Disjoint G-Designs and the Intersection Problem for Some Seven Edge Graphs

by

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Abstract

A G-Design of order n is a decomposition of the edges of K_n (the complete graph on n vertices) into subgraphs of K_n isomorphic to G, called blocks, in which each edge of K_n appears in exactly one block. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary block sets corresponding to two G-designs of order n, one might ask under what circumstances can a G-design $\mathcal{B}_2'$ of order n be found such that $\mathcal{B}_2$ and $\mathcal{B}_2'$ are isomorphic and $\mathcal{B}_1$ and $\mathcal{B}_2'$ are disjoint. Luc Teirlinck found that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are K_3-designs of order n, then such a design $\mathcal{B}_2'$ can be found for all $n \geq 7$. In the first part of this dissertation, a similar result is shown for all graphs G with the exception of four small graphs (i.e. some graphs with less than 5 vertices and 3 edges). That is, if $\mathcal{B}_1$ and $\mathcal{B}_2$ is a pair of arbitrary G-designs of order n, then there is a G-design $\mathcal{B}_2'$ such that $\mathcal{B}_2$ and $\mathcal{B}_2'$ are isomorphic and $\mathcal{B}_1$ and $\mathcal{B}_2'$ are disjoint provided n is sufficiently large.

The remainder of the dissertation focuses on the intersection problem for some selected graphs G. The intersection problem is concerned with determining the positive integer pairs n,k for which there are G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ of order n such that $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$. In a landmark result, C.C. Lindner and A. Rosa solved the intersection problem for Steiner Triple Systems (which are K_3-designs). Since then the intersection problem has been solved for various combinatorial designs among which are cycle systems where the cycles have length less than 10, star designs, and various other simple connected graphs with no more than 6 edges. The results presented here solve the intersection problem for 7 bipartite graphs each of which has exactly 7 vertices and 7 edges.

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Preliminaries

1.1 G-Designs

Let us begin by establishing some notation that will be in effect for the remainder of discussion unless stated otherwise. For a graph G with vertex set $V(G)$ and edge set $E(G)$,

$$v_G = |V(G)| \qquad \text{and} \qquad e_G = |E(G)|.$$

Definition 1.1. For a graph G with $v_G = n$, the *automorphism group of G*, denoted $\text{Aut}(G)$, is the subgroup of S_n, the group of permutations on $V(G)$, such that for each edge $\{u, v\} \in E(G)$ and $\sigma \in \text{Aut}(G)$, $\{\sigma(u), \sigma(v)\} \in E(G)$. Let $a_G = |\text{Aut}(G)|$.

For everything else, the notation and definitions of [8] will be used unless specifically stated otherwise. Moreover, all graphs are simple, finite, and nontrivial, that is, they have at least one edge, unless stated otherwise, and n will refer to a positive integer unless stated otherwise.

Definition 1.2. If G and H are graphs, then a *G-design on H* is an ordered pair $(V, \mathcal{B})$ such that V is the vertex set of H and $\mathcal{B}$ is a collection of subgraphs of H, each isomorphic to G, called blocks, where each edge in H is in exactly one block of $\mathcal{B}$. In practice, a G-design on H is named by the set of blocks $\mathcal{B}$; that is, $\mathcal{B}$ is referred to as a G-design on H.

If $\mathcal{B}$ is a G-design on K_n, then $\mathcal{B}$ is called a *G-design of order n*. For future reference, the vertex set of the complete graph on n vertices is $V(K_n) = \{1, 2, \ldots, n\}$ unless stated otherwise.

Observation. Some obvious necessary conditions for the existence of a nontrivial G-design $\mathcal{B}$ on H are

1. $v_G \leq v_H$

2. $e_G | e_H$ since the number of blocks $|\mathcal{B}| = \dfrac{e_H}{e_G}$

3. $d_H \equiv 0 \pmod{d_G}$ where d_G is the greatest common divisor of the degrees in G and d_H is the greatest common divisor of the degrees in H.

Definition 1.3. Letting $V = \{1, \ldots, n\}$ be the vertex set for a graph H, two G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ on H are *isomorphic* if $\mathcal{B}_1 = \sigma \mathcal{B}_2$ for some permutation $\sigma \in \mathrm{Aut}(H)$ with $\sigma \mathcal{B}_2 = \{\sigma b \mid b \in \mathcal{B}_2\}$ where σb is the graph obtained by applying sigma to the vertices in b. If two G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ on H are isomorphic, we denote it

$$\mathcal{B}_1 \cong \mathcal{B}_2.$$

Definition 1.4. Two G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ on a graph H are *disjoint* if $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.

Example 1.1. Figure 1.1 shows disjoint K_3-designs of order 9.

Figure 1.1: Disjoint K_3-designs of Order 9
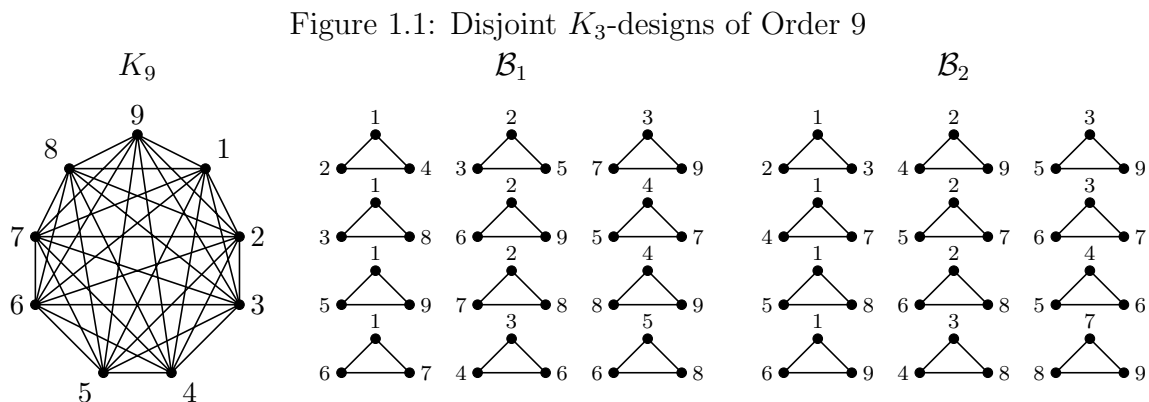
Definition 1.5. The *spectrum of* G, denoted $\mathrm{Spec}(G)$, is the set of all n for which there is a G-design of order n. Note that $1 \in \mathrm{Spec}(G)$ vacuously for any graph G.

Remark. When discussing G-designs, we only consider the positive integers $n \in \mathrm{Spec}(G)$ unless stated otherwise.

The following gives some possible conditions on the graph G:

- $\underline{\mathbf{T}_1}$ – For $n \in \mathrm{Spec}(G)$, there is a pair $\mathcal{B}_1$, $\mathcal{B}_2$ of G- designs of order n with $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.

- $\underline{\mathbf{T}_2}$ – Given a G-design $\mathcal{B}_1$ of order n, there is a G-design $\mathcal{B}_2$ of order n such that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.

- $\underline{\mathbf{T}_3}$ – Given G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ of order n, there is a G- design $\mathcal{B}_2'$, isomorphic to $\mathcal{B}_2$, such that $\mathcal{B}_1 \cap \mathcal{B}_2' = \emptyset$.

For the sake of simplicity, if the statement about G in $\mathbf{T}_i$ ($i \in \{1, 2, 3\}$) holds for a particular integer n, we say G is $\mathbf{T}_i$ for order n. Notice that the conditions ascend in order of strength; that is,

$$G \text{ is } \mathbf{T}_3 \text{ for order } n \ \Rightarrow \ G \text{ is } \mathbf{T}_2 \text{ for order } n \ \Rightarrow \ G \text{ is } \mathbf{T}_1 \text{ for order } n.$$

Condition $\mathbf{T}_3$ is a generalization to G-designs of a problem posed in [11] concerning Steiner triple systems. It was shown in [17] that K_3 is $\mathbf{T}_3$ for all orders $n \geq 7$. Incidentally, the $\mathbf{T}_i$ naming scheme for the conditions on a graph G is for Luc Teirlinck whose work on K_3 inspired further investigation into other graphs. Consequently, graphs that satisfy condition $\mathbf{T}_3$ for all sufficiently large n may also be referred to as *Teirlinck graphs*. Interestingly, each graph G (with the exception of a few small graphs) is $\mathbf{T}_3$ for all sufficiently large n. In order to show this, we need some results from group theory.

1.2 Group Actions, Orbits, and Stabilizers

Much of the following discussion about group actions follows the development and proof techniques of [2] except notation has been changed and some additional results are included.

Although many of the definitions and theorems apply when the group or set is infinite, we assume that all groups and sets are finite unless explicitly stated otherwise.

Definition 1.6. Let $(\Gamma, \bullet)$ be a group and let X be a set. Γ *acts on* X if for each $g \in \Gamma$ and each $x \in X$, there is defined an element $g \cdot x \in X$ such that:

i) For each $x \in X$, $\iota_\Gamma \cdot x = x$, where ι_Γ is the identity element of Γ.

ii) For all $g_1, g_2 \in \Gamma$ and each $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 \bullet g_2) \cdot x$.

If a group acts on a set, then it is called a *group action.*

Remark. When it is clear that two elements of a group Γ are being combined under the group's binary operation, it is customary to omit the symbol for the binary operation. This practice will be adhered to from now on.

Example 1.2. Let X be the set of all subgraphs of K_4 isomorphic to K_3, i.e.

$$X = \left\{ {}_1^2 \underset{3}{\triangle}, {}_1^2 \underset{4}{\triangle}, {}_1^3 \underset{4}{\triangle}, {}_2^3 \underset{4}{\triangle} \right\}$$

The permutation group S_4 acts on X by applying $\sigma \in S_4$ as follows:

$$\sigma \cdot \left({}_a^b \underset{c}{\triangle} \right) = {}_{\sigma(a)}^{\sigma(b)} \underset{\sigma(c)}{\triangle} \text{ for } 1 \le a < b < c \le 4.$$

Remark. If G is a graph such that $v_G \le n$ and X is the set of all subgraphs of K_n isomorphic to G, then the permutation group S_n acts on X by applying the permutations in S_n to the vertices of K_n similarly to Example 1.2.

Observation. Let Γ be a group that acts on a set X. Define the relation $\sim_\Gamma$ on X by

$$x \sim_\Gamma y \text{ if and only if there is some } g \in \Gamma \text{ such that } g \cdot x = y$$

for all $x, y \in X$.

Using the properties of groups and group actions, it is easily shown that $\sim_\Gamma$ is an equivalence relation on X. Thus $\sim_\Gamma$ partitions the elements of X into disjoint equivalence classes.

Definition 1.7. Let Γ be a group that acts on a set X. The equivalence classes of the relation $\sim_\Gamma$ are called *orbits*, and for each $x \in X$, $\mathrm{Orb}(x)$ is the orbit to which x belongs. Alternatively,

$$\mathrm{Orb}(x) = \{g \cdot x \mid g \in \Gamma\}.$$

Definition 1.8. Let Γ be a group that acts on a set X. The group action of Γ on X is called *transitive* if for each pair $x, y \in X$ there is some $g \in \Gamma$ such that $g \cdot x = y$. If the group action of Γ on X is transitive, we say that Γ *acts transitively on* X.

Theorem 1.1. *A group Γ acts transitively on a set X if and only if* $\mathrm{Orb}(x) = X$ *for each* $x \in X$.

Example 1.3. The group action given in Example 1.2 is transitive. In fact, the action of S_n on the set X of all subgraphs of K_n isomorphic to a graph G is transitive.

Definition 1.9. Let Γ be a group that acts on a set X. For each $x \in X$, the *stabilizer* of x is the set of elements of Γ that fix x. We denote the stabilizer of x by $\mathrm{Stab}(x)$; thus,

$$\mathrm{Stab}(x) = \{g \in \Gamma \mid g \cdot x = x\}.$$

Lemma 1.2. *Suppose G is a group that acts on a set X. Then for each $x \in X$, $\mathrm{Stab}(x)$ is a subgroup of G.*

Proof. The proof of this lemma follows easily by using the properties of group actions to check that $\mathrm{Stab}(x)$ satisfies the conditions for a subgroup of G. $\square$

Theorem 1.3 (The Orbit-Stabilizer Theorem). *Let Γ be a group that acts on a set X. Then for each $x \in X$,*

$$|\operatorname{Orb}(x)| = [\Gamma : \operatorname{Stab}(x)] \quad or \quad |\Gamma| = |\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)|.$$

Proof. Let $x \in X$ and $\mathcal{C} = \{g \operatorname{Stab}(x) \mid g \in \Gamma\}$ be the collection of left cosets of $\operatorname{Stab}(x)$ in Γ. Then by definition $|\mathcal{C}| = [\Gamma : \operatorname{Stab}(x)]$. Let us define a function $\varphi : \mathcal{C} \to \operatorname{Orb}(x)$ by

$$\varphi(g \operatorname{Stab}(x)) = g \cdot x \qquad \text{for each } g \in \Gamma.$$

To see that φ is well defined, suppose $g_1, g_2 \in \Gamma$ with $g_1 \operatorname{Stab}(x) = g_2 \operatorname{Stab}(x)$. Then by left cancellation we have $\operatorname{Stab}(x) = g_1^{-1} g_2 \operatorname{Stab}(x)$ which means $g_1^{-1} g_2 \in \operatorname{Stab}(x)$. Thus

$$g_1 \cdot x = g_1 \cdot (g_1^{-1} g_2 \cdot x) = g_1 \cdot [g_1^{-1} \cdot (g_2 \cdot x)] = g_1 g_1^{-1} \cdot (g_2 \cdot x) = \iota_\Gamma \cdot (g_2 \cdot x) = g_2 \cdot x.$$

We now want to show that φ is bijective.

Suppose $g \cdot x \in \operatorname{Orb}(x)$. Then $\varphi(g \operatorname{Stab}(x)) = g \cdot x$; hence, φ is an onto function.

Furthermore, suppose $\varphi(g_1 \operatorname{Stab}(x)) = \varphi(g_2 \operatorname{Stab}(x))$ for some $g_1, g_2 \in \Gamma$. Then $g_1 \cdot x = g_2 \cdot x$, and $g_1^{-1} g_2 x = x$ which means $g_1^{-1} g_2 \in \operatorname{Stab}(x)$ (or $g_1^{-1} g_2 \operatorname{Stab}(x) = \operatorname{Stab}(x)$). Consequently, $g_1 \operatorname{Stab}(x) = g_2 \operatorname{Stab}(x)$ implying that φ is a one-to-one function.

Therefore, $|\mathcal{C}| = |\operatorname{Orb}(x)|$ proving the desired result. Lagrange's Theorem shows that $[\Gamma : \operatorname{Stab}(x)] = \dfrac{|\Gamma|}{|\operatorname{Stab}(x)|}$ which means

$$|\operatorname{Orb}(x)| = \frac{|\Gamma|}{|\operatorname{Stab}(x)|}.$$

$\square$

Lemma 1.4. *Let Γ be a group that acts on a set X, and define $[x \to y] = \{g \in \Gamma | g \cdot x = y\}$ for any pair $x, y \in X$. If $g \in [x \to y]$, then*

$$[x \to y] = g \operatorname{Stab}(x).$$

Proof. Suppose $g \in [x \to y]$ for some pair $x, y \in X$ and $g \operatorname{Stab}(x)$ is the left coset of $\operatorname{Stab}(x)$ in Γ containing g. By definition,

$$h \in [x \to y] \iff h \cdot x = y = g \cdot x \iff (g^{-1}h) \cdot x = x \iff g^{-1}h \in \operatorname{Stab}(x) \iff h \in g \operatorname{Stab}(x).$$

Therefore $[x \to y] = g \operatorname{Stab}(x)$. $\qquad\square$

Definition 1.10. Let Γ be a group that acts on a set X. For each $g \in \Gamma$ and $S \subseteq X$ define

$$gS = \{g \cdot s \mid s \in S\}.$$

Lemma 1.5. *If Γ is a group that acts transitively on a set X, $A \subseteq X$, and $B = \{b\} \subseteq X$, then*
$$\sum_{g \in \Gamma} |A \cap gB| = |A| \cdot |\operatorname{Stab}(b)|.$$

Proof. Suppose $|A| = n$, $A = \{a_1, \ldots, a_n\}$ and for simplicity $A_i = \{a_i\}$ for $1 \le i \le n$. Then for $1 \le i \le n$, there is a $g_i \in \Gamma$ such that $g_i \cdot b = a_i$ (or $g_i B = A_i$), and $[b \to a_i] = g_i \operatorname{Stab}(b)$ by Lemma 1.4. Consequently, for $1 \le i \le n$,

$$|A_i \cap gB| = \begin{cases} 1 & \text{if } g \in g_i \operatorname{Stab}(b) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\sum_{g\in\Gamma}|A_i \cap gB| = |g_i\operatorname{Stab}(b)| = |\operatorname{Stab}(b)|$ for each $i \in \{1,\ldots,n\}$. Moreover, since $[b \to a_i] \cap [b \to a_j] = \emptyset$ for $1 \le i < j \le n$,

$$\sum_{g\in\Gamma}|A \cap gB| = \sum_{g\in\Gamma}\left|\left(\bigcup_{i=1}^{n}A_i\right) \cap gB\right| = \sum_{g\in\Gamma}\left|\bigcup_{i=1}^{n}(A_i \cap gB)\right|$$

$$= \sum_{i=1}^{n}\left(\sum_{g\in\Gamma}|A_i \cap gB|\right) = \sum_{i=1}^{n}|\operatorname{Stab}(b)| = n|\operatorname{Stab}(b)|$$

$$= |A| \cdot |\operatorname{Stab}(b)|.$$

$\square$

Theorem 1.6. *Suppose Γ is a group that acts transitively on a nonempty set X. If $A, B \subseteq X$, then*

$$\frac{1}{|\Gamma|}\sum_{g\in\Gamma}|A \cap gB| = \frac{|A| \cdot |B|}{|X|}.$$

Proof. Suppose $|B| = m$, $B = \{b_1,\ldots,b_m\}$, and for ease of notation $B_i = \{b_i\}$ for each $i \in \{1,\ldots,m\}$. For each $g \in \Gamma$, $(A \cap gB_i) \cap (A \cap gB_j) = \emptyset$ for $1 \le i < j \le m$. Hence

$$\sum_{g\in\Gamma}|A \cap gB| = \sum_{g\in\Gamma}\left|A \cap \left(g\bigcup_{i=1}^{m}B_i\right)\right|$$

$$= \sum_{g\in\Gamma}\left|A \cap \left(\bigcup_{i=1}^{m}gB_i\right)\right|$$

$$= \sum_{g\in\Gamma}\left|\bigcup_{i=1}^{m}(A \cap gB_i)\right|$$

$$= \sum_{i=1}^{m}\left(\sum_{g\in\Gamma}|A \cap gB_i|\right)$$

$$= \sum_{i=1}^{m}(|A| \cdot |\operatorname{Stab}(b_i)|) \qquad \text{by Lemma 1.5}$$

$$= \sum_{i=1}^{m}\left(|A| \cdot \frac{|\Gamma|}{|X|}\right) \qquad \text{by the Orbit-Stabilizer Theorem}$$

$$= m \frac{|A|}{|X|} |\Gamma|$$

$$= \frac{|A| \cdot |B|}{|X|} |\Gamma|.$$

Therefore, $\dfrac{1}{|\Gamma|} \displaystyle\sum_{g \in \Gamma} |A \cap gB| = \dfrac{|A| \cdot |B|}{|X|}.$ $\square$

Corollary 1.7. *Suppose Γ is a group that acts transitively on a nonempty set X. If $A, B \subseteq X$ such that $|A| \cdot |B| < |X|$, then there is some $g \in \Gamma$ such that $A \cap gB = \emptyset$.*

Proof. From Theorem 1.6,

$$\sum_{g \in \Gamma} |A \cap gB| = |\Gamma| \frac{|A| \cdot |B|}{|X|} < |\Gamma|.$$

Hence, for some $g \in \Gamma$, $A \cap gB = \emptyset$; otherwise, the sum on the left side of the inequality is too large. $\square$

Chapter 2

Disjoint G-Designs

From the necessary conditions mentioned previously, a G-design of order $n \geq 2$, say $\mathcal{B}$, exists for a graph G only if

$$v_G \leq n$$

$$|\mathcal{B}| = \frac{e_{K_n}}{e_G} = \frac{\binom{n}{2}}{e_G} = \frac{n(n-1)}{2e_G}.$$

For a graph G, let X_G^n be the set of all subgraphs of K_n that are isomorphic to G.

Lemma 2.1. *For a graph G,*

$$|X_G^n| = \frac{n(n-1)\cdots(n-v_G+1)}{a_G}.$$

Proof. The number of ways of labelling the vertices of G with vertices from $V(K_n)$ is $n(n-1)\cdots(n-v_G+1)$ because there are n choices for the first vertex, $n-1$ choices for the second vertex, and continuing on $n-v_G+1$ choices for the v_Gth vertex. These labellings will represent distinct subgraphs of K_n except when there is a permutation of the vertices of G that gives the same labelling of G; that is, the permutation has the same adjacencies between vertices as the original labelling. The set of these permutations is $\mathrm{Aut}(G)$. Therefore

$$|X_G^n| = \frac{n(n-1)\cdots(n-v_G+1)}{a_G}.$$

$\square$

Theorem 2.2. *Let G be a graph and*

$$G \notin \left\{ \begin{array}{c} \cdot \cdot \end{array}, \begin{array}{c} \cdot \\ \cdot \cdot \end{array}, \begin{array}{c} \cdot \cdot \\ \cdot \cdot \end{array}, < \begin{array}{c} \cdot \\ \cdot \end{array} \right\}$$

Then there is a positive integer N_G such that G is $\mathbf{T}_3$ for $n \geq N_G$.

Proof. For a graph G to have at least one edge and not be in the set of excluded graphs, $v_G \geq 3$. If $v_G = 3$ and G is not in the set of excluded graphs, then G is K_3. This graph was shown to be $\mathbf{T}_3$ for each $n \geq 7$ in [17].

For a pair $\mathcal{B}_1$ and $\mathcal{B}_2$ of G-designs of order n, we have $\mathcal{B}_1, \mathcal{B}_2 \subseteq X_G^n$,

$$|\mathcal{B}_1| = |\mathcal{B}_2| = \frac{n(n-1)}{2e_G}, \qquad \text{and} \qquad \frac{|\mathcal{B}_1| \cdot |\mathcal{B}_2|}{|X_G^n|} = \frac{a_G n^2 (n-1)^2}{4e_G^2 n(n-1)\cdots(n-v_G+1)}.$$

To proceed, we consider two cases for a graph with at least 4 vertices:

(i) Suppose $v_G = 4$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ are G-designs of order n. Then

$$\lim_{n\to\infty} \frac{|\mathcal{B}_1| \cdot |\mathcal{B}_2|}{|X_G^n|} = \lim_{n\to\infty} \frac{a_G n(n-1)}{4e_G^2(n-2)(n-3)} = \frac{a_G}{4e_G^2}.$$

As summarized in Table 2.1, $\dfrac{a_G}{4e_G^2} < 1$ for all 4-vertex graphs except for the one in the set of excluded graphs.

(ii) Suppose $v_G \geq 5$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ are G-designs of order n. Then

$$\lim_{n\to\infty} \frac{|\mathcal{B}_1| \cdot |\mathcal{B}_2|}{|X_G^n|} = \lim_{n\to\infty} \frac{a_G n(n-1)}{4e_G^2(n-2)(n-3)\cdots(n-v_G+1)} = 0.$$

In either case, $\lim\limits_{n\to\infty} \dfrac{|\mathcal{B}_1| \cdot |\mathcal{B}_2|}{|X_G^n|} < 1$ which means there is a positive integer N_G such that for all $n \geq N_G$ and $n \in \mathrm{Spec}(G)$, $|\mathcal{B}_1| \cdot |\mathcal{B}_2| < |X_G^n|$. From Theorem 1.6, there is a $\sigma \in S_n$ such that

$$\mathcal{B}_1 \cap \sigma\mathcal{B}_2 = \emptyset.$$

$\square$

The above proof does not show that the excluded graphs are not $\mathbf{T}_3$ for all orders n such that a design exists. However, there is exactly one way to decompose a graph into its component edges which means K_2 is clearly not $\mathbf{T}_3$ for any order n.

