Widths of Finite Posets Under the Majorization Ordering

by

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Abstract

This dissertation focuses on the structural properties of two types of posets, $P(n,m)$ and $P'(n,m)$, both of which are ordered by the majorization ordering. Specifically, we consider those cases where $1 \leq n \leq 4$. The poset $P(n,m)$ consists of sequences of non-negative integers of length n that sum to m, more formally defined as $P(n,m) = \{x \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} x_i = m\}$. The second poset, $P'(n,m)$, is a subposet of $P(n,m)$ where we restrict the sequences to be decreasing, i.e., $P'(n,m) = \{x \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} = m \text{ and } x_i \geq x_{i+1}\}$.

We define the majorization ordering to be: for any two sequences $x, y \in (\mathbb{Z}_{\geq 0})^n$ we say that x is majorized by y if the following conditions hold:

$$\sum_{i=0}^{j-1} x_i \leq \sum_{i=0}^{j-1} y_i \quad \text{for } 0 \leq j-1 < n-1, \; \sum_{i=0}^{n-1} x_i = \sum_{i=0}^{n-1} y_i$$

We demonstrate that these posets exhibit Sperner-like properties. In particular, we show that the largest antichain in $P(n,m)$ and $P'(n,m)$ for $1 \leq n \leq 4$ is realized by a “middle” “level”, similar to that of the classical Sperner theorem. However since $P'(n,m)$ is not a graded poset, it does not have true levels, which is why we refer to these properties as “Sperner-like”. We also use the term “middle” loosely here, as there may be many levels or induced levels which are maximal, and they all generally occur in the middle section of these posets. Despite this, many of the structural properties of $P(n,m)$ are inherited by $P'(n,m)$.

In the case of $P(n,m)$, we provide explicit chain decompositions, while for $P'(n,m)$ we give explicit chain decompositions for $n \in \{1,2\}$. For $P'(n,m)$ when $n \in \{3,4\}$, we give an inductive proof of the existence of a minimal chain decomposition on the outer layer(s), with induction handling the smaller poset.

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Chapter 1

Introduction

Extremal set theory is a branch of mathematics, more specifically a branch of extremal combinatorics, which is concerned with determining the maximum or minimum size of a collection of sets that satisfy specified constraints, and more generally problems of this form for arbitrary partially ordered sets. Classical results from extremal set theory include Dilworth's theorem [2], which proves that the maximum size of an antichain equals the minimum number of chains needed to cover the set. Other notable results are Sperner's theorem [14], which proves that the largest antichain in the Boolean lattice $\{0,1\}^n$ has size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$; and the Kruskal-Katona theorem [7],[6], known to be equivalent to the isoperimetric inequality in the hypercube graph [3] which characterizes the smallest possible shadow of a set family, or in other words, determines the smallest possible neighborhood among all sets of a given number of vertices in one part of the bipartite graph induced between two consecutive layers of the Boolean lattice.

In this dissertation we study the structures of two types of posets both of which use the majorization ordering, where (using Dilworth's theorem) we obtain analogues for Sperner's theorem in some highly nontrivial special cases. The first set of elements we study is, for $n, m \in \mathbb{Z}_{\geq 0}$, let $P(n,m) = \left\{x \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} x_i = m\right\}$. In the second type we restrict those elements of $P(n,m)$ to just those that are decreasing and denote this restriction with $P'(n,m)$. That is for $n, m \in \mathbb{Z}_{\geq 0}$, let $P'(n,m) = \left\{x \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} x_i = m \text{ and } x_i \geq x_{i+1}, 0 \leq i \leq n-2\right\}$. We then can define the majorization ordering to be, for any two sequences $x, y \in (\mathbb{Z}_{\geq 0})^n$ we say that x is majorized by y denoted by $x \leq_M y$ if

$$\sum_{i=0}^{j-1} x_i \leq \sum_{i=0}^{j-1} y_i \quad \text{for } 0 \leq j-1 < n-1$$

and

$$\sum_{i=0}^{n-1} x_i = \sum_{i=0}^{n-1} y_i$$

With the two sets $P(n,m)$ and $P'(n,m)$ the second condition is redundant, since both sets already have $\sum x_i$ constant. Our aim is to determine whether these posets are Sperner, that is whether the size of a largest antichain is realized by a "middle" level, as is the case with the Boolean lattice, as well as determine the sizes of such antichains and corresponding chain decompositions according to Dilworth's theorem. Rigorous definitions for these concepts can be found at the end of this chapter.

Motivation for studying the width of these particular posets comes from the classical Muirhead inequality (see for example [5] for a more detailed exposition). Muirhead can be viewed as a vast generalization of the AM-GM inequality $\frac{1}{2}(X_1^2 + X_2^2) \geq X_1 X_2$, or more generally, $\frac{1}{n}(X_1^{nd} + X_2^{nd} + \cdots + X_n^{nd}) \geq \big(X_1 X_2 \ldots X_n\big)^d$ (for any assignment of positive real numbers to the X_is). Note that all these polynomials are *symmetric* (invariant under swapping any two X_i's) and in fact *monomial symmetric* (generated from averaging a single monomial under the action of the symmetric group S_n on $\{1, 2, \ldots, n\}$): in fact, any monomial symmetric polynomial Q is uniquely determined by its degree sequence y_Q written in (without loss of generality) non-increasing order. Muirhead says that to determine whether two monomial symmetric multivariate polynomials P and Q of the same degree satisfy $P[\vec{X}] \leq Q[\vec{X}]$ for every $\vec{X} = (X_1, X_2, \ldots, X_n)$ is equivalent to determining whether their respective degree sequences satisfy $y_P \leq_M y_Q$ in the majorization order. In this sense, the versions of AM-GM described above can arise from the majorizing relations $(2,0) \geq_M (1,1)$ and $(nd, 0, \ldots, 0) \geq_M (d, d, \ldots, d)$. In other words, the AM-GM inequality is simply comparing the very topmost element to the bottom element of $P'(n,m)$. In full generality, the elements of our poset $P'(n,m)$ represent all possible sequences of exponents of the symmetric polynomials, and the width of the poset represents the largest number of these symmetric polynomials of degree m which are incomparable according to the Muirhead inequality.

For example, Figure A.10 in the Appendix shows the width of $P'(4,9)$ is 3; by Muirhead's inequality, this is equivalent to the statement that the most mutually incomparable monomial symmetric polynomials possible in four variables of total degree 9 is 3, as attained (uniquely) by:

$$\left\{ \sum_{\sigma \in S_4} X^6_{\sigma(1)} X^1_{\sigma(2)} X^1_{\sigma(3)} X^1_{\sigma(4)}, \sum_{\sigma \in S_4} X^5_{\sigma(1)} X^2_{\sigma(2)} X^2_{\sigma(3)}, \sum_{\sigma \in S_4} X^4_{\sigma(1)} X^4_{\sigma(2)} X^1_{\sigma(3)} \right\}.$$

Our results form the next piece in a long history of attempts at trying to establish Sperner-type properties for different posets. One major result of this type can be found in O'Hara's combinatorial proof of the unimodality of the Gaussian binomial coefficients or Gaussian polynomials [9], which are an alternative description of the levels L_k of the poset $P(n,m)$. The unimodality of Gaussian binomial coefficients was originally a famous open problem in algebraic combinatorics, and its original proofs came before O'Hara's. The first of these was due to Sylvester [15] (20 years after it was originally conjectured), and later Proctor who used linear algebra [11] and White found a proof using Pólya theory [17]. O'Hara's proof is remarkable for being the first purely combinatorial approach. She defines a poset similar to $P(n,m)$ where the size of each level is equal to the corresponding coefficient in the Gaussian polynomial, and inductively constructs a symmetric chain decomposition. In this way, the symmetric chain decomposition algorithm is on some suitably-defined "outer layer" only and the induction takes care of the remaining smaller poset.

A similar history can be seen in Zhong's work in proving the unimodality and symmetry of weak composition rank sequences [18]. As with O'Hara's result, Zhong was also not the first to discover the proof of the unimodality weak composition, this was done by Sagan [13]. Technically Sagan proved the unimodality of just the composition, but as Zhong points out in [18] there is a simple bijection between compositions and weak compositions. But again, Zhong was the first to provide a combinatorial proof of this result. In Zhong's proof, the strategy provides an algorithm to construct all the chains and then to prove that those chains give a symmetric chain decomposition.

For our results, the proofs we provide for $P(n, m)$ consist of a construction of chains and then a proof of those chains decomposing our poset, loosely following the structure of Zhong's proof. Our results for $P'(n, m)$ align more with O'Hara's approach; our proofs are inductive and provide a chain decomposition only on a suitably carved "outer layer" with induction taking care of the smaller inside poset.

1.1 Definitions

Definition 1.1. A relation $R \subset P \times P$ on a set P is called a *partial order* on P if R is reflexive, transitive, and antisymmetric. Then P equipped with R is called a *partially ordered set* or poset. We typically use the notation "$x \preceq y$" to mean $(x, y) \in R$, and denote the poset by $(P, \preceq)$, or simply P if the relation is understood. Two posets $(P, \preceq)$ and $(P', \preceq')$ are *isomorphic* if there is a bijective map $f : P \to P'$ such that $x \preceq y$ in P if and only if $f(x) \preceq' f(y)$ in P'. When such an f exists, we write $(P, \preceq) \cong (P', \preceq')$.

All posets in this thesis will be finite.

Definition 1.2. Two elements a and b in a partially ordered set $(P, \preceq)$ are called *comparable* if either $a \preceq b$ or $b \preceq a$. If neither $a \preceq b$ nor $b \preceq a$, then a and b are *incomparable*.

Definition 1.3. A *Hasse Diagram* of a poset $(P, \preceq)$ consists of a collection of points in the plane (vertices), one vertex for each element of P, such that

1) if $a \prec b$, then the vertex b is placed higher than a in the diagram, and

2) if $a \prec b$ and there is no element $c \in P$ such that $a \prec c \prec b$, then a line is drawn joining a and b. (We then also say b *covers* a).

We note here that for finite sets P, the cover relations actually determine the whole poset up to isomorphism (hence a finite poset is determined by its Hasse diagram).

Definition 1.4. Let $(P, \preceq)$ be a poset, and let $S \subseteq P$. Then $(S, \preceq)$ is a *subposet*.

Definition 1.5. A subposet $(C, \preceq)$ in which every two elements of C are comparable is called a *chain*.

Definition 1.6. A chain is *maximal* if it is contained in no strictly larger chain.

Definition 1.7. The *height* of a poset $(P, \preceq)$ is the size of a maximal chain.

Definition 1.8. A subposet $(A, \preceq)$ in which every two elements of A are incomparable is called an *antichain*.

Definition 1.9. The *width* of the poset $(P, \preceq)$ is the size of a largest antichain.

Hopcroft and Karp [4] proved that the width of a poset can be computed in polynomial time, via a reduction to the problem of finding a maximum size matching in a bipartite graph. This should not be too surprising, given the equivalence between Dilworth and Hall's marriage theorem.

Definition 1.10. A *graded poset* is a poset $(P, \preceq)$ equipped with a rank function ρ from P to $\mathbb{Z}_{\geq 0}$. ρ such that for all $x, y \in P$, if y covers x, then $\rho(y) = \rho(x) - 1$. [1]

Equivalently, a poset is *graded* if every maximal chain has the same length. Note that the roles of x and y in the above are reversed compared to the usual definition of a rank function ρ' in the literature. But they are readily seen to be equivalent, by taking $\rho'(x) := \rho(P) - \rho(x)$, where $\rho(P)$ is the maximum rank attained in P. We employ this "upside down" definition as it will simplify some of our algebra later.

For example, the usual rank function for $x = (x_1, x_2, \ldots, x_n)$ the Boolean lattice $\{0,1\}^n$, is defined as $\rho'(x) = \sum_i x_i$, that is it counts the number of ones which corresponds to the Hamming weight of x. For our definition, a rank function would be $\rho(x) = n - \sum_i x_i$, which counts the number of zeroes.

In contrast, below is an example of a poset generated by the covering relations $a > b > c > d$ and $a > e > d$. This is not a graded poset since there are two maximal chains $C_1 = \{a, b, c, d\}$ and $C_2 = \{a, e, d\}$ which do not have the same length.

---

[1] Note that the additional condition "if $x \preceq y$ then $\rho(x) \geq \rho(y)$", often included in the definition of rank, is redundant here as we are only considering finite posets (so any comparable elements can be connected by a sequence of covers).

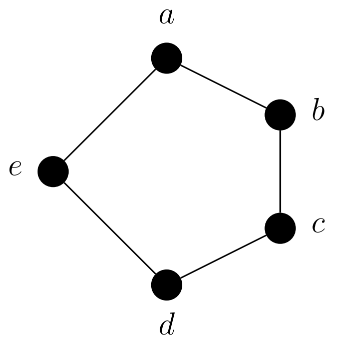


Figure 1.1: Example of a non-graded Poset

Definition 1.11. We say that the chain $C = x_0 \preceq x_1 \preceq \cdots \preceq x_k$ is a graded poset is a symmetric chain if $\rho(x_i) + \rho(x_{k-i}) = \rho(P)$ for $0 \leq i \leq k$ where $\rho(P)$ is the maximum rank.

Definition 1.12. A poset admits a *symmetric chain decomposition (SCD)* if the poset can be expressed as the disjoint union of symmetric chains.

Definition 1.13. A *level* in a graded poset is defined to be $L_k = \{x \in P : \rho(x) = k\}$.

Definition 1.14. A poset is said to have the Sperner property, or to be a *Sperner poset*, if the poset is graded and contains no antichain which is larger than the size of a largest level.

Definition 1.15. A sequence of non-negative integers $\{a_0, a_1, \ldots, a_k\}$ is said to be *unimodal* if there exists a $0 \leq j \leq k$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_k$$

Definition 1.16. For $n, k \in \mathbb{Z}_{\geq 0}$ with $n \geq k$, the Gaussian Polynomial or q-binomial coefficient $\binom{n}{k}_q$ or $\left\{ {n \atop k} \right\}_q$ is defined as the following rational function of the variable q:

$$\begin{aligned}\binom{n}{k}_q &= \frac{\prod_{i=1}^{n}(q^i - 1)}{\prod_{j=1}^{k}(q^j - 1)\prod_{i=1}^{n-k}(q^i - 1)} \\ &= \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)\cdot(q^{n-k} - 1)(q^{n-k-1} - 1)\cdots(q - 1)}.\end{aligned}$$

If $k = 0$ or $n = k$ we interpret the Gaussian Polynomial to be 1, as one of the products in the denominator above is empty and the other matches the numerator.

Note that evaluating the above expression for real-valued q and letting $q \to 1$ gives the usual binomial coefficients, namely

$$\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k}$$

by (for example) n applications of l'Hôpital's rule. In fact, since $\binom{n}{k}_q$ turns out to be a polynomial (see Lemma 4.21), the above equality can be simply interpreted as $\binom{n}{k}_1 = \binom{n}{k}$.

Definition 1.17. For $n, m \in \mathbb{Z}_{\geq 0}$, let $P(n, m) = \{x \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} x_i = m\}$.

Definition 1.18. For $n, m \in \mathbb{Z}_{\geq 0}$, let $P'(n, m) = \{x \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} x_i = m \text{ and } x_i \geq x_{i+1}\ \forall i\}$.

Throughout this thesis, $P(n, m)$ and $P'(n, m)$ will always be posets equipped with the majorization order $\leq_M$:

Definition 1.19. For any two sequences $x, y \in (\mathbb{Z}_{\geq 0})^n$, we say x is *majorized* by y if

$$\sum_{i=0}^{j} x_i \leq \sum_{i=0}^{j} y_i \quad \text{for } 0 \leq j \leq n-2$$

and

$$\sum_{i=0}^{n-1} x_i = \sum_{i=0}^{n-1} y_i$$

Definition 1.20. For each nonnegative integer $i \leq m$ we define a *slice* to be $S_i = \{x \in P(n, m) \text{ or } P'(n, m) : x_{n-1} = i\}$. That is, the set of vectors whose last entry is i.

Definition 1.21. The rank function for $P(n, m)$ is defined to be $\rho(x) = \sum_{i=0}^{n-1} i x_i$.

We will see in Proposition 1.23 below that this is indeed a rank function.

Definition 1.22. The induced rank function for $P'(n, m)$ is also defined to be $\rho(x) = \sum_{i=0}^{n-1} i x_i$.

Again we note here that $P'(n, m)$ is not a graded poset and so to use the term level or to define a rank function is not accurate. However since $P'(n, m) \subseteq P(n, m)$, $P'(n, m)$ inherits some of the structure from $P(n, m)$, hence we use induced rank and induced level (or often just “level”) in the same way as rank and level for $P(n, m)$. *In particular, whenever Hasse diagrams of* $P(n, m)$ *and (*$P'(n, m)$*) are illustrated in figures, they will be depicted so that elements of constant (respectively, induced) rank are depicted along the same horizontal line across the page.* See for example the alignment of $(5, 1, 1, 1)$ and $(4, 2, 2, 0) \in P'(4, 8)$, are both of induced rank 6, in Figure 2.2.

While it is true that *any* poset is a subposet of a graded poset so can inherit some level structure (if $P = \{a_i : i \in \{1, 2, \ldots, N\}$, then $f : P \to \{0, 1\}^N$ given by $f(a_i)_j := 1$ if and only if $a_j \preceq a_i$ makes $f(P)$ into a subposet of the Boolean lattice which is isomorphic to P), this inherited level-structure for $P'(n, m)$ is special as it retains a “Sperner” property with respect to this induced rank function for $n \leq 4$. This will be covered in detail in Chapter 5.

We close this introduction by checking that the above function $\rho(x) := \sum_{i=0}^{n-1} ix_i$ is indeed a rank function for $P(n, m)$.

Proposition 1.23. *$\rho(x) := \sum_{i=1}^{n-1} ix_i$ is a rank function on $(P(n, m), \leq_M)$.*

Proof. First claim that if y covers x, then there exists $t < n$ such that $x = (y_0, y_1, \ldots, y_{t-1} - 1, y_t + 1, y_{t+1}, \ldots, y_{n-1})$. The rank function condition follows because then $\rho(x) - \rho(y) = \sum_{i=1}^{n-1} i(x_i - y_i) = 0 + \cdots + 0 + (t - 1)(-1) + t(1) + 0 + \cdots = 1$. To see the claim, let t be the first entry in which x and y differ; as $x \leq_M y$ we must have $x_t < y_t$. Now let $z = (y_0, y_1, \ldots, y_t - 1, y_{t+1} + 1, y_{t+2}, \ldots, y_{n-1})$, so certainly, $z <_M y$. But also, as $x \leq_M y$, $\sum_{i=0}^{j} x_i \leq \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} z_i$ for any j except possibly $j = t$, and here $\sum_{i=0}^{t} x_i = \sum_{i=0}^{t-1} y_i + x_t < \sum_{i=0}^{t} y_i = (\sum_{i=0}^{t} z_i) + 1$, so as they are both integers it still follows $\sum_{i=0}^{t} x_i \leq \sum_{i=0}^{t} z_i$ too. So we’ve shown $x \leq_M z$. Since y covers x, it follows $x = z$, as claimed. $\square$

Chapter 2

Hasse Diagrams and Construction of $P(n,m)$ and $P'(n,m)$

In this chapter we will focus on the structures of $P(n,m)$ and $P'(n,m)$ and how to construct them iteratively, using decompositions into slices.

In $P(n,m)$ if we consider the element $(m,0,0,\ldots,0)$, it has rank 0 and is thus in level L_0, and is in fact the only such (as it is the unique nonnegative solution to $\sum_{i=0}^{n-1} ix_i = 0, \sum_{i=0}^{n-1} x_i = m$).

To construct a new poset say $P(n,m)$, if we look at $P(n-1,m)$ all of these elements still sum to m as desired, but they need to be lengthened by one space. To preserve the sum we can simply append a 0 to the end and now all elements are of the correct length and sum. These elements then become a part of slice 0 in $P(n,m)$ since their last digit is 0. To construct slice 1 we can follow a similar process but we instead start with $P(n-1,m-1)$. This way when we append 1 to each element, we not only lengthen it but we also have to increase the sum. So to create slice i in $P(n,m)$ we take $P(n-1,m-i)$ for $0 \leq i \leq m$ and append i to each of the elements. When we do this we get the correct elements, we now have to shift the elements into the appropriate levels. For $x \in P(n,m)$, when we appended i we increased the level by $i(n-1)$, since $\sum_{j=0}^{n-2} jx_j$ is not affected, then to get $\sum_{j=0}^{n-1} jx_j = \sum_{j=0}^{n-2} jx_j + i(n-1)$. So each element in $P(n,m)$ is shifted by exactly $i(n-1)$ from the $P(n-1,m-i)$ from which it came.

For example to construct $P(4,3)$, we take the posets $P(3,3)$, $P(3,2)$, $P(3,1)$, and $P(3,0)$ and append $0, 1, 2$, and 3 to their respective elements so that each smaller poset now has length 4 and all elements sum to 3. All the elements of $P(3,3)$ remain in the same levels since we appended 0. For $P(3,2)$ each element is shifted down $1(3) = 3$ levels (that is to say, all ranks have increased by 3), for $P(3,1)$ each element is shifted by $2(3) = 6$, and for $P(3,0)$ each element is shifted by $3(3) = 9$.

The Hasse diagram in figure 2.1 shows the slices which consist of the smaller posets, as well as how each new slice is shifted by $0, 3, 6$, and 9.

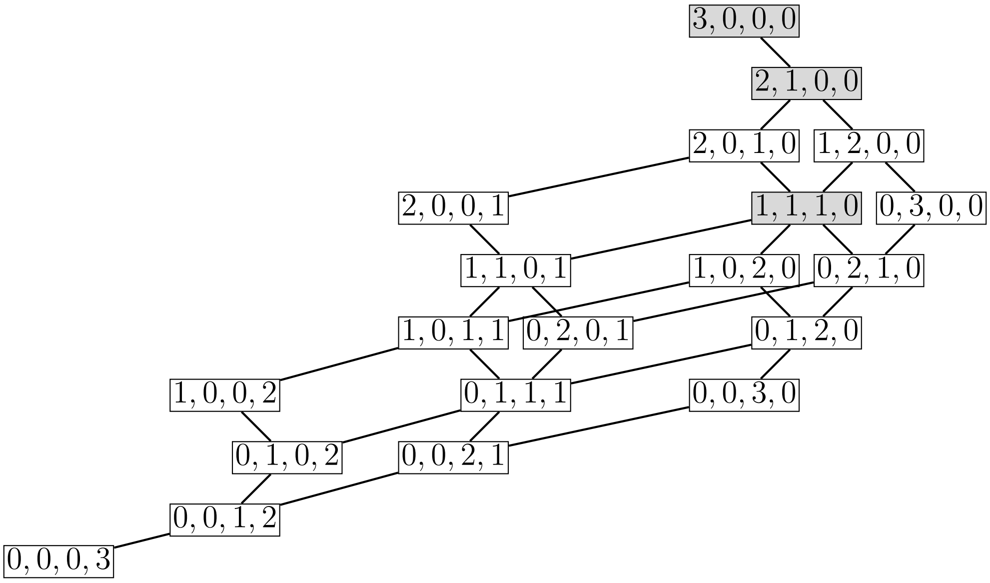


Figure 2.1: Hasse Diagram for $P(4,3)$. [1]

So with this construction we obtain the following set decomposition of the levels:

$$L(k,n,m) = \bigsqcup_{i=0}^{m} L(k - i(n-1), n-1, m-i) \times \{i\}$$

which in particular gives a recursion for their sizes

$$|L(k,n,m)| = \sum_{i=0}^{m} \Big| L\big(k - i(n-1), n-1, m-i\big) \Big|,$$

where $k, m, n \in \mathbb{Z}_{\geq 0}$ and $k \leq m(n-1)$. Furthermore, for the base cases:

- if $k < 0$ then $L(k,n,m) = \emptyset$ (by definition) so the last few terms in the above sum will disappear,
- if $k = 0$ and $n > 0$ then $L(k,n,m) = \{(m, 0, \ldots, 0)\}$ so $|L(k,n,m)| = 1$,

---

[1] Note that the subposet (in gray) $P'(4,3) \subseteq P(4,3)$ only consists of $(3,0,0,0)$, $(2,1,0,0)$ and $(1,1,1,0)$.

