Edge-Regular Graphs and Uniform Shared Neighborhood Structures

by

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Abstract

The definition of edge-regularity in graphs is a relaxation of the definition of strong regularity, so strongly regular graphs are edge-regular and, not surprisingly, the family of edge-regular graphs is much larger and more diverse than that of the strongly regular.

A shared neighborhood structure (SNS) in a graph is a subgraph induced by the intersection of the open neighbor sets of two adjacent vertices. If a SNS is the same for all adjacent vertices in an edge-regular graph, call the SNS a uniform shared neighborhood structure (USNS). USNS-forbidden graphs (graphs which cannot be a USNS of an edge-regular graph) and USNS in graph products of edge-regular graphs are examined.

Additionally, a few methods of constructing new graphs from old are of use. One of these is the unary “graph shadow” operation. Here, this operation is generalized, and then generalized again, and conditions are given under which application of the new operations to edge-regular graphs result in edge-regular graphs. Also, some attention to strongly regular graphs is given.

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Chapter 1

Introduction

Let $G = (V, E)$ be a finite, simple *graph* with vertex set $V = V(G)$ and edge set $E = E(G)$. If $uv \in E(G)$ for vertices $u, v \in V(G)$, then their adjacency is denoted $u \sim v$. The *degree* of a vertex is the number of edges it is incident to. Because G is simple, the degree of $v \in V(G)$ is also the number of vertices it is adjacent to. A graph G is *regular* if the degrees of the vertices in $V(G)$ are all the same. The *open neighborhood* of a vertex u in G, denoted $N_G(u)$, is the set of vertices u is adjacent to. If G is understood, this open neighborhood will be denoted $N(u)$.

A graph G is *edge-regular* if G is both regular and, for some λ, every pair of adjacent vertices in G have exactly λ common (or shared) neighbors. If G is edge-regular, we say $G \in ER(n, d, \lambda)$, where $|V(G)| = n$, G is regular of degree d, and $|N(u) \cap N(v)| = \lambda$ for all $uv \in E(G)$.

Further, a *strongly regular* graph is an edge-regular graph G in which for some $\mu \geq 0$, for all $x, y \in V(G), x \neq y$, such that $xy \notin E(G)$, $|N(x) \cap N(y)| = \mu$. The set of graphs in $ER(n, d, \lambda)$ satisfying the additional strong regularity requirement with parameter μ will be denoted $SR(n, d, \lambda, \mu)$.

An *induced subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, $E(H)$ contains all of the edges of G among the vertices of $V(H)$, and only those edges. The induced subgraph H of G is denoted as $G[V(H)]$. If $G[N_G(u) \cap N_G(v)] \cong H$ for all $u \sim v; u, v \in V(G)$, where $\cong$ denotes a graph isomorphism, then G has a *uniform shared neighborhood structure*, abbreviated *USNS*. For instance, letting K_n denote the complete graph on n vertices, $G = K_3 \in ER(3, 2, 1)$ has USNS K_1.

For graphs G and H, define $G + H$ to be the graph formed from G and H where $V(G + H) = V(G) \cup V(H)$ (such that $V(G)$ and $V(H)$ are disjoint) and $E(G+H) = E(G) \cup E(H)$. Further, for a graph G and positive integer m, define mG to be the union, or sum, of m disjoint copies of G. That is, $mG = G + G + \cdots + G$.

Edge-regular graphs do not need to have a USNS. If G is the Cartesian product of K_4 and $K_6 \setminus \{\text{a perfect matching in } K_6\}$, $G \in ER(24, 7, 2)$ has two different shared neighborhood structures (SNS): K_2 and $2K_1$. Also, a SNS for one pair of adjacent vertices may also be the SNS for a different pair of adjacent vertices. Suppose G is $K_6 \setminus \{\text{a perfect matching in } K_6\}$ as in Fig. 1.1. Then $G \in ER(6, 4, 2)$ has $2K_1$ as a USNS, and each of the three $2K_1$'s in G is the SNS of two disjoint pairs of adjacent vertices.

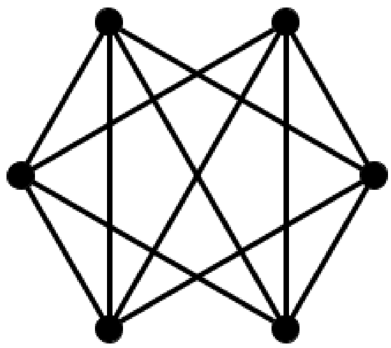

Figure 1.1: K_6 with a perfect matching removed

Edge-regular graphs with fixed λ have been studied by Glorioso [1] (when $\lambda = 2$), Bragan [2] (when $\lambda = 1$), and Guest et al. [3] (when $\lambda = 1$ and some cases when $\lambda > 1$). All of these publications include constructions for edge-regular graphs with the given λ value, with a particular emphasis on RCA graphs, which are edge-regular graphs in which every maximal clique is maximum. Further, edge-regular graphs satisfying $d - \lambda \leq 3$ have been fully characterized by Johnson, Myrvold, and Roblee [4].

The research presented in Chapter 2 will pertain to the structure of edge-regular graphs, akin to the research presented in [5], which explores constructions of a specific type of edge-regular graph, a *Neumaier* graph. There is an emphasis on families of graphs that cannot be a USNS in any edge-regular graph (USNS-forbidden graphs), as well as corresponding constructions of graphs in these families. This naturally leads to a question that this research works toward: what graphs are USNS-forbidden, and does there exist a characterization of these graphs?

Another topic in interest of edge-regular graphs is using graph products to produce new edge-regular graphs from old. Glorioso [1] characterized for which edge-regular graphs the Cartesian, Tensor, Strong, and Lexicographic products of edge-regular graphs is edge-regular. The author expanded upon the Cartesian and Tensor products in [6] to characterize preservation by these products of edge-regular graphs with a uniform shared neighborhood structure (USNS). We explore these extended results of the Cartesian and Tensor products in Chapter 2.

A different type of graph operation, the *shadow* of a graph, is formally defined and partially studied by the author in [6] and by Asmiati et al. in [7]. In Chapter 3, the goal is to generalize the definition of the shadow of a graph as a graph operation, and generalized again, and to determine when these generalized operations preserve regularity, edge-regularity, and strong regularity of finite, simple graphs. Special attention is given to cycle graphs; there is a characterization for when the shadow graph of a cycle graph is edge-regular.

A brief summary of open-ended questions and possibilities for further research are provided in Chapter 4. We begin by building up the class of USNS-forbidden graphs of edge-regular graphs.

Chapter 2

USNS of edge-regular graphs

2.1 USNS-forbidden graphs

There are families of graphs that cannot be a USNS in any edge-regular graph; call these *USNS-forbidden* graphs. Our results about such graphs will be proved by contradiction. For a graph G and $u, v \in V(G)$, let $A(u, v)$ denote the set of vertices in G that are adjacent to u but not to v, and let $B(u, v)$ denote the set of vertices in G that are adjacent to v but not to u. Finally, let $X(u, v)$ denote the set of vertices in G that are adjacent neither to u nor v.

Path graphs are one such family that provide insight into USNS-forbidden graphs. However, as it will be mentioned after the proof of Theorem 2.4, it is not settled if the entire family of path graphs are USNS-forbidden. We start with a path on 3 vertices.

2.1.1 P_3

Let P_m be the path graph on m vertices. Note that we begin by proving a path on 3 vertices is USNS-forbidden, as a path on either 1 or 2 vertices is possible as a USNS.

Theorem 2.1. *If $G \in ER(n, d, 3)$ with a USNS, then the USNS $\ncong P_3$.*

Proof. By way of contradiction, let $u \sim v$, and let $N(u) \cap N(v) = \{w_1, w_2, w_3\}$, where $G[N(u) \cap N(v)] \cong P_3$. Without loss of generality, let $w_1 \sim w_2 \sim w_3$ and $w_1 \nsim w_3$. Then as $w_1 \sim w_2$, $G[N(w_1) \cap N(w_2)] \cong P_3$. As two of w_1 and w_2's common neighbors are u and v, there must exist a third vertex, say z, such that $N(w_1) \cap N(w_2) = \{u, v, z\}$ and $G[N(w_1) \cap N(w_2)] \cong P_3$.

Without loss of generality, suppose $z \sim u$. Then $\{w_1, v, w_3, z\} \subseteq N(u) \cap N(w_2)$, contradicting $\lambda = 3$. Thus, $G[N(w_1) \cap N(w_2)] \not\cong P_3$. □

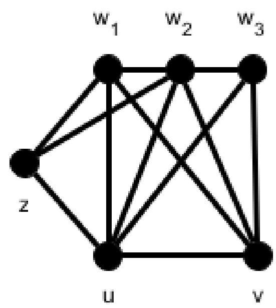

Figure 2.1: Construction of the proof of Theorem 2.1

It should be noted that Theorem 2.1 is a special case of Theorem 2.5, found later in the paper. As an aside, it is possible for a different graph on 3 vertices to be a USNS: $K_2 + K_1$, which is the USNS of the complement of the Petersen graph.

Naturally, there are a variety of graphs to sum with P_3 to see if it is a possible USNS for some edge-regular graph. There is a partial result for $P_3 + H$ where H is an arbitrary graph.

Theorem 2.2. *Suppose that G is edge-regular with USNS $P_3 + H$ for some graph H. Then H has at least one edge. Further: if, for some $u, v \in V(G)$, $u \sim v$, and $G[N(u) \cap N(v)]$ contains a P_3 component with vertices $w_1 \sim w_2 \sim w_3$, and if the edge uv is an edge of a P_3 component of $G[N(w_1) \cap N(w_2)]$, then H has a P_4 subgraph and a K_2 component.*

Proof. Suppose $u, v \in V(G)$, $u \sim v$, and $w_1w_2w_3$ is a P_3 component of $G[N(u) \cap N(v)]$. Then uv is an edge of $G[N(w_1) \cap N(w_2)] \cong P_3 + H$. If uv is not an edge of a P_3 component of $P_3 + H$ then $uv \in E(H)$. Therefore, the theorem will be proven if we prove that H contains a P_4 and a K_2 component, under the assumption that uv is an edge of a P_3 component of $P_3 + H = G[N(w_1) \cap N(w_2)]$.

The third vertex of P_3 in $G[N(w_1) \cap N(w_2)]$ must be an element of $A(u, v)$ or $B(u, v)$. Without loss of generality, suppose the remaining vertex is $a_1 \in A(u, v)$ (that is, a_1 is adjacent to u but not to v). Then every vertex of H in $G[N(w_1) \cap N(w_2)]$ must be in $X(u, v)$, as any vertex in $A(u, v)$ or $B(u, v)$ would have an adjacency to u or v, respectively. Thus, $N(w_1) \cap N(w_2) = \{a_1, u, v, x_1, \ldots, x_{|H|}\}$.