As it turns out, all of the excluded graphs except possibly the path of length 2 are not $\mathbf{T}_3$ for any order n.

Example 2.1. Let $G = \overset{\bullet}{\bullet\!-\!\bullet}$, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the following G-designs of order $n \geq 3$:

$$\mathcal{B}_1 = \left\{ \overset{1}{\underset{i\,\bullet\!-\!\bullet\,j}{\bullet}} \,\middle|\, 2 \leq i < j \leq n \right\} \cup \left\{ \overset{i+1}{\underset{1\,\bullet\!-\!\bullet\,i}{\bullet}} \,\middle|\, 2 \leq i \leq n-1 \right\} \cup \left\{ \overset{2}{\underset{1\,\bullet\!-\!\bullet\,n}{\bullet}} \right\}$$

$$\mathcal{B}_2 = \left\{ \overset{j+1}{\underset{i\,\bullet\!-\!\bullet\,j}{\bullet}} \,\middle|\, 1 \leq i < j \leq n-1 \right\} \cup \left\{ \overset{i+1}{\underset{i\,\bullet\!-\!\bullet\,n}{\bullet}} \,\middle|\, 1 \leq i \leq n-2 \right\} \cup \left\{ \overset{1}{\underset{n-1\,\bullet\!-\!\bullet\,n}{\bullet}} \right\}$$

Notice that in $\mathcal{B}_1$ every edge of K_n not incident on vertex 1 is in a block where 1 is the isolated vertex. In $\mathcal{B}_2$, each vertex of K_n is the isolated vertex in some block of the design. For any $\sigma \in S_n$, there is an $i \in \{1, 2, \ldots, n\}$ such that $\sigma(i) = 1$. By construction there must be some block in $\mathcal{B}_2$ such that i is the isolated vertex along with some edge $\{j, k\}$. Applying σ to that block gives

$$\sigma \cdot \left(\overset{i}{\underset{j\,\bullet\!-\!\bullet\,k}{\bullet}} \right) = \overset{1}{\underset{\sigma(j)\,\bullet\!-\!\bullet\,\sigma(k)}{\bullet}} \in \mathcal{B}_1.$$

Thus $\mathcal{B}_1 \cap \sigma \mathcal{B}_2 \neq \emptyset$ for each $\sigma \in S_n$ which means G is not $\mathbf{T}_3$ for any order n.

Example 2.2. Let $G = \overset{\bullet\;\;\bullet}{\bullet\!-\!\bullet}$, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the following G-designs of order $n \geq 4$:

$$\mathcal{B}_1 = \left\{ \overset{1\;\;2}{\underset{i\;\;\;j}{\bullet\!-\!\bullet}} \,\middle|\, 3 \leq i < j \leq n \right\} \cup \left\{ \overset{2\;\;3}{\underset{1\;\;\;i}{\bullet\!-\!\bullet}}, \overset{1\;\;3}{\underset{2\;\;\;i}{\bullet\!-\!\bullet}} \,\middle|\, 4 \leq i \leq n \right\} \cup \left\{ \overset{3\;\;4}{\underset{1\;\;\;2}{\bullet\!-\!\bullet}}, \overset{2\;\;4}{\underset{1\;\;\;3}{\bullet\!-\!\bullet}}, \overset{1\;\;4}{\underset{2\;\;\;3}{\bullet\!-\!\bullet}} \right\}.$$

$$\mathcal{B}_2 = \left\{ \overset{i+2\;\;i+3}{\underset{i\;\;\;\;i+1}{\bullet\!-\!\bullet}} \,\middle|\, 1 \leq i \leq n \right\} \cup \left\{ \overset{i+1\;\;i+j+1}{\underset{i\;\;\;\;\;i+j}{\bullet\!-\!\bullet}} \,\middle|\, 1 \leq i \leq n, 2 \leq j \leq \left\lfloor \tfrac{n}{2} \right\rfloor \right\}.$$

(The addition in $\mathcal{B}_2$ is done modulo n with n replacing 0.) Each edge of K_n not incident on vertex 1 or 2 is in a block in $\mathcal{B}_1$ in which the isolated vertices are 1 and 2. In $\mathcal{B}_2$, each pair of vertices in K_n is in some block as the isolated vertex pair; that is, for each $i, j \in V(K_n)$ $(i \neq j)$, there is some block

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$$\overset{i\bullet\quad\bullet j}{\underset{k\bullet\!\!-\!\!\bullet\ell}{}}\in \mathcal{B}_2.$$

Suppose $\sigma \in S_n$. Then there is a pair $i, j \in \{1, 2, \ldots, n\}$ such that $\sigma(i) = 1$ and $\sigma(j) = 2$. Applying σ to the block in $\mathcal{B}_2$ that contains i and j as isolated vertices gives

$$\sigma \cdot \left(\overset{i\bullet\quad\bullet j}{\underset{k\bullet\!\!-\!\!\bullet\ell}{}}\right) = \overset{1\bullet\quad\bullet 2}{\underset{\sigma(k)\bullet\!\!-\!\!\bullet\sigma(\ell)}{}} \in \mathcal{B}_1.$$

Hence $\mathcal{B}_1 \cap \sigma\mathcal{B}_2 \neq \emptyset$ for each $\sigma \in S_n$ meaning G is not $\mathbf{T}_3$ for any order n.

As shown in Theorem 2.2, if G is a graph with $v_G \geq 4$, $e_G \geq 2$, and $n \in \mathrm{Spec}(G)$ is sufficiently large, then we can find a pair of disjoint G-designs of order n. Moreover, if G is a graph with $v_G \geq 5$ and $n \in \mathrm{Spec}(G)$ is sufficiently large, then we can find k pairwise disjoint G-designs of order n for any integer $k \geq 2$.

Theorem 2.3. *Let G be a graph with $v_G \geq 5$, and let $k \geq 2$ be an integer. There is a positive integer $N_{G,k}$ such that if $n \geq N_{G,k}$ and $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ are G-designs of order n, then there are G-designs $\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_k$ of order n such that $\mathcal{B}_i \cong \mathcal{B}_i{}'$ for $1 \leq i \leq k$ and*

$$\mathcal{B}'_i \cap \mathcal{B}'_j = \emptyset \qquad \text{for } 1 \leq i < j \leq k.$$

Proof. Suppose G is a graph such that $v_G \geq 5$. The proof proceeds by induction on the number of G-designs k.

(i) Suppose $k = 2$. By Theorem 2.2 there is an integer N_G such that for all $n \geq N_G$, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are G-designs of order n, then for some $\sigma \in \mathrm{Aut}(K_n)$, $\mathcal{B}_1 \cap \sigma\mathcal{B}_2 = \emptyset$. Let $\mathcal{B}'_1 = \mathcal{B}_1$ and $\mathcal{B}'_2 = \sigma\mathcal{B}_2$.

(ii) Suppose for $2 \leq r < k$ that there is an integer $N_{G,r}$ such that for all $n \geq N_{G,r}$ if $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ are G-designs of order n, then there are G-designs $\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_r$ such that $\mathcal{B}'_i \cong \mathcal{B}_i$ for $1 \leq i \leq r$ and $\mathcal{B}'_i \cap \mathcal{B}'_j = \emptyset$ for $1 \leq i < j \leq r$.

(iii) Suppose $k \geq 3$. By the induction hypothesis there is a positive integer $N_{G,k-1}$ such that for all $n \geq N_{G,k-1}$ if $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ are G-designs of order n, then there are G-designs $\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_{k-1}$ such that $\mathcal{B}'_i \cong \mathcal{B}_i$ for $1 \leq i \leq k-1$ and $\mathcal{B}'_i \cap \mathcal{B}'_j = \emptyset$ for $1 \leq i < j \leq k-1$. Since $\mathcal{B}'_1, \mathcal{B}'_2, \ldots, \mathcal{B}'_{k-1}$ are pairwise disjoint G-designs of order n,

$$\left| \bigcup_{i=1}^{k-1} \mathcal{B}'_i \right| = (k-1) \frac{n(n-1)}{2e_G}.$$

Furthermore, $\displaystyle\bigcup_{i=1}^{k-1} \mathcal{B}'_i \subseteq X_G^n$ and $\mathcal{B}_k \subseteq X_G^n$ and

$$\lim_{n \to \infty} \frac{\left| \bigcup_{i=1}^{k-1} \mathcal{B}'_i \right| \cdot |\mathcal{B}_k|}{|X_G^n|} = \lim_{n \to \infty} \frac{a_G(k-1)n(n-1)}{4e_G^2(n-2)(n-3)\cdots(n-v_G+1)} = 0.$$

Thus there is a positive integer $N_{G,k}$ such that $\dfrac{\left| \bigcup_{i=1}^{k-1} \mathcal{B}'_i \right| \cdot |\mathcal{B}_k|}{|X_G^n|} < 1$ for all $n \geq N_{G,k}$ with $n \in \mathrm{Spec}(G)$. By Theorem 1.6, there is a $\sigma \in S_n$ such that $\displaystyle\bigcup_{i=1}^{k-1} \mathcal{B}'_i \cap \sigma \mathcal{B}_k = \emptyset$. Let $\mathcal{B}'_k = \sigma \mathcal{B}_k$ which completes the proof.

$\square$

For a graph G with $v_G = 4$, the limit in the proof of Theorem 2.3 is

$$\lim_{n \to \infty} \frac{a_G(k-1)n(n-1)}{4e_G^2(n-2)(n-3)} = \frac{a_G(k-1)}{4e_G^2}.$$

Provided the limit is less than 1, then the conclusion of the theorem should hold as well. Let

$$K_G = \max\left\{ k \in \mathbb{Z} \ \middle| \ \frac{a_G(k-1)}{4e_G^2} < 1 \right\} = \max\left\{ k \in \mathbb{Z} \ \middle| \ k < \frac{4e_G^2}{a_G} + 1 \right\}.$$

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Then for the 4-vertex graphs, K_G is the maximum number of disjoint G-designs that can be found using the techniques of Theorem 2.3. The values of a_G and K_G for all 4-vertex graphs are summarized in Table 2.1.

Table 2.1: Summary of 4-Vertex Graphs

G	a_G	$\dfrac{a_G}{4e_G^2}$	K_G
	4	1	1
	8	$\dfrac{1}{2}$	2
	2	$\dfrac{1}{8}$	8
	6	$\dfrac{1}{6}$	6
	2	$\dfrac{1}{18}$	18
	6	$\dfrac{1}{6}$	6
	2	$\dfrac{1}{32}$	32
	8	$\dfrac{1}{8}$	8
	4	$\dfrac{1}{25}$	25
	24	$\dfrac{1}{6}$	6

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Chapter 3

The Intersection Problem for 7 Graphs with 7 Vertices and 7 Edges

Throughout this chapter, we assume that all graphs have nontrivial components unless stated otherwise. Before we solve the intersection problem for some select 7 vertex, 7 edge graphs, we first need some definitions. Among these definitions is that of the intersection problem itself.

Definition 3.1. For a graph G, the *intersection problem for* G is the problem of determining all integer pairs n, k for which there exist G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ of order n such that $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$.

For a graph G, we need to know $\mathrm{Spec}(G)$ in order to solve the intersection problem. Then for each $n \in \mathrm{Spec}(G)$ we can define two sets that are closely related to the intersection problem.

Definition 3.2. For a graph G, we define the following two sets. Suppose there exists a G-design on a graph H.

- Let $I_G(H)$ be the set of all k such that there are G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ on H with $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$. Clearly, if $\mathcal{B}$ is a G-design on H, then $|\mathcal{B}| \in I_G(H)$.

- Let $J_G(H)$ be the set of all non-negative integers k such that $ke_G \leq e_H$ except for $k = \dfrac{e_H}{e_G} - 1$.

If $H = K_n$, then we write these sets as $I_G(n)$ and $J_G(n)$ respectively.

The set $I_G(n)$ is the set of all *realizable* k for which there exist G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ of order n such that $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$. Thus the solution to the intersection problem for a graph G

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is to find $I_G(n)$ for each $n \in \mathrm{Spec}(G)$. Since the size of the intersection of two G-designs of order n must be nonnegative and cannot exceed the number of blocks in either design, the set $J_G(n)$ gives us the set of *possible* integers that two G-designs of order n could have in common. We can exclude the integer one less than the number of blocks in the G-design, for if two G-designs have that many blocks in common, the two designs are forced to have the remaining block in common as well. Thus, $I_G(n) \subseteq J_G(n)$ for any graph G and $n \in \mathrm{Spec}(G)$.

The intersection problem was first solved for Steiner Triple Systems (K_3-designs) by C.C. Lindner and A. Rosa in [15]. Since then, the intersection problem has been solved for various other graphs including cycles of length less than 10 [3,12], connected graphs with at most 4 vertices or 4 edges [4,6,9] (except there are a few undetermined values for K_4-designs of order $n = 25, 28, 37$), star graphs with m edges [5], a 4-cycle with a pendant edge [16], and graphs with 6 vertices, 6 edges, and a 4-cycle subgraph [13]. The type of graphs for which we will be solving the intersection problem have at least 7 vertices and 7 edges. But first, several useful definitions and results must be introduced.

Solving the intersection problem involves working with various sets of non-negative integers which necessitates defining some binary operations on these sets.

Definition 3.3. Let A and B be nonempty sets of non-negative integers, and let p be a positive integer. Then

1. $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$

2. $p * A = A_1 + A_2 + \ldots + A_p$ where $A_i = A$ for $1 \leq i \leq p$. There should be no ambiguity in this definition due to the associativity of addition.

3. For convenience, let $0 * A = \{0\}$.

Example 3.1. Let $A = \{0, 1, 3\}$ and $B = \{0, 4, 5\}$. Then $A + B = \{0, 1, 3, 4, 5, 6, 7, 8\}$ and $2 * A = \{0, 1, 2, 3, 4, 6\}$.

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Observation. Let A_1, A_2, B_1, and B_2 be nonempty sets of non-negative integers such that $A_i \subseteq B_i$ for each i, and let p be a positive integer. Then

1. $A_1 + A_2 \subseteq B_1 + B_2$

2. $p * A_i \subseteq p * B_i$ for each $i \in \{1, 2\}$.

The following generalization of G-designs is another useful definition.

Definition 3.4. Let $\mathcal{K}$ be a collection of graphs. A $\mathcal{K}$-*decomposition* of a graph H is an ordered pair $(V, \mathcal{B})$ where V is the vertex set of H and $\mathcal{B}$ is a collection of subgraphs of H called blocks, each of which is isomorphic to some graph in $\mathcal{K}$, such that each edge of H is in exactly one block of $\mathcal{B}$. Notice that if $\mathcal{K} = \{G\}$, then a $\mathcal{K}$-decomposition of H is a G-design on H.

Proposition 3.1. *If a $\mathcal{K}$-decomposition of a graph H exists and there exists a G-design on K for each $K \in \mathcal{K}$, then there exists a G-design on H.*

Proof. Suppose H is a graph with vertex set V and $(V, \mathcal{B})$ is a $\mathcal{K}$-decomposition of H for some collection of graphs $\mathcal{K}$. Furthermore, suppose G is a graph such that for each graph $K \in \mathcal{K}$ there is a G-design on K. Construct a G-design on B with block set Γ_B for each $B \in \mathcal{B}$. Then $\bigcup_{B \in \mathcal{B}} \Gamma_B$ is a G-design on H. $\qquad\square$

Lemma 3.2. *Let n and t be positive integers with $t \geq 2$.*

(a) There exists a $\{K_n, K_{n,n}\}$-decomposition of K_{nt}.

(b) There exists a $\{K_{n+1}, K_{n,n}\}$-decomposition of K_{nt+1}.

*Proof.* (a) There are t subgraphs of K_{nt} induced by the sets $\{ni + r \mid 1 \leq r \leq n\}$ where $0 \leq i \leq t - 1$ that are isomorphic to K_n. Note that the edges of K_{nt} in these induced subgraphs must be of the form $\{u, v\}$ where $u = ni+r$ and $v = ni+s$ where $0 \leq i \leq t - 1$ and $1 \leq r < s \leq n$. There are $\binom{t}{2}$ subgraphs of K_{nt} isomorphic to $K_{n,n}$ formed by taking

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vertices u and v such that $u \in \{ni + r \mid 1 \le r \le n\}$ and $v \in \{nj + s \mid 1 \le s \le n\}$ where $0 \le i < j \le t - 1$ and taking all edges $\{u, v\} \in E(K_{nt})$ such that $u = ni + r$ and $v = nj + s$ with $0 \le i < j \le t - 1$ and $1 \le r, s \le n$. Clearly, none of the edges of K_{nt} in the subgraphs isomorphic to K_n overlap with those in the subgraphs isomorphic to $K_{n,n}$. Counting the number of edges in these subgraphs gives

$$ t\binom{n}{2} + \binom{t}{2}n^2 = \frac{nt(n-1)}{2} + \frac{n^2 t(t-1)}{2} = \frac{nt(nt-1)}{2} = \binom{nt}{2} = e_{K_{nt}}. $$

Thus a $\{K_n, K_{n,n}\}$-decomposition of K_{nt} exists. (For a visual representation of the proof, see Figure 3.1.)

Figure 3.1: $\{K_n, K_{n,n}\}$-Decomposition of K_{nt} with $t \ge 2$
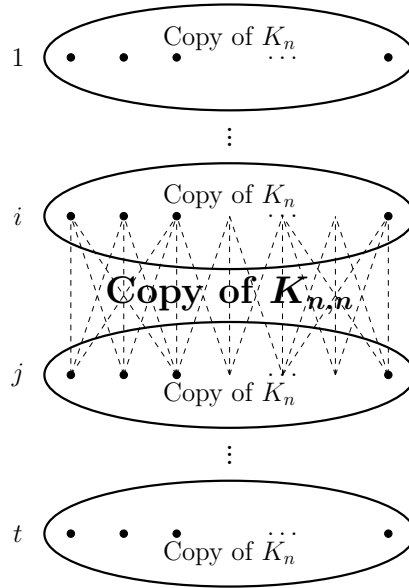
(b) Let $V(K_{nt+1}) = V(K_{nt}) \cup \{\infty\}$. Extend a $\{K_n, K_{n,n}\}$-decomposition of K_{nt} to a $\{K_{n+1}, K_{n,n}\}$-decomposition of K_{nt+1} by adding the vertex ∞ to each of the t copies of K_n in the $\{K_n, K_{n,n}\}$-decomposition of K_{nt} to get t copies of K_{n+1}. Hence a $\{K_{n+1}, K_{n,n}\}$-decomposition of K_{nt+1} with t copies of K_{n+1} and $\binom{t}{2}$ copies of $K_{n,n}$ exists. (For a visual representation of the proof, see Figure 3.2.)

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□

Figure 3.2: $\{K_{n+1}, K_{n,n}\}$-Decomposition of K_{nt+1} with $t \geq 2$
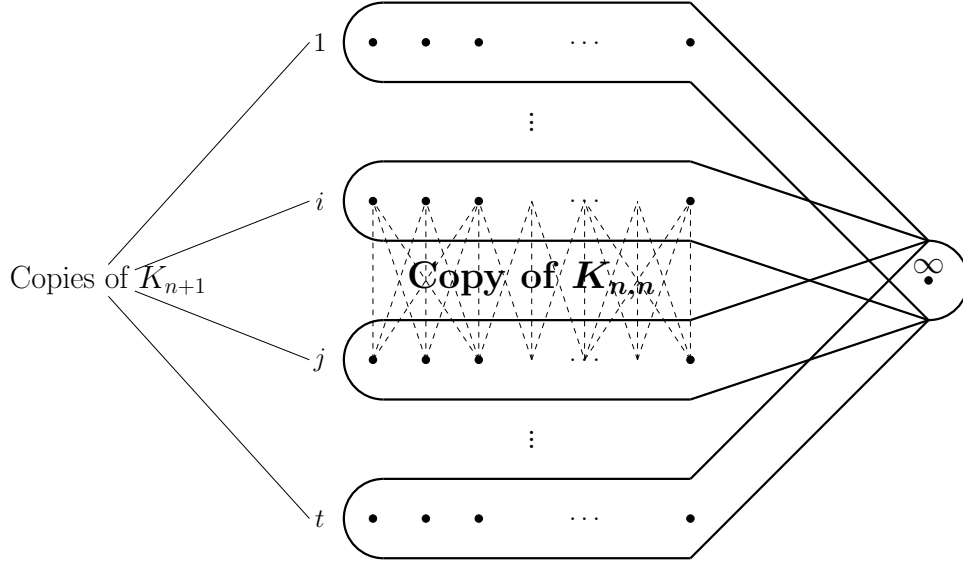
The following lemma will be used frequently for solving the intersection problem for several upcoming graphs.

Lemma 3.3. *If there exists an $\{H_1, H_2, \ldots, H_m\}$-decomposition of K_n with r_i blocks of the decomposition isomorphic to H_i for $1 \leq i \leq m$ and there exists a G-design on H_i for $1 \leq i \leq m$, then*

$$I_G(n) \supseteq r_1 * I_G(H_1) + r_2 * I_G(H_2) + \ldots + r_m * I_G(H_m).$$

Proof. Suppose $k \in r_1 * I_G(H_1) + r_2 * I_G(H_2) + \ldots + r_m * I_G(H_m)$. Then
$k = s_1 + s_2 + \ldots + s_m$ where $s_i \in r_i * I_G(H_i)$ for $1 \leq i \leq m$. Furthermore, each
$s_i = t_{i,1} + t_{i,2} + \ldots + t_{i,r_i}$ where $\{t_{i,1}, t_{i,2}, \ldots, t_{i,r_i}\} \subseteq I_G(H_i)$ for $1 \leq i \leq m$. Thus there exist
r_i pairs of G-designs on the r_i distinct blocks in the decomposition of K_n isomorphic to H_i,

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say $\mathcal{B}_{t_{i,j}}$ and $\mathcal{B}'_{t_{i,j}}$, such that $|\mathcal{B}_{t_{i,j}} \cap \mathcal{B}'_{t_{i,j}}| = t_{i,j}$ for $1 \le j \le r_i$. Let

$$\mathcal{B}_{H_i} = \bigcup_{j=1}^{r_i} \mathcal{B}_{t_{i,j}} \quad \text{and} \quad \mathcal{B}'_{H_i} = \bigcup_{j=1}^{r_i} \mathcal{B}'_{t_{i,j}} \quad \text{for} \quad 1 \le i \le m.$$

Then $|\mathcal{B}_{H_i} \cap \mathcal{B}'_{H_i}| = s_i$ for $1 \le i \le m$. Let

$$\mathcal{B} = \bigcup_{i=1}^{m} \mathcal{B}_{H_i} \quad \text{and} \quad \mathcal{B}' = \bigcup_{i=1}^{m} \mathcal{B}'_{H_i}.$$

Then $|\mathcal{B} \cap \mathcal{B}'| = \sum_{i=1}^{m} s_i = k$. Since $\mathcal{B}$ and $\mathcal{B}'$ are unions of G-designs on H_i for $1 \le i \le m$ which combined appropriately (i.e. taking the union of the blocks in each design) comprise $\{H_1, H_2, \ldots, H_m\}$-decompositions of K_n, they must both be G-designs of order n. Therefore, $k \in I_G(n)$. $\qquad\square$

Finally, a concept that will be used in solving the intersection problems for several upcoming graphs is that of a trade.

Definition 3.5. A pair of G-designs $\mathcal{T}_1$ and $\mathcal{T}_2$ on the same graph H are called *mates* if $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$. If $\mathcal{T}$ is a G-design with at least one mate and $|\mathcal{T}| = t$, then $\mathcal{T}$ is said to be a *trade of volume t*. It should be noted that no trades of volume 1 exist.

As outlined in the following lemma from [5], the intersection problem for a graph G is closely related to finding trades.

