A Brief Survey of Hyperspaces and a Construction of a Whitney Map

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A Brief Survey of Hyperspaces and a Construction of a Whitney Map

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Vita

Sean Andrew O'Neill was born. After being born, he grew and grew and grew, until one day, he stopped growing. Along the way, he befriended a green iguana. He has a mother, a father, a brother and a girlfriend.

Thesis Abstract

A Brief Survey of Hyperspaces and a Construction of a Whitney Map

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This paper is a brief survey of hyperspaces of topological spaces. In particular, the hyperspace of all nonempty compact subsets of a space and the hyperspace of all nonempty subcontinua of a space with the Vietoris topology. An example of a Whitney map on the hyperspace of any metric space is then constructed.

Acknowledgments

Many are those whom I would like to acknowledge in this section. In fact, it would be more prudent to try to list those which I would not like to acknowledge. However, I am a traditionalist and as such will adhere to the custom of naming those to be recognized.

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The employees of Little Italy Pizzeria.

301D

My friends, past, present, and future, must all must be thanked for participating in and/or putting up with my antics.

Finally, Christi Morrow is very deserving of acknowledgment. Thank you for putting up with the distance and being loving and understanding these recent years.

Please note that this is only a partial list. Add yourself if you feel that you are deserving. I have left you space:

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

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Chapter 1

Preliminaries

Definition 1.1 A *topological space*, $(X, \mathcal{T})$, is a set X together with collection, $\mathcal{T}$, of subsets of X such that:

i. $\emptyset, X \in \mathcal{T}$,

ii. If $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$,

iii. If $\mathcal{U} \subset \mathcal{T}$ then $\cup \mathcal{U} \in \mathcal{T}$. The elements of $\mathcal{T}$ are called the *open sets*.

Definition 1.2 A collection of subsets, $\mathcal{B}$, of X is a *base* for a topology, T, on X if:

i. For every $x \in X$ there is some $B \in \mathcal{B}$ such that $x \in B$,

ii. If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

The base $\mathcal{B}$ is said to *generate* the topology on X.

Definition 1.3 $C \subset X$ is *closed* if $X \setminus C$ is open.

Definition 1.4 $x \in X$ is a *limit point* of $Y \subset X$ if every open U with $x \in U$ has $U \cap Y \neq \emptyset$.

Definition 1.5 The *closure* of $M \subset X$, denoted $\overline{M}$, is the set M together with its limit points.

Definition 1.6 A topological space is *Hausdorff* if for every pair of distinct points there exists a pair of disjoint open sets, each containing one point respectively.

Definition 1.7 A topological space is *regular* if for every point and closed subset not containing that point, there exist disjoint open sets such that one contains the point and the other contains the closed subset.

Definition 1.8 A topological space is *normal* if for every pair of disjoint closed subsets there exist disjoint open sets such that each contain one of the sets.

Definition 1.9 A function $d : X \times X \to \mathbb{R}$ is called a *metric* if:

i. $d(x, y) \geq 0$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

ii. $d(x, y) = d(y, x)$ for every $x, y \in X$,

iii. (Triangle Inequality) For every $x, y, z \in X$, $d(x, z) + d(z, y) \geq d(x, y)$.

Definition 1.10 The *ϵ-ball* about $x \in X$, denoted $b(x, \epsilon)$ is the set of all points $y \in Y$ such that $d(x, y) < \epsilon$.

Definition 1.11 A topological space X is said to be *metrizable* with metric d if the set of ϵ-balls generated by d form a base for the topology. X is also said to be a *metric space*.

Definition 1.12 An *open cover* of a topological space X is a collection, $mathcalU$, of open subsets of X such that $X \subset \cup\mathcal{U}$. A subset of $\mathcal{U}$ that contains X in its union is called a *subcover* of $\mathcal{U}$.

Definition 1.13 A topological space X is *compact* if every open cover has a finite subcover.

Definition 1.14 Let X be a topological space. Two subsets H and K of X are called *mutually separated* if neither set contains a point or a limit point of the other.

Definition 1.15 A topological space X is *connected* if it is not the union of two non-empty mutually separated subsets.

Definition 1.16 A topological space X is a *continuum* if X is Hausdorff and both connected and compact.

Definition 1.17 A topological space X is a *metric continuum* if it is a continuum and metrizable.

Definition 1.18 If X and Y are topological spaces, a function $f : X \to Y$ is *continuous* if for every open $U \subset Y$ and $x \in X$ with $f(x) \in U$, there exists an open $V \subset X$ with $x \in V$ such that $f(V) \subset U$. Equivalently, for every open $U \subset Y$, $f^{-1}(U)$ is open in X. Also, for metric spaces in particular, for every $x \in X$ and $\epsilon > o$ there exists $\delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$, $f(x)$ and $f(y)$ are within ϵ of each other in Y.

Definition 1.19 A function $f : X \to Y$ is *open* if for every open $V \subset X$, $f(V)$ is open in the image of X.

Definition 1.20 A function $f : X \to Y$ is a *homeomorphic embedding* of X into Y if f is one-to-one, continuous and open. X is said to be *homeomorphic* to its image in Y. If f is onto then X and Y are said to be *homeomorphic*.

Most of the following basic theorems may be found in one or more of [1], [2], and [3]. The proofs of these theorems are omitted, but may be found in [1].

Theorem 1.21 A closed subset of a compact space is compact.

Theorem 1.22 A compact subset of a Hausdorff space is closed.

Theorem 1.23 If H and K are disjoint compact subsets of a Hausdorff space X then there is a pair of disjoint open sets, each containing one.

Theorem 1.24 The continuous image of a compact set is compact.

Theorem 1.25 If f is a continuous one-to-one map from a compact space to a Hausdorff space then f is a homeomorphic embedding.

Theorem 1.26 (Tychonoff) Any product of compact sets is compact.

Theorem 1.27 Let B be a basis for a topological space X. Then every open set of X is a union of members of B.

