling out" vertices one by one, at each stage producing a graph H for which A is an amalgamation. H is said to be a *disentanglement* of A. So, every disentanglement H of A has an associated amalgamation function f of H. Furthermore, if A is edge-colored, then this disentanglement naturally induces an edge-coloring of H.

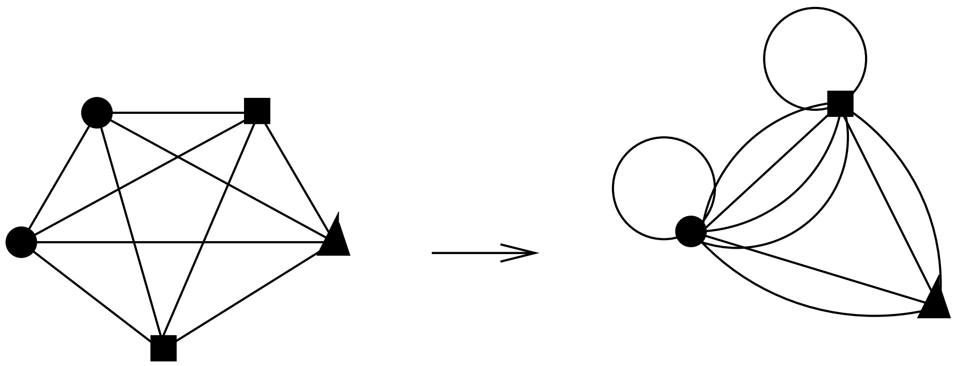

Figure 3.1: An amalgamation of K_5 in which vertices have been partitioned into three parts: circle, square, and triangle

A disentanglement H of A is said to be *regular* if each color class of H is regular, and a disentanglement H of A is said to be *final* if $\eta(h) = 1$ for all $h \in V(H)$.

In Chapter 4, we want to color the edges of an amalgamation of K_n so that when we disentangle the amalgamation, color class 0 will induce a primitive graph and each other color class will induce a Hamilton cycle. In Chapter 5, we use the same technique for complete multipartite graphs. The crucial tool for the proofs is Theorem 3.3, which says that we can disentangle the amalgamation of K_n and that the colored edges incident to a vertex in the amalgamation will split up evenly among the corresponding vertices in the disentanglement. What we need to do is to show that the conditions of Theorem 3.3 hold.

Now, let n and d be fixed. For every graph $G \in G(n, d)$, let $K(G)$ be a 2-edge colored copy of K_n with colors 0 and α in which the edges colored 0 induce a copy of G (in a later proof, the edges colored α will be partitioned into several color classes).

Notice that given n and d, any two graphs in $G(n, d)$

(a) have the same number of edges in C_i, for $1 \leq i \leq d$, and

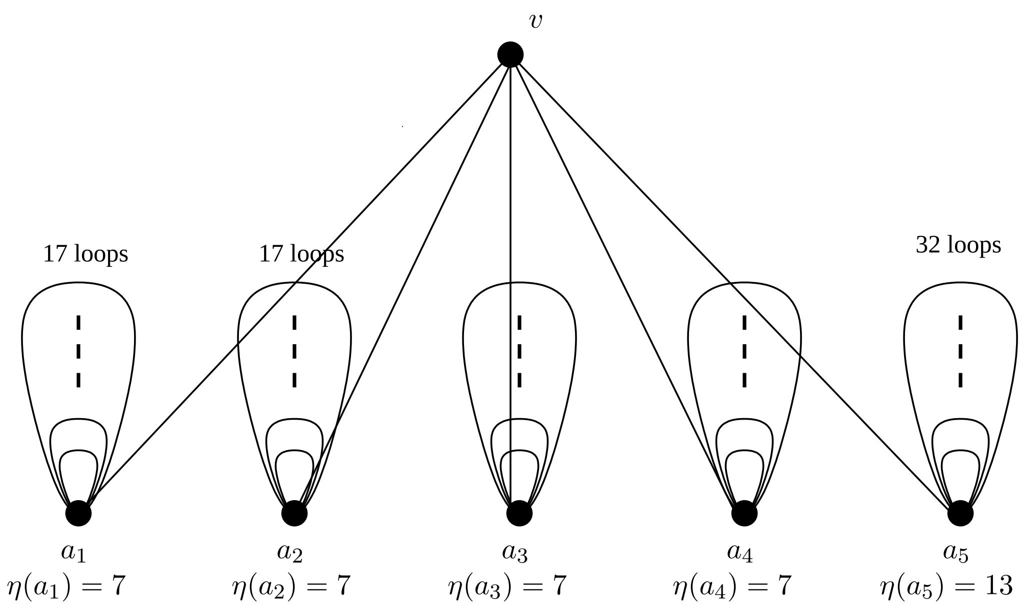

Figure 3.2: An amalgamation of all graphs in $G(42, 5)$ in which vertices of the same component are amalgamated together

(b) have the same number of edges joining the cut-vertex v to the vertices in C_i, for $1 \leq i \leq d$.

Let $K'(G)$ be the amalgamation formed from $K(G)$ using the partition $\{\{v\}, V(C_1), \ldots, V(C_d)\}$. Properties (a) and (b) imply that for any two graphs $G_1, G_2 \in G(n, d)$, the amalgamations $K'(G_1)$ and $K'(G_2)$ are isomorphic. Thus, we let $K(n, d)$ be the unique edge-colored amalgamated graph formed like this. Note that $V(K(n, d)) = \{a_i \mid 0 \leq i \leq d\}$, and that

$$\eta(a_i) = \begin{cases} 1 & \text{if } i = 0, \\ d + 2 & \text{if } 1 \leq i \leq d - 1, \text{ and} \\ n - (d-1)(d+2) - 1 & \text{if } i = d. \end{cases} \qquad (*)$$

Then $K(n, d)$ has the following properties:

(a') There are no edges colored 0 joining a_i and a_j, for $1 \leq i < j \leq d$,

(b') There is exactly one edge colored 0 joining a_i and a_0, for $1 \leq i \leq d$,

(c') $d_{A(0)}(a_i) = \eta(a_i)d$, for $0 \leq i \leq d$, and

(d') $d_{A(\alpha)}(a_i) = (n-1-d)\eta(a_i)$, for $0 \leq i \leq d$.

Lemma 3.1 *Every regular final disentanglement H of $K(n,d)$ has the property that $H(0) \in G(n,d)$.*

Proof Let H be a regular final disentanglement of $K(n,d)$. We check to see that H satisfies properties $(a)-(d)$ in the definition of $G(n,d)$. By (a'), there are no edges colored 0 joining a_i and a_j, for $1 \leq i < j \leq d$, so there is a cut-vertex in $H(0)$ and this satisfies (a). By $(*)$ and (b'), each of $C_1, \ldots, C_{d-1}$ contains exactly $d+2$ vertices and C_d contains $n-(d-1)(d+2)-1$ vertices; in each case exactly one vertex of which is adjacent to cut-vertex v. This proves (b) and (c).

By (c'), $d_{A(0)}(a_i) = \eta(a_i)d$. Since H is regular and it is a final disentanglement, $\eta(h) = 1$ for each $h \in H$. This says that $H(0)$ is d-regular, proving (d). Hence, $H(0) \in G(n,d)$. ∎

We will use the following two results. We will color the edges with colors $0, 1, \ldots, \ell$ and $s = \ell + 1$ in Chapter 4 and with colors $0, 1, \ldots, \ell, \alpha$ and $s = \ell + 2$ in Chapter 5. So, we state the results here for s-edge-colorings.

Lemma 3.2 ([13]) *Let $H \cong K_n$ be an s-edge-colored graph where each color class i is d_i-regular, and let $f : V(H) \to V(G)$ be an amalgamation function with amalgamation numbers given by the function $\eta : V(G) \to \mathbb{N}$. The following conditions hold for any pair of vertices $w, v \in V(G)$:*

(1) $d(w) = \eta(w)(n-1)$,

(2) the number of edges between w *and* v *is* $m(w,v) = \eta(w)\eta(v)$ *if* $w \neq v$,

(3) w *has* $\eta(w)(\eta(w)-1)/2$ *loops, and*

(4) $d_{G(i)}(w) = \eta(w_i)d_i$ *for each color* $i \in \{0,1,\ldots,\ell\}$.

Theorem 3.3 ([13]) *Let* A *be an* s*-edge-colored graph satisfying conditions* $(1)-(4)$ *of Lemma 3.2 for the function* $\eta : V(A) \to \mathbb{N}$*. Then there exists a disentanglement* H *of* A *with amalgamation function* $f(H)$ *such that* $H \cong K_n$ *and the following two conditions hold:*

(i) For any $z \in V(A)$*, degree* $d_{H(i)}(u) \in \{\left\lfloor \frac{d_{A(i)}(z)}{\eta(z)} \right\rfloor, \left\lceil \frac{d_{A(i)}(z)}{\eta(z)} \right\rceil\}$ *for all* $i \in 0,\ldots,\ell$ *and all* $u \in f^{-1}(z)$*, and*

(ii) If $\frac{d_{A(i)}(z)}{\eta(z)}$ *is an even integer for all* $z \in V(A)$*, then* $\omega(A(i)) = \omega(H(i))$.