Consider the adjacent vertices u and w_2. Notice that $\{a_1, w_1, v, w_3\} \subseteq N(u) \cap N(w_2)$, and $G[a_1, w_1, v, w_3]$ is connected. As these four vertices are part of the same component in $N(w_2) \cap N(u)$, then they cannot contain the P_3 component and thus are contained in the H component so H must contain a P_4.

Now consider the adjacent vertices v and w_1. As $\{w_2, u\} \subseteq N(v) \cap N(w_1)$ and $w_2 \sim u$, then w_2 and u are in the same component of $G[N(v) \cap N(w_1)]$. The only other vertices in $N(v) \cap N(w_1)$ are in $B(u, v)$, and none of these can be adjacent to u, nor to w_2, since $N(w_1) \cap N(w_2)$ has no elements in $B(u, v)$ and no vertices in $B(u, v)$ are adjacent to u.

Consequently, the single edge uw_2 is a component of $G[N(v) \cap N(w_1)] \cong P_3 + H$, and is obviously not a P_3. Therefore H has a K_2 component. □

A natural corollary follows from the above theorem to forbid a union of isolated vertices with P_3.

Corollary 2.1. *If $G \in ER(n, d, 3 + \ell)$, $l \geq 1$, with a USNS, then the USNS $\ncong P_3 + \ell K_1$.*

Corollary 2.2. *Suppose m is a positive integer. Then mP_3 is USNS-forbidden.*

Proof. For $m = 1$, see Theorem 2.1. Assume that $m > 1$. If G is edge-regular with USNS mP_3, then because every $uv \in E(G)$ is in a component of $G[N(x) \cap N(y)]$ for any $x \sim y$ in $N(u) \cap N(v)$, every $uv \in E(G)$ is in a P_3 component of two adjacent vertices in a P_3 component of $G[N(u) \cap N(v)]$. Therefore, by Theorem 2.2, $G[N(u) \cap N(v)] \cong P_3 + (m - 1)P_3$ contains a P_4 subgraph. Obviously, this is impossible. □

2.1.2 P_4

Since P_3 is a forbidden USNS, it is natural to ask if longer paths are also forbidden. The theorem below asserts that P_4, like P_3, is USNS-forbidden.

Theorem 2.3. *If $G \in ER(n, d, 4)$ with a USNS, then the USNS $\ncong P_4$.*

Proof. Suppose for contradiction $\exists\, G \in ER(n, d, 4)$ with USNS $\cong P_4$. Let $u \sim v$, and let $N(u) \cap N(v) = \{w_1, w_2, w_3, w_4\}$, where $G[N(u) \cap N(v)] \cong P_4$ with endpoints w_1 and w_4 and $w_1 \sim w_2$. $G[N(w_1) \cap N(w_2)] \cong P_4$, as G has a P_4 USNS.

Case 1. $N(w_1) \cap N(w_2) = \{a_1, u, v, b_1\}$, such that $G[N(w_1) \cap N(w_2)] \cong P_4$ having endpoints a_1 and b_1, with $a_1 \in A(u,v)$ and $b_1 \in B(u,v)$. See Fig. 2.2 for reference.

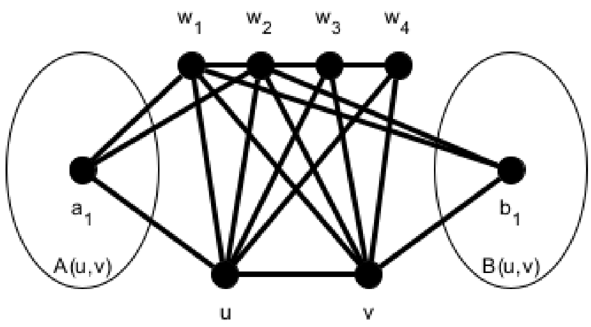

Figure 2.2: Beginning of case 1 in the proof of Theorem 2.3

Consider the vertices u and w_1, which are adjacent by assumption. As vertices u and w_1 have common neighbors a_1, w_2, and v, then there must exist another vertex in their shared neighborhood adjacent either to a_1 or v. As w_1 is not adjacent to w_3 and w_4, and u is only adjacent to v, the w_i vertices, and vertices in $A(u,v)$, then the 4th vertex in this common neighborhood must be some $a_2 \in A(u,v)$. So $a_2 \sim a_1$, and $G[N(u) \cap N(w_1)] \cong P_4$ with endpoints a_2 and v.

Now consider adjacent vertices u and w_2. $N(u) \cap N(w_2) = \{a_1, w_1, v, w_3\}$ is completely determined from previous assumptions. As $G[N(u) \cap N(w_2)] \cong P_4$, then this must have endpoints w_3 and a_1, so $w_3 \nsim a_1$.

Now consider the adjacent vertices v and w_1. Then $N(v) \cap N(w_1) = \{b_2, b_1, w_2, u\}$, where $b_1, b_2 \in B(u,v)$. Using similar logic to how $N(u) \cap N(w_1)$ was constructed, then we conclude that $G[N(v) \cap N(w_1)] \cong P_4$ having endpoints b_2 and u, with $b_2 \nsim w_2$ and $b_2 \sim b_1$.

Now consider the adjacent vertices v and w_2. $N(v) \cap N(w_2) = \{b_1, w_1, u, w_3\}$ is completely determined from previous assumptions. As $G[N(v) \cap N(w_2)] \cong P_4$, then this must have endpoints w_3 and b_1, so $w_3 \nsim b_1$.

Lastly, consider the adjacent vertices w_2 and w_3. As $\{u, v\} \in N(w_2) \cap N(w_3)$, $\exists\ z \in \{N(w_2) \cap N(w_3)\} \setminus \{u, v\}$ such that $z \in A(u,v)$ or $z \in B(u,v)$. As $w_2 \nsim a_2$ and $w_2 \nsim b_2$ (from $N(u) \cap N(w_2)$ and $N(v) \cap N(w_2)$, respectively), then $z \neq a_2$ and $z \neq b_2$. As $w_3 \nsim a_1$ and $w_3 \nsim b_1$ (implied from $N(u) \cap N(w_2)$ and $N(v) \cap N(w_2)$, respectively), then $z \neq a_1$ and $z \neq b_1$.

Without loss of generality, say $z \in A(u,v)$. Then $N(u) \cap N(w_2)$ contains $z \in A(u,v) \setminus \{a_1\}$, a contradiction. Thus, $N(w_1) \cap N(w_2) \neq \{a_1, u, v, b_1\}$.

Case 2. $N(w_1) \cap N(w_2) = \{v, u, a_1, x_1\}$, where $a_1 \in A(u,v)$ and $x_1 \in X(u,v)$. By assumption, $u \sim a_1$, $u \nsim x_1$, and $v \nsim x_1$, so v and x_1 are endpoints of $G[N(w_1) \cap N(w_2)]$.

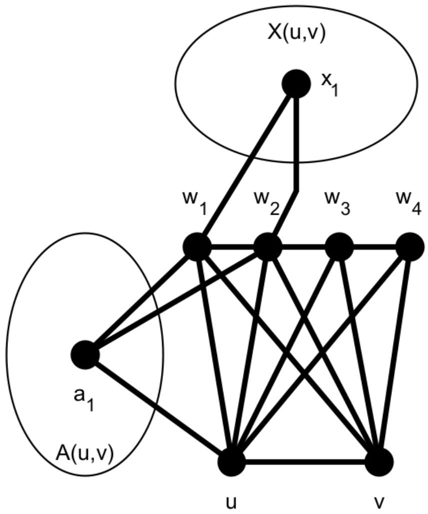

Figure 2.3: Beginning of case 2 in the proof of Theorem 2.3

Consider adjacent vertices w_1 and v. Then $N(w_1) \cap N(v) = \{u, w_2, b_2, b_3\}$ for some $b_2, b_3 \in B(u,v)$. This follows from the facts that v has no neighbors in $A(u,v) \cup X(u,v)$ and w_1 is adjacent to no w_j; $j > 2$. Therefore, the two vertices in $N(w_1) \cap N(v)$ other than u and w_2 must be in $B(u,v)$. By assumption, u is not adjacent to any vertex in $B(u,v)$, so w_2 must be adjacent to one of $\{b_2, b_3\}$. Without loss of generality, $w_2 \sim b_2$. However, this implies $N(w_1) \cap N(w_2)$ contains b_2, a contradiction. So $N(w_1) \cap N(w_2) \neq \{v, u, a_1, x_1\}$.

Case 3. $N(w_1) \cap N(w_2) = \{a_2, a_1, u, v\}$, where $a_1, a_2 \in A(u,v)$. By assumption, $u \sim a_2$ and $u \sim a_1$, so u is not an endpoint of $G[N(w_1) \cap N(w_2)]$. v must be an endpoint, as v is only adjacent to u. Without loss of generality, say a_2 is an endpoint and a_1 is not an endpoint in $G[N(w_1) \cap N(w_2)]$. As $a_2 \sim u$, then $G[N(w_1 \cap N(w_2)] \ncong P_4$, a contradiction. So $N(w_1) \cap N(w_2) \neq \{a_2, a_1, u, v\}$.

This exhausts all possibilities for $N(w_1) \cap N(w_2)$, so G cannot have P_4 as a USNS. $\square$

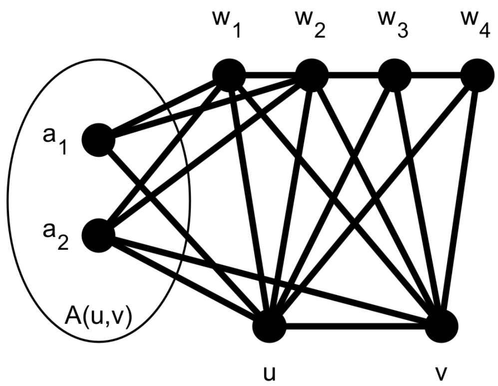

Figure 2.4: Beginning of case 3 in the proof of Theorem 2.3

2.1.3 P_5

Theorem 2.4. *Let $G \in ER(n, d, \lambda)$ with a P_λ USNS for $\lambda \geq 5$, and let $u \sim v$ in G with $N(u) \cap N(v) = \{w_1, w_2, \ldots, w_\lambda\}$, where w_1 is an endpoint of $G[N(u) \cap N(v)]$. If $w_1 \sim w_2$, then $N(w_1) \cap N(w_2)$ contains exactly one vertex from $N(u) \setminus (N(u) \cap N(v))$ and exactly one vertex from $N(v) \setminus (N(u) \cap N(v))$.*

Proof. **Case 1.** We first assume that $N(w_1) \cap N(w_2)$ contains no vertex from $A(u, v)$. So $N(w_1) \cap N(w_2)$ contains u, v, a vertex in $B(u, v)$, and $\lambda - 3$ vertices in $X(u, v)$.