Lemma 3.4. *Let $\mathcal{B}_1$ be a G-design of order n. Then there is a G-design $\mathcal{B}_2$ of order n with $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$ if and only if $\mathcal{B}_1$ contains a trade of volume $\binom{n}{2}/e_G - k$.*

Proof. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are G-designs of order n with $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$. Let H be the subgraph of K_n formed by removing each edge found in some block contained in $\mathcal{B}_1 \cap \mathcal{B}_2$ from K_n. Then $\mathcal{T}_1 = \mathcal{B}_1 - (\mathcal{B}_1 \cap \mathcal{B}_2)$ is a G-design on H as is $\mathcal{T}_2 = \mathcal{B}_2 - (\mathcal{B}_1 \cap \mathcal{B}_2)$. Furthermore, $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ meaning $\mathcal{T}_1$ and $\mathcal{T}_2$ are mates with $|\mathcal{T}_1| = |\mathcal{T}_2| = \binom{n}{2}/e_G - k$. Thus $\mathcal{T}_1$ is a trade of volume $\binom{n}{2}/e_G - k$ contained in $\mathcal{B}_1$.

Suppose $\mathcal{B}_1$ is a G-design of order n containing a trade $\mathcal{T}_1$ of volume $\binom{n}{2}/e_G - k$. Then $\mathcal{T}_1$ is a G-design on some subgraph H of K_n with a mate $\mathcal{T}_2$. Consequently, $\mathcal{T}_1$ and $\mathcal{T}_2$ are G-designs on the same subgraph H with $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ and $|\mathcal{T}_1| = |\mathcal{T}_2| = \binom{n}{2}/e_G - k$. Moreover, there must be a G-design $\mathcal{S}$ on $\overline{H}$ (the complement of H in K_n) with $|\mathcal{S}| = k$ since $\mathcal{T}_1$ is contained in a G-design of order n; that is, $\mathcal{B}_1 = \mathcal{T}_1 \cup \mathcal{S}$ with $\mathcal{T}_1 \cap S = \emptyset$. Let $\mathcal{B}_2 = \mathcal{T}_2 \cup \mathcal{S}$. Then $\mathcal{B}_2$ is a G-design of order n since $\mathcal{T}_2$ is a G-design on H and $\mathcal{S}$ is a G-design on $\overline{H}$. Additionally,

$$|\mathcal{B}_1 \cap \mathcal{B}_2| = |(\mathcal{T}_1 \cup \mathcal{S}) \cap (\mathcal{T}_2 \cup \mathcal{S})| = |(\mathcal{T}_1 \cap \mathcal{T}_2) \cup \mathcal{S}| = |\mathcal{S}| = k.$$

Hence there is a G-design $\mathcal{B}_2$ of order n with $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$ \qquad $\square$

For each graph G that will be discussed in the subsequent sections, there exists a G-design of orders n and $n + 1$ as well as a G-design on $K_{n,n}$ for some positive integer n which essentially reduces the intersection problem for these graphs to a few small cases. (See Figures 3.3 and 3.4.)

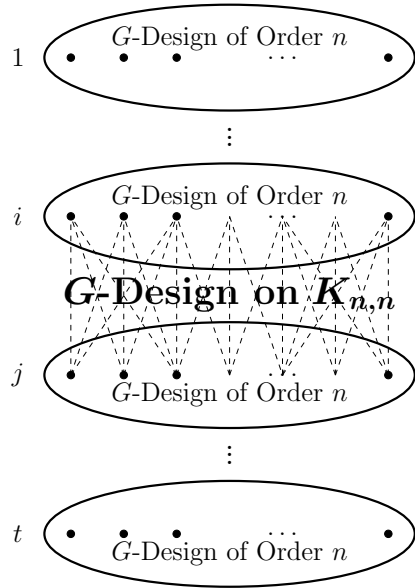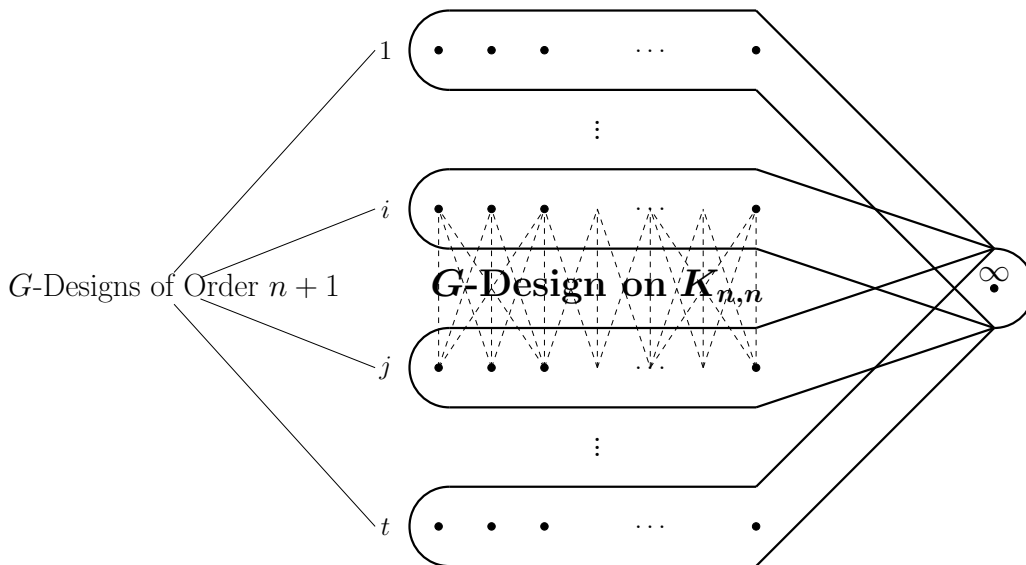Figure 3.3: G-Design of Order nt with $t \geq 2$

Figure 3.4: G-Design of Order $nt + 1$ with $t \geq 2$

G-Designs of Order $n + 1$ **G-Design on $K_{n,n}$**

3.1 A Pair of Dragons with 7 Edges

Definition 3.6. A *dragon* $D_\ell(m)$ $(\ell < m)$ is a graph with m edges consisting of a cycle of length ℓ and an attached path called the tail. (See Figure 3.5 for a picture of the dragon $D_4(7)$.)

Figure 3.5: The Dragon $D_4(7)$
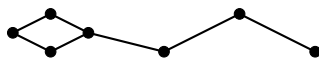
The definition and notation describing dragons are a generalization of that given in [14]. In particular, a dragon $D_3(m)$ containing a triangle with vertices $\{a, b, c\}$ is denoted by $(a, b, c; v_1, v_2, \ldots, v_{m-3})$ where the tail is attached to vertex c. Similarly, a dragon $D_4(m)$ containing a 4-cycle with vertices $\{a, b, c, d\}$ and edges $\{ab, bc, cd, da\}$ is denoted by $(a, b, c, d; v_1, v_2, \ldots, v_{m-4})$ where the tail is attached to vertex d. As for the intersection problem, several of the dragon graphs have been completely solved. $D_3(4)$ (also known as

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a 3-kite or just a kite) is solved in [6]. The dragons $D_4(5)$ (also called a 4-kite) and $D_4(6)$ are solved in [16] and [13] respectively. In solving the intersection problem for $D_4(7)$, the spectrum of $D_4(7)$ must first be found. In order to find the spectrum, we first look at a few small $D_4(7)$-designs.

Example 3.2. A $D_4(7)$-design of order 7 is given by

$$\mathcal{B}_7 = \{(3, 6, 7, 1; 4, 5, 2), (1, 5, 7, 2; 6, 4, 3), (2, 4, 7, 3; 5, 6, 1)\}.$$

See Figure 3.6 for a picture of the blocks as graphs.

Figure 3.6: $\mathcal{B}_7$, a $D_4(7)$-Design of Order 7
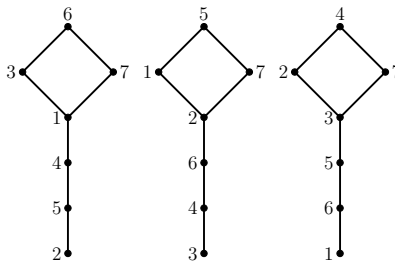
Example 3.3. A $D_4(7)$-design of order 8 is given by

$$\mathcal{B}_8 = \{(4, 8, 6, 1; 7, 3, 5), (5, 7, 8, 2; 3, 4, 6), (1, 5, 8, 3; 6, 2, 7), (5, 6, 7, 4; 2, 1, 8)\}.$$

The blocks can be seen as graphs in Figure 3.7.

Figure 3.7: $\mathcal{B}_8$, a $D_4(7)$-Design of Order 8

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Example 3.4. If $V(K_{7,7}) = \mathbb{Z}_7 \times \{1, 2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$, then
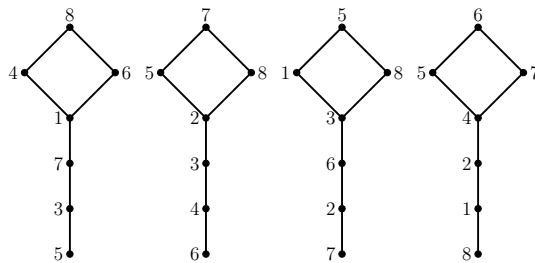
$$\mathcal{B}_{7,7} = \{((i,2),(i+2,1),(i+1,2),(i,1);(i+3,2),(i+1,1),(i+5,2) \mid 0 \le i \le 6\},$$

where addition in the first coordinates is done modulo 7, is a $D_4(7)$-design on $K_{7,7}$. To see that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x,1),(y,2)\}$ in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$, and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in an $D_4(7)$-design on $K_{7,7}$. Also, see Figure 3.8.

Figure 3.8: $\mathcal{B}_{7,7}$, a Cyclic $D_4(7)$-Design on $K_{7,7}$
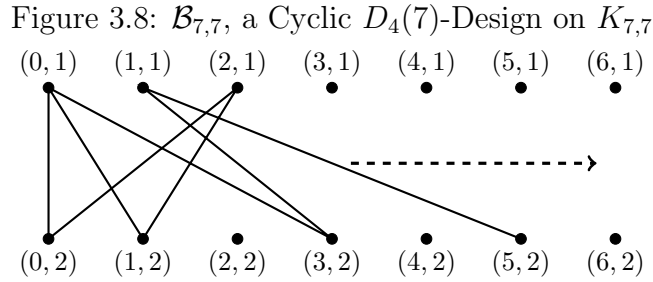
For a view of each of the blocks, see Figure A.1 in the appendix.

Now for the spectrum of $D_4(7)$.

Theorem 3.5. *There exists a $D_4(7)$-design of order n if and only if $n \equiv 0$ or $1 \pmod 7$.*

Proof. The necessity of $n \equiv 0$ or $1 \pmod 7$ is obvious since those are the only orders such that $7 | \binom{n}{2}$. The proof of sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$, a $D_4(7)$-design of order n exists as shown in Example 3.2. Also, a $D_4(7)$-design on $K_{7,7}$ exists as shown in Example 3.4. By Lemma 3.2, a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} exists for each $t \ge 2$; hence, a $D_4(7)$-design of order $7t$ exists for each positive integer t according to Proposition 3.1.

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, a $D_4(7)$-design of order 8 exists as shown in Example 3.3. Once again, a $D_4(7)$-design on $K_{7,7}$ exists as shown in Example 3.4. By Lemma 3.2, a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} exists for each $t \geq 2$; hence, a $D_4(7)$-design of order $7t + 1$ exists for each positive integer t according to Proposition 3.1.

Therefore, a $D_4(7)$-design of order n exists for each $n \equiv 0$ or $1 \pmod 7$. $\qquad \square$

Directly solving the intersection problem for $D_4(7)$ begins with looking at small cases. The techniques that will prove most useful in determining $I_{D_4(7)}(n)$ will be permutation of vertices and finding trades.

Example 3.5. Starting with $\mathcal{B}_7$ from Example 3.2, consider the following three $D_4(7)$-designs of order 7.

$$\mathcal{B}_7 = \{(3, 6, 7, 1; 4, 5, 2), (1, 5, 7, 2; 6, 4, 3), (2, 4, 7, 3; 5, 6, 1)\}$$

Transposing vertices 3 and 4 in $\mathcal{B}_7$ yields the design

$$\mathcal{B}_7^0 = \{(4, 6, 7, 1; 3, 5, 2), (1, 5, 7, 2; 6, 3, 4), (2, 3, 7, 4; 5, 6, 1)\}$$

which is disjoint from $\mathcal{B}_7$.

Transposing vertices 3 and 7 in $\mathcal{B}_7$ yields a design

$$\mathcal{B}_7^1 = \{(3, 6, 7, 1; 4, 5, 2), (1, 5, 3, 2; 6, 4, 7), (2, 4, 3, 7; 5, 6, 1)\}$$

that shares exactly 1 block with $\mathcal{B}_7$. Thus $I_{D_4(7)}(7) = \{0, 1, 3\} = J_{D_4(7)}(7)$.

Example 3.6. Starting with $\mathcal{B}_8$ from Example 3.3, consider the following three $D_4(7)$-designs of order 8.

$$\mathcal{B}_8 = \{(4, 8, 6, 1; 7, 3, 5), (5, 7, 8, 2; 3, 4, 6), (1, 5, 8, 3; 6, 2, 7), (5, 6, 7, 4; 2, 1, 8)\}$$

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Transposing vertices 7 and 8 in $\mathcal{B}_8$ yields the following design that is disjoint from $\mathcal{B}_8$.

$$\mathcal{B}_8^0 = \{(4, 7, 6, 1; 8, 3, 5), (5, 8, 7, 2; 3, 4, 6), (1, 5, 7, 3; 6, 2, 8), (5, 6, 8, 4; 2, 1, 7)\}$$

Transposing vertices 4 and 6 in $\mathcal{B}_8$ yields the following design that has exactly one block in common with $\mathcal{B}_8$.

$$\mathcal{B}_8^1 = \{(4, 8, 6, 1; 7, 3, 5), (5, 7, 8, 2; 3, 6, 4), (1, 5, 8, 3; 4, 2, 7), (5, 4, 7, 6; 2, 1, 8)\}$$

The last two blocks in $\mathcal{B}_8$ form a trade of volume 2 in the design. The trade and a mate are illustrated in the subgraph of K_8 shown in Figure 3.9.

Figure 3.9: A Trade of Volume 2 and a Mate in a $D_4(7)$-Design of Order 8
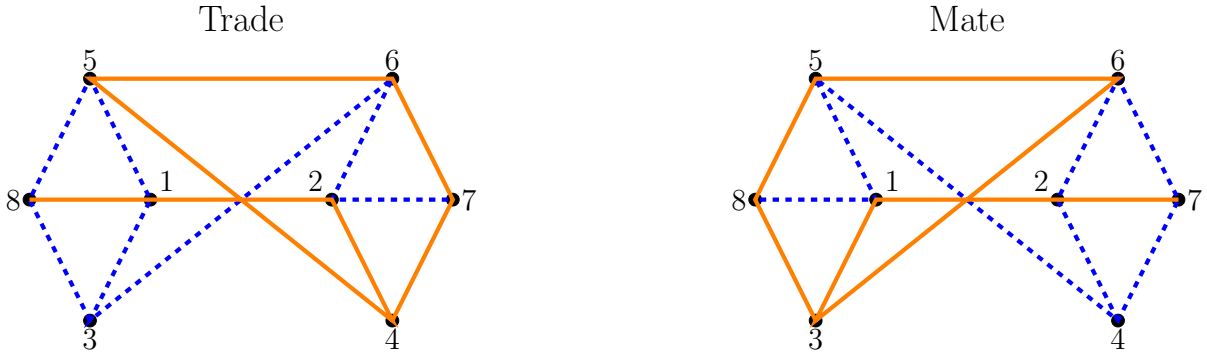
Thus $I_{D_4(7)}(8) = \{0, 1, 2, 4\} = J_{D_4(7)}(8)$.

For the purposes of solving the intersection problem of $D_4(7)$ for orders larger than 8, the existence of some elements of $I_{D_4(7)}(K_{7,7})$ also need to be shown.

Example 3.7. Looking at the $D_4(7)$-design on $K_{7,7}$ of Example 3.4,

$$\mathcal{B}_{7,7} = \{((i, 2), (i + 2, 1), (i + 1, 2), (i, 1); (i + 3, 2), (i + 1, 1), (i + 5, 2) \mid 0 \le i \le 6\},$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{((i,1),(i+2,2),(i+1,1),(i,2);(i+3,1),(i+1,2),(i+5,1) \mid 0 \le i \le 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(0,2)$ and $(1,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^1$ on $K_{7,7}$ that has exactly one block in common with $\mathcal{B}_{7,7}$. (See Example A.1 in the appendix for a list of the blocks of $\mathcal{B}_{7,7}^1$ which indicates the block that $\mathcal{B}_{7,7}$ and $\mathcal{B}_{7,7}^1$ have in common.)

Hence $I_{D_4(7)}(K_{7,7}) \supseteq \{0,1,7\}$.

Example 3.8. From Lemma 3.2 and Theorem 3.5, a $D_4(7)$-design of order 14 can be constructed using 2 $D_4(7)$-designs of order 7, a $D_4(7)$-design on $K_{7,7}$, and a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.5 and 3.7,

$$
\begin{aligned}
I_{D_4(7)}(14) &\supseteq 2 * I_{D_4(7)}(7) + I_{D_4(7)}(K_{7,7}) \\
&\supseteq 2 * \{0,1,3\} + \{0,1,7\} \\
&\supseteq \{0,1,2,3,4,6\} + \{0,1,7\} \\
&\supseteq \{0,1,2,3,4,5,6,7,8,9,10,11,13\} \\
&\supseteq J_{D_4(7)}(14).
\end{aligned}
$$

Hence $I_{D_4(7)}(14) = J_{D_4(7)}(14)$.

Finally the remaining orders $n \in \mathrm{Spec}(D_4(7))$ are solved in the following theorem.

Theorem 3.6. *If $n \in \mathrm{Spec}(D_4(7))$, then*

$$I_{D_4(7)}(n) = J_{D_4(7)}(n).$$

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Proof. As shown in Theorem 3.5, the values of $n \in \text{Spec}(D_4(7))$ are $n \equiv 0$ or $1 \pmod 7$. Hence the proof proceeds by considering two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$ and $n = 14$, $I_{D_4(7)}(n) = J_{D_4(7)}(n)$ as shown in Examples 3.5 and 3.8 respectively. For $n = 7t$ with $t \geq 3$ an integer, it was shown in Theorem 3.5 that a $D_4(7)$-design of order n can be constructed using a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} with t blocks isomorphic to K_7 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$. From Lemma 3.3,

$$
\begin{aligned}
I_{D_4(7)}(7t) &\supseteq t * I_{D_4(7)}(7) + \binom{t}{2} * I_{D_4(7)}(K_{7,7}) \\
&\supseteq t * \{0, 1, 3\} + \binom{t}{2} * \{0, 7\} \\
&\supseteq \{0, 1, 2, \ldots, 3t - 3, 3t - 2, 3t\} + \{7r \mid 0 \leq r \leq \binom{t}{2}\} \\
&\supseteq \{0, 1, 2, \ldots, 7\binom{t}{2} + 3t - 3, 7\binom{t}{2} + 3t - 2, 7\binom{t}{2} + 3t\} \\
&\supseteq \{0, 1, 2, \ldots, 7\binom{t}{2} + 3t - 3, 7\binom{t}{2} + 3t - 2, \tfrac{7t^2 - 7t}{2} + \tfrac{6t}{2}\} \\
&\supseteq \{0, 1, 2, \ldots, 7\binom{t}{2} + 3t - 3, 7\binom{t}{2} + 3t - 2, \tfrac{1}{7}\tfrac{7t(7t-1)}{2}\} \\
&\supseteq \{0, 1, 2, \ldots, \tfrac{1}{7}\binom{7t}{2} - 3, \tfrac{1}{7}\binom{7t}{2} - 2, \tfrac{1}{7}\binom{7t}{2}\} \\
&\supseteq J_{D_4(7)}(7t).
\end{aligned}
$$

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, $I_{D_4(7)}(n) = J_{D_4(7)}(n)$ as shown in Example 3.6. For $n = 7t + 1$ with $t \geq 2$ an integer, it was shown in Theorem 3.5 that a $D_4(7)$-design of order n can be constructed using a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} with t blocks isomorphic to K_8 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$.

Consequently

$$
\begin{aligned}
I_{D_4(7)}(7t + 1) &\supseteq t * I_{D_4(7)}(8) + \binom{t}{2} * I_{D_4(7)}(K_{7,7}) \\
&\supseteq t * \{0, 1, 2, 4\} + \{7r \mid 0 \leq r \leq \binom{t}{2}\} \\
&\supseteq \{0, 1, 2, \ldots, 4t - 3, 4t - 2, 4t\} + \{7r \mid 0 \leq r \leq \binom{t}{2}\}
\end{aligned}
$$

$$\supseteq \left\{0, 1, 2, \ldots, 7\binom{t}{2} + 4t - 3, 7\binom{t}{2} + 4t - 2, 7\binom{t}{2} + 4t\right\}$$

$$\supseteq \left\{0, 1, 2, \ldots, 7\binom{t}{2} + 4t - 3, 7\binom{t}{2} + 4t - 2, \frac{7t^2 - 7t}{2} + \frac{8t}{2}\right\}$$

$$\supseteq \left\{0, 1, 2, \ldots, 7\binom{t}{2} + 4t - 3, 7\binom{t}{2} + 4t - 2, \frac{1}{7}\frac{7t(7t+1)}{2}\right\}$$

$$\supseteq \left\{0, 1, 2, \ldots, \frac{1}{7}\binom{7t+1}{2} - 3, \frac{1}{7}\binom{7t+1}{2} - 2, \frac{1}{7}\binom{7t+1}{2}\right\}$$

$$\supseteq J_{D_4(7)}(7t + 1).$$

Therefore, $I_{D_4(7)}(n) = J_{D_4(7)}(n)$ for each $n \in \mathrm{Spec}(D_4(7))$. $\qquad \square$

Using the similar techniques, the intersection problem for $D_6(7)$ can be solved as well. Also, similar notation will be used when denoting $D_6(7)$ as a block in a design.

Example 3.9. If the graph $D_6(7)$ has vertex set $V(D_6(7)) = \{a, b, c, d, e, f, g\}$, then its edge set is $E(D_6(7)) = \{ab, bc, cd, de, ef, af, fg\}$. For brevity, this graph will be denoted by the vector $(a, b, c, d, e, f; g)$ henceforth. Figure 3.10 gives an illustration of this graph.

Figure 3.10: The Dragon $D_6(7)$
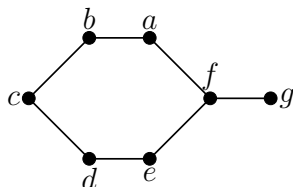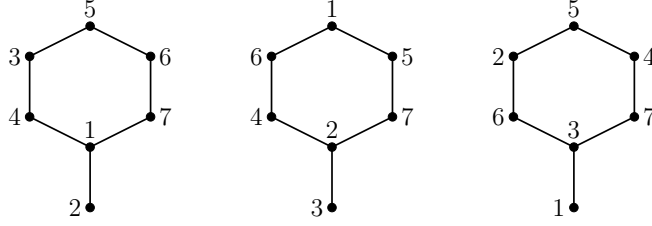

Once again, the spectrum of the graph needs to be found before solving the intersection problem which will be done through several examples and Lemma 3.2.

Example 3.10. The following set forms a $D_6(7)$-design of order 7:

$$\mathcal{B}_7 = \{(4, 3, 5, 6, 7, 1; 2), (4, 6, 1, 5, 7, 2; 3), (6, 2, 5, 4, 7, 3; 1)\}.$$

See Figure 3.11 to see how the blocks look as graphs.