This result will be used in the following way in Chapter 4: We will color the edges of $K(n,d)$ with 2 colors; 0 and α. Then, we will recolor the edges colored α with $(n-d-1)/2$ colors in such a way that each color class produces a Hamilton cycle in K_n when Theorem 3.3 is applied to the recolored graph. To do this, we will need Lemma 3.4. It will also be clear that the edges colored 0 in K_n induce a copy of G for some $G \in G(n,d)$. An edge-coloring of G is said to be *evenly-equitable* if for each pair of colors i and j and for each $v \in V(G)$, $d_{G(i)}(v)$ is even and $|d_{G(i)}(v) - d_{G(j)}(v)| \in \{0,2\}$.

Lemma 3.4 ([6]) *For each* $m \geq 1$*, each finite eulerian graph has an evenly-equitable edge-coloring with* m *colors.*

Chapter 4

Hamilton decompositions of graphs with primitive complements

In this chapter, we will give a proof to the following theorem.

Theorem 4.1 *There exists a set S of x edge-disjoint Hamilton cycles in K_n such that $K_n - E(S)$ is primitive if and only if*

1. $x = \frac{n-1}{2}$ *if n is odd, and*

2. $\frac{n-\sqrt{n}}{2} \leq x \leq \frac{n-2}{2}$ *if n is even.*

We begin by proving the following theorem, which implies the sufficiency for even n.

Theorem 4.2 *For each odd $d \geq 3$ and each even $n \geq (d+1)^2$, there exists a $G \in G(n,d)$ such that $K_n - E(G)$ has Hamilton decomposition.*

Proof We begin with the 2-edge-colored graph $K(n,d)$ on $d+1$ vertices, which is an amalgamation of K_n and has the amalgamation numbers given in $(*)$.

Let $A = K(n,d)$ for convenience. By multiplying the amalgamation numbers in $(*)$ by $(n-1)$, we get:

$$\begin{aligned} d_A(a_0) &= n-1, \\ d_A(a_i) &= (d+2)(n-1) \quad \text{for } 1 \leq i \leq d-1, \text{ and} \\ d_A(a_d) &= (n-1)(n-(d-1)(d+2)-1), \end{aligned}$$

where $a_0 \in V(A)$ corresponds to the cut-vertex v in $G \in G(n,d)$, and $a_i \in V(A)$ corresponds to the vertices in C_i for $1 \leq i \leq d$.

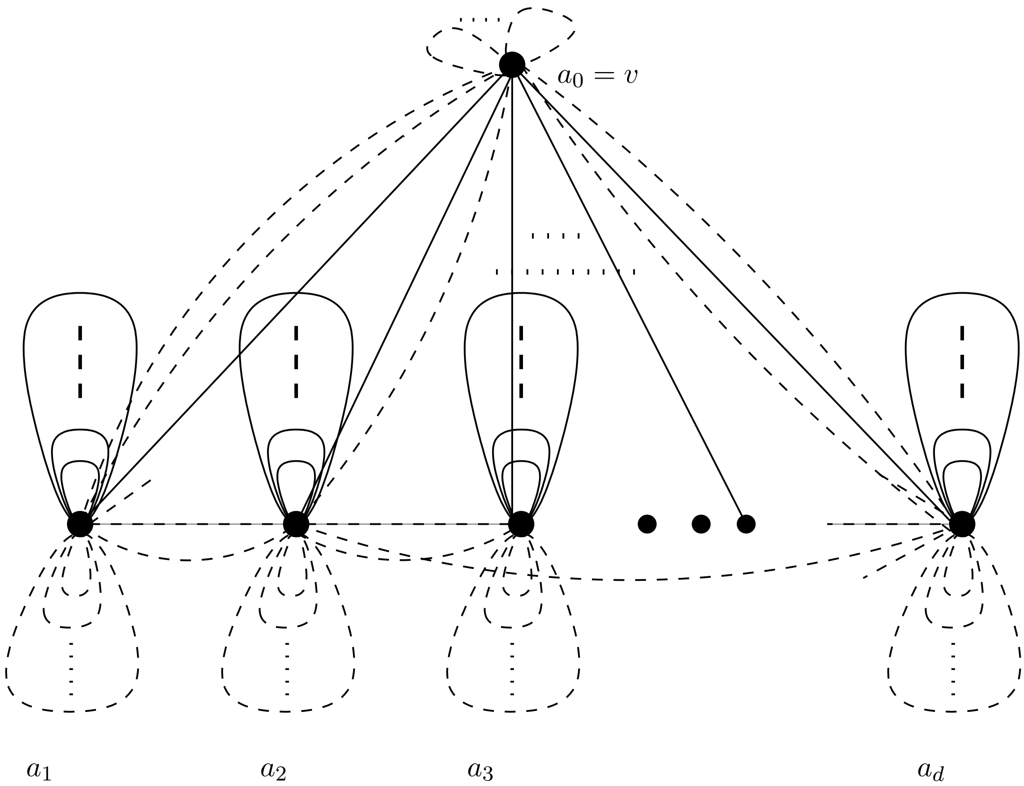

Figure 4.1: $A = K(n, d)$

Next, we recolor the edges of $A(\alpha)$ with colors $1, \ldots, \ell = (n-d-1)/2$ so that, for each color $k \in \{1, 2, \ldots, \ell\}$ and each vertex $z \in V(A)$:

(a) $A(k)$ is connected, and

(b) $d_{A(k)}(z) = 2\eta(z)$ (we already know $d_{A(0)}(z) = d\eta(z)$).

Then, we can apply Theorem 3.3 to obtain the graph $H \cong K_n$ satisfying

(i) for all $u \in f^{-1}(z)$, $d_{H(k)}(u) = \frac{d_{A(k)}(z)}{\eta(z)} = \begin{cases} 2 & \text{for } 1 \leq k \leq \ell, \\ d & \text{for } k = 0, \end{cases}$

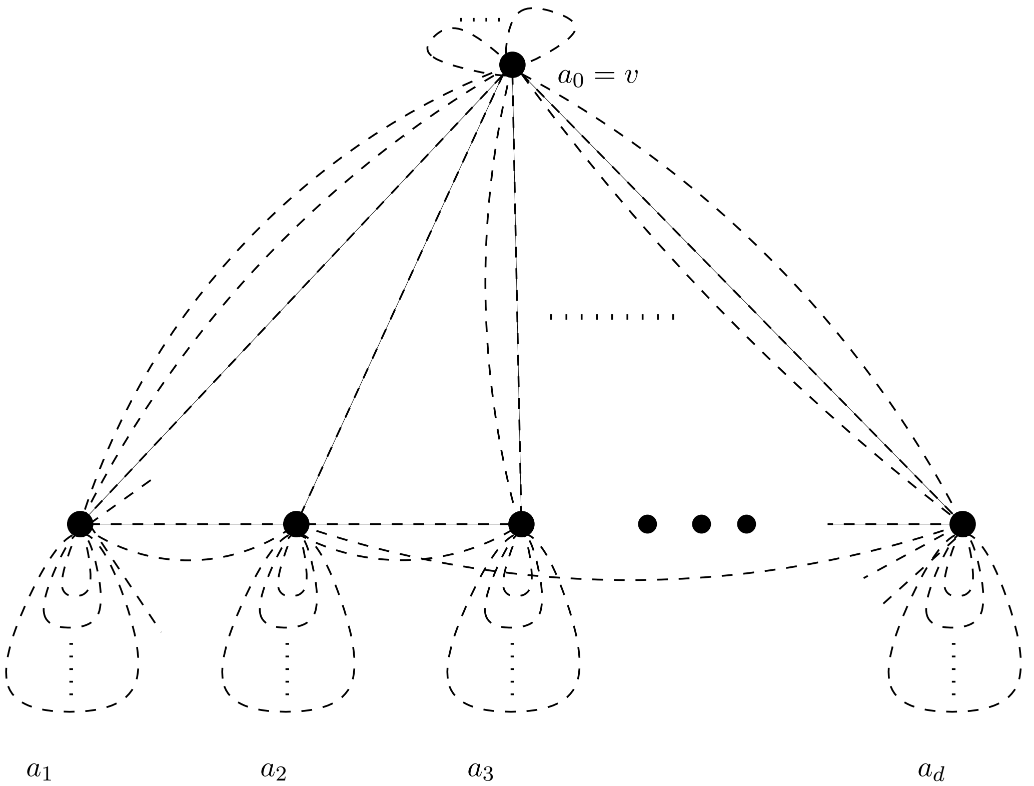

Figure 4.2: $A(\alpha)$ after we removed the edges of the primitive graph

(ii) for $1 \le k \le \ell$, $H(k)$ is connected (since $\frac{d_{A(k)}(z)}{\eta(z)} = 2$ is even for all $z \in V(A)$).