Consider adjacent vertices u and w_1. Then $N(u) \cap N(w_1)$ contains v and w_2. But as u is not adjacent to any vertex in the set $B(u, v)$ nor $X(u, v)$, the remainder of the vertices in this common neighborhood must be elements of $A(u, v)$. Yet there is no adjacency from these vertices in $A(u, v)$ to v. If any of these vertices in $A(u, v)$ were to be adjacent to w_2, then $N(w_1) \cap N(w_2)$ would contain a vertex from $A(u, v)$, contradicting our case assumption. As $\lambda \geq 5$, then $G[N(u) \cap N(w_1)] \ncong P_\lambda$, a contradiction.

Case 2. We assume that $N(w_1) \cap N(w_2)$ contains more than one vertex from $A(u, v)$, say m vertices from $A(u, v)$. Then u in $G[N(w_1) \cap N(w_2)]$ has degree $m + 1$. As $m \geq 2$, then $G[N(w_1) \cap N(w_2)] \ncong P_\lambda$, a contradiction.

Thus, $N(w_1) \cap N(w_2)$ must contain exactly one vertex from $A(u, v)$ and exactly one vertex from $B(u, v)$. $\square$

While paths are far from completely decided upon as a family of USNS-forbidden graphs, there are other families of graphs that are. The following theorems tackle a few of these families, namely the family of complete bipartite graphs of different partition sizes, star graphs, and wheel graphs. We begin with bipartite graphs.

2.1.4 Bipartite and Stars

Theorem 2.5. *If $G \in ER(n, d, m_1 + m_2)$ with a USNS, and $m_1 \neq m_2$, then the USNS $\ncong K_{m_1,m_2}$.*

Proof. Let $u \sim v$, and let $N(u) \cap N(v) = \{w_1, w_2, \ldots, w_{m_1}, z_1, z_2, \ldots, z_{m_2}\}$, where $G[w_1, w_2, \ldots, w_{m_1}, z_1, z_2, \ldots, z_{m_2}] \cong K_{m_1,m_2}$ with $w_1, \ldots, w_{m_1}$ in one part and $z_1, \ldots, z_{m_2}$ in the other part.

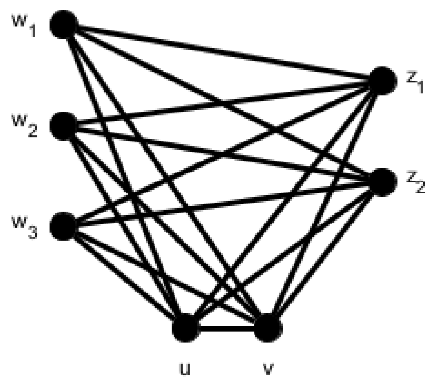

Figure 2.5: A $K_{3,2}$ shared neighborhood of vertices u and v

Consider the adjacent vertices w_1 and z_1. Then without loss of generality, $N(w_1) \cap N(z_1) = \{u, v, a_1, \ldots, a_{m_2-1}, b_1, \ldots, b_{m_1-1}\}$, where $a_1, \ldots, a_{m_2-1} \in A(u, v)$ and $b_1, \ldots, b_{m_1-1} \in B(u, v)$. So $G[N(w_1) \cap N(z_1)] \cong K_{m_1,m_2}$, where $v, a_1, \ldots, a_{m_2-1}$ are in one part and $u, b_1, \ldots, b_{m_1-1}$ are in the other part.

Now consider the adjacent vertices u and w_1. Then by previous assumptions, $N(u) \cap N(w_1)$ contains $\{z_1, \ldots, z_{m_2}, a_1, \ldots, a_{m_2-1}, v\}$. Further, as $\lambda = m_1 + m_2$ by assumption and $|N(u) \cap N(w_1)| \geq 2m_2$, then $m_2 \leq m_1$. By symmetry, $m_1 \leq m_2$, so $m_1 = m_2$. Thus, K_{m_1,m_2} is only possible as a USNS when $m_1 = m_2$. □

A graph such as the one in Fig. 2.5 is USNS-forbidden, where $m_1 = 3$ and $m_2 = 2$. What immediately follows from Theorem 2.5 is a fact about the *star graph* S_ℓ, which is a graph with one central vertex and $\ell - 1$ vertices adjacent to it, but not to each other, see Figure 2.6 for an example.

Corollary 2.3. *If $G \in ER(n, d, \ell)$ with a USNS, then for all $\ell \geq 3$, the USNS $\ncong S_\ell$.*

Proof. Let $m_1 = 1$ and $m_2 = \ell - 1$. Then $K_{m_1,m_2} \cong S_\ell$. So S_ℓ cannot be a USNS by Theorem 2.5. □

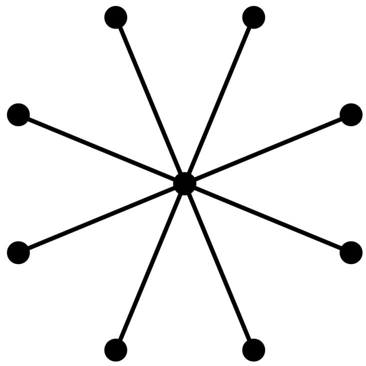

Figure 2.6: S_9, a star graph on 9 vertices

As noted earlier, Theorem 2.5 generalizes Theorem 2.1, as $P_3 \cong K_{1,2}$.

This is not to suggest that complete bipartite graphs with equal part sizes are also USNS-forbidden. On the contrary, consider K_4, which has a $K_2 \cong K_{1,1}$ USNS.

2.1.5 Wheels

In the following result, let the *wheel graph* W_m to be a connected graph on $m+1$ vertices, such that m vertices induce a cycle, and the $(m+1)^{st}$ vertex is adjacent to all vertices of the cycle. See Figure 2.7 for an example.

Theorem 2.6. *If $G \in ER(n, d, m + 1)$, $m \geq 4$, has a USNS, then the USNS $\ncong W_m$.*

Proof. Suppose for contradiction $u \sim v$ such that $G[N(u) \cap N(v)] \cong W_m$ consisting of vertices $w_1, \ldots, w_{m+1}$ such that $w_2, \ldots, w_{m+1}$ are the vertices in the cycle and w_1 is adjacent to the vertices in the cycle.

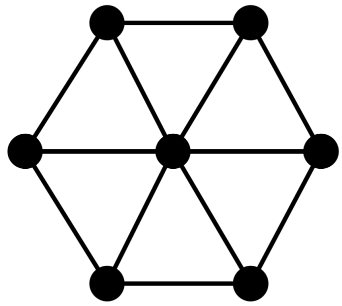

Figure 2.7: W_6, a wheel graph on 7 vertices

Consider adjacent vertices u and w_1. $N(u) \cap N(w_1) = \{w_2, w_3, \ldots, w_{m+1}, v\}$. So w_1 is not adjacent to any vertex in $A(u, v)$.

Similarly, w_1 is not adjacent to any vertex in $B(u, v)$.

As $G[u, v, w_1] \cong K_3$ and $N(w_2) \cap N(w_3)$ contain u, v, w_1, then this K_3 is an induced subgraph of $G[N(w_2) \cap N(w_3)]$. As $m \geq 4$, one of u, v, w_1 must be the center of this wheel.

If u is the center, the other $m - 2$ vertices in $N(w_2) \cap N(w_3)$ besides u, v, w_1 must be in $A(u, v)$, so $w_1 \sim a_i$ for some $a_i \in A(u, v)$, a contradiction.

If v is the center, the other $m - 2$ vertices in $N(w_2) \cap N(w_3)$ besides u, v, w_1 must be in $B(u, v)$, so $w_1 \sim b_i$ for some $b_i \in B(u, v)$, a contradiction.

If w_1 is the center, then as $m - 2 > 0$, u and v are adjacent vertices on a cycle C_m in $G[N(w_2) \cap N(w_3)]$ of length $m \geq 4$ which cannot contain any w_j, $j > 3$ (because $w_2 \not\sim w_j$). Then there is a P_4 $auvb$ on C_m with $a \in A(u, v)$, $b \in B(u, v)$. But then w_1, as the center of the wheel, is adjacent in G to both a and b, whereas either adjacency contradicts a previous inference.

Thus, W_m is not a possible USNS when $m \geq 4$. $\square$

2.2 Constructions of $ER(n, d, \lambda)$ with USNS

Now, we shift focus towards using graph products to construct edge-regular graphs with a USNS. We begin with the Cartesian product.

2.2.1 Cartesian product

Given graphs G_1 and G_2, the *Cartesian product* of G_1 and G_2 is denoted $G_1 \square G_2$. The vertex set is defined by $V(G_1 \square G_2) = V(G_1) \times V(G_2)$. The edge set is defined by, given two vertices (u, u') and $(v, v') \in V(G_1 \square G_2)$, $(u, u') \sim (v, v')$ if and only if either $u = v$ and $u' \sim v'$ (in G_2) or $u \sim v$ (in G_1) and $u' = v'$. Refer to Figure 2.8 for an example.

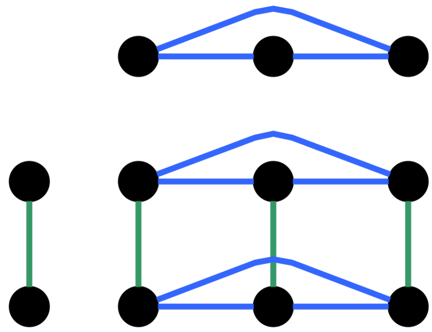

Figure 2.8: $K_2 \square K_3$, the Cartesian product of K_2 (left) and K_3 (top)

It was shown in [1] that if $G_1 \in ER(n_1, d_1, \lambda)$ and $G_2 \in ER(n_2, d_2, \lambda)$, then $G_1 \square G_2 \in ER(n_1 n_2, d_1 + d_2, \lambda)$. However, it is rare that the Cartesian product of two edge-regular graphs that each have a USNS will have a USNS.

Theorem 2.7. *Let $n_1, n_2 \geq 1$ and $d_1, d_2 \geq 0$. If $G_1 \in ER(n_1, d_1, \lambda)$ with a USNS $\cong X$ and $G_2 \in ER(n_2, d_2, \lambda)$ with a USNS $\cong Y$, then $G_1 \square G_2 \in ER(n_1 n_2, d_1 + d_2, \lambda)$ has a USNS if and only if $X \cong Y$, in which case the USNS of $G_1 \square G_2$ is $X (\cong Y)$.*

Proof. Let $G_1 \in ER(n_1, d_1, \lambda)$ with USNS $\cong X$ and $G_2 \in ER(n_2, d_2, \lambda)$ with USNS $\cong Y$.