- if $k = n = 0$ but $m > 0$ then $|L(k, n, m)| = 0$,
- if $k = n = m = 0$ then $|L(0, 0, 0)| = 1$ as it consists just of the empty string, which has 0 entries and the empty sum 0,
- if $k > 0$ but $n = 0$ then $|L(k, n, m)| = 0$.

These establish all $|L(k, n, m)|$ when k or n is 0, and all others can be found inductively using the above relation.

In $P'(n, m)$ to construct larger posets from smaller ones, we cannot just append values to the end of our elements since the elements of $P'(n, m)$ must be non-increasing. To ensure the elements remain non-increasing when we append say i to lengthen an element, we must also fix that value of i in all of the prior spaces as well. So for slice 0 this is the same as just appending a 0, but not so for the rest of the slices. To construct slice 1 we need to append a 1 and also fix 1 in all of the n spaces, meaning every entry is at least 1. This leaves us with a sum of $m - n$ which is unfixed. Thus slice 1 is constructed from $P'(n - 1, m - n)$ rather than just $m - 1$ as it was with $P(n, m)$. And similarly for a general slice i we take $P'(n - 1, m - in)$ for $0 \leq i \leq \lfloor \frac{m}{n} \rfloor$ and append i and fix i in each of the prior spaces again meaning that each entry is at least i. When we do this we get the correct elements, but again we need to shift the elements into the appropriate levels. For $x \in P'(n, m)$ when we append and fix i we are effectively increasing the level by the rank of $(i, i, \ldots, i)$ which is $\sum_{j=0}^{n-1} ji = i\frac{n(n-1)}{2}$. So each element in $P'(n, m)$ is shifted in rank by exactly $i\frac{n(n-1)}{2}$ from the $P'(n - 1, m - in)$ from which it came.

For example to construct $P'(4, 8)$ we take the posets $P'(3, 8), P'(3, 4)$ and $P'(3, 0)$ and append and fix $0, 1,$ and 2 respectively so that each element in the smaller posets now have length 4 and all the elements sum to 8. All the elements of $P'(3, 8)$ remain in the same level since we appended and fixed 0. For $P'(3, 4)$ each element's rank is shifted by $1 \cdot \frac{4 \cdot 3}{2} = 6$, and for $P'(3, 0)$ each element's rank is shifted by $2 \cdot \frac{4 \cdot 3}{2} = 12$.

As you can see in the Hasse diagram in figure 2.2 we see that the slices consist of the smaller posets, as well as how each new slice is shifted by $0, 6,$ and 12.

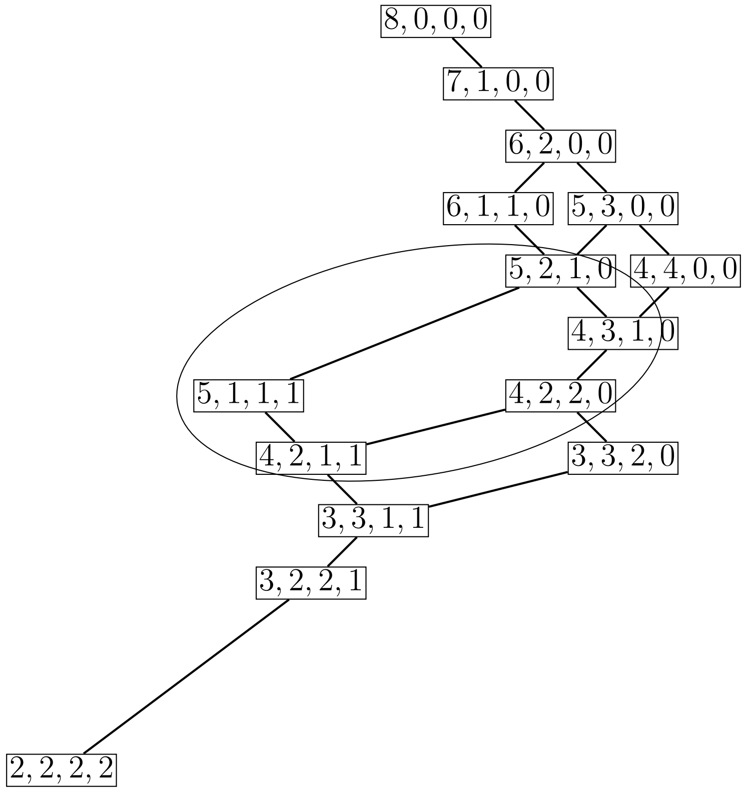


Figure 2.2: Hasse Diagram for $P'(4,8)$[2]

So again with this construction we also obtain a decomposition and hence recursive formula for the sizes of each of the L' levels. This one however is a bit more particular.

$$L'(k,n,m) = \bigsqcup_{i=0}^{\left\lfloor \frac{2k}{n(n-1)} \right\rfloor} \left(L'\Big(k - i\Big(\frac{n(n-1)}{2}\Big), n-1, m-ni\Big) + \underbrace{(i,i,\ldots,i)}_{n-1} \right) \times \{i\}, \text{hence}$$

$$|L'(k,n,m)| = \sum_{i=0}^{\left\lfloor \frac{2k}{n(n-1)} \right\rfloor} \left| L'\Big(k - i\Big(\frac{n(n-1)}{2}\Big), n-1, m-ni\Big) \right|$$

where $k, m, n \in \mathbb{Z}_{\geq 0}$ and, as in the $L(k,n,m)$ case:

- if $k = 0$ and $n > 0$ then $|L'(k,n,m)| = 1$,
- if $k = n = 0$ but $m > 0$ then $|L'(k,n,m)| = 0$,
- if $k = n = m = 0$ then $|L'(0,0,0)| = 1$,

---

[2] The circled part, a subposet isomorphic to the poset from Figure 1.1, illustrates why $P'(n,m)$ is not a graded poset in general, as one can use this subposet to build maximal chains in $P'(4,8)$ of sizes 10 and 11.

- if $k > 0$ but $n = 0$ then $|L'(k, n, m)| = 0$.

- As with $P(n, m)$ the largest element is $(m, 0, 0, \ldots, 0)$ and is in level $1(0) + 2(0) + \cdots + (n-1)(0) = 0$.

- The unique smallest element in $P'(n, m)$ is $(\lceil \frac{m}{n} \rfloor, \lceil \frac{m}{n} \rfloor, \ldots, \lceil \frac{m}{n} \rfloor)$ which would be in level $\lceil \frac{m}{n} \rfloor + 2(\lceil \frac{m}{n} \rfloor) + \cdots + (n-1)(\lceil \frac{m}{n} \rfloor) = \frac{n(n-1)}{2} \cdot \lceil \frac{m}{n} \rfloor$.

Chapter 3

$P(n,m)$ and $P'(n,m)$ Cases for $n \leq 3$

In this chapter we focus on the widths, w, of $P(n,m)$ and $P'(n,m)$ when $n \leq 3$. The proofs for the two $n = 3$ cases help provide the framework for the length 4 cases which become more much tedious. We will see the proofs of those in chapters 4 and 5.

Theorem 3.1. *For $P(1,m)$ and $P(2,m)$ the width is $w = 1$.*

Proof. If $n = 1$, then there is only one possible element in $P(1,m)$ hence the length of a longest antichain is one.

Now suppose that $n = 2$, and let $x = x_1x_2$, $y = y_1y_2$ be distinct elements in $P(2,m)$. Since x and y have fixed sum m, then

$$x_1 + x_2 = m = y_1 + y_2.$$

If $x_1 = y_1$, then $x_2 = y_2$, and thus $x = y$. Since x and y are distinct then $x_1 \neq y_1$, and moreover either $x_1 < y_1$ or $x_1 > y_1$ and similarly for x_2 and y_2. This along with the above equation gives us that either $x \leq_M y$ or $y \leq_M x$. Thus these elements actually form a chain of length $n = 2$. Thus we have that the length of a longest antichain again is one. $\square$

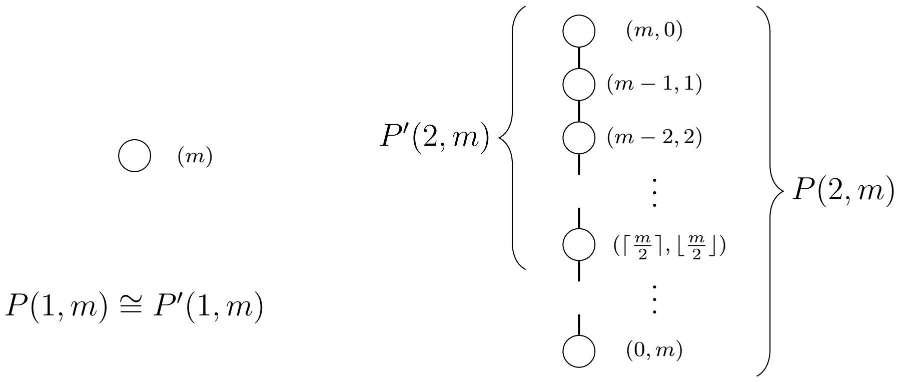


Figure 3.1: Hasse Diagrams of $P(n,m)$ and $P'(n,m)$ when $n = 1$ and $n = 2$

Corollary 3.2. *Since* $P'(n,m) \subseteq P(n,m)$ *then the widths of* $P'(1,m)$ *and* $P'(2,m)$ *are also* $w = 1$.

For $n \geq 3$ we start to see more interesting structures. For example $P(3,2)$ is the smallest poset in which we see incomparable elements. That is $(1,0,1) \nleq_M (0,2,0)$, since $1 \geq 0$ but $1+0 \leq 0+2$. With these more interesting structures we now introduce an important subset of $P(n,m)$ which we claim is an antichain.

Lemma 3.3. *Let* $A_k = \{x \in P(n,m) : \sum_{i=0}^{n-1} ix_i = k\}$*, then* A_k *is an antichain.*

While this also follows from $P(n,m)$ being ranked, see Proposition 1.23, it is insightful to see a more concrete algebraic proof.

Proof. Let $x, y \in (P(n,m), \leq_M)$ and $x \neq y$ and let $x, y \in A_k$ for some fixed $k \in \mathbb{Z}_{\geq 0}$. Then

$$\sum_{i=0}^{n-1} x_i = m = \sum_{i=0}^{n-1} y_i \tag{3.1}$$

$$\sum_{i=0}^{n-1} ix_i = k = \sum_{i=0}^{n-1} iy_i. \tag{3.2}$$

Suppose for the sake of contradiction that x and y are comparable, and without loss of generality suppose that $x \geq_M y$. That is,

$$\sum_{i=0}^{j} x_i \geq \sum_{i=0}^{j} y_i \qquad \text{for all} \quad 0 \leq j \leq n-1. \tag{3.3}$$

Expanding equation 3.3 gives us the following inequalities:

$$\begin{aligned} x_0 &\geq y_0 \\ x_0 + x_1 &\geq y_0 + y_1 \\ x_0 + x_1 + x_2 &\geq y_0 + y_1 + y_2 \\ \vdots \quad &\quad \vdots \\ x_0 + x_1 + x_2 + \cdots + x_{n-2} &\geq y_0 + y_1 + y_2 + \cdots + y_{n-2}. \end{aligned}$$

Subtracting each of these partial sums from m then gives:

$$\begin{aligned} m - x_0 &\leq m - y_0 \\ m - (x_0 + x_1) &\leq m - (y_0 + y_1) \\ \vdots \quad &\quad \vdots \\ m - (x_0 + x_1 + \cdots + x_{n-2}) &\leq m - (y_0 + y_1 + \cdots + y_{n-2}) \end{aligned} \tag{3.4}$$

We can expand $\sum_{i=0}^{n-1} i x_i$ as

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_{n-1} &= m - x_0 \\ +x_2 + x_3 + \cdots + x_{n-1} &= m - (x_0 + x_1) \\ +x_3 + \cdots + x_{n-1} &= m - (x_0 + x_1 + x_2) \\ \vdots \\ +x_{n-1} &= m - (x_0 + x_1 + x_2 + \cdots + x_{n-2}), \end{aligned}$$

and we can do the same thing with $\sum_{i=0}^{n-1} iy_i$. Thus we can rewrite equation 3.2 as:

$$\begin{aligned} m - x_0 &\quad m - y_0 \\ +m - (x_0 + x_1) &\quad +m - (y_0 + y_1) \\ +m - (x_0 + x_1 + x_2) &\quad +m - (y_0 + y_1 + y_2) \\ &\vdots \\ +m - (x_0 + x_1 + x_2 + \cdots + x_{n-2}) &\quad +m - (y_0 + y_1 + y_2 + \cdots + y_{n-2}) \\ &= k = \end{aligned}$$

Putting these together with the inequalities from 3.4 gives us:

$$\begin{aligned} m - x_0 &\leq m - y_0 \\ +m - (x_0 + x_1) &\leq +m - (y_0 + y_1) \\ +m - (x_0 + x_1 + x_2) &\leq +m - (y_0 + y_1 + y_2) \\ &\vdots \\ +m - (x_0 + x_1 + x_2 + \cdots + x_{n-2}) &\leq +m - (y_0 + y_1 + y_2 + \cdots + y_{n-2}) \\ &= k = \end{aligned}$$

But since $x \neq y$ then one of these inequalities must be false and therefore x and y are indeed incomparable. Thus we have that A_k is an antichain. $\square$

Corollary 3.4. *The definition of A_k is exactly that for the definition of level k, thus every level of $P(n, m)$ is an antichain and every induced level of $P'(n, m)$ is an antichain.*

If we already knew what the width of these posets are, then by Dilworth's theorem we would know there exists a chain decomposition of the same size. And so while we don't use Dilworth's Theorem exactly, the theorem does provide a helpful upper bound to w while any A_k from lemma 3.3 will be an antichain and thus a lower bound on w.

Theorem 3.5 (Dilworth's Theorem). *Let $(P, \preceq)$ be a finite poset of width w. Then there exists a partition of P into w chains.*

Our strategy for providing the exact width, w, of $P(3, m)$ is to first provide a chain decomposition of size $\lfloor \frac{m}{2} \rfloor + 1$, which would give an upper bound according to Dilworth's Theorem. We then provide a lower bound by showing there exists an antichain of size $\lfloor \frac{m}{2} \rfloor + 1$, and moreover we claim that $|A_{\frac{2m}{2}}|$ is such an antichain.

To show that our chain decomposition does in fact capture all the elements in $P(n, m)$ we provide a useful lemma about the size of $P(n, m)$.

Lemma 3.6. *Let $P(n, m)$ be the set of all sequences x of length n such that for all $x_i \in x$, $x_i \geq 0$ and $\sum_{i=0}^{n-1} x_i = m$, then $|P(n, m)| = \binom{m+n-1}{n-1}$.*

Proof. This is equivalent to saying there are m stars and $n - 1$ bars to arrange, thus there are a total of $m + n - 1$ many objects and we can choose where the $n - 1$ bars go. Thus we have $|P(n, m)| = \binom{m+n-1}{n-1}$. $\square$

Theorem 3.7. *For $P(3, m)$ the width is $w = \lfloor \frac{m}{2} \rfloor + 1$.*

Proof. For $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, we let chain C_i be defined as follows:

$$C_i = \{(m - i, 0, i), (m - i - 1, 1, i), (m - i - 2, 2, i), \ldots (i, m - 2i, i),$$
$$(i, m - 2i - 1, i + 1), (i, m - 2i - 2, i + 2) \ldots (i, 0, m - i)\}$$

Note that:

$$|C_i| = (m - 2i + 1) + (m - 2i + 1) - 1 = 2m + 1 - 4i.$$

So summing over all i, the lengths of each chain we have:

$$\begin{aligned}
\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \big(2m+1-4i\big) &= (2m+1)\Big(\Big\lfloor \frac{m}{2} \Big\rfloor + 1\Big) - 4\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} i \\
\text{for } m \text{ odd } &\Rightarrow (2m+1)\Big(\frac{m-1}{2}+1\Big) - 4\sum_{i=0}^{\frac{m-1}{2}} i \\
&= m^2 + m + \frac{m}{2} + \frac{1}{2} - \frac{4}{2}\Big(\frac{m-1}{2}\Big)\Big(\frac{m-1}{2}+1\Big) \\
&= m^2 + \frac{3m}{2} + \frac{1}{2} - 2\Big(\frac{m^2}{4} - \frac{1}{4}\Big) \\
&= \frac{m^2}{2} + \frac{3m}{2} + 1 \\
&= \frac{(m+2)(m+1)}{2} \\
&= \binom{m+2}{2} \\
&= |P(3,m)| \\
\text{for } m \text{ even } &\Rightarrow (2m+1)\Big(\frac{m}{2}+1\Big) - 4\sum_{i=0}^{\frac{m}{2}} i \\
&= m^2 + 2m + \frac{m}{2} + 1 - \frac{4}{2}\Big(\frac{m}{2}\Big)\Big(\frac{m}{2}+1\Big) \\
&= m^2 + \frac{5m}{2} + 1 - 2\Big(\frac{m^2}{4} + \frac{m}{2}\Big) \\
&= \frac{m^2}{2} + \frac{3m}{2} + 1 \\
&= \frac{(m+2)(m+1)}{2} \\
&= \binom{m+2}{2} \\
&= |P(3,m)|
\end{aligned}$$

Now if we can verify that each element in $P(3,m)$ is in exactly one chain, then the C_is form a chain decomposition.

Let $x = (x_0, x_1, x_2) \in P(3,m)$, and evaluate $x_0 + x_1$ and $x_1 + x_2$. If $x_0 + x_1$ is larger then $x \in C_{x_2}$, and similarly if $x_1 + x_2$ is larger then $x \in C_{x_0}$. If $x_0 + x_1 = x_1 + x_2$ then $x_0 = x_2$ and thus $C_{x_0} = C_{x_2}$. Thus every element in $P(3,m)$ can be assigned to exactly one chain, and the union of all the chains is the entire poset. Thus it remains to show that there are at most $\lfloor \frac{m}{2} \rfloor + 1$ many elements in a largest antichain.

Since $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, then there are $\lfloor \frac{m}{2} \rfloor + 1$ many chains. Thus taking one element from each chain we can have at most $\lfloor \frac{m}{2} \rfloor + 1$ many elements in a longest antichain.

By Lemma 3.3, we know that A_k is an antichain. So to bound w below, we now claim that A_k with $k = \frac{2m}{2} = m$ has the correct size, that is $\lfloor \frac{m}{2} \rfloor + 1$. Let C_i be defined as above, and let $A_m = D_1 \cup D_2$. Where D_1 and D_2 are defined as:

$$D_1 = \{x \in A_m \cap C_i : x_2 = i\}$$

$$D_2 = \{x \in A_m \cap C_i : x_0 = i\}.$$

Suppose then $x \in D_1$, then

$$x_0 + x_1 + i = m \quad \text{and} \quad x_1 + 2i = k = m.$$

This gives us that $x_0 = i = x_2$ and hence $D_1 \subseteq D_2$. Similarly if $x \in D_2$, then

$$i + x_1 + x_2 = m \quad \text{and} \quad x_1 + 2x_2 = k = m.$$

Which gives us that $x_2 = i = x_0$ and hence $D_2 \subseteq D_1$. Thus $D_1 = D_2$ and since $A_m = D_1 \cup D_2$ we have $A_m = D_1$. So

$$A_m = \Big\{(0, m, 0), (1, m-2, 1) \ldots, \Big(\Big\lfloor \frac{m}{2} \Big\rfloor, m - 2\Big\lfloor \frac{m}{2} \Big\rfloor, \Big\lfloor \frac{m}{2} \Big\rfloor\Big)\Big\},$$

and $|A_m| = \lfloor \frac{m}{2} \rfloor + 1$ as desired. □

Since we do not have a nice formula for the general size of $P'(3, m)$, our strategy here will be a little different. We will instead provide an inductive proof that the width of $P'(3, m)$ is $\lfloor \frac{m}{6} \rfloor + 1$. We note here that one could provide a similar inductive proof for the $P(3, m)$ case with some careful alterations of the proof provided for $P'(3, m)$. We gave the explicit proof for $P(3, m)$ because again this will be important later for the proof of the width of $P(4, m)$.

Lemma 3.8. *There is a chain of length m in $P'(3,m)$ consisting of all $(x_0,x_1,x_2) \in P'(3,m)$ such that $x_2 = 0$ or $x_0 - x_1 \leq 2$ or both.*

Proof. The last element of $P'(3,m)$ is $\big(\frac{m}{3},\frac{m}{3},\frac{m}{3}\big)$ if $m \equiv 0 \bmod 3$, $\big(\lceil\frac{m}{3}\rceil,\lfloor\frac{m}{3}\rfloor,\lfloor\frac{m}{3}\rfloor\big)$ if $m \equiv 1 \bmod 3$, and $\big(\lceil\frac{m}{3}\rceil,\lceil\frac{m}{3}\rceil,\lfloor\frac{m}{3}\rfloor\big)$ if $m \equiv 2 \bmod 3$. When $m \equiv 1 \bmod 3$ then the last element is in level $\frac{m+1}{3} + 2\big(\frac{m-2}{3}\big) = m-1$ and similarly for $m \equiv 2 \bmod 3$ then the last element is in level $\frac{m-1}{3} + 2\big(\frac{m-1}{3}\big) = m-1$. Therefore when $m \equiv 1$ or $2 \bmod 3$ then the length of a longest chain is at most m. For $m \equiv 0 \bmod 3$ the last element is in level $\frac{m}{3} + 2\big(\frac{m}{3}\big) = m$. However there is nothing in level $m-1$. The element directly above $\big(\frac{m}{3},\frac{m}{3},\frac{m}{3}\big)$ is $\big(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1\big)$ which is in level $\frac{m}{3} + 2\big(\frac{m}{3}-1\big) = m-2$. So again the length of a longest chain when $m \equiv 0 \bmod 3$ is at most $m-1+1 = m$.