Notice that (i) and (ii) imply that, for each color $k \in \{1, 2, \ldots, \ell\}$, the color class $H(k)$ induces a Hamilton cycle. By Lemma 3.1, the edges colored 0 in H induce a primitive graph. We only need to specify the $(\ell + 1)$-edge-coloring of A.

We now start recoloring the edges of $A(\alpha)$. In the first step, we will guarantee the connectivity of each color class. In the second step, we will boost the degree of each vertex a_i in each color class to $2\eta(a_i)$.

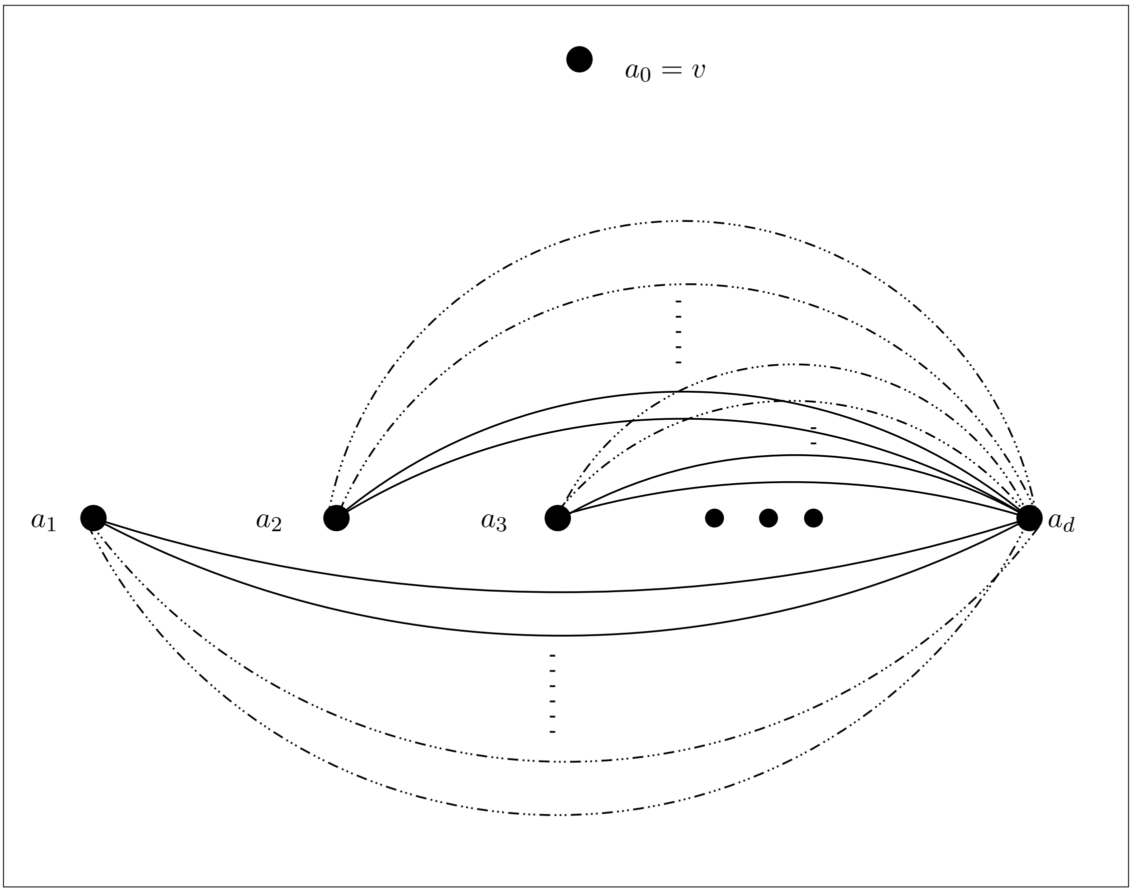

Figure 4.3: First step in recoloring: recolor 2 edges between a_i and a_d with k, for $1 \le k \le \ell$, and $1 \le i \le d-1$

First, for $1 \le i \le d-1$ and for $1 \le k \le \ell$, recolor two edges joining vertices a_i and a_d with color k. To do this, we should check if there are at least $2\ell = n-d-1$ edges colored α between a_i and a_d to ensure this first step is possible. Suppose $1 \le i \le d-1$. All the edges between a_i and a_d are in $A(\alpha)$. Since A is an amalgamation of K_n, there are $\eta(a_i)\eta(a_d) = (d+2)(n-d^2-d+1)$ edges between a_i and a_d. So, we now show that $(d+2)(n-d^2-d+1) \ge n-d-1$.

Recall that by the hypothesis, $n \geq (d+1)^2$ and $d \geq 3$. So,

$$\begin{aligned} n(d+1) \geq (d+1)^3 &= d^3 + 3d^2 + 3d + 1 \\ &> d^3 + 3d^2 - 3, \qquad \text{since } d \geq 3. \end{aligned}$$

Therefore,

$$\begin{aligned} (d+2)(n - d^2 - d + 1) &= nd - d^3 - d^2 + d + 2n - 2d^2 - 2d + 2 \\ &= nd + n + n - d^3 - 3d^2 - d + 2 \\ &= (n - d - 1) + nd + n - d^3 - 3d^2 + 3 \\ &= (n - d - 1) + n(d+1) - (d^3 + 3d^2 - 3) \\ &> n - d - 1. \end{aligned}$$

Hence, we have enough edges in $A(\alpha)$ to recolor two edges between a_i and a_d with color k, for each color $k \in \{1, 2, \ldots, \ell\}$ and each $i \in \{1, 2, \ldots, d-1\}$.

Now, in our second step, we recolor the remaining edges colored α with the same ℓ colors. Let $\bar{A}(\alpha)$ be a graph induced by the remaining edges. Then, $\bar{A}(\alpha)$ is connected since for each $i \in \{1, 2, \ldots, d-1\}$, vertex a_i is joined to a_d with $(d+2)(n-d^2-d+1)-(n-d-1) > 0$ edges and the degree $d_{\bar{A}(\alpha)}(a_i)$ is even:

$$d_{\bar{A}(\alpha)}(a_i) = \begin{cases} 2\ell & \text{for } i = 0, \\ \eta(a_i)2\ell - 2\ell & \text{for } 1 \leq i \leq d-1, \\ \eta(a_i)2\ell - 2\ell(d-1) & \text{for } i = d. \end{cases} \qquad (**)$$

So, $\bar{A}(\alpha)$ is eulerian and, by Lemma 3.4 we can give $\bar{A}(\alpha)$ an evenly equitable edge-coloring with ℓ colors. So, for each $a_i \in V(\bar{A}(\alpha))$, and each $1 \leq k \leq \ell$, $d_{A(k)}(a_i)$ is either $2\left\lfloor \frac{d_{\bar{A}(\alpha)}(a_i)}{2\ell} \right\rfloor$

or $2\left\lceil \frac{d_{\bar{A}(\alpha)}(a_i)}{2\ell} \right\rceil$. Since 2ℓ is a factor of $d_{\bar{A}(\alpha)}(a_i)$ for each vertex $a_i \in V(\bar{A}(\alpha))$, we have

$$d_{A(k)}(a_i) \quad = 2\left\lfloor \frac{d_{\bar{A}(\alpha)}(a_i)}{2\ell} \right\rfloor = 2\left\lceil \frac{d_{\bar{A}(\alpha)}(a_i)}{2\ell} \right\rceil = \frac{d_{\bar{A}(\alpha)}(a_i)}{\ell}.$$

Substituting from $(**)$, we get

$$d_{A(k)}(a_i) = \begin{cases} 2 & \text{for } i = 0, \\ 2(\eta(a_i) - 1) & \text{for } 1 \leq i \leq d-1, \\ 2(\eta(a_i) - d + 1) & \text{for } i = d. \end{cases}$$

For $1 \leq i \leq d-1$, and for $1 \leq k \leq \ell$, a_i is incident with two edges colored k that were recolored in step 1 and is incident with $2\eta(a_i) - 2$ edges colored k that were recolored in step 2; a_i is incident with $2\eta(a_i)$ edges colored k, as required by (b). Similarly, $d_{A(k)}(a_i) = 2\eta(a_i)$ for $i \in \{0, d\}$.

Hence, we have the desired $(\ell+1)$-coloring of $K(n,d)$. So, Theorem 3.3 provides an $(\ell+1)$-edge-coloring of K_n where color 0 induces a primitive graph G and each of colors 1 to ℓ induces a Hamilton cycle in $K_n - E(G)$. ∎

Now, we prove the converse.

Theorem 4.3 *If there exists a set S of x edge-disjoint Hamilton cycles such that $K_n - E(S)$ is primitive, then $x \geq (n - \sqrt{n})/2$ when n is even, and $x = (n-1)/2$ when n is odd.*

Proof If $K_n - E(S)$ is primitive, then it must be regular; say it is d-regular. We consider the cases when n is odd and when n is even.

If n is odd and $x < (n-1)/2$, then since $K_n - E(S)$ is regular of even degree, Petersen's Theorem [16] guarantees that it contains a 2-factor. Hence $K_n - E(S)$ is not primitive.