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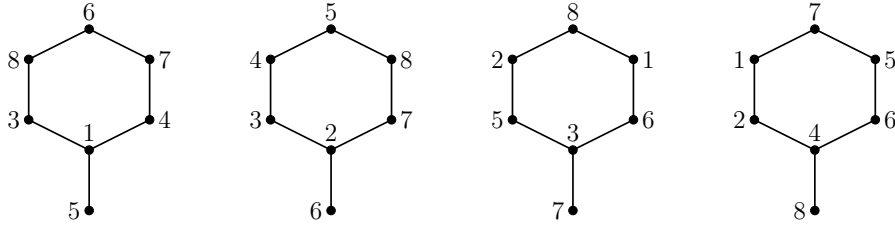Figure 3.11: $\mathcal{B}_7$, a $D_6(7)$-Design of Order 7

Example 3.11. The following set forms a $D_6(7)$-design of order 8:

$$\mathcal{B}_8 = \{(3, 8, 6, 7, 4, 1; 5), (3, 4, 5, 8, 7, 2; 6), (5, 2, 8, 1, 6, 3; 7), (2, 1, 7, 5, 6, 4; 8)\}.$$

To see the blocks as graphs, view Figure 3.12.

Figure 3.12: $\mathcal{B}_8$, a $D_6(7)$-Design of Order 8

Example 3.12. Let $V(K_{7,7}) = \mathbb{Z}_7 \times \{1, 2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$. Then

$$\mathcal{B}_{7,7} = \{((i, 2), (i + 4, 1), (i + 6, 2), (i + 2, 1), (i + 1, 2), (i, 1); (i + 5, 2)) \mid 0 \leq i \leq 6\}$$

where addition in the first coordinates is done modulo 7 is a $D_6(7)$-design on $K_{7,7}$. To see that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x, 1), (y, 2)\}$ in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$, and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in a $D_6(7)$-design on $K_{7,7}$. Also, see Figure 3.13.

Figure 3.13: $\mathcal{B}_{7,7}$, a Cyclic $D_6(7)$-Design on $K_{7,7}$
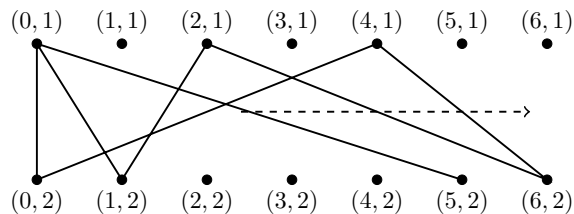
For an illustration of the individual blocks of $\mathcal{B}_{7,7}$, see Figure A.3 in the appendix.

The spectrum of $D_6(7)$ is outlined in the next theorem.

Theorem 3.7. *There exists a $D_6(7)$-design of order n if and only if $n \equiv 0$ or $1 \pmod{7}$.*

Proof. The necessity of the orders $n \equiv 0$ or $1 \pmod{7}$ is clear given that these are the only orders such that $7 \,|\, \binom{n}{2}$. Showing their sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0 \pmod{7}$. For $n = 7$, a $D_6(7)$-design of order n has been exhibited in Example 3.10. For $n = 7t$ $(t \geq 2)$, a $D_6(7)$-design of order n exists according to Proposition 3.1 and Lemma 3.2 because there exists both a $D_6(7)$-design of order 7 and a $D_6(7)$-design on $K_{7,7}$ (Example 3.12). Hence there is a $D_6(7)$-design of order $7t$ for each positive integer t.

(ii) Suppose $n \equiv 1 \pmod{7}$. For $n = 8$, a $D_6(7)$-design of order n has been exhibited in Example 3.11. For $n = 7t + 1$ $(t \geq 2)$, a $D_6(7)$-design of order n exists according to Proposition 3.1 and Lemma 3.2 because there exists both a $D_6(7)$-design of order 8 and a $D_6(7)$-design on $K_{7,7}$. Thus there is an $D_6(7)$-design of order $7t + 1$ for each positive integer t.

Therefore a $D_6(7)$-design of order n exists if and only if $n \equiv 0$ or $1 \pmod{7}$. $\qquad \square$

Similarly to $D_4(7)$, solving the intersection problem for $D_6(7)$ proceeds by considering a few small cases.

Example 3.13. Starting with the $D_6(7)$-design of order 7 given in Example 3.10, permutation of select vertices will yield a disjoint design and a design that shares one block in common with the original design. As a reminder, the original design is

$$\mathcal{B}_7 = \{(4, 3, 5, 6, 7, 1; 2), (4, 6, 1, 5, 7, 2; 3), (6, 2, 5, 4, 7, 3; 1)\}.$$

Transposing vertices 3 and 4 gives the following $D_6(7)$-design of order 7 which is disjoint from the original design.

$$\mathcal{B}_7^0 = \{(3, 4, 5, 6, 7, 1; 2), (3, 6, 1, 5, 7, 2; 4), (6, 2, 5, 3, 7, 4; 1)\}.$$

Transposing vertices 3 and 6 and transposing vertices 4 and 7 gives a design shown below that shares exactly one block in common with $\mathcal{B}_7$.

$$\mathcal{B}_7 = \{(4, 3, 5, 6, 7, 1; 2), (7, 3, 1, 5, 4, 2; 6), (3, 2, 5, 7, 4, 6; 1)\}.$$

Consequently $I_{D_6(7)}(7) = J_{D_6(7)}(7)$.

Example 3.14. For the $D_6(7)$-design of order 8 of Example 3.11

$$\mathcal{B}_8 = \{(3, 8, 6, 7, 4, 1; 5), (3, 4, 5, 8, 7, 2; 6), (5, 2, 8, 1, 6, 3; 7), (2, 1, 7, 5, 6, 4; 8)\},$$

a disjoint design can be found by transposing vertices 5 and 7 as shown in the design below.

$$\mathcal{B}_8^0 = \{(3, 8, 6, 5, 4, 1; 7), (3, 4, 7, 8, 5, 2; 6), (7, 2, 8, 1, 6, 3; 5), (2, 1, 5, 7, 6, 4; 8)\}.$$

To obtain a design that shares exactly one block with $\mathcal{B}_8$, transpose vertices 3 and 4, and transpose vertices 7 and 8 as seen in the design below.

$$\mathcal{B}_8^1 = \{(3, 8, 6, 7, 4, 1; 5), (4, 3, 5, 7, 8, 2; 6), (2, 1, 8, 5, 6, 3; 7), (5, 2, 7, 1, 6, 4; 8)\}$$

Notice that the last two blocks of $\mathcal{B}_8$ form a trade of volume 2 in the design as shown in Figure 3.14.

Figure 3.14: A Trade of Volume 2 and a Mate in a $D_6(7)$-Design of Order 8
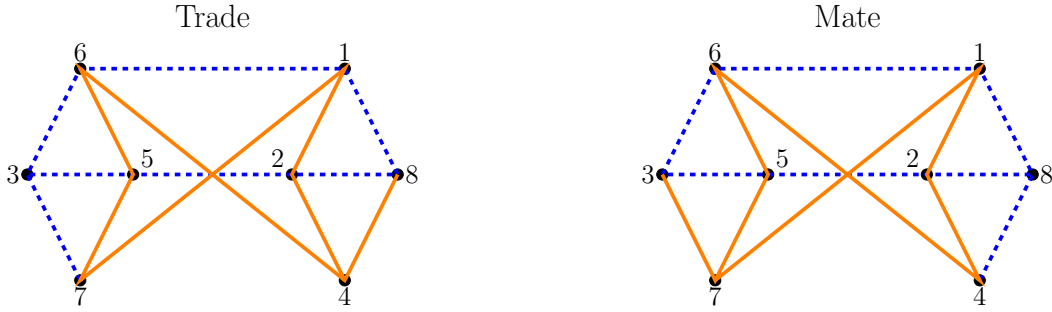
Thus $I_{D_6(7)}(8) = J_{D_6(7)}(8)$.

To solve the intersection problem for larger orders in $\mathrm{Spec}(D_6(7))$, the existence of some of the elements in $I_{D_6(7)}(K_{7,7})$ needs to be shown.

Example 3.15. Looking at the $D_6(7)$-design on $K_{7,7}$ of Example 3.12,

$$\mathcal{B}_{7,7} = \{((i,2),(i+4,1),(i+6,2),(i+2,1),(i+1,2),(i,1);(i+5,2)) \mid 0 \le i \le 6\},$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{((i,1),(i+4,2),(i+6,1),(i+2,2),(i+1,1),(i,2);(i+5,1)) \mid 0 \le i \le 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(2,2)$ and $(3,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. (See Example A.2.)

Hence $I_{D_6(7)}(K_{7,7}) \supseteq \{0,2,7\}$.

The intersection problem for order 14 is more easily handled separately from the other higher orders. The solution is shown in the next example.

Example 3.16. From Lemma 3.2 and Theorem 3.7, a $D_6(7)$-design of order 14 can be constructed by combining two $D_6(7)$-designs of order 7 and one $D_6(7)$-design on $K_{7,7}$ in a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.13 and 3.15,

$$I_{D_6(7)}(14) \supseteq 2 * I_{D_6(7)}(7) + I_{D_6(7)}(K_{7,7})$$

$$\supseteq 2 * \{0, 1, 3\} + \{0, 2, 7\}$$

$$\supseteq \{0, 1, 2, 3, 4, 6\} + \{0, 2, 7\}$$

$$\supseteq \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$$

$$\supseteq J_{D_6(7)}(14).$$

Thus $I_{D_6(7)}(14) = J_{D_6(7)}(14)$.

The solution to the intersection problem for $D_6(7)$ is completed in the following theorem.

Theorem 3.8. If $n \in \operatorname{Spec}(D_6(7))$, then

$$I_{D_6(7)}(n) = J_{D_6(7)}(n).$$

Proof. For $n = 7$, $n = 8$, and $n = 14$, $I_{D_6(7)}(n) = J_{D_6(7)}(n)$ as shown in Examples 3.13, 3.14, and 3.16 respectively. For the remaining orders in $\operatorname{Spec}(D_6(7))$, two cases are considered.

(i) Suppose $n \equiv 0 \pmod 7$ and $n \geq 21$; that is, $n = 7t$ for some integer $t \geq 3$. From Theorem 3.7, a $D_6(7)$-design of order $7t$ ($t \geq 3$) can be constructed containing t $D_6(7)$-designs of order 7 and $\binom{t}{2}$ $D_6(7)$-designs on $K_{7,7}$. According to Lemma 3.3, if $t \geq 3$,

$$I_{D_6(7)}(7t) \supseteq t * I_{D_6(7)}(7) + \binom{t}{2} * I_{D_6(7)}(K_{7,7})$$

$$\supseteq t * \{0, 1, 3\} + \binom{t}{2} * \{0, 7\}$$

$$\supseteq \left\{0, 1, 2, \ldots, \tfrac{1}{7}\tbinom{7t}{2} - 3, \tfrac{1}{7}\tbinom{7t}{2} - 2, \tfrac{1}{7}\tbinom{7t}{2}\right\} \qquad \text{(See proof of Theorem 3.6.)}$$

$$\supseteq J_{D_6(7)}(7t).$$

(ii) Suppose $n \equiv 1 \pmod 7$ and $n \geq 15$; that is, $n = 7t + 1$ for some integer $t \geq 2$. From Theorem 3.7, a $D_6(7)$)-design of order $7t + 1$ ($t \geq 2$) can be constructed containing t $D_6(7)$-designs of order 8 and $\binom{t}{2}$ $D_6(7)$-designs on $K_{7,7}$. By Lemma 3.3, if $t \geq 2$,

$$I_{D_6(7)}(7t + 1) \supseteq t * I_{D_6(7)}(8) + \tbinom{t}{2} * I_{D_6(7)}(K_{7,7})$$

$$\supseteq t * \{0, 1, 2, 4\} + \left\{7r \mid 0 \leq r \leq \tbinom{t}{2}\right\}$$

$$\supseteq \left\{0, 1, 2, \ldots, \tfrac{1}{7}\tbinom{7t+1}{2} - 3, \tfrac{1}{7}\tbinom{7t+1}{2} - 2, \tfrac{1}{7}\tbinom{7t+1}{2}\right\} \quad \text{(See Theorem 3.6.)}$$

$$\supseteq J_{D_6(7)}(7t + 1).$$

Therefore $I_{D_6(7)}(n) = J_{D_6(7)}(n)$ for each $n \in \text{Spec}(D_6(7))$. $\qquad \square$

3.2 A Graph Containing a 4-Cycle, 2 Pendant Edges on 1 Vertex in the Cycle, and a Single Pendant Edge on an Adjacent Vertex in the Cycle

Since there is no well known convention for naming the graph of interest, one is given in the following definition.

Definition 3.7. The graph $R_k(\ell, m)$ ($\ell \leq m$) is a graph containing a cycle of length k, $\ell > 0$ pendant edges on one vertex in the k-cycle, and $m > 0$ pendant edges on an adjacent vertex in the k-cycle.

Example 3.17. The graph $R_4(1, 2)$ has vertex set $V(R_4(1, 2)) = \{a, b, c, d, e, f, g\}$ and edge set $E(R_4(1, 2)) = \{ab, bc, cd, de, be, ef, eg\}$. For convenience, this graph will be denoted by the vector $R(a; b, c, d, e; f, g)$ from now on. This graph is illustrated in Figure 3.15.

Figure 3.15: The Graph $R_4(1, 2)$
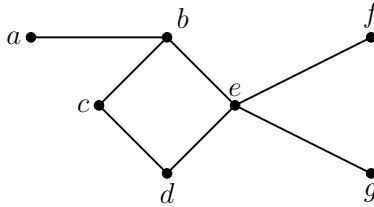
The intersection problem for $R_4(1, 1)$ is solved in [13]. In the following discussion, the intersection problem for $R_4(1, 2)$ is solved, but first the spectrum of $R_4(1, 2)$ must be established. As usual, some small designs are given first.

Example 3.18. The following block set is an $R_4(1, 2)$-design of order 7.

$$\mathcal{B}_7 = \{R(3; 4, 5, 7, 1; 2, 6), R(1; 5, 6, 7, 2; 3, 4), R(2; 6, 4, 7, 3; 1, 5)\}$$

The blocks are illustrated as graphs in Figure 3.16.

Figure 3.16: $\mathcal{B}_7$, an $R_4(1, 2)$-Design of Order 7
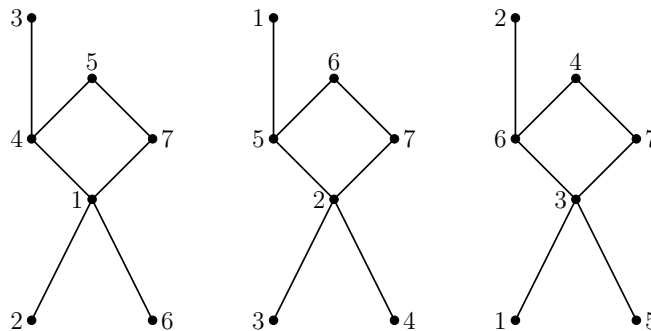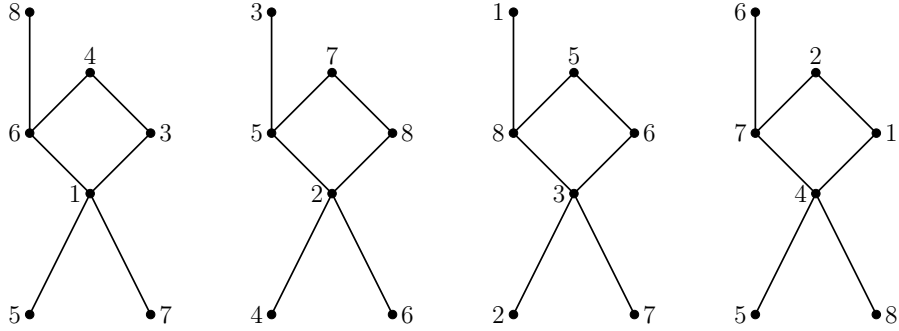

Example 3.19. The following block set is an $R_4(1, 2)$-design of order 8.

$$\mathcal{B}_8 = \{R(8; 6, 4, 3, 1; 5, 7), R(3; 5, 7, 8, 2; 4, 6), R(1; 8, 5, 6, 3; 2, 7), R(6; 7, 2, 1, 4; 5, 8)\}$$

The blocks are shown as graphs in Figure 3.17.

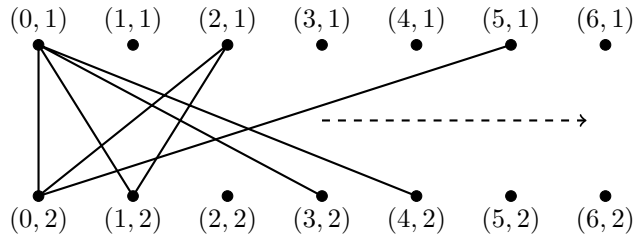Figure 3.17: $\mathcal{B}_8$, an $R_4(1,2)$-Design of Order 8

Example 3.20. Let $V(K_{7,7}) = \mathbb{Z}_7 \times \{1,2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$. Then the following set is an $R_4(1,2)$-design on $K_{7,7}$.

$$\mathcal{B}_{7,7} = \{R((i+5,1);(i,2),(i+2,1),(i+1,2),(i,1);(i+3,2),(i+4,2)) \mid 0 \le i \le 6\}$$

The addition in the first coordinates is done modulo 7. To see that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x,1),(y,2)\}$ in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$, and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in an $R_4(1,2)$-design on $K_{7,7}$. Also, see Figure 3.18.

Figure 3.18: $\mathcal{B}_{7,7}$, a Cyclic $R_4(1,2)$-Design on $K_{7,7}$

Look at Figure A.5 in the appendix to see each of the blocks in $\mathcal{B}_{7,7}$.

Theorem 3.9. *There exists an $R_4(1,2)$-design of order n if and only if $n \equiv 0$ or $1 \pmod 7$.*

Proof. The necessity of $n \equiv 0$ or $1 \pmod 7$ is obvious since those are the only orders such that $7 | \binom{n}{2}$. The proof of sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$, an $R_4(1,2)$-design of order n exists as shown in Example 3.18. Also, an $R_4(1,2)$-design on $K_{7,7}$ exists as shown in Example 3.20. By Lemma 3.2, a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} exists for each $t \geq 2$; consequently, an $R_4(1,2)$-design of order $7t$ exists for each positive integer t according to Proposition 3.1.

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, an $R_4(1,2)$-design of order 8 exists as shown in Example 3.19. Once again, an $R_4(1,2)$-design on $K_{7,7}$ exists as shown in Example 3.20. By Lemma 3.2, a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} exists for each $t \geq 2$; thus, an $R_4(1,2)$-design of order $7t + 1$ exists for each positive integer t according to Proposition 3.1.

Therefore, an $R_4(1,2)$-design of order n exists for each $n \equiv 0$ or $1 \pmod 7$. $\qquad \square$

With the spectrum of $R_4(1,2)$ established, the intersection problem is now solved by considering a few small designs.

Example 3.21. Starting with $\mathcal{B}_7$ from Example 3.18, consider the following three $R_4(1,2)$-designs of order 7.

$$\mathcal{B}_7 = \{R(3; 4, 5, 7, 1; 2, 6), R(1; 5, 6, 7, 2; 3, 4), R(2; 6, 4, 7, 3; 1, 5)\}$$

Transposing vertices 6 and 7 in $\mathcal{B}_7$ yields the design

$$\mathcal{B}_7^0 = \{R(3; 4, 5, 6, 1; 2, 7), R(1; 5, 7, 6, 2; 3, 4), R(2; 7, 4, 6, 3; 1, 5)\}$$

which is disjoint from $\mathcal{B}_7$.

Transposing vertices 2 and 6 in $\mathcal{B}_7$ yields a design

$$\mathcal{B}_7^1 = \{R(3; 4, 5, 7, 1; 2, 6), R(1; 5, 2, 7, 6; 3, 4), R(6; 2, 4, 7, 3; 1, 5)\}$$

that shares exactly 1 block with $\mathcal{B}_7$. Thus $I_{R_4(1,2)}(7) = J_{R_4(1,2)}(7)$.

Example 3.22. Starting with $\mathcal{B}_8$ from Example 3.19, consider the following three $R_4(1,2)$-designs of order 8.

$$\mathcal{B}_8 = \{R(8; 6, 4, 3, 1; 5, 7), R(3; 5, 7, 8, 2; 4, 6), R(1; 8, 5, 6, 3; 2, 7), R(6; 7, 2, 1, 4; 5, 8)\}$$

Transposing vertices 7 and 8 in $\mathcal{B}_8$ yields the following design that is disjoint from $\mathcal{B}_8$.

$$\mathcal{B}_8^0 = \{R(7; 6, 4, 3, 1; 5, 8), R(3; 5, 8, 7, 2; 4, 6), R(1; 7, 5, 6, 3; 2, 8), R(6; 8, 2, 1, 4; 5, 7)\}$$

Transposing vertices 5 and 7 in $\mathcal{B}_8$ yields the following design that has exactly one block in common with $\mathcal{B}_8$.

$$\mathcal{B}_8^1 = \{R(8; 6, 4, 3, 1; 5, 7), R(3; 7, 5, 8, 2; 4, 6), R(1; 8, 7, 6, 3; 2, 5), R(6; 5, 2, 1, 4; 7, 8)\}$$

The last two blocks in $\mathcal{B}_8$ form a trade of volume 2 in the design. The trade and a mate are illustrated in the subgraph of K_8 shown in Figure 3.19.

Figure 3.19: A Trade of Volume 2 and a Mate in an $R_4(1,2)$-Design of Order 8
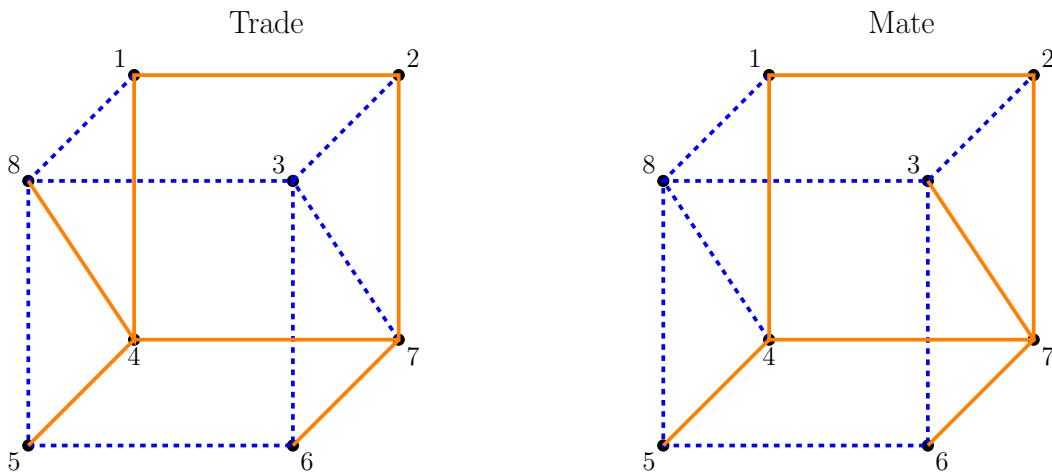
Thus $I_{R_4(1,2)}(8) = J_{R_4(1,2)}(8)$.

As with the previous graphs, some elements of $I_{R_4(1,2)}(K_{7,7})$ must be shown in order to solve the intersection problem for $n \geq 14$ in the spectrum of $R_4(1,2)$.

Example 3.23. Looking at the $R_4(1,2)$-design on $K_{7,7}$ of Example 3.20,

$$\mathcal{B}_{7,7} = \{R((i+1,2),(i+2,1),(i,2),(i,1);(i+3,2),(i+4,2),(i+5,1)) \mid 0 \leq i \leq 6\}$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{R((i+1,1),(i+2,2),(i,1),(i,2);(i+3,1),(i+4,1),(i+5,2)) \mid 0 \leq i \leq 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(3,2)$ and $(4,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. For more details, see Example A.3 in the appendix.

Hence $I_{R_4(1,2)}(K_{7,7}) \supseteq \{0,2,7\}$.

Once again, the solution for order 14 is more easily handled separately from the higher orders in the spectrum of $R_4(1,2)$.

Example 3.24. From Lemma 3.2 and Theorem 3.9, an $R_4(1,2)$-design of order 14 can be constructed using 2 $R_4(1,2)$-designs of order 7, an $R_4(1,2)$-design on $K_{7,7}$, and a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.21 and 3.23,

$$\begin{aligned}
I_{R_4(1,2)}(14) &\supseteq 2 * I_{R_4(1,2)}(7) + I_{R_4(1,2)}(K_{7,7}) \\
&\supseteq 2 * \{0,1,3\} + \{0,2,7\} \\
&\supseteq \{0,1,2,3,4,6\} + \{0,2,7\} \\
&\supseteq \{0,1,2,3,4,5,6,7,8,9,10,11,13\} \\
&\supseteq J_{R_4(1,2)}(14).
\end{aligned}$$

Hence $I_{R_4(1,2)}(14) = J_{R_4(1,2)}(14)$.

