Applications of Stationary Sets in Set Theoretic Topology

by

Steven Clontz

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Approved by

Gary Gruenhage, Chair, Professor of Mathematics
Michel Smith, Professor of Mathematics
Stewart Baldwin, Professor of Mathematics
Andras Bezdek, Professor of Mathematics

Abstract

The notion of a stationary subset of a regular cardinal, a set which intersects any closed unbounded subset of that cardinal, is a useful tool in investigating certain properties of topological spaces. In this paper we utilize stationary sets to achieve an interesting characterization of paracompactness of a linearly ordered topological space. We also use stationary sets to find a pair of Baire spaces whose product is not Baire.

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Chapter 1

Introduction

A stationary subset of a regular cardinal is defined to be any subset of that cardinal which intersects every closed and unbounded subset of that cardinal. Stationary sets are useful tools in investigating properties of linearly ordered topological spaces. An example of this is the following theorem characterizing paracompactness in a linearly ordered topological space (often abbreviated LOTS).

Theorem. *A linearly ordered topological space X is paracompact iff X does not contain a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

This result was first discovered by Engelking and Lutzer in 1977 [3], for a larger class of "generalized ordered spaces", spaces which are a subspace of another linearly ordered topological space. However, as we will see, the proof developed in his paper works in a way that could easily be extended to cover these generalized spaces as well.

Of course, stationary sets can be used in scenarios beyond LOTS. In perhaps a more surprising application, stationary subsets of ω_1 can also be used to construct a pair of non-linearly ordered sets which are Baire, but whose product is not Baire.

Theorem. *There are metrizable Baire spaces X and Y such that $X \times Y$ is not Baire.*

The first discovery of this result is due to Fleissner and Kunen in 1978 [4].

The goal of this paper is to develop the tools necessary for the proof of both theorems. In the next chapter, we will outline the basics of topology and relevant basic results in the field, in order to accommodate the reader and serve as a reference. Readers familiar with the field may begin at Chapter 3, wherein the basic definitions and results concerning stationary subsets of the first uncountable ordinal ω_1. In Chapter 4, these results are generalized to

any regular cardinal by introducing some set-theoretic results. In Chapter 5, we investigate the compactness of linearly ordered topological spaces, and in Chapter 6 we use the results developed thus far to prove the above theorem on paracompactness. Chapter 7 changes pace and quickly develops the tools to prove the second theorem concerning the Baire property.

It should be noted that all the theorems and lemmas in this paper were taken from the class notes of Dr. Gary Gruenhage's set-theoretic topology course held in Fall 2008 and Spring 2009 at Auburn University, which the author regrettably was unable to take part in. The proofs to these, however, are all due to the author, with no reference to the original proofs of these results, but with the obvious assistance of Dr. Gruenhage. The basics covered in Chapter 2 are based on the second edition of *Topology* by James R. Munkres [1] and *Set Theory: An Introduction to Independence Proofs* by Kenneth Kunen [2].

Chapter 2

The Basics

To begin, let's define the basic concept of a topological space.

Definition 2.1. A **topology** on a set X is a collection $\tau \subseteq \mathcal{P}(X)$ having the following properties:

1. $\emptyset, X \in \tau$.

2. If $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$.

3. If $U, V \in \tau$, then $U \cap V \in \tau$.

   (Or equivalently, any finite intersection of sets in τ is in τ.)

An ordered pair $\langle X, \tau \rangle$ where τ is a topology on X is known as a **topological space**, although we often refer to it as simply X when τ is known by context.

The basic concepts of topology are the notions of open and closed sets.

Definition 2.2. For a topological space $\langle X, \tau \rangle$, $U \subseteq X$ is said to be **open in** $\langle X, \tau \rangle$ (or simply **open** or **open in** X) if $U \in \tau$.

$K \subseteq X$ is said to be **closed in** $\langle X, \tau \rangle$ (or simply **closed** or **closed in** X) if $X \setminus K \in \tau$.

A set which is both closed and open is referred to as **clopen**.

Proposition 2.3. *Any finite union of closed sets is closed, and any arbitrary intersection of closed sets is closed.*

The concept of a limit point is often used to identify closed sets.

Definition 2.4. x is a **limit point** of a set A in a topological space X if every open set containing x intersects A at a point distinct from x.

Proposition 2.5. *A set K in a topological space X is closed if and only if K contains all its limit points.*

Any set is contained within a minimal closed set, which we call its closure.

Definition 2.6. The **closure** $\bar{A}$ of a set A in a topological space X is the intersection of all closed sets containing A.

Two basic topologies for any arbitrary set X are the discrete and indiscrete topologies.

Definition 2.7. The **discrete topology** on a set X is $\tau = \mathcal{P}(X)$. The **indiscrete** (or **trivial**) **topology** on a set X is $\tau = \{\emptyset, X\}$.

It is easily seen that these indeed satisfy the criteria in Definition 2.1.

If we have a topological space and wish to investigate a subset of that space, we may easily apply a "subspace" topology.

Definition 2.8. Let $\langle X, \tau \rangle$ be a topological space and $Y \subseteq X$. $\tau_Y = \{U \cap Y : U \in \tau\}$ is known as the **subspace topology** on Y with respect to X.

Listing every open set in a topology can be tedious, so often topologies are described using simpler collections called bases.

Definition 2.9. A **basis** on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

1. For each $x \in X$ there is a basis element $B \in \mathcal{B}$ with $x \in B$.

2. If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there is a basis element $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The **topology τ generated by $\mathcal{B}$** is

$$\{U \in \mathcal{P}(X) : \text{ for all } x \in U \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}.$$

It can be seen that any τ generated by a basis which satisfies these requirements satisfies the requirements of a topology. A basis can be thought of as a collection of the basic elements of a topology on X. Certainly, any basis element is an open set in the space. Indeed, it is a fact that every open set in the generated topology is a union of basis elements.

Proposition 2.10. *An open set of a topological space generated by the basis $\mathcal{B}$ is the union of elements of $\mathcal{B}$.*

Definition 2.11. A **local base** at a point x in a topological space X is a collection of open sets $\mathcal{B}_x$, each of which contains x, such that for every open set U containing x, there is some $B \in \mathcal{B}_x$ with $x \in B \subseteq U$.

Another example of a topological space is a product space of various topological spaces.

Definition 2.12. The **Cartesian product** $\prod_{i \in I} X_i$ is the set of functions $f : I \to \bigcup_{i \in I} X_i$ where $f(i) \in X_i$.

In the case that there is some X with $X_i = X$ for all $i \in I$, the cartesian product is often written as X^I.

Definition 2.13. The **product topology** on the cartesian product of topological spaces $\prod_{i \in I} X_i$ is the topology generated by the basis of sets of the form $\prod_{i \in I} U_i$, where U_i is open in X_i for all $i \in I$, and $U_i = X_i$ for all but finite $i \in I$.

Certainly, the topological spaces $X = \{0, 1, 2\}$ and $Y = \{a, b, c\}$ with topologies $\tau_X = \{\emptyset, \{0\}, X\}$ and $\tau_Y = \{\emptyset, \{a\}, X\}$ have no interesting differences other than the labels we give the elements. We use the concept of a homeomorphism to link two topologically equivalent spaces.

Definition 2.14. If $f : X \to Y$ is a function, $S \subseteq X$, and $T \subseteq Y$, then $f''(S) = \{f(s) \in Y : s \in S\}$ is the **image of S under f**, and $f^{-1}(T) = \{s \in X : f(s) \in T\}$ is the **inverse image of T under f**.

Definition 2.15. For two topological spaces X, Y, the function $h : X \to Y$ is a **homeomorphism** if h is a bijection and for every set U open in X and V open in Y, $f''(U)$ is open in Y (making it an **open map**) and $f^{-1}(V)$ is open in X (making it **continuous**).

Proposition 2.16. *A map with the property that for every point y in its range and open set V containing y, there is an open set U in the domain with $f''(U) \subseteq V$, is continuous.*

Definition 2.17. Two topological spaces X, Y are said to be **homeomorphic** if there exists a homeomorphism $h : X \to Y$. We write $X \cong Y$.

Following are some properties of various topological spaces which will be referenced or investigated in this paper.

Definition 2.18. A topological space X is said to be T_3 if for every point $x \in X$, $\{x\}$ is closed, and for every open set U containing x, there is another open set V with $x \subseteq V \subseteq \overline{V} \subseteq U$.

Definition 2.19. An **open cover** $\mathcal{U}$ of a topological space X is a collection of open sets such that $\bigcup \mathcal{U} = X$.

Definition 2.20. A topological space X is said to be **compact** if for every open cover $\mathcal{U}$ of X, there exists some finite subcover $\mathcal{U}^* \subseteq \mathcal{U}$.

Definition 2.21. A **refinement** $\mathcal{V}$ **of** $\mathcal{U}$ is a set such that for every $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ with $V \subseteq U$.

Definition 2.22. A collection of sets $\mathcal{A}$ in a topological space X is said to be **locally finite** if for every point $x \in X$ and there exists some open set U containing x such that U intersects only finitely many members of $\mathcal{A}$.

Definition 2.23. A topological space X is said to be **paracompact** if for every open cover $\mathcal{U}$ of X there exists some refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is an open cover of X and is locally finite.

Proposition 2.24. *Any closed subspace of a paracompact space is paracompact.*

Definition 2.25. A topological space X is said to be **connected** if there does not exist a nonempty proper clopen subset $A \subset X$.

Definition 2.26. A **metric** on a space X is a function $d : X \times X \to \mathbb{R}$ where:

1. $d(x, y) = 0 \Leftrightarrow x = y$

2. $d(x, y) \geq 0$ for all $x, y \in X$

3. $d(x, y) = d(y, x)$ for all $x, y \in X$

4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

An **open ball of radius** r with respect to a metric d, written $B_r(x)$, is the set $\{y : d(x, y) < r\}$.