If n is even, then Hoffman et al. [7] showed that $K_n - E(S)$ can be primitive with degree d if and only if d is odd and $n \geq (d+1)^2$. So, $\sqrt{n} - 1 \geq d$. Since S contains $x = (n-1-d)/2$ edge-disjoint Hamilton cycles, substituting for d gives us:

$$x \geq \frac{n - 1 - \sqrt{n} + 1}{2} = \frac{n - \sqrt{n}}{2}.$$

Hence, we are done. ∎

Theorem 4.2 and Theorem 4.3 together prove Theorem 4.1.

We conclude this chapter with the following avenue for future research! Let $G'(n,d)$ be the more general family of graphs defined by all the properties of graphs in $G(n,d)$ except that properties (b) and (c) are relaxed to allow $C_1, ..., C_d$ to contain any odd number of vertices. It is easy to see that graphs in $G'(n,d)$ are primitive.

Conjecture: *There exists a Hamilton decomposition of* $K_n - E(G)$ *for all* $G \in G'(n,d)$.

Chapter 5

Hamilton decompositions with primitive complements in K_m^p

In this chapter, we give sufficient conditions to find a set of edge-disjoint Hamilton cycles in K_m^p where the complement is primitive. Let's start with giving the preliminary results we will use in the proof of the main theorem.

We will use the following results to partition the vertices of the primitive graph into p parts, each of size m, then use it as the vertex set for K_m^p. A vertex coloring c of a graph G is said to be *equitable* if $|c_i - c_j| \leq 1$ for all colors $1 \leq i, j \leq p$, where c_i is the number of vertices in G colored i.

Theorem 5.1 ([18]) *If G is a graph satisfying $\Delta(G) \leq r$, then G has an equitable $(r+1)$-coloring.*

Lemma 5.2 *Let $d \geq 3$ be odd and $n = mp \geq (d+1)^2$ be even, for some odd m. For $p \geq d+1$, we can give any $G \in G(n,d)$ an equitable p-vertex coloring which induces an equitable vertex coloring in C_d and satisfying $|X_1| = |X_2| = \ldots = |X_p| = m$, where X_k is the set of vertices in G colored k, for $1 \leq k \leq p$.*

Proof By Theorem 5.1, we know that if G is a graph satisfying $\Delta(G) \leq r$, then G has an equitable $(r+1)$-coloring.

Let $G \in G(n,d)$. Since C_d is d-regular, $\Delta(C_d) = d$. Then, for $p \geq d+1$, we can give C_d an equitable p-coloring. Since $G - C_d$ is also d-regular, similarly we can give $G - C_d$ an equitable p-coloring. Since in G these two subgraphs are joined buy a cut-edge, the colors can be named so that the union of these two colorings gives us an equitable p-vertex coloring of G with all color classes of size $\lfloor \frac{n}{p} \rfloor$ or $\lceil \frac{n}{p} \rceil$. But p divides $n = mp$ and the size of all color classes is $\lfloor \frac{n}{p} \rfloor = \lceil \frac{n}{p} \rceil = m$. Hence, we are done. ∎

The next Lemma will help us in the proof of the main result.

Lemma 5.3 *For any graph G, if $\lceil \frac{\Delta(G)-\delta(G)}{k} \rceil = 1$, then in any equitable edge-coloring of G with k-colors, $|c_i(u) - c_i(v)| \leq 2$ for any $u, v \in G$, and any color i, $1 \leq i \leq k$.*

Proof Let $c : E(G) \longmapsto \{1, 2, \ldots, k\}$ be an equitable k-edge-coloring of G. Then, for any $a \in V(G)$, and any $i \in \{1, 2, \ldots, k\}$

$$|c_i(a)| = \lceil \frac{d_G(a)}{k} \rceil \text{ or } \lfloor \frac{d_G(a)}{k} \rfloor$$

Then, for any $u, v \in V(G)$ and any $i \in \{1, 2, \ldots, k\}$,

$$\begin{aligned} |c_i(u) - c_i(v)| &\leq \lceil \tfrac{d_G(u)}{k} \rceil - \lfloor \tfrac{d_G(v)}{k} \rfloor \\ &\leq \lceil \tfrac{\Delta(G)}{k} \rceil - \lfloor \tfrac{\delta(G)}{k} \rfloor \\ &= \lceil \tfrac{\Delta(G)}{k} \rceil - \lceil \tfrac{\delta(G)}{k} \rceil + \lceil \tfrac{\delta(G)}{k} \rceil - \lfloor \tfrac{\delta(G)}{k} \rfloor \\ &\leq \lceil \tfrac{\Delta(G)-\delta(G)}{k} \rceil + 1 \\ &= 2 \end{aligned}$$

∎

We will use Tutte's f-factor Theorem in the proof of Theorem 5.5. Before stating Tutte's f-factor Theorem, let's give necessary definitions. To assist the reader, throughout the section we adopt Tutte's notation [20].

The *valency* of a vertex x in a graph G is the degree of x in G and is denoted by $\text{val}(G, x)$. If f is a function from the vertex set $V(G)$ of G into the set of integers, define another function f' by the rule $f'(x) = \mathrm{val}(G, x) - f(x)$ for each vertex x of G. Given such a function f, an *f-factor* is a spanning subgraph F of G satisfying $\mathrm{val}(F, x) = f(x)$ for each vertex x of G.

A *G-triple* is an ordered triple (S, T, U) where $\{S, T, U\}$ partitions $V(G)$. For any subset S of $V(G)$, $f(S) = \sum_{v \in S} f(v)$. For any disjoint subsets S and T of $V(G)$, $\lambda(S, T)$ denotes the number of edges of G joining S to T (in other sections this would be represented by $m(S, T)$).

If $B = (S, T, U)$ is a G-triple and C is any component of U in G, then define

$$J(B, f, C) = f(C) + \lambda(V(C), T).$$

We say that C is an ODD component if $J(B, f, C)$ is an odd integer. Note that we use capital letters to distinguish it from "odd component" where the number of vertices in the component is odd. The number of ODD components of U in G with respect to B and f is denoted by $h(B, f)$. Now, we define the *deficiency* $\delta(B, f)$ of the G-triple $B = (S, T, U)$ with respect to f, as follows:

$$\delta(B, f) = h(B, f) - f(S) - f'(T) + \lambda(S, T).$$

An *f-barrier* of G is a G-triple $B = (S, T, U)$ such that $\delta(B, f) > 0$. We can now state Tutte's f-factor Theorem.

Theorem 5.4 ([20]) *Given G and f, exactly one of the following statements is true:*

(1) G has an f-factor.

(2) G has an f-barrier.

In other words, if we let f be a vertex-function of a graph G, then G has an f-factor or there exists a G-triple $B = (S, T, U)$ of G with $\delta(B, f) > 0$, but not both.

Now, we can state our theorem which is a generalization of the Erdős-Gallai Theorem. A *multigraph* is a graph in which multiple edges between two vertices are allowed, and a degree sequence is called λ-*multigraphic* if there is a multigraph of index λ with this degree sequence.

Theorem 5.5 *A sequence $\pi = (d_1, d_2, \ldots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \ldots \geq d_p$ and an even sum is multigraphic with multiplicity at most λ if and only if*

$$\sum_{i=1}^{k} d_i \leq \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\} \text{ , for every } k,\ 1 \leq k \leq p.$$

Proof Let's first assume that the sequence $\pi = (d_1, d_2, \ldots, d_p)$ of non-negative integers with $d_1 \geq d_2 \geq \ldots \geq d_p$ and an even sum is multigraphic with multiplicity at most λ and let G be a graph realizing this degree sequence. Then for any set S of k vertices in G, the total degree of the vertices in S is equal to the twice the number of edges in S plus the number of edges between the sets S and $G - S$. The maximum number of edges in S is $\binom{\lambda k}{2}$ and the maximum number of edges between S and $G - S$ is $\sum_{i=k+1}^{p} min\{d_i, \lambda k\}$. Hence, $\sum_{i=1}^{k} d_i \leq \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\}$ follows for every k, $1 \leq k \leq p$.

Now, assume the inequality $\sum_{i=1}^{k} d_i \leq \lambda k(k-1) + \sum_{i=k+1}^{p} min\{d_i, \lambda k\}$ holds for every k, $1 \leq k \leq p$. We want to show that $H = \lambda K_p$ has an f-factor with $f(v_i) = d_i$ for all $v_i \in H$. We will use Tutte's f-factor Theorem and show that $\delta(B, f) \leq 0$ for all $B = (S, T, U)$, where $\{S, T, U\}$ is a partition of $V(H)$.