We assume that $G_1 \square G_2 \in ER(n_1 n_2, d_1 + d_2, \lambda)$ has a USNS. Suppose $(u, v) \sim (x, y)$ in $G_1 \square G_2$. Then by the definition of the Cartesian product, either $u = x$ in G_1 and $v \sim y$ in G_2 or $u \sim x$ in G_1 and $v = y$ in G_2.

If $u = x$ in G_1 and $v \sim y$ in G_2, then $N_{G_1 \square G_2}(u, v) \cap N_{G_1 \square G_2}(x, y) = \{(u, z) | z \in N_{G_2}(v) \cap N_{G_2}(y)\}$ which induces, in $G_1 \square G_2$, a graph isomorphic to Y.

Similarly, if $u \sim x$ in G_1 and $v = y$ in G_2, then $N_{G_1 \square G_2}((u, v), (x, y))$ induces, in $G_1 \square G_2$, a graph isomorphic to X. However, $G_1 \square G_2$ has a USNS, by assumption. Thus, $X \cong Y$.

In the other direction, we assume that $X \simeq Y$. Then the argument above about SNS's in $G_1 \square G_2$ shows that $G_1 \square G_2$ has $X \simeq Y$ as USNS. $\square$

2.2.2 Tensor product

The *tensor product* of G_1 and G_2 is denoted $G_1 \otimes G_2$. The vertex set is $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$. The edge set is defined by, given two vertices (u, u') and $(v, v') \in V(G_1 \otimes G_2)$, $(u, u') \sim (v, v')$ if and only if $u \sim v$ in G_1 and $u' \sim v'$ in G_2. By previous work in [1], if $G_1 \in ER(n_1, d_1, \lambda_1)$ and $G_2 \in ER(n_2, d_2, \lambda_2)$, then $G_1 \otimes G_2 \in ER(n_1 n_2, d_1 d_2, \lambda_1 \lambda_2)$. The following theorem extends the work in [1] to include the preservation and structure of the USNS in $G_1 \otimes G_2$. Refer to Figure 2.9 for an example.

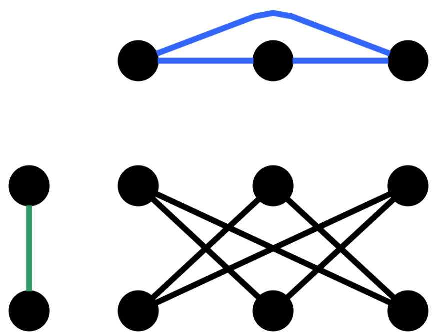

Figure 2.9: $K_2 \otimes K_3$, the tensor product of K_2 (left) and K_3 (top)

Theorem 2.8. *If $G_1 \in ER(n_1, d_1, \lambda_1)$ with a USNS $\cong H_1$ and $G_2 \in ER(n_2, d_2, \lambda_2)$ with a USNS $\cong H_2$, then $G_1 \otimes G_2 \in ER(n_1 n_2, d_1 d_2, \lambda_1 \lambda_2)$ with a USNS $\cong H_1 \otimes H_2$.*

Proof. Suppose that $(u, v) \sim (x, y)$ in $G_1 \otimes G_2$. Then $(s, t) \in N_{G_1 \otimes G_2}(u, v) \cap N_{G_1 \otimes G_2}(x, y)$ if and only if $u \sim s$, $x \sim s$ in G_1 and $v \sim t$, $y \sim t$ in G_2. Thus, $N_{G_1 \otimes G_2}(u, v) \cap N_{G_1 \otimes G_2}(x, y) = (N_{G_1}(u) \cap N_{G_1}(x)) \times (N_{G_1}(v) \cap N_{G_1}(y)) = V(H_1) \times V(H_2)$, and this set induces $H_1 \otimes H_2$ in $G_1 \otimes G_2$. $\square$

For example, $K_n \otimes K_m \cong K_{m,m,\ldots,m} \setminus \{(n-1)\text{-factor edges}\}$, an n-partite graph with uniform part size m where the edges of a $(n-1)$-factor are the column edges when the vertices are arranged in a $n \times m$ matrix. Therefore, the USNS of $(K_n \otimes K_m) \cong K_{n-2} \otimes K_{m-2} \cong K_{m-2,m-2,\ldots,m-2} \setminus \{(n-3)\text{-factor edges}\}$, an $(n-2)$-partite graph with uniform part size

$m-2$, where the edges of a $(n-3)$-factor are column edges when the vertices are arranged in a $(n-2)\times(m-2)$ matrix.

From this, it follows that given $G \in ER(n,d,\lambda)$ with USNS $\cong H$ where $|H| = \lambda$, $K_3 \otimes G$ has a USNS of $|H|K_1$. In other words, the tensor product of an edge-regular graph G with some USNS and a K_3 removes all of the edges of the USNS of G as a new USNS.

Another example: $G_1 \otimes G_2$, where $G_1 \in ER(n,d,\lambda)$ and G_2 is a triangle-free regular graph, has an empty graph USNS. That is, $G_1 \otimes G_2$ is also triangle-free.

2.2.3 Shadow operation

We now turn our attention to a lesser-known graph operation, the *shadow* of a graph. In Chapter 3, we will define a generalized version of the shadow of a graph as well as proving necessary conditions on when the shadow of a graph is edge-regular. For now, we prove some basic facts using a simple shadow definition.

For any positive integer n, let $[n] = \{1,\dots,n\}$. Enlarging the definition in [7], given a graph G, define $D_m(G)$ to be the m^{th} *shadow graph* of G, by $V(D_m(G)) = \{v_j^i | i \in [m]; j \in [n]\}$, given that $V(G) = \{v_1,\dots,v_n\}$; for $j,l \in [n]$ and $i,k \in [m]$, the vertices v_j^i and v_l^k are adjacent in $D_m(G)$ if $v_j \sim v_l$ in G. See Fig. 2.10 for an example.

Theorem 2.9. *If $G \in ER(n,d,\lambda)$ with a USNS $\cong H$, then $D_m(G) \in ER(mn, md, m\lambda)$ with a USNS $\cong D_m(H)$.*

Proof. Let $G \in ER(n,d,\lambda)$. Then by construction the m^{th} shadow of G contains m copies of every vertex of G, so $|D_m(G)| = mn$.

Now suppose $N_G(v_i) = \{u_1,\dots,u_d\}$. Then v_i^k is adjacent to each of $\{u_1^1,\dots,u_d^1,u_1^2,\dots,u_d^2,\dots,u_1^m,\dots,u_d^m\}$ for $k \in [m]$. So $D_m(G)$ is regular of degree md.

Using similar logic, say $v_i \sim v_j$ in G such that $N(v_i)\cap N(v_j) = \{u_1,\dots,u_\lambda\}$. Then $N(v_i^k)\cap N(v_j^l) = \{u_\beta^\alpha | \alpha \in [m]; \beta \in [\lambda]\}$ for $k,l \in [m]$. Thus, every pair of adjacent vertices in $D_m(G)$ share exactly $m\lambda$ vertices.

Further, as $G[\{v_1,\dots,v_\lambda\}] \cong H$, then $N(v_i^k)\cap N(v_j^l)$ contains exactly m copies of H, one in each shadow. The edge set among these m copies of H are as defined in the m^{th} shadow graph. Thus, $D_m(G)$ has a USNS $\cong D_m(H)$. □

Iteration of a USNS with the shadow graph function allows for additional infinite families of USNS.

Theorem 2.10. $D_q(D_m(G)) \cong D_{qm}(G)$ *for integers* $q, m \geq 2$.

Proof. Suppose $V(G) = \{v_1, \ldots, v_n\}$ and $V(D_m(G)) = \{v_j^i | i = 1, \ldots, m; j = 1, \ldots, n\}$. Then $V(D_q(D_m(G))) = \{v_j^{i,k} | i = 1, \ldots, m; j = 1, \ldots, n; k = 1, \ldots, q\}$; for all $1 \leq i \leq m, 1 \leq k \leq q, 1 \leq j \leq n$, $v_j^{i,k}$ is adjacent in $D_q(D_m(G))$ to every $v_r^{s,t}$ such that $v_j \sim v_r$ in G.

Arrange the qm copies of G in an $n \times m \times q$ array and label the vertices so that, with reference to a fixed list $v_1, \ldots, v_n$ of the vertices of G, for $(s,t) \in [m] \times [q]$, the appearance of v_i in the line of the array consisting of places with coordinates $(-, s, t)$ is $v_i^{s,t}$. Now it is clear that adjacency in this incarnation of $D_{qm}(G)$ is the same as in $D_q(D_m(G))$.

In both cases, $v_i^{m_1,q_1} \sim v_j^{m_2,q_2}$ if $v_i \sim v_j$ in G for $i \neq j; 1 \leq i,j \leq n; 1 \leq m_1, m_2 \leq m; 1 \leq q_1, q_2 \leq q$. Then $E(D_q(D_m(G))) = E(D_{qm}(G))$. So $D_q(D_m(G)) \cong D_{qm}(G)$ for all $q, m \geq 2$. $\square$

For example, $D_m(K_n) \cong K_{m,m,\ldots,m} \cong T_{mn,n} \in ER(mn, m(n-1), m(n-2))$, a complete n-partite graph with uniform partition size m, commonly known as a (regular) *Turán graph*. $D_3(K_3) \cong T_{9,3}$ is shown in Fig. 2.10. So the USNS of $D_m(K_n)$ is $D_m(K_{n-2}) \cong K_{m,m,\ldots,m} \cong T_{m(n-2),n-2}$, the Turán graph on $m(n-2)$ vertices with partition size m and $n-2$ parts.

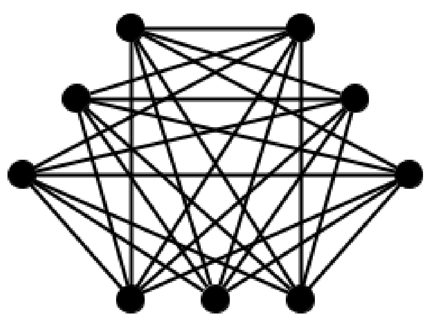

Figure 2.10: $D_3(K_3) \cong T_{9,3}$ with USNS of $D_3(K_1) \cong T_{3,1} = \overline{K_3}$

As stated in the preliminaries of the paper, not all edge-regular graphs have a connected USNS.