Definition 2.27. A topological space X is said to be **metrizable** if there exists a metric d such that $\{B_r(x) : x \in X \text{ and } r > 0\}$ forms a basis generating the topology on X.

Proposition 2.28. *Any subspace of a metrizable space is metrizable.*

Definition 2.29. A collection of sets $\mathcal{A}$ is said to be σ**-locally finite** if $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ where $\mathcal{A}_n$ is locally finite for all $n \in \mathbb{N}$.

The following two results are not trivial, but are basic results from introductory topology needed in this paper.

Proposition 2.30. *Every metrizable space is paracompact.*

Proposition 2.31. *Every space which is T_3 and has a σ-locally finite basis is metrizable.*

The following definitions are needed in the final chapter.

Definition 2.32. A subset A of a topological space X is said to be **dense** in the space if it intersects every open set in the space.

Proposition 2.33. *A subset A of a topological space X is dense in the space if and only if* $\overline{A} = X$.

Definition 2.34. A topological space X is said to be **Baire** if every countable intersection of dense open sets in the space is dense.

We now turn our attention to linearly ordered sets. If a set has a linear order on it, there is a natural topology which arises from this order.

Definition 2.35. A relation $<$ on X is called a **linear order** on X if it has the following properties for all $a, b, c \in X$:

- Either $a = b$, $a < b$, or $b < a$. (called **comparability**)

- $a \not< a$. (called **nonreflexivity**)

- If $a < b$ and $b < c$, then $a < c$. (called **transitivity**)

The ordered pair $\langle X, < \rangle$ is called a **linearly ordered set**, or just X if the order $<$ is implied.

Definition 2.36. An **upper bound** of a subset S of a linearly ordered set $\langle X, < \rangle$ is a point $u \in X$ such that $s \leq u$ for all $s \in S$.

Similarly, a **lower bound** of a subset S of a linearly ordered set $\langle X, < \rangle$ is a point $l \in X$ such that $s \geq l$ for all $s \in S$.

Definition 2.37. The **least upper bound** of a subset S of a linearly ordered set $\langle X, < \rangle$ is a point $u \in X$ such that u is an upper bound of S and $u \leq v$ for all upper bounds v of S. It is sometimes referred to as the **supremum** of S and written $\sup(S)$.

The **greatest lower bound** of a subset S of a linearly ordered set $\langle X, < \rangle$ is a point $l \in X$ such that l is an lower bound of S and $l \geq m$ for all lower bounds m of S. It is sometimes referred to as the **infimum** of S and written $\inf(S)$.

Proposition 2.38. *Let X be a linearly ordered set. If $p \in S \subseteq X$ is an upper bound (resp. lower bound) of S, then it is the least upper bound (resp. greatest lower bound) of S. It is also the maximum (resp. minimum) element of S.*

Of course, it need not be that every subset of a linearly ordered set have a supremum or infimum.

Definition 2.39. A subset S of a linearly ordered set $\langle X, < \rangle$ is **bounded** if it has a lower bound and an upper bound. Otherwise, it is said to be **unbounded**.

Definition 2.40. Two linearly ordered sets $\langle X, <_X \rangle, \langle Y, <_Y \rangle$ are said to be **order isomorphic** if there exists a bijection $f : X \to Y$ which is **order preserving**, that is, $x_1 <_X x_2 \Leftrightarrow f(x_1) <_Y f(x_2)$. This bijection is called an **order isomorphism**.

Proposition 2.41. *Any order-preserving map is injective.*

Definition 2.42. For a linearly ordered set $\langle X, < \rangle$ and points $a, b \in X$, let the following sets denote **intervals** of X:

- $(a, b) = \{x \in X : a < x < b\}$ is an **open interval** of X

- $[a, b) = \{x \in X : a \leq x < b\}$ is a **left-closed interval** of X

- $(a, b] = \{x \in X : a < x \leq b\}$ is a **right-closed interval** of X

- $[a, b] = \{x \in X : a \leq x \leq b\}$ is a **closed interval** of X

Definition 2.43. For a linearly ordered set $\langle X, < \rangle$ and points $a, b \in X$, let the following sets denote **rays** of X:

- $(a, \to) = \{x \in X : a < x\}$

- $[a, \to) = \{x \in X : a \leq x\}$

- $(\leftarrow, b) = \{x \in X : x < b\}$

- $(\leftarrow, b] = \{x \in X : x \leq b\}$

The first and third rays are called **open rays** and the second and fourth rays are called **closed rays**.

It can be shown that $\{(a,b) : a, b \in X\}$ forms a basis for a topology on X, and we call this the order topology.

Definition 2.44. For a linearly ordered set $\langle X, < \rangle$, the **order topology on** $\langle X, < \rangle$ is the topology generated by the basis of open intervals and rays $\{(a,b) : a, b \in X\} \cup \{(\leftarrow, b) : b \in X\} \cup \{(a, \rightarrow) : a \in X\}$. If τ is this topology, then the topological space $\langle X, \tau \rangle$ is then called a **linearly ordered topological space** or **LOTS**.

The majority of this paper investigates the order topology as applied to ordinal numbers. We will need some (largely glossed over) set theory background.

Definition 2.45. A set α is **transitive** if $\beta \in \alpha$ implies $\beta \subseteq \alpha$.

Definition 2.46. A linear order $<$ on X is a **well-order** if for every $Y \subseteq X$, there is a $<$-least element $y \in Y$. (y is $<$-least in Y if for every $z \in Y$, $y \leq z$.)

Proposition 2.47. *For any bounded subset S of a well-ordered space X, $\sup(S)$ exists.*

Proposition 2.48. *Induction and definition by recursion may be performed on any well-ordered set.*

Definition 2.49. For a linearly ordered set $\langle X, < \rangle$, the **lexicographical order** $<_L$ on $X \times X$ is given by $\langle x_1, x_2 \rangle <_L \langle y_1, y_2 \rangle$ if one of the following holds:

- $x_1 < y_1$

- $x_1 = y_1$ and $x_2 < y_2$

Proposition 2.50. $<_L$ *is a linear order. If $<$ is a well-order, then $<_L$ is also a well-order.*

Definition 2.51. A set α is an **ordinal** if α is transitive and well-ordered by $\in$.

For two ordinals α, β, if $\beta \in \alpha$, we often say $\beta < \alpha$.

Example 2.52. The following are examples of ordinals:

- $0 = \emptyset$

- $1 = \{0\}$

- $2 = \{0, \{0\}\} = \{0, 1\}$

- $n = \{0, 1, \ldots, n-1\}$

- $\omega = \{0, 1, \ldots, n, \ldots\} = \mathbb{N}$

- $\omega + 1 = \{0, 1, \ldots, n, \ldots, \omega\}$

- $\omega \cdot 2 = \{0, 1, \ldots, n, \ldots, \omega, \omega + 1, \ldots, \omega + n, \ldots\}$

- $\omega^2 = \{0, \ldots, \omega, \ldots, \omega \cdot 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot n, \ldots\}$

- $\omega^\omega = \{0, \ldots, \omega, \ldots, \omega^2, \ldots, \omega^n, \ldots\}$

While the later ordinals in that list may seem to contain many "more" ordinals than those above, every ordinal in that list is countable, in that there is an onto function from ω to any ordinal in that list.

Definition 2.53. An ordinal κ is called a **cardinal** if for all ordinals $\beta < \kappa$, there is no onto function from β to κ.

Definition 2.54. The **cardinality** of a set S, $|S|$, is the unique cardinal κ such that there is a bijection from κ to S.

Example 2.55. The natural numbers $0, 1, \ldots, n$ and the collection of natural numbers ω are all cardinals.

Definition 2.56. An ordinal α is called **countable** if there is an onto function from ω to α. Otherwise it's called **uncountable**.

Proposition 2.57. $\omega_1 = \{\alpha : \alpha \text{ is a countable ordinal}\}$ *is the least uncountable ordinal (and cardinal).*

Chapter 3

ω_1, the First Uncountable Ordinal

We begin by first investigating this first uncountable ordinal. As we will see in the next chapter, these results will be easily generalized to certain higher cardinals as well. We first establish the following convention.

Definition 3.1. A subset C of a limit ordinal α is said to be **club** if it is closed and unbounded in α.

It should be noted that for the majority of this paper (until Chapter 7), all ordinals are assumed to have the order topology.

Theorem 3.2. *Let $\mathcal{C} = \{C_n : n < \omega\}$ be a collection of club sets in ω_1. Then $\bigcap \mathcal{C}$ is club.*

Proof. Certainly, any intersection of closed sets is closed. Let $<_L$ be the lexicographic order on $\omega \times \omega$. To see that $\bigcap \mathcal{C}$ is unbounded, we take any $\alpha \in \omega_1$ and fix a point $c_{0,0} \in C_0$ greater than c'. We then define $c_{m,n}$ (for $m, n < \omega$) to be the minimum element of C_n strictly greater than $c_{m',n'}$ for all $(m', n') <_L (m, n)$.

We then set c to be the least upper bound of $\{c_{m,n} : m, n < \omega\}$. Fixing $n < \omega$, c is a limit point of $\{c_{m,n} : m < \omega\} \subseteq C_n$, so $c \in C_n$ for all $n < \omega$, which gives a point $c > \alpha$ in $\bigcap \mathcal{C}$. □

Theorem 3.3. *For any closed unbounded subset C of ω_1, there is a strictly increasing homeomorphism from C to ω_1.*

Proof. Define $f : \omega_1 \to C$ such that

- $f(0) = \min(C)$

- $f(\alpha) = \min(C \setminus f''(\alpha))$.