Case 1. *If $m \equiv 0 \bmod 6$ then $C = \{(m,0,0),(m-1,1,0),\ldots(\frac{m}{2},\frac{m}{2},0),(\frac{m}{2},\frac{m}{2}-1,1),(\frac{m}{2},\frac{m}{2}-2,2),(\frac{m}{2}-1,\frac{m}{2}-1,2),(\frac{m}{2}-1,\frac{m}{2}-2,3),\ldots,(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1)\} \cup \{(\frac{m}{3},\frac{m}{3},\frac{m}{3})\}$ is a chain of length m.*

Proof. There are $\frac{m}{2}+1$ elements with $x_2 = 0$. Starting with $(\frac{m}{2},\frac{m}{2},0)$ if $x_0 - x_1 \leq 1$ then we move one from x_1 to x_2. If $x_0 - x_1 = 2$ we move one from x_0 to x_2. Thus the remaining elements all have $x_0 - x_1 \leq 2$ and these elements follow the pattern $0,1,2,0,1,2,0,\ldots$ for what $x_0 - x_1$ is equal to. We repeat this until we reach $(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1)$ which has $x_0 - x_1 = 1$ but cannot have anything moved from x_1 to x_2 and maintain decreasing-ness, so this last repetition only has two elements in it rather than 3. So rewriting $(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1)$ to $(\frac{m}{2}-(\frac{m}{6}-1),\frac{m}{2}-\frac{m}{6},\frac{m}{2}-(\frac{m}{6}+1))$, we see that there are

$$3\Big(\frac{m}{6}-1\Big)+2 = \frac{m}{2}-1$$

many elements in this second half of the chain. And so including the very last element which is $(\frac{m}{3},\frac{m}{3},\frac{m}{3})$, and removing the element we counted twice we have,

$$\underbrace{\frac{m}{2}+1}_{x_2=0} + \underbrace{\frac{m}{2}-1}_{x_0-x_1\leq 2} - \underbrace{1}_{\text{overlap}} + \underbrace{1}_{\text{last element}} = m$$

many elements in this chain as desired. ■

Case 2. *If* $m \equiv 1 \bmod 6$ *then* $C = \{(m,0,0),(m-1,1,0),\dots(\frac{m+1}{2},\frac{m+1}{2}-1,0),(\frac{m+1}{2},\frac{m+1}{2}-2,1),(\frac{m+1}{2}-1,\frac{m+1}{2}-1,1),(\frac{m+1}{2}-1,\frac{m+1}{2}-2,2),(\frac{m+1}{2}-1,\frac{m+1}{2}-3,3),\dots,(\frac{m+2}{3},\frac{m-1}{3},\frac{m-1}{3})\}$ *is a chain of length* m*.*

Proof. There are $\frac{m+1}{2}$ elements with $x_2 = 0$. Starting with $(\frac{m+1}{2},\frac{m+1}{2}-1,0)$ if $x_0 - x_1 \leq 1$ then we move one from x_1 to x_2. If $x_0 - x_1 = 2$ we move one from x_0 to x_2. Thus the remaining elements all have $x_0 - x_1 \leq 2$ and these elements follow the pattern $1,2,0,1,2,0,1,\dots$ for what $x_0 - x_1$ is equal to. We repeat this until we reach $(\frac{m+2}{3},\frac{m-1}{3},\frac{m-1}{3})$ which has $x_0 - x_1 = 1$ but nothing can be moved and still maintain decreasing-ness, so this last repetition only has one element in it rather than 3. So rewriting $(\frac{m+2}{3},\frac{m-1}{3},\frac{m-1}{3})$ to $(\frac{m+1}{2}-\frac{m-1}{6},\frac{m+1}{2}-1-\frac{m-1}{6},\frac{m+1}{2}-1-\frac{m-1}{6})$, we see that there are

$$3\Big(\frac{m-1}{6}\Big)+1=\frac{m+1}{2}$$

many elements in this second half of the chain. So removing the element we counted twice we have,

$$\underbrace{\frac{m+1}{2}}_{x_2=0}+\underbrace{\frac{m+1}{2}}_{x_0-x_1\leq 2}-\underbrace{1}_{\text{overlap}}=m$$

many elements in this chain as desired. ■

Case 3. *If* $m \equiv 2 \bmod 6$ *then* $C = \{(m,0,0),(m-1,1,0),\dots(\frac{m}{2},\frac{m}{2},0),(\frac{m}{2},\frac{m}{2}-1,1),(\frac{m}{2},\frac{m}{2}-2,2),(\frac{m}{2}-1,\frac{m}{2}-1,2),(\frac{m}{2}-1,\frac{m}{2}-2,3),\dots,(\frac{m+1}{3},\frac{m+1}{3},\frac{m-2}{3})\}$ *is a chain of length* m*.*

Proof. There are $\frac{m}{2}+1$ elements with $x_2 = 0$. Starting with $(\frac{m}{2},\frac{m}{2},0)$ if $x_0 - x_1 \leq 1$ then we move one from x_1 to x_2. If $x_0 - x_1 = 2$ we move one from x_0 to x_2. Thus the remaining elements all have $x_0 - x_1 \leq 2$ and these elements follow the pattern $0,1,2,0,1,2,0,\dots$ for what $x_0 - x_1$ is equal to. We repeat this until we reach $(\frac{m+1}{3},\frac{m+1}{3},\frac{m-2}{3})$ which has $x_0 - x_1 = 0$ but nothing can be moved and still maintain decreasing-ness, so this last repetition only has one element in it rather than 3. So rewriting $(\frac{m+1}{3},\frac{m+1}{3},\frac{m-2}{3})$ to $(\frac{m}{2}-\frac{m-2}{6},\frac{m}{2}-\frac{m-2}{6},\frac{m}{2}-\frac{m-2}{6}-1)$, we see that there are

$$3\Big(\frac{m-2}{6}\Big)+1=\frac{m}{2}$$

many elements in this second half of the chain. So removing the element we counted twice we have,

$$\underbrace{\frac{m}{2}+1}_{x_2=0}+\underbrace{\frac{m}{2}}_{x_0-x_1\leq 2}-\underbrace{1}_{\text{overlap}}=m$$

many elements in this chain as desired. ■

Case 4. *If* $m\equiv 3 \bmod 6$ *then* $C=\{(m,0,0),(m-1,1,0),\dots(\frac{m+1}{2},\frac{m+1}{2}-1,0),(\frac{m+1}{2},\frac{m+1}{2}-2,1),(\frac{m+1}{2}-1,\frac{m+1}{2}-1,1),(\frac{m+1}{2}-1,\frac{m+1}{2}-2,2),(\frac{m+1}{2}-1,\frac{m+1}{2}-3,3),\dots,(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1)\}\cup\{(\frac{m}{3},\frac{m}{3},\frac{m}{3})\}$ *is a chain of length* m.

Proof. There are $\frac{m+1}{2}$ elements with $x_2=0$. Starting with $(\frac{m+1}{2},\frac{m+1}{2}-1,0)$ if $x_0-x_1\leq 1$ then we move one from x_1 to x_2. If $x_0-x_1=2$ we move one from x_0 to x_2. Thus the remaining elements all have $x_0-x_1\leq 2$ and these elements follow the pattern $1,2,0,1,2,0,1,\dots$ for what x_0-x_1 is equal to. We repeat this until we reach $(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1)$ which has $x_0-x_1=1$ but cannot have anything moved from x_1 to x_2 and maintain decreasing-ness, so this last repetition only has one element in it rather than 3. So rewriting $(\frac{m}{3}+1,\frac{m}{3},\frac{m}{3}-1)$ to $(\frac{m+1}{2}-(\frac{m+3}{6})+1,\frac{m+1}{2}-\frac{m+3}{6},\frac{m+1}{2}-\frac{m+3}{6}-1)$, we see that there are

$$3\Big(\frac{m+3}{6}-1\Big)+1=\frac{m-1}{2}$$

many elements in this second half of the chain. And so including the very last element which is $(\frac{m}{3},\frac{m}{3},\frac{m}{3})$, and removing the element we counted twice we have,

$$\underbrace{\frac{m+1}{2}}_{x_2=0}+\underbrace{\frac{m-1}{2}}_{x_0-x_1\leq 2}-\underbrace{1}_{\text{overlap}}+\underbrace{1}_{\text{last element}}=m$$

many elements in this chain as desired. ■

Case 5. *If* $m\equiv 4 \bmod 6$ *then* $C=\{(m,0,0),(m-1,1,0),\dots(\frac{m}{2},\frac{m}{2},0),(\frac{m}{2},\frac{m}{2}-1,1),(\frac{m}{2},\frac{m}{2}-2,2),(\frac{m}{2}-1,\frac{m}{2}-1,2),(\frac{m}{2}-1,\frac{m}{2}-2,3),\dots,(\frac{m+2}{3},\frac{m-1}{3},\frac{m-1}{3})\}$ *is a chain of length* m.

Proof. There are $\frac{m}{2}+1$ elements with $x_2 = 0$. Starting with $(\frac{m}{2}, \frac{m}{2}, 0)$ if $x_0 - x_1 \leq 1$ then we move one from x_1 to x_2. If $x_0 - x_1 = 2$ we move one from x_0 to x_2. Thus the remaining elements all have $x_0 - x_1 \leq 2$ and these elements follow the pattern $0, 1, 2, 0, 1, 2, 0, \ldots$ for what $x_0 - x_1$ is equal to. We repeat this until we reach $(\frac{m+2}{3}, \frac{m-1}{3}, \frac{m-1}{3})$ which has $x_0 - x_1 = 1$ but nothing can be moved and still maintain decreasing-ness, so this last repetition only has two elements in it rather than 3. So rewriting $(\frac{m+2}{3}, \frac{m-1}{3}, \frac{m-1}{3})$ to $(\frac{m}{2} - (\frac{m-4}{6}), \frac{m}{2} - (\frac{m+2}{6}), \frac{m}{2} - (\frac{m+2}{6}))$, we see that there are

$$3\Big(\frac{m-4}{6}\Big) + 2 = \frac{m}{2}$$

many elements in this second half of the chain. So removing the element we counted twice we have,

$$\underbrace{\frac{m}{2} + 1}_{x_2=0} + \underbrace{\frac{m}{2}}_{x_0 - x_1 \leq 2} - \underbrace{1}_{\text{overlap}} = m$$

many elements in this chain as desired. ■

Case 6. *If* $m \equiv 5 \bmod 6$ *then* $C = \{(m, 0, 0), (m-1, 1, 0), \ldots (\frac{m+1}{2}, \frac{m+1}{2} - 1, 0), (\frac{m+1}{2}, \frac{m+1}{2} - 2, 1), (\frac{m+1}{2} - 1, \frac{m+1}{2} - 1, 1), (\frac{m+1}{2} - 1, \frac{m+1}{2} - 2, 2), (\frac{m+1}{2} - 1, \frac{m+1}{2} - 3, 3), \ldots, (\frac{m+1}{3}, \frac{m+1}{3}, \frac{m-2}{3})\}$ *is a chain of length* m.

Proof. There are $\frac{m+1}{2}$ elements with $x_2 = 0$. Starting with $(\frac{m+1}{2}, \frac{m+1}{2} - 1, 0)$ if $x_0 - x_1 \leq 1$ then we move one from x_1 to x_2. If $x_0 - x_1 = 2$ we move one from x_0 to x_2. Thus the remaining elements all have $x_0 - x_1 \leq 2$ and these elements follow the pattern $1, 2, 0, 1, 2, 0, 1, \ldots$ for what $x_0 - x_1$ is equal to. We repeat this until we reach $(\frac{m+1}{3}, \frac{m+1}{3}, \frac{m-2}{3})$ which has $x_0 - x_1 = 0$ but nothing can be moved and still maintain decreasing-ness. So rewriting $(\frac{m+1}{3}, \frac{m+1}{3}, \frac{m-2}{3})$ to $(\frac{m+1}{2} - \frac{m+1}{6}, \frac{m+1}{2} - \frac{m+1}{6}, \frac{m+1}{2} - \frac{m+1}{6} - 1)$, we see that there are

$$3\Big(\frac{m+1}{6}\Big) = \frac{m+1}{2}$$

many elements in this second half of the chain. So removing the element we counted twice we have,

$$\underbrace{\frac{m+1}{2}}_{x_2=0}+\underbrace{\frac{m+1}{2}}_{x_0-x_1\leq 2}-\underbrace{1}_{\text{overlap}}=m$$

many elements in this chain as desired. ■

And from these six cases we can see that these are in fact all the elements of $P'(3,m)$ such that either $x_2=0$ or $x_1-x_2\leq 2$ or both. □

As previously mentioned our strategy for proving what the width of $P'(3,m)$ is will be an inductive approach. In other words we will "peel" off a chain and show that the remaining elements indeed form a smaller poset.

Theorem 3.9. *For $P'(3,m)$, the width, w, is $\left\lfloor\frac{m}{6}\right\rfloor+1$. Moreover the Hasse Diagram can be decomposed into a single chain of length m and the Hasse Diagram for $P'(3,m-6)$.*

Proof. Suppose that $m\leq 5$, we see from the Hasse Diagrams in figure 3.2 that $P'(3,m)$ consists of a single chain. Thus the width, w, of each of these is $w=\left\lfloor\frac{m}{6}\right\rfloor+1$.

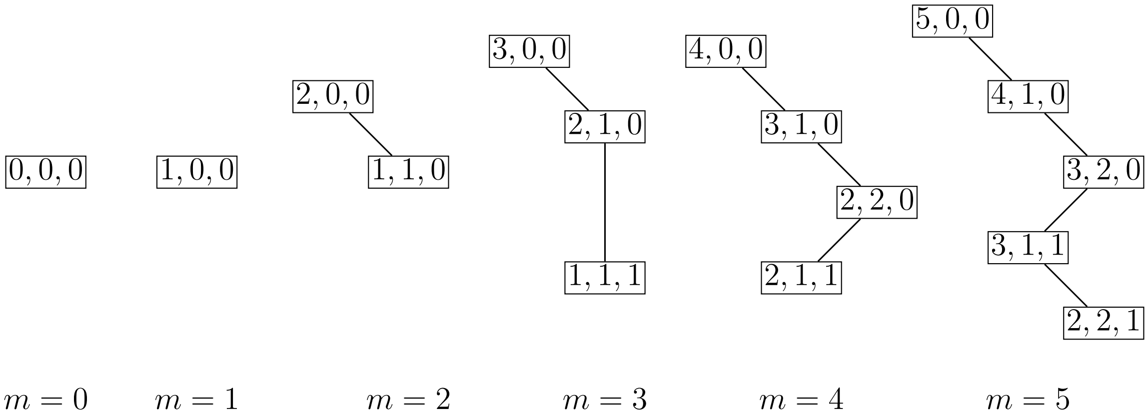


Figure 3.2: Hasse Diagrams for $P'(3,m)$ with $m\leq 5$

Now assume that $m\geq 6$ and suppose that $P'(3,m)$ is the smallest poset for which the theorem is false. If we let C be the chain, as described in Lemma 3.8, then the theorem holds for $P'(3,m)\setminus C$ provided it is isomorphic to a smaller poset. We claim that $P'(3,m)\setminus C\cong P'(3,m-6)$ via $f:(x_0,x_1,x_2)\mapsto(x_0-4,x_1-1,x_2-1)$. In particular to check this is well defined and surjective we show that $p = (p_0, p_1, p_2) \in C$ if and only if $(p_0 - 4, p_1 - 1, p_2 - 1) \notin P'(3, m-6)$.

Indeed if $p = (p_0, p_1, p_2) \in P'(3, m)$, then $p_0 \geq p_1 \geq p_2 \geq 0$. If $p \in C$, then either $p_2 = 0$ or $p_0 - p_1 \leq 2$. If $p_2 = 0$, then $(p_0 - 4, p_1 - 1, p_2 - 1) \notin P'(3, m-6)$ since $p_2 - 1 < 0$. So suppose then that $p_2 > 0$ and $p_0 - p_1 \leq 2$. Then $p_0 - 4 \leq (p_1 + 2) - 4 = p_1 - 2 < p_1 - 1$ and thus $(p_0 - 4, p_1 - 1, p_2 - 1)$ is not decreasing so $(p_0 - 4, p_1 - 1, p_2 - 1) \notin P'(3, m-6)$. So $f\big(P'(3, m) \setminus C\big) \subseteq P'(3, m-6)$

Now suppose that $q = (q_0, q_1, q_2) \in P'(3, m)$ but $(q_0 - 4, q_1 - 1, q_2 - 1) \notin P'(3, m-6)$. Thus we have that either $q_2 - 1 < 0$ or or $q_0 - 4 < q_1 - 1$, (note: if $q_1 - 1 < q_2 - 1$ then $q_1 < q_2$ which is a contradiction). If $q_2 - 1 < 0$, then $q_2 < 1$ which implies that $q_2 = 0$ and thus $q \in C$. So now suppose that $q_0 - 4 < q_1 - 1$ this gives us that $q_0 - q_1 < 3$ that is $q_0 - q_1 \leq 2$ and thus again we have that $q \in C$. So $f\big(P'(3, m) \setminus C\big) \supseteq P'(3, m-6)$. Thus f is a bijection.

Now suppose that $r, s \in P'(3, m) \setminus C$ and let $r \geq_M s$. That is

$$r_0 \geq s_0, \qquad r_0 + r_1 \geq s_0 + s_1, \quad \text{and} \quad r_0 + r_1 + r_2 \geq s_0 + s_1 + s_2.$$

Then we have that $f(r) = (r_0 - 4, r_1 - 1, r_2 - 1 \in P'(3, m-6)$ and $f(s) = (s_0 - 4, s_1 - 1, s_2 - 1) \in P'(3, m-6)$. So using the above inequalities we see that

$$r_0{-}4 \geq s_0{-}4, \qquad r_0{-}4{+}r_1{-}1 \geq s_0{-}4{+}s_1{-}1, \quad \text{and} \quad r_0{-}4{+}r_1{-}1{+}r_2{-}1 \geq s_0{-}4{+}s_1{-}1{+}s_2{-}1.$$

So we have that $f(r) \geq_M f(s)$. Now suppose that $t, v \in P'(3, m-6)$ and let $t \geq_M v$. That is

$$t_0 \geq v_0, \qquad t_0 + t_1 \geq v_0 + v_1, \quad \text{and} \quad t_0 + t_1 + t_2 \geq v_0 + v_1 + v_2.$$

Then we have that $f^{-1}(t) = (t_0 + 4, t_1 + 1, t_2 + 1) \in P'(3, m) \setminus C$ and $f(v) = (v_0 + 4, v_1 + 1, v_2 + 1) \in P'(3, m) \setminus C$. So using the above inequalities we see that

$$t_0{+}4 \geq v_0{+}4, \qquad t_0{+}4{+}t_1{+}1 \geq v_0{+}4{+}v_1{+}1, \quad \text{and} \quad t_0{+}4{+}t_1{+}1{+}t_2{+}1 \geq v_0{+}4{+}v_1{+}1{+}v_2{+}1.$$

So we have that $f^{-1}(t) \geq_M f^{-1}(v)$.

So we have that $P'(3,m) \setminus C \cong P'(3,m-6)$ and thus we have that the width of $P'(3,m)$ is the width of $P'(3,m-6)+1 = \left(\left\lfloor \frac{m-6}{6} \right\rfloor + 1\right) + 1 = \left\lfloor \frac{m}{6} \right\rfloor + 1$ as desired. $\square$

The strategy of bounding the width above and below used to prove $P(3,m)$ will be used again to prove the width of $P(4,m)$ and similarly the inductive strategy used for $P'(3,m)$ will be used again in the $P'(4,m)$ case. However the proofs for $P(4,m)$ and $P'(4,m)$ are much more involved than those of $P(3,m)$ and $P'(3,m)$, so they will each receive their own subsequent chapters.

Chapter 4

The Width of $P(4, m)$

The strategy for proving $P(4, m)$ is similar to the strategy for the $P(3, m)$ case. That is we will bound w above by giving a chain decomposition and then below by showing there exists a level with the same number of elements as the number of chains. But unlike with the $n = 3$ case, the width depends on the parity of m.

Theorem 4.1. *For $P(4, m)$ and m odd, the width is $w = \binom{\frac{m+3}{2}}{2}$.*

Proof. To bound w above we provide a chain decomposition of size $\binom{\frac{m+3}{2}}{2}$. For $i, j \geq 0$ and $0 \leq i + j \leq \frac{m-1}{2}$, let chain $C_{i,j}$ be defined as:

$$\begin{aligned}
C_{i,j} = \{&(m-(i+j), 0, i, j), (m-(i+j)-1, 1, i, j), (m-(i+j)-2, 2, i, j), \ldots, \\
&(i, m-(2i+j), i, j), (i, m-(2i+j)-1, i+1, j), (i, m-(2i+j)-2, i+2, j), \ldots, \\
&(i, j, m-(i+2j), j), (i, j, m-(i+2j)-1, j+1), (i, j, m-(i+2j)-2, j+2), \ldots, \\
&(i, j, 0, m-(i+j))\}
\end{aligned}$$

Note that:

$$\begin{aligned}
|C_{i,j}| &= \big(m-(2i+j)+1\big) + \big(m-(2i+j)-j+1\big) + \big(m-(i+2j)+1\big) - 2 \\
&= 3m + 1 - 5i - 5j
\end{aligned}$$

To ensure our chains capture all elements in the poset we show each element can be assigned to exactly one chain, and that when we sum over the sizes of all the chains we get the entire poset. So summing over all the sizes of all the chains and letting $x = \frac{m-1}{2}$, we have

$$
\begin{aligned}
\sum_{j=0}^{\frac{m-1}{2}} \sum_{i=0}^{\frac{m-1}{2}-j} \big(3m+1-5i-5j\big) &= \sum_{j=0}^{x} \left((x-j+1)(3m+1-5j) - \frac{5}{2}(x-j)(x-j+1) \right) \\
&= \sum_{j=0}^{x} \Bigg(3mx + x - 5xj - 3mj - j + 5j^2 + 3m + 1 - 5j \\
&\qquad\qquad - \frac{5}{2}(x^2 - 2xj + x + j^2 - j) \Bigg) \\
&= \sum_{j=0}^{x} \left(3mx + x + 3m + 1 - \frac{5}{2}x^2 - \frac{5}{2}x - 3mj - \frac{7}{2}j + \frac{5}{2}j^2 \right) \\
&= (x+1)\Big(3mx + x + 3m + 1 - \frac{5}{2}x^2 - \frac{5}{2}x\Big) + \sum_{j=0}^{x} \Big(-3mj - \frac{7}{2}j + \frac{5}{2}j^2 \Big) \\
&= (x+1)\Big(3mx + x + 3m + 1 - \frac{5}{2}x^2 - \frac{5}{2}x\Big) - \Big(3m + \frac{7}{2}\Big)\Big(\frac{x(x+1)}{2}\Big) \\
&\qquad + \frac{5x(x+1)(2x+1)}{12} \\
&= -\frac{5}{2}x^3 + 3mx^2 - 4x^2 + 6mx - \frac{1}{2}x + 3m + 1 + \frac{5}{6}x^3 - \frac{1}{2}x^2 - \frac{4}{3}x \\
&\qquad - \frac{3}{2}mx^2 - \frac{3}{2}mx \\
&= -\frac{5}{3}x^3 + \frac{3}{2}mx^2 - \frac{9}{2}x^2 + \frac{9}{2}mx - \frac{11}{6}x + 3m + 1
\end{aligned}
$$

Now plugging in $x = \dfrac{m-1}{2}$ we get:

$$
\begin{aligned}
&= -\frac{5}{24}m^3 + \frac{15}{24}m^2 - \frac{15}{24}m + \frac{5}{24} + \frac{3}{8}m^3 - \frac{6}{8}m^2 + \frac{3}{8}m - \frac{9}{8}m^2 + \frac{18}{8}m - \frac{9}{8} \\
&\qquad + \frac{9}{4}m^2 - \frac{9}{4}m - \frac{11}{12}m + \frac{11}{12} + 3m + 1 \\
&= \frac{m^3}{6} + m^2 + \frac{11m}{6} + 1 \\
&= \frac{(m+3)(m+2)(m+1)}{6} \\
&= \binom{m+3}{3} \\
&= |P(4,m)|.
\end{aligned}
$$

Now if we can verify that each element in $P(4,m)$ is in exactly one chain, then the $C_{i,j}$s form a chain decomposition. To do this we take an $x = x_0x_1x_2x_3 \in P(4,m)$, and evaluate the following sums, $x_0 + x_1$, $x_1 + x_2$, and $x_2 + x_3$. If there is a unique largest sum $x_a + x_b$, then:

- if $x_0 + x_1$ is the largest, $x \in C_{x_2 x_3}$.
- if $x_1 + x_2$ is the largest, $x \in C_{x_0 x_3}$.
- if $x_2 + x_3$ is the largest, $x \in C_{x_0 x_1}$.

If there is not a unique largest sum, then

- If $x_0 + x_1 = x_1 + x_2$ is the largest then $x_0 = x_2$ and $x \in C_{x_2 x_3} = C_{x_0 x_3}$.
- If $x_1 + x_2 = x_2 + x_3$ is the largest then $x_1 = x_3$ and $x \in C_{x_0 x_3} = C_{x_0 x_1}$.
- Since m is odd then $x_0 + x_1 \neq x_2 + x_3$.

Thus every element in $P(4, m)$ can be assigned to exactly one chain, and the union of all the chains is the entire poset, thus it remains to show that there are at most $\binom{\frac{m+3}{2}}{2}$ many elements in a largest antichain.

To see this we can use a stars and bars argument. Since $i, j \geq 0$ and $0 \leq i + j \leq \frac{m-1}{2}$. There are at most $\frac{m-1}{2}$ stars to be distributed to i and j, and since $i + j$ could be less than $\frac{m-1}{2}$ we can place the remaining stars in a third part to be "deleted" or ignored. Thus there are $\frac{m-1}{2}$ stars and 2 bars giving us $\binom{\frac{m-1}{2}+2}{2} = \binom{\frac{m+3}{2}}{2}$ many chains in our decomposition.