Consider $\mathcal{P}$, the Petersen graph; $\mathcal{P} \in ER(10, 3, 0)$. As stated earlier, the complement of the Petersen graph, as in Figure 2.11, is an interesting case. This graph is also already known to be edge-regular, as discussed in the $d = \lambda + 3$ case of [4]: $\overline{\mathcal{P}} \in ER(10, 6, 3)$ with a USNS of $K_2 + K_1$.

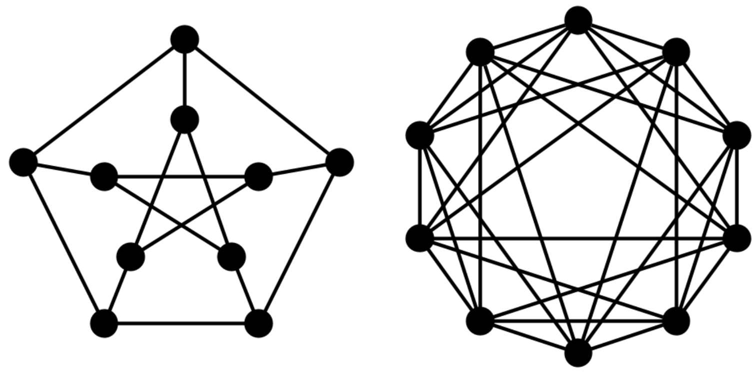

Figure 2.11: The Petersen graph (left) and its complement (right)

2.3 Conway's 99-graph problem

A strongly regular graph in $SR(n, d, \lambda, \mu)$ is a graph in $ER(n, d, \lambda)$ such that every pair of non-adjacent vertices share exactly μ common neighbors. Conway's *99-graph problem* is an open problem that asks about the existence of a graph in $SR(99, 14, 1, 2)$ [8]. Here we will show the non-existence of the 99-graph among Cartesian or tensor products of two edge-regular graphs.

First, define a graph G as a *regular clique assembly*, denoted $RCA(n, d, k)$, as a graph on n vertices, regular of degree d, and $k = \omega(G)$ (the clique number of G). An RCA graph G has three distinct properties: $\omega(G) \geq 2$, every maximal clique of G is maximum, and each edge of G is in exactly one maximum clique of G [3].

Next, we list results from [3] to use in the following theorem as lemmas.

Lemma 2.1. $RCA(n, d, k) \subseteq ER(n, d, k - 2)$*, with equality when* $k \in \{2, 3\}$.

Lemma 2.2. *Suppose* $ER(n, d, 1) \neq \emptyset$*. Then*

1. d *is even;*
2. $3|nd$*;*

3. for each $G \in ER(n,d,1)$ and $v \in V(G)$, $N_G[v]$ induces in G an friendship graph, $\{v\} \vee \frac{d}{2}K_2$;

4. if $d > 2$, each $G \in ER(n,d,1)$ is the clique graph of its clique graph, $CL(G) \in RCA(\frac{nd}{6}, \frac{3}{2}(d-2), \frac{d}{2})$.

Lemma 2.3. *$ER(3(d-1), d, 1) \neq \emptyset$ if and only if $d \in \{2, 4, 6, 10\}$.*

If the 99-graph G exists, then it is necessarily an edge-regular graph in $ER(99, 14, 1)$. This is equivalent to a regular clique assembly on the parameters $RCA(99, 14, 3)$ by Lemma 2.1. The idea here is to try to construct $RCA(99, 14, 3)$ by a product of two graphs G_1 and G_2, and to show that there is no such combination if the product is either the Cartesian or the tensor product.

Theorem 2.11. *If Conway's 99-graph exists, then it cannot be constructed as the Cartesian product of two RCA graphs.*

Proof. Suppose $G_1 \in RCA(n_1, d_1, 3)$ and $G_2 \in RCA(n_2, d_2, 3)$. Then $G_1 \square G_2 \in RCA(n_1 n_2, d_1 + d_2, 3)$ by Theorem 2.7. Since G_1 and G_2 are regular graphs of odd order, $2 \mid d_i$, $i = 1, 2$. There are only two options for n_1 and n_2, namely the pairs $\{33, 3\}$ and $\{11, 9\}$.

Let $n_1 = 3$ and $n_2 = 33$. As with all regular graphs, $n > d$, so G_1 must have degree 2. So, $G_1 \in RCA(3, 2, 3) = ER(3, 2, 1) \cong K_3$. Then $G_2 \in RCA(33, 12, 3) = ER(33, 12, 1)$. By Lemma 2.3, $ER(33, 12, 1) = \emptyset$. So $\{33, 3\}$ is not a possible pair of orders of G_1 and G_2.

Let $n_1 = 9$ and $n_2 = 11$. Then for G_1 the only possible d_1 are $\{2, 4, 6, 8\}$ since $n_1 = 9 > d_1$.

If $G_1 \in RCA(9, 2, 3)$, then $G_2 \in RCA(11, 12, 3)$, impossible as $n_2 < d_2$. If $G_1 \in RCA(9, 4, 3)$, then $G_2 \in RCA(11, 10, 3) = ER(11, 10, 1)$. Given that $n_2 = d_2 + 1$, then G_2 would need to be K_{11}, of which $\lambda = 9 \neq 1$, so $RCA(11, 10, 3) = ER(11, 10, 1) = \emptyset$. If $G_1 \in RCA(9, 6, 3)$, then $G_2 \in RCA(11, 8, 3) = ER(11, 8, 1)$. By Lemma 2.2, since $3 \nmid nd = 88$, it follows that $ER(11, 8, 1) = \emptyset$.

Finally, if $G_1 \in RCA(9, 8, 3) = ER(9, 8, 1)$, then as $n_1 = d_1 + 1$, G_1 is K_9. Yet $K_9 = ER(9, 8, 7)$, so $ER(9, 8, 1) = \emptyset$.

Thus, the 99-graph cannot be the Cartesian product of two RCA graphs. □

Using similar logic, it is straightforward to show that the tensor product of two edge-regular graphs cannot yield Conway's 99-graph.

Theorem 2.12. *If Conway's 99-graph exists, then it cannot be constructed with the tensor product of edge-regular graphs.*

Proof. Suppose $G_1 \in ER(n_1, d_1, \lambda_1)$ and $G_2 \in ER(n_2, d_2, \lambda_2)$ such that $G_1 \otimes G_2 \in ER(99, 14, 1)$. It is straightforward to see that if G_1 or G_2 is disconnected, then $G_1 \otimes G_2$ is disconnected, so we may assume that both G_1 and G_2 are connected graphs. By Theorem 2.8, $n_1 n_2 = 99$, $d_1 d_2 = 14$, and $\lambda_1 \lambda_2 = 1$. Thus, $\lambda_1 = \lambda_2 = 1$. Further, $d_1 d_2 = 1 \cdot 14$ or $d_1 d_2 = 2 \cdot 7$.

Suppose $d_1 d_2 = 1 \cdot 14$ and without loss of generality, $d_1 = 1$. Then $\lambda_1 = 1 = d_1$, a contradiction as $d > \lambda$ for all edge-regular graphs. Thus, $\{d_1, d_2\} \neq \{1, 14\}$.

Suppose $\{d_1, d_2\} = \{2, 7\}$ and without loss of generality, $d_1 = 2$. Then $\lambda_1 = 1$ and $d_1 = 2$ imply $n_1 = 3$. So $n_2 = 33$, $d_2 = 7$, and $\lambda_2 = 1$. An edge-regular graph $ER(33, 7, 1) = RCA(33, 7, 3)$ by Lemma 2.1. Yet $RCA(33, 7, 3)$ would be a regular graph of odd order and odd degree, an impossibility. □

Chapter 3

Shadow Operator

3.1 (m, x)-Shadows

We define a more generalized graph shadow operation below than the one defined in Section 2.2.3. In this definition, and throughout the remainder of the paper, when $u \in V(G)$ and $G_1, \ldots, G_m$ are copies of G, the vertex of G_i playing the role of u will be denoted u_i.

3.1.1 $x \neq 0$

For a positive integer x and $v \in V(G)$, $N_G^x(v)$ is the x-distance neighborhood of vertex v in G. If no graph G is specified, then it will be apparent from the context in what graph the x-distance neighborhood is being considered. If no x is specified, then x is understood to be 1. For an example, refer to figure 3.1.

Definition 3.1. Given a finite graph G and $x \geq 1$, $m \geq 2$, the (m, x)-*shadow of* G, denoted $D_m^x(G)$, is the simple graph whose vertices are in m distinct copies of G, say $G_1, G_2, \ldots, G_m$; $V(D_m^x(G)) = \bigcup_{i=1}^m V(G_i)$ and the edge set is $E(D_m^x(G)) = \{u_i v_j | 1 \leq i < j \leq m, u_i \in V(G_i), v_j \in V(G_j), u \in N_G^x(v)\} \cup \{u_i v_i | 1 \leq i \leq m, u_i, v_i \in V(G_i), uv \in E(G)\}$.

Theorem 3.1. *Given $G \in ER(n, d, \lambda)$, then for $x > 0$ and $m > 1$, $D_m^x(G)$ is edge- regular if and only if the following conditions hold for some nonnegative integers d_x, λ_x:*

1. *For all $v \in V(G)$, $|N^x(v)| = d_x$.*
2. *For all $u, v \in V(G)$ such that $u \sim v$ in G, $|N^x(u) \cap N^x(v)| = \lambda_x$.*

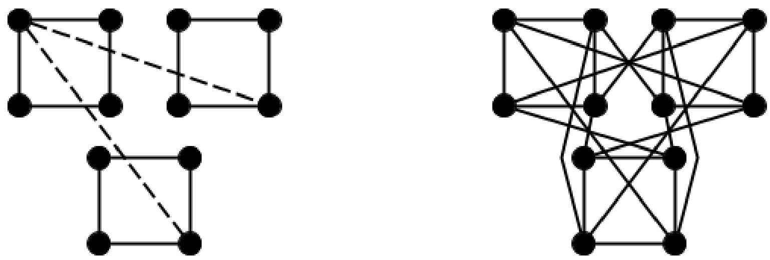

Figure 3.1: Distance 2 vertices from one vertex of C_4 to 2 other copies of C_4 (left). Edges are added between distance 2 vertices in different copies of C_4 to obtain $D_3^2(C_4)$ (right).

3. For all $v, w \in V(G)$ *such that* $w \in N^x(v)$,

$|N^x(v) \cap N(w)| + |N(v) \cap N^x(w)| + (m-2)|N^x(v) \cap N^x(w)| = \lambda + (m-1)\lambda_x$

Proof. Suppose $G \in ER(n, d, \lambda)$ and $D_m^x(G)$ is edge-regular, where $m \geq 2$ and $x \geq 1$. Define $d_x(v) = |N^x(v)|$. Then for a vertex $v \in V(D_m^x(G))$, $\deg(v) = d + (m-1)d_x(v)$. As $D_m^x(G)$ is edge-regular by assumption, then $\deg(u) = \deg(v)$ for all $u, v \in V(D_m^x(G))$. So, $d + (m-1)d_x(u) = d + (m-1)d_x(v)$ implies that $d_x(u) = d_x(v) = d_x$ for some constant d_x, so condition 1 is met.