The value of $\delta(B,f) = h(B,f) - f(S) - f'(T) + \lambda(S,T)$ is greater when $f(S)$ and $f'(T)$ are smaller. We can make $f(S)$ small by putting the vertices with the smallest f value in S and we can make $f'(T)$ small by putting the vertices with greatest f value in T. Since our integer sequence is in a decreasing order, there exist two numbers $t = |T|$ and $s = |S|$ such that δ takes its maximum value when $T = \{v_1, \ldots, v_t\}$ and $S = \{v_{p-s+1}, \ldots, v_p\}$. Then, letting $h = h(B,f)$ and $\delta = \delta(B,f)$ we get

$$\begin{aligned}
\delta &= h - \textstyle\sum_{i=p-s+1}^{p} d_i - \lambda t(p-1) + \sum_{i=1}^{t} d_i + \lambda ts \\
&\leq h - \textstyle\sum_{i=p-s+1}^{p} d_i - \lambda t(p-1) + \lambda t(t-1) + \sum_{i=t+1}^{p} min\{d_i, \lambda t\} + \lambda ts \qquad (1) \\
&= h - \textstyle\sum_{i=p-s+1}^{p} d_i - \lambda t(p-t-s) + \sum_{i=t+1}^{p-s} min\{d_i, \lambda t\} + \sum_{i=p-s+1}^{p} min\{d_i, \lambda t\} \\
&= h - \lambda t(p-t-s) + \textstyle\sum_{i=t+1}^{p-s} min\{d_i, \lambda t\} + \sum_{i=p-s+1}^{p} (min\{d_i, \lambda t\} - d_i) \\
&\leq h \quad (\text{since } -\lambda t(p-t-s) + \textstyle\sum_{i=t+1}^{p-s} min\{d_i, \lambda t\} \leq 0 \qquad (2) \\
&\qquad\quad \text{and } \textstyle\sum_{i=p-s+1}^{p} (min\{d_i, \lambda t\} - d_i) \leq 0) \\
&\leq 1 \quad (\text{since } H[U], \text{ being a subgraph of } \lambda K_p, \text{ is connected, so } h \in \{0,1\}) \qquad (3)
\end{aligned}$$

with $\delta = 1$ if and only if

(4) $\sum_{i=1}^{t} d_i = \lambda t(t-1) + \sum_{i=t+1}^{p} min\{d_i, \lambda t\}$ (from (1)), and

(5) $\sum_{i=t+1}^{p-s} min\{d_i, \lambda t\} - \lambda t(p-t-s) = 0$ (from (2)), and

(6) $\sum_{i=p-s+1}^{p} (min\{d_i, \lambda k\} - d_i) = 0$ (from (2)), and

(7) $h = 1$ (from (3)).

So the result is proved unless (4)-(7) all hold. Now, we will show that if (4)-(6) are true, then $h = 0$.

Condition (4) implies $\lambda t(p-t-s) = \sum_{i=t+1}^{p-s} min\{d_i, \lambda t\}$. Note that this also implies that $d_i \geq \lambda t$ for $i \in \{t+1, \ldots, p-s\}$. Condition (5) implies $\sum_{i=p-s+1}^{p} min\{d_i, \lambda t\} = \sum_{i=p-s+1}^{p} d_i$.

We know $\sum_{i=1}^{p} d_i = f(T) + f(U) + f(S)$ is even. We have

$$\begin{aligned}
\sum_{i=1}^{p} d_i &= f(T) + f(U) + f(S) \\
&= \sum_{i=1}^{t} d_i + f(U) + f(S) \\
&= \lambda t(t-1) + \sum_{i=t+1}^{p} min\{d_i, \lambda t\} + f(U) + f(S) \\
&= \lambda t(t-1) + \sum_{i=t+1}^{p-s} min\{d_i, \lambda t\} + \sum_{i=p-s+1}^{p} min\{d_i, \lambda t\} + f(U) + f(S) \\
&= \lambda t(t-1) + \lambda t(p-t-s) + \sum_{i=p-s+1}^{p} min\{d_i, \lambda t\} + f(U) + f(S) \\
&= \lambda t(t-1) + \lambda t(p-t-s) + \sum_{i=p-s+1}^{p} d_i + f(U) + f(S) \\
&= \lambda t(t-1) + \lambda t(p-t-s) + 2f(S) + f(U).
\end{aligned}$$

Since all of $\lambda t(t-1)$, $2f(S)$, and the left hand side are even, $\lambda t(p-t-s) + f(U) = \lambda(U,T) + f(U)$ is even, so U is EVEN. Therefore $h = 0$. Hence, we have $\delta \leq 0$ for every case and $H = \lambda K_p$ has an f-factor where $f(v_i) = d_i$ for all $v_i \in H$. ∎

We will use the Theorem 5.5 in the proof of the following lemma.

Lemma 5.6 *Let p be even. Suppose $\pi = (d_1, d_2, \ldots, d_p)$ is a sequence of integers with $6 \geq d_1 \geq d_2 \geq \ldots \geq d_p \geq 1$, satisfying*

(i) $d_1 - d_p \leq 4$,

(ii) $\sum_{i=1}^{p} d_i$ *is even, and*

(iii) $d_1 \leq \sum_{i=2}^{p} d_i$.

Then π is λ-multigraphic, where λ can be chosen to satisfy

$$\lambda = \begin{cases} 2 & \textit{if } p \geq 6 \\ 3 & \textit{if } p = 6 \textit{ and } \pi : (5,5,1,1,1,1) \\ 4 & \textit{if } p = 4 \end{cases}$$

Proof Case 1: First, let $p = 4$ and $\lambda = 4$. It is enough to show that we have

$$\sum_{i=1}^{k} d_i \leq 4k(k-1) + \sum_{i=k+1}^{4} min\{d_i, 4k\} \text{ , } \textit{for every } k\textit{, } 1 \leq k \leq 4$$

by Theorem 5.5. We will denote the left hand side of the inequality by LHS and right hand side of the inequality by RHS for simplicity. We will proceed case by case for each k:

$\underline{k = 4}$: LHS ≤ 24 since each vertex may have degree at most 6. RHS $\geq 4.4.3 = 48 > 24$.

$\underline{k = 3}$: Similarly, LHS ≤ 18 and RHS $\geq 4.3.2 + min\{d_4, 12\} > 24 > 18$.

$\underline{k = 2}$: LHS ≤ 12. Clearly RHS $\geq 4.2 + (1 + 1) = 10$, so we only need to consider the case where LHS ≥ 11 but this means at least one of the vertices has degree 6. Since the difference between the degrees can not be more than 4, all other vertices must have degrees at least 2, implying that RHS $\geq 4.2 + (2 + 2) = 12$.

$\underline{k = 1}$: Since each vertex has degree at most 6, LHS ≤ 6. We will analyze this case in two subcases.

If $d_i \leq 4$ for $i \in \{2, 3, 4\}$, then $\sum_{i=2}^{4} min\{d_i, 4k\} = \sum_{i=2}^{4} d_i \geq d_1$ by condition (iii). Hence LHS $\leq$ RHS.

If there is at least one $d_i > 4$ for $i \in \{2, 3, 4\}$, then since other 2 vertices have degree at least 1, RHS $\geq 7 \geq$ LHS.

In all cases when $p = 4$, we have shown that the above inequality holds.

Case 2: Now, let $\lambda = 3$, $p = 6$, and the degree sequence be (5,5,1,1,1,1).

The below figure gives a realization of this degree sequence with $\lambda = 3$.

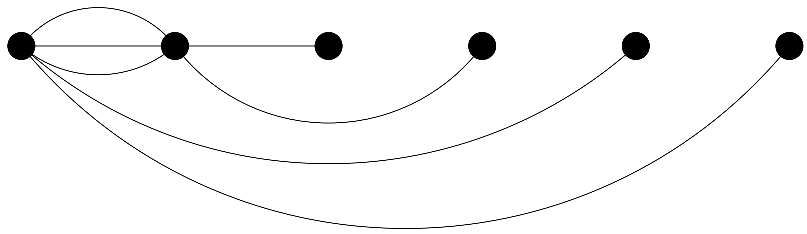

Figure 5.1: A graph with degree sequence $(5, 5, 1, 1, 1, 1)$

Case 3: Now, let $\lambda = 2$, $p \geq 6$, and $\pi \neq (5, 5, 1, 1, 1, 1)$.

It is enough to show that we have

$$\sum_{i=1}^{k} d_i \leq 2k(k-1) + \sum_{i=k+1}^{4} min\{d_i, 2k\} \text{ , } \textit{for every } k, \ 1 \leq k \leq p$$

by Theorem 5.5. We will again proceed case by case for each k.

$\underline{k \geq 4}$: Since each vertex has degree at most 6, we have LHS $\leq 6k$. Also, RHS $\geq 2k(k-1) = k(2k-2) \geq 6k$ for $k \geq 4$.

$\underline{k = 3}$: LHS ≤ 18.

If there is at least one vertex with degree 6, there can not be any vertices with degree 1. So, RHS $\geq 2.3.2 + 3.2 = 18$ since $p \geq 6$. If there is no vertex with degree 6, then LHS ≤ 15, and RHS $\geq 2.3.2 + (1 + 1 + 1) = 15$.

$\underline{k = 2}$: LHS ≤ 12.

If there is at least one vertex with degree 6, there can not be any vertices with degree 1. So, RHS $\geq 2.2.1 + (2 + 2 + 2 + 2) = 12$ since $p \geq 6$.

Suppose $d_1 \leq 5$; So, LHS ≤ 10.