We can also achieve this same result algebraically by summing over 1 for the , as shown below.

$$\begin{aligned}
\sum_{j=0}^{\frac{m-1}{2}} \sum_{i=0}^{\frac{m-1}{2}-j} (1) &= \sum_{j=0}^{\frac{m-1}{2}} \frac{m-1}{2} - j + 1 = \sum_{j=0}^{\frac{m-1}{2}} \frac{m+1}{2} - j = \frac{m+1}{2}\Big(\frac{m-1}{2} + 1\Big) - \sum_{j=0}^{\frac{m-1}{2}} j \\
&= \Big(\frac{m+1}{2}\Big)^2 - \frac{1}{2}\Big(\frac{m-1}{2}\Big)\Big(\frac{m-1}{2} + 1\Big) = \frac{m^2 + 2m + 1}{4} - \frac{1}{2}\Big(\frac{m-1}{2}\Big)\Big(\frac{m+1}{2}\Big) \\
&= \frac{m^2 + 2m + 1}{4} - \frac{1}{2}\Big(\frac{m^2 - 1}{4}\Big) = \frac{2m^2 + 4m + 2 - m^2 + 1}{8} = \frac{m^2 + 4m + 3}{8} \\
&= \frac{1}{2}\Big(\frac{m+3}{2}\Big)\Big(\frac{m+1}{2}\Big) = \binom{\frac{m+3}{2}}{2}
\end{aligned}$$

Thus by taking one element from each chain we can have at most $\binom{\frac{m+3}{2}}{2}$ many elements in a longest antichain.

We now bound w below by showing that there exists an antichain of size $\binom{\frac{m+3}{2}}{2}$. We claim that level $k = \frac{3m-1}{2}$ is such an antichain. By Lemma 3.3 we already know that level $k = \frac{3m-1}{2} = A_k$ is indeed an antichain, thus it remains to show that $|A_k| = \binom{\frac{m+3}{2}}{2}$.

Let $k = \frac{3m-1}{2}$ and let $A_k = D_3 \cup D_2 \cup D_1$ with D_3, D_2, and D_1 defined as:

$$D_3 = \{x \in A_k \cap C_{ij} : i = x_0 \text{ and } j = x_1\}$$

$$D_2 = \{x \in A_k \cap C_{ij} : i = x_0 \text{ and } j = x_3\}$$

$$D_1 = \{x \in A_k \cap C_{ij} : i = x_2 \text{ and } j = x_3\}.$$

Claim 4.2. $D_3 \cap D_1 = \varnothing$.

Subproof: Suppose $D_3 \cap D_1 \neq \varnothing$, then there exists an $x \in D_3 \cap D_1$, but $x \in D_3$ implies $x_0+x_1 = i+j \leq \frac{m-1}{2}$ and $x \in D_1$ implies $x_2+x_3 = i+j \leq \frac{m-1}{2}$. But then $x_0+x_1+x_2+x_3 \leq 2 \cdot \frac{m-1}{2} = m-1 < m$. Which contradicts $x \in P(4, m)$. ■

Claim 4.3. *If* $D_3 \cap D_2 \neq \varnothing$*, then* $m \equiv 3 \bmod 4$.

Subproof: Let $x \in D_3 \cap D_2$ and suppose $m \not\equiv 3 \bmod 4$, that is $m \equiv 1 \bmod 4$ or $m = 4z+1$ for some $z \in \mathbb{Z}$. $x \in D_3$ implies $x \in C_{x_0x_1}$ and $x \in D_2$ implies $x \in C_{x_0x_3}$. From a previous Lemma we know that all $x \in P(n, m)$ are in exactly one C_{ij} thus $x_1 = x_3$. This gives us

$$m = x_0 + x_1 + x_2 + x_3 = x_0 + x_2 + 2x_3$$

and

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= \frac{3m-1}{2} = \frac{3(4z+1)-1}{2} \\ x_1 + x_2 + x_3 + x_2 + 2x_3 &= \frac{12z+2}{2} \\ m - x_0 + m - x_0 &= 6z+1 \\ 2m - 2x_0 &= 6z+1 \qquad \text{a contradiction.} \end{aligned}$$

Thus $m \equiv 3 \bmod 4$ as desired. ■

Claim 4.4. *If* $D_1 \cap D_2 \neq \varnothing$*, then* $m \equiv 1 \bmod 4$.

Subproof: Let $x \in D_1 \cap D_2$ and suppose $m \not\equiv 1 \bmod 4$, that is $m \equiv 3 \bmod 4$ or $m = 4z+3$ for some $z \in \mathbb{Z}$. $x \in D_1$ implies $x \in C_{x_2x_3}$ and $x \in D_2$ implies $x \in C_{x_0x_3}$. From a previous lemma we know that all $x \in P(n,m)$ are in exactly one C_{ij} thus $x_0 = x_2$. This gives us

$$m = x_0 + x_1 + x_2 + x_3 = x_1 + 2x_2 + x_3$$

and

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= \frac{3m-1}{2} = \frac{3(4z+3)-1}{2} \\ m - x_3 + 3x_3 &= \frac{12z+8}{2} \\ 4k + 3 + 2x_3 &= 6z + 4 \qquad \text{a contradiction.} \end{aligned}$$

Thus $m \equiv 1 \bmod 4$ as desired. ■

Now let $\Delta = \{(i,j) \in \mathbb{Z}^2 : 0 \le i + j \le \frac{m-1}{2}\}$, and define G, R, and B to be

$$\begin{aligned} G &= \left\{(i,j) \in \Delta : j \ge \left\lfloor \frac{m+1}{4} \right\rfloor\right\} \\ R &= \left\{(i,j) \in \Delta : i \le \left\lfloor \frac{m+1}{4} \right\rfloor \text{ and } j \le \left\lfloor \frac{m}{4} \right\rfloor\right\} \\ B &= \left\{(i,j) \in \Delta : i \ge \left\lfloor \frac{m+3}{4} \right\rfloor\right\} \end{aligned}$$

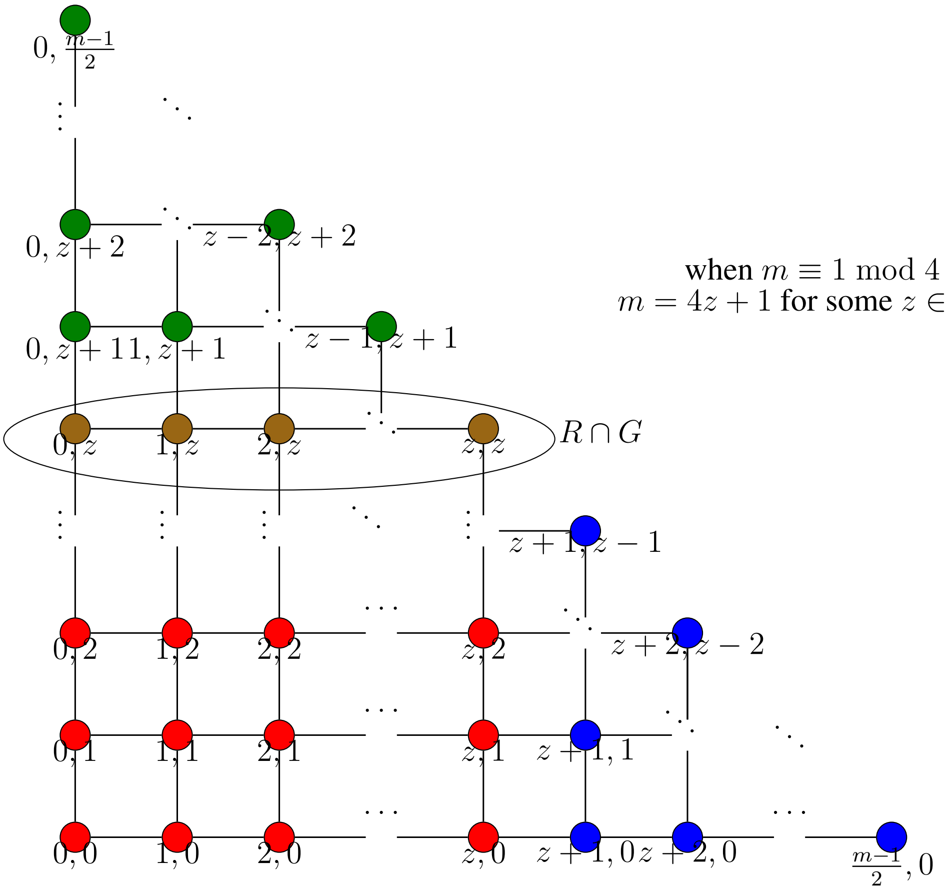
Note that $|\Delta| = \binom{\frac{m-1}{2}+2}{2} = \binom{\frac{m+3}{2}}{2}$.

When $m \equiv 1 \bmod 4$, we have $m = 4z+1$ for some $z \in \mathbb{Z}_{\ge 0}$, thus G, R, and B are

$$G = \{(i,j) \in \Delta : j \ge z\}$$

$$R = \{(i,j) \in \Delta : i \le z \text{ and } j \le z\}$$

$$B = \{(i,j) \in \Delta : i \ge z+1\}$$

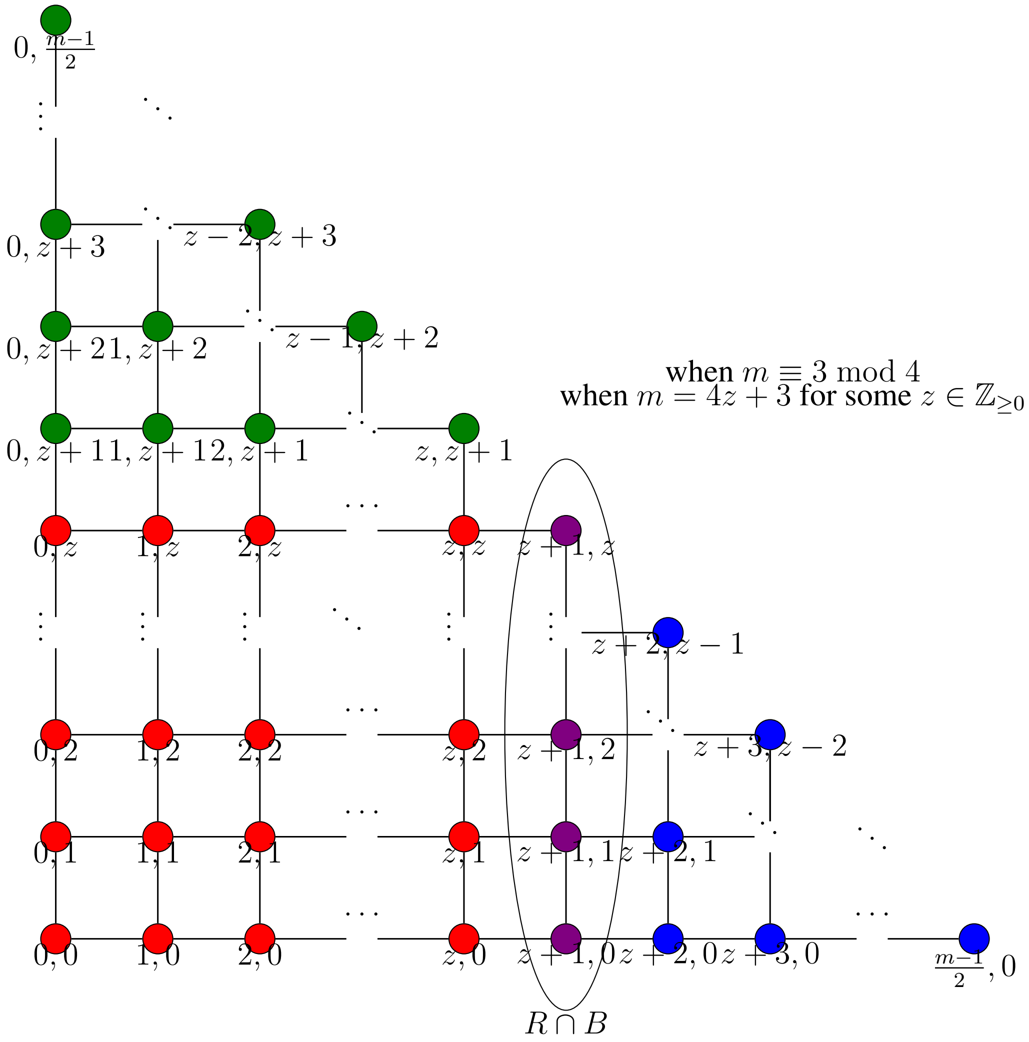


When $m \equiv 3 \bmod 4$, we have $m = 4z + 3$ for some $z \in \mathbb{Z}_{\geq 0}$, thus G, R, and B are

$$G = \{(i,j) \in \Delta : j \geq z+1\}$$

$$R = \{(i,j) \in \Delta : i \leq z+1 \text{ and } j \leq z\}$$

$$B = \{(i,j) \in \Delta : i \geq z+1\}$$



when $m \equiv 3 \bmod 4$
when $m = 4z + 3$ for some $z \in \mathbb{Z}_{\geq 0}$

Now let

$$D_1^* = \{(i,j) : \exists x \in D_1 \text{ with } i = x_2 \text{ and } j = x_3\}$$

$$D_2^* = \{(i,j) : \exists x \in D_2 \text{ with } i = x_0 \text{ and } j = x_3\}$$

$$D_3^* = \{(i,j) : \exists x \in D_3 \text{ with } i = x_0 \text{ and } j = x_1\}$$

We now claim that $D_1^* = G$, $D_2^* = R$, and $D_3^* = B$.

Claim 4.5. $D_1^* = G$.

Subproof: Let $(i, j) \in D_1^*$. Thus there exists an $x \in D_1$ with $i = x_2$ and $j = x_3$. Since x is in D_1 we know that $x_0 + x_1 \geq \{x_1 + x_2, x_2 + x_3\}$, in particular $x_0 \geq x_2$. Thus

$$\begin{aligned} m &= x_0 + x_1 + x_2 + x_3 \\ m - x_0 &= x_1 + x_2 + x_3 \\ &\text{and} \\ k &= x_1 + 2x_2 + 3x_3 \\ &= x_1 + x_2 + x_3 + x_2 + 2x_3 \\ &= m - x_0 + x_2 + 2x_3 \\ &\leq m + 2x_3 \end{aligned}$$

If $m = 4z + 1$ then $k = 6z + 1$, and we have

$$\begin{aligned} 6z + 1 &\leq 4z + 1 + 2x_3 \\ 2z &\leq 2x_3 \\ z &\leq x_3 = j. \end{aligned}$$

And if $m = 4z + 3$ then $k = 6z + 4$, and we have

$$\begin{aligned} 6z + 4 &\leq 4z + 3 + 2x_3 \\ 2z + 1 &\leq 2x_3 \\ z + \frac{1}{2} &\leq x_3 = j. \end{aligned}$$

So in either case we have that $(i, j) \in G$. Now suppose $(i, j) \in G$, then $i \geq 0$, $\left\lfloor \frac{m+1}{4} \right\rfloor \leq j$, and $0 \leq i + j \leq \frac{m-1}{2}$. Let $x = x_0, x_1, x_2, x_3$ with $x_0 = i + 2j - \frac{m-1}{2}$, $x_1 = k - 2i - 3j$, $x_2 = i$ and

$x_3 = j$. Consider

$$\begin{aligned}\sum x_i &= i + 2j - \frac{m-1}{2} + k - 2i - 3j + i + j \\ &= -\frac{m-1}{2} + k \\ &= -\frac{m}{2} + \frac{1}{2} + \frac{3m-1}{2} \\ &= m\end{aligned}$$

and

$$\sum ix_i = k - 2i - 3j + 2(i) + 3(j) = k.$$

Thus it remains to show that each of the $x_i \geq 0$. Now $i, j \geq 0$ by assumption so $x_2, x_3 \geq 0$. If $m \equiv 1 \bmod 4$ then

$$\begin{aligned}x_0 &= i + 2j - \frac{m-1}{2} \\ &\geq i + 2z - 2z \\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}x_1 &= 6z + 1 - 2i - 3j \\ &= 6z + 1 - 2(i + j) - j \\ &\geq 6z + 1 - 2(2z) - j \\ &\geq 2z + 1 - 2z \\ &\geq 1.\end{aligned}$$

If $m \equiv 3 \bmod 4$ then

$$\begin{aligned} x_0 &= i + 2j - \frac{m-1}{2} \\ &\geq i + 2(z+1) - 2z - 1 \\ &\geq i + 1 \\ &\geq 1 \end{aligned}$$

and

$$\begin{aligned} x_1 &= 6z + 4 - 2i - 3j \\ &= 6z + 4 - 2(i+j) - j \\ &\geq 6z + 4 - 2(2z+1) - j \\ &\geq 2z + 2 - 2(z+1) \\ &\geq 0. \end{aligned}$$

Thus for any $(i, j) \in G$, the above defined x will be in D_1, hence $(i, j) \in D_1^*$ as desired. ■

Claim 4.6. $D_2^* = R$.

Subproof: Let $(i, j) \in D_2^*$. Thus there exists an $x \in D_2$ with $i = x_0$ and $j = x_3$. Since x is in D_2 we know that $x_1 + x_2 \geq \{x_0 + x_1, x_2 + x_3\}$, in particular $x_2 \geq x_0$ and $x_1 \geq x_3$. Thus

$$\begin{aligned} m &= x_0 + x_1 + x_2 + x_3 \\ m - x_0 &= x_1 + x_2 + x_3 \end{aligned}$$

$$\begin{aligned} k &= x_1 + 2x_2 + 3x_3 \\ &= x_1 + x_2 + x_3 + x_2 + 2x_3 \\ &= m - x_0 + x_2 + 2x_3 \\ &\geq m + 2x_3 \end{aligned} \qquad \begin{aligned} k &= x_1 + 2x_2 + 3x_3 \\ &\leq 2x_1 + 2x_2 + 2x_3 \\ &= 2(m - x_0) \end{aligned}$$

If $m = 4z + 1$, then $k = 6z + 1$, and we have

$$\begin{aligned} 6z + 1 &\geq 4z + 1 + 2x_3 & \qquad 6z + 1 &\leq 2(4z + 1 - x_0) \\ 2z &\geq 2x_3 & 2x_0 &\leq 2z + 1 \\ z &\geq x_3 & x_0 &\leq z + \frac{1}{2} \end{aligned}$$

If $m = 4z + 3$, then $k = 6z + 4$, and we have

$$\begin{aligned} 6z + 4 &\geq 4z + 3 + 2x_3 & \qquad 6z + 4 &\leq 2(4z + 3 - x_0) \\ 2z + 1 &\geq 2x_3 & 2x_0 &\leq 2z + 2 \\ z + \frac{1}{2} &\geq x_3. & x_0 &\leq z + 1. \end{aligned}$$

So in either case we have that $(i, j) \in R$. Now suppose $(i, j) \in R$, then $0 \leq i \leq \lfloor \frac{m}{4} \rfloor$, $0 \leq j \leq z$, and $0 \leq i + j \leq \frac{m-1}{2}$. Let $x = x_0, x_1, x_2, x_3$ with $x_0 = i$, $x_1 = \frac{m-1}{2} + 1 - 2i + j$, $x_2 = \frac{m-1}{2} + i - 2j$, and $x_3 = j$. Consider

$$\begin{aligned} \sum x_i &= i + \frac{m-1}{2} + 1 - 2i + j + \frac{m-1}{2} + i - 2j + j \\ &= 2\Big(\frac{m-1}{2}\Big) + 1 \\ &= m \end{aligned}$$

and

$$\begin{aligned} \sum i x_i &= \frac{m-1}{2} + 1 - 2i + j + 2\Big(\frac{m-1}{2} + i - 2j\Big) + 3(j) \\ &= 3\Big(\frac{m-1}{2}\Big) + 1 \\ &= \frac{3m-1}{2} \\ &= k. \end{aligned}$$

Thus it remains to show that each of the $x_i \geq 0$. Now $i, j \geq 0$ by assumption so $x_0, x_3 \geq 0$. If $m \equiv 1 \bmod 4$ then

$$\begin{aligned} x_1 &= \frac{m-1}{2} + 1 - 2i + j \\ &\geq 2z + 1 - 2z + j \\ &= j + 1 \\ &\geq 1 \end{aligned} \qquad \begin{aligned} x_2 &= \frac{m-1}{2} + i - 2j \\ &\geq 2z + i - 2z \\ &\geq 0. \end{aligned}$$

and

If $m \equiv 3 \bmod 4$ then

$$\begin{aligned} x_1 &= \frac{m-1}{2} + 1 - 2i + j \\ &\geq 2z + 2 - 2(z+1) + j \\ &= j \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} x_2 &= \frac{m-1}{2} + i - 2j \\ &\geq 2z + 1 + i - 2z \\ &\geq 1 + i \\ &\geq 1. \end{aligned}$$

Thus for any $(i, j) \in R$, the above defined x will be in D_2, and hence $(i, j) \in D_2^*$ as desired.

$\blacksquare$

Claim 4.7. $D_3^* = B$.