We first claim that f is an order isomorphism. It's obviously order-preserving (and thus injective). To see that it is onto, suppose by way of contradiction that there is some $\gamma \in C$ such that $\gamma \notin \mathrm{ran}(f)$. By the definition of f, $f(\alpha) < \gamma$ for all $\alpha < \omega_1$. Thus $f''(\omega_1) \subseteq \gamma$, a countable set. But $f''(\omega_1)$ is the 1-1 image of an uncountable set, and thus is an uncountable subset of a countable set - contradiction.

Any order isomorphism is an open map. We now show that f is also a homeomorphism by showing its continuity. Let $\gamma < \omega_1$ and $\beta < f(\gamma)$. If γ is not a limit ordinal, then $\{\gamma\}$ is open and $f''(\{\gamma\}) \subseteq (\beta, f(\gamma) + 1)$. Otherwise, there is a strictly increasing sequence of ordinals converging to γ, and their image under f is also a strictly increasing sequence of ordinals converging into some γ'. As C is closed, $\gamma' \in C$ and thus $f(\gamma) = \gamma'$. So $f(\gamma)$ is a limit point of $f''(\gamma)$, and there is some $\delta < \gamma$ such that $\beta < f(\delta) \leq f(\gamma)$. Thus $f''[(\delta, \gamma + 1)] \subseteq (\beta, f(\gamma) + 1)$. $\qquad\square$

Theorem 3.4. *Let $f : \omega_1 \to \omega_1$ be a function and $C = \{\alpha : \beta < \alpha \Rightarrow f(\beta) < \alpha\}$ be a subset of ω_1. Then C is closed and unbounded.*

Proof. Let γ be a limit point of C, and $\beta < \gamma$. As γ is a limit point of C, there must be some $\alpha \in C$ with $\beta < \alpha \leq \gamma$. Note then that $f(\beta) < \alpha \leq \gamma$, so $\gamma \in C$. Thus C is closed.

Now suppose by way of contradiction that σ is the supremum of C. Define $g : \omega_1 \to \omega_1$ such that $g(\alpha) = \sup(f''(\alpha)) + 1$. Consider the sequence $\{\sigma + 1, g(\sigma + 1), g^2(\sigma + 1), ...\}$. Note that $\sigma + 1 \notin C$ so $g(\sigma + 1) = \sup(f''(\sigma + 1)) + 1 \geq (\sigma + 1) + 1 > \sigma + 1$. Inductively, it is easily seen that as $g^n(\sigma + 1) \notin C$, $g^{n+1}(\sigma + 1) > g^n(\sigma + 1)$. This is a strictly increasing sequence, so it converges to a limit ordinal ρ.

Consider any $\delta < \rho$. There is a minimum $g^n(\sigma + 1)$ such that $\delta < g^n(\sigma + 1)$. It follows that $f(\delta) < \sup(f''(g^n(\sigma + 1))) + 1 = g(g^n(\sigma + 1)) = g^{n+1}(\sigma + 1) < \rho$. So $\rho \in C$, a contradiction. Thus C is unbounded. $\qquad\square$

The construction in Theorem 3.4 (and later in Theorem 4.8) is often useful in constructing club sets.

In this paper we are largely concerned with the idea of a stationary set.

Definition 3.5. A subset S of a cardinal κ is said to be **stationary** if it intersects every club subset of κ.

Two facts about stationary sets are evident: all club sets are stationary, and any stationary set must be unbounded. But it need not be true that every stationary set be closed, as there are two disjoint stationary subsets of ω_1, as we will see in the next chapter.

The following result is classic, and very useful.

Theorem 3.6 (Pressing Down Lemma Lite for ω_1). *Let S be a stationary set in ω_1. If for each ordinal $\alpha \in S \setminus \{0\}$, we choose an ordinal $\beta_\alpha < \alpha$, then there is some $\beta < \omega_1$ such that $\beta = \beta_\alpha$ for uncountably many $\alpha \in S$.*

Proof. Suppose by way of contradiction that for all $\beta < \omega_1$, $\{\alpha \in S : \beta_\alpha = \beta\}$ is bounded. Then, let $f : \omega_1 \to \omega_1$ be defined such that $f(\alpha) = \sup(\{\gamma \in S : \beta_\gamma = \alpha\})$.

By the above theorem, $C = \{\alpha : f(\beta) < \alpha \text{ for all } \beta < \alpha\}$ is closed and unbounded. Let $\alpha \in S$. $\beta_\alpha < \alpha$, and $f(\beta_\alpha) = \sup(\{\gamma \in S : \beta_\gamma = \beta_\alpha\}) \geq \alpha$. So $\alpha \notin C$ and $S \cap C = \emptyset$. Contradiction. $\qquad\square$

The "full" result states that there is a stationary set T of ω_1 such that $\beta = \beta_\alpha$ for all $\alpha \in T$. However, this stronger version is not needed for any result in this paper.

A direct application of stationary sets is the fact that they are never paracompact.

Theorem 3.7. *If S is a stationary subset of ω_1, then S is not paracompact.*

Proof. Consider $\{(\leftarrow, \alpha+1) : \alpha \in S\} = \{[0, \alpha] : \alpha \in S\}$, an open cover of S. Suppose by way of contradiction that there exists a locally finite open refinement $\mathcal{U}$. For each $\alpha < \omega_1$ there exists a β_α such that $(\beta_\alpha, \alpha]$ intersects only finitely many sets in $\mathcal{U}$. By the Pressing Down

Lemma, there is a β such that $\beta = \beta_\alpha$ for uncountably many α. Thus (β, ω_1) intersects only finitely many sets in $\mathcal{U}$.

This means that $\mathcal{U}$ must be a countable collection. And as every set in $\mathcal{U}$ is a subset of $[0, \alpha]$ for some $\alpha < \omega_1$, $\mathcal{U}$ is a countable collection of countable sets. But this is a contradiction as a countable collection of countable sets cannot cover an uncountable set. $\qquad\square$

The following lemma proves our intuition that any open set of a LOTS can be constructed by unioning some disjoint convex open spaces.

Lemma 3.8. *Let U be an open subset of a linearly ordered topological space X. Then U is the union of a disjoint collection of convex open spaces.*

Proof. We begin by defining a relation $\sim$ on U. We say $a \sim b$ if and only if the closed interval $[\min(a, b), \max(a, b)] \subseteq U$. We should show this is an equivalence relation.

- For $a \in U$, we note $[\min(a, a), \max(a, a)] = \{a\} \subseteq U$, so $a \sim a$.

- If $a \sim b$, we note $U \supseteq [\min(a, b), \max(a, b)] = [\min(b, a), \max(b, a)] \subseteq U$, so $b \sim a$.

- If $a \sim b$ and $b \sim c$, we note $[\min(a, c), \max(a, c)] \subseteq [\min(a, b), \max(a, b)] \cup [\min(b, c), \max(b, c)] \subseteq U \cup U = U$. Thus $a \sim c$.

So $\sim$ partitions U. We should show $[x]$, the equivalence class of some $x \in U$, is a convex open set. Take $a < b$ in $[x]$. $a \sim x \sim b$, so $[\min(a, b), \max(a, b)] = [a, b] \subseteq U$. Thus for any $c \in (a, b)$, $[\min(a, c), \max(a, c)] = [a, c] \subseteq [a, b] \subseteq U$, so $c \sim a \sim x$ and $c \in [x]$, making $[x]$ convex.

Lastly, note that for $c \in [x]$, $a \in U$, and there is an open interval $(a, b) \subseteq U$ about c. For all $y \in (a, c)$, $[y, c] \subseteq (a, b) \subseteq U$, and for all $y \in (c, b)$, $[c, y] \subseteq (a, b) \subseteq U$. Either way, $y \sim c \sim x$, so $y \in [x]$, and $[x]$ is open. $\qquad\square$

Using this lemma, we may characterize all nonstationary subsets of ω_1 as metrizable and paracompact.

Theorem 3.9. *A subspace S of ω_1 is metrizable iff S is paracompact iff S is non-stationary.*

Proof. It is a well-known result that all metrizable spaces are paracompact (cited in Chapter 2), whether they are a subspace of ω_1 or not. In addition, paracompact implying non-stationary is the result of Theorem 3.7.

It remains to be shown that non-stationary implies metrizable. Let Y be a nonstationary subset of ω_1. Note that it is routine to show that ω_1 is T_3, so Y is also T_3. We should show Y has a σ-locally finite basis.

There exists a closed, unbounded set C disjoint from Y. Let $\omega_1 \setminus C = D$. Thus $Y \subseteq D$, an open set which is the disjoint collection of convex open spaces. We partition D by these convex open spaces, so that $D = \bigcup_{\alpha < \omega_1} D_\alpha$. (We may index these spaces by ω_1 as they are disjoint, and if there were only countably many D_α then Y is countable and the result is trivial.)

Note that each D_α has an element $\delta_\alpha \in C$ as an upper bound, and thus has a basis $\mathcal{B}_\alpha = \{(\beta, \gamma) \cap D_\alpha : \beta < \gamma < \delta_\alpha\}$ (as D_α is convex) which is countable. Let us rename these countable intervals to be $U_{n,\alpha}$ for $n < \omega$.

We then find that $\bigcup_{n<\omega} \{U_{n,\alpha} : \alpha < \omega_1\}$ is a basis for the entire space D. Likewise, $\bigcup_{n<\omega} \{U_{n,\alpha} \cap Y : \alpha < \omega_1\}$ is a basis for the subspace Y of D. We should show that for any $n < \omega$, $\{U_{n,\alpha} \cap Y : \alpha < \omega_1\}$ is locally finite.

Let $n < \omega$ and $\xi \in Y$. As the D_α partition $D \supseteq Y$, $\xi \in D_\alpha \cap Y$ for some $\alpha < \omega_1$. The only element of $\{U_{n,\alpha} \cap Y | \alpha < \omega_1\}$ intersecting D_α is precisely $U_{n,\alpha} \cap Y$. Thus every element of Y has a neighborhood (specifically, D_α for an appropriate $\alpha < \omega_1$) intersecting finite elements of $\{U_{n,\alpha} \cap Y | \alpha < \omega_1\}$.