Theorem 1.28 The following are equivalent:

i. $f : X \to Y$ is a continuous function from topological space X to topological space Y.

ii. If O is a (basic) open set in Y, then $f^{-1}(O)$ is open in X.

Theorem 1.29 If X is a compact Hausdorff space, then X is regular.

Theorem 1.30 If X is a compact Hausdorff space, then X is normal.

Theorem 1.31 If X is metrizable, then X is Hausdorff, regular, and normal.

Theorem 1.32 If X is regular, then X is Hausdorff.

Theorem 1.33 If X is normal, then X is regular.

Theorem 1.34 (Zorn's Lemma) Let A be a nonempty partially ordered set. Then if every chain has an upper bound, A has a maximal element.

Theorem 1.35 If X is a metric space and every sequence has a convergent subsequence, then X is compact.

Chapter 2

Basic Properties of Hyperspaces

Definition 2.1 Suppose that X is a topological space. Then the *hyperspace* of X, denoted by 2^X is the space of nonempty compact subsets of X together with the following types of sets forming a base for its topology. Suppose that $\{U_1, U_2, .., U_n\} = \mathcal{U}$ is a finite collection of open subsets of X, then $R(\mathcal{U}) = R(U_1, U_2, ...U_n) = \{K \in 2^X : K \subset \cup_{i=1}^n U_i$ and for all $1 \leq i \leq n, K \cap U_i \neq \emptyset\}$. The topology which this base generates is called the *Vietoris* topology.

Before moving on, let us indeed see that these sets form a base for a topology on 2^X.

Theorem 2.2 The sets of the form $R(\mathcal{U}) = \{K \in 2^X : K \subset \cup_{i=1}^n U_i$ and for all $1 \leq i \leq n, K \cap U_i \neq \emptyset\}$ where $\mathcal{U} = \{U_1, U_2, ...U_n\}$ is a finite collection of open sets of X form a base on 2^X.

Proof: First, observe that every $K \in 2^X$ is in $R(X)$. Now let $\mathcal{U} = \{U_1, ..., U_n\}$ and $\mathcal{V} = \{V_1, ..., V_m\}$ be finite collections of open subsets of X and suppose $K \in 2^X$ with $K \subset R(\mathcal{U}) \cap R(\mathcal{V})$. From this we can see that $K \subset [\cup_{i=1}^n U_i] \cap [\cup_{j=1}^m V_j]$. For every $i \leq n$ and $j \leq m$ let $O_{i,j} = U_i \cap V_j$. Define $\mathcal{O}$ to be those $O_{i,j}$ that have $K \cap O_{i,j} \neq \emptyset$. It is clear that $R(\mathcal{O})$ is a member of the collection of subsets of 2^X in the hypothesis. To see that $K \in R(\mathcal{O})$, first note that $K \cap O \neq \emptyset$ for all $O \in \mathcal{O}$ by definition. Next, $K \subset \cup\mathcal{O}$ for if $x \in K$ then for some $i \leq n$ and $j \leq m$, $x \in U_i$ and $x \in V_j$, hence $x \in K \cap O_{i,j}$ which implies that $O_{i,j} \in \mathcal{O}$, and so $x \in O_{i,j} \subset \cup\mathcal{O}$. Finally, we must see that $R(\mathcal{O}) \subset R(\mathcal{U}) \cap R(\mathcal{V})$. First show $R(\mathcal{O}) \subset R(\mathcal{U})$. Let $H \in R(\mathcal{O})$ then $H \subset \cup\mathcal{O} \subset \cup\mathcal{U}$ since each $O \in \mathcal{O}$ is a subset of some $V \in \mathcal{V}$. Now, if $i \leq n$ then since $K \cap U_i \neq \emptyset$ there exists $j \leq m$ so that $K \cap U_i \cap V_j \neq \emptyset$

and so $O_{i,j} \in \mathcal{O}$. Since $H \in R(\mathcal{O})$, $\emptyset \neq H \cap O_{i,j} \subset H \cap U_i$, and so $H \in R(\mathcal{U})$. By a similar argument, $H \in R(\mathcal{V})$ and so $R(\mathcal{O}) \subset R(\mathcal{U}) \cap R(\mathcal{V})$.

Theorem 2.3 If X is Hausdorff then 2^X is Hausdorff.

Proof: Suppose $H, K \in 2^X$ such that $H \neq K$. Without loss of generality, assume there exists $x \in H \setminus K$. Then there exist disjoint open $U, V \subset X$ with $x \in U$ and $K \subset V$. Then $H \in R(U, X)$ and $K \in R(V)$. If $L \in R(U, X)$ then $L \cap U \neq \emptyset$ and $L \not\subset V$, hence $L \notin R(V)$. If $L \in R(V)$ then $L \subset V$ implying that $L \cap U = \emptyset$, hence $L \notin R(U, X)$. From this it can be concluded that $R(U, X) \cap R(V) = \emptyset$, thus 2^X is Hausdorff.

Example 2.4 2^I contains a Hilbert Cube.