Now define $\lambda_x(u, v) = |N^x(u) \cap N^x(v)|$. Then for $u, v \in V(G)$ such that u and v are adjacent, $|N_{D_m^x(G)}(u) \cap N_{D_m^x(G)}(v)| = \lambda + (m-1)\lambda_x(u, v)$. As $D_m^x(G)$ is edge-regular by assumption, then for all $u, v, y, z \in V(D_m^x(G))$ such that u is adjacent to v and y is adjacent to z, $|N_{D_m^x(G)}(u) \cap N_{D_m^x(G)}(v)| = |N_{D_m^x(G)}(y) \cap N_{D_m^x(G)}(z)|$. So, $\lambda + (m-1)\lambda_x(u, v) = \lambda + (m-1)\lambda_x(y, z)$ implies that $\lambda_x(u, v) = \lambda_x(y, z) = \lambda_x$ for some constant λ_x, so condition 2 is met.

Now consider adjacent vertices in $D_m^x(G)$ in different copies of G, say v and w', where w' is a copy of a vertex $w \in N_G^x(v)$. Then v and w' share $|N_G(v) \cap N_G^x(w)|$ vertices in the copy of G containing v. Likewise, $|N_G^x(v) \cap N_G(w)|$ vertices are shared in the copy of G containing w'. In each of the remaining $m-2$ copies of G, v and w' share $|N_G^x(v) \cap N_G^x(w)|$ vertices. As $D_m^x(G)$ is edge-regular by assumption, then $|N_{D_m^x(G)}(v) \cap N_{D_m^x(G)}(w')|$ is equal to the number of vertices shared in $D_m^x(G)$ by two adjacent vertices in the same copy of G. So, $|N_{D_m^x(G)}(v) \cap N_{D_m^x(G)}(w')| = |N_G(v) \cap N_G^x(w)| + |N_G^x(v) \cap N_G(w)| + (m-2)|N_{D_m^x(G)}(v) \cap N_{D_m^x(G)}(w')| = \lambda + (m-1)\lambda_x$ by condition 2. Thus, all conditions are met.

The proof of the converse is straightforward, using the same arguments as in the forward direction. □

Theorem 3.1 generalizes one implication of a result in [6], which asserts that when $x = 1$, $D^1_m(G)$ is edge-regular if G is edge-regular. When $x = 1$ and $G \in ER(n, d, \lambda)$, clearly conditions 1, 2, and 3 hold for any m with $d_1 = d$ and $\lambda_1 = \lambda$.

But for $m, x > 1$, the edge-regularity of G does not imply the edge-regularity of $D^x_m(G)$. An example is given in figure 3.2. In this example, conditions 3 of Theorem 3.1 does not hold.

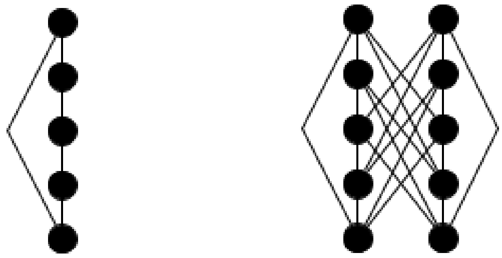

Figure 3.2: $C_5 \in ER(5, 2, 0)$ (left) and $D^2_2(C_5)$ which is not edge-regular (right).

3.1.2 $x = 0$

In the case $x = 0$, if we agree that for each $v \in V(G)$, $N^0(v) = \{v\}$, and read the definition of adjacency in $D^0_m(G)$ as it was for $D^x_m(G)$, $x > 0$, then we see that for each $v \in V(G)$, each of its clones in any of the m copies of G constituting $D^0_m(G)$ is adjacent to each of the $m - 1$ other clones of v in the other copies of G, and to no other vertices in those other copies. We enshrine this description in the following proposition.

Proposition 3.1. *For each graph G and integer $m \geq 1$, $D^0_m(G) \cong G \square K_m$.*

Corollary 3.1. *If $G \in ER(n, d, \lambda)$ then $H = D^0_m(G)$ is of order mn and regular of degree $m - 1 + d$. If $m > 1$ then H is edge-regular if and only if $m - 2 = \lambda$, in which case $H \in ER(mn, m - 1 + d, m - 2) = ER(mn, m - 1 + d, \lambda)$.*

Surprisingly, Theorem 3.1 holds when $x = 0$. Clearly, for any G, conditions 1 and 2 are met with $d_0 = 1$ and $\lambda_0 = 0$. For condition 3, observe that $w \in N^0(v)$ implies that $w = v$. Then the equation in condition 3 collapses to $0 + 0 + (m - 2)(1) = \lambda + 0$, or $m - 2 = \lambda$, which is precisely the necessary and sufficient condition for $D_m^0(G)$ to be edge-regular when G is edge regular (Corollary 3.1).

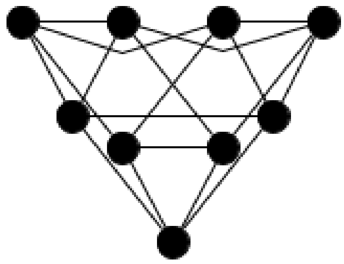

Figure 3.3: An example of a 0-distance shadow, $D_3^0(K_3) \in ER(9, 4, 1)$.

3.1.3 C_n

We now show how a class of edge-regular graphs, under the shadow graph operation, is used to build larger edge-regular graphs. Let C_n denote the cycle graph on n vertices, $d(u, v)$ denote the distance between two vertices u and v, and let $r(G)$ denote the *radius* of a graph G. $r(G)$ is the minimum *eccentricity* among the vertices of G. The eccentricity $\epsilon(v)$ of $v \in V(G)$ is the maximum of the distances in G from v to vertices of G (these definitions require G to be connected). It is well known that, for $n \geq 3$, $r(C_n) = \lfloor \frac{n}{2} \rfloor$.

Corollary 3.2. *$D_m^x(C_n)$ is edge-regular for all integers $n \geq 3$, $m \geq 2$, $1 \leq x \leq r(C_n)$ except in either of the following cases for some integer $k \geq 2$:*

1. *$n = 3k, x = k, m \neq 2$.*
2. *$n = 2k + 1, x = k, m \neq 3$.*

Proof. Consider $n \equiv 0 \pmod 2$, so C_n is an even cycle with radius $\frac{n}{2}$. Then for $1 \leq x \leq \frac{n}{2}$, $v \in V(C_n)$, $|N^x(v)| = 1$ if $x = \frac{n}{2}$, or $|N^x(v)| = 2$ if $x < \frac{n}{2}$. Further, for $uv \in E(C_n)$, $|N^x(u) \cap N^x(v)| = 0$. Additionally, for $w \in N^x(v)$, notice that $|N^x(v) \cap N(w)| = |N(v) \cap N^x(w)| = 0$; $|N^x(v) \cap N^x(w)| = 1$ if $x = \frac{n}{3}$, and $|N^x(v) \cap N^x(w)| = 0$ if $x \neq \frac{n}{3}$.

Then by Theorem 3.1, for $n \equiv 0 \pmod 2$, $D_m^x(C_n)$ is edge-regular if $x \neq \frac{n}{3}$. When $x = \frac{n}{3}$, $D_m^x(C_n)$ is edge-regular only when $m = 2$.

Now consider $n \equiv 1 \pmod 2$, so C_n is an odd cycle with radius $\lfloor \frac{n}{2} \rfloor$. Then for $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$, $v \in V(C_n)$, $|N^x(v)| = 2$. Further, for $uv \in E(C_n)$, $|N^x(u) \cap N^x(v)| = 1$ if $x = \lfloor \frac{n}{2} \rfloor$, or $|N^x(u) \cap N^x(v)| = 0$ if $x < \lfloor \frac{n}{2} \rfloor$. Additionally, for $w \in N^x(v)$, notice that $|N^x(v) \cap N(w)| = |N(v) \cap N^x(w)| = 1$ if $x = \lfloor \frac{n}{2} \rfloor$, or $|N^x(v) \cap N(w)| = |N(v) \cap N^x(w)| = 0$ if $x \neq \lfloor \frac{n}{2} \rfloor$; $|N^x(v) \cap N^x(w)| = 1$ if $x = \frac{n}{3}$, and $|N^x(v) \cap N^x(w)| = 0$ if $x \neq \frac{n}{3}$.

Then by Theorem 3.1, for $n \equiv 1 \pmod 2$, $D^x_m(C_n)$ is edge-regular if $x \notin \{\frac{n}{3}, \lfloor \frac{n}{2} \rfloor\}$. If $x = \frac{n}{3}$, $D^x_m(C_n)$ is edge-regular only when $m = 2$. If $x = \lfloor \frac{n}{2} \rfloor$, $D^x_m(C_n)$ is edge-regular only when $m = 3$. □

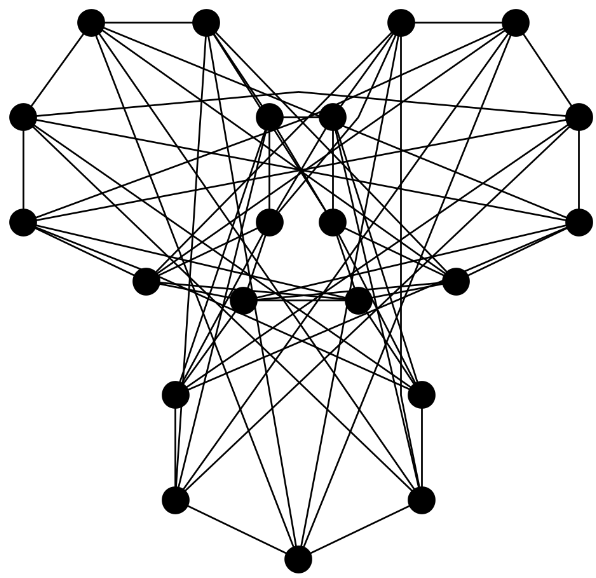

Figure 3.4: $D^3_3(C_7)$, an edge-regular graph by Corollary 3.2

3.1.4 Regular, Edge-Regular, and Strongly Regular conditions

Theorem 3.2. *Suppose $m, x \geq 1$ are integers, G is a simple, connected graph of order n and $D^x_m(G)$ is the (m, x)-shadow of G. Consider the following conditions for some nonnegative integers d'_x, λ'_x, and μ'_x:*

1. For all $v \in V(G)$, $|N(v)| + (m-1)|N^x(v)| = d'_x$

2. For all $u, v \in V(G)$ such that $u \sim v$ in G,

$$|N(u) \cap N(v)| + (m-1)|N^x(u) \cap N^x(v)| = \lambda'_x$$

3. For all $v, w \in V(G)$ such that $w \in N_G^x(v)$,

$$|N(v) \cap N^x(w)| + |N^x(v) \cap N(w)| + (m-2)|N^x(v) \cap N^x(w)| = \lambda'_x$$

4. For all $u, v \in V(G)$ such that $u \not\sim v$ in G,

$$|N(u) \cap N(v)| + (m-1)|N^x(u) \cap N^x(v)| = \mu'_x$$

5. For all $v, w \in V(G), w \notin N^x(v)$,

$$|N(v) \cap N^x(w)| + |N^x(v) \cap N(w)| + (m-2)|N^x(v) \cap N^x(w)| = \mu'_x$$

Condition 1 is met if and only if $D_m^x(G)$ regular of degree d'_x.