We conclude that Y is σ-locally finite and T_3, so Y is metrizable. $\square$

Chapter 4

Generalizing to Any Regular Cardinal κ

First, a classic cardinality result:

Lemma 4.1. *For any infinite cardinal, $|\kappa \times \kappa| = \kappa$.*

Proof. Certainly, $|\kappa \times \kappa| \geq \kappa$, so we need only show $|\kappa \times \kappa| \leq \kappa$ by constructing an onto function from κ to $\kappa \times \kappa$.

We define a well-order $<_\star$ on $\kappa \times \kappa$ as such. $(\alpha_1, \beta_1) <_\star (\alpha_2, \beta_2)$ if one of the following holds:

1. $\max(\alpha_1, \beta_1) < \max(\alpha_2, \beta_2)$

2. $\max(\alpha_1, \beta_1) = \max(\alpha_2, \beta_2)$ and $(\alpha_1, \beta_1) <_L (\alpha_2, \beta_2)$

   (where $<_L$ is the lexicographical order)

It is easily seen that this is a well-order, as it was defined based on the well orders $<, <_L$. We may now define a function $f : \kappa \to \kappa \times \kappa$ where $f(0) = (0,0)$ and $f(\alpha)$ is the $<_\star$-least element of $\kappa \times \kappa \setminus f''(\alpha)$. Note that it is order-preserving, and thus injective. We should show it's an onto function.

Consider first the case that $\kappa = \omega$. Suppose by way of contradiction that (a, b) is not in the range of f for $a, b < \omega$. Then we have an injective map from ω to some $<_\star$-predecessors of (a, b), which in turn are a subset of $(\max(a, b) + 1) \times (\max(a, b) + 1)$, a finite set, which gives us our contradiction.

Now, let us assume inductively that $|\lambda \times \lambda| \leq \lambda$ for all $\lambda < \kappa$. Suppose by way of contradiction that (α, β) is not in the range of f for $\alpha, \beta < \kappa$. Then we have a bijection from κ to some $<_\star$-predecessors of (α, β), which in turn are a subset of $(\max(\alpha, \beta) + 1) \times$

$(\max(\alpha, \beta) + 1)$. As $\max(\alpha, \beta) + 1 < \kappa$, $|\max(\alpha, \beta) + 1| = \lambda < \kappa$ for some cardinal λ. Thus $|(\max(\alpha, \beta) + 1) \times (\max(\alpha, \beta) + 1)| = |\lambda \times \lambda| \leq \lambda$. But in that case, f is a bijection from κ to a set of cardinality $\leq \lambda < \kappa$, a contradiction. We've thus proven $|\kappa \times \kappa| \leq \kappa$, and thus $|\kappa \times \kappa| = \kappa$. $\qquad\square$

While the results of Chapter 3 are nice, the methods of these proofs can be generalized by introducing the idea of a "regular" cardinal.

Definition 4.2. The **cofinality** of an ordinal α, written $cf(\alpha)$, is the least cardinal such that there exists an unbounded function f from $cf(\alpha)$ to α. (That is, there is no element of α which is strictly greater than every ordinal in the range of f.)

Definition 4.3. A cardinal κ is said to be **regular** if $\kappa = cf(\kappa)$.

We follow with a few basic cardinality results.

Theorem 4.4.

(i) *If κ is any infinite cardinal, then the union of $\leq \kappa$-many sets, each of cardinality $\leq \kappa$, has cardinality $\leq \kappa$.*

(ii) *If κ is a successor ordinal, then κ is regular.*

(iii) *For any limit ordinal λ, $cf(\lambda)$ is a regular cardinal.*

Proof.

(i) Consider $\bigcup_{\alpha \in \lambda} S_\alpha$ where $\lambda \leq \kappa$ and $|S_\alpha| \leq \kappa$. Let f_α be an injection from S_α to κ. We define $f : \bigcup_{\alpha \in \lambda} S_\alpha \to \kappa \times \kappa$ so that $f(s)$ maps to $(\alpha, f_\alpha(s))$ where α is the least ordinal where $s \in S_\alpha$. f is an injection from $\bigcup_{\alpha \in \lambda} S_\alpha$ into $\kappa \times \kappa$, so $\left| \bigcup_{\alpha \in \lambda} S_\alpha \right| \leq |\kappa \times \kappa| = \kappa$.

(ii) Suppose by way of contradiction that $cf(\kappa^+) \leq \kappa$. There is a function $f : \kappa \to \kappa^+$ whose range is unbounded in κ^+. Then let $g_\alpha : \kappa \to \alpha$ be an onto function for all

$\alpha < \kappa^+$. (This is possible as $|\alpha| < \kappa^+ \Rightarrow |\alpha| \leq \kappa$.) We use these to define a function $g : \kappa \times \kappa \to \kappa^+$ where $g(\alpha, \beta) = g_{f(\alpha)}(\beta)$. Since for any $\gamma < \kappa^+$ we may find a δ_γ such that $f(\delta_\gamma) > \gamma$, we know $g_{f(\delta_\gamma)}$ maps onto γ. Thus our g is an onto function from a set of cardinality κ to κ^+, a contradiction.

(iii) Let γ be a limit ordinal, and $\kappa = cf(\gamma)$. There is a strictly increasing function $f : \kappa \to \gamma$ which is unbounded in γ.

Suppose by way of contradiction that there exists an unbounded strictly increasing function $g : \lambda \to \kappa$ for $\lambda < \kappa$. Composing f and g gives us an unbounded increasing function $f \circ g : \lambda \to \gamma$, a contradiction of the definition of κ as the *least* cardinal which has an unbounded map onto γ. Thus $cf(\kappa) = \kappa$ and $cf(\gamma)$ is regular.

$\square$

As is shown in the following theorem, regular cardinals have many of the properties commonly associated with "uncountable" versus "countable" sets.

Theorem 4.5. *For an infinite cardinal κ, the following are equivalent:*

(i) *κ is regular.*

(ii) *For any $A \subseteq \kappa$, if $|A| < \kappa$, then $\sup(A) < \kappa$.*

(iii) *The union of $< \kappa$-many sets, each of cardinality $< \kappa$, has cardinality $< \kappa$.*

Proof.

(i) $\Rightarrow$ (ii) (Shown by contrapositive.) Let $A \subseteq \kappa$ such that $|A| = \kappa_A < \kappa$ and $\sup(A) = \kappa$. Then $i : A \to \kappa$ where i is the inclusion map $(i(\alpha) = \alpha)$ has an unbounded range. Let $\theta : \kappa_A \to A$ be a bijection. Then $i \circ \theta : \kappa_A \to \kappa$ has an unbounded range, showing $cf(\kappa) \leq \kappa_A < \kappa$ and thus κ is not regular.

(ii) $\Rightarrow$ (i) Let $\lambda < \kappa$. Suppose by way of contradiction that there is a function $f : \lambda \to \kappa$ whose range was unbounded in κ, that is, $\sup(A) = \kappa$ for $A = \{f(\alpha) : \alpha \in \lambda\}$; contradiction. Thus $cf(\kappa) = \kappa$.

(i) $\Rightarrow$ (iii) Let $\lambda < \kappa$, and assume by induction that for $\beta < \lambda$, the union of β-many sets, each of cardinality $< \kappa$, has cardinality $< \kappa$. Assume that for each $\alpha < \lambda$, U_α is a set with $|U_\alpha| < \kappa$. As κ is regular, it follows that the function $f : \lambda \to \kappa$ where $f(\beta) = \left| \bigcup_{\alpha < \beta} U_\alpha \right|$ is bounded by some cardinal $\mu < \kappa$. As $\bigcup_{\alpha < \lambda} U_\alpha = \bigcup_{\beta < \lambda} \left(\bigcup_{\alpha < \beta} U_\alpha \right)$, it follows that the cardinality of the union of λ-many sets of cardinality $\leq \mu$ must be of cardinality $\max(\lambda, \mu) < \kappa$.

(iii) $\Rightarrow$ (i) Let $\lambda < \kappa$. Suppose by way of contradiction that there is a function $f : \lambda \to \kappa$ whose range is unbounded. Then $\bigcup_{\alpha < \lambda} f(\alpha) = \kappa$, and the union of less than κ-many sets each of cardinality less than κ has cardinality κ, a contradiction.

$\square$

Now that we've established the rules for regular cardinals, we observe that they behave in nice ways, that is, similar to the relationship between ω and ω_1. From this we can generalize many of the theorems from the previous chapter by merely replacing ω_1 with any regular cardinal κ, replacing ω with any cardinal $\lambda < \kappa$, assuming "uncountable" to mean "of cardinality κ", and assuming "countable" to mean "of cardinality $< \kappa$".

Theorem 4.6. *Let κ be an uncountable regular cardinal and $\lambda < \kappa$. Let $\mathcal{C} = \{C_\alpha : \alpha < \lambda\}$ be a collection of club sets in κ. Then $\bigcap \mathcal{C}$ is club.*

Proof. See the proof of Theorem 3.2. $\square$

Theorem 4.7. *Let κ be an uncountable regular cardinal. For any closed unbounded subset C of κ, there is a strictly increasing homeomorphism from C to κ.*

Proof. See the proof of Theorem 3.3. $\square$

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Theorem 4.8. *Let $f : \kappa \to \kappa$ be a function and $C = \{\alpha : \beta < \alpha \Rightarrow f(\beta) < \alpha\}$ be a subset of κ. Then C is closed and unbounded.*

Proof. See the proof of Theorem 3.4. $\square$

Theorem 4.9 (Pressing Down Lemma Lite). *Let S be a stationary set in κ. If for each ordinal $\alpha \in S \setminus \{0\}$, we choose an ordinal $\beta_\alpha < \alpha$, then there is some $\beta < \kappa$ such that $\beta = \beta_\alpha$ for κ-many $\alpha \in S$.*

Proof. See the proof of Theorem 3.6. $\square$

Theorem 4.10. *If S is a stationary subset of a regular cardinal κ, then S is not paracompact.*

Proof. See the proof of Theorem 3.7. $\square$

These final results about regular cardinals, specifically ω_1, will be needed in the final chapter.