It can be shown that $X = \prod_{i=1}^{\infty}[\frac{1}{2i}, \frac{1}{2i-1}]$ is embeddable in 2^I. One might ask, "How?" Well this is how: define $f : X \to 2^I$ by $f(x) = \{x_i\}_{i=1}^{\infty} \cup \{0\}$ where $x = (x_i)_{i=1}^{\infty} \in X$. It is clear that $f(x) \subset I$ for each $x \in X$ and is closed since $0 \in f(x)$ and 0 being its only limit point. Hence $f(x)$ is compact for every $x \in X$ and is in 2^I. It is also clear that f is one-to-one. To see that f is continuous, let $\mathcal{U} = \{U_1, U_2, ..., U_n\}$ and suppose that $f(x) \in R(\mathcal{U})$. Assume $0 \in U_1$, hence there exists an N such that if $i \geq N$ then $[\frac{1}{2i}, \frac{1}{2i-1}] \subset U_1$. For each $i < N$ define $\mathcal{U}_i = \{U \in \mathcal{U} : x_i \in U\}$ and $X_i = [\frac{1}{2i}, \frac{1}{2i-1}] \cap [\cap \mathcal{U}]$. Now, if for some $1 < j \leq n$, $U_j \notin \mathcal{U}_i$ for any $i < N$, then since $f(x) \in R(\mathcal{U})$ there is some i such that $x_i \in U_j$ since the $x_i's$ converge to 0, so in this case define $X_i = [\frac{1}{2i}, \frac{1}{2i-1}] \cap U_j$. For all other i's define $X_i = [\frac{1}{2i}, \frac{1}{2i-1}]$. So, $\prod_{i=1}^{\infty} X_i \subset X$ is open and if $y \in \prod_{i=1}^{\infty} X_i$, $f(y) \in R(\mathcal{U})$ based on the construction, hence f is continuous. Since X is compact and 2^I is Hausdorff, f is a homeomorphic embedding.

Definition 2.5 Let $C(X)$ denote the subspace of 2^X consisting of all nonempty subcontinua in X.

Example 2.6 $C(I)$, where I is the unit interval.

Note that every nonempty subcontinuum of I is of the form $[x, y]$ where $x \leq y$. So, it makes sense that perhaps $C(I)$ might be homeomorphic to $X = \{(x, y) \in I \times I : x \leq y\} \subset I \times I$, and hence be homeomorphic to $I \times I$. To see this, define $f : X \to C(I)$ by $f((x, y)) = [x, y]$ for each $(x, y) \in I \times I$. It is clear that f is one-to-one and onto. To see that f is continuous, let $R(\mathcal{U}) \subset C(I)$ be a basic open set and $[x, y] \in R(\mathcal{U})$. Let $U_x \subset \cap\{U \in \mathcal{U} : x \in U\}$ and $U_y \subset \cap\{U \in \mathcal{U} : y \in U\}$, be connected open intervals in I each containing x or y respectively. Further, assume the upper boundary of U_x is less than some element in $[x, y] \cap U$ and the lower boundary of U_y is greater than the same element in $[x, y] \cap U$ for every $U \in \mathcal{U}$. Now, $(x, y) \in U_x \times U_y \subset I \times I$ which is open. Let $(w, z) \in U_x \times U_y$. If $t \in [w, z]$ then at least one of the following is true: $t \in [w, x]$, $t \in [x, y]$, $t \in [y, z]$, all of which are covered by $\mathcal{U}$ and so $[w, z] \subset \cup\mathcal{U}$. Next, for every $U \in \mathcal{U}$ such that either x or $y \in U$ then $U \cap [w, z] \neq \emptyset$. If $U \in \mathcal{U}$ contains neither x nor y, then by the construction of U_x and U_y, there is some $t \in U$ such that $w < t < z$, i.e. $t \in [w, z]$ hence $[w, z] \cap U \neq \emptyset$ and $[w, z] \in R(\mathcal{U})$. From this we can conclude that f is continuous and thus a homeomorphism.

Example 2.7 $C(S^1)$

If each proper subcontinuum is identified with its length and midpoint and the entire circle as a point at the tip it is not too difficult to see that $C(S^1)$ is a cone, and thus homeomorphic to $I \times I$.

Theorem 2.8 $C(X)$ is a closed subset of 2^X.

Proof: We shall see this by showing that $2^X \setminus C(X)$ is open. Let $K \in 2^X \setminus C(X)$, then since K is not connected, there exist $U, V \subset X$ open and disjoint such that $K \subset U \cup V$ and

K has nonempty intersection with each. It is clear that $K \in R(U, V)$ and that $R(U, V) \subset 2^X \setminus C(X)$, hence $2^X \setminus C(X)$ is open, and $C(X)$ is closed.

Theorem 2.9 If X is a metric space then 2^X is also a metric space.

Proof: Suppose X is a metric space with bounded metric d. Define D on 2^X by $D(H, K) = max\{sup_{x \in H}\{d(x, K)\}, sup_{x \in K}\{d(x, H)\}\}$ for every $H, K \in 2^X$. Hence forth, B_ϵ will denote a ball in 2^X and b_ϵ will denote a ball in the underlying space X.

First let us see that D defines a metric on 2^X. If $H = K$, $d(x, K) \leq d(x, x) = 0$ for every $x \in H$ and $d(x, H) \leq d(x, x) = 0$ for every $x \in K$, so $D(H, K) = 0$. Since each is compact and hence closed, the implication reverses, so $D(H, K) = 0$ if and only if $H = K$. It is also clear that symmetry holds from the definition of D. Now, suppose $H, K, G \in 2^X$, and assume $D(H, K) = \epsilon_1$ and $D(K, G) = \epsilon_2$. Since $D(H, K) = \epsilon_1$, $d(h, K) \leq \epsilon_1$ for every $h \in H$. So for every $h \in H$ there exists $k_h \in K$ with $d(h, k_h) \leq \epsilon_1$. Similarly, there exists $g_h \in G$ with $d(k_h, g_h) \leq \epsilon_2$. So for every $h \in H$ there exists $g_h \in G$ such that $d(h, g_h) \leq \epsilon_1 + \epsilon_2$ and so $d(h, G) \leq \epsilon_1 + \epsilon_2$ for every $h \in H$. Similarly $d(g, H) \leq \epsilon_1 + \epsilon_2$ for every $g \in G$ and so $D(H, G) \leq \epsilon_1 + \epsilon_2$. Hence D is a metric.