Conditions 1, 2, and 3 are met if and only if $D_m^x(G) \in ER(mn, d'_x, \lambda'_x)$.

All conditions are met if and only if $D_m^x(G) \in SR(mn, d'_x, \lambda'_x, \mu'_x)$.

Proof. Suppose condition 1 holds. For all $v_i \in V(D_m^x(G))$, v_i is adjacent to $N_G(v_i)$ in one copy of G and adjacent to $N_G^x(v_i)$ in $m-1$ copies of G. Then $|N_{D_m^x(G)}(v_i)| = |N(v_i)| + (m-1)|N^x(v_i)| = d'_x$. So $D_m^x(G)$ is regular of degree d'_x.

Now suppose conditions 1, 2, and 3 hold. As stated previously, if condition 1 holds, then $D_m^x(G)$ is regular. There are two "types" of adjacencies in $D_m^x(G)$. One type are adjacencies between vertices in the same copy of G. Consider two vertices of this type, say u and v. Then as u and v share λ neighbors in the same copy of G and x-distance neighbors in $m-1$ copies of G, $|N_{D_m^x(G)}(u) \cap N_{D_m^x(G)}(v)| = \lambda + (m-1)|N^x(u) \cap N^x(v)| = \lambda'_x$ by condition 2.

The second type of adjacencies are between vertices in distinct copies of G. Consider two vertices of this type, say v and w', where w' is a copy of w, which is in the same copy of G as v. That is, $w \in N^x(v)$. Then in the copy of G containing v, the number of common neighbors of

v and w' is the number of neighbors of v that are x-distance neighbors of w. In the copy of G containing w', the number of common neighbors is neighbors of w that are x-distance neighbors of v. In the remaining $m-2$ copies of G not containing v or w', the common neighbors are x-distance neighbors of both v and w. Then $|N_{D_m^x(G)}(v) \cap N_{D_m^x(G)}(w')| = |N^x(v) \cap N(w)| + |N(v) \cap N^x(w)| + (m-2)|N^x(v) \cap N^x(w)| = \lambda_x'$ by condition 3.

Thus, as all pairs of adjacent vertices have λ_x' common neighbors, $D_m^x(G)$ is edge-regular.

Now suppose all conditions are met. As stated previously, since conditions 1, 2, and 3 are met, $D_m^x(G) \in ER(mn, d_x', \lambda_x')$.

Consider non-adjacent vertices in the same copy of G, say u and v. In the same copy of G, the number of common neighbors is $|N(u) \cap N(v)|$. In the other $m-1$ copies of G, the number of common neighbors is $|N^x(u) \cap N^x(v)|$. So $|N_{D_m^x(G)}(u) \cap N_{D_m^x(G)}(v)| = |N(u) \cap N(v)| + (m-1)|N^x(u) \cap N^x(v)| = \mu_x'$ by condition 4.

Consider non-adjacent vertices in distinct copies of G, say v and w', where w' is a copy of $w \in V(G)$. That is, $w \notin N_G^x(v)$. In the copy of G containing v, the number of common neighbors of v and w' is the number of neighbors of v in the x-distance neighborhood of w. In the copy of G containing w', the number of common neighbors is the number of neighbors of w in the x-distance neighborhood of v. In the remaining $m-2$ copies of G, the number of common neighbors is the number of x-distance neighbors of v in the x-distance neighborhood of w. So $|N_{D_m^x(G)}(v) \cap N_{D_m^x(G)}(w')| = |N^x(v) \cap N(w)| + |N(v) \cap N^x(w)| + (m-2)|N^x(v) \cap N^x(w)| = \mu_x'$ by condition 5. As all non-adjacent vertices in $D_m^x(G)$ share μ_x' common neighbors, $G \in SR(mn, d_x', \lambda_x', \mu_x')$.

The converse is straightforward. □

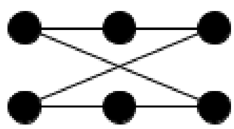

Figure 3.5: $D_2^2(P_3) \cong C_6 \in ER(6, 2, 0)$

Theorem 3.2 removes the restriction that G is edge-regular, and allows for any simple, connected graph. Consider the $(2,2)$-shadow of the path graph on 3 vertices, which is non-regular, shown in figure 3.5. $D_2^2(P_3) \cong C_6 \in ER(6,2,0)$, which satisfies conditions 1, 2, and 3 in the above theorem.

While the above edge-regular example is an immediate result of Theorem 3.2, it is unknown for what graphs G and parameters m, x yield strongly regular graphs from $D_m^x(G)$.

3.2 (m, X)-Shadows

To further generalize, we consider multiple distances to be used in the definition of the shadow of the graph as opposed to a single distance. This new multi-distance shadow generalizes definition 3.1, formally defined below.

Definition 3.2. Given a finite graph G, $X = \{x_1, \ldots, x_p\}$, $1 \leq x_i < \cdots < x_p$, and $m \geq 2$, the (m, X)-*shadow of* G, denoted $D_m^X(G)$, is the simple graph whose vertices are in m distinct copies of G, say $G_1, G_2, \ldots, G_m$; $V(D_m^X(G)) = \bigcup_{i=1}^m V(G_i)$ and $E(D_m^X(G)) = \{u_iv_j | i \neq j, u_i \in V(G_i), v_j \in V(G_j), u \in \bigcup_{k=1}^p N_G^{x_k}(v)\} \cup \{u_iv_i | u_i, v_i \in V(G_i), uv \in E(G)\}$.

As with the case of $D_m^x(G)$ in Theorem 3.1, an immediate question arises as to when edge-regularity is preserved in $D_m^X(G)$. The following theorem characterizes when $D_m^X(G)$ is edge-regular, given that G is edge-regular.

Theorem 3.3. *Given $G \in ER(n, d, \lambda)$ and $X = \{x_1, x_2, \ldots, x_p\}$, then $D_m^X(G)$ is edge-regular if and only if the following conditions are met for some integers $\bar{d}$ and $\bar{\lambda}$:*

1. *For all $v \in V(G)$,* $\displaystyle\sum_{i=1}^{p} |N^{x_i}(v)| = \bar{d}$
2. *For all $u, v \in V(G)$ such that $u \sim v$ in G,* $\displaystyle\sum_{i=1}^{p}\sum_{j=1}^{p} |N^{x_i}(u) \cap N^{x_j}(v)| = \bar{\lambda}$
3. *For all $v, w \in V(G)$ such that* $w \in \displaystyle\bigcup_{q=1}^{p} N^{x_q}(v)$,

$$\sum_{i=1}^{p} |N(v) \cap N^{x_i}(w)| + \sum_{j=1}^{p} |N^{x_j}(v) \cap N(w)| + (m-2)\sum_{k=1}^{p}\sum_{l=1}^{p} |N^{x_k}(v) \cap N^{x_l}(w)| = \lambda + (m-1)\bar{\lambda}.$$

Proof. Suppose G and $D_m^X(G)$ are edge-regular, where $m \geq 1$ and $X = \{x_1, \ldots, x_p\}$. Then in $D_m^X(G)$,

$$\begin{aligned}\deg(v) &= d + (m-1)|N_G^{x_1}(v)| + (m-1)|N_G^{x_2}(v)| + \cdots + (m-1)|N_G^{x_p}(v)| \\ &= d + (m-1)\sum_{i=1}^{p}|N^{x_i}(v)|\end{aligned}$$

As $D_m^X(G)$ is regular by assumption, $\deg(u) = \deg(v)$ for all $u, v \in V(D_m^X(G)$. So,

$$\begin{aligned}d + (m-1)\sum_{i=1}^{p}|N^{x_i}(u)| &= d + (m-1)\sum_{i=1}^{p}|N^{x_i}(v)| \\ \sum_{i=1}^{p}|N^{x_i}(u)| &= \sum_{i=1}^{p}|N^{x_i}(v)|\end{aligned}$$

Thus, $\sum_{i=1}^{p}|N^{x_i}(v)| = \bar{d}$ for all $v \in V(D_m^X(G))$, so condition 1 is met.

Now define $\lambda_{i,j}(u,v) = |N_G^{x_i}(u) \cap N_G^{x_j}(v)|$. Then for $uv \in E(G)$,

$$\begin{aligned}|N_{D_m^X(G)}(u) \cap N_{D_m^X(G)}(v)| &= \lambda + (m-1)\lambda_{1,1}(u,v) + \cdots + (m-1)\lambda_{1,p}(u,v) \\ &\quad + \cdots \\ &\quad + (m-1)\lambda_{p,1}(u,v) + \cdots + (m-1)\lambda_{p,p}(u,v) \\ &= \lambda + (m-1)\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_{i,j}(u,v)\end{aligned}$$

As $D_m^X(G)$ is edge-regular by assumption,

$$|N_{D_m^X(G)}(u) \cap N_{D_m^X(G)}(v)| = |N_{D_m^X(G)}(y) \cap N_{D_m^X(G)}(z)|$$

for all $uv, yz \in E(G)$. So,

$$\begin{aligned}\lambda + (m-1)\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_{i,j}(u,v) &= \lambda + (m-1)\sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_{i,j}(y,z) \\ \sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_{i,j}(u,v) &= \sum_{i=1}^{p}\sum_{j=1}^{p}\lambda_{i,j}(y,z)\end{aligned}$$

Thus, for all $uv \in E(G)$, $\sum_{i=1}^{p}\sum_{j=1}^{p} \lambda_{i,j}(u,v) = \bar{\lambda}$, so condition 2 is met.