Subproof: Let $(i,j) \in D_3^*$. Thus there exists an $x \in D_3$ with $i = x_0$ and $j = x_1$. Since x is in D_3 we know that $x_2 + x_3 \geq \{x_0 + x_1, x_1 + x_2\}$, in particular $x_3 \geq x_1$. Thus

$$\begin{aligned} m &= x_0 + x_1 + x_2 + x_3 \\ m - x_0 &= x_1 + x_2 + x_3 \\ &\text{and} \\ k &= x_1 + 2x_2 + 3x_3 \\ &\geq 2x_1 + 2x_2 + 2x_3 \\ &= 2(m - x_0). \end{aligned}$$

If $m = 4z + 1$ then $k = 6z + 1$ and we have

$$\begin{aligned} 6z + 1 &\geq 2(4z + 1 - x_0) \\ 6z + 1 &\geq 8z + 2 - 2x_0 \\ 2x_0 &\geq 2z + 1 \\ x_0 &\geq z + \frac{1}{2}. \end{aligned}$$

If $m = 4z + 3$ then $k = 6z + 4$ and we have

$$\begin{aligned} 6z + 4 &\geq 2(4z + 3 - x_0) \\ 6z + 4 &\geq 8z + 6 - 2x_0 \\ 2x_0 &\geq 2z + 2 \\ x_0 &\geq z + 1. \end{aligned}$$

So in either case we have that $(i,j) \in B$. Now suppose $(i,j) \in B$, then $\left\lfloor \frac{m+3}{4} \right\rfloor \leq i \leq \frac{m-1}{2}$, and $0 \leq i + j \leq \frac{m-1}{2}$. Let $x = x_0, x_1, x_2, x_3$ with $x_0 = i$, $x_1 = j$, $x_2 = k + 1 - 3i - 2j$, and

$x_3 = 2i + j - \frac{m-1}{2} - 1)$. Consider

$$\begin{aligned}
\sum x_i &= i + j + k + 1 - 3i - 2j + 2i + j - \frac{m-1}{2} - 1 \\
&= k - \frac{m-1}{2} \\
&= \frac{3m-1}{2} - \frac{m-1}{2} \\
&= m
\end{aligned}$$

and

$$\begin{aligned}
\sum ix_i &= j + 2(k + 1 - 3i - 2j) + 3\Big(2i + j - \frac{m-1}{2} - 1\Big) \\
&= 2k + 2 - \frac{3m-3}{2} - 3 \\
&= 2k - \frac{3m-1}{2} \\
&= k.
\end{aligned}$$

Thus it remains to show that each of the $x_i \geq 0$. Now $i, j \geq 0$ by assumption so $x_0, x_1 \geq 0$. If $m \equiv 1 \bmod 4$ then

$$\begin{aligned}
x_2 &= k + 1 - 3i - 2j \\
&= 6z + 2 - 2(i + j) - i \\
&\geq 6z + 2 - 2\Big(\frac{m-1}{2}\Big) - i \\
&\geq 6z + 3 - 4z - (z + 1) \\
&\geq z + 2 \\
&\geq 2
\end{aligned}
\qquad
\begin{aligned}
x_3 &= 2i + j - \frac{m-1}{2} - 1 \\
&\geq 2(z + 1) + j - 2z - 1 \\
&\geq j + 1 \\
&\geq 1
\end{aligned}$$

If $m \equiv 3 \bmod 4$ then

$$\begin{aligned} x_2 &= k+1-3i-2j \\ &= 6z+5-2(i+j)-i \\ &\geq 6z+5-2\Big(\frac{m-1}{2}\Big)-i \\ &\geq 6z+5-(4z+2)-(z+1) \\ &\geq z+2 \\ &\geq 2 \end{aligned}$$

$$\begin{aligned} x_3 &= 2i+j-\frac{m-1}{2}-1 \\ &\geq 2(z+1)+j-(2z+1)-1 \\ &\geq j \\ &\geq 0 \end{aligned}$$

Thus for any $(i, j) \in B$, the above defined x will be in D_3, and hence $(i, j) \in D_3^*$ as desired. ■

So using the principle of inclusion exclusion and the claims we have

$$\begin{aligned} |A_k| &= |D_3 \cup D_2 \cup D_1| \\ &= |D_3| + |D_2| + |D_1| - |D_3 \cap D_2| - |D_3 \cap D_1| - |D_2 \cap D_1| \\ &+ |D_3 \cap D_2 \cap D_1| \\ &= |D_3^*| + |D_2^*| + |D_1^*| - |D_3^* \cap D_2^*| - 0 - |D_2^* \cap D_1^*| + 0 \\ &= |B| + |R| + |G| - |B \cap R| - |R \cap G| \\ &= |\Delta| \end{aligned}$$

Since $\Delta = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i + j \leq \frac{m-1}{2}\}$, then the same stars and bars argument from before works to show that this is $\binom{\frac{m+3}{2}}{2}$ as desired. □

Theorem 4.8. *For $P(4, m)$ and m even, the width is $w = \binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$.*

Proof. To bound w above we provide a chain decomposition of size $\binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$. For $i, j \geq 0$ and $0 \leq i + j \leq \frac{m}{2} - 1$, we define a "regular chain" $C_{i,j}$ as we did with the odd case:

$$\begin{aligned}
C_{i,j} = \{&(m-(i+j),0,i,j),(m-(i+j)-1,1,i,j),(m-(i+j)-2,2,i,j),\ldots,\\
&(i,m-(2i+j),i,j),(i,m-(2i+j)-1,i+1,j),(i,m-(2i+j)-2,i+2,j),\ldots,\\
&(i,j,m-(i+2j),j),(i,j,m-(i+2j)-1,j+1),(i,j,m-(i+2j)-2,j+2),\ldots,\\
&(i,j,0,m-(i+j))\}
\end{aligned}$$

and the size of each of these chains is given by:

$$\begin{aligned}
|C_{i,j}| &= \big(m-(2i+j)+1\big) + \big(m-(2i+j)-j+1\big) + \big(m-(i+2j)+1\big) - 2\\
&= 3m+1-5i-5j.
\end{aligned}$$

Now summing over all the sizes of each chain and letting $x = \frac{m}{2} - 1$ we have

$$\begin{aligned}
\sum_{j=0}^{\frac{m}{2}-1} \sum_{i=0}^{\frac{m}{2}-1-j} (3m+1-5i-5j) &= -\frac{5}{3}x^3 + \frac{3}{2}mx^2 - \frac{9}{2}x^2 + \frac{9}{2}mx - \frac{11}{6}x + 3m + 1\\
&= -\frac{5}{24}m^3 + \frac{15}{12}m^2 - \frac{15}{6}m + \frac{5}{3} + \frac{3}{8}m^3 - \frac{3}{2}m^2 + \frac{3}{2}m - \frac{9}{8}m^2 + \frac{9}{2}m - \frac{9}{2} + \frac{9}{4}m^2\\
&\qquad - \frac{9}{2}m - \frac{11}{12}m + \frac{11}{6} + 3m + 1\\
&= \frac{1}{6}m^3 + \frac{7}{8}m^2 + \frac{13}{12}m\\
&= \frac{1}{6}m^3 + m^2 + \frac{11}{6}m + 1 - \Big(\frac{1}{8}m^2 + \frac{3}{4}m + 1\Big)\\
&= \frac{m^3+6m^2+11m+6}{6} - \left(\frac{\frac{m^2}{2} + \frac{m}{2} + m + 2}{2}\right)\\
&= \frac{(m+3)(m+2)(m+1)}{6} - \left(\frac{\big(\frac{m}{2}+2\big)\big(\frac{m}{2}+1\big)}{2}\right)\\
&= \binom{m+3}{3} - \binom{\frac{m}{2}+2}{2}
\end{aligned}$$

Now for $0 \leq j \leq i$ and $i + j = \frac{m}{2}$ we define a “leftover chain” $C_{i,j}$ as

$$\begin{aligned} C_{i,j} = \{&(i,j,i,j),(i,j,i-1,j+1),(i,j,i-2,j+2),\dots,\\ &(i,j,j,i),(i-1,j+1,j,i),(i-2,j+2,j,i),\dots,(j,i,j,i)\} \end{aligned}$$

and the size of each of these chains is given by:

$$\begin{aligned} |C_{i,j}| &= ((i-j)+1) + ((i-j)+1) - 1 \\ &= 2i - 2j + 1 \\ &= 2\left(\frac{m}{2} - j\right) - 2j + 1 \\ &= 2\left(\frac{m}{2}\right) - 4j + 1 \\ &= m - 4j + 1 \end{aligned}$$

Now summing over all the sizes of each of these chains we have

$$\begin{aligned} &\sum_{j=0}^{\lfloor \frac{m}{4} \rfloor} m - 4j + 1 = \\ \text{for } m \equiv 0 \bmod 4 \ &\Rightarrow (m+1)\left(\frac{m}{4}+1\right) - \frac{4}{2}\left(\frac{m}{4}\right)\left(\frac{m}{4}+1\right) \\ &= \frac{m^2}{4} + \frac{5m}{4} + 1 - \frac{m^2}{8} - \frac{m}{2} \\ &= \frac{m^2}{8} + \frac{3m}{4} + 1 \\ &= \frac{\left(\frac{m}{2}\right)^2}{2} + \frac{3\left(\frac{m}{2}\right)}{2} + 1 \\ &= \frac{\left(\frac{m}{2}+2\right)\left(\frac{m}{2}+1\right)}{2} \\ &= \binom{\frac{m}{2}+2}{2} \end{aligned}$$

$$\begin{aligned}
&\sum_{j=0}^{\lfloor \frac{m}{4} \rfloor} m-4j+1 = \\
&\text{for } m \equiv 2 \bmod 4 \Rightarrow (m+1)\Big(\frac{m-2}{4}+1\Big) - \frac{4}{2}\Big(\frac{m-2}{4}\Big)\Big(\frac{m-2}{4}+1\Big) \\
&\qquad = (m+1)\Big(\frac{m}{4}+\frac{1}{2}\Big) - 2\Big(\frac{m}{4}-\frac{1}{2}\Big)\Big(\frac{m}{4}+\frac{1}{2}\Big) \\
&\qquad = \frac{m^2}{4}+\frac{3m}{4}+\frac{1}{2}-2\Big(\frac{m^2}{16}-\frac{1}{4}\Big) \\
&\qquad = \frac{m^2}{8}+\frac{3m}{4}+1 \\
&\qquad = \binom{\frac{m}{2}+2}{2}
\end{aligned}$$

Now adding the sizes of each of the two different types of chains it is easy to see that we get $\binom{m+3}{3} = |P(4,m)|$ as desired. Now if we can verify that each element is in exactly one chain, then the $C_{i,j}$s will form a chain decomposition.

Let $x = x_0x_1x_2x_3 \in P(4,m)$ and evaluate the following sums, x_0+x_1, x_1+x_2, and x_2+x_3. If there is a unique largest sum (which can only happen in the normal chains since by construction the leftover chains always have $x_0+x_1 = \frac{m}{2} = x_2+x_3$ with $x_1+x_2 \leq \frac{m}{2}$.) then:

- if x_0+x_1 is the largest $x \in C_{x_2,x_3}$,
- if x_1+x_2 is the largest $x \in C_{x_0,x_3}$,
- and if x_2+x_3 is the largest $x \in C_{x_0,x_1}$.

If there is not a unique largest sum then we have the following cases $x_0+x_1 = x_1+x_2$ or $x_1+x_2 = x_2+x_3$ or $x_0+x_1 = x_2+x_3$ or $x_0+x_1 = x_1+x_2 = x_2+x_3$. Note that if $x_0+x_1 = x_1+x_2 = x_2+x_3$ then certainly $x_1+x_2 = x_2+x_3$ so we will treat these two cases at the same time.

- If $x_0+x_1 = x_2+x_3$ is the largest sum, then x is in one of our leftover chains, and by construction $x_0 \geq x_2$ so $x_1+x_2 \leq x_0+x_1 = x_2+x_3$
    - If $x_0 \geq x_3$, then $x \in C_{x_0x_1}$ otherwise $x_0 \leq x_3$ in which case $x \in C_{x_3x_2}$,
    - and if $x_0 = x_3$, then $x_1 = x_2$ and $x \in C_{x_0x_1} = C_{x_3x_2}$.

  - Note that: $C_{x_3x_2}$ is not a typo as $x_3 = i$ and $x_2 = j$ when $x_0 \leq x_3$. So to maintain the ij order in C_{ij} we write x_3 first in the subscript.

- If $x_0 + x_1 = x_1 + x_2$, then $x_0 = x_2$ and so $x \in C_{x_2,x_3} = C_{x_0,x_3}$.

- If $x_1 + x_2 = x_2 + x_3$, then $x_1 = x_3$ and so $x \in C_{x_0,x_3} = C_{x_0,x_1}$.

Thus every element in $P(4, m)$ can be assigned to exactly one chain, and the union of all the chains is the entire post. Thus it remains to show that there are at most $\binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$ many elements in a largest antichain.

For the leftover chains we know that $0 \leq j \leq \lfloor \frac{m}{4} \rfloor$, so the leftover chains give $\lfloor \frac{m}{4} \rfloor + 1$ many chains. For the regular chains we can use a similar stars and bars argument from the odd case with $\frac{m}{2} - 1$ many stars and 2 bars. Therefore there are $\binom{\frac{m}{2}-1+2}{2} = \binom{\frac{m}{2}+1}{2}$ many of the regular chains. Putting this together with the number of leftover chains gives $\binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$ total chains. So we can select at most 1 element from each of these chains to form our antichain so $w \leq \binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$ as desired.

We now bound w below by showing that there exists an antichain of size $\binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$. We claim that level $k = \frac{3m}{2}$ is such an antichain. By Lemma 3.3 we already know that level $k = \frac{3m}{2} = A_k$ is indeed an antichain, thus it remains to show that $|A_k| = \binom{\frac{m}{2}+1}{2} + \lfloor \frac{m}{4} \rfloor + 1$.

Let $k = \frac{3m}{2}$ and let $A_k = D_1 \cup D_2 \cup D_3 \cup D_{-1} \cup D_{-2}$ where D_1, D_2, D_3, D_{-1}, and D_{-2} are defined as follows:

$$\begin{aligned}
D_1 &= \{x \in A_k \cap C_{ij} : i = x_2, j = x_3, \text{ and } 0 \leq i + j \leq \frac{m}{2} - 1\} \\
D_2 &= \{x \in A_k \cap C_{ij} : i = x_0, j = x_3, \text{ and } 0 \leq i + j \leq \frac{m}{2} - 1\} \\
D_3 &= \{x \in A_k \cap C_{ij} : i = x_0, j = x_1, \text{ and } 0 \leq i + j \leq \frac{m}{2} - 1\} \\
D_{-1} &= \{x \in A_k \cap C_{ij} : i = x_0, j = x_1, \text{ and } 0 \leq i, j \text{ and } i + j = \frac{m}{2}\} \\
D_{-2} &= \{x \in A_k \cap C_{ij} : i = x_3, j = x_2, \text{ and } 0 \leq i, j \text{ and } i + j = \frac{m}{2}\}
\end{aligned}$$

Note that by definition D_1, D_2, and D_3 are disjoint from D_{-1} and D_{-2}.

Claim 4.9. $D_{-1} = D_{-2}$.

Subproof: Let $x \in D_{-1}$, then $i + j = x_0 + x_1 = \frac{m}{2}$. But we know that $x_0 + x_1 + x_2 + x_3 = m$ so $x_2 + x_3 = \frac{m}{2}$ as well. Thus

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= k \\ j + 2(x_2 + x_3) + x_3 &= \frac{3m}{2} \\ j + 2(\frac{m}{2}) + x_3 &= \frac{3m}{2} \\ j + x_3 &= \frac{m}{2} \end{aligned}$$

Thus $x_3 = i$ so $x_2 = j$ and thus $x \in D_{-2}$. Now suppose $x \in D_{-2}$, then $i + j = x_2 + x_3 = \frac{m}{2}$. But we know that $x_0 + x_1 + x_2 + x_3 = m$ so $x_0 + x_1 = \frac{m}{2}$ as well. Thus

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= k \\ x_1 + 2(x_2 + x_3) + x_3 &= \frac{3m}{2} \\ x_1 + 2(\frac{m}{2}) + i &= \frac{3m}{2} \\ x_1 + i &= \frac{m}{2} \end{aligned}$$

Thus $x_1 = j$ so $x_0 = i$ and thus $x \in D_{-1}$. ■

Claim 4.10. $D_1 \cap D_3 = \varnothing$.

Subproof: Suppose $D_1 \cap D_3 \neq \varnothing$, then there exists an $x \in D_1 \cap D_3$, but $x \in D_1$ implies $x_2 + x_3 = i + j \leq \frac{m}{2} - 1$ and $x \in D_3$ implies $x_0 + x_1 = i + j \leq \frac{m}{2} - 1$. But then $x_0 + x_1 + x_2 + x_3 \leq 2 \cdot \frac{m}{2} - 2 < m$. Which contradicts $x \in P(4, m)$. ■

Claim 4.11. If $D_1 \cap D_2 \neq \varnothing$, then $m \equiv 0 \bmod 4$.

Subproof: Let $x \in D_1 \cap D_2$ and suppose $m \not\equiv 0 \bmod 4$, then $m \equiv 2 \bmod 4$ and $m = 4z + 2$ for some $z \in \mathbb{Z}$. If $x \in D_1$ then $x \in C_{x_2x_3}$ and if $x \in D_2$ then $x \in C_{x_0x_3}$. From a previous lemma we know each $x \in P(4,m)$ are in exactly one C_{ij} thus $x_0 = x_2$. This gives us

$$m = x_0 + x_1 + x_2 + x_3 = x_1 + 2x_2 + x_3$$

and

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= k = \frac{3m}{2} \\ m - x_3 + 3x_3 &= \frac{3(4z+2)}{2} \\ m - 2x_3 &= 6z + 3 \\ 4z + 2 - 2x_3 &= 6z + 3 \qquad \Rightarrow\!\Leftarrow \end{aligned}$$

Thus $m \equiv 0 \bmod 4$ as desired. ■

Claim 4.12. If $D_2 \cap D_3 \neq \varnothing$, then $m \equiv 2 \bmod 4$.

Subproof: Let $x \in D_2 \cap D_3$ and suppose $m \not\equiv 0 \bmod 4$, then $m \equiv 2 \bmod 4$ and $m = 4z + 2$ for some $z \in \mathbb{Z}$. If $x \in D_2$ then $x \in C_{x_0 x_3}$ and if $x \in D_3$ then $x \in C_{x_0 x_1}$. From a previous lemma we know each $x \in P(4,m)$ are in exactly one C_{ij} thus $x_1 = x_3$. This gives us

$$m = x_0 + x_1 + x_2 + x_3 = x_0 + x_2 + 2x_3$$

and

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= k = \frac{3m}{2} \\ x_1 + x_2 + x_3 + x_2 + 2x_3 &= \frac{3(4z+2)}{2} \\ x_1 + x_2 + x_3 + x_2 + 2x_3 &= 6z + 3 \\ 2(m - x_0) &= 6z + 3 \qquad \text{a contradiction.} \end{aligned}$$

Thus $m \equiv 2 \bmod 4$ as desired. ■

Now let $\Delta = \{(i,j) \in \mathbb{Z}^2 : 0 \leq i+j \leq \frac{m}{2} - 1\}$ and define G, R, and B to be

$$G = \Big\{(i,j) \in \Delta : j \geq \Big\lfloor \frac{m+2}{4} \Big\rfloor \Big\}$$
$$R = \Big\{(i,j) \in \Delta : i \leq \Big\lfloor \frac{m}{4} \Big\rfloor \text{ and } j \leq \Big\lfloor \frac{m}{4} \Big\rfloor \Big\}$$
$$B = \Big\{(i,j) \in \Delta : i \geq \Big\lfloor \frac{m+2}{4} \Big\rfloor \Big\}$$

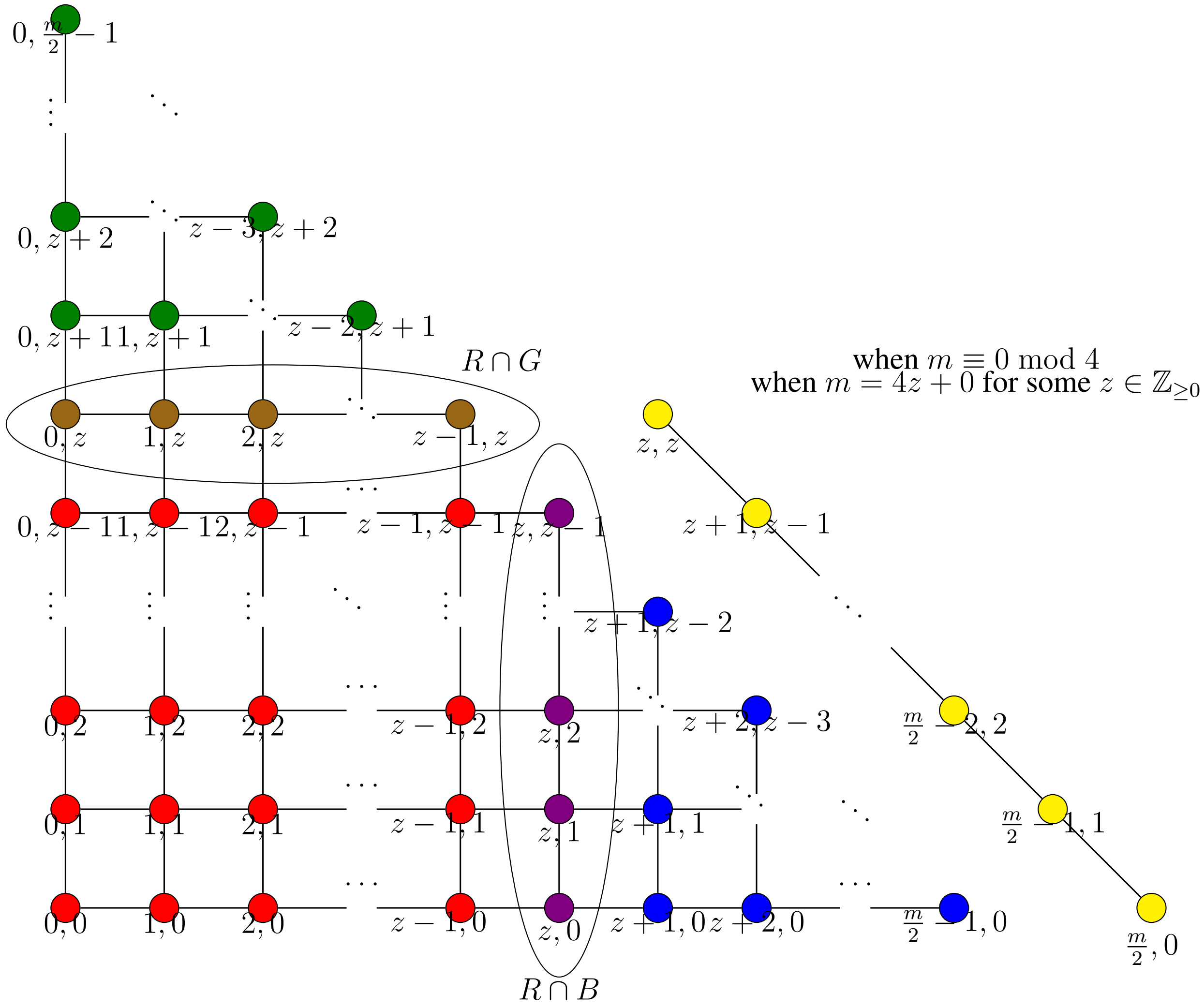
and define Y as

$$Y = \big\{(i,j) : i+j = \frac{m}{2} \text{ and } 0 \leq j \leq i\big\}$$

Note that $|\Delta| = \binom{\frac{m}{2}-1+2}{2} = \binom{\frac{m}{2}+1}{2}$, and $|Y| = \lfloor \frac{m}{4} \rfloor + 1$

When $m \equiv 0 \bmod 4$, we have $m = 4z + 0$ for some $z \in \mathbb{Z}_{\geq 0}$, thus G, R, and B are

$$G = \big\{(i,j) \in \Delta : j \geq z\big\}$$
$$R = \big\{(i,j) \in \Delta : i \leq z \text{ and } j \leq z\big\}$$
$$B = \big\{(i,j) \in \Delta : i \geq z\big\}$$

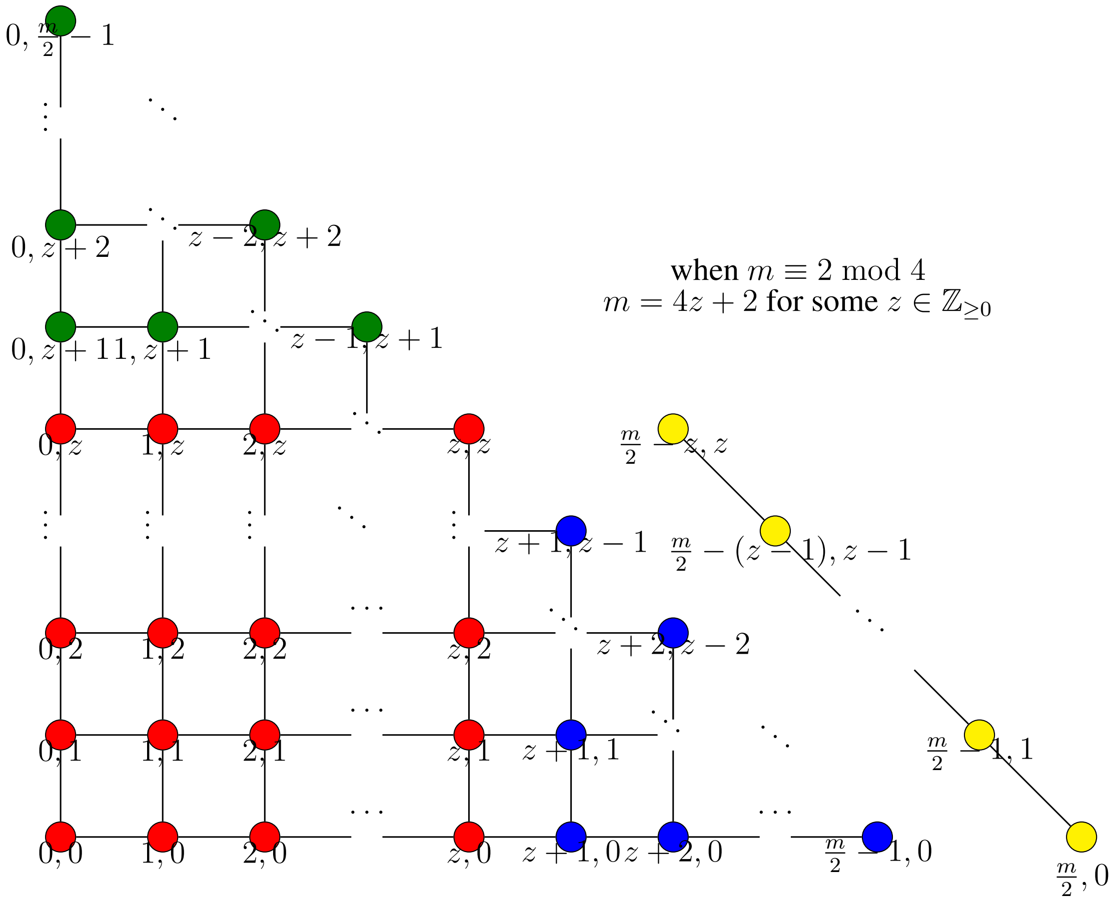


When $m \equiv 2 \bmod 4$, we have $m = 4z + 2$ for some $z \in \mathbb{Z}_{\geq 0}$, thus G, R, and B are

$$G = \Big\{(i, j) \in \Delta : j \geq z + 1\}$$

$$R = \Big\{(i, j) \in \Delta : i \leq z \text{ and } j \leq z\Big\}$$

$$B = \Big\{(i, j) \in \Delta : i \geq z + 1\}$$



when $m \equiv 2 \bmod 4$
$m = 4z + 2$ for some $z \in \mathbb{Z}_{\geq 0}$

Now let

$$D_1^* = \{(i, j) : \exists x \in D_1 \text{ with } i = x_2 \text{ and } j = x_3\}$$

$$D_2^* = \{(i, j) : \exists x \in D_2 \text{ with } i = x_0 \text{ and } j = x_3\}$$

$$D_3^* = \{(i, j) : \exists x \in D_3 \text{ with } i = x_0 \text{ and } j = x_1\}$$

$$D_{-1}^* = \{(i, j) : \exists x \in D_{-1} \text{ with } i = x_0 \text{ and } j = x_1\}$$

We now claim that $D_1^* = G$, $D_2^* = R$, $D_3^* = B$, and $D_{-1}^* = Y$.