Now to see that D generates the topology on 2^X, let $U \subset 2^X$ be open in the metric topology, $K \in U$ and $\epsilon > 0$ such that $B_\epsilon(K) \subset U$. Then $\{b_{\epsilon/3}(k) : k \in K\}$ is an open cover of K in X. By compactness of K, there exist $k_1, k_2, ..., k_n \in K$ such that $\{b_{\epsilon/3}(k_1), b_{\epsilon/3}(k_2), ..., b_{\epsilon/3}(k_n)\} = \mathcal{V}$ cover K. Clearly $K \in R(\mathcal{V}) = R$ which is a basic open set in the topology on 2^X. Now, let $H \in R$ then if $k \in K$, $k \in b_{\epsilon/3}(k_i)$ for some $i \leq n$. Since $b_{\epsilon/3}(k_1) \cap H \neq \emptyset$ let $h \in b_{\epsilon/3}(k_1) \cap H$ then $d(k, h) \leq d(k, k_i) + d(k_i, h) < \epsilon/3 + \epsilon/3 = 2\epsilon/3$ and so $sup_{k \in K}\{d(k, H)\} \leq 2\epsilon/3 < \epsilon$. Also, since $H \subset \cup_{i=0}^n b_{\epsilon/3}(k_i)$, $sup_{h \in H}\{d(h, K)\} < \epsilon$ hence $D(H, K) < \epsilon$. So, $H \in B_\epsilon(K) \subset U$ which means U is open in the Vietoris topology.

Next let $U \subset 2^X$ be open in the Vietoris topology and let $K \in U$. Then there is a finite collection, $\mathcal{U}$, of open subsets of X so that $K \in R(\mathcal{U}) \subset U$. For every $k \in K$ let $\epsilon_k > 0$ such that $b_{\epsilon_k}(k) \subset U$ for every $U \in \mathcal{U}$ containing k. Since K is compact let $k_1, k_2, ..., k_n \in K$

such that $\{b_{\epsilon_{k_i}/2}(k) : i \leq n\}$ cover K and without loss of generality, assume that for every $V \in \mathcal{U}$ that for some $i \leq n$, $b_{\epsilon_{k_i}/2}(k_i) \subset V$. Let $\epsilon = min\{\frac{\epsilon_{k_i}}{4} : i \leq n\}$. If $D(H, K) < \epsilon$ then for every $i \leq n$ there is an $h \in H$ such that $d(h, k_i) < \epsilon < \frac{\epsilon_{k_i}}{2}$ and so $V \cap H \neq \emptyset$ for every $V \in \mathcal{U}$. Also, for every $h \in H$ there exists a $k \in K$ such that $d(h, k) < \epsilon$. Since $k \in b_{\epsilon_{k_i}/2}(k_i)$ for some $i \leq n$, $d(h, k_i) \leq d(h, k) + d(k, k_i) < \epsilon + \frac{\epsilon_{k_i}}{2} < \epsilon_i$, which implies that $H \subset \cup_{i=0}^{n} b_{\epsilon_{k_i}}(k_i) \subset \cup\mathcal{U}$, hence $H \in R(\mathcal{U}) \subset U$. So U is open with respect to the metric D, and so D generates the topology on 2^X.

Note that this also makes $C(X)$ a metric space if X is.

Example 2.10 The following metrics are equivalent to that defined in Theorem 2.9: Suppose that X is a metric space with a bounded metric d . If $H, K \in 2^X$ define

$$D_1(H, K) = inf\{\epsilon | H \subset \cup_{x \in K} b_\epsilon(x) \text{ and } K \subset \cup_{x \in H} b_\epsilon(x)\},$$

and

$D_2(H, K) = sup\{\epsilon|$ there is a point p $\in H$ so that $B_\epsilon(p) \cap K = \emptyset$ or there is a point $p \in K$ so that $B_\epsilon(p) \cap H = \emptyset\}$.

Not only do these metrics generate the same topology on 2^X as D, but in fact for each pair $H, K \in 2^X$, these metrics produce the exact same value as D.

Suppose $D(H, K) = \epsilon$. Without loss of generality, assume there is some $h \in H$ such that $d(h, K) = \epsilon$. Then for every $k \in K$, $h \notin b_\epsilon(k)$ hence, $D_1(H, K) \geq \epsilon = D(H, K)$.

Next, suppose $D_1(H, K) = \epsilon$. Without loss of generality, assume for every $\delta < \epsilon$, $H \nsubseteq \cup_{x \in K} b_\delta(x)$. Then for each $\delta < \epsilon$ there is some $h \in H$ such that $d(h, K) \geq \delta$ and so $b_\delta(h) \cap K = \emptyset$. Hence $D_2(H, K) \geq \delta$ for each $\delta < \epsilon$, so $D_2(H, K) \geq \epsilon = D_1(H, K)$.

Now, suppose $D_2(H, K) = \epsilon$. Without loss, assume that for every $\delta < \epsilon$, there is some $h \in H$ such that $b_\delta(h) \cap K = \emptyset$. Then for each $\delta < \epsilon$, there is some $h \in H$ such that $d(h, K) \geq \delta$, and so $D(H, K) \geq \delta$ for every $\delta < \epsilon$, hence $D(H, K) \geq \epsilon = D_2(H, K)$.

Theorem 2.11 If F_1 is the subset of 2^X consisting of all the singleton sets, then F_1 is homeomorphic to X.

Proof: Define $f : X \to 2^X$ by $f(x) = \{x\}$ for every $x \in X$. f is clearly one to one and $f(X) = F_1$. Now, if $U \subset X$ is open, $f(U) = R(U) \cap F_1$ which is open in $f(X)$, hence f is open. Also, if $R(\mathcal{U})$ is a basic open set in 2^X then $R(\mathcal{U}) \cap F_1 = \{\{x\} : x \in U$ for every $U \in \mathcal{U}\}$ and so $f^{-1}(R(\mathcal{U})) = \cap\mathcal{U}$ which is open X. So f is continuous and thus an embedding of X.

Corollary 2.12 If 2^X is metrizable, then so is X.

Proof: Follows from Theorem 2.11.

Note that this also holds if 2^X is replaced by $C(X)$ since $F_1 \subset C(X)$.