The intersection problem for the remaining orders $n \in \text{Spec}(R_4(1,2))$ are solved in the following theorem.

Theorem 3.10. *If $n \in \text{Spec}(R_4(1,2))$, then*

$$I_{R_4(1,2)}(n) = J_{R_4(1,2)}(n).$$

Proof. As shown in Theorem 3.9, the values of $n \in \text{Spec}(R_4(1,2))$ are $n \equiv 0$ or $1 \pmod 7$. Thus the proof proceeds by considering two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$ and $n = 14$, $I_{R_4(1,2)}(n) = J_{R_4(1,2)}(n)$ as shown in Examples 3.21 and 3.24 respectively. For $n = 7t$ with $t \geq 3$ an integer, it was shown in Theorem 3.9 that a $R_4(1,2)$-design of order n can be constructed using a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} with t blocks isomorphic to K_7 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$. From Lemma 3.3,

$$
\begin{aligned}
I_{R_4(1,2)}(7t) &\supseteq t * I_{R_4(1,2)}(7) + \binom{t}{2} * I_{R_4(1,2)}(K_{7,7}) \\
&\supseteq t * \{0,1,3\} + \binom{t}{2} * \{0,7\} \\
&\supseteq \left\{0,1,2,\ldots,\tfrac{1}{7}\binom{7t}{2} - 3, \tfrac{1}{7}\binom{7t}{2} - 2, \tfrac{1}{7}\binom{7t}{2}\right\} \qquad \text{(See proof of Theorem 3.6.)} \\
&\supseteq J_{R_4(1,2)}(7t).
\end{aligned}
$$

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, $I_{R_4(1,2)}(n) = J_{R_4(1,2)}(n)$ as shown in Example 3.22. For $n = 7t+1$ with $t \geq 2$ an integer, it was shown in Theorem 3.9 that a $R_4(1,2)$-design of order n can be constructed using a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} with t blocks isomorphic to K_8 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$.

Consequently

$$I_{R_4(1,2)}(7t+1) \supseteq t * I_{R_4(1,2)}(8) + \binom{t}{2} * I_{R_4(1,2)}(K_{7,7})$$

$$\supseteq t * \{0, 1, 2, 4\} + \left\{7r \mid 0 \leq r \leq \binom{t}{2}\right\}$$

$$\supseteq \left\{0, 1, 2, \ldots, \tfrac{1}{7}\binom{7t+1}{2} - 3, \tfrac{1}{7}\binom{7t+1}{2} - 2, \tfrac{1}{7}\binom{7t+1}{2}\right\} \quad \text{(See Theorem 3.6.)}$$

$$\supseteq J_{SE_4(1,2)}(7t+1).$$

Therefore, $I_{R_4(1,2)}(n) = J_{R_4(1,2)}(n)$ for each $n \in \text{Spec}(R_4(1,2))$. $\qquad\square$

3.3 The "Starship Enterprise" Graph Containing 7 Vertices and a 4-Cycle

Once again, the graph under discussion does not appear to have a set notation; thus, notation will be given in the following definition.

Definition 3.8. Let $SE_k(\ell, m)^1$ ($\ell \leq m$) be the graph with a cycle of length k and two edge-disjoint paths of length $\ell > 0$ and $m > 0$ incident on one vertex in the cycle.

Example 3.25. The graph $SE_4(1,2)$ has vertex set $V(SE_4(1,2)) = \{a, b, c, d, e, f, g\}$ and edge set $E(SE_4(1,2)) = \{ab, bc, cd, ad, de, df, fg\}$. For brevity, this graph will be denoted by the vector $SE(a, b, c, d; e, f, g)$ henceforth. This graph can be seen in Figure 3.20.

Figure 3.20: The "Starship Enterprise" Graph $SE_4(1,2)$
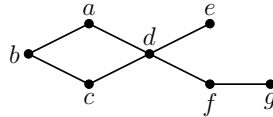
The intersection problem for the graph $SE_4(1,1)$ is solved in [13], and the following discussion solves the intersection problem for $SE_4(1,2)$. First the spectrum needs to be shown for this graph which will be done through several examples and Lemma 3.2.

---

[1]By appropriately orienting the vertices, these graphs look like a top-down view of the Starship Enterprise from the various *Star Trek* films and television series. Hence the initialism "*SE*" is used.

Example 3.26. The following set forms an $SE_4(1, 2)$-design of order 7:

$$\mathcal{B}_7 = \{SE(6, 5, 7, 1; 2, 4, 3), SE(4, 6, 7, 2; 3, 5, 1), SE(5, 4, 7, 3; 1, 6, 2)\}.$$

As graphs, the blocks are shown in Figure 3.21.

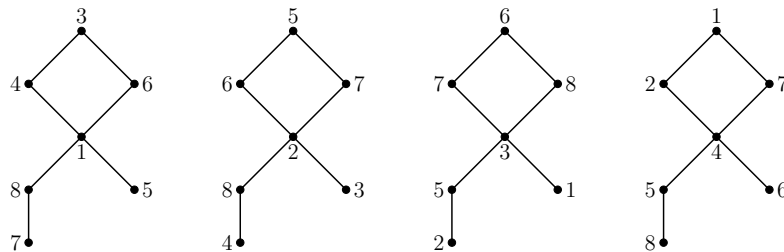Figure 3.21: $\mathcal{B}_7$, an $SE_4(1, 2)$-Design of Order 7
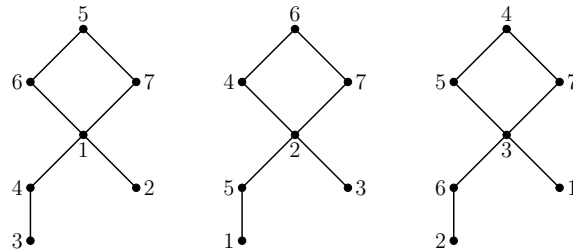
Example 3.27. The following set forms an $SE_4(1, 2)$-design of order 8:

$$\mathcal{B}_8 = \{SE(4, 3, 6, 1; 5, 8, 7), SE(6, 5, 7, 2; 3, 8, 4), SE(7, 6, 8, 3; 1, 5, 2), SE(2, 1, 7, 4; 6, 5, 8)\}.$$

As graphs, the blocks can be seen in Figure 3.22.

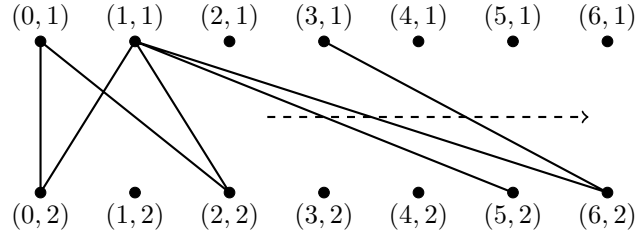Figure 3.22: $\mathcal{B}_8$, an $SE_4(1, 2)$-Design of Order 8

Example 3.28. Let $V(K_{7,7}) = \mathbb{Z}_7 \times \{1, 2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$. Then the following set is an $SE_4(1, 2)$-design on $K_{7,7}$.

$$\mathcal{B}_{7,7} = \{SE((i, 2), (i, 1), (i + 2, 2), (i + 1, 1); (i + 5, 2), (i + 6, 2), (i + 3, 1)) \mid 0 \leq i \leq 6\}$$

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The addition in the first coordinates is done modulo 7. To see that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x,1),(y,2)\}$ in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$, and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in an $SE_4(1,2)$-design on $K_{7,7}$. Also, see Figure 3.23.

Figure 3.23: $\mathcal{B}_{7,7}$, a Cyclic $SE_4(1,2)$-Design on $K_{7,7}$

To see each of the blocks in $\mathcal{B}_{7,7}$, view Figure A.7 in the appendix.

The spectrum of $SE_4(1,2)$ is outlined in the next theorem.

Theorem 3.11. *There exists an $SE_4(1,2)$-design of order n if and only if $n \equiv 0$ or $1 \pmod 7$.*

Proof. The necessity of the orders $n \equiv 0$ or $1 \pmod 7$ is clear given that these are the only orders such that $7 | \binom{n}{2}$. Showing their sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$, an $SE_4(1,2)$-design of order n has been exhibited in Example 3.26. For $n = 7t$ ($t \geq 2$), an $SE_4(1,2)$-design of order n exists according to Proposition 3.1 and Lemma 3.2 because there exists both an $SE_4(1,2)$-design of order 7 and an $SE_4(1,2)$-design on $K_{7,7}$ (Example 3.28). Hence there is an $SE_4(1,2)$-design of order $7t$ for each positive integer t.

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, an $SE_4(1,2)$-design of order n has been exhibited in Example 3.27. For $n = 7t+1$ ($t \geq 2$), an $SE_4(1,2)$-design of order n exists according to Proposition 3.1 and Lemma 3.2 because there exists both an $SE_4(1,2)$-design of order

8 and an $SE_4(1,2)$-design on $K_{7,7}$. Thus there is an $SE_4(1,2)$-design of order $7t + 1$ for each positive integer t.

Therefore an $SE_4(1,2)$-design of order n exists if and only if $n \equiv 0$ or $1 \pmod 7$. $\square$

As usual, solving the intersection problem for $SE_4(1,2)$ proceeds by considering a few small cases.

Example 3.29. Starting with the $SE_4(1,2)$-design of order 7 given in Example 3.26, permutation of select vertices will yield a disjoint design and a design that shares one block in common with the original design. As a reminder, the original design is

$$\mathcal{B}_7 = \{SE(6,5,7,1;2,4,3), SE(4,6,7,2;3,5,1), SE(5,4,7,3;1,6,2)\}.$$

Transposing vertices 3 and 4 gives the following $SE_4(1,2)$-design of order 7 which is disjoint from the original design.

$$\mathcal{B}_7^0 = \{SE(6,5,7,1;2,3,4), SE(3,6,7,2;4,5,1), SE(5,3,7,4;1,6,2)\}.$$

Transposing vertices 6 and 7 gives a design shown below that shares exactly one block in common with $\mathcal{B}_7$.

$$\mathcal{B}_7^1 = \{SE(6,5,7,1;2,4,3), SE(4,7,6,2;3,5,1), SE(5,4,6,3;1,7,2)\}.$$

Consequently $I_{SE_4(1,2)}(7) = J_{SE_4(1,2)}(7)$.

Example 3.30. For the $SE_4(1,2)$-design of order 8 of Example 3.27

$$\mathcal{B}_8 = \{SE(4,3,6,1;5,8,7), SE(6,5,7,2;3,8,4), SE(7,6,8,3;1,5,2), SE(2,1,7,4;6,5,8)\},$$

a disjoint design can be found by transposing vertices 5 and 7 as shown in the design below.

$$\mathcal{B}_8^0 = \{SE(4,3,6,1;7,8,5), SE(6,7,5,2;3,8,4), SE(5,6,8,3;1,7,2), SE(2,1,5,4;6,7,8)\}$$

To obtain a design that shares exactly one block with $\mathcal{B}_8$, transpose vertices 4 and 6 as seen in the design below.

$$\mathcal{B}_8^1 = \{SE(4,3,6,1;5,8,7), SE(4,5,7,2;3,8,6), SE(7,4,8,3;1,5,2), SE(2,1,7,6;4,5,8)\}$$

Notice that the last two blocks of $\mathcal{B}_8$ form a trade of volume 2 in the design as illustrated in Figure 3.24.

Figure 3.24: A Trade of Volume 2 and a Mate in an $SE_4(1,2)$-Design of Order 8
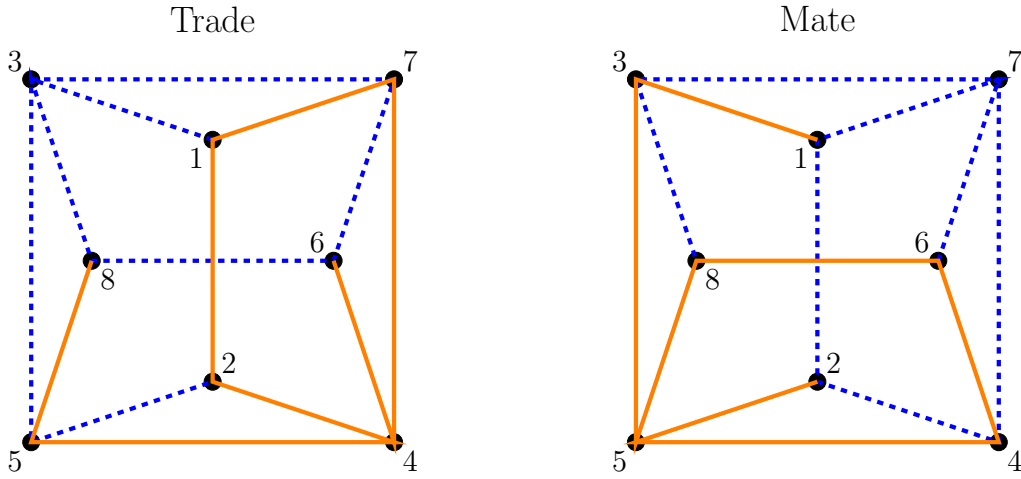
Thus $I_{SE_4(1,2)}(8) = J_{SE_4(1,2)}(8)$.

To solve the intersection problem for larger orders in $\text{Spec}(SE_4(1,2))$, the existence of some of the elements in $I_{SE_4(1,2)}(K_{7,7})$ needs to be shown.

Example 3.31. Looking at the $SE_4(1,2)$-design on $K_{7,7}$ of Example 3.28,

$$\mathcal{B}_{7,7} = \{SE((i,2),(i,1),(i+2,2),(i+1,1);(i+5,2),(i+6,2),(i+3,1)) \mid 0 \le i \le 6\},$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{SE((i,1),(i,2),(i+2,1),(i+1,2);(i+5,1),(i+6,1),(i+3,2)) \mid 0 \le i \le 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(0,2)$ and $(2,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. (For full details, see Example A.4.)

Hence $I_{SE_4(1,2)}(K_{7,7}) \supseteq \{0,2,7\}$.

The intersection problem for order 14 is more easily handled separately from the other higher orders. The solution is shown in the next example.

Example 3.32. From Lemma 3.2 and Theorem 3.11, an $SE_4(1,2)$-design of order 14 can be constructed by combining two $SE_4(1,2)$-designs of order 7 and one $SE_4(1,2)$-design on $K_{7,7}$ in a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.29 and 3.31,

$$
\begin{aligned}
I_{SE_4(1,2)}(14) &\supseteq 2 * I_{SE_4(1,2)}(7) + I_{SE_4(1,2)}(K_{7,7}) \\
&\supseteq 2 * \{0,1,3\} + \{0,2,7\} \\
&\supseteq \{0,1,2,3,4,6\} + \{0,2,7\} \\
&\supseteq \{0,1,2,3,4,5,6,7,8,9,10,11,13\} \\
&\supseteq J_{SE_4(1,2)}(14).
\end{aligned}
$$

Thus $I_{SE_4(1,2)}(14) = J_{SE_4(1,2)}(14)$.

The solution to the intersection problem for $SE_4(1,2)$ is completed in the following theorem.

Theorem 3.12. *If $n \in \mathrm{Spec}(SE_4(1,2))$, then*

$$I_{SE_4(1,2)}(n) = J_{SE_4(1,2)}(n).$$

Proof. For $n = 7$, $n = 8$, and $n = 14$, $I_{SE_4(1,2)}(n) = J_{SE_4(1,2)}(n)$ as shown in Examples 3.29, 3.30, and 3.32 respectively. For the remaining orders in $\mathrm{Spec}(SE_4(1,2))$, two cases are considered.

(i) Suppose $n \equiv 0 \pmod 7$ and $n \geq 21$; that is, $n = 7t$ for some integer $t \geq 3$. From Theorem 3.11, an $SE_4(1,2)$-design of order $7t$ ($t \geq 3$) can be constructed containing t $SE_4(1,2)$-designs of order 7 and $\binom{t}{2}$ $SE_4(1,2)$-designs on $K_{7,7}$. According to Lemma 3.3, if $t \geq 3$,

$$
\begin{aligned}
I_{SE_4(1,2)}(7t) &\supseteq t * I_{SE_4(1,2)}(7) + \binom{t}{2} * I_{SE_4(1,2)}(K_{7,7}) \\
&\supseteq t * \{0,1,3\} + \binom{t}{2} * \{0,7\} \\
&\supseteq \left\{0,1,2,\ldots,\tfrac{1}{7}\binom{7t}{2} - 3, \tfrac{1}{7}\binom{7t}{2} - 2, \tfrac{1}{7}\binom{7t}{2}\right\} \qquad \text{(See proof of Theorem 3.6.)} \\
&\supseteq J_{SE_4(1,2)}(7t).
\end{aligned}
$$

(ii) Suppose $n \equiv 1 \pmod 7$ and $n \geq 15$; that is, $n = 7t + 1$ for some integer $t \geq 2$. From Theorem 3.11, an $SE_4(1,2)$-design of order $7t+1$ ($t \geq 2$) can be constructed containing t $SE_4(1,2)$-designs of order 8 and $\binom{t}{2}$ $SE_4(1,2)$-designs on $K_{7,7}$. By Lemma 3.3, if $t \geq 2$,

$$
\begin{aligned}
I_{SE_4(1,2)}(7t+1) &\supseteq t * I_{SE_4(1,2)}(8) + \binom{t}{2} * I_{SE_4(1,2)}(K_{7,7}) \\
&\supseteq t * \{0,1,2,4\} + \left\{7r \mid 0 \leq r \leq \binom{t}{2}\right\} \\
&\supseteq \left\{0,1,2,\ldots,\tfrac{1}{7}\binom{7t+1}{2} - 3, \tfrac{1}{7}\binom{7t+1}{2} - 2, \tfrac{1}{7}\binom{7t+1}{2}\right\} \quad \text{(See Theorem 3.6.)} \\
&\supseteq J_{SE_4(1,2)}(7t+1).
\end{aligned}
$$

Therefore $I_{SE_4(1,2)}(n) = J_{SE_4(1,2)}(n)$ for each $n \in \mathrm{Spec}(SE_4(1,2))$. $\qquad \square$

3.4 A Graph Containing a 4-Cycle with a Pendant Edge on One Vertex and a Path of Length Two on the Opposite Vertex in the Cycle

As with previous graphs, there is no standard convention for naming the graph of interest so one is given in the following definition.

Definition 3.9. The graph $T(\ell, m)$ $(0 < \ell \leq m)$ is a graph containing a cycle of length 4, a path of length ℓ incident on one vertex in the 4-cycle, and a path of length m incident on the opposite vertex in the 4-cycle.

Example 3.33. The graph $T(1, 2)$ has vertex set $V(T(1, 2)) = \{a, b, c, d, e, f, g\}$ and edge set $E(T(1, 2)) = \{ab, bc, cd, de, be, df, fg\}$. For convenience, this graph will be denoted by the vector $T(a; b, c, d, e; f, g)$ from now on. The graph is illustrated in Figure 3.25.

Figure 3.25: The Graph $T(1, 2)$
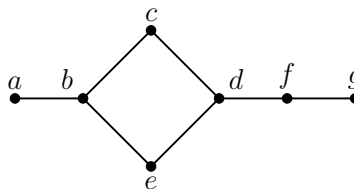
The intersection problem for $T(1, 1)$ is solved in [13]. In the following discussion, the intersection problem for $T(1, 2)$ is solved, but first the spectrum of $T(1, 2)$ must be established. As usual, some small designs are given first.

Example 3.34. The following block set is a $T(1, 2)$-design of order 7.

$$\mathcal{B}_7 = \{T(5; 1, 2, 4, 7; 6, 3), T(6; 2, 3, 5, 7; 4, 1), T(4; 3, 1, 6, 7; 5, 2)\}$$

The blocks are shown as graphs in Figure 3.26.

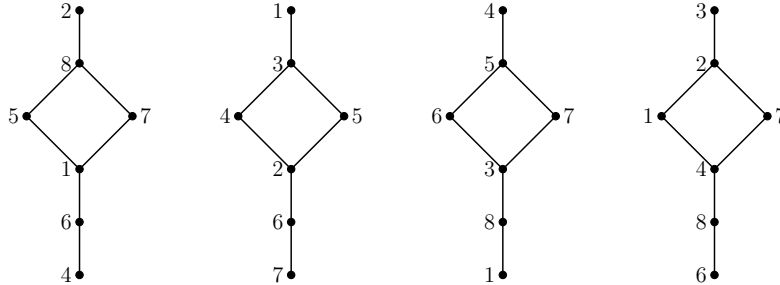Figure 3.26: $\mathcal{B}_7$, a $T(1,2)$-Design of Order 7
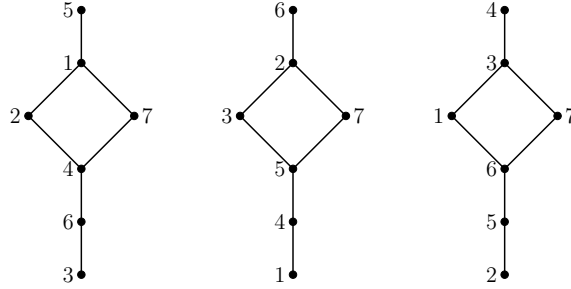
Example 3.35. The following block set is a $T(1,2)$-design of order 8.

$$\mathcal{B}_8 = \{T(2; 8, 5, 1, 7; 6, 4), T(1; 3, 4, 2, 5; 6, 7), T(4; 5, 6, 3, 7; 8, 1), T(3; 2, 1, 4, 7; 8, 6)\}$$

The blocks are illustrated as graphs in Figure 3.27.

Figure 3.27: $\mathcal{B}_8$, a $T(1,2)$-Design of Order 8

Example 3.36. Let $V(K_{7,7}) = \mathbb{Z}_7 \times \{1,2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$. Then

$$\mathcal{B}_{7,7} = \{T((i+6,2); (i+2,1), (i,2), (i,1), (i+1,2); (i+2,2), (i+6,1)) \mid 0 \le i \le 6\}$$

where the addition in the first coordinates is done modulo 7 is a $T(1,2)$-design on $K_{7,7}$. To see that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x,1),(y,2)\}$ in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$,

and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in an $T(1,2)$-design on $K_{7,7}$. Also, see Figure 3.28.

Figure 3.28: $\mathcal{B}_{7,7}$, a Cyclic $T(1,2)$-Design on $K_{7,7}$
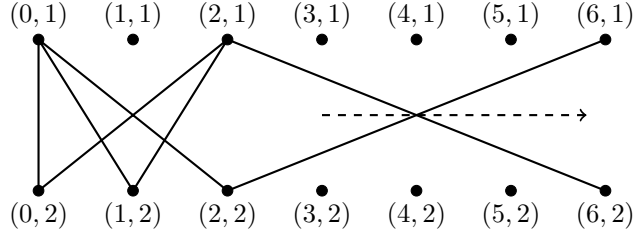
For a complete list of the blocks as graphs, see Figure A.9 in the appendix.

Theorem 3.13. *There exists a $T(1,2)$-design of order n if and only if $n \equiv 0$ or $1 \pmod 7$.*

Proof. The necessity of $n \equiv 0$ or $1 \pmod 7$ is obvious since those are the only orders such that $7 | \binom{n}{2}$. The proof of sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$, a $T(1,2)$-design of order n exists as shown in Example 3.34. Also, a $T(1,2)$-design on $K_{7,7}$ exists as shown in Example 3.36. By Lemma 3.2, a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} exists for each $t \geq 2$; consequently, a $T(1,2)$-design of order $7t$ exists for each positive integer t according to Proposition 3.1.

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, a $T(1,2)$-design of order 8 exists as shown in Example 3.35. Once again, a $T(1,2)$-design on $K_{7,7}$ exists as shown in Example 3.36. By Lemma 3.2, a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} exists for each $t \geq 2$; thus, a $T(1,2)$-design of order $7t + 1$ exists for each positive integer t according to Proposition 3.1.