Claim 4.13. *$D_1^* = G$.*

Subproof: Let $(i,j) \in D_1^*$. Thus there exists an $x \in D_1$ with $i = x_2$ and $j = x_3$. Since x is in D_1 we know that $x_0 + x_1 \geq \{x_1 + x_2, x_2 + x_3\}$, in particular $x_0 \geq x_2$. Thus

$$\begin{aligned} m &= x_0 + x_1 + x_2 + x_3 \\ m - x_0 &= x_1 + x_2 + x_3 \\ &\text{and} \\ k &= x_1 + 2x_2 + 3x_3 \\ &= x_1 + x_2 + x_3 + x_2 + 2x_3 \\ &= m - x_0 + x_2 + 2x_3 \\ &\leq m + 2x_3 \end{aligned}$$

If $m = 4z + 0$ then $k = 6z$, and we have

$$\begin{aligned} 6z &\leq 4z + 2x_3 \\ 2z &\leq 2x_3 \\ z &\leq x_3 = j. \end{aligned}$$

And if $m = 4z + 2$ then $k = 6z + 3$, and we have

$$\begin{aligned} 6z + 3 &\leq 4z + 2 + 2x_3 \\ 2z + 1 &\leq 2x_3 \\ z + \frac{1}{2} &\leq x_3 = j. \end{aligned}$$

So in either case we have that $(i,j) \in G$. Now suppose $(i,j) \in G$, then $i \geq 0$, $\left\lfloor \frac{m+2}{4} \right\rfloor \leq j$, and $0 \leq i + j \leq \frac{m}{2} - 1$. Let $x = x_0, x_1, x_2, x_3$ with $x_0 = i + 2j - \frac{m}{2}$, $x_1 = k - 2i - 3j$, $x_2 = i$ and

$x_3 = j$. Consider

$$\begin{aligned}\sum x_i &= i + 2j - \frac{m}{2} + k - 2i - 3j + i + j \\ &= -\frac{m}{2} + k \\ &= -\frac{m}{2} + \frac{3m}{2} \\ &= m\end{aligned}$$

and

$$\sum ix_i = k - 2i - 3j + 2(i) + 3(j) = k.$$

Thus it remains to show that each of the $x_i \geq 0$. Now $i, j \geq 0$ by assumption so $x_2, x_3 \geq 0$. If $m \equiv 0 \bmod 4$ then

$$\begin{aligned}x_0 &= i + 2j - \frac{m}{2} \\ &\geq i + 2z - 2z \\ &\geq i \\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}x_1 &= 6z - 2i - 3j \\ &= 6z - 2(i + j) - j \\ &\geq 6z - 2(2z - 1) - j \\ &\geq 2z + 2 - 2z \\ &\geq 2.\end{aligned}$$

If $m \equiv 2 \bmod 4$ then

$$\begin{aligned} x_0 &= i + 2j - \frac{m}{2} \\ &\geq i + 2(z+1) - 2z - 1 \\ &\geq i + 1 \\ &\geq 1 \end{aligned}$$

and

$$\begin{aligned} x_1 &= 6z + 3 - 2i - 3j \\ &= 6z + 3 - 2(i+j) - j \\ &\geq 6z + 3 - 2(2z) - j \\ &\geq 2z + 3 - \left(\frac{m}{2} - 1\right) \\ &\geq 2z + 3 - 2z \\ &\geq 3. \end{aligned}$$

Thus for any $(i, j) \in G$, the above defined x will be in D_1, hence $(i, j) \in D_1^*$ as desired. ■

Claim 4.14. $D_2^* = R$.

Subproof: Let $(i, j) \in D_2^*$. Thus there exists an $x \in D_2$ with $i = x_0$ and $j = x_3$. Since x is in D_2 we know that $x_1 + x_2 \geq \{x_0 + x_1, x_2 + x_3\}$, in particular $x_2 \geq x_0$ and $x_1 \geq x_3$. Thus

$$\begin{aligned} m &= x_0 + x_1 + x_2 + x_3 \\ m - x_0 &= x_1 + x_2 + x_3 \end{aligned}$$

$$\begin{aligned} k &= x_1 + 2x_2 + 3x_3 \\ &= x_1 + x_2 + x_3 + x_2 + 2x_3 \\ &= m - x_0 + x_2 + 2x_3 \\ &\geq m + 2x_3 \end{aligned} \qquad \begin{aligned} k &= x_1 + 2x_2 + 3x_3 \\ &\leq 2x_1 + 2x_2 + 2x_3 \\ &= 2(m - x_0) \end{aligned}$$

If $m = 4z + 0$, then $k = 6z$, and we have

$$\begin{aligned} 6z &\geq 4z + 2x_3 & 6z &\leq 2(4z - x_0) \\ 2z &\geq 2x_3 & 2x_0 &\leq 2z \\ z &\geq x_3 & x_0 &\leq z \end{aligned}$$

If $m = 4z + 2$, then $k = 6z + 3$, and we have

$$\begin{aligned} 6z + 3 &\geq 4z + 2 + 2x_3 & 6z + 3 &\leq 2(4z + 2 - x_0) \\ 2z + 1 &\geq 2x_3 & 2x_0 &\leq 2z + 1 \\ z + \frac{1}{2} &\geq x_3. & x_0 &\leq z + \frac{1}{2}. \end{aligned}$$

So in either case we have that $(i, j) \in R$. Now suppose $(i, j) \in R$, then $0 \leq i \leq \lfloor \frac{m}{4} \rfloor$, $0 \leq j \leq \lfloor \frac{m}{4} \rfloor$, and $0 \leq i + j \leq \frac{m}{2} - 1$. Let $x = x_0, x_1, x_2, x_3$ with $x_0 = i$, $x_1 = \frac{m}{2} - 2i + j$, $x_2 = \frac{m}{2} + i - 2j$, and $x_3 = j$. Consider

$$\begin{aligned} \sum x_i &= i + \frac{m}{2} - 2i + j + \frac{m}{2} + i - 2j + j \\ &= m \end{aligned}$$

and

$$\begin{aligned} \sum ix_i &= \frac{m}{2} - 2i + j + 2\Big(\frac{m}{2} + i - 2j\Big) + 3(j) \\ &= 3\Big(\frac{m}{2}\Big) \\ &= k. \end{aligned}$$

Thus it remains to show that each of the $x_i \geq 0$. Now $i, j \geq 0$ by assumption so $x_0, x_3 \geq 0$. If $m \equiv 0 \bmod 4$ or if $m \equiv 2 \bmod 4$ we have that $0 \leq i, j \leq z$ thus

$$\begin{aligned} x_1 &= \frac{m}{2} - 2i + j & \qquad x_2 &= \frac{m}{2} + i - 2j \\ &\geq 2z - 2z + j & &\geq 2z + i - 2z \\ &= j + 1 & &\geq 0. \\ &\geq 1 & & \end{aligned}$$

and

Thus for any $(i, j) \in R$, the above defined x will be in D_2, and hence $(i, j) \in D_2^*$ as desired.

■

Claim 4.15. $D_3^* = B$.

Subproof: Let $(i, j) \in D_3^*$. Thus there exists an $x \in D_3$ with $i = x_0$ and $j = x_1$. Since x is in D_3 we know that $x_2 + x_3 \geq \{x_0 + x_1, x_1 + x_2\}$, in particular $x_3 \geq x_1$. Thus

$$\begin{aligned} m &= x_0 + x_1 + x_2 + x_3 \\ m - x_0 &= x_1 + x_2 + x_3 \end{aligned}$$

and

$$\begin{aligned} k &= x_1 + 2x_2 + 3x_3 \\ &\geq 2x_1 + 2x_2 + 2x_3 \\ &= 2(m - x_0). \end{aligned}$$

If $m = 4z$ then $k = 6z$ and we have

$$\begin{aligned} 6z &\geq 2(4z - x_0) \\ 2x_0 &\geq 2z \\ x_0 &\geq z. \end{aligned}$$

If $m = 4z + 2$ then $k = 6z + 3$ and we have

$$\begin{aligned} 6z + 3 &\geq 2(4z + 2 - x_0) \\ 2x_0 &\geq 2z + 1 \\ x_0 &\geq z + \frac{1}{2}. \end{aligned}$$

So in either case we have that $(i, j) \in B$. Now suppose $(i, j) \in B$, then $\lfloor \frac{m+2}{4} \rfloor \leq i \leq \frac{m}{2} - 1$, and $0 \leq i + j \leq \frac{m}{2} - 1$. Let $x = x_0, x_1, x_2, x_3$ with $x_0 = i$, $x_1 = j$, $x_2 = k - 3i - 2j$, and $x_3 = 2i + j - \frac{m}{2})$. Consider

$$\begin{aligned} \sum x_i &= i + j + k - 3i - 2j + 2i + j - \frac{m}{2} \\ &= k - \frac{m}{2} \\ &= \frac{3m}{2} - \frac{m}{2} \\ &= m \end{aligned}$$

and

$$\begin{aligned} \sum ix_i &= j + 2(k - 3i - 2j) + 3\Big(2i + j - \frac{m}{2}\Big) \\ &= 2k - \frac{3m}{2} \\ &= 2k - k \\ &= k. \end{aligned}$$

Thus it remains to show that each of the $x_i \geq 0$. Now $i, j \geq 0$ by assumption so $x_0, x_1 \geq 0$. If $m \equiv 0 \bmod 4$ then

$$\begin{aligned} x_2 &= k - 3i - 2j \\ &= 6z - 2(i+j) - i \\ &\geq 6z - 2\left(\frac{m}{2} - 1\right) - i \\ &\geq 6z - 4z + 2 - (2z - 1) \\ &\geq 2z + 2 - 2z + 1 \\ &\geq 3 \end{aligned} \qquad \begin{aligned} x_3 &= 2i + j - \frac{m}{2} \\ &\geq 2(z+1) + j - 2z \\ &\geq j + 2 \\ &\geq 2 \end{aligned}$$

If $m \equiv 2 \bmod 4$ then

$$\begin{aligned} x_2 &= k - 3i - 2j \\ &= 6z + 3 - 2(i+j) - i \\ &\geq 6z + 3 - 2\left(\frac{m}{2} - 1\right) - i \\ &\geq 6z + 3 - 4z - (2z + 1) \\ &\geq 2z + 3 - 2z - 1 \\ &\geq 2 \end{aligned} \qquad \begin{aligned} x_3 &= 2i + j - \frac{m}{2} \\ &\geq 2(z+1) + j - 2z \\ &\geq j + 2 \\ &\geq 2 \end{aligned}$$

Thus for any $(i,j) \in B$, the above defined x will be in D_3, and hence $(i,j) \in D_3^*$ as desired. ■

So using the principle of inclusion exclusion and recalling that the D_as are by definition disjoint from the D_{-a}s the above claims we have

$$\begin{aligned}
|A_k| &= |D_3 \cup D_2 \cup D_1 \cup D_{-1} \cup D_{-2}| \\
&= |D_1| + |D_2| + |D_3| + |D_{-1}| + |D_{-2}| - |D_1 \cap D_2| - |D_2 \cap D_3| \\
&\quad - |D_1 \cap D_3| - |D_{-1} \cap D_{-2}| + |D_1 \cap D_2 \cap D_3| \\
&= |D_1^*| + |D_2^*| + |D_3^*| + |D_{-1}| + |D_{-2}| - |D_1^* \cap D_2^*| - |D_2^* \cap D_3^*| \\
&\quad - 0 - |D_{-2}| + 0 \\
&= |G| + |R| + |B| + |D_{-1}| - |G \cap R| - |R \cap B| \\
&= |\Delta| + |Y|
\end{aligned}$$

■

And as we have already noted we have that $|\Delta| = \binom{\frac{m}{2}-1+2}{2} = \binom{\frac{m}{2}+1}{2}$, and $|Y| = \left\lfloor \frac{m}{4} \right\rfloor + 1$. Thus $|A_k| = \binom{\frac{m}{2}+1}{2} + \left\lfloor \frac{m}{4} \right\rfloor + 1$ as desired. □

Definition 4.16. *For positive integers m and n let $L(m, n)$ denote the set of all m-tuples $(a_1, a_2, \ldots, a_m)$ of integers with $0 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n$. The set $L(m, n)$ is a poset such that $(a_1, a_2, \ldots, a_m) \preceq (b_1, b_2, \ldots, b_m)$ holds precisely when $a_i \leq b_i$ for $i = i, 2, \ldots, m$.*

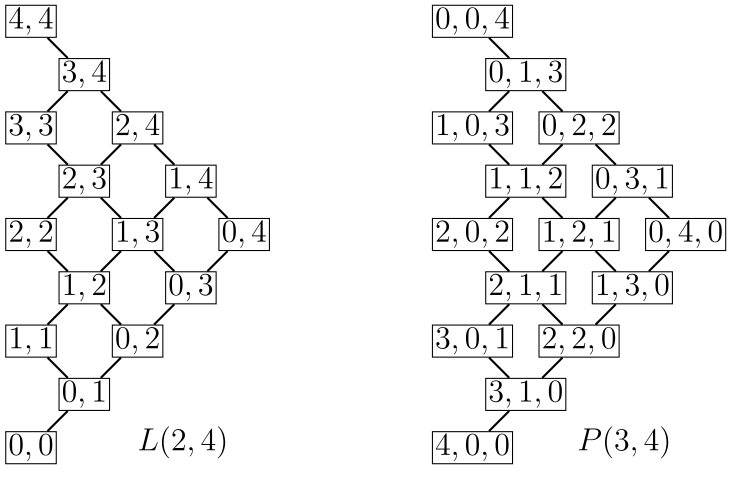


Figure 4.1: Hasse Diagrams for $L(2, 4) \cong P(3, 4)$

Theorem 4.17. $P(m+1,n) \cong L(m,n)$.

Proof. Let $x = (x_0, x_1, \dots x_m) \in P(m+1,n)$ and define $f : P(m+1,n) \to L(m,n)$ to be $f(x) = (s_m, s_{m-1}, \dots, s_2, s_1)$ where $s_i = x_i + x_{i+1}, + \cdots + x_m$ for $1 \leq i \leq m$. Thus the length of $f(x) = m$, and since $s_i \leq x_0 + x_1 + \cdots + x_m = n$, then $f(x) \in L(m,n)$.

Now suppose for $x = (x_0, x_1, \dots, x_m)$ and $y = (y_0, y_1, \dots, y_m) \in P(m+1,n)$ that $f(x) = f(y)$. That is

$$(s_m, s_{m-1}, \dots, s_2, s_1) = f(x) = f(y) = (t_m, t_{m-1}, \dots, t_2, t_1)$$

where $s_i = x_i + x_{i+1}, + \cdots + x_m$ and $t_i = y_i + y_{i+1}, + \cdots + y_m$ for $1 \leq i \leq m$. Since $f(x) = f(y)$ then

$$x_m = s_m = t_m = y_m$$

and using substitution we also get that

$$\begin{array}{cc}
x_{m-1} + x_m = s_{m-1} = t_{m-1} = y_{m-1} + y_m & \Rightarrow x_{m-1} = y_{m-1} \\
x_{m-2} + x_{m-1} + x_m = s_{m-2} = t_{m-2} = y_{m-2} + y_{m-1} + y_m & \Rightarrow x_{m-2} = y_{m-2} \\
\vdots & \vdots \\
x_1 + x_2 + \cdots + x_m = s_1 = t_1 = y_1 + y_2 + \cdots + y_m & \Rightarrow x_1 = y_1
\end{array}$$

And since $x, y \in P(m+1,n)$ then

$$x_0 + x_1 + \cdots + x_m = n = y_0 + y_1 + \cdots + y_m$$

and so again by substitution we get that $x_0 = y_0$ and so $x = y$.

Now suppose that $a \in L(m,n)$ and consider the element $x = (n - a_m, a_m - a_{m-1}, a_{m-1} - a_{m-2}, a_{m-2} - a_{m-3}, \dots, a_3 - a_2, a_2 - a_1, a_1)$. It is easy to see that $\sum_i x_i = n$ and that the length of x is $m+1$. Thus it remains to show that $f(x) = a$, that is we want to show that

$(s_m, s_{m-1}, \ldots, s_1) = (a_1, a_2, \ldots, a_m)$.

$$\begin{aligned}
s_m &= x_m && = a_1\\
s_{m-1} &= x_{m-1} + x_m = a_2 - a_1 + a_1 && = a_2\\
s_{m-2} &= x_{m-2} + x_{m-1} + x_m = a_3 - a_2 + a_2 - a_1 + a_1 && = a_3\\
&\vdots && \vdots\\
s_1 &= x_1 + x_2 + \cdots + x_{m-1} + x_m = a_m - a_{m-1} + \cdots a_2 - a_1 + a_1 && = a_m
\end{aligned}$$

Therefore $f(x) = a$.

Now to show that the structure is preserved we want to show that for $x, y \in P(m+1, n)$ if $y \geq_M x$ then $f(x) \geq f(y)$ in $L(m, n)$. So suppose that y covers x then $y \geq_M x$ and if y is in level k then x is in level $k+1$. So since $y \geq_M x$ then,

$$\begin{aligned}
x_0 &\leq y_0\\
x_0 + x_1 &\leq y_0 + y_1\\
x_0 + x_1 + x_2 &\leq y_0 + y_1 + y_2\\
\vdots\quad &\quad\vdots\\
x_0 + x_1 + x_2 + \cdots + x_{m-1} &\leq y_0 + y_1 + y_2 + \cdots + y_{m-1}\\
x_0 + x_1 + x_2 + \cdots + x_m &\leq y_0 + y_1 + y_2 + \cdots + y_m.
\end{aligned}$$

rewriting this gives

$$\begin{gathered} n - s_1 \leq n - t_1 \\ n - s_2 \leq n - t_2 \\ n - s_3 \leq n - t_3 \\ \vdots \quad \vdots \\ n - s_m \leq n - t_m \\ n \leq n \end{gathered}$$

thus

$$\begin{gathered} s_1 \geq t_1 \\ s_2 \geq t_2 \\ s_3 \geq t_3 \\ \vdots \quad \vdots \\ s_m \geq t_m \\ 0 \geq 0. \end{gathered}$$

So $f(x) \geq f(y) \in L(m, n)$. Moreover since these steps are easily reversible then $f(x) \geq f(y)$ if and only if $y \geq_M x$. □

Theorem 4.18. *The partially ordered set* $L(3, n) \cong P(4, m)$ *has a partition into saturated symmetric chains when* $n \geq 0$. *[8]*

In [8], Lindström does not explicitly enumerate the number of chains, as we have. Therefore he does not give an exact width for the poset.

Theorem 4.19. $L(4, n) \cong P(5, m)$ *is a symmetric chain order. [16]*

In [16], West does explicitly enumerate the number of chains.

Independently from Lindström and West, in [12] Riess also proved the existence of a symmetric chain decomposition of $L(3, n)$ and $L(4, n)$. Our chain decomposition is not symmetric and thus is genuinely different from what Lindström, West, and Riess have done.

As stated in the introduction O'Hara in [9] gave the first combinatorial proof of the unimodality of the Gaussian polynomials. Providing such a proof had been an open problem for a long time. We recall their definition below.

Definition 4.20. *For natural numbers n, k the Gaussian Polynomial or q-binomial coefficient $\binom{n}{k}_q$, sometimes written $\left\{ {n \atop k} \right\}_q$, is the following rational function of the variable q:*

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1) \cdot (q^{n-k} - 1)(q^{n-k-1} - 1) \cdots (q - 1)}$$

If we exclude from possible assignments of q all i-th roots of unity for every $i \leq n$, then $\binom{n}{k}_q$ becomes

$$\begin{aligned} \binom{n}{k}_q &= \frac{(q^n - 1)(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)} \\ &= \frac{(q^{n-1} + \cdots + q + 1) \cdots (q^2 + q + 1)(q + 1)}{(q^{k-1} + \cdots + q + 1) \cdots (q + 1) \cdot (q^{n-k-1} + \cdots + q + 1) \cdots (q + 1)} \end{aligned} \tag{4.1}$$

We now introduce the q-analog of the positive integer n, denoted by $[n]$ or $[n]_q$ as $(q^{n-1} + q^{n-2} + \cdots + q + 1)$ and the q-analog of the factorial, denoted $[n]!$ or $[n]_q!$ as $[1]_q \cdot [2]_q \cdots [n]_q$. So with this new notation we see that equation 4.1 becomes

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n - k]_q!}$$

Which should look familiar! This is similar to the usual definition of the binomial coefficients and is why these Gaussian polynomials are sometimes referred to as the Gaussian binomial polynomials or q-binomial coefficients. Moreover, as mentioned in Chapter 1, letting $q = 1$ gives rise to the usual definition of the binomial coefficients.

Lemma 4.21 (Folklore)**.** *The Gaussian Polynomial is not just a rational function of q but is a polynomial with integer coefficients.*

We provide a proof for sake of completeness.

Proof. We prove this by induction on n. As mentioned in Chapter 1, We have that $\binom{n}{0}_q = \binom{n}{n}_q = 1$. So since 1 is a polynomial in q then the claim holds for $k \in \{0, n\}$. We now assume instead that $0 < k < n$ and that the claim holds for $i = n - 1$. Observe that $\binom{i}{k}_q$ and $\binom{i}{k-1}_q$ are polynomials by induction (note that $0 \leq k - 1 < n - 1 = i$, $0 < k \leq n - 1 = i$ so the coefficients are defined), and therefore so is $q^{i-k+1}\binom{i}{k-1}_q$. We now consider the following:

$$\begin{aligned}
\binom{i}{k}_q + q^{i-k+1}\binom{i}{k-1}_q &= \frac{[i]_q!}{[k]_q![i-k]_q!} + q^{i-k+1}\frac{[i]_q!}{[k-1]_q![i-k+1]_q!} \\
&= \frac{[i]_q!}{[k-1]_q![i-k]_q!}\Big(\frac{1}{q^{k-1}+\cdots+q+1} + \frac{q^{i-k+1}}{q^{i-k}+\cdots+q+1}\Big) \\
&= \frac{[i]_q!}{[k-1]_q![i-k]_q!}\Big(\frac{(q^{i-k}+\cdots+q+1) + q^{i-k+1}(q^{k-1}+\cdots+q+1)}{(q^{k-1}+\cdots+q+1)(q^{i-k}+\cdots+q+1)}\Big) \\
&= \frac{[i]_q!}{[k-1]_q![i-k]_q!}\Big(\frac{q^{i}+\cdots+q+1}{(q^{k-1}+\cdots+q+1)(q^{i-k}+\cdots+q+1)}\Big) \\
&= \frac{[i+1]_q!}{[k]_q![i-k+1]_q!} \\
&= \binom{i+1}{k}_q.
\end{aligned}$$

That is, we have written $\binom{n}{k}_q$ as a $\mathbb{Z}[q]$-linear combination of two polynomials, so it is therefore also a polynomial as desired. □

The main relevance of these Gaussian polynomials to combinatorics is that $\binom{n+m}{m}_q$ is the generating function which gives the number of ways, $B(n, m, r)$, of throwing r identical balls into m identical boxes such that each box holds at most n balls. For example when $r = 4$, $m = 2$, and $n = 4$, we have the following polynomial:

$$\begin{aligned}
\binom{4+2}{2}_q &= \frac{(q^6-1)(q^5-1)(q^4-1)(q^3-1)(q^2-1)(q-1)}{(q^2-1)(q-1)\cdot(q^4-1)(q^3-1)(q^2-1)(q-1)} \\
&= \frac{(q^6-1)(q^5-1)}{(q^2-1)((q-1)} \\
&= (q^2+q+1)(q^2-q+1)(q^4+q^3+q^2+q+1) \\
&= q^8+q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1
\end{aligned}$$

From this we see that $B(4,2,4) = 3$ since the coefficient of q^4 is 3. That is there are 3 ways of throwing 4 balls into 2 boxes such that each box can hold at most 4 balls, namely $\{2,2\}$, $\{1,3\}$, and $\{0,4\}$. Which is precisely the middle level, level 4, of the Poset $L(2,4)$ from figure 4.1. Alternatively, this generating function gives the number of partitions of r into m parts such that each part has size at most n.