Chapter 3

More Properties of Hyperspaces

Definition 3.1 Suppose that $\{M_i\}_{i\in\mathbb{N}}$ is a sequence of nonempty sets. Let M be the set to which the point p belongs if and only if every open set containing p intersects infinitely many sets of the sequence $\{M_i\}_{i\in\mathbb{N}}$, then M is called the *limiting set* of the sequence $\{M_i\}_{i\in\mathbb{N}}$.

Definition 3.2 The set M is called the *sequential limiting set* of the sequence of $\{M_i\}_{i\in\mathbb{N}}$ if and only if it is the limiting set of every infinite subsequence of $\{M_i\}_{i\in\mathbb{N}}$.

Lemma 3.3 If M is the limiting set of a sequence $\{M_i\}_{i\in\mathbb{N}}$ in 2^X then M is closed

Proof: Let $x \in \overline{M}$ and $U \subset X$ be open with $x \in U$. Then, there is some $m \in M$ such that $m \in U$, hence U intersects infinitely many members of $\{M_i\}_{i\in\mathbb{N}}$ and $x \in M$. Thus, M is closed.

Theorem 3.4 Let X be a compact space and suppose that M is an element of 2^X. Then the sequence of elements $\{M_i\}_{i\in\mathbb{N}}$ of 2^X converges to M in the Vietoris topology on 2^X if and only if M is the sequential limiting set of the sequence $\{M_i\}_{i\in\mathbb{N}}$ of sets in the topology of X.

Proof: Suppose that $\{M_i\}_{i\in\mathbb{N}}$ converges to M in the topology of 2^X, and let $\{M_{i_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{M_i\}_{i\in\mathbb{N}}$ with limiting set M'. If $p \in M$ and $U \subset X$ is open with $p \in U$ then $R(U, X) \subset 2^X$ and $M \in R(U, X)$. By hypothesis, $\{M_{i_k}\}_{k\in\mathbb{N}}$ converges to M in 2^X, hence there exists a $J \in \mathbb{N}$ such that if $j \geq J$ $M_{i_j} \in R(U, X)$, in particular $M_{i_j} \cap U \neq \emptyset$ for every $j \geq J$. Hence $p \in M'$ and so $M \subset M'$. Now suppose that $p \in M' \setminus M$. Since

M is compact there are disjoint open $U_1, U_2 \subset X$ such that $p \in U_1$ and $M \subset U_2$. Clearly $M \in R(U_2) \subset 2^X$ which is open, hence there exists a $J \in \mathbb{N}$ such that for every $j \geq J$, $M_{i_j} \in R(U_2)$, in particular $M_{i_j} \subset U_2$ for every $j \geq J$. From this, $U_1 \cap M_{i_j} = \emptyset$ for every $j \geq J$ and $p \in M'$, hence $M' \subset M$ thus $M' = M$ and so M is the sequential limiting set of $\{M_i\}_{i\in\mathbb{N}}$. Conversely, assume M is the sequential limiting set of $\{M_i\}_{i\in\mathbb{N}}$ in the topology of X. It is also clear the M contains all of it's limit points, hence M is closed, and since X is compact, M is compact and so $M \in 2^X$. Now let $R(\mathcal{U}) \subset 2^X$ be open with $M \in R(\mathcal{U})$. Note that if $p \in M$ and $U \subset X$ is open with $p \in U$ then there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $U_n \cap M_n \neq \emptyset$ since M is the sequential limiting set of the sequence. So, since for every $U \in \mathcal{U}$, $U \cap M \neq \emptyset$ there is some $N_U \in \mathbb{N}$ such that if $n \geq N_U$ then $M_n \cap U \neq \emptyset$. Further, there exists an $N' \in \mathbb{N}$ such that if $n \geq N'$ then $M_n \subset \cup\mathcal{U}$. If not, then there exists an infinite subsequence $\{M_{i_k}\}_{k\in\mathbb{N}}$ of $\{M_i\}_{i\in\mathbb{N}}$ with $M_{i_k} \not\subset \cup\mathcal{U}$ for every k. For every k pick $x_k \in M_{i_k} \setminus \cup\mathcal{U}$. Then $\{x_{i_k}\}_{k\in\mathbb{N}}$ is a sequence in X, and by compactness has a limit point or point that repeats infinitely many times, $x \in X$, not in $\cup\mathcal{U}$. It is clear that x would then be in the limiting set of $\{M_{i_k}\}_{k\in\mathbb{N}}$ contradicting M being the sequential limiting set of our sequence. Let $N = \max\{\{N_U : U \in \mathcal{U}\} \cup \{N'\}\}$. Then, if $n \geq N$, $M_n \in R(\mathcal{U})$ and so $\{M_i\}_{i\in\mathbb{N}}$ converges to M in 2^X.

Lemma 3.5 If X is a compact space and $\{M_i\}_{i\in\mathbb{N}}$ is a sequence in 2^X, then the limiting set M of $\{M_i\}_{i\in\mathbb{N}}$ is nonempty.

Proof: Choose $x_i \in M_i$ for every $i \in \mathbb{N}$. If one x_i is repeated infinitely many times, then it is in M and we are done. Suppose this is not the case. Then $\{x_i\}_{i\in\mathbb{N}}$ is infinite and has a limit point $x \in X$. Then $x \in M$, in particular $M \neq \emptyset$.

Lemma 3.6 If X is a compact space and $\{M_i\}_{i\in\mathbb{N}}$ is a sequence in 2^X with limiting set M. Then if no proper subset of M is the limiting set for some subsequence of $\{M_i\}_{i\in\mathbb{N}}$, $\{M_i\}_{i\in\mathbb{N}}$ converges to M in 2^X.

Proof: Let $\{M_{i_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{M_i\}_{i\in\mathbb{N}}$ with limiting set M'. Then $M' \neq \emptyset$ and it is clear that $M' \subset M$. By hypothesis the containment cannot be proper so $M' = M$, hence M is the sequential limiting set of $\{M_i\}_{i\in\mathbb{N}}$. By Theorem, $\{M_i\}_{i\in\mathbb{N}}$ converges to M.