Lemma 4.11. *Let C, D be closed unbounded subsets of a regular cardinal κ, and $\phi : \kappa \to D$ be a strictly increasing homeomorphism. $\phi''(C)$ is closed unbounded in κ.*

Proof. Fix $\alpha \in \kappa$. As D is unbounded, we may fix $\beta \in D$ such that $\beta > \alpha$. There is a $\gamma \in \kappa$ such that $\phi(\gamma) = \beta$. As C is unbounded, we may fix $\delta \in C$ such that $\delta > \gamma$, yielding $\phi(\delta) > \phi(\gamma) = \beta > \alpha$. Thus $\phi''(C)$ is unbounded in κ.

As ϕ is a homeomorphism, $\phi''(C)$ is a closed subset of D. This means there is a closed set E of κ such that $E \cap D = \phi''(C)$. As $\phi''(C)$ is the intersection of closed sets, $\phi''(C)$ is closed. $\square$

Lemma 4.12. *There are two disjoint stationary subsets of ω_1.*

Proof. Suppose not. Let $\mathbb{Q}$ be the set of rational numbers. Let $f : \omega_1 \to (0, 1) \setminus \mathbb{Q}$ be injective. $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ cannot both contain the image of stationary sets in ω_1, so there is a closed unbounded set C_1 of ω_1 that maps by f into an open interval of irrational numbers of length $\frac{1}{2}$.

Now assume we have, by way of induction, a chain of club sets $C_1 \supseteq \ldots \supseteq C_n$ such that C_i maps by f into an open interval of irrational numbers with rational endpoints of length $\frac{1}{2^i}$. Let $h : \omega_1 \to C_n$ be a strictly increasing homeomorphism and $g = f \circ h$. Assume $f''(C_n) \subseteq (a, b) \setminus \mathbb{Q}$ where $a, b \in \mathbb{Q}$ and $b - a = \frac{1}{2^n}$. $g : \omega_1 \to (a, b) \setminus \mathbb{Q}$ is injective. $(a, \frac{a+b}{2})$ and $(\frac{a+b}{2}, b)$ cannot both contain the image of stationary sets in ω_1, so there is a closed unbounded set C^* of ω_1 that maps by g to either $(a, \frac{a+b}{2})$ and $(\frac{a+b}{2}, b)$, which are both have rational endpoints and are of length $\frac{1}{2^{n+1}}$. It then follows by Lemma 4.11 that $C_{n+1} = h''(C^*) \subseteq C_n$ is also a club set, and $f''(C_{n+1}) = f''(h''(C^*)) = g''(C^*)$ is a subset of an open interval of irrational numbers with rational endpoints of length $\frac{1}{2^{n+1}}$.

We then observe that for all $n < \omega$, $C = \bigcap_{i < \omega} C_i \subseteq C_n$ maps injectively into a subset of an open interval of length $\frac{1}{2^n}$, a contradiction. $\square$

Chapter 5

Some Theorems on the Compactness of Linearly Ordered Topological Spaces

Before moving on to the first main result of this paper, we should investigate a characterization of compactness in the context of a LOTS.

Theorem 5.1. *A linearly ordered topological space X is compact iff every nonempty subset of X has a least upper bound and a greatest lower bound.*

Proof. If X is compact, then suppose by way of contradiction that $S \subseteq X$ is a set with no least upper bound. Let $T = \{t \in X : \forall s \in S(t \geq s)\}$. We claim

$$\mathcal{U} = \{(\leftarrow, s) : s \in S\} \cup \{(t, \rightarrow) : t \in T\}$$

is an open cover of X. If so, then there is a finite subcover $\{(\leftarrow, s_i) : 0 < i < m\} \cup \{(t_i, \rightarrow); 0 < i < n\}$, but this cannot cover the rightmost s_i (as $t_j \geq s_i$ for any $t_j \in T$), which would yield our contradiction.

To see that $\{(\leftarrow, s) : s \in S\} \cup \{(t, \rightarrow) : t \in T\}$ is an open cover, suppose by way of contradiction that $x \in X$ was not covered by $\mathcal{U}$. As x is not covered by $\{(\leftarrow, s) : s \in S\}$, $x \geq s$ for all $s \in S$, which makes x an upper bound of S and places $x \in T$. It then follows that $x \leq t$ for all $t \in T$ as it is not covered by $\{(t, \rightarrow) : t \in T\}$. This makes it a lower bound of T, and since it belongs to T, x is the greatest lower bound of T.

Now note that there must be some $y < x$ that is also an upper bound of S since S has no least upper bound. y is not the greatest lower bound of T (as $x > y$ is), so it must not lie in T. This mean that there is some $s_y \in S$ such that $s_y > y$. But that contradicts the fact that y was an upper bound for S, which proves that every subset of X must have a least upper bound.

To see that every subset also must have a greatest lower bound, simply reverse all the orders in the above argument. Alternately, note that we merely need to consider a "mirrored" linear space $X_M = \{x_M : x \in X\}$ with the order $x_M <_M x_M \Leftrightarrow x > y$. There is an obvious homeomorphism between the two spaces, so X_M is also compact. Any subset without a greatest lower bound in X would yield a reflection in X_M without a least upper bound, a contradiction. This finishes the forward implication.

Now conversely, assume that every subset of X has a least upper bound and greatest lower bound. This means that X has a greatest lower bound, and minumum element, a and a least upper bound, and maximum element, b. Let $\mathcal{U} = \{(a_\alpha, b_\alpha) : \alpha < \kappa\}$ be an open interval cover of X for some cardinal κ. Let $S = \{s \in X : \text{there is a finite subcover of } \mathcal{U} \text{ for}$ $[a, s]\}$. S is nonempty as there is a finite subcover of $\mathcal{U}$ for $[a, a] = \{a\}$, placing $a \in S$. We note $\sup(S) \in S$ as there is an interval $(a_{\alpha'}, b_{\alpha'}) \in \mathcal{U}$ containing $\sup(S)$, and a finite subcover of $\mathcal{U}$ covering $[a, a_{\alpha'}]$ (since $a_{\alpha'} < \sup(S) \Rightarrow a_{\alpha'} \in S$), so by combining them we find a finite subcover of $\mathcal{U}$ covering $[a, \sup(S)]$.

We claim that $\sup(S) = b$. To see this, assume by way of contradiction that $\sup(S) < b$. Let $(a_{\alpha'}, b_{\alpha'})$ continue to denote a set in $\mathcal{U}$ covering $\sup(S)$. We now note that if $(a_{\alpha'}, b_{\alpha'})$ contains no element of X greater than $\sup(S)$, then there are no elements in X between $\sup(S)$ and $b_{\alpha'}$. Thus we can use any interval in $\mathcal{U}$ which contains $b_{\alpha'}$, and combine that with the finite subcover for $[a, \sup(S)]$ to get a finite subcover of $\mathcal{U}$ covering $[a, b_{\alpha'}] \supset [a, \sup(S)]$, a contradiction.

We've thus seen that $(a_{\alpha'}, b_{\alpha'})$ must contain an element of X greater than $\sup(S)$, which we will call t. Of course this then means that $[a, t] \supset [a, \sup(S)]$ can be covered by the finite subcover of $[a, \sup(S)]$ combined with $(a_{\alpha'}, b_{\alpha'})$, which places $t > \sup(S)$ in S and yields our contradiction.

As $\sup(S) = b$, there exists a finite subcover of $\mathcal{U}$ for $[a, b] = X$, demonstrating the compactness of X. $\qquad\square$

Even if a linearly ordered set is not compact we can easily compactify it, that is, embed it densely in a compact linearly ordered topological space.

Theorem 5.2. *Every linearly ordered topological space X is a dense subset of a compact linearly ordered topological space $\hat{X}$.*

Proof. We define $\hat{X}$ as such. $\hat{X} \subseteq \mathbb{P}(X)$ where $A \in \hat{X}$ iff all of the following holds:

1. A is closed in X

2. $a \in A$ and $b < a \Rightarrow b \in A$

3. X has a least element $\Rightarrow A \neq \emptyset$

We shall call a set that holds for the first two conditions "left-closed". We should show that the partial order $\subseteq$ is a linear order on $\hat{X}$.

Let $A \neq B$. Without loss of generality, we can assume there is a $b \in B \setminus A$. We note there is no element of A greater than or equal to b, as that would imply $b \in A$. So for all $a \in A$, $a < b$, and thus $a \in B$, giving us $A \subseteq B$.

So the linear order $\subseteq$ generates a topology on $\hat{X}$. We then note that the subspace made up of sets $A_x = \{a \in X | a \leq x\}$ for all $x \in X$ is homeomorphic to the space X in the natural way: let $\phi(A_x) = x$. Then $\phi[(A_x, A_y)] = (x, y)$, and images and inverse images preserve basic open sets, making ϕ a homeomorphism.

We now show that $\{A_x : x \in X\}$ is dense in $\hat{X}$. Take a set $A \in \hat{X}$. Consider a basic open set about A, (A^-, A^+), noting $A^- \subset A \subset A^+$. Let $x \in A \setminus A^-$. We shall show that $A_x \in (A^-, A^+)$. Let $a \in A_x$. $x \in A$ and $a \leq x$ implies $a \in A$, so $A_x \subseteq A \subset A^+$. Now let $a \in A^-$. As $x \notin A^-$, it follows that $x > a$. It then follows that as $a < x$, $a \in A_x$. This means $A^- \subseteq A_x$, and as $x \in A_x \setminus A^-$, we see that $A^- \subset A_x$, which places $A_x \in (A^-, A^+)$. This makes the closure of $\{A_x : x \in X\}$ to be $\hat{X}$, and $\{A_x : x \in X\}$ is dense.