Now consider adjacent vertices of $D_m^X(G)$ in different copies of G, say v and w', where w' is a copy of a vertex $w \in \bigcup_{q=1}^{p} N_G^{x_q}(v)$. Then v and w' share $\sum_{i=1}^{p} |N_G(v) \cap N_G^{x_i}(w)|$ vertices in the copy of G containing v. Likewise, v and w' share $\sum_{j=1}^{p} |N_G^{x_j}(v) \cap N_G(w)|$ vertices in the copy of G containing w'. In each of the remaining $m-2$ copies of G, v and w' share $\sum_{k=1}^{p}\sum_{l=1}^{p} |N_G^{x_k}(v) \cap N_G^{x_l}(w)|$ vertices. As $D_m^X(G)$ is edge-regular by assumption, $|N_{D_m^X(G)}(v) \cap N_{D_m^X(G)}(w')|$ is equal to the number of vertices shared by two adjacent vertices in the same copy of G. Thus,

$$\begin{aligned}|N_{D_m^X(G)}(v) \cap N_{D_m^X(G)}(w')| &= \sum_{i=1}^{p} |N_G(v) \cap N_G^{x_i}(w)| + \sum_{j=1}^{p} |N_G^{x_j}(v) \cap N_G(w)| \\ &\quad + (m-2)\sum_{k=1}^{p}\sum_{l=1}^{p} |N_G^{x_k}(v) \cap N_G^{x_l}(w)| \\ &= \lambda + (m-1)\bar{\lambda} \quad \text{(by condition 2)}\end{aligned}$$

Thus, condition 3 is met.

The converse is straightforward. □

Figure 3.6: Distance 1 and 2 vertices from one vertex of P_4 to another copy of P_4 (left). Edges added between distance 1 and 2 vertices in different copies of P_4 to obtain $D_2^{1,2}(P_4)$ (right).

3.3 Generalized Graph Shadows

Definition 3.3. Given finite graphs G, H and $x \geq 1$, $m = |V(H)|$; let the vertices of H be ordered $w_1, \ldots, w_m$. The (H,x)-*shadow of* G, denoted $D_m^x(G,H)$, is the simple graph whose vertices are in m distinct copies of G, say $G_1, G_2, \ldots, G_m$. $V(D_m^x(G)) = \bigcup_{i=1}^{m} V(G_i)$ and edge set $E(D_m^x(G)) = \{u_iv_j | i \neq j, u_i \in V(G_i), v_j \in V(G_j), w_i, w_j \in V(H), w_iw_j \in E(H), u \in N_G^x(v)\} \cup \{u_iv_i | u_i, v_i \in V(G_i), uv \in E(G)\}$.

The definition of $D^x_m(G,H)$ refers to an ordering of $V(H)$, but the isomorphism class of $D^x_m(G,H)$ does not depend on the ordering. Permuting the vertex set of H results in a rearrangement of the list $G_1,\ldots,G_m$; while the new $D^x_m(G,H)$ obtained thereby is not identical to the old, the two are obviously isomorphic, as the ends of the edges between G_i and G_j, $i\neq j$, in the old are dragged along to the new positions of these copies of G.

Note that by this definition, if $|V(G)|=n$, $D^x_m(G)=D^x_m(G,K_n)$. For an example, refer to figure 3.7. The following theorem gives a characterization for when $D^x_m(G,G)$ is edge-regular.

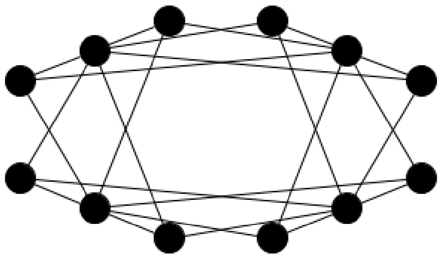

Figure 3.7: $D^1_4(P_3,C_4)$

Theorem 3.4. *Given $G\in ER(n,d,\lambda)$, then $D^x_m(G,G)$ is edge-regular if and only if the following conditions are met:*

1. *For all $v\in V(G)$, $d(1+|N^x(v)|)=\bar{d}$.*
2. *For all $u,v\in V(G)$ such that $u\sim v$ in G, $\lambda+d|N^x(u)\cap N^x(v)|=\bar{\lambda}$.*
3. *For all $v,w\in V(G)$ such that $w\in N^x_G(v)$,*

$$|N^x(v)\cap N(w)|+|N(v)\cap N^x(w)|+\lambda|N^x(v)\cap N^x(w)|=\bar{\lambda}.$$

Proof. To prove the forward direction, suppose $G\in ER(n,d,\lambda)$. Then $v\in V(G)$ is adjacent to d vertices in its own copy of G and is adjacent to $|N^x(v)|$ vertices in d other copies of G. Then for $v_i\in V(G)$, $|N_{D^x_m(G,G)}(v_i)|=d+d|N^x(v_i)|=d(1+|N^x(v_i)|)=\bar{d}_i$. As G is edge-regular, $\bar{d}_i=\bar{d}$ for all $i\in[n]$. Thus, condition 1 is met.

Consider adjacent vertices u,v in the same copy of G. Then u and v have λ common neighbors in the copy of G containing them, and have $|N^x(u)\cap N^x(v)|$ common neighbors in

d copies of G. So $|N_{D^x_m(G,G)}(u) \cap N_{D^x_m(G,G)}(v)| = \lambda + d|N^x(u) \cap N^x(v)| = \bar{\lambda}_{u,v}$. As G is edge-regular, $\bar{\lambda}_{u,v} = \bar{\lambda}$ for all pairs $u, v \in V(G)$ such that $u \sim v$. Thus, condition 2 is met.

Consider adjacent vertices in distinct copies of G, say v and w', where w' is a copy of w, which is in the same copy of G as v. That is, $w \in N^x(v)$. Then in the copy of G containing v, the number of common neighbors of v and w' is $|N^x(v) \cap N(w)|$. In the copy of G containing w', the number of common neighbors is $|N(v) \cap N^x(w)|$. These distinct copies of G are mutually adjacent to λ other copies of G. In these λ copies of G, v and w' have $|N^x(v)\cap N^x(w)|$ common neighbors. So $|N_{D^x_m(G,G)}(v) \cap N_{D^x_m(G,G)}(w')| = |N^x(v) \cap N(w)| + |N(v) \cap N^x(w)| + \lambda|N^x(v) \cap N^x(w)| = \bar{\lambda}_{v,w'}$. As G is edge-regular, then $\bar{\lambda}_{v,w'} = \bar{\lambda}$ for all pairs of adjacent vertices v, w' in distinct copies of G. Thus, condition 3 is met.

The converse is straightforward. □

Corollary 3.3. *If $G \in ER(n, d, \lambda)$, then $D_m(G, G)$ is edge-regular.*

Proof. Given $G \in ER(n, d, \lambda)$, let $x = 1$. Note then that $d_x = d$ and $\lambda_x = \lambda$. Then for a vertex $v \in V(G)$, $d(1 + |N(v)|) = d + (d)^2 = \bar{d}$ for some integer $\bar{d}$. So condition 1 of Theorem 3.4 is met. For adjacent vertices $u, v \in V(G)$, $\lambda + d|N(u) \cap N(v)| = \lambda + d\lambda = \bar{\lambda}$ for some integer $\bar{\lambda}$. So condition 2 of Theorem 3.4 is met. Finally, condition 3 of Theorem 3.4 is trivially met when $x = 1$. Thus, $D_m(G, G)$ is edge-regular. □

Corollary 3.3 justifies a way to construct edge-regular graphs which resembles a recursive process in the generalized graph shadow.

Chapter 4

Conclusions and future work

4.1 USNS

A *component*-regular graph is a graph such that each component is regular. Every known USNS graph is component-regular, and every aforementioned USNS-forbidden graph is not component-regular. Is it true that every USNS graph is component-regular? Could it be the case that a graph H is a USNS of some edge-regular graph G if and only if H is component-regular? Regarding graph operations, it would be of interest to discover other graph operations that create (or preserve) USNS of edge-regular graphs.

4.2 Conway's 99-graph problem

Can any more facts regarding edge-regular graphs or their behavior in graph products be stated to help the search for the answer to Conway's 99-graph question? This graph, if it exists, would necessarily be a regular clique assembly (as defined in Section 2.3) by Lemma 2.1. It is not apparent from the scaffolding methods provided in [2] and [3] that the 99-graph is possible as an edge-regular graph; more can hopefully be explored here.

4.3 Shadow characterizations

It remains to be seen, for any edge-regular graph H and some other graph G, when $D^x_m(G, H)$ is edge-regular. Characterizing when $D^x_m(G, H)$ is edge-regular for any simple, connected graphs G and H would generalize a number of results in this paper and would provide a framework for construction of regular and strongly-regular graphs. Under the assumption that some useful characterization exists, extending it to strongly regular graphs would also be of interest.

References

[1] V. Glorioso, *Edge-Regular Graphs with* $\lambda = 2$. ProQuest LLC, Ann Arbor, MI, 2019, thesis (Ph.D.)–Auburn University. [Online]. Available: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:30265876

[2] K. Bragan, *Topics in Edge Regular Graphs*. ProQuest LLC, Ann Arbor, MI, 2014, thesis (Ph.D.)–Auburn University. [Online]. Available: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:30261616

[3] K. B. Guest, J. M. Hammer, P. D. Johnson, and K. Roblee, "Regular clique assemblies, configurations, and friendship in edge-regular graphs," *Tamkang J. Math.*, vol. 48, no. 4, pp. 301–320, 2017. [Online]. Available: https://doi.org/10.5556/j.tkjm.48.2017.2237

[4] P. D. Johnson, Jr., W. Myrvold, and K. J. Roblee, "More extremal problems for edge-regular graphs," *Util. Math.*, vol. 73, pp. 159–168, 2007.

[5] R. J. Evans, S. Goryainov, E. V. Konstantinova, and A. D. Mednykh, "A general construction of strictly Neumaier graphs and a related switching," *Discrete Math.*, vol. 346, no. 7, pp. Paper No. 113 384, 11, 2023. [Online]. Available: https://doi.org/10.1016/j.disc.2023.113384

[6] J. DeLeo, "Uniform shared neighborhood structures in edge-regular graphs," 2024, to appear in the Australasian Journal of Combinatorics. [Online]. Available: https://arxiv.org/abs/2409.00268

[7] Asmiati, W. Okzarima, Notiragayu, and L. Zakaria, "Upper bounds of the locating chromatic numbers of shadow cycle graphs," *Int. J. Math. Comput. Sci.*, vol. 19, no. 1, pp. 239–248, 2024.

[8] J. H. Conway, "Five \$1,000 problems (update 2017)," *OEIS*, p. 1, 2014. [Online]. Available: https://oeis.org/A248380/a248380.pdf