Therefore, a $T(1,2)$-design of order n exists for each $n \equiv 0$ or $1 \pmod 7$. $\qquad \square$

With the spectrum of $T(1,2)$ established, the intersection problem is now solved by considering a few small designs.

Example 3.37. Starting with $\mathcal{B}_7$ from Example 3.34, consider the following three $T(1,2)$-designs of order 7.

$$\mathcal{B}_7 = \{T(5; 1, 2, 4, 7; 6, 3), T(6; 2, 3, 5, 7; 4, 1), T(4; 3, 1, 6, 7; 5, 2)\}$$

Transposing vertices 6 and 7 in $\mathcal{B}_7$ yields the design

$$\mathcal{B}_7^0 = \{T(5; 1, 2, 4, 6; 7, 3), T(7; 2, 3, 5, 6; 4, 1), T(4; 3, 1, 7, 6; 5, 2)\}$$

which is disjoint from $\mathcal{B}_7$.

Transposing vertices 2 and 7 in $\mathcal{B}_7$ yields a design

$$\mathcal{B}_7^1 = \{T(5; 1, 2, 4, 7; 6, 3), T(6; 7, 3, 5, 2; 4, 1), T(4; 3, 1, 6, 2; 5, 7)\}$$

that shares exactly 1 block with $\mathcal{B}_7$. Thus $I_{T(1,2)}(7) = J_{T(1,2)}(7)$.

Example 3.38. Starting with $\mathcal{B}_8$ from Example 3.35, consider the following three $T(1,2)$-designs of order 8.

$$\mathcal{B}_8 = \{T(2; 8, 5, 1, 7; 6, 4), T(1; 3, 4, 2, 5; 6, 7), T(4; 5, 6, 3, 7; 8, 1), T(3; 2, 1, 4, 7; 8, 6)\}$$

Transposing vertices 7 and 8 in $\mathcal{B}_8$ yields the following design that is disjoint from $\mathcal{B}_8$.

$$\mathcal{B}_8^0 = \{T(2; 7, 5, 1, 8; 6, 4), T(1; 3, 4, 2, 5; 6, 8), T(4; 5, 6, 3, 8; 7, 1), T(3; 2, 1, 4, 8; 7, 6)\}$$

Transposing vertices 5 and 7 in $\mathcal{B}_8$ yields the following design that has exactly one block in common with $\mathcal{B}_8$.

$$\mathcal{B}_8^1 = \{T(2; 8, 5, 1, 7; 6, 4), T(1; 3, 4, 2, 7; 6, 5), T(4; 7, 6, 3, 5; 8, 1), T(3; 2, 1, 4, 5; 8, 6)\}$$

The last two blocks in $\mathcal{B}_8$ form a trade of volume 2 in the design. The trade and a mate are illustrated in the subgraph of K_8 shown in Figure 3.29.

Figure 3.29: A Trade of Volume 2 and a Mate in a $T(1,2)$-Design of Order 8
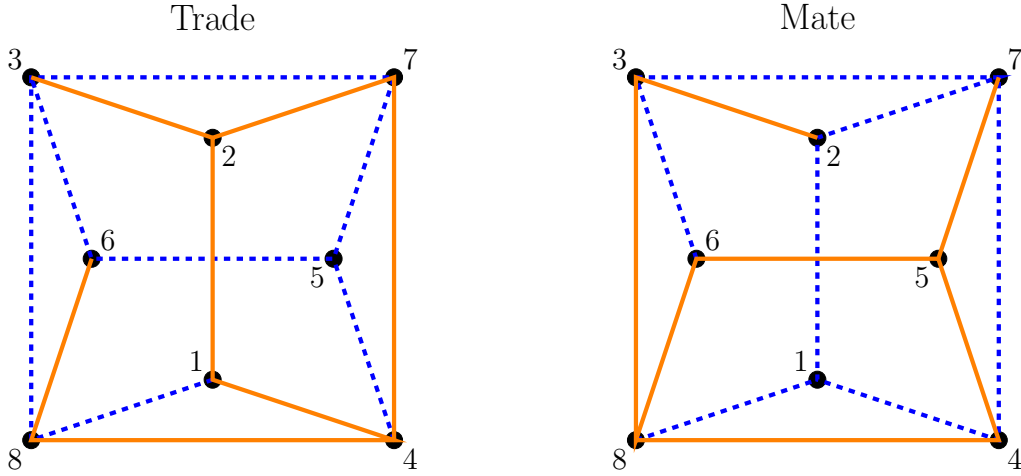
Thus $I_{T(1,2)}(8) = J_{T(1,2)}(8)$.

As with the previous graphs, some elements of $I_{T(1,2)}(K_{7,7})$ must be shown in order to solve the intersection problem for $n \geq 14$ in the spectrum of $T(1,2)$.

Example 3.39. Looking at the $T(1,2)$-design on $K_{7,7}$ of Example 3.36,

$$\mathcal{B}_{7,7} = \{T((i+6,2); (i+2,1), (i,2), (i,1), (i+1,2); (i+2,2), (i+6,1)) \mid 0 \leq i \leq 6\}$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{T((i+6,1); (i+2,2), (i,1), (i,2), (i+1,1); (i+2,1), (i+6,2)) \mid 0 \leq i \leq 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(3,2)$ and $(4,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^3$ on $K_{7,7}$ that has exactly three blocks in common with $\mathcal{B}_{7,7}$. For more insight, view Example A.5.

Hence $I_{T(1,2)}(K_{7,7}) \supseteq \{0, 3, 7\}$.

Once again, the solution for order 14 is more easily handled separately from the higher orders in the spectrum of $T(1,2)$.

Example 3.40. From Lemma 3.2 and Theorem 3.13, a $T(1,2)$-design of order 14 can be constructed using 2 $T(1,2)$-designs of order 7, a $T(1,2)$-design on $K_{7,7}$, and a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.37 and 3.39,

$$
\begin{aligned}
I_{T(1,2)}(14) &\supseteq 2 * I_{T(1,2)}(7) + I_{T(1,2)}(K_{7,7}) \\
&\supseteq 2 * \{0, 1, 3\} + \{0, 3, 7\} \\
&\supseteq \{0, 1, 2, 3, 4, 6\} + \{0, 3, 7\} \\
&\supseteq \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\} \\
&\supseteq J_{T(1,2)}(14).
\end{aligned}
$$

Hence $I_{T(1,2)}(14) = J_{T(1,2)}(14)$.

The intersection problem for the remaining orders $n \in \mathrm{Spec}(T(1,2))$ are solved in the following theorem.

Theorem 3.14. *If* $n \in \mathrm{Spec}(T(1,2))$, *then*

$$
I_{T(1,2)}(n) = J_{T(1,2)}(n).
$$

Proof. As shown in Theorem 3.13, the values of $n \in \mathrm{Spec}(T(1,2))$ are $n \equiv 0$ or $1 \pmod 7$. Thus the proof proceeds by considering two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$ and $n = 14$, $I_{T(1,2)}(n) = J_{T(1,2)}(n)$ as shown in Examples 3.37 and 3.40 respectively. For $n = 7t$ with $t \geq 3$ an integer, it was shown in Theorem 3.13 that a $T(1,2)$-design of order n can be constructed using a

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$\{K_7, K_{7,7}\}$-decomposition of K_{7t} with t blocks isomorphic to K_7 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$. From Lemma 3.3,

$$
\begin{aligned}
I_{T(1,2)}(7t) &\supseteq t * I_{T(1,2)}(7) + \tbinom{t}{2} * I_{T(1,2)}(K_{7,7}) \\
&\supseteq t * \{0, 1, 3\} + \tbinom{t}{2} * \{0, 7\} \\
&\supseteq \left\{0, 1, 2, \ldots, \tfrac{1}{7}\tbinom{7t}{2} - 3, \tfrac{1}{7}\tbinom{7t}{2} - 2, \tfrac{1}{7}\tbinom{7t}{2}\right\} \qquad \text{(See proof of Theorem 3.6.)} \\
&\supseteq J_{T(1,2)}(7t).
\end{aligned}
$$

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, $I_{T(1,2)}(n) = J_{T(1,2)}(n)$ as shown in Example 3.38. For $n = 7t+1$ with $t \geq 2$ an integer, it was shown in Theorem 3.13 that a $T(1,2)$-design of order n can be constructed using a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} with t blocks isomorphic to K_8 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$.

Consequently

$$
\begin{aligned}
I_{T(1,2)}(7t + 1) &\supseteq t * I_{T(1,2)}(8) + \tbinom{t}{2} * I_{T(1,2)}(K_{7,7}) \\
&\supseteq t * \{0, 1, 2, 4\} + \left\{7r \mid 0 \leq r \leq \tbinom{t}{2}\right\} \\
&\supseteq \left\{0, 1, 2, \ldots, \tfrac{1}{7}\tbinom{7t+1}{2} - 3, \tfrac{1}{7}\tbinom{7t+1}{2} - 2, \tfrac{1}{7}\tbinom{7t+1}{2}\right\} \qquad \text{(See Theorem 3.6.)} \\
&\supseteq J_{T(1,2)}(7t + 1).
\end{aligned}
$$

Therefore, $I_{T(1,2)}(n) = J_{T(1,2)}(n)$ for each $n \in \mathrm{Spec}(T(1, 2))$. $\qquad\square$

3.5   A Graph Containing a 4-Cycle with a Pendant Edge on One Vertex and a Path of Length Two on an Adjacent Vertex in the Cycle

As with the previous graphs, some notation needs to be introduced before the solution to the intersection problem is given.

Definition 3.10. Let $U(\ell, m)$ $(\ell \leq m)$ be the graph with a cycle of length 4, a path of length $\ell > 0$ incident on one vertex in the cycle, and a path of length $m > 0$ incident on an adjacent vertex in the cycle.

Example 3.41. The graph $U(1,2)$ has vertex set $V(U(1,2)) = \{a, b, c, d, e, f, g\}$ and edge set $E(U(1,2)) = \{ab, bc, cd, de, be, ef, fg\}$. For brevity, this graph will be denoted by the vector $U(a; b, c, d, e; f, g)$ henceforth. The graph can be seen in Figure 3.30.

Figure 3.30: The Graph $U(1,2)$
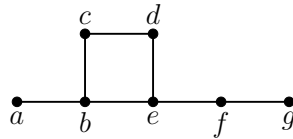
The intersection problem for the graph $U(1,1)$ is solved in [13], and the following discussion solves the intersection problem for $U(1,2)$. First the spectrum needs to be shown for this graph which will be done through several examples and Lemma 3.2.

Example 3.42. The following set forms a $U(1,2)$-design of order 7:

$$\mathcal{B}_7 = \{U(5; 4, 7, 3, 1; 6, 2), U(6; 5, 7, 1, 2; 4, 3), U(4; 6, 7, 2, 3; 5, 1)\}.$$

As graphs, the blocks are shown in Figure 3.31.

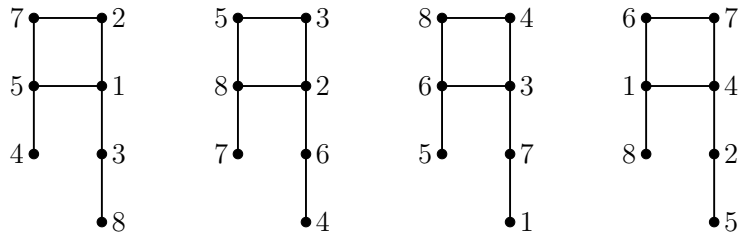Figure 3.31: $\mathcal{B}_7$, a $U(1,2)$-Design of Order 7
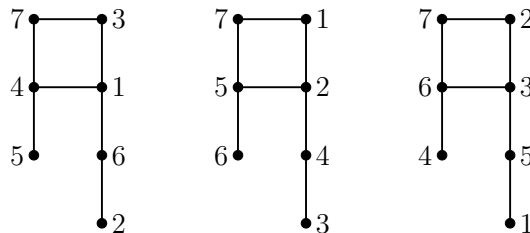
Example 3.43. The following set forms a $U(1,2)$-design of order 8:

$$\mathcal{B}_8 = \{U(4; 5, 7, 2, 1; 3, 8), U(7; 8, 5, 3, 2; 6, 4), U(5; 6, 8, 4, 3; 7, 1), U(8; 1, 6, 7, 4; 2, 5)\}.$$

As graphs, the blocks are shown in Figure 3.32.

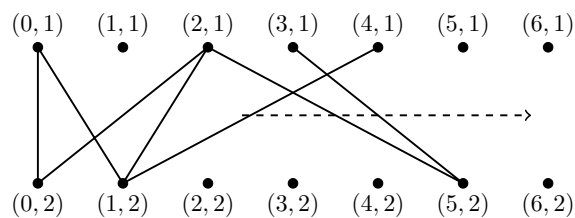Figure 3.32: $\mathcal{B}_8$, a $U(1,2)$-Design of Order 8

Example 3.44. If $V(K_{7,7}) = \mathbb{Z}_7 \times \{1, 2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$, then

$$\mathcal{B}_{7,7} = \{U((i+4, 1); (i+1, 2), (i, 1), (i, 2), (i+2, 1); (i+5, 2), (i+3, 1)) \mid 0 \leq i \leq 6\},$$

where the addition in the first coordinates is done modulo 7, is a $U(1,2)$-design on $K_{7,7}$. To see that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x, 1), (y, 2)\}$ in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$, and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in a $U(1,2)$-design on $K_{7,7}$. Also, see Figure 3.33.

Figure 3.33: $\mathcal{B}_{7,7}$, a Cyclic $U(1,2)$-Design on $K_{7,7}$

For a complete listing of the blocks in $\mathcal{B}_{7,7}$, see Figure A.11 in the appendix.

The spectrum of $U(1,2)$ is outlined in the next theorem.

Theorem 3.15. *There exists a $U(1,2)$-design of order n if and only if $n \equiv 0$ or 1 (mod 7).*

Proof. The necessity of the orders $n \equiv 0$ or 1 (mod 7) is clear given that these are the only orders such that $7 | \binom{n}{2}$. Showing their sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0$ (mod 7). For $n = 7$, a $U(1,2)$-design of order n has been exhibited in Example 3.42. For $n = 7t$ ($t \geq 2$), a $U(1,2)$-design of order n exists according to Proposition 3.1 and Lemma 3.2 because there exists both a $U(1,2)$-design of order 7 and a $U(1,2)$-design on $K_{7,7}$ (Example 3.44). Hence there is a $U(1,2)$-design of order $7t$ for each positive integer t.

(ii) Suppose $n \equiv 1$ (mod 7). For $n = 8$, a $U(1,2)$-design of order n has been exhibited in Example 3.43. For $n = 7t + 1$ ($t \geq 2$), a $U(1,2)$-design of order n exists according to Proposition 3.1 and Lemma 3.2 because there exists both a $U(1,2)$-design of order 8 and a $U(1,2)$-design on $K_{7,7}$. Thus there is a $U(1,2)$-design of order $7t + 1$ for each positive integer t.

Therefore a $U(1,2)$-design of order n exists if and only if $n \equiv 0$ or 1 (mod 7). $\qquad\square$

As usual, solving the intersection problem for $U(1,2)$ proceeds by considering a few small cases.

**Example 3.45.** Starting with the $U(1,2)$-design of order 7 given in Example 3.42,

$$\mathcal{B}_7 = \{U(5; 4, 7, 3, 1; 6, 2), U(6; 5, 7, 1, 2; 4, 3), U(4; 6, 7, 2, 3; 5, 1)\},$$

transposing vertices 6 and 7 yields the following design that is disjoint from $\mathcal{B}_7$.

$$\mathcal{B}_7^0 = \{U(5; 4, 6, 3, 1; 7, 2), U(7; 5, 6, 1, 2; 4, 3), U(4; 7, 6, 2, 3; 5, 1)\}$$

There is not an easily observable permutation of the vertices that gives a design with one block in common with $\mathcal{B}_7$. Instead a demonstration that the last two blocks in $\mathcal{B}_7$ form a trade of volume 2 is given in Figure 3.34.

Figure 3.34: A Trade of Volume 2 and a Mate in a $U(1,2)$-Design of Order 7
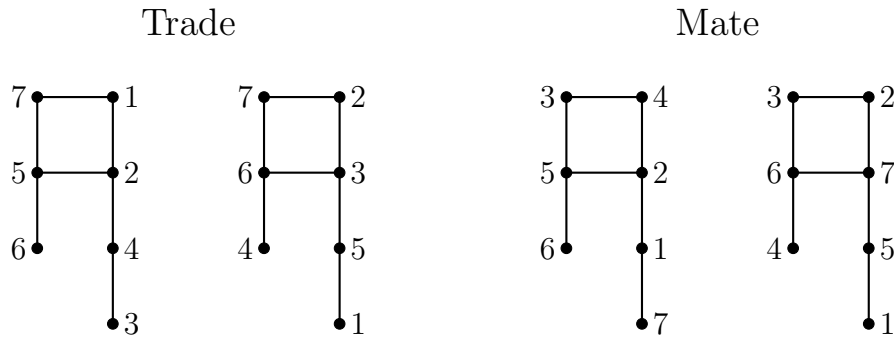
Consequently $I_{U(1,2)}(7) = J_{U(1,2)}(7)$.

Example 3.46. For the $U(1,2)$-design of order 8 of Example 3.43

$$\mathcal{B}_8 = \{U(4;5,7,2,1;3,8), U(7;8,5,3,2;6,4), U(5;6,8,4,3;7,1), U(8;1,6,7,4;2,5)\},$$

a disjoint design can be found by transposing vertices 7 and 8 as shown in the design below.

$$\mathcal{B}_8^0 = \{U(4;5,8,2,1;3,7), U(8;7,5,3,2;6,4), U(5;6,7,4,3;8,1), U(7;1,6,8,4;2,5)\}.$$

Due to a lack of symmetry in the graph $U(1,2)$, there is no obvious permutation of vertices that generates a design with one block in common with $\mathcal{B}_8$. A design with one block in common with $\mathcal{B}_8$ does exist however and is given as follows.

$$\mathcal{B}_8^1 = \{U(4;5,7,2,1;3,8), U(2;8,7,4,6;3,5), U(8;4,1,7,3;2,6), U(7;6,1,8,5;2,4)\}.$$

Lastly, observe that the last two blocks of $\mathcal{B}_8$ form a trade of volume 2 in the design as illustrated in Figure 3.35.

Figure 3.35: A Trade of Volume 2 and a Mate in a $U(1,2)$-Design of Order 8
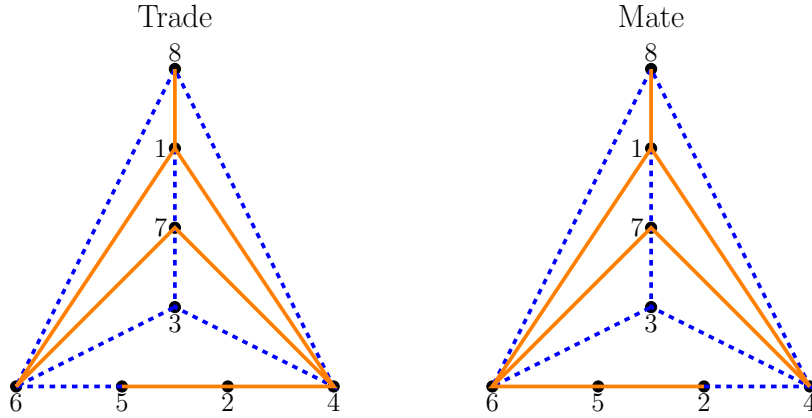
Thus $I_{U(1,2)}(8) = J_{U(1,2)}(8)$.

To solve the intersection problem for larger orders in $\text{Spec}(U(1,2))$, the existence of some of the elements in $I_{U(1,2)}(K_{7,7})$ needs to be shown.

Example 3.47. Looking at the $U(1,2)$-design on $K_{7,7}$ of Example 3.44,

$$\mathcal{B}_{7,7} = \{U((i+4,1);(i+1,2),(i,1),(i,2),(i+2,1);(i+5,2),(i+3,1)) \mid 0 \leq i \leq 6\},$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{U((i+4,2);(i+1,1),(i,2),(i,1),(i+2,2);(i+5,1),(i+3,2)) \mid 0 \leq i \leq 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(3,2)$ and $(4,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. For more information, see Example A.6.

Hence $I_{U(1,2)}(K_{7,7}) \supseteq \{0,2,7\}$.

The intersection problem for order 14 is more easily handled separately from the other higher orders. The solution is shown in the next example.

Example 3.48. From Lemma 3.2 and Theorem 3.15, a $U(1,2)$-design of order 14 can be constructed by combining two $U(1,2)$-designs of order 7 and one $U(1,2)$-design on $K_{7,7}$ in a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.45 and 3.47,

$$
\begin{aligned}
I_{U(1,2)}(14) &\supseteq 2 * I_{SE_4(1,2)}(7) + I_{SE_4(1,2)}(K_{7,7}) \\
&\supseteq 2 * \{0, 1, 3\} + \{0, 2, 7\} \\
&\supseteq \{0, 1, 2, 3, 4, 6\} + \{0, 2, 7\} \\
&\supseteq \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\} \\
&\supseteq J_{U(1,2)}(14).
\end{aligned}
$$

Thus $I_{U(1,2)}(14) = J_{U(1,2)}(14)$.

The solution to the intersection problem for $U(1,2)$ is completed in the following theorem.

Theorem 3.16. *If $n \in \mathrm{Spec}(U(1,2))$, then*

$$
I_{U(1,2)}(n) = J_{U(1,2)}(n).
$$

Proof. For $n = 7$, $n = 8$, and $n = 14$, $I_{U(1,2)}(n) = J_{U(1,2)}(n)$ as shown in Examples 3.45, 3.46, and 3.48 respectively. For the remaining orders in $\mathrm{Spec}(U(1,2))$, two cases are considered.

(i) Suppose $n \equiv 0 \pmod 7$ and $n \geq 21$; that is, $n = 7t$ for some integer $t \geq 3$. From Theorem 3.15, a $U(1,2)$-design of order $7t$ $(t \geq 3)$ can be constructed containing t $U(1,2)$-designs of order 7 and $\binom{t}{2}$ $U(1,2)$-designs on $K_{7,7}$. According to Lemma 3.3, if $t \geq 3$,

$$
\begin{aligned}
I_{U(1,2)}(7t) &\supseteq t * I_{U(1,2)}(7) + \binom{t}{2} * I_{U(1,2)}(K_{7,7}) \\
&\supseteq t * \{0, 1, 3\} + \binom{t}{2} * \{0, 7\}
\end{aligned}
$$

$$\supseteq \left\{ 0, 1, 2, \dots, \tfrac{1}{7}\binom{7t}{2} - 3, \tfrac{1}{7}\binom{7t}{2} - 2, \tfrac{1}{7}\binom{7t}{2} \right\} \qquad \text{(See proof of Theorem 3.6.)}$$

$$\supseteq J_{U(1,2)}(7t).$$

(ii) Suppose $n \equiv 1 \pmod 7$ and $n \geq 15$; that is, $n = 7t + 1$ for some integer $t \geq 2$. From Theorem 3.15, a $U(1,2)$-design of order $7t + 1$ ($t \geq 2$) can be constructed containing t $U(1,2)$-designs of order 8 and $\binom{t}{2}$ $U(1,2)$-designs on $K_{7,7}$. By Lemma 3.3, if $t \geq 2$,

$$I_{U(1,2)}(7t + 1) \supseteq t * I_{U(1,2)}(8) + \binom{t}{2} * I_{U(1,2)}(K_{7,7})$$

$$\supseteq t * \{0, 1, 2, 4\} + \left\{ 7r \mid 0 \leq r \leq \binom{t}{2} \right\}$$

$$\supseteq \left\{ 0, 1, 2, \dots, \tfrac{1}{7}\binom{7t+1}{2} - 3, \tfrac{1}{7}\binom{7t+1}{2} - 2, \tfrac{1}{7}\binom{7t+1}{2} \right\} \qquad \text{(See Theorem 3.6.)}$$

$$\supseteq J_{U(1,2)}(7t + 1).$$

Therefore $I_{U(1,2)}(n) = J_{U(1,2)}(n)$ for each $n \in \mathrm{Spec}(U(1,2))$. $\qquad\qquad \square$

## 3.6 The "Viper" Graph Containing a 4-Cycle and 7 Edges

One last time, some notation must be introduced for the graph whose intersection problem is to be solved.