To conclude, we show that $\hat{X}$ is compact. Let $\hat{S}$ be a nonempty subset of $\hat{X}$. We note that $\cap \hat{S}$ is a closed set in X and that $a \in \cap \hat{S}$ and $b < a$ implies $a \in S$ for all $S \in \hat{S}$, and

thus $b \in S$ for all $S \in \hat{S}$ yielding $b \in \cap\hat{S}$, and thus $\cap\hat{S} \in \hat{X}$. Also, if $S \in \hat{S}$, then certainly $\cap\hat{S} \subseteq S$, so $\cap\hat{S}$ is a lower bound of $\hat{S}$. For any lower bound T of $\hat{S}$, we note that $t \in T$ implies $t \in S$ for any $S \in \hat{S}$, and thus $t \in \cap\hat{S}$, which shows $T \subseteq \cap\hat{S}$. Thus $\cap\hat{S}$ is the greatest lower bound of $\hat{S}$.

Now we note that $\overline{\cup\hat{S}}$ is a closed set in X and that $a \in \overline{\cup\hat{S}}$ and $b < a$ implies either

1. $a \in S$ for some $S \in \hat{S}$, so $b \in S \subseteq \cup\hat{S} \subseteq \overline{\cup\hat{S}}$ or

2. a is a limit point of $\cup\hat{S}$ but not in that union, so $a \geq s$ for all $s \in \cup\hat{S}$, which requires (b, a) to intersect $\cup\hat{S}$ at some $s > b$ in some $S \in \hat{S}$, which then implies that $b \in S \subseteq \cup\hat{S} \subseteq \overline{\cup\hat{S}}$.

So we have that $\overline{\cup\hat{S}} \in \hat{X}$.

Certainly $\overline{\cup\hat{S}}$ is a superset of all $S \in \hat{S}$, so $\overline{\cup\hat{S}}$ is an upper bound of $\hat{S}$. And if T is any upper bound of $\hat{S}$, then it is closed in X and is a superset of $\cup\hat{S}$, and as $\overline{\cup\hat{S}}$ is the intersection of all such sets, we see that $\overline{\cup\hat{S}} \subseteq T$ and $\overline{\cup\hat{S}}$ is the least upper bound of $\hat{S}$.

As any arbitrary $\hat{S} \subseteq \hat{X}$ has both a least upper bound and greatest lower bound, we know that $\hat{X}$ is compact, finishing the proof. $\square$

It's often a useful trick to compactify a space in order to gain some extra structure, as we will see in a later proof.

Chapter 6

A Characterization of the Paracompactness of a Linearly Ordered Topological Space

We're about ready to tackle the first main result of this paper. First, we introduce another sense of connectedness in the sense of a collection of subsets of a topological space.

Definition 6.1. For a topological space X, a collection $\mathcal{U} \subseteq \mathcal{P}(X)$, and two points $a, b \in X$, the finite sequence $\langle U_0, ..., U_{n-1} \rangle$ of sets in $\mathcal{U}$ is called a **finite linked chain** joining a, b if $a \in U_0$, $b \in U_{n-1}$, and for all $0 \leq i < n - 1$, $U_i \cap U_{i+1} \neq \emptyset$.

Definition 6.2. A collection $\mathcal{U}$ of subsets of a topological space X is said to be **connected** if every pair of sets in $\mathcal{U}$ is connected by a finite linked chain in $\mathcal{U}$.

Definition 6.3. For a linearly ordered topological space X, a cover $\mathcal{U}$ of X by open intervals, and two points $a < b \in X$, a finite linked chain $\langle (l_0, r_0), ..., (l_{n-1}, r_{n-1}) \rangle$ of intervals in $\mathcal{U}$ joining a, b is called a **progressive finite linked chain** if for all $0 < i \leq n - 1$, $r_{i-1} < r_i$.

Lemma 6.4. *For a linearly ordered topological space X and an open cover $\mathcal{U}$ of X by intervals, there exists a finite linked chain in $\mathcal{U}$ connecting two points $a < b \in X$ if and only if there exists a progressive finite linked chain joining them.*

Proof. The backwards implication is trivial. If $\langle (l_0, r_0), ..., (l_{n-1}, r_{n-1}) \rangle$ is a chain joining a to b and isn't already progressive, then let (l_i, r_i) be the first link in the chain which does not satisfy the progressive requirement, that is, $r_i \leq r_{i-1}$. If $b \in (l_{i-1}, r_{i-1})$ then $\langle (l_0, r_0), ..., (l_{i-1}, r_{i-1}) \rangle$ is our progressive finite linked chain. Otherwise, as b is covered by some (l_k, r_k) for $k \geq i$, there must be some least j with $r_{i-1} < r_j$. We note $(l_{i-1}, r_{i-1}) \cap (l_j, r_j) \neq \emptyset$ since $l_j < r_{j-1} \leq r_{i-1}$, so $\langle (l_0, r_0), ..., (l_{i-1}, r_{i-1}), (l_j, r_j), ..., (l_{n-1}, r_{n-1}) \rangle$ is another finite linked chain with at least one less nonprogression. We may then complete this process finitely many times until we have our progressive linked chain. $\square$

The reader may note that the reuse of the term "connected" is appropriate as there is a strong connection between a connected topological space and the connectedness of an open cover of that space.

Theorem 6.5. *A space X is connected iff every cover of X by nonempty open sets is connected.*

Proof. Let X be disconnected, so $X = A \cup B$ with A, B disjoint and clopen. Then $\{A, B\}$ is an open cover and A, B cannot be joined by a finite linked chain in that cover.

Now assume X is a topological space with an open cover $\mathcal{U}$ with sets A, B which cannot be joined by a finite linked chain in $\mathcal{U}$. Then let $\mathcal{U}_0 = \{A\}$ and for all $0 < i < \omega$ let $\mathcal{U}_{i+1} = \{U \in \mathcal{U} : U \cap (\bigcup \mathcal{U}_i) \neq \emptyset\}$. We note that $B \notin \mathcal{U}_i$ for any $i < \omega$ as that would give us a finite linked chain from A to B.

Then we note that $\bigcup(\bigcup_{i<\omega} \mathcal{U}_i)$ is closed, for if l is a limit point of $\bigcup(\bigcup_{i<\omega} \mathcal{U}_i)$, then any open set $U_l \in \mathcal{U}$ containing l intersects $\bigcup(\bigcup_{i<\omega} \mathcal{U}_i)$, so it intersects a member of $\mathcal{U}_i$ for some $i < \omega$, putting U_l in $\mathcal{U}_{i+1}$ and $l \in \bigcup(\bigcup_{i<\omega} \mathcal{U}_i)$.

As $\bigcup(\bigcup_{i<\omega} \mathcal{U}_i)$ is clopen and a strict subset of X, X is not connected. $\square$

In order to find a stationary subset of a regular cardinal, we first note that we may compactify a LOTS to obtain a subspace homeomorphic to a regular cardinal.

Lemma 6.6. *Let X be a compact linearly ordered topological space, and $p \in X$. Let κ be the cofinality of $L_p = \{y \in X : y < p\}$. Then L_p contains a closed cofinal set homeomorphic to the cardinal κ.*

Proof. Let $y' : \kappa \to L_p$ be a strictly increasing cofinal map. We may define a new cofinal map y by letting $y(\alpha) = y'(\alpha)$ for successor αs, and $y(\alpha) = \sup(\{y(\beta) : \beta < \alpha\})$ for limit α. This is well-defined as the compact X must contain the supremum of any nonempty set, and this supremum must be less than p. We claim y is a homeomorphism onto its range Y.

It is certainly an order isomorphism, and thus open. All that is left to show it is a homeomorphism is to see is that it is continuous. Let $\alpha < \kappa$ and $x < y(\alpha) < z$ in Y. If α

is not a limit ordinal, then $\{\alpha\}$ is open and $y''(\{\alpha\}) \subseteq (x, z)$. Otherwise, we note that $y(\alpha)$ is the supremum of $\{y(\beta) : \beta < \alpha\}$, so we may pick $\beta < \alpha$ with $x < y(\beta) < y(\alpha)$, giving us $y''[(\beta, \alpha + 1)] \subseteq (x, z)$.

Lastly, we should show that Y is closed. Let $x < p$ be a limit point of Y. There is a least α such that $x < y(\alpha)$, so we may assume (z, x) intersects Y for any $z < x$. We then note that $x = \sup(\{y(\beta) : \beta < \alpha\}$, and α must be a limit ordinal, so $x \in Y$. $\qquad\square$

For convenience, we will partition our linearly ordered set into portions based on the open cover of our space.

Lemma 6.7. *If for an open cover $\mathcal{U}$ of a topological space X and sets $U, V \in \mathcal{U}$ we have a relation $\sim$ such that $U \sim V$ if they are joined by a finite linked chain in $\mathcal{U}$, then $\sim$ is an equivalence relation.*

Proof. We note $U \sim U$ as $\langle U \rangle$ is a finite linked chain from U to U.

If $U \sim V$, then there is a finite linked chain $\langle U, ..., V \rangle$, which by reversal gives a finite linked chain $\langle V, ..., U \rangle$, showing $V \sim U$.

Lastly, if $U \sim V$ and $V \sim W$, then by combining the finite linked chains $\langle U, ..., V \rangle$ and $\langle V, ..., W \rangle$ we get the chain $\langle U, ..., V, ..., W \rangle$, which shows $U \sim W$.