O'Hara's strategy in [9], was to define a poset $U(a,b)$ such that its elements are the partitions into b parts each of size at most a. Namely

$$U(a,b) = \{(p_1, p_2, \ldots, p_b) : 0 \leq p_1 \leq p_2 \leq \cdots \leq p_b \leq a\}.$$

And the levels of the posets are those elements in $U(a,b)$ such that $\sum p_i = k$, that is a level is

$$U_k(a,b) = \{(p_1, p_2, \ldots, p_b) : 0 \leq p_1 \leq p_2 \leq \cdots \leq p_b \leq a \text{ and } p_1 + p_2 + \cdots + p_a = k\}.$$

It is easy to see that the elements of $U(a,b)$ and $L(m,n)$ are the same and are even arranged into levels containing the same elements, so unimodality of one implies unimodality of the other. However the ordering of the $L(m,n)$ posets from Lindström, West, and Riess are not the same as the $U(a,b)$ ordering O'Hara used. Lindström, West, and Riess's ordering was that $x \geq y$ if $x_i \geq y_i$ for all i, under this ordering the Hasse diagrams give the Young's lattices for the integer partitions. O'Hara, on the other hand, used the ordering that $x \geq y$ iff $\sum x_i \geq \sum y_i$ (which simply orders the elements by their levels). So the element $(2,3)$ and the element $(0,4)$ are incomparable in $(L(2,4), \leq)$ since $2 \geq 0$ but $3 \leq 4$ but are comparable in $(U(2,4), \leq)$ since $2 + 3 \geq 0 + 4$, as shown in Figure 4.2.

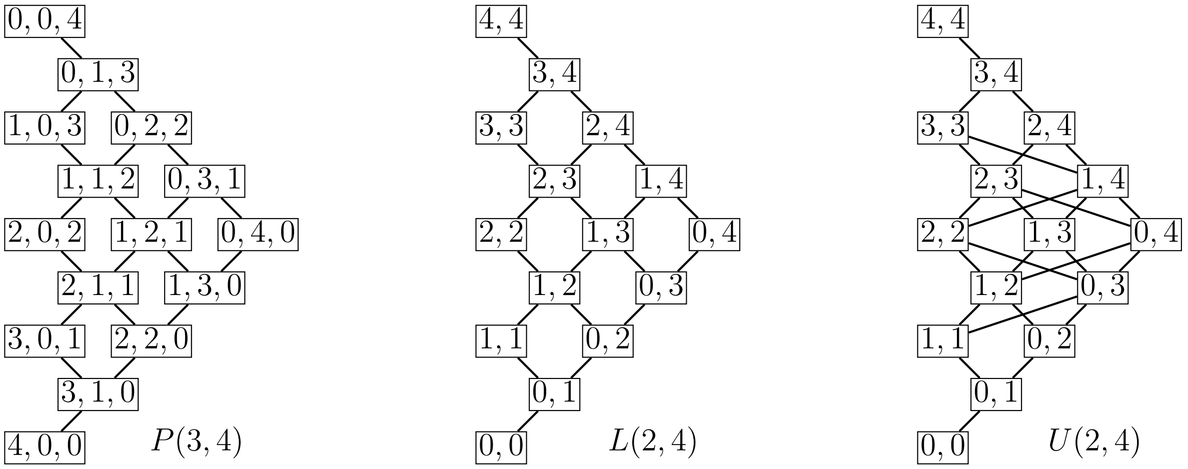


Figure 4.2: Hasse Diagrams for $P(3,4) \cong L(2,4) \subsetneq U(2,4)$

So while O'Hara's symmetric chain decomposition of $U(m,n)$ gives a beautiful proof of the unimodality of the levels of $L(m,n)$ (in other words, that $P(m+1,n)$ is Sperner), we conjecture the following combinatorial strengthening:

Conjecture 4.22. *$P(n,m)$ under the majorization ordering has a symmetric chain decomposition for all $n, m \geq 0$.*

Chapter 5

The Existence of a Chain Decomposition for$P'(4, m)$

To prove the existence of a minimal chain decomposition for $P'(4, m)$, it would be natural to try and extend the mapping used in the inductive proof for $P'(3, m)$. In that proof, recall that we "peeled off" a chain, C, of length m and then showed that under the mapping $f : (x_0, x_1, x_2) \to (x_0 - 4, x_1 - 1, x_2 - 1)$, $P'(3, m) \setminus C \cong P'(3, m - 6)$. The natural extension then is to use the mapping $g : (x_0, x_1, x_2, x_3) \to (x_0 - 4, x_1 - 1, x_2 - 1, x_3 - 1)$. Unlike with the $P'(3, m)$ case we cannot simply peel off a single chain to get to $P'(4, m - 7)$ since g will necessarily peel off all of slice 0, which is isomorphic to $P'(3, m)$, as well as other elements. So the parts that get peeled off will themselves need to be decomposed into chains. So under the mapping g, when we decompose the elements that do get peeled off into chains and then induct, in most cases we will get a chain decomposition that results in more chains than the minimum number required. For example in Figure 5.1 we see that $g((5, 1, 1, 1)) = (1, 0, 0, 0)$ is the only element that gets mapped to $P'(4, 8 - 7)$, while all other elements get peeled off. It is easy to see in this small example that we need two chains to decompose everything except for $(5, 1, 1, 1)$, namely $C_1 = \{(8, 0, 0, 0), (7, 1, 0, 0), (6, 2, 0, 0), (5, 3, 0, 0), (4, 4, 0, 0), (4, 3, 1, 0), (4, 2, 2, 0), (3, 3, 2, 0)\}$ and $C_2 = \{(6, 1, 1, 0), (5, 2, 1, 0), (4, 2, 1, 1), (3, 3, 1, 1), (3, 2, 2, 1), (2, 2, 2, 2)\}$. So when we induct we get $2 + 1$ many chains. But again it is easy to see that the entire poset could have been decomposed into two chains by including $5, 1, 1, 1$ in C_2 since $(5, 2, 1, 0) \geq_M (5, 1, 1, 1) \geq_M (4, 2, 1, 1)$. So the mapping g, does not provide the best result. By the same argument neither does the mapping which removes $(5, 1, 1, 1)$.

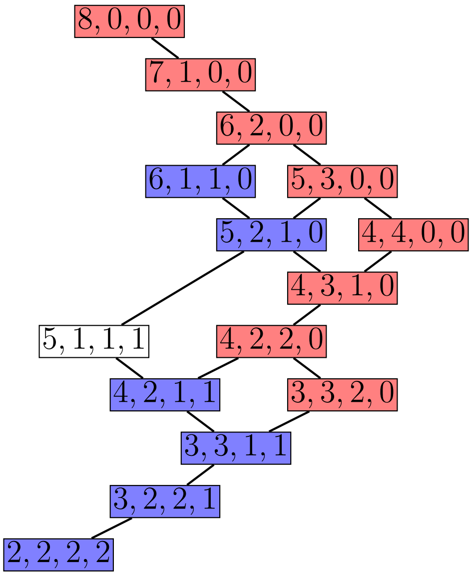


Figure 5.1: The mapping from the $P'(3,m)$ case does not extend to to $P'(4,m)$

It turns out that inducting using the mapping $f : (x_0, x_1, x_2, x_3) \to (x_0 - 10, x_1 - 2, x_2 - 2, x_3 - 2)$ which subtracts $(10, 2, 2, 2)$ from all elements possible is the one that works.[1] So we claim that $P'(4,m)\setminus R \cong P'(4,m-16)$ via the mapping f where R consists of all the elements such that $x_3 \in \{0,1\}$ or $x_0 - x_1 \leq 7$ or both. We also show that R can be decomposed into as few chains as possible. To do this we will prove what the structure of R looks like as well as how to decompose R into chains such that the maximal chains in R align with the maximal chains in $P'(4, m - 16)$.

Definition 5.1. For a given level k and slice i we define a level-slice, $L_{i,k}$ to be all the elements in $P'(n,m)$ such that p is in level k and slice i.

Lemma 5.2. *In a given level-slice, $L_{i,k}$ the first differences, $x_0 - x_1$, are congruent modulo* 3.

---

[1] We do not have at present any heuristic reason why $(10, 2, 2, 2)$ should work special for $P'(4,m)$ in the way that subtracting $(4, 1, 1)$ drove the induction for $P'(3,m)$ in Chapter 3, and since these results do not extend to further n (see Chapter 6) such an explanation is unlikely to ever present itself.

Proof. Let $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3) \in L_{i,k}$ for some i, k. Thus we have that $x_3 = y_3 = i$ and the following equations:

$$x_0 + x_1 + x_2 + i = y_0 + y_1 + y_2 + i \tag{5.1}$$

$$x_1 + 2x_2 + 3i = y_1 + 2y_2 + 3i \tag{5.2}$$

If we let $\lambda = y_0 - x_0$, then this together with equation 5.1 becomes

$$x_1 + x_2 = y_1 + y_2 + \lambda \tag{5.3}$$

and now equations 5.2 and 5.3 gives us

$$\begin{aligned} (x_1 + x_2) + x_2 &= y_1 + 2y_2 \\ (y_1 + y_2 + \lambda) + x_2 &= y_1 + 2y_2 \\ x_2 &= y_2 - \lambda. \end{aligned} \tag{5.4}$$

$\lambda = y_0 - x_0$ together with equations 5.1 and 5.4 we have

$$\begin{aligned} x_0 + x_1 + x_2 &= y_0 + y_1 + y_2 \\ x_1 &= (y_0 - x_0) + y_1 + (y_2 - x_2) \\ x_1 &= y_1 + 2\lambda. \end{aligned}$$

Thus we have that if x and y are in the same level-slice, then $x = y + \lambda(-1, 2, -1, 0)$. Thus we can see that the first differences among elements in the same level-slice will be congruent modulo 3. □

Corollary 5.3. *In R, all the elements of a given level-slice, $L_{i,k}$, will have either $0 \leq x_0 - x_1 \leq 7$ or $x_3 \leq 1$. So by lemma 5.2 for $i \geq 2$ the maximum size of a level-slice in R is 3 if $x_0 - x_1 \equiv 0$ or $1 \bmod 3$ and is 2 if $x_0 - x_1 \equiv 2 \bmod 3$.*

Definition 5.4. In R for a fixed slice, i with $i \geq 2$, and a fixed level, k, we define a level-slice to be maximal if $|L_{i,k}| = 3$ when $x_0 - x_1 \equiv 0$ or $1 \bmod 3$ or if $|L_{i,k}| = 2$ when $x_0 - x_1 \equiv 2 \bmod 3$.

Corollary 5.5. *For a fixed slice, i, and level, k, $x_0 - x_2$ is constant for all $x \in L_{i,k}$. That is,*

$$\begin{aligned} x_0 - x_2 &= x_0 + x_1 + x_2 + x_3 + 2x_3 - x_1 - 2x_2 - 3x_3 \\ &= m + 2i - k. \end{aligned}$$

Thus $m + 2i - k$ is an alternative parameter to determine the the level (given a particular slice). We next characterize how small or large this parameter can be for a given slice, i, while maintaining maximality as stated in definition 5.4. This will help us to start to get an idea for what the structures in R are whenever $i \geq 2$. Again by corollary 5.3 we know that the maximum size of a level-slice is 3 so we in a sense have to peel off 3 chains from each of the slices $i \geq 2$ in R. But again to ensure minimality of chains its not that simple.

Lemma 5.6. *In R for a fixed slice i with $i \geq 2$, the maximal level-slices are bounded in general by*

$$5 \leq m + 2i - k \leq \left\lfloor \frac{m}{2} \right\rfloor - 2i + 1$$

But we can tighten these bounds if we case on the parity of m as well as what $x_0 - x_1$ is congruent to modulo 3 for all x in the given level-slice.

1. *if $x_0 - x_1 \equiv 0 \bmod 3$ then*
$$6 \leq m + 2i - k \leq \begin{cases} \frac{m}{2} - 2i & \text{if } m \text{ is even} \\ \frac{m-1}{2} - 2i - 1 & \text{if } m \text{ is odd} \end{cases}$$
2. *if $x_0 - x_1 \equiv 1 \bmod 3$ then*
$$7 \leq m + 2i - k \leq \begin{cases} \frac{m}{2} - 2i - 1 & \text{if } m \text{ is even} \\ \frac{m-1}{2} - 2i + 1 & \text{if } m \text{ is odd} \end{cases}$$
3. *if $x_0 - x_1 \equiv 2 \bmod 3$ then*
$$5 \leq m + 2i - k \leq \begin{cases} \frac{m}{2} - 2i + 1 & \text{if } m \text{ is even} \\ \frac{m-1}{2} - 2i & \text{if } m \text{ is odd} \end{cases}$$

Proof. Suppose $m + 2i - k \le 4$ that is $x_0 - x_2 \le 4$ for all $x \in L_{i,k}$. Since $x_0 - x_2 \le 4$ and $x_1 \ge x_2$ then $x_0 - x_1 \le 4$. By Corollary 5.3 we have that the largest $L_{i,k}$ can be now is 2 if $x_0 - x_1 \equiv 0$ or $1 \bmod 3$ with those values being $\{0, 3\}$ and $\{1, 4\}$ respectively. And there is only 1 element if $x_0 - x_1 \equiv 2 \bmod 3$, which has $x_0 - x_1 = 2$.

Now suppose $\lfloor \frac{m}{2} \rfloor - 2i + 1 < m + 2i - k$. If m is even then we have:

$$\begin{aligned}
\frac{m}{2} - 2i + 2 &\le m + 2i - k \\
2 &\le \frac{m}{2} + 4i - k \\
4 &\le m + 8i - 2k \\
4 &\le m + 2i - k + 6i - k \\
4 &\le x_0 - x_2 + 6i - k \\
4 - 6i + k &\le x_0 - x_2 \\
4 - 6x_3 + x_1 + 2x_2 + 3x_3 &\le x_0 - x_2 \\
4 - 3x_3 + 3x_2 &\le x_0 - x_1 \le \{6, 7, 5\} \\
4 - 3x_3 + 3x_2 &\le \{6, 7, 5\} \\
-3x_3 + 3x_2 &\le \{2, 3, 1\} \\
x_2 - x_3 &\le \left\{\frac{2}{3}, 1, \frac{1}{3}\right\} \\
x_2 - x_3 &\le \{0, 1, 0\} \text{ since } x_2 - x_3 \text{ is an integer.}
\end{aligned}$$

So by 5.2 we have that there can be at most 1 element if $x_0 - x_1 \equiv 0$ or $2 \bmod 3$ and at most 2 elements if $x_0 - x_1 \equiv 1 \bmod 3$.

And similarly if m is odd we have:

$$\begin{aligned}
\frac{m}{2} - \frac{1}{2} - 2i + 2 &\leq m + 2i - k \\
\frac{m}{2} - 2i + \frac{3}{2} &\leq m + 2i - k \\
\frac{3}{2} &\leq \frac{m}{2} + 4i - k \\
3 &\leq m + 8i - 2k \\
3 &\leq m + 2i - k + 6i - k \\
3 &\leq x_0 - x_2 + 6i - k \\
3 - 6i + k &\leq x_0 - x_2 \\
3 - 6x_3 + x_1 + 2x_2 + 3x_3 &\leq x_0 - x_2 \\
-3x_3 + 3x_2 &\leq x_0 - x_1 \leq \{6, 7, 5\} \\
3 - 3x_3 + 3x_2 &\leq \{6, 7, 5\} \\
-3x_3 + 3x_2 &\leq \{3, 4, 2\} \\
-3x_3 + 3x_2 &\leq \left\{1, \frac{4}{3}, \frac{2}{3}\right\} \\
x_2 - x_3 &\leq \{1, 1, 0\} \text{ since } x_2 - x_3 \text{ is an integer.}
\end{aligned}$$

So again by 5.2 we have that there can be at most 1 element if $x_0 - x_1 \equiv 2 \bmod 3$ and at most 2 elements if $x_0 - x_1 \equiv 0$ or $1 \bmod 3$. Thus if $m + 2i - k$ lies outside of these bounds then the level-slice cannot be maximal.

To tighten these bounds for the different cases of $x_0 - x_1 \bmod 3$, we first prove that the cases are in fact distinct. First assume that m is even. If $m + 2i - k = \frac{m}{2} - 2i + E$, with E

being an “error” term, then we claim that $x_0 - x_1 \equiv 2E \bmod 3$.

$$\begin{aligned} m + 2i - k &= \frac{m}{2} - 2i + E \\ 2m + 4i - 2k &= m - 4i + 2E \\ m + 8i &= 2k + 2E \\ x_0 + x_1 + x_2 + x_3 + 8x_3 &= 2x_1 + 4x_2 + 6x_3 + 2E \\ x_0 - x_1 &= 3x_2 - 3x_3 + 2E \\ x_0 - x_1 &\equiv 2E \bmod 3 \end{aligned}$$

So if m is even and $x_0 - x_1 \equiv 0 \bmod 3$ then the largest E is allowed to be and still have $m + 2i - k \leq \frac{m}{2} - 2i + 1$ would be $E = 0$. So we can tighten the upper bound for this case to $m + 2i - k \leq \frac{m}{2} - 2i + 0$. And similarly for $x_0 - x_1 \equiv 1 \bmod 3$ we have that $E = -1$ is largest and for $x_0 - x_1 \equiv 2 \bmod 3$ we have that $E = 1$ is largest.

Now if we assume that m is odd. If $m + 2i - k = \frac{m-1}{2} - 2i + E$ then we claim that $x_0 - x_1 \equiv 2E - 1 \bmod 3$.

$$\begin{aligned} m + 2i - k &= \frac{m-1}{2} - 2i + E \\ 2m + 4i - 2k &= m - 1 - 4i + 2E \\ m + 8i &= 2k + 2E - 1 \\ x_0 + x_1 + x_2 + x_3 + 8x_3 &= 2x_1 + 4x_2 + 6x_3 + 2E - 1 \\ x_0 - x_1 &= 3x_2 - 3x_3 + 2E - 1 \\ x_0 - x_1 &\equiv 2E - 1 \bmod 3 \end{aligned}$$

So if m is odd and $x_0 - x_1 \equiv 0 \bmod 3$ then the largest E is allowed to be and still have $m + 2i - k \leq \frac{m}{2} - 2i + 1$ would be $E = -1$. So we can tighten the upper bound for this case to $m + 2i - k \leq \frac{m}{2} - 2i - 1$. And similarly for $x_0 - x_1 \equiv 1 \bmod 3$ we have that $E = 1$ is largest and for $x_0 - x_1 \equiv 2 \bmod 3$ we have that $E = 0$ is largest.

We can also tighten up the lower bounds as well. Since $x_1 \geq x_2$ then $x_0 - x_1 \leq x_0 - x_2$. So if $x_0 - x_1 \equiv 0 \bmod 3$ then the smallest $x_0 - x_2$ is allowed to be and still have $5 \geq m + 2i - k$ would be 6. And similarly for $x_0 - x_1 \equiv 1 \bmod 3$ and $x_0 - x_1 \equiv 2 \bmod 3$ we can tighten the lower bounds to 7 and 5 respectively.

Now suppose $m + 2i - k = c$ is bounded by $5 \leq m + 2i - k \leq \lfloor \frac{m-4i}{2} \rfloor + 1$. Since the x_i need to be integers, we let s be an integer such that the x_i are integers. Thus we claim that all elements which satisfy $5 \leq m + 2i - k \leq \lfloor \frac{m-4i}{2} \rfloor + 1$ are of the following form and that x is indeed a valid element after removal.

$$x = \left(\frac{k - 3i + 2c + s}{3}, \frac{k - 3i + 2c - 2s}{3}, \frac{k - 3i - c + s}{3}, i \right)$$

Thus we need to show that $\sum x_i = m$, $\sum ix_i = k$, $x_0 \geq x_1 \geq x_2 \geq x_3$, and that $0 \leq x_0 - x_1 \leq 7$. We see that

$$\begin{aligned}
\sum x_i &= \frac{k - 3i + 2c + s}{3} + \frac{k - 3i + 2c - 2s}{3} + \frac{k - 3i - c + s}{3} + i \\
&= k - 2i + c \\
&= k - 2i + m + 2i - k \\
&= m
\end{aligned}$$

and

$$\begin{aligned}
\sum ix_i &= \frac{k - 3i + 2c - 2s}{3} + 2\frac{k - 3i - c + s}{3} + 3i \\
&= \frac{k - 3i + 2c - 2s}{3} + \frac{2k - 6i - 2c + 2s}{3} + 3i \\
&= k
\end{aligned}$$

We can quickly see that $x_0 \geq x_1$ and that $x_0 - x_1 = s$ thus we only allow those s such that $0 \leq s \leq 7$. We now check to see if $x_1 \geq x_2 \geq x_3$. We assumed that $m + 2i - k \geq \{6, 7, 5\}$

thus

$$\begin{aligned}
m + 2i - k &\geq \{6, 7, 5\} \\
m + 2i - k &\geq s \\
c &\geq s \\
3(c - s) &\geq 0 \\
2c - 2s &\geq -c + s \\
\frac{k - 3i + 2c - 2s}{3} &\geq \frac{k - 3i - c + s}{3} \\
x_1 &\geq x_2
\end{aligned}$$

We also assumed that $m + 2i - k \leq \frac{m}{2} - 2i + \{0, -1, 1\}$ if m is even and $m + 2i - k \leq \frac{m-1}{2} - 2i + \{-1, 1, 0\}$ if m is odd. But in any case we have that $\frac{m}{2} - 2i + \frac{s}{2}$ is larger, since again we note that $s = x_0 - x_1$. So

$$\begin{aligned}
\frac{m}{2} - 2i + \frac{s}{2} &\geq m + 2i - k \\
0 &\geq \frac{m}{2} + 4i - k - \frac{s}{2} \\
0 &\geq m + 8i - 2k - s \\
2k - 5i - m + s &\geq 3i \\
k - 3i - (m + 2i - k) + s &\geq 3i \\
k - 3i - c + s &\geq 3i \\
\frac{k - 3i - c + s}{3} &\geq i \\
x_2 &\geq x_3
\end{aligned}$$

Thus the existence of such elements only depends on carefully chosen s. So for each level-slice we can pick s to be either $\{0, 3, 6\}$, $\{1, 4, 7\}$, or $\{2, 5\}$ and we will get all the valid elements after removal from a given level-slice. $\square$

Now that we have bounds on where our level-slices are maximal we see what lies just above these maximal level-slices. That what is the smallest (top-most) non-empty level-slice for a given slice, and how many levels above is it from the smallest (top-most) maximal level-slice.