If $d_1 = d_2 = 5$, then since we exclude the case where the sequence is $(5,5,1,1,1,1)$, $\sum_{i=3}^{p} min\{d_i, 2\} \geq 5$. But we can not have odd number of vertices with odd degree, implying that $\sum_{i=3}^{p} min\{d_i, 2\} \geq 6$. So, RHS $\geq 2.2.1 + 6 = 10 =$ LHS.

If LHS $= 9$ with $d_1 = 5$, again since we can not have odd number of vertices with odd degree, $\sum_{i=3}^{p} min\{d_i, 2\} \geq 5$. So, RHS $\geq 2.2.1 + 5 = 9$. If LHS ≤ 8, RHS $\geq 2.2.1 + (1 + 1 + 1 + 1) = 8$ since we have at least 6 parts, and each vertex has degree at least 1.

$\underline{k = 1}$: LHS ≤ 6 since each vertex has degree at most 6.

If $d_1 = 6$, then the rest of the vertices must have degree at least 2. So, RHS $\geq 0 + (2 + 2 + 2 + 2 + 2) = 10 > 6$. If $d_1 \leq 5$, then RHS $\geq 0 + (1 + 1 + 1 + 1 + 1) = 5 \geq$ LHS.

For all the cases we have shown that the inequality holds. Hence, we are done. ∎

We are now ready to prove an important technical result, which is used to obtain memorable corollaries. We first define a graph Γ' that will eventually be shown to be an amalgamation of Hamilton cycles in K_m^p.

Let $G \in G(n,d)$. By Lemma 5.2, we can give G a proper p-vertex coloring c in which $|X_1| = |X_2| = \ldots = |X_p| = m$ where X_i is the set of vertices colored i. Now, let $H(G,c) = K_m^p - E(G)$ and $X_1, X_2, \ldots, X_p$ be the parts of K_m^p.

Recall from Chapter 2 that if $G \in G(n,d)$, then G has a cut-vertex v; one subgraph C_d of $G - v$ may be larger than the other components. Let $X = V(C_d)$, $Y = V(G(n,d)) - X$, and $\epsilon = |E(H(G,c)[Y])|$. Let $\Gamma'(T', B)$ be the multigraph formed from $H(G,c)$ by applying the amalgamation function

$$f(v) = \begin{cases} t_i & \text{if } v \in X \cap X_i \\ b_i & \text{if } v \in Y \cap X_i \end{cases}$$

where $T' = \bigcup_{i=1}^{p} t_i$ and $B = \bigcup_{i=1}^{p} b_i$ (think of t_i and b_i as top and bottom vertices respectively colored i).

By using Lemma 5.2 to color the vertices of G, it follows immediately that $\eta(t_i) \in \{\lfloor \frac{|X|}{p} \rfloor, \lceil \frac{|X|}{p} \rceil\}$ and $\eta(b_i) \in \{\lfloor \frac{|Y|}{p} \rfloor, \lceil \frac{|Y|}{p} \rceil\}$.

Let $\Gamma(T = T' \cup \{\infty\}, B)$ be formed from $\Gamma'(T', B)$, by

1. deleting the edges joining two vertices in X, then
2. adding a new vertex ∞ to T', deleting each edge $\{b_i, b_j\}$, and joining both b_i and b_j to ∞ with an edge instead.

So,

$$\begin{aligned} d_\Gamma(v) &\leq d_{\Gamma'}(v) \text{ for all } v \in T', \\ d_\Gamma(v) &= d_{\Gamma'}(v) \text{ for all } v \in B, \text{ and} \qquad (*) \\ d_\Gamma(\infty) &= 2e(\Gamma'_B). \end{aligned}$$

Now, $\Gamma(T, B)$ is a bipartite graph. Let $\Gamma = \Gamma(T, B)$, $\Gamma_T = \Gamma(T, B)[T]$, and $\Gamma_B = \Gamma(T, B)[B]$ for simplicity.

The following result will be used in the proof of Theorem 5.8, and it is also a vital tool used in proving Theorem 3.3.

Theorem 5.7 ([19]) *Every bipartite multigraph has an equitable k-edge-coloring for all $k \geq 1$.*

We are now ready to prove the main result. The following builds upon Γ' and $H(G, c)$ defined above.

Theorem 5.8 *Let m and $d \geq 3$ be odd, and $p \geq d+1$ be even such that $mp \geq (d+1)^2$. Then there exists a set S of ℓ Hamilton cycles in K_m^p such that $K_m^p - E(S)$ is primitive, if*

$$\begin{cases} \ell \geq \epsilon, \\ \eta(t_i)\eta(t_j) \geq \lambda\ell + 2\lfloor\frac{\ell}{p-1}\rfloor \quad \textit{for any } t_i, t_j \in T', \textit{ and} \\ \frac{\Delta(\Gamma_{T'}) - \delta(\Gamma_{T'})}{2} \leq \ell, \end{cases}$$

where $\lambda \leq 4$ when $p = 4$, $\lambda \leq 3$ when $p = 6$, and $\lambda = 2$ when $p \geq 8$.

Proof Since $n = mp \geq (d+1)^2$, there exists a primitive graph $G \in G(n, d)$ with m, p, and d as assumed. Let $H(G, c)$, Γ, and Γ' be as described in the previous paragraphs. We need to give an ℓ-edge-coloring to the edges of $H(G, c)$ where each color induces an Hamilton cycle. This is done in 3 steps:

(1) the edges of Γ' except for the ones joining two vertices in T are equitably colored;

(2) the remaining edges in Γ' are colored in 3 steps:

  (i) color some edges with to boost the degree of each vertex to an even number in each color class; then

  (ii) ensure each color class is connected; and thirdly

  (iii) color the remaining edges;

(3) Γ' is disentangled.

In order for each color class eventually induce a Hamilton cycle, we need each color k, $1 \leq k \leq \ell$ to appear $2\eta(v)$ times at each vertex $v \in T' \cup B$ in Γ'. Since Γ is bipartite, by Theorem 5.7 we can begin with an equitable 2ℓ-edge-coloring of the edges of Γ (*i.e*: using twice as many colors as we end up with). By $(*)$, we have $d_\Gamma(\infty) = 2\epsilon$. Also, the assumption $\ell \geq \epsilon$ implies $2\ell \geq 2\epsilon$. This implies that every color k, $1 \leq k \leq 2\ell$, appears on at most one of the edges incident with ∞, and appears on exactly $\eta(B)$ edges in Γ (by Theorem 5.7).

Now, define a coloring of the edges of Γ' from Γ as follows:

(a) for each edge $\{b_i, b_j\}$ in Γ', if $\{\infty, b_i\}$ and $\{\infty, b_j\}$ are colored t and k respectively, then color $\{b_i, b_j\}$ with t and recolor all edges colored k with t;

(b) arbitrarily pair the remaining colors and recolor the edges joining the vertices of T to the vertices of B one of the paired colors.

This completes step (1). Notice that, for $1 \leq k \leq \ell$, the number of edges colored k joining vertices in T' to the vertices in B is

$$\begin{cases} 2\eta(B) - 2 & \text{if } k \text{ is a color on an edge incident with } \infty \text{ in } \Gamma \text{, and} \\ 2\eta(B) & \text{otherwise.} \end{cases}$$

Now we can start step (2) in coloring edges of Γ'. Since we assumed $\Delta(\Gamma_{T'}) - \delta(\Gamma_{T'}) \leq 2\ell$, for any vertex $t_i, t_j \in T'$ and for any color k, $1 \leq k \leq 2\ell$, in the 2ℓ-edge coloring of Γ we have $|c_k(t_i) - c_k(t_j)| \leq 2$ by Lemma 5.3. So, in the ℓ-edge coloring of Γ' we get $|c_k(t_i) - c_k(t_j)| \leq 4$. For any color $1 \leq k \leq \ell$ and any vertex $t_i \in T'$, let $c_k = max\{c_k(t_i)\}_{i=1}^{p}$, and define

$$d_k = \begin{cases} c_k + 3 & \text{if } c_k \text{ is odd and } p = 4, \\ c_k + 2 & \text{if } c_k \text{ is even, and} \\ c_k + 1 & \text{otherwise.} \end{cases}$$

Then define a function $DIF_k(t_i) = d_k - c_k(t_i)$. To boost each vertex $t_i \in T'$ to have even degree in each color class, the edges of a subgraph with degree sequence $\{DIF_k(t_i)\}_{i=1}^{p}$ are colored k for $1 \leq k \leq \ell$. First, we need to show $(DIF_k(t_1), DIF_k(t_2), \ldots, DIF_k(t_p))$ is a degree sequence by using Lemma 5.6. We can relabel the vertices so that $DIF_k(t_1) \geq DIF_k(t_2) \geq, \ldots, \geq DIF_k(t_p)$.

Since, for each k, $\sum_{i=1}^{p} c_k(t_i)$ is equal to $2\eta(B)$ or $2\eta(B) - 2$, this sum is even. So, there are even number of vertices with odd $c_k(t_i)$. Clearly $DIF_k(t_i)$ is odd if $c_k(t_i)$ is odd, implying that there are also even number of odd $DIF_k(t_i)$'s, and so $\sum_{i=1}^{p} DIF_k(t_i)$ is even as well. So, Lemma 5.6 (ii) is satisfied.