Lemma 3.7 Let X be a compact metric space and $\{M_i\}_{i\in\mathbb{N}}$ be a sequence in X. Then $\mathbb{M} = \{M : M$ is the limiting set for some subsequence of $\{M_i\}_{i\in\mathbb{N}}\}$ ordered by reverse inclusion is a partially ordered set and contains a maximal element, i.e. there exists a subsequence of $\{M_i\}_{i\in\mathbb{N}}$ with limiting set satisfying the hypothesis of the preceding lemma and hence is convergent.

Proof: Let $\{M_i\}_{i\in\mathbb{N}}$ be a sequence in X and $\mathbb{M} = \{M : M$ is the limiting set for some subsequence of $\{M_i\}_{i\in\mathbb{N}}\}$. Then reverse inclusion is reflexive, anti-symmetric and transitive, hence $(\mathbb{M}, \supset)$ is a partially ordered set. Let $\mathcal{C} \subset \mathbb{M}$ be a chain. Then $C = \cap\mathcal{C} \neq \emptyset$ and for every $n \in \mathbb{N}$ there is some $C_n \in \mathcal{C}$ such that $C_n \subset \cup_{x\in C} b(x, \frac{1}{n}) = B_n$. Then $C \subset \cap_{n\in\mathbb{N}} C_n \subset \cap_{n\in\mathbb{N}} B_n = C$, i.e. $\cap_{n\in\mathbb{N}} C_n = C$. For every $n \in \mathbb{N}$ let $K_n \subset \mathbb{N}$ such that $\mathcal{M}_n = \{M_k\}_{k\in K_n}$ is a subsequence of $\{M_i\}_{i\in\mathbb{N}}$ with limiting set C_n. Now, for every $n \in \mathbb{N}$ since $C_n \subset B_n$ and B_n open, there exists $N_n \in \mathbb{N}$ such that if $k \in K_n$ with $k \geq N_n$ and $M_k \in \mathcal{M}_n$ then $M_k \subset B_n$. Choose $M_{k_1} \in \mathcal{M}_1$ such that $k_1 \geq N_1$. For $n > 1$ pick $M_{k_n} \in \mathcal{M}_n$ such that $k_n \geq N_n$. Let C' be the limiting set of $\{M_{k_n}\}_{n\in\mathbb{N}}$. Note that $\overline{B_{n-1}} \subset B_n$ for each $n \in \mathbb{N}$ so if $x \notin B_n$ then $x \in X \setminus \overline{B_{n+1}}$ an open set missing M_{k_j} for every $j \geq n+1$ and so $C' \subset \cap_{n\in\mathbb{N}} B_n = C$. So $C' \in \mathbb{M}$ with $C' \subset N$ for every $N \in \mathcal{C}$, i.e. C' is an upper bound for $\mathcal{C}$. By Zorn's lemma, there exists a maximal $M \in \mathbb{M}$ with associated subsequence $\mathcal{M}_M$ satisfying the hypothesis of the previous lemma.

Theorem 3.8 If X is a compact metric space then 2^X is compact.

Proof: Follows from the preceding lemmas and since sequential compactness is equivalent to compactness in metric spaces.

Note that since $C(X)$ is closed in 2^X, it is also compact.

Definition 3.9 If n is an integer then F_n denotes the set of all elements of 2^X that have at most n points.

Theorem 3.10 $F_n \subset 2^X$ is closed for every n.

Proof: We shall show that $2^X \setminus F_n$ is open in 2^X. Let $K \in 2^X \setminus F_n$, then K has at least $n+1$ distinct points, $x_1, ..., x_{n+1}$. Since X is Hausdorf, there exist disjoint open $U_1, .., U_{n+1} \subset 2^X$ each containing the respective points in K. Then $K \in R(U_1, .., U_{n+1}, X)$ and it is clear that if $H \in R(U_1, .., U_{n+1}, X)$, then H has at least $n+1$ points and so $R(U_1, .., U_{n+1}, X) \subset 2^X \setminus F_n$, i.e. $2^X \setminus F_n$ is open.

Corollary 3.11 If 2^X is compact, so is X.

Proof: X is homeomorphic to F_1.

Note that this also holds if 2^X is replaced by $C(X)$.

Definition 3.12 If n is an integer then K_n denotes the set of all elements of 2^X that have at most n components.

Theorem 3.13 $K_n \subset 2^X$ is closed for every n.

Proof: We shall again show that $2^X \setminus K_n$ is open in 2^X. Let $H \in 2^X \setminus K_n$, then K has at least $n+1$ different components, $H_1, ..., H_{n+1}$, each closed as a subset of H and hence compact. Since X is Hausdorff and H_i is compact for every $i \leq n+1$, there exist disjoint open $U_1, .., U_{n+1} \subset 2^X$ each containing the respective components of H. Then $H \in R(U_1, .., U_{n+1}, X)$ and it is clear that if $J \in R(U_1, .., U_{n+1}, X)$, then J has at least $n+1$ components and so $R(U_1, .., U_{n+1}, X) \subset 2^X \setminus K_n$, i.e. $2^X \setminus K_n$ is open.

Theorem 3.14 The set $F = \cup_{n=1}^{\infty} F_n$ is dense (F_σ) in 2^X.

Proof: Let $R \subset 2^X$ be open. Then there exists a finite collection $\mathcal{U}$ of open subsets of X such that $R(\mathcal{U}) \subset R$. Now, for each $U \in \mathcal{U}$ let $x_U \in U$. Then $Y = \{x_U : U \in \mathcal{U}\} \in R(\mathcal{U}) \subset R$, and $Y \in F_{|\mathcal{U}|} \subset F$, thus F is dense in 2^X.