This proves that $\sim$ is an equivalence relation. $\qquad\square$

Definition 6.8. If $\mathcal{U}$ is an open cover of a topological space X and $U \in \mathcal{U}$, let $[U]$ be the equivalence class with respect to the above-defined $\sim$ for U. We call $[U]$ the **connected extension of U from $\mathcal{U}$**. $\bigcup[U] \subseteq X$ is said to be the U-**component** of X.

Lemma 6.9. *For an open cover $\mathcal{U}$ of a topological space X, $\sim$ partitions X into clopen U-components $\bigcup[U]$. That is, for each $x \in X$ there is a unique $[U]$ such that $x \in \bigcup[U]$.*

Proof. Certainly, for any $x \in X$ there is a set U in the open cover $\mathcal{U}$ which covers x, so $x \in \bigcup[U]$.

Now, if $x \in U$ and $x \in V$ for $U, V \in \mathcal{U}$, then $U \cap V \neq \emptyset$. Thus $\langle U, V \rangle$ is a finite linked chain, and $U \sim V \Rightarrow [U] = [V]$. Thus the $\bigcup[U]$ covering x is unique.

Lastly, we note that $\bigcup[U]$ is open as it is the union of open sets. We then note that $X \setminus \bigcup[U] = \bigcup \mathcal{U} \setminus [U]$ as $\mathcal{U}$ must cover all of X and any element $V \in \mathcal{U}$ which covers $\bigcup[U]$ must be in $[U]$. This complement is also open, so $\bigcup[U]$ is also closed. $\qquad\square$

When considering paracompactness, we may focus our attention onto the $[U]$-components of the space.

Lemma 6.10. *A topological space X is paracompact iff for any open cover $\mathcal{U}$ and any set $U \in \mathcal{U}$, there is a locally finite open refinement of $[U]$ covering $\bigcup[U]$.*

Proof. For the forward implication, start with the open cover $\mathcal{U}$ and fix $\mathcal{U}^*$ as its locally finite refinement covering X. Fix $U \in \mathcal{U}$. The subset $\mathcal{V} = \{V : V \in \mathcal{U}^* \text{ and } V \subseteq \bigcup[U]\}$ of $\mathcal{U}$ is also a locally finite refinement. In addition $\mathcal{V}$ covers $\bigcup[U]$, as for $x \in \bigcup[U]$, we have a $W^* \in \mathcal{U}^*$ covering it, and since $W^* \subseteq W$ for some $W \in \mathcal{U}$, and W intersects $\bigcup[U]$, $W \subseteq \bigcup[U]$ and $W^* \subseteq \bigcup[U]$, so $W^* \in \mathcal{V}$.

Conversely, we may assume that we may find a locally finite refinement $[U]^*$ of each $[U]$. We've seen that for each $x \in X$, x is covered by a unique $[U] \subseteq \mathcal{U}$. So we may find an open set W containing x within $\bigcup[U]$ that intersects only finitely many elements of $[U]^*$. If W intersects any element of some $[V]^*$, that admits a finite linked chain $\langle U, ..., W, ..., V \rangle$ which means $[V]^* = [U]^*$. Thus $\bigcup_{U \in \mathcal{U}}[U]^*$ is a locally finite refinement of $\mathcal{U}$, and covers X. $\qquad\square$

With this we are finally ready to approach the first main result. It should be noted that the forward implication is largely trivial as we observe that any closed subspace of a paracompact space is paracompact. The other direction is much less obvious; however, by carefully examining the $[U]$-components generated by a particular cover of the space, we can construct a locally finite refinement by grabbing a copy of a regular uncountable cardinal in the compactification of the $[U]$-component and using the fact that within a $[U]$-component, any two sets in the cover is connected by a progressive finite linked chain.

Theorem 6.11. *A linearly ordered topological space X is paracompact iff X does not contain a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

Proof. We first note that if X contains a closed subspace Y homeomorphic to a stationary subset of a regular uncountable cardinal, then Y is a closed subspace of X which is not paracompact. Thus X cannot be paracompact.

We now assume that X does not contain a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal. Let $\mathcal{U}$ be a collection of open intervals covering X. Fix $U \in \mathcal{U}$. By Lemma 6.10, we need only show a locally finite refinement of the connected extension $[U]$ covering $\bigcup[U]$. Let $Y = \bigcup[U]$ and $x_0 \in Y$.

If the cofinality of Y is $n < \omega$, then there is a greatest element $y \in Y$, and we may pick a progressive finite linked chain in $[U]$ beginning with a set which covers x_0 and ending with a set which covers y, which is a finite refinement of $[U]$ covering $[x_0, \rightarrow) \cap Y$. Call this refinement $\mathcal{Y}^{\rightarrow}$.

Now, if the cofinality of Y is $\kappa = \omega$, there is some increasing map f with $f(0) = x_0$ which is cofinal in Y. We may then, for each $n < \omega$, find a progressive finite linked chain C_n in $[U]$ joining $f(n)$ and $f(n+1)$. We may then define $C_n' = \{L \cap (f(n-1), f(n+2)) : L$ is a link in $C_n\}$ where $f(-1)$ is assumed to represent $\leftarrow$. $\bigcup_{n<\omega} C_n'$ then covers $[x_0, \rightarrow) \cap Y$. In addition, each point in $(x_0, \rightarrow) \cap Y$ lies in some $[f(n), f(n+1)]$ and thus the open set $(f(n-1), f(n+2))$ could only intersect the finite elements of the five sets C_{n-2}' through C_{n+2}', making the union a locally finite refinement. Again, call this refinement $\mathcal{Y}^{\rightarrow}$.

Finally, consider when the cofinality of Y is $\kappa > \omega$. Let $\hat{Y}$ be the compactification of Y from Theorem 5.2. $\hat{Y}$ adds a greatest element p not in Y since Y had uncountable cofinality. Lemma 6.6 gives us that $\hat{Y} \setminus \{p\}$ contains some closed cofinal subset K homeomorphic to the cofinality κ' of $\hat{Y} \setminus \{p\}$.

We note that if $f : \kappa' \rightarrow \hat{Y} \setminus \{p\}$ is a cofinal map, then the density of Y in $\hat{Y} \setminus \{p\}$ gives a point of Y in the open set $(f(\alpha), f(\alpha+1))$ in $\hat{Y}$, so the cofinality κ of Y is $\leq \kappa'$. Of course, if $g : \kappa \rightarrow Y$ is cofinal, then its inclusion map $i_g : \kappa \rightarrow \hat{Y} \setminus \{p\}$ is cofinal in $\hat{Y}$, so $\kappa' \leq \kappa$. Thus $\kappa' = \kappa$.

Now observe that $K \cap Y$ cannot be a stationary subset of K by our assumption that Y does not contain a closed subset homeomorphic to a stationary subset of any regular uncountable cardinal such as κ. This gives us a subset of $K \backslash Y$ which is closed and unbounded in K, call it $\hat{K}$. We may assume without loss of generality that all elements of $\hat{K}$ are greater than x_0 as any final interval of a closed unbounded set is also a closed unbounded set. $\hat{K}$ is homeomorphic to κ, so suppose $\phi : \kappa \to \hat{K}$ is a homeomorphism. We may first construct a finite refinement of $[U]$ on $[x_0, \phi(0)) \cap Y$ by using any progressive finite linked chain connecting x_0 to any point of Y greater than $\phi(0)$, with each element intersected with the open set $(\leftarrow, \phi(0)) \cap Y$ in the subspace Y of $\hat{Y}$ (dropping any empty sets). Similarly, for each $\alpha < \kappa$, we can similarly construct a finite refinement of $[U]$ for $(\phi(\alpha), \phi(\alpha+1))$ by using any progressive finite linked chain connecting x_0 to any point of Y greater than $\phi(\alpha + 1)$, with each element of the chain intersected with $(\phi(\alpha), \phi(\alpha+1)) \cap Y$. The union of these is a locally finite refinement of $[U]$ covering $[x_0, \to) \cap Y$. Of course we'll call this refinement $\mathcal{Y}^{\to}$ as well.

We note that $\mathcal{Y}^{\to}$ is locally finite at x_0, so only finite elements of it can extend left of x_0. We can then use similar arguments to generate a locally finite refinement $\mathcal{Y}^{\leftarrow}$ which covers $(\leftarrow, x_0] \cap Y$ and only has finitely many elements which extend right of x_0. Lastly, $\mathcal{Y} = \mathcal{Y}^{\leftarrow} \cup \mathcal{Y}^{\to}$ is then a locally finite refinement of $[U]$ covering Y, finishing the proof. $\quad \square$

Chapter 7

Stationary Sets and the Baire Property

Slightly changing pace, we shall investigate how we may use stationary subsets of ω_1 to construct two Baire spaces whose product is not Baire. It should be noted that in this chapter we assume ω_1 has the discrete topology.

Definition 7.1. For $\sigma \in \omega_1^{<\omega}$, let $[\sigma] = \{f \in \omega_1^\omega : \sigma \subseteq f\}$

Definition 7.2. For any countable α and $f \in \omega_1^\alpha$, let $f^* = \sup(\mathrm{ran}(f))$. For $A \subseteq \omega_1$, let $A^* = \{f \in \omega_1^\omega : f^* \in A\}$.

Lemma 7.3. *For all $f \in \omega_1^\omega$, $\{[f \upharpoonright n] : n < \omega\}$ is a local base at f. $\{[f \upharpoonright n] : n < \omega, f \in \omega_1^\omega\}$ is a basis for ω_1^ω.*

Proof. By the definition of the product topology, a basic open set in our space is

$$\{\alpha_0\} \times ... \times \{\alpha_{n-1}\} \times \omega_1 \times ...$$

where $\alpha_i \in \omega_1$.