Definition 3.11. The *viper graph* $V_\ell(m)^2$ is the graph with m vertices and m edges, containing a cycle of length ℓ, a path of length $m - \ell - 2$ incident on a vertex in the cycle, and two pendant edges at the opposite end of the path.

Example 3.49. The viper graph $V_4(7)$ is the graph with vertex set $V(V_4(7)) = \{a, b, c, d, e, f, g\}$ and edge set $E(V_4(7)) = \{ab, bc, cd, ad, de, ef, eg\}$. An example of this graph is shown in Figure 3.36. From now on a graph with these vertex and edge sets will be denoted $V(a, b, c, d; e, f, g)$.

---

[2]Depending on the orientation of the vertices, the graph looks like a diamond shaped head with a forked tongue which are traits common to the family of snakes known as vipers.

Figure 3.36: The "Viper" Graph $V_4(7)$
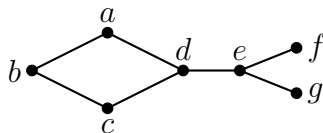
In the following discussion, the intersection problem for $V_4(7)$ is solved, but first the spectrum of $V_4(7)$ must be established. As usual, some small designs are given first.

Example 3.50. The following block set is a $V_4(7)$-design of order 7.

$$\mathcal{B}_7 = \{V(2,7,6,1;4,3,5), V(3,7,4,2;5,1,6), V(1,7,5,3;6,2,4)\}$$

The blocks are illustrated as graphs in Figure 3.37.
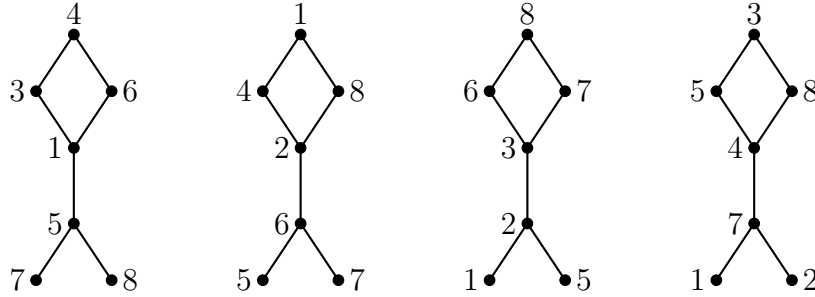
Figure 3.37: $\mathcal{B}_7$, a $V_4(7)$-Design of Order 7
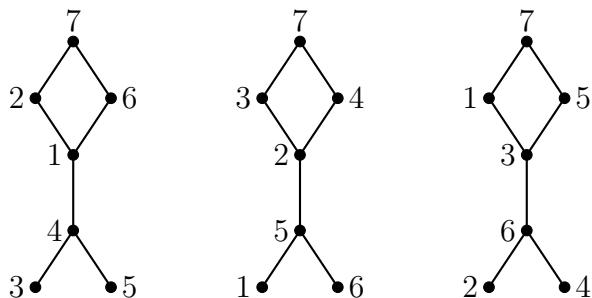
Example 3.51. The following block set is a $V_4(7)$-design of order 8.

$$\mathcal{B}_8 = \{V(3,4,6,1;5,7,8), V(4,1,8,2;6,5,7), V(6,8,7,3;2,1,5), V(5,3,8,4;7,1,2)\}$$

The blocks are illustrated as graphs in Figure 3.38.

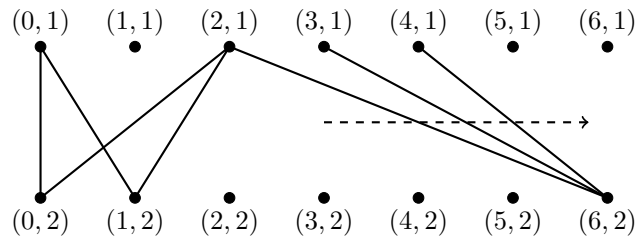Figure 3.38: $\mathcal{B}_8$, a $V_4(7)$-Design of Order 8

Example 3.52. Let $V(K_{7,7}) = \mathbb{Z}_7 \times \{1, 2\}$ with partitions $A = \mathbb{Z}_7 \times \{1\}$ and $B = \mathbb{Z}_7 \times \{2\}$.

Then

$$\mathcal{B}_{7,7} = \{V((i,2),(i,1),(i+1,2),(i+2,1);(i+6,2),(i+3,1),(i+4,1)) \mid 0 \le i \le 6\}$$

where the addition in the first coordinates is done modulo 7 is a $V_4(7)$-design on $K_{7,7}$. To see

that this is a design on $K_{7,7}$, notice that for each $j \in \mathbb{Z}_7$ there is some edge $\{(x,1),(y,2)\}$

in each block of $\mathcal{B}_{7,7}$ such that $y - x = j$. Thus each edge of $K_{7,7}$ is in some block of $\mathcal{B}_{7,7}$,

and $|\mathcal{B}_{7,7}| = 7$ which is the number of blocks expected in an $V_4(7)$-design on $K_{7,7}$. Also, see

Figure 3.39.

Figure 3.39: $\mathcal{B}_{7,7}$, a Cyclic $V_4(7)$-Design on $K_{7,7}$

To see each of the blocks in $\mathcal{B}_{7,7}$ as graphs, look at Figure A.13 in the appendix.

Theorem 3.17. *There exists a $V_4(7)$-design of order n if and only if $n \equiv 0$ or $1 \pmod 7$.*

Proof. The necessity of $n \equiv 0$ or $1 \pmod 7$ is obvious since those are the only orders such that $7 | \binom{n}{2}$. The proof of sufficiency proceeds by checking two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$, a $V_4(7)$-design of order n exists as shown in Example 3.50. Also, a $V_4(7)$-design on $K_{7,7}$ exists as shown in Example 3.52. By Lemma 3.2, a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} exists for each $t \geq 2$; consequently, a $V_4(7)$-design of order $7t$ exists for each positive integer t according to Proposition 3.1.

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, a $V_4(7)$-design of order 8 exists as shown in Example 3.51. Once again, a $V_4(7)$-design on $K_{7,7}$ exists as shown in Example 3.52. By Lemma 3.2, a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} exists for each $t \geq 2$; thus, a $V_4(7)$-design of order $7t+1$ exists for each positive integer t according to Proposition 3.1.

Therefore, a $V_4(7)$-design of order n exists for each $n \equiv 0$ or $1 \pmod 7$. $\qquad\square$

With the spectrum of $V_4(7)$ established, the intersection problem is now solved by considering a few small designs.

Example 3.53. Starting with $\mathcal{B}_7$ from Example 3.50, consider the following three $V_4(7)$-designs of order 7.

$$\mathcal{B}_7 = \{V(2,7,6,1;4,3,5), V(3,7,4,2;5,1,6), V(1,7,5,3;6,2,4)\}$$

Transposing vertices 6 and 7 in $\mathcal{B}_7$ yields the design

$$\mathcal{B}_7^0 = \{V(2,6,7,1;4,3,5), V(3,6,4,2;5,1,7), V(1,6,5,3;7,2,4)\}$$

which is disjoint from $\mathcal{B}_7$.

Transposing vertices 3 and 5 in $\mathcal{B}_7$ yields a design

$$\mathcal{B}_7^1 = \{V(2,7,6,1;4,3,5), V(5,7,4,2;3,1,6), V(1,7,3,5;6,2,4)\}$$

that shares exactly 1 block with $\mathcal{B}_7$. Thus $I_{V_4(7)}(7) = J_{V_4(7)}(7)$.

Example 3.54. Starting with $\mathcal{B}_8$ from Example 3.51, consider the following three $V_4(7)$-designs of order 8.

$$\mathcal{B}_8 = \{V(3,4,6,1;5,7,8), V(4,1,8,2;6,5,7), V(6,8,7,3;2,1,5), V(5,3,8,4;7,1,2)\}$$

Transposing vertices 6 and 8 in $\mathcal{B}_8$ yields the following design that is disjoint from $\mathcal{B}_8$.

$$\mathcal{B}_8^0 = \{V(3,4,8,1;5,7,6), V(4,1,6,2;8,5,7), V(8,6,7,3;2,1,5), V(5,3,6,4;7,1,2)\}$$

Transposing vertices 7 and 8 in $\mathcal{B}_8$ yields the following design that has exactly one block in common with $\mathcal{B}_8$.

$$\mathcal{B}_8^1 = \{V(3,4,6,1;5,7,8), V(4,1,7,2;6,5,8), V(6,7,8,3;2,1,5), V(5,3,7,4;8,1,2)\}$$

The last two blocks in $\mathcal{B}_8$ form a trade of volume 2 in the design. The trade and a mate are illustrated in the subgraph of K_8 shown in Figure 3.40.

Figure 3.40: A Trade of Volume 2 and a Mate in a $V_4(7)$-Design of Order 8
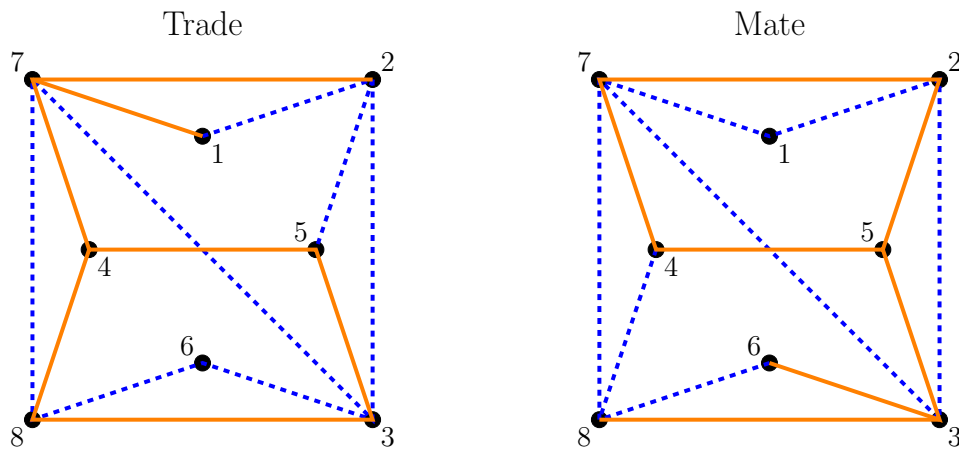
Thus $I_{V_4(7)}(8) = J_{V_4(7)}(8)$.

As with the previous graphs, some elements of $I_{V_4(7)}(K_{7,7})$ must be shown in order to solve the intersection problem for $n \geq 14$ in the spectrum of $V_4(7)$.

Example 3.55. Looking at the $V_4(7)$-design on $K_{7,7}$ of Example 3.52,

$$\mathcal{B}_{7,7} = \{V((i,2),(i,1),(i+1,2),(i+2,1);(i+6,2),(i+3,1),(i+4,1)) \mid 0 \leq i \leq 6\}$$

a disjoint design is readily found by changing the second coordinates of each vertex in each block as shown below.

$$\mathcal{B}_{7,7}^0 = \{V((i,1),(i,2),(i+1,1),(i+2,2);(i+6,1),(i+3,2),(i+4,2)) \mid 0 \leq i \leq 6\}$$

(Note that addition is done modulo 7 in both designs.)

Transposing vertices $(4,2)$ and $(5,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^4$ on $K_{7,7}$ that has exactly four blocks in common with $\mathcal{B}_{7,7}$. For more details, see Example A.7.

Hence $I_{V_4(7)}(K_{7,7}) \supseteq \{0,4,7\}$.

Once again, the solution for order 14 is more easily handled separately from the higher orders in the spectrum of $V_4(7)$.

Example 3.56. From Lemma 3.2 and Theorem 3.17, a $V_4(7)$-design of order 14 can be constructed using 2 $V_4(7)$-designs of order 7, a $V_4(7)$-design on $K_{7,7}$, and a $\{K_7, K_{7,7}\}$-decomposition of K_{14}. From Lemma 3.3 and Examples 3.53 and 3.55,

$$I_{V_4(7)}(14) \supseteq 2 * I_{V_4(7)}(7) + I_{V_4(7)}(K_{7,7})$$

$$\supseteq 2 * \{0,1,3\} + \{0,4,7\}$$

$$\supseteq \{0,1,2,3,4,6\} + \{0,4,7\}$$

$$\supseteq \{0,1,2,3,4,5,6,7,8,9,10,11,13\}$$

$$\supseteq J_{V_4(7)}(14).$$

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Hence $I_{V_4(7)}(14) = J_{V_4(7)}(14)$.

The intersection problem for the remaining orders $n \in \mathrm{Spec}(V_4(7))$ are solved in the following theorem.

Theorem 3.18. *If $n \in \mathrm{Spec}(V_4(7))$, then*

$$I_{V_4(7)}(n) = J_{V_4(7)}(n).$$

Proof. As shown in Theorem 3.17, the values of $n \in \mathrm{Spec}(V_4(7))$ are $n \equiv 0$ or $1 \pmod 7$. Thus the proof proceeds by considering two cases.

(i) Suppose $n \equiv 0 \pmod 7$. For $n = 7$ and $n = 14$, $I_{V_4(7)}(n) = J_{V_4(7)}(n)$ as shown in Examples 3.53 and 3.56 respectively. For $n = 7t$ with $t \geq 3$ an integer, it was shown in Theorem 3.17 that a $V_4(7)$-design of order n can be constructed using a $\{K_7, K_{7,7}\}$-decomposition of K_{7t} with t blocks isomorphic to K_7 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$. From Lemma 3.3,

$$
\begin{aligned}
I_{V_4(7)}(7t) &\supseteq t * I_{V_4(7)}(7) + \binom{t}{2} * I_{V_4(7)}(K_{7,7}) \\
&\supseteq t * \{0, 1, 3\} + \binom{t}{2} * \{0, 7\} \\
&\supseteq \left\{ 0, 1, 2, \ldots, \tfrac{1}{7}\binom{7t}{2} - 3, \tfrac{1}{7}\binom{7t}{2} - 2, \tfrac{1}{7}\binom{7t}{2} \right\} \qquad \text{(See proof of Theorem 3.6.)} \\
&\supseteq J_{V_4(7)}(7t).
\end{aligned}
$$

(ii) Suppose $n \equiv 1 \pmod 7$. For $n = 8$, $I_{V_4(7)}(n) = J_{V_4(7)}(n)$ as shown in Example 3.54. For $n = 7t + 1$ with $t \geq 2$ an integer, it was shown in Theorem 3.17 that a $V_4(7)$-design of order n can be constructed using a $\{K_8, K_{7,7}\}$-decomposition of K_{7t+1} with t blocks isomorphic to K_8 and $\binom{t}{2}$ blocks isomorphic to $K_{7,7}$.

Consequently

$$I_{V_4(7)}(7t+1) \supseteq t * I_{V_4(7)}(8) + \binom{t}{2} * I_{V_4(7)}(K_{7,7})$$

$$\supseteq t * \{0, 1, 2, 4\} + \left\{7r \mid 0 \leq r \leq \binom{t}{2}\right\}$$

$$\supseteq \left\{0, 1, 2, \ldots, \tfrac{1}{7}\binom{7t+1}{2}-3, \tfrac{1}{7}\binom{7t+1}{2}-2, \tfrac{1}{7}\binom{7t+1}{2}\right\} \quad \text{(See Theorem 3.6.)}$$

$$\supseteq J_{SE_4(1,2)}(7t+1).$$

Therefore, $I_{V_4(7)}(n) = J_{V_4(7)}(n)$ for each $n \in \text{Spec}(V_4(7))$. $\qquad\qquad \square$

3.7 Comments on the Remaining Bipartite Graphs with Seven Edges

At first glance, there is nothing special about the bipartite graphs for which the intersection problem has been solved in this chapter. That is, for any bipartite graph G with $e_G = 7$, $\text{Spec}(G) = \{n \mid n \equiv 0 \text{ or } 1 \pmod 7\}$, $I_G(7) = J_G(7)$, $I_G(8) = J_G(8)$, and $I_G(K_{7,7}) \supseteq \{0, k, 7\}$ $(k \in \{1, 2, 3, 4, 5\})$, the solution to the intersection problem is

$$I_G(n) = J_G(n) \quad \text{for each } n \equiv 0 \text{ or } 1 \pmod 7.$$

However, the remaining connected bipartite graphs with 7 edges do not satisfy all of these conditions. There are no simple bipartite graphs with 7 edges and fewer than 6 vertices. Those with 6 vertices are listed in Figure 3.41.

Figure 3.41: Bipartite Graphs with 6 Vertices and 7 Edges

$G_1 \qquad G_2 \qquad G_3$
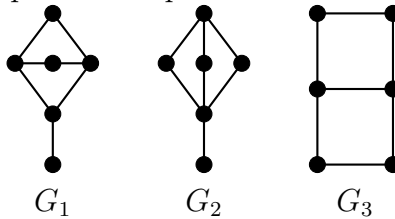
The graphs G_1 and G_2 are shown to have spectra $\{n \,|\, n \equiv 0 \text{ or } 1 \pmod 7, n \neq 7, 8\}$ by constructions in [1]. The authors of that paper give credit for this result to [10], but that thesis could not be obtained for verification at this time. For G_3, the spectrum is $\{n \,|\, n \equiv 0 \text{ or } 1 \pmod 7, n \neq 7\}$ as shown in [7]. Thus for each of these graphs no design of order 7 exists.

The connected bipartite graphs with 7 vertices and 7 edges for which the intersection problem has not been solved in this chapter are listed in Figure 3.42.

Figure 3.42: Unsolved Bipartite Graphs with 7 Vertices and 7 Edges
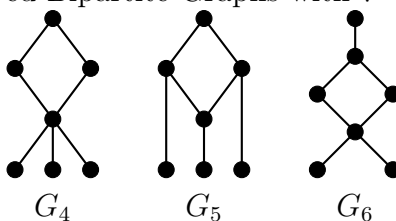
Proposition 3.19. *There is no design of order 7 for G_4, G_5, or G_6.*

Proof. Since each of these graphs contain 7 vertices, every vertex of K_7 must appear exactly once in each block of a design of order 7. Moreover, there must be exactly 6 total edges incident on any given vertex in the design because $d_{K_7}(v) = 6$ for each $v \in V(K_7)$.

In G_4 there is a vertex with degree 5. In a proposed G_4-design of order 7, a vertex that has degree 5 in one of the blocks cannot possibly appear in both of the remaining blocks because $\delta(G_4) = 1$. Thus there is no G_4-design of order 7.

In G_5 there are 3 vertices with degree 3. In a proposed G_5-design of order 7, there will be 9 total vertices with degree 3 among the blocks. Consequently, any labeling of these vertices using the vertices in K_7 will have at least one repetition. However any vertex that appears twice on these vertices of degree 3 cannot appear in all 3 blocks, because G_5 is connected and all of the edges incident on that vertex in K_7 have already been used in two of the blocks. Hence there is no G_5-design of order 7.
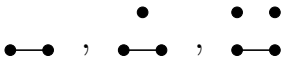
For a proposed G_6-design of order 7, in each of the blocks we must label the vertex with degree 3 with some $a \in V(K_7)$ and the vertex with degree 4 with some $b \in V(K_7)$ where a and b are distinct. Once this labeling is made, a must appear on vertices with degrees 2 and 1 in the remaining two blocks while b must appear on vertices both with degree 1 in the two remaining blocks. In G_6 no vertices with degree 1 have a neighbor with degree 2 or less which means the edge $\{a, b\} \in E(K_7)$ will not be an edge in any block of such a design. Thus a G_6-design of order 7 does not exist. $\qquad\square$

If a graph has at least 8 vertices and has 7 edges, then there is certainly no design of order 7 for that graph.

Since no design of order 7 exists for any of the graphs mentioned in this section, the solution to the intersection problem is not readily accessible using the techniques from earlier in this chapter. For the seven vertex graphs, the spectrum has not even been determined yet. However, the minimum possible value of n that satisfies the necessary condition of $n \equiv 0 \pmod 7$ for which a design of order n can exist for these graphs is 14. Solving the intersection problem for a design of order 14 for these graphs will be much harder because there will be 13 blocks in such a design which means 13 intersection values must be determined. Few cases where a graph has at least 8 vertices and has 7 edges have been investigated, but one such graph has been solved previously, that is, the star with seven edges S_7 (or $K_{1,7}$) in [5].

Chapter 4

Summary of Results and Discussion

In the preceding chapters, several results regarding intersections of G-designs were shown. Chapter 2 establishes that for a graph G and two arbitrary G-designs $\mathcal{B}_1$ and $\mathcal{B}_2$ of order n there is a G-design $\mathcal{B}_2'$ of order n such that $\mathcal{B}_2 \cong \mathcal{B}_2'$ and $\mathcal{B}_1 \cap \mathcal{B}_2' = \emptyset$ for sufficiently large n with the exception of the graphs ●—● , ●—● , ●—● , and possibly ●—●. In Chapter 1 a graph that satisfies this condition for an order n was called $\boldsymbol{T_3}$ for order n. Moreover, if a graph G is $\boldsymbol{T_3}$ for all n that are sufficiently large, it is called a Teirlinck graph.

For the first three of the excluded graphs it was shown by either a simple argument or a counterexample that they are not $\boldsymbol{T_3}$ for each n for which a nontrivial design of order n exists. However, for the last graph (the path of length 2) no such counterexamples could be found, but the graph does not fall under the purview of the results found in Chapter 2. For completeness, either a counterexample to show that P_2 is not $\boldsymbol{T_3}$ for sufficiently large n or a proof that it is a Teirlinck graph is still an open problem.

Luc Teirlinck's result for K_3-designs, which inspired the work presented in Chapter 2, shows that K_3 is $\boldsymbol{T_3}$ for each $n \geq 7$. His result gives a best case scenario, that is, $n = 7$ is the smallest possible value for which the graph can be $\boldsymbol{T_3}$. All other graphs besides the four listed above are shown to be Teirlinck graphs in Chapter 2, but the proof techniques used do not necessarily provide the smallest possible integers n for which the graphs are $\boldsymbol{T_3}$ for order n. Instead the values N_G for each graph G that can be found using the techniques of Chapter 2 are merely lower bounds for G to be $\boldsymbol{T_3}$ for orders $n \geq N_G$. This raises the question of what is the least upper bound on N_G for a graph G; that is, what is the smallest value of N_G for which G is $\boldsymbol{T_3}$ for each $n \geq N_G$? Furthermore, for what graphs, if any, does

this least upper bound match the lowest number that can be found by the techniques of Chapter 2?

In the end of Chapter 2, the notion of Teirlinck graphs is extended to determining for a graph G with $k \geq 2$ arbitrary G-designs of order n if k corresponding isomorphic designs which are pairwise disjoint can be found provided n is sufficiently large. What was found is that if G is nontrivial and $v_G \geq 5$, then for any $k \geq 2$ and a collection of k arbitrary G-designs of order n, there is a positive integer $N_{G,k}$ such that the k isomorphic designs that are pairwise disjoint could be found for each order $n \geq N_{G,k}$. The technique for proving this result is similar to that for showing the result for Teirlinck graphs in the preceding paragraphs. So a natural question for a graph G and an integer $k \geq 2$ is what is the smallest positive integer $N_{G,k}$ for which k pairwise disjoint G-designs of order n can be found that are correspondingly isomorphic to an arbitrary set of k G-designs for each order $n \geq N_{G,k}$. Additionally, it is of interest when the techniques for proving the general result yield these optimum values.

Lastly, the extension of the $\boldsymbol{T}_3$ condition to k arbitrary G-designs using the techniques presented in Chapter 2 does not allow for arbitrarily large k if $v_G = 4$. Instead there is an upper bound on k for guaranteeing to find the k pairwise disjoint designs using those techniques. However, other techniques not used in this dissertation might yield larger upper bounds on the values of k or even that there is no upper bound provided the order of the G-design is large enough. Moreover, the techniques of this dissertation tell us nothing about P_2-designs or K_3-designs. The proof of Teirlinck's result about K_3-designs is different from the proofs for G-designs in Chapter 2. Thus, it is possible that Teirlinck's result could be extended for $k > 2$ arbitrary K_3-designs for sufficiently large orders n.