Theorem 3.15 If $W \subset C(X)$ is a continuum then $\cup\{H|H \in W\}$ is a continuum.

proof: Let $\mathcal{U}$ be an open cover of $W' = \cup\{H|H \in W\}$. Then for every $H \in W$ there exists a finite $\mathcal{U}_H \subset \mathcal{U}$ covering H and so that each member of $\mathcal{U}_H$ has nonempty intersection with H. So, $\mathcal{U}_{2^X} = \{R(\mathcal{U}_H) : H \in W$ is an open cover of W in 2^X. Since W is compact, there exist $H_1, H_2, .., H_n$ so that $W \subset \cup_{i=1}^n R(\mathcal{U}_{H_i})$. Clearly, $\cup_{i=1}^n \mathcal{U}_{H_i}$ is a subset of $\mathcal{U}$ that is finite and covers W', hence W' is compact. To see that W' is connected, assume by way of contradiction that it is not. So, let $W' = A \cup B$ where A and B are non-empty and mutually separated. Notice that each A and B are closed, and hence, compact and so there exist disjoint open $U, V \subset X$ such that $A \subset U$ and $B \subset V$. Since each $H \in W$ is connected, either $H \subset U$ or $H \subset V$ and so $W \subset R(U) \cup R(V)$ each open and having non-empty intersection with W, a contradiction. Hence, W' is connected.

Chapter 4

Construction of a Whitney Map

Lemma 4.1 Let $\phi : Y \to X$ be continuous. Then $\Phi : 2^Y \to 2^X$, defined by

$$\Phi(K) = \phi(K) = \{\phi(k) : k \in K\}$$

for each $K \in 2^Y$, is continuous.

Proof: For each $K \in 2^Y$, $\Phi(K) = \phi(K)$ is compact and hence in 2^X. Let $V \subset 2^X$ be open, $\Phi(K) \in V$ and $\Phi(K) \in B(U_1, ..., U_2) \subset V$. Since $\phi(K) = \Phi(K) \subset \cup_{i=1}^n U_i$, $K \subset \cup_{i=1}^n \phi^{-1}(U_i)$, each of which is open in Y. Also, since for each $i \leq n$, $U_i \cap \Phi(K) = U_i \cap \phi(K) \neq \emptyset$, we have $\phi^{-1}(U_i) \cap K \neq \emptyset$, i.e. $K \in B(\phi^{-1}(U_1), .., \phi^{-1}(U_n))$. Further, it is clear that if $H \in B(\phi^{-1}(U_1), .., \phi^{-1}(U_n))$ then $\Phi(H) \in B(U_1, ..., U_2)$.

Lemma 4.2 If $K \subset X$ is compact, then 2^K is homeomorphic to $\{H \in 2^X : H \subset K\}$.

Proof: Let ϕ be the identity map. Then Φ is clearly one to one, onto and continuous by Lemma 4.1. Since K is compact, 2^K is compact and so Φ is an embedding.

In the following, let (X, d) be a metric space and assume $d \leq 1$

Definition 4.3 For every $n \geq 2$ define $w_n : F_n \to [0, \infty)$ by $w_n(K) = 0$ if $|K| < n$ or $w_n(K) = \min\{d(x, y) : x, y \in K \text{ and } x \neq y\}$ if $|K| = n$.

Lemma 4.4 If $H, K \in F_n$ and $D(H, K) < \frac{\epsilon}{2}$ then $|w_n(H) - w_n(K)| < \epsilon$

Proof: Case 1: $|H|, |K| < n$, then the lemma is clear.

Case 2: $|H| = n$ and $|K| < n$. Since $D(H, K) < \frac{\epsilon}{2}$, for each $h \in H$ there is some $k \in K$ such that $d(h, k) < \frac{\epsilon}{2}$. Since $|K| < |H|$ there are distinct $h, h' \in H$ such that $d(k, h) < \frac{\epsilon}{2}$ and $d(k, h') < \frac{\epsilon}{2}$ for some $k \in K$. Hence

$$|w_n(H) - w_n(K)| = w_n(H) \leq d(h, h') \leq d(k, h) + d(k, h') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Case 3: $|H| = n, |K| = n$. Let $h, h' \in H$ be distinct such that $w_n(H) = d(h, h')$. If a) there exist distinct $k, k' \in K$ such that $d(k, h) < \frac{\epsilon}{2}$ and $d(k', h') < \frac{\epsilon}{2}$, then

$$w_n(K) \leq d(k, k') \leq d(k, h) + d(h, h') + d(k', h') < w_n(H) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = w_n(H) + \epsilon.$$

If b) there exist distinct $k, k' \in K$ such that for some $h \in H$, $d(k, h) < \frac{\epsilon}{2}$ and $d(k', h) < \frac{\epsilon}{2}$ and so

$$w_n(K) \leq d(k, k') \leq d(k, h) + d(k', h) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = w_n(H) + \epsilon.$$

Similarly, $w_n(H) < w_n(K) + \epsilon$ and so $|w_n(H) - w_n(K)| < \epsilon$.

Theorem 4.5 w_n is continuous on F_n.

Proof: follows from Lemma 4.4.

Definition 4.6 Define $w_n : 2^X \to [0, \infty)$ by $w_n(K) = \max\{w_n(H) : H \subset K \text{ and } H \in F_n\}$.

Note: This definition matches the above definition for every $K \in F_n$ because every proper subset H of K has $w_n(H) = 0$. Also, since F_n is closed and $\{H \in 2^X : H \subset K\}$ is compact for every $K \in 2^X$ and so their intersection is compact, and hence achieves a maximum value since w_n is continuous on F_n.

Theorem 4.7 w_n is continuous on 2^X.