Thus for an arbitrary function $f \in \omega_1^\omega$ contained in that open set, $f(i) = \alpha_i$ for $i < n$ and $[f \upharpoonright n]$ is exactly that set, establishing our local base.

In addition, for any such basic open set, we may pick any function f such that $f(i) = \alpha_i$ for $i < n$, yielding a $[f \upharpoonright n]$ which is exactly equal to that set. Thus the collection $\{[f \upharpoonright n] : n < \omega, f \in \omega_1^\omega\}$ is exactly the normal basis for the product topology. $\square$

Theorem 7.4. *If $A \subseteq \omega_1$ is uncountable, then A^* is dense in ω_1^ω.*

Proof. For any basic open set $[g \upharpoonright n]$, there is an $\alpha \in A$ with $(g \upharpoonright n)^* \leq \alpha$. Thus $(g \upharpoonright n)^\frown \langle \alpha, \alpha, ... \rangle \in A^* \cap [g \upharpoonright n]$. $\square$

Theorem 7.5. *If A and B are uncountable disjoint subsets of ω_1, then $A^* \times B^*$ is not Baire.*

Proof. For any two functions $\alpha, \beta \in \omega_1^{<\omega}$ and any uncountable $X \subseteq \omega_1$, let $\sigma_{\alpha,\beta}^n \in \omega_1^n$ be the constant function mapping to $\max(\alpha^*, \beta^*)$, and let $\tau_{\alpha,\beta}^X \in \omega_1^\omega$ be the constant function mapping to the least element of X greater than $\max(\alpha^*, \beta^*)$. Then, let

$$E_n = \bigcup_{0 \neq \alpha, \beta \in \omega_1^{<\omega}} [\alpha ^\frown \sigma_{\alpha,\beta}^n] \times [\beta ^\frown \sigma_{\alpha,\beta}^n]$$

for each $n < \omega$. As it is the union of basic open sets, E_n is open.

To see that E_n is dense in $A^* \times B^*$, consider a basic open set $[f \upharpoonright i] \times [g \upharpoonright j]$ for $f, g \in \omega_1^\omega$, and the ordered pair of functions

$$\left\langle (f \upharpoonright i) ^\frown (\sigma_{f\upharpoonright i, g\upharpoonright j}^n) ^\frown (\tau_{f\upharpoonright i, g\upharpoonright j}^A), (g \upharpoonright j) ^\frown (\sigma_{f\upharpoonright i, g\upharpoonright j}^n) ^\frown (\tau_{f\upharpoonright i, g\upharpoonright j}^B) \right\rangle.$$

This ordered pair of functions lies in E_n, $A^* \times B^*$, and $[f \upharpoonright i] \times [g \upharpoonright j]$, showing that E_n is dense.

If our space were Baire, it would follow that the intersection of countably many of the E_n's would be itself an open dense set. However, we will show a countable intersection which is in fact completely empty. Consider $\bigcap_{n<\omega} E_{2n}$. Indeed, if $\langle f, g \rangle$ was in $\bigcap_{n<\omega} E_{2n}$, then consider f^* and g^*. We note that for each $n < \omega$, there are functions $\alpha_n, \beta_n \in \omega_1^{<\omega}$ such that $\alpha_n ^\frown \sigma_{\alpha_n,\beta_n}^{2n}$ is an initial restriction of f of domain $> 2n$ and $\beta_n ^\frown \sigma_{\alpha_n,\beta_n}^{2n}$ is an initial restriction of g of domain $> 2n$. Thus $f(n) = (\alpha_n ^\frown \sigma_{\alpha_n,\beta_n}^{2n})(n) \leq \max(\alpha_n^*, \beta_n^*)$ and $g(n) = (\beta_n ^\frown \sigma_{\alpha_n,\beta_n}^{2n})(n) \leq \max(\alpha_n^*, \beta_n^*)$. This tells us that $f^* = \sup(\{\max(\alpha_n^*, \beta_n^*) | n < \omega\}) = g^*$, giving that $\langle f, g \rangle$ cannot be in $A^* \times B^*$ as A, B are disjoint. $\qquad \square$

We have thus observed a product which is not Baire, regardless of whether or not its component spaces are Baire. We proceed to show that, indeed, we may find two uncountable subsets A, B of ω_1 such that A^*, B^* are Baire, by utilizing stationary sets.

Lemma 7.6. *Let U be dense open in ω_1^ω. For each $\sigma \in \omega_1^{<\omega}$, there exists an extension $\sigma_U \in \omega_1^{<\omega}$ such that $[\sigma_U] \subseteq U$.*

Proof. As U is dense, the open set $[\sigma]$ intersects U at some function Σ. As U is open, there is some basic open neighborhood $[\Sigma \upharpoonright n]$ which is a subset of U. Let $\sigma_U = (\Sigma \upharpoonright n) \cup \sigma$. $\sigma_U \supseteq \sigma$ and $[\sigma_U] \subseteq [\Sigma \upharpoonright n] \subseteq U$. $\qquad\square$

Lemma 7.7. *Let U be dense open in ω_1^ω. For each $\sigma \in \omega_1^{<\omega}$, let σ_U be defined as above. Then*

$$C_U = \{\alpha < \omega_1 : \sigma \in \alpha^{<\omega} \Rightarrow \sigma_U \in \alpha^{<\omega}\}$$

is closed and unbounded.

Proof. Let γ be a limit point of C_U. If $\sigma \in \gamma^{<\omega}$, $\sigma \in \delta^{<\omega}$ for some $\delta < \gamma$. Thus $\sigma \in \alpha^{<\omega}$ for some $\alpha \in (\delta, \gamma) \cap C_U$. As $\alpha \in C_U$, $\sigma_U \in \alpha^{<\omega} \subset \gamma^{<\omega}$. Thus $\gamma \in C_U$ and C_U is closed.

Let $f : \omega_1 \to \omega_1$ be defined such that $f(\beta) = \sup(\{\sigma_U^* : \sigma \in \beta^{<\omega}\})$. Let D be the set of limit ordinals in ω_1. Then (by Theorems 4.8 and 4.6)

$$C = \{\alpha < \omega_1 : \beta < \alpha \Rightarrow f(\beta) < \alpha\} \cap D$$

is unbounded (and closed). Let $\alpha \in C$. If $\sigma \in \alpha^{<\omega}$, then as α is a limit ordinal, $\sigma \in \beta^{<\omega}$ for some $\beta < \alpha$. $\sigma_U^* \leq \sup(\{\sigma_U^* : \sigma \in \beta^{<\omega}\}) = f(\beta) < \alpha$, so $\sigma_U \in \alpha^{<\omega}$ and $\alpha \in C_U$. $C \subseteq C_U$ implies C_U is unbounded. $\qquad\square$

Theorem 7.8. *Let A be a stationary subset of ω_1. Then A^* is Baire.*

Proof. Let V_n be dense open in A^* for $n < \omega$. V_n is dense in ω_1^ω. In addition, for each $n < \omega$, $V_n = U_n \cap A^*$ for some open U_n in ω_1^ω. Since V_n is dense in ω_1^ω and $V_n \subseteq U_n$, U_n is dense as well as open. Let C_{U_n} be defined as above.

Let $\sigma \in \omega_1^{<\omega}$ and consider the basic open set $[\sigma] \cap A^*$ in A^*. To show A^* is Baire, we must show that $\bigcap_{n<\omega} V_n$ is dense in A^*, that is, there exists a function in

$$[\sigma] \cap A^* \cap \left(\bigcap_{n<\omega} V_n \right) = [\sigma] \cap A^* \cap \left(\bigcap_{n<\omega} U_n \right).$$

This function must then have an initial segment of σ, a supremum in A, and be in U_n for all $n < \omega$.

Start by fixing an ordinal α in the intersection of the closed unbounded set $[\sigma^*+1, \omega_1) \cap \bigcap_{n<\omega} C_{U_n}$ and the stationary set A and an increasing sequence $\alpha_n \to \alpha$. Note that $\sigma \in \alpha^{<\omega}$. Let $\sigma^0 = (\sigma_{U_0}) ^\frown \langle \alpha_0 \rangle$ and in general, $\sigma^{n+1} = ((\sigma^n)_{U_{n+1}}) ^\frown \langle \alpha_{n+1} \rangle$ for all $n < \omega$. It follows by the definition of α that each $\sigma^n \in \alpha^{<\omega}$.

Consider $\Sigma = \bigcup_{n<\omega} \sigma^n$. $\Sigma \in \omega_1^\omega$ since the length of the σ^n was increased by at least one at each step. Σ has σ as an initial segment. Also, $\Sigma \in [\sigma_{U_0}] \subseteq U_0$ and $\Sigma \in [\sigma^n_{U_{n+1}}] \subseteq U_{n+1}$ for all $n < \omega$. Lastly, $\alpha \in A$ is an upper bound of the range of Σ as α is an upper bound for the range of σ^n, and thus an upper bound of the range of $\sigma^{n+1} = (\sigma^n_{U_{n+1}}) ^\frown \langle \alpha_{n+1} \rangle$ as $\alpha \in C_{n+1}$ (and similarly for σ and σ^0). α is the least upper bound of the range of Σ as for any ordinal $\beta < \alpha$, we may find some α_n with $\beta < \alpha_n < \alpha$ in the range of Σ. $\qquad \square$

Observing from Chapter 4 that we may find two disjoint stationary subsets of ω_1, this wraps up our final result.

Corollary 7.9. *There are metrizable Baire spaces X and Y such that $X \times Y$ is not Baire.*

Proof. Let A, B be disjoint stationary subsets of ω_1. By Theorem 7.8, A^*, B^* are Baire spaces, and by Theorem 7.5 $A^* \times B^*$ is not Baire. Finally, we note that A^*, B^* are subspaces of the metrizable space ω_1^ω and are thus are metrizable themselves. $\qquad \square$

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