4.1 Results for the Intersection Problem

In Chapter 3, a couple of key results are given for each of the graphs in the set

$$\mathcal{G} = \{D_4(7), D_6(7), R_4(1,2), SE_4(1,2), T(1,2), U(1,2), V_4(7)\}.$$

For each graph $G \in \mathcal{G}$, the Theorems 4.1 and 4.2 were proven in Chapter 3. A summary of these results is given in Table 4.1.

Theorem 4.1. *A G-design of order n exists if and only if $n \equiv 0$ or $1 \pmod{7}$.*
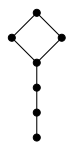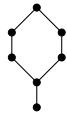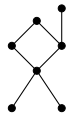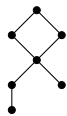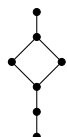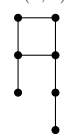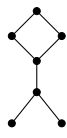
Theorem 4.2. *For each $n \in \mathrm{Spec}(G)$,*

$$I_G(n) = J_G(n).$$

The graphs in $\mathcal{G}$ are each bipartite, have 7 vertices, and 7 edges. Furthermore, for each of the graphs $G \in \mathcal{G}$, there exists a G-design of order 7, a G-design of order 8, and a G-design on $K_{7,7}$. These commonalities among the graphs in $\mathcal{G}$ lead to a common proof technique for establishing the spectrum of each of the graphs which is to combine the smaller designs into a larger design of order $7t$ or $7t + 1$ for positive integers t.

Likewise, for each graph $G \in \mathcal{G}$ the intersection problem for orders 7 and 8 are solved with results $I_G(7) = J_G(7)$ and $I_G(8) = J_G(8)$. Showing that $I_G(K_{7,7}) \supseteq \{0, 7\}$ (and that the intersection set contains one other value between 0 and 7 which is possibly different for each graph G) solves the intersection problem for the remaining $n \in \mathrm{Spec}(G)$ by combining appropriate smaller design into a pair of larger designs.

Table 4.1: Summary of Intersection Problem Results for 7-Edge Graphs

G	$\mathrm{Spec}(G)$	$I_G(n)$
$D_4(7)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$
$D_6(7)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$
$R_4(1,2)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$
$SE_4(1,2)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$
$T(1,2)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$
$U(1,2)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$
$V_4(7)$	$n \equiv 0$ or $1 \pmod 7$	$J_G(n)$

4.2 Further Inquiries into the Intersection Problem

The number of open intersection problems is too numerous to cover in this discussion, but some specific cases closely related to the results of this dissertation may be of interest.

Prior to the results found in this paper, the intersection problem has been solved for very few graphs with 7 or more edges. In fact, the only graphs with 7 or more edges for which the intersection problem has been solved are the cycles C_7, C_8, and C_9 and the star graphs S_n (i.e. $K_{1,n}$) where $n \geq 7$. As stated before, the results presented here extend this collection of solved intersection problems by 7 graphs with 7 vertices and 7 edges each of which is bipartite. The most accessible graphs using the techniques presented in Chapter 3 seem to be the remaining connected bipartite graphs with 7 edges and no more than 7 vertices. (The reason for wanting 7 vertices or fewer is the possibility of having a design of order 7.) Pictures of these graphs are given in Figure 4.1.

Figure 4.1: Unsolved Bipartite Graphs with 7 Edges and No More Than 7 Vertices

$G_1 \qquad G_2 \qquad G_3 \qquad G_4 \qquad G_5 \qquad G_6$
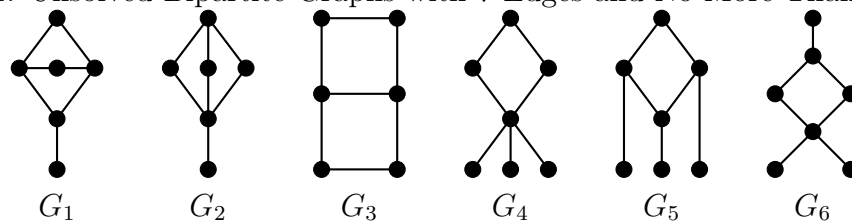
Unfortunately, as shown in [1], [7], and Proposition 3.19, there is no design of order 7 for any of these remaining graphs. Thus the smallest possible order n divisible by 7 for which there could be a design of order n for these graphs is $n = 14$. Finding the spectrum of these graphs may or may not be overly difficult, but the intersection problem certainly is if trying to employ the techniques of this dissertation. Since $n = 14$ is the smallest possible order divisible by 7 for designs of order n and the number of blocks in these designs of order 14, if they exist for these graphs, is 13, the intersection problem for even small designs proves quite difficult. Consequently, the complete solution to the intersection problem for any of these graphs (and other bipartite graphs with 7 edges) will prove much harder to find than

the solutions for the other graphs found in the results of Chapter 3 because of the lack of designs of order 7.

Indeed, finding solutions to some bipartite graphs with 9 or fewer vertices and 9 edges using similar techniques to those shown in Chapter 3 may actually be easier than finding the solutions to the remaining bipartite graphs with 7 edges due to the fact that small designs, that is, designs of order 9 and 10 may exist. (Solving the intersection problem for graphs with 8 edges using similar techniques is somewhat difficult because the smallest possible order for a design is 16.) If the designs of order 9 and 10 exist along with a design on $K_{9,9}$ and we have the necessary intersection values for these small designs, then the techniques used in Chapter 3 can be used to solve the intersection problem for these graphs quite easily.

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Appendix A

Intersections of Designs on $K_{7,7}$

In the solutions to the intersection problem for each graph G presented in Chapter 3, the existence of some values in $I_G(K_{7,7})$ needed to be shown. In order to avoid being bogged down in figures, some details of finding those values were omitted. The details are developed fully in the following examples.

Example A.1. Recall that the $D_4(7)$-design on $K_{7,7}$ from Example 3.4 is

$$\mathcal{B}_{7,7} = \{((i,2),(i+2,1),(i+1,2),(i,1);(i+3,2),(i+1,1),(i+5,2) \mid 0 \le i \le 6\},$$

where addition in the first coordinates is done modulo 7. The blocks of the design can be viewed in Figure A.1.

Figure A.1: $\mathcal{B}_{7,7}$, a $D_4(7)$-Design on $K_{7,7}$
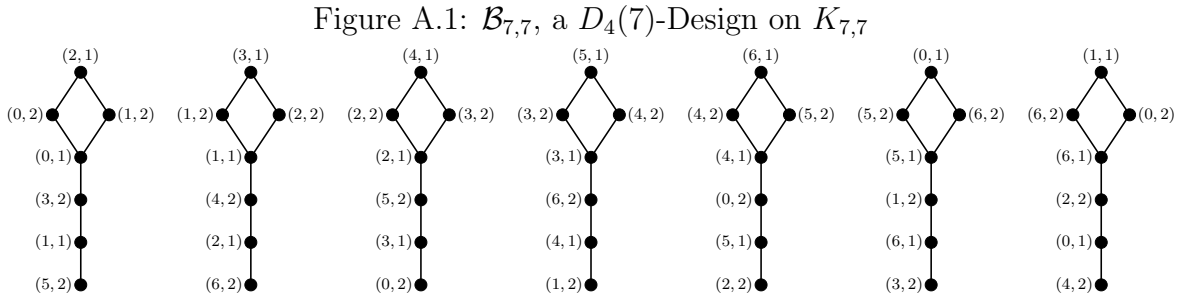
As claimed in Example 3.7, swapping vertices $(0,2)$ and $(1,2)$ in $\mathcal{B}_{7,7}$ creates a design $\mathcal{B}^1_{7,7}$ on $K_{7,7}$ that has exactly one block in common with $\mathcal{B}_{7,7}$. The blocks of $\mathcal{B}^1_{7,7}$ are listed in Figure A.2 with the shared block having solid edges and the other blocks having dashed edges.

Figure A.2: $\mathcal{B}_{7,7}^1$, a $D_4(7)$-Design on $K_{7,7}$ with Exactly One Block in Common with $\mathcal{B}_{7,7}$
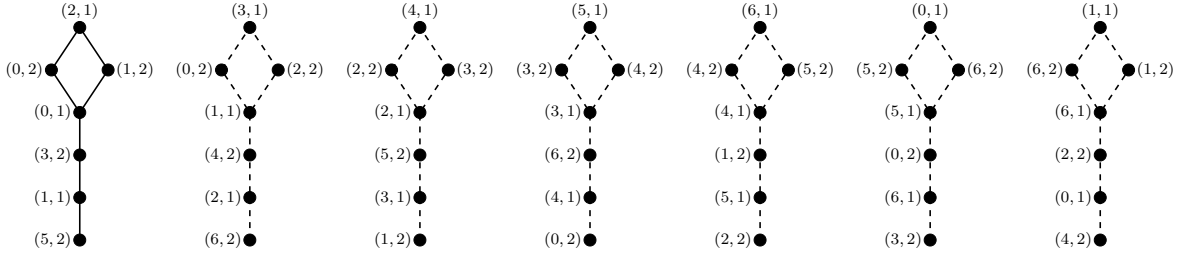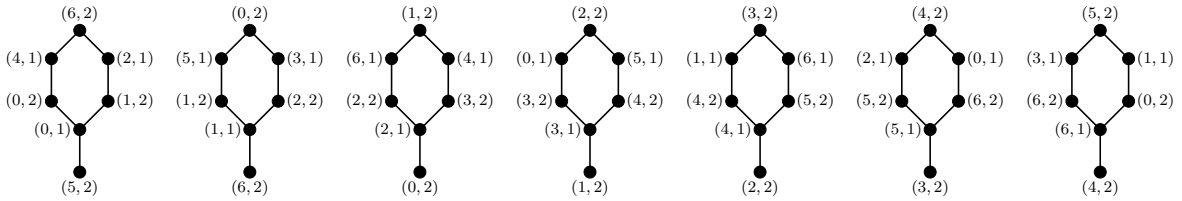

Example A.2. In Example 3.12, a $D_6(7)$-design on $K_{7,7}$ is found cyclically as

$$\mathcal{B}_{7,7} = \{((i,2),(i+4,1),(i+6,2),(i+2,1),(i+1,2),(i,1);(i+5,2)) \mid 0 \leq i \leq 6\}.$$

The listing of the blocks as graphs in Figure A.3 gives more insight into finding a second design that has exactly two blocks in common with $\mathcal{B}_{7,7}$.

Figure A.3: $\mathcal{B}_{7,7}$, a $D_6(7)$-Design on $K_{7,7}$

As stated in Example 3.15, transposing vertices $(2,2)$ and $(3,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. In Figure A.4 the blocks of $\mathcal{B}_{7,7}^2$ are shown with the common blocks having solid edges and the remaining blocks having dashed edges.
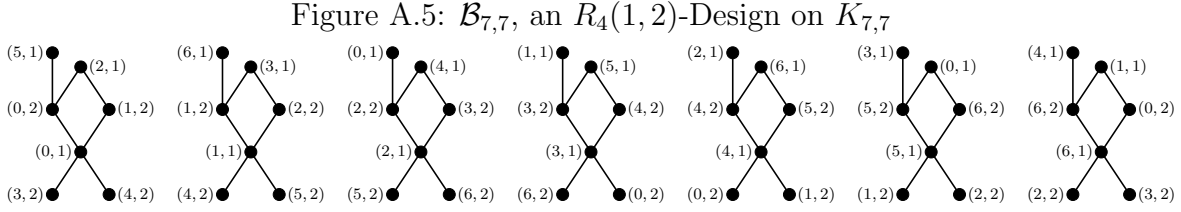
Figure A.4: $\mathcal{B}_{7,7}^2$, a $D_6(7)$-Design on $K_{7,7}$ with Exactly Two Blocks in Common with $\mathcal{B}_{7,7}$
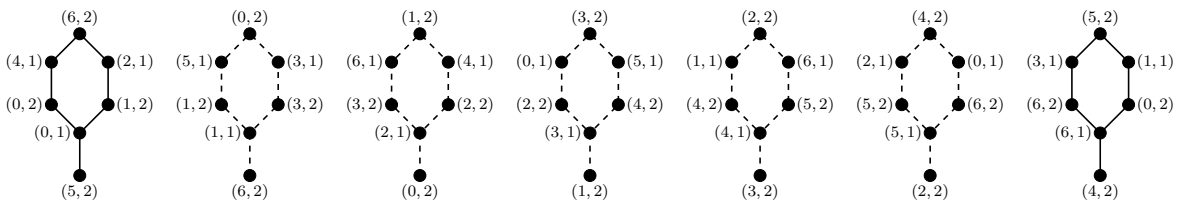
**Example A.3.** As a reminder, the $R_4(1, 2)$-design on $K_{7,7}$ from Example 3.20 is

$$\mathcal{B}_{7,7} = \{R((i + 5, 1); (i, 2), (i + 2, 1), (i + 1, 2), (i, 1); (i + 3, 2), (i + 4, 2)) \mid 0 \leq i \leq 6\}.$$

The blocks of this design are shown individually in Figure A.5.

Figure A.5: $\mathcal{B}_{7,7}$, an $R_4(1, 2)$-Design on $K_{7,7}$

As proposed in Example 3.23, switching the vertices $(3, 2)$ and $(4, 2)$ in each block of $\mathcal{B}_{7,7}$ creates a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. In Figure A.6, the blocks of $\mathcal{B}_{7,7}^2$ are shown with the blocks that are also in $\mathcal{B}_{7,7}$ having solid edges and those that are not in $\mathcal{B}_{7,7}$ having dashed edges.

Figure A.6: $\mathcal{B}_{7,7}^2$, an $R_4(1, 2)$-Design on $K_{7,7}$ with Exactly Two Blocks in Common with $\mathcal{B}_{7,7}$
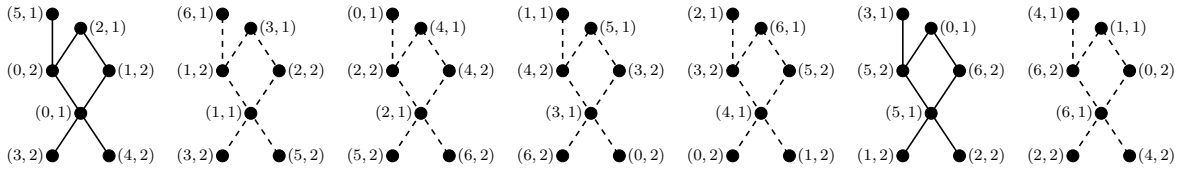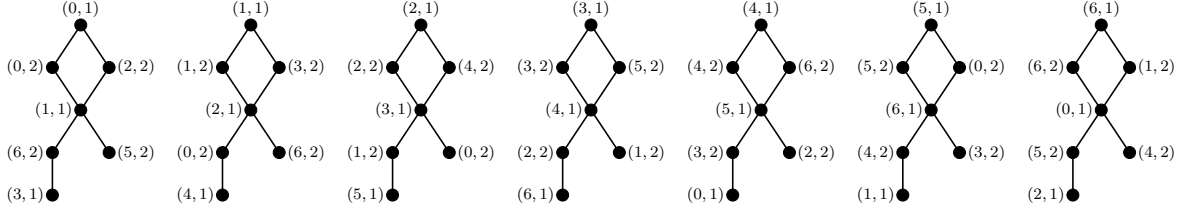

Example A.4. The set

$$\mathcal{B}_{7,7} = \{SE((i, 2), (i, 1), (i + 2, 2), (i + 1, 1); (i + 5, 2), (i + 6, 2), (i + 3, 1)) \mid 0 \leq i \leq 6\}$$

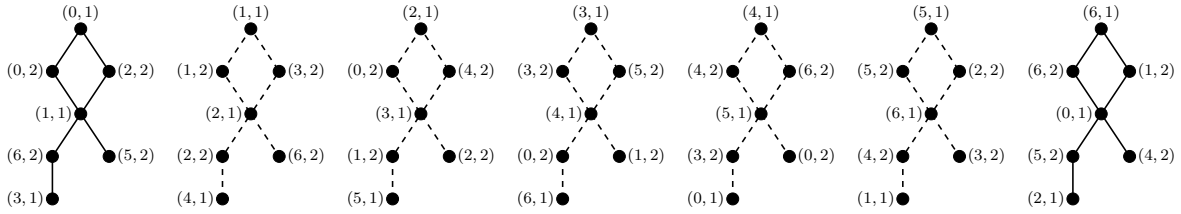where addition in the first coordinates is done modulo 7 is an $SE_4(1, 2)$-design on $K_{7,7}$. (See Example 3.28.) Figure shows each of the blocks in $\mathcal{B}_{7,7}$.

Figure A.7: $\mathcal{B}_{7,7}$, an $SE_4(1,2)$-Design on $K_{7,7}$

As stated in Example 3.31, swapping vertices $(0,2)$ and $(2,2)$ in $\mathcal{B}_{7,7}$ gives a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that shares exactly two blocks with $\mathcal{B}_{7,7}$. The blocks of $\mathcal{B}_{7,7}^2$ are listed in Figure A.8 with the shared blocks of the two designs having solid edges and the remaining blocks having dashed edges.

Figure A.8: $\mathcal{B}_{7,7}^2$, an $SE_4(1,2)$-Design on $K_{7,7}$ with Exactly Two Blocks in Common with $\mathcal{B}_{7,7}$
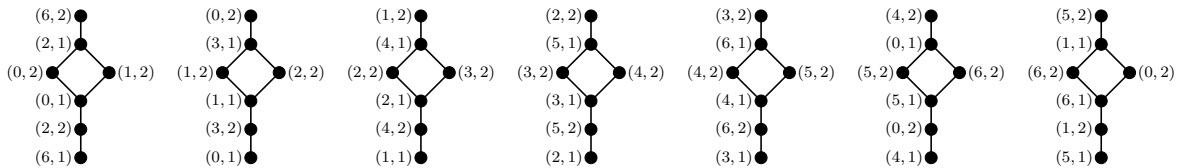

Example A.5. Recall that the $T(1,2)$-design on $K_{7,7}$ from Example 3.36 is

$$\mathcal{B}_{7,7} = \{T((i+6,2); (i+2,1), (i,2), (i,1), (i+1,2); (i+2,2), (i+6,1)) \mid 0 \leq i \leq 6\},$$

where addition in the first coordinates is done modulo 7. The blocks of the design can be viewed in Figure A.9.

Figure A.9: $\mathcal{B}_{7,7}$, a $T(1,2)$-Design on $K_{7,7}$

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As claimed in Example 3.39, switching vertices $(3,2)$ and $(4,2)$ in $\mathcal{B}_{7,7}$ creates a design $\mathcal{B}_{7,7}^3$ on $K_{7,7}$ that has exactly three blocks in common with $\mathcal{B}_{7,7}$. The blocks of $\mathcal{B}_{7,7}^3$ are listed in Figure A.10 with the shared blocks having solid edges and the other blocks having dashed edges.

Figure A.10: $\mathcal{B}_{7,7}^3$, a $T(1,2)$-Design on $K_{7,7}$ with Exactly Three Blocks in Common with $\mathcal{B}_{7,7}$
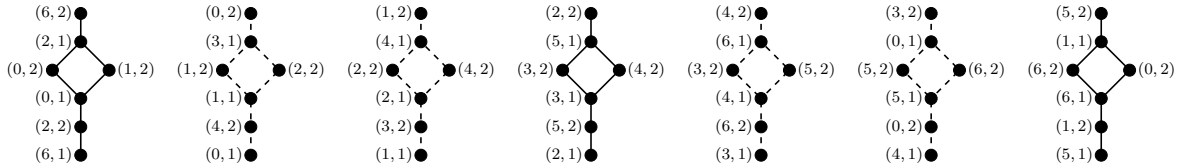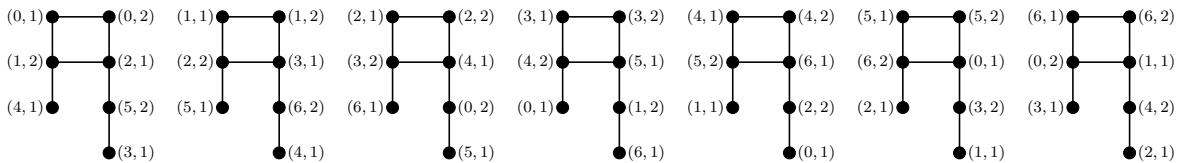

Example A.6. In Example 3.44, a $U(1,2)$-design on $K_{7,7}$ is found cyclically as

$$\mathcal{B}_{7,7} = \{U((i+4,1);(i+1,2),(i,1),(i,2),(i+2,1);(i+5,2),(i+3,1)) \mid 0 \le i \le 6\}.$$

The listing of the blocks as graphs in Figure A.11 gives more insight into finding a second design that has exactly two blocks in common with $\mathcal{B}_{7,7}$.

Figure A.11: $\mathcal{B}_{7,7}$, a $U(1,2)$-Design on $K_{7,7}$

As proposed in Example 3.47, transposing the vertices $(3,2)$ and $(4,2)$ in each block of $\mathcal{B}_{7,7}$ creates a design $\mathcal{B}_{7,7}^2$ on $K_{7,7}$ that has exactly two blocks in common with $\mathcal{B}_{7,7}$. In Figure A.12, the blocks of $\mathcal{B}_{7,7}^2$ are shown with the blocks that are also in $\mathcal{B}_{7,7}$ having solid edges and those that are not in $\mathcal{B}_{7,7}$ having dashed edges.

Figure A.12: $\mathcal{B}_{7,7}^2$, a $U(1,2)$-Design on $K_{7,7}$ with Exactly Two Blocks in Common with $\mathcal{B}_{7,7}$
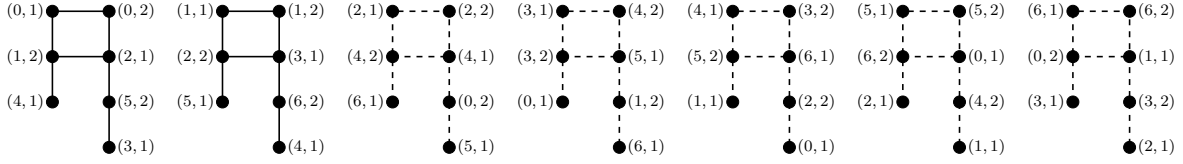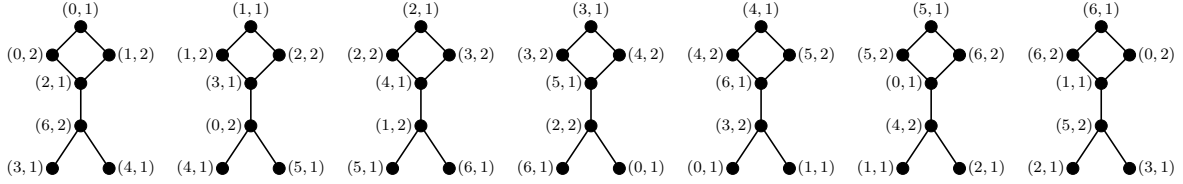

Example A.7. As a reminder, the $V_4(7)$-design on $K_{7,7}$ from Example 3.52 is

$$\mathcal{B}_{7,7} = \{V((i,2),(i,1),(i+1,2),(i+2,1);(i+6,2),(i+3,1),(i+4,1)) \mid 0 \le i \le 6\}.$$

The blocks of this design are shown individually in Figure A.13.

Figure A.13: $\mathcal{B}_{7,7}$, a $V_4(7)$-Design on $K_{7,7}$

As stated in Example 3.55, switching vertices $(4,2)$ and $(5,2)$ in $\mathcal{B}_{7,7}$ yields a design $\mathcal{B}_{7,7}^4$ on $K_{7,7}$ that shares exactly four blocks with $\mathcal{B}_{7,7}$. The blocks of $\mathcal{B}_{7,7}^4$ are listed in Figure A.14 with the shared blocks of the two designs having solid edges and the remaining blocks having dashed edges.

Figure A.14: $\mathcal{B}_{7,7}^4$, a $V_4(7)$-Design on $K_{7,7}$ with Exactly Four Blocks in Common with $\mathcal{B}_{7,7}$