Since $|c_k(t_i) - c_k(t_j)| \leq 4$ and we add at most 2 to the $c_k(t_i)$'s for $p \geq 6$, the largest possible value of $DIF_k(t_i)$ is 6 when $p \geq 6$.

Suppose $p \geq 6$ and there is at least one vertex t_j with $DIF_k(t_j) = 6$. Then $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq 10 \geq 6$ since the $DIF_k(t_i) \geq 2$ for each vertex. Therefore $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq max\{DIF_k(t_i)\}_{i=1}^{p}$, and so Lemma 5.6 (iii) is satisfied.

Next suppose $p \geq 6$ and $max\{DIF_k(t_i)\}_{i=1}^{p} \leq 5$. Then we have $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq 5$ since the $DIF_k(t_i) \geq 1$ for each vertex. Again, this implies that $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq max\{DIF_k(t_i)\}_{i=1}^{p}$, so Lemma 5.6 (iii) is satisfied.

Similarly, for $p = 4$ and odd c_k, the largest possible $DIF_k(t_i)$ is 7 since we add 3 to the c_k. So, we have $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq 9$ since the $DIF_k(t_i) \geq 3$ for each vertex. Therefore $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq max\{DIF_k(t_i)\}_{i=1}^{p}$ since the $DIF_k(t_i) \leq 7$ for each vertex. For even c_k the largest possible $DIF_k(t_i)$ is 6 since we add 2 to the c_k. So, $max\{DIF_k(t_i)\}_{i=1}^{p} \leq 6$, implying $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq 6$; $\sum_{i=1}^{p} DIF_k(t_i) - max\{DIF_k(t_i)\}_{i=1}^{p} \geq max\{DIF_k(t_i)\}_{i=1}^{p}$. Therefore Lemma 5.6 (iii) is satisfied for every case. Hence, $(DIF_k(t_1), DIF_k(t_2), \ldots, DIF_k(t_p))$ is a λ-multigraphic degree sequence.

Since we assumed that $\eta(t_i)\eta(t_j) > \lambda\ell$ for every $t_i, t_j \in T'$, we have enough edges between any $t_i, t_j \in T$ to realize this degree sequence with multigraph of index λ, for each color k, $1 \leq k \leq \ell$. So, for each $1 \leq i \leq p$ and $1 \leq k \leq \ell$, we can increase the degree of t_i in the k'th color class from $c_k(t_i)$ to d_k by adding this graph. This completes step 2-(i).

Next, to ensure that each color class is connected, we add an Hamilton cycle on p vertices to $\Gamma'_{T'}$ for each color k, for $1 \leq k \leq \ell$. Since p is even, there are $p-1$ Hamilton cycles in a Hamilton decomposition of $2K_p$. Since we need one Hamilton cycle for each color class, we need to have $\lfloor \frac{\ell}{p-1} \rfloor$ copies of $2K_p$ to have enough Hamilton cycles to ensure each of the ℓ color class is connected. The condition $\eta(t_i)\eta(t_j) \geq \lambda\ell + 2\lfloor \frac{\ell}{p-1} \rfloor$ guarantees there are still enough edges between every $t_i, t_j \in T'$ to do that. Now, the degree of each $t_i \in T'$ at each color class is $d_k + 2$, and step 2-(ii) is completed.

By the assumption that m and d are odd and p is even, the degree of each vertex in $K_m^p - E(G)$ is $m(p-1) - d = 2\ell$ is even, so the degree of each vertex a_i of Γ', namely $\eta(a_i)2\ell$, is also even. If we take out the colored edges from Γ', the degree of each vertex in the new graph $\bar{\Gamma}$ is

$$\begin{cases} \eta(a_i)2\ell - \ell(d_k + 2) & \text{for } a_i \in T, \quad \text{and} \\ 0 & \text{for } a_i \in B. \end{cases}$$

Since d_k is even, $d_{\bar{\Gamma}}(t_i) = \eta(a_i)2\ell - \ell(d_k+2)$ is even. So, the graph $\bar{\Gamma}$ is eulerian. By Lemma 3.4, we can give $\bar{\Gamma}$ an evenly equitable edge coloring with ℓ colors. So, for each $t_i \in T'$, and each $1 \le k \le \ell$, $d_{\bar{\Gamma}(k)}(t_i)$ is either $2\left\lfloor \frac{d_{\bar{\Gamma}}(t_i)}{2\ell} \right\rfloor$ or $2\left\lceil \frac{d_{\bar{\Gamma}}(t_i)}{2\ell} \right\rceil$. But d_k is even, so 2ℓ divides $d_{\bar{\Gamma}}(t_i)$. Therefore, for each $a_i \in V(\Gamma')$,

$$d_{\bar{\Gamma}(k)}(a_i) = \begin{cases} 2\eta(a_i) - d_k - 2 & \text{for } a_i \in T', \quad \text{and} \\ 0 & \text{for } a_i \in B. \end{cases}$$

We now gather together all we know. For all $t_i \in T'$ and $1 \le k \le \ell$, t_i is incident with c_k edges colored with color k in step 1, $DIF_k + 2$ edges colored with color k in step 2-(i) and 2-(ii), and $2\eta(t_i) - (c_k + DIF_k(t_i) + 2) = 2\eta(t_i) - (d_k + 2)$ edges colored with color k in step 2-(iii). So now, when we add them up, each vertex t_i is incident with $2\eta(t_i)$ edges colored with k. Because of step 2-(ii), each color class is connected. Similarly, $d_{\bar{\Gamma}(k)}(b_i) = 2\eta(b_i)$ for each $b_i \in B$.

Now, for $1 \le i \le p$ we can add $\binom{\eta(a_i)}{2}$ loops to each $a_i \in V(\Gamma')$ and $\eta(t_i)\eta(b_i)$ edges between $t_i \in T'$ and $b_i \in B$, coloring the new edges and loops with color α. Also, add the edges colored 0 corresponding to $E(G)$, the edges in the primitive graph $G \in G(n,d)$. Now we have the amalgamated graph A described in the Lemma 3.2 where for $1 \le k \le \ell$

(a) $A(k)$ is connected, and

(b) $d_{A(k)}(a_i) = 2\eta(a_i)$, for all $a_i \in A$

where $A(k)$ is the subgraph of A induced by the edges colored k, and $d_{A(k)}(a_i)$ is the degree of a_i in $A(k)$.

We can now apply Theorem 3.3 to obtain the graph H, satisfying

(i) $H \cong K_n$

(ii) for all $u \in f^{-1}(a_i)$,

$$d_{H(k)}(u) = \frac{d_{A(k)}(a_i)}{\eta(a_i)} = \begin{cases} 2 & \text{for } 1 \leq k \leq \ell \\ m-1 & \text{for } k = \alpha \end{cases}$$

(iii) for $1 \leq k \leq \ell$, $H(k)$ is connected, since $\frac{d_{A(k)}(a_i)}{\eta(a_i)} = 2$ is even for all $a_i \in V(A)$.

Hence, we have the desired $s = \ell + 2$ coloring, in which

(1) removing the edges colored α converts K_{mp} to K_m^p,

(2) the edges colored 0 induce a graph in $G(n, d)$, and

(3) each of the other colors induces a Hamilton cycle. ∎

Theorem 5.8 leads to the following Corollary.

Corollary 5.9 *Let p be fixed. Then there exists a set S of ℓ Hamilton cycles in K_m^p for all $m \geq m_d$ such that $K_m^p - E(S)$ is primitive, where m_d is a function of d.*

Proof We will show that for fixed p there exists a constant m_d for each d such that the conditions of the Theorem 5.8 are satisfied for all $m \geq m_d$.

Let's first consider the condition $\ell \geq \epsilon$. We can write it as $\ell - \epsilon = \frac{m(p-1)-d}{2} - \epsilon \geq 0$. Then $f(m) = \frac{m(p-1)-d}{2} - \epsilon$ is an increasing function of m since ϵ is fixed for a fixed d, and f is linear on m with positive coefficient since $p \geq 4$. So, there exists a constant m_1 such that $f(m) \geq 0$ for all $m \geq m_1$.

Now, let's consider the second condition, $\eta(t_i)\eta(t_j) \geq \lambda\ell + 2\lfloor\frac{\ell}{p-1}\rfloor$ for any $t_i, t_j \in T'$. Since $\eta(t_i)\eta(t_j) \geq \lfloor\frac{V(C_d)}{p}\rfloor^2$, it is enough to show $g(m) = \lfloor\frac{V(C_d)}{p}\rfloor^2 - \lambda\ell - 2\lfloor\frac{\ell}{p-1}\rfloor$ is an increasing function.

$$
\begin{aligned}
g(m) &= \lfloor \tfrac{V(C_d)}{p} \rfloor^2 - \lambda \ell - 2\lfloor \tfrac{\ell}{p-1} \rfloor \\
&= \lfloor \tfrac{mp-d^2-d+1}{p} \rfloor^2 - \lambda \tfrac{m(p-1)-d}{2} + 2\lfloor \tfrac{m(p-1)-d}{p-1} \rfloor \\
&= \lfloor m - \tfrac{(d^2+d-1)}{p} \rfloor^2 - \lambda \tfrac{m(p-1)-d}{2} + 2\lfloor m - \tfrac{d}{p-1} \rfloor
\end{aligned}
$$

Obviously, g is a quadratic function on m and concave up. So, there exists a m_2 such that $g(m) \geq 0$ for all $m \geq m_2$.