Lemma 5.7. *Given a slice i with $i \geq 2$, the smallest (top-most) level-slice which is not empty is determined by:*

1. *if* $x_0 - x_1 \equiv 0 \bmod 3$ *then* $m + 2i - k = \begin{cases} \frac{m}{2} - 2i + 3 \text{ \textit{if} } m \text{ \textit{is even}} \\ \frac{m-1}{2} - 2i + 2 \text{ \textit{if} } m \text{ \textit{is odd}} \end{cases}$

2. *if* $x_0 - x_1 \equiv 1 \bmod 3$ *then* $m + 2i - k = \begin{cases} \frac{m}{2} - 2i + 2 \text{ \textit{if} } m \text{ \textit{is even}} \\ \frac{m-1}{2} - 2i + 4 \text{ \textit{if} } m \text{ \textit{is odd}} \end{cases}$

3. *if* $x_0 - x_1 \equiv 2 \bmod 3$ *then* $m + 2i - k = \begin{cases} \frac{m}{2} - 2i + 1 \text{ \textit{if} } m \text{ \textit{is even}} \\ \frac{m-1}{2} - 2i + 3 \text{ \textit{if} } m \text{ \textit{is odd}} \end{cases}$

Proof. Assume first that m is even. Then we have that:

$$\begin{aligned}
m + 2i - k &= \frac{m}{2} - 2i + \{3, 2, 1\} \\
\frac{m}{2} + 4i - k &= \{3, 2, 1\} \\
m + 8i - 2k &= \{6, 4, 2\} \\
m + 2i - k + 6i - k &= \{6, 4, 2\} \\
x_0 - x_2 + 6i - k &= \{6, 4, 2\} \\
x_0 - x_2 &= \{6, 4, 2\} - 6i + k \\
x_0 - x_2 &= \{6, 4, 2\} - 6x_3 + x_1 + 2x_2 + 3x_3 \\
x_0 - x_1 &= \{6, 4, 2\} - 3x_3 + 3x_2 \\
\{6, 7, 5\} \geq x_0 - x_1 &= \{6, 4, 2\} - 3x_3 + 3x_2 \\
\{6, 7, 5\} &\geq \{6, 4, 2\} - 3x_3 + 3x_2 \\
\{0, 3, 3\} &\geq 3x_2 - 3x_3 \\
\{0, 1, 1\} &\geq x_2 - x_3. \qquad (5.5)
\end{aligned}$$

So by the $\lambda \cdot (-1, 2, -1, 0)$ operation given in lemma 5.2, we have that $|L_{i,k}| \leq 1$ if $x_0 - x_1 \equiv 0 \bmod 3$ and $|L_{i,k}| \leq 2$ if $x_0 - x_1 \equiv 1$ or $2 \bmod 3$.

And similarly if we assume that m is odd, then we have

$$\begin{aligned}
m+2i-k &= \frac{m-1}{2} - 2i + \{2,4,3\} \\
\frac{m+1}{2} + 4i - k &= \{2,4,3\} \\
m+1+8i-2k &= \{4,8,6\} \\
m+8i-2k &= \{3,7,5\} \\
m+2i-k+6i-k &= \{3,7,5\} \\
x_0 - x_2 + 6i - k &= \{3,7,5\} \\
x_0 - x_2 &= \{3,7,5\} - 6i + k \\
x_0 - x_2 &= \{3,7,5\} - 6x_3 + x_1 + 2x_2 + 3x_3 \\
x_0 - x_1 &= \{3,7,5\} - 3x_3 + 3x_2 \\
\{6,7,5\} \geq x_0 - x_1 &= \{3,7,5\} - 3x_3 + 3x_2 \\
\{6,7,5\} &\geq \{3,7,5\} - 3x_3 + 3x_2 \\
\{3,0,0\} &\geq 3x_2 - 3x_3 \\
\{1,0,0\} &\geq x_2 - x_3 \qquad (5.6)
\end{aligned}$$

So again by the $\lambda \cdot (-1,2,-1,0)$ operation given in lemma 5.2, we have that $|L_{i,k}| \leq 2$ if $x_0 - x_1 \equiv 0 \bmod 3$ and $|L_{i,k}| \leq 1$ if $x_0 - x_1 \equiv 1$ or $2 \bmod 3$.

If any of the bounds were increased by 3 for any of these cases, then the values we would initially add would be $\{1+3, 2+3, 3+3, 4+3\} = \{4,5,6,7\}$ and then when we double this in line 3 of the argument we get $\{8,10,12,14\}$ which are all bigger than $\{5,6,7\}$ so we would end up with $x_2 - x_3 < 0$ which is a contradiction.

We now argue that the above levels are in fact not empty.

Case 7. *Suppose m is even and $x_0 - x_1 \equiv 0 \bmod 3$. Then the first non-empty level-slice is given by $m + 2i - k = \frac{m}{2} - 2i + 3$. That is $k = \frac{m}{2} + 4i - 3$.*

Then by equation 5.5 we have that $x_2 - x_3 \leq 0$, and since $x_2 \geq x_3$ this gives us that $x_2 = x_3$. We claim that there is only one x which satisfies $m + 2i - k = \frac{m}{2} - 2i + 3$, is $\left(\frac{m}{2} - i + 3, \frac{m}{2} - i - 3, i, i\right)$. We need to check that $\sum x_i = m$, $\sum ix_i = k$, and that $x_0 \geq x_1 \geq x_2 \geq x_3$.

$$\sum x_i = \frac{m}{2} - i + 3 + \frac{m}{2} - i - 3 + i + i = m$$
$$\sum ix_i = \frac{m}{2} - i - 3 + 2i + 3i = \frac{m}{2} + 4i - 3 = k$$

It is clear that $x_0 \geq x_1$ and $x_2 \geq x_3$ so we need only check that $x_1 \geq x_2$. We assumed that $\frac{m}{2} - 2i + 3 = m + 2i - k \geq 6$ so we have that

$$\begin{aligned} \frac{m}{2} - 2i + 3 &\geq 6 \\ \frac{m}{2} - 2i - 3 &\geq 0 \\ \frac{m}{2} - i - 3 &\geq i \\ x_1 &\geq x_2 \end{aligned}$$

We note that applying any multiple of $(-1, +2, -1, 0)$ to this x would either cause $x_0 - x_1 \geq 9$ or $x_2 < x_3$ which neither are possible in the level-slice after removal of $(10, 2, 2, 2)$. So this is the only x in this level-slice.

Case 8. *Suppose m is even and $x_0 - x_1 \equiv 1 \bmod 3$. Then the first non-empty level-slice is given by $m + 2i - k = \frac{m}{2} - 2i + 2$. That is $k = \frac{m}{2} + 4i - 2$.*

Then by equation 5.5 we have that $x_2 - x_3 \leq 1$, and since $x_2 \geq x_3$ this gives us that $x_2 = x_3$ or $x_2 = x_3 + 1$. We claim that there are only two elements x and y which satisfy $m + 2i - k = \frac{m}{2} - 2i + 3$. They are $x = \left(\frac{m}{2} - i + 2, \frac{m}{2} - i - 2, i, i\right)$ and $y = \left(\frac{m}{2} - i + 3, \frac{m}{2} - i - 4, i + 1, i\right)$. We now again need to check that $\sum x_i = m$, $\sum ix_i = k$, and that $x_0 \geq x_1 \geq x_2 \geq x_3$, and similarly for y.

$$\sum x_i = \frac{m}{2} - i + 2 + \frac{m}{2} - i - 2 + i + i = m$$
$$\sum ix_i = \frac{m}{2} - i - 2 + 2i + 3i = \frac{m}{2} + 4i - 2 = k$$

and

$$
\begin{aligned}
\sum y_i &= \frac{m}{2} - i + 3 + \frac{m}{2} - i - 4 + i + 1 + i = m \\
\sum iy_i &= \frac{m}{2} - i - 4 + 2(i+1) + 3i = \frac{m}{2} + 4i - 2 = k
\end{aligned}
$$

It is clear that $x_0 \geq x_1$ and $x_2 \geq x_3$ so we need only check that $x_1 \geq x_2$. We assumed that $\frac{m}{2} - 2i + 2 = m + 2i - k \geq 7$ so we have that

$$
\begin{gathered}
\frac{m}{2} - 2i + 2 \geq 7 \\
\frac{m}{2} - 2i - 5 \geq 0 \\
\frac{m}{2} - 2i - 2 \geq 0 \\
\frac{m}{2} - i - 2 \geq i \\
x_1 \geq x_2
\end{gathered}
$$

It is clear that $y_0 \geq y_1$ and $y_2 \geq y_3$ so we need only check that $y_1 \geq y_2$. We assumed that $\frac{m}{2} - 2i + 2 = m + 2i - k \geq 7$ so we have that

$$
\begin{gathered}
\frac{m}{2} - 2i + 2 \geq 7 \\
\frac{m}{2} - 2i - 5 \geq 0 \\
\frac{m}{2} - i - 4 \geq i + 1 \\
y_1 \geq y_2
\end{gathered}
$$

We note that applying any multiple of $(-1, +2, -1, 0)$ to obtain a third element, z, would either the cause $z_0 - z_1 \geq 10$ or $z_2 < z_3$ which neither are possible in the level-slice after removal of $(10, 2, 2, 2)$. So these are the only elements in this level-slice.

Case 9. *Suppose m is even and $x_0 - x_1 \equiv 2 \bmod 3$. Then the first non-empty level-slice is given by $m + 2i - k = \frac{m}{2} - 2i + 1$. That is $k = \frac{m}{2} + 4i - 1$.*

Then by equation 5.5 we have that $x_2 - x_3 \leq 1$, and since $x_2 \geq x_3$ this gives us that $x_2 = x_3$ or $x_2 = x_3 + 1$. We claim that there are only two elements x and y which satisfy $m + 2i - k = \frac{m}{2} - 2i + 1$. They are $x = \left(\frac{m}{2} - i + 1, \frac{m}{2} - i - 1, i, i\right)$ and $y = \left(\frac{m}{2} - i + 2, \frac{m}{2} - i - 3, i + 1, i\right)$. We now again need to check that $\sum x_i = m$, $\sum i x_i = k$, and that $x_0 \geq x_1 \geq x_2 \geq x_3$, and similarly for y.

$$\begin{aligned}
\sum x_i &= \frac{m}{2} - i + 1 + \frac{m}{2} - i - 1 + i + i = m \\
\sum i x_i &= \frac{m}{2} - i - 1 + 2i + 3i = \frac{m}{2} + 4i - 1 = k
\end{aligned}$$

and

$$\begin{aligned}
\sum y_i &= \frac{m}{2} - i + 2 + \frac{m}{2} - i - 3 + i + 1 + i = m \\
\sum i y_i &= \frac{m}{2} - i - 3 + 2(i + 1) + 3i = \frac{m}{2} + 4i - 1 = k
\end{aligned}$$

It is clear that $x_0 \geq x_1$ and $x_2 \geq x_3$ so we need only check that $x_1 \geq x_2$. We assumed that $\frac{m}{2} - 2i + 1 = m + 2i - k \geq 5$ so we have that

$$\begin{aligned}
\frac{m}{2} - 2i + 1 &\geq 5 \\
\frac{m}{2} - 2i - 4 &\geq 0 \\
\frac{m}{2} - 2i - 1 &\geq 0 \\
\frac{m}{2} - i - 1 &\geq i \\
x_1 &\geq x_2
\end{aligned}$$

It is clear that $y_0 \geq y_1$ and $y_2 \geq y_3$ so we need only check that $y_1 \geq y_2$. We assumed that $\frac{m}{2} - 2i + 1 = m + 2i - k \geq 5$ so we have that

$$\begin{aligned}
\frac{m}{2} - 2i + 1 &\geq 5 \\
\frac{m}{2} - 2i - 4 &\geq 0 \\
\frac{m}{2} - i - 3 &\geq i + 1 \\
y_1 &\geq y_2
\end{aligned}$$

We note that applying any multiple of $(-1, +2, -1, 0)$ to obtain a third element, z, would either the cause $z_0 - z_1 \geq 8$ or $z_2 < z_3$ which neither are possible in the level-slice after removal of $(10, 2, 2, 2)$. So these are the only elements in this level-slice.

Case 10. *Suppose m is odd and $x_0 - x_1 \equiv 0 \bmod 3$. Then the first non-empty level-slice is given by $m + 2i - k = \frac{m-1}{2} - 2i + 2$. That is $k = \frac{m+1}{2} + 4i - 2$.*

Then by equation 5.6 we have that $x_2 - x_3 \leq 1$, and since $x_2 \geq x_3$ this gives us that $x_2 = x_3$ or $x_2 = x_3 + 1$. We claim that there are only two elements x and y which satisfy $m + 2i - k = \frac{m-1}{2} - 2i + 2$. They are $x = \left(\frac{m+1}{2} - i + 1, \frac{m+1}{2} - i - 2, i, i\right)$ and $y = \left(\frac{m+1}{2} - i + 2, \frac{m+1}{2} - i - 4, i + 1, i\right)$. We now again need to check that $\sum x_i = m$, $\sum ix_i = k$, and that $x_0 \geq x_1 \geq x_2 \geq x_3$, and similarly for y.

$$\begin{aligned}
\sum x_i &= \frac{m+1}{2} - i + 1 + \frac{m+1}{2} - i - 2 + i + i = m \\
\sum ix_i &= \frac{m+1}{2} - i - 2 + 2i + 3i = \frac{m+1}{2} + 4i - 2 = k
\end{aligned}$$

and

$$\begin{aligned}
\sum y_i &= \frac{m+1}{2} - i + 2 + \frac{m+1}{2} - i - 4 + i + 1 + i = m \\
\sum iy_i &= \frac{m+1}{2} - i - 4 + 2(i+1) + 3i = \frac{m+1}{2} + 4i - 2 = k
\end{aligned}$$

It is clear that $x_0 \geq x_1$ and $x_2 \geq x_3$ so we need only check that $x_1 \geq x_2$. We assumed that $\frac{m-1}{2} - 2i + 2 = m + 2i - k \geq 6$ so we have that

$$\begin{aligned}
\frac{m-1}{2} - 2i + 2 &\geq 6 \\
\frac{m-1}{2} - 2i - 4 &\geq 0 \\
\frac{m+1}{2} - 2i - 2 &\geq 0 \\
\frac{m+1}{2} - i - 2 &\geq i \\
x_1 &\geq x_2
\end{aligned}$$

It is clear that $y_0 \geq y_1$ and $y_2 \geq y_3$ so we need only check that $y_1 \geq y_2$. We assumed that $\frac{m-1}{2} - 2i + 2 = m + 2i - k \geq 6$ so we have that

$$\begin{aligned}
\frac{m-1}{2} - 2i + 2 &\geq 6 \\
\frac{m-1}{2} - 2i - 4 &\geq 0 \\
\frac{m+1}{2} - 2i - 5 &\geq 0 \\
\frac{m+1}{2} - i - 4 &\geq i + 1 \\
y_1 &\geq y_2
\end{aligned}$$

We note that applying any multiple of $(-1, +2, -1, 0)$ to obtain a third element, z, would either the cause $z_0 - z_1 \geq 8$ or $z_2 < z_3$ which neither are possible in the level-slice after removal of $(10, 2, 2, 2)$. So these are the only elements in this level-slice.

Case 11. *Suppose m is odd and $x_0 - x_1 \equiv 1 \bmod 3$. Then the first non-empty level-slice is given by $m + 2i - k = \frac{m-1}{2} - 2i + 4$. That is $k = \frac{m+1}{2} + 4i - 4$.*

Then by equation 5.6 we have that $x_2 - x_3 \leq 0$, and since $x_2 \geq x_3$ this gives us that $x_2 = x_3$. We claim that there is only one element x which satisfies $m + 2i - k = \frac{m-1}{2} - 2i + 4$. That is $x = \left(\frac{m+1}{2} - i + 3, \frac{m+1}{2} - i - 4, i, i\right)$. We now again need to check that $\sum x_i = m$,

$\sum ix_i = k$, and that $x_0 \geq x_1 \geq x_2 \geq x_3$.

$$\sum x_i = \frac{m+1}{2} - i + 3 + \frac{m+1}{2} - i - 4 + i + i = m$$
$$\sum ix_i = \frac{m+1}{2} - i - 4 + 2i + 3i = \frac{m+1}{2} + 4i - 4 = k$$

It is clear that $x_0 \geq x_1$ and $x_2 \geq x_3$ so we need only check that $x_1 \geq x_2$. We assumed that $\frac{m-1}{2} - 2i + 4 = m + 2i - k \geq 7$ so we have that

$$\begin{aligned} \frac{m-1}{2} - 2i + 4 &\geq 7 \\ \frac{m-1}{2} - 2i - 3 &\geq 0 \\ \frac{m+1}{2} - 2i - 4 &\geq 0 \\ \frac{m+1}{2} - i - 4 &\geq i \\ x_1 &\geq x_2 \end{aligned}$$

We note that applying any multiple of $(-1, +2, -1, 0)$ to obtain another element, z, would either the cause $z_0 - z_1 \geq 10$ or $z_2 < z_3$ which neither are possible in the level-slice after removal of $(10, 2, 2, 2)$. So this is the only element in this level-slice.

Case 12. *Suppose* m *is odd and* $x_0 - x_1 \equiv 2 \bmod 3$. *Then the first non-empty level-slice is given by* $m + 2i - k = \frac{m-1}{2} - 2i + 3$. *That is* $k = \frac{m+1}{2} + 4i - 3$.

Then by equation 5.6 we have that $x_2 - x_3 \leq 0$, and since $x_2 \geq x_3$ this gives us that $x_2 = x_3$. We claim that there is only one element x which satisfies $m + 2i - k = \frac{m-1}{2} - 2i + 3$. That is $x = \left(\frac{m+1}{2} - i + 2, \frac{m+1}{2} - i - 3, i, i\right)$. We now again need to check that $\sum x_i = m$, $\sum ix_i = k$, and that $x_0 \geq x_1 \geq x_2 \geq x_3$.

$$\sum x_i = \frac{m+1}{2} - i + 2 + \frac{m+1}{2} - i - 3 + i + i = m$$
$$\sum ix_i = \frac{m+1}{2} - i - 3 + 2i + 3i = \frac{m+1}{2} + 4i - 3 = k$$

It is clear that $x_0 \geq x_1$ and $x_2 \geq x_3$ so we need only check that $x_1 \geq x_2$. We assumed that $\frac{m-1}{2} - 2i + 3 = m + 2i - k \geq 5$ so we have that

$$\begin{aligned}
\frac{m-1}{2} - 2i + 3 &\geq 5 \\
\frac{m-1}{2} - 2i - 2 &\geq 0 \\
\frac{m+1}{2} - 2i - 3 &\geq 0 \\
\frac{m+1}{2} - i - 3 &\geq i \\
x_1 &\geq x_2
\end{aligned}$$

We note that applying any multiple of $(-1, +2, -1, 0)$ to obtain another element, z, would either the cause $z_0 - z_1 \geq 8$ or $z_2 < z_3$ which neither are possible in the level-slice after removal of $(10, 2, 2, 2)$. So this is the only element in this level-slice. $\square$

Putting together the results from Lemmas 5.6 and 5.7 we see what the top two level-slices (with $x_0 - x_1$ congruent in each level-slice) look like, as shown in Figures 5.2 and 5.3.

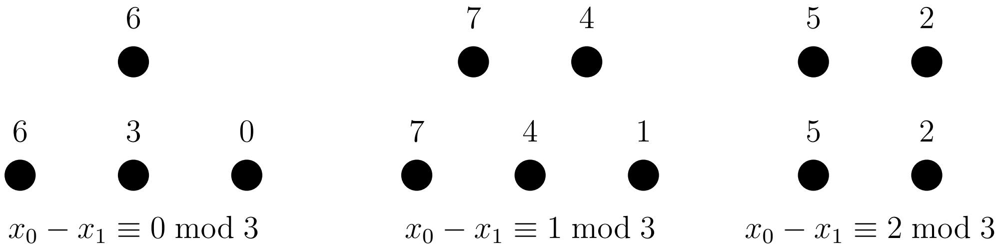


Figure 5.2: Topmost Non-Empty Level-Slice Pairs when m is Even

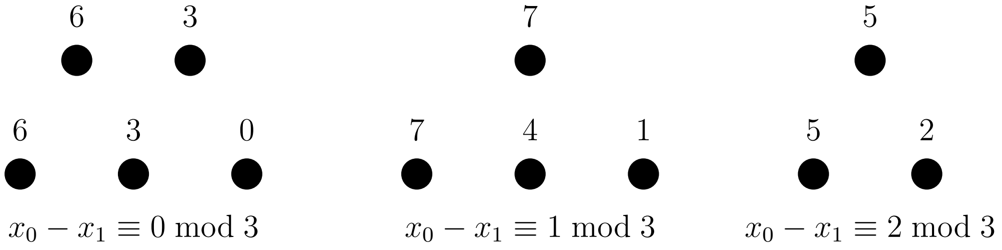


Figure 5.3: Topmost Non-Empty Level-Slice Pairs when m is Odd

Now that we know that the top-most non-empty level-slice pairs look like. We shift our focus to what is to the left of these structures. That is if these structures are in slice i what does the structure look like in slice $i+1$ while maintaining the level.

Lemma 5.8. *Let $Y_{i,k}$ and $X_{i-1,k-3}$ be level-slices with $i \geq 1$. For every $y = (y_0, y_1, y_2, y_3) \in Y_{i,k}$ there exists an $x = (x_0, x_1, x_2, x_3) \in X_{i-1,k-3}$ with $x \geq_M y$. Moreover if $y_0 - y_1 \leq 6$ or $y_3 \in \{1, 2\}$ then we define $x = f(y) = (y_0 + 1, y_1, y_2, y_3 - 1)$ or if $7 \geq y_0 - y_1 \geq 2$ then $x = g(y) = (y_0, y_1 + 2, y_2 - 1, y_3 - 1)$.*

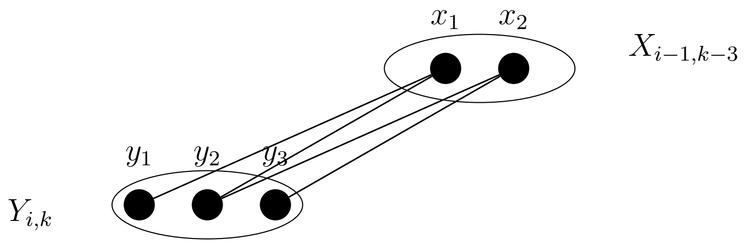


Figure 5.4: Edges from $Y_{i,k}$ to $X_{i-1,k-3}$

We warn that the converse of Lemma 5.8 is not true in general. For example $(m, 0, 0, 0)$ is in level-slice $L_{0,0}$. For slice 1 the topmost element (provided $m \geq 4$) is $(m-3, 1, 1, 1)$ which is in level 6 so the level-slice $L_{1,3} = \varnothing$. Note that Lemma 5.10 below will provide a "close enough" converse that will suffice for our purposes.

Proof. Let $y \in Y_{i,k}$. We claim that if $y_0 - y_1 \leq 6$ or $y_3 \in \{1, 2\}$, then $x = f(y) = (y_0 + 1, y_1, y_2, y_3 - 1) \in X_{i-1,k-3}$. Certainly $x_3 = y_3 - 1 = i - 1$, and if $y_3 \in \{1, 2\}$ then $x_3 \in \{0, 1\}$ so x is not removed. And $x_0 - x_1 = y_0 + 1 - y_1 = y_0 - y_1 + 1 \leq 7$, so again x is not removed. And clearly $x \geq_M y$. Thus we need to show that if y is in level k, then x is in level $k - 3$.

$$\sum ix_i = y_1 + 2y_2 + 3(y_3 - 1) = y_1 + 2y_2 + 3y_3 - 3 = k - 3.$$

Now assume that $y_0 - y_1 \geq 2$, then $x = g(y) = (y_0, y_1 + 2, y_2 - 1, y_3 - 1) \in X_{i-1}$. Again certainly $x_3 = y_3 - 1 = i - 1$, so x is in slice $i - 1$. Since $y_0 - y_1 \geq 2$, then

$x_0 - x_1 = y_0 - (y_1 + 2) \geq y_0 - y_1 - 2 \geq 0$ so x is indeed decreasing. And since y is not removed then x is also not removed since its first difference is smaller than that of y. So again x is not removed. And again it is clear that $x \geq_M y$