Proof: Let $K \in 2^X$ and $\epsilon > 0$. Suppose $H \in 2^X$ with $D(H,K) < \frac{\epsilon}{2}$. Let $K' \subset K$ such that $K' \in F_n$ and $w_n(K') = w_n(K)$. Since $D(H,K) < \frac{\epsilon}{2}$, for each $k \in K'$ there exists $h \in H$ such that $d(h,k) < \frac{\epsilon}{2}$. Let H' be the set of those elements in H. Then $H' \in F_n$ and $D(H',K') < \frac{\epsilon}{2}$, so by preceding lemma,

$$w_n(K) = w_n(K') < w_n(H') + \epsilon \leq w_n(H) + \epsilon.$$

Similarly,

$$w_n(H) < w_n(K) + \epsilon$$

and so

$$|w_n(H) - w_n(K)| < \epsilon.$$

Hence, w_n is continuous on 2^X.

Definition 4.8 Let $w : 2^X \to \mathbb{R}$ be a continuous function. Then if

i. $w(K) \geq 0$ for every $K \in 2^X$, with $w(K) = 0$ if and only if $|K| = 1$,

ii. If $H \subset K$, then $w(H) < w(K)$.

w is called a *Whitney map.* If w is continuous and has the properties that $w(K) = 0$ for every $K \in 2^X$ with $|K| = 1$ and for any pair, $H, K \in 2^X$ such that $H \subset K$, $w(H) \leq (K)$ then w is called a *semi-Whitney map.*

Theorem 4.9 w_n is a semi-Whitney map for every $n \geq 2$

Proof: It is clear that for any $K \in F_1$, $w_n(K) = 0$. Now, let $H \subset K$ and $H' \in F_n$ such that $H' \subset H$ and $w_n(H') = w_n(H)$. Then $H' \subset K$ and so

$$w_n(H) = w_n(H') \leq w_n(K).$$

Definition 4.10 $w : 2^X \to [0, \infty)$ by

$$w(K) = \sum_{n \geq 2} w_n(K) 2^{-n}$$

for every $K \in 2^X$. This exists for each $K \in 2^X$ since $w_n(K) \leq 1$ for every $n \geq 2$.

Lemma 4.11 If f_n is a semi-whitney map on 2^X with $ran(f_n) \subset [0, 2^{-n}]$ for every $n \in \mathbb{N}$ then $f = \sum_{n \in \mathbb{N}} f_n$ is a semi-whitney map. Further, if for each pair $H, K \in 2^X$ with $H \subsetneq K$, there is some $m \in \mathbb{N}$ such that $f_m(H) < f_m(K)$, then f is a whitney map.

Proof: It is clear that $f(K)$ exists for every $K \in 2^X$. Since each f_n is continuous, f is the uniform limit of continuous functions and hence is continuous. For every $K \in F_1$,

$$f(K) = \sum_{n \in \mathbb{N}} f_n(K) = \sum_{n \in \mathbb{N}} 0 = 0.$$

Next, if $H, K \in 2^X$ with $H \subset K$ since each f_n is a semi-whitney map, then $f_n(H) \leq f_n(K)$ for each n, and so

$$f(H) = \sum_{n \in \mathbb{N}} f_n(H) \leq \sum_{n \in \mathbb{N}} f_n(K) = f(K),$$

hence f is semi-whitney. Further, if $H \subsetneq K$ and it is assumed that there exists an $m \in \mathbb{N}$ such that $f_m(H) < f_m(K)$, then since the series defining $f(H)$ and $f(K)$ are absolutely convergent

$$f(H) = f_m(H) + \sum_{n \neq m} f_n(H) < f_m(K) + \sum_{n \neq m} f_n(K) = f(K),$$

i.e. f would be a whitney map.

Corollary 4.12 w is a semi-whitney map.

Proof: Multiplying w_n by 2^{-n} preserves the semi-whitney properties and makes the functions fulfill the first part of the lemma above.

It takes a little more to show that w is a whitney map.

Lemma 4.13 If $K \in 2^X$ then $\{w_n(K)\}_{n \in \mathbb{N}}$ is monotonic decreasing and $lim_{n \to \infty} w_n(K) = 0$.

Proof: That the sequence is monotonic decreasing is clear. If $K \in 2^X$ is finite, then it is clear that $lim_{n \to \infty} w_n(K) = 0$. Let $\epsilon > 0$ and let $K \in 2^X$ be infinite. Then since K is compact in X there is a finite cover, $\mathcal{U}$, of K by balls of diameter less than ϵ. Then if $H \subset K$ is finite and $|H| = n > |\mathcal{U}| = N$, there exist distinct $h, h' \in H$ such that h, h' are in the same member of $\mathcal{U}$ and so $w_n(H) < d(h, h') < \epsilon$. Then $w_n(K) \leq \epsilon$.

Lemma 4.14 If $H, K \in 2^X$ and $H \subsetneqq K$, then there exists a $m \in \mathbb{N}$ such that $w_m(H) < w_m(K)$.

Proof: Let $H, K \in 2^X$ and $H \subsetneqq K$. If H is finite, let $m - 1 = |H| < |K|$ and clearly $w_m(H) = 0 < w_m(K)$. Now, assume that H is infinite. Since $H \subsetneqq K$, there exists $k \in K \backslash H$, and so there exists an $\epsilon > 0$ so that $b(k, \epsilon) \cap H = \emptyset$. By preceding lemma, $w_n(H) < \epsilon$ for all n sufficiently large. Note that $w_n(H) > 0$ for every n since H is infinite, this coupled with the preceding lemma give us that for some $m, m-1 \in \mathbb{N}$, $w_m(H) < w_{m-1}(H) < \epsilon$. Let $H' \subset H$ be in F_{m-1} such that $w_{m-1}(H') = w_{m-1}(H)$. Then $K' = H' \cup \{k\} \subset K$ is in F_m. Also, since $d(k, h) \geq \epsilon$ for each $h \in H'$,

$$w_m(H) < w_{m-1}(H') = w_m(K') \leq w_m(K).$$

Theorem 4.15 w is a Whitney map.

Proof: The product of w_m with 2^{-m} preserves the inequality.

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