Lastly, let's consider $\frac{\Delta(\Gamma_{T'})-\delta(\Gamma_{T'})}{2} \leq \ell$. We will show that $h(m) = 2\ell - \Delta(\Gamma_{T'}) + \delta(\Gamma_{T'})$ is an increasing function on m. Since we partitioned the vertices into p parts by giving an equitable p-vertex coloring to C_d and an equitable p-vertex coloring to the rest of the vertices of the primitive graph on mp vertices, if we let $a = \lfloor \frac{|V(C_1) \cup V(C_2) \cup \cdots \cup V(C_{d-1} \cup v)|}{p} \rfloor$, then $\eta(b_i)$ is either a or $a+1$ for all $b_i \in B$. So, for all $t_i \in T'$, $\eta(t_i)$ is either $m-a$ or $m-a-1$. If we let $b = |V(C_1) \cup V(C_2) \cup \cdots \cup V(C_{d-1} \cup v)|$, we get

$$
\begin{aligned}
\Delta(\Gamma_{T'}) &= (m-a)(b-a) - (m-a)d \\
&= mb - ma - ab - a^2 - md + ad
\end{aligned}
$$

and

$$
\begin{aligned}
\delta(\Gamma_{T'}) &= (m-a-1)(b-a-1) - (m-a-1)d \\
&= mb - ma - m - ab + a^2 + 2a - b + 1 - md + ad + d.
\end{aligned}
$$

So,

$$
\begin{aligned}
h(m) &= 2\ell - \Delta(\Gamma_{T'}) + \delta(\Gamma_{T'}) \\
&= m(p-1) - m + 2a - b + 1 \\
&= m(p-2) + 2a - b + 1.
\end{aligned}
$$

Since a and b do not depend on m, and p and d are fixed, $h(m)$ is a linear function of m. Also since $p \geq 4$, $(p-2)$ is positive. Therefore, there exists a constant m_3 such that $h(m) \geq 0$ for all $m \geq m_3$.

Hence, for fixed p, there exists a constant $m_d = max\{m_1, m_2, m_3\}$ for each d so that for all $m \geq m_d$ all three conditions of Theorem 5.8 is satisfied. ∎

The following corollary is an example of the use of Corollary 5.9, specifically evaluating m_d in some cases.

Corollary 5.10 *Let $p = 6$. Then there exists an Hamilton decomposition of a graph G on mp vertices with primitive complement in K_m^p for all $m \geq m_d$ where*

$$m_d = \begin{cases} 15 & \textit{if} \quad d = 3, \quad \textit{and} \\ 113 & \textit{if} \quad d = 5. \end{cases}$$

Proof First, let $d = 3$:

We will first show that $f(m) \geq 0$ for all $m \geq 15$, where $f(m) = \frac{m(p-1)-d}{2} - \epsilon$. For $p = 6$ and $d = 3$, we have $\epsilon = 34$, and so we have $f(m) = \frac{5m-3}{2} - 34$. $f(15) = 36 - 34 = 2 \geq 0$ and since f is an increasing function we have $f(m) \geq 0$ for all $m \geq 15$.

Second, we will show that $g(m) \geq 0$ for all $m \geq 15$, where

$$\begin{aligned} g(m) &= \lfloor \tfrac{V(C_d)}{p} \rfloor^2 - \lambda\ell - 2\lfloor \tfrac{\ell}{p-1} \rfloor \\ &= \lfloor \tfrac{mp-d^2-d+1}{p} \rfloor^2 - \lambda\tfrac{m(p-1)-d}{2} + 2\lfloor \tfrac{m(p-1)-d}{p-1} \rfloor. \end{aligned}$$

Since $\lambda \in \{2, 3\}$ for $p = 6$, and $b = 11$ for $d = 3$, we have $g(m) = \lfloor \frac{6m-11}{6} \rfloor^2 - (\frac{5m-3}{2})3 - 2\lfloor \frac{5m-3}{10} \rfloor$. So, $g(15) = 13^2 - 108 - 14 = 47$ and $g(13) = 11^2 - 117 - 12 = 16$. Since g is quadratic on m with positive coefficient where $g(15) > g(13)$ it is increasing around 15. Hence, $g(15) \geq 0$ implies that $g(m) \geq 0$ for all $m \geq 15$.

Next, let's show $h(m) \geq 0$ for all $m \geq 15$, where

$$\begin{aligned} h(m) &= 2\ell - \Delta(\Gamma_{T'}) + \delta(\Gamma_{T'}) \\ &= m(p-1) - m + 2a - b + 1 \\ &= m(p-2) + 2a - b + 1. \end{aligned}$$

For p = 6 and d = 3 we have $a = 1$ and $b = 11$. So, we have $h(m) = 4m+2-11+1 = 4m-8$ and $h(15) = 52 \geq 0$. Since h is an increasing function, we can say $h(m) \geq 0$ for all $m \geq 15$.

Hence, we are done with the $d = 3$ case.

Now, for $d = 5$, we can proceed similarly.

For $p = 6$ and $d = 5$, we have $a = 4$, $b = 29$, and $\epsilon = 278$. So, we have $f(m) = \frac{5m-5}{2} - 278$. $f(113) = 280 - 278 = 2 \geq 0$ and f is increasing.

Next, we have $g(m) = \lfloor \frac{6m-29}{6} \rfloor^2 - (\frac{5m-5}{2})3 - 2\lfloor \frac{5m-5}{10} \rfloor$. Also, $g(113) = 108^2 - (280)3 - (52)2 = 10,712$ and $g(111) = 106^2 - (275)3 - 2(55) = 10,311$. Since g is quadratic, $g(113) > g(111)$, and $g(113) \geq 0$, $g(m) \geq 0$ for all $m \geq 113$.

Lastly, we have $h(m) = m(p-2) + 2a - b + 1$. For $p = 6$ and $d = 5$, we have $h(m) = 4m + 8 - 29 + 1 = 4m - 20$. So, $h(113) = 432$. Since h is an increasing function, we have $h(m) \geq 0$ for all $m \geq 113$.

Hence, we are done. ∎

Remark: Notice that $f(m_d - 2) < 0$ in both cases, so we can not lower m_d with this approach.

Chapter 6

Conclusion

In this dissertation, we used a very powerful graph homomorphism tool called amalgamation. Also the results of Leach and Rodger [13] we mentioned in Chapter 3 are used in proving our technical results.

In Chapter 4, the problem of Hamilton decompositions of graphs with primitive complements in K_n was completely solved by the following result.

Theorem 4.1 *There exists a set S of x edge-disjoint Hamilton cycles in K_n such that $K_n - E(S)$ is primitive if and only if*

$$\begin{cases} x = \frac{n-1}{2} & \textit{if } n \textit{ is odd, and} \\ \\ x \geq \frac{n-\sqrt{n}}{2} & \textit{if } n \textit{ is even.} \end{cases}$$

And at the and of the Chapter 4, we conjectured that if some conditions on $G(n,d)$ are relaxed to allow another family of primitive graphs $G'(n,d)$, then there still exists a Hamilton decomposition of $K_n - E(G)$ for all $G \in G'(n,d)$.

In Chapter 5, we worked on Hamilton decompositions of graphs with primitive complements in complete multipartite graphs K_m^p. We proved that Erdős-Gallai Theorem can be modified for multigraphs. Then, we proved the following theorem.

Theorem 5.8 *Let m and $d \geq 3$ be odd, and $p \geq d+1$ be even such that $mp \geq (d+1)^2$. Then there exists a set S of ℓ Hamilton cycles in K_m^p such that $K_m^p - E(S)$ is primitive, if*

$$\begin{cases} \ell \geq \epsilon \\ \eta(t_i)\eta(t_j) \geq \lambda\ell + 2\lfloor\frac{\ell}{p-1}\rfloor \quad \text{for any } t_i, t_j \in T', \text{ and} \\ \frac{\Delta(\Gamma_{T'})-\delta(\Gamma_{T'})}{2} \leq \ell \end{cases}$$

where $\lambda \leq 4$ when $p = 4$, $\lambda \leq 3$ when $p = 6$, and $\lambda = 2$ when $p \geq 8$.

Theorem 5.8 leaded us stronger results for multipartite graphs with p fixed and n large enough.

This work also leads us other interesting questions; since the chromatic number of the primitive graph is important to partition its vertices, it would be interesting to know what the smallest primitive graph with given chromatic number is. This is an open question that is likely to be challenging to solve. It is one that needs to be addressed if one is to tackle the case where $p \leq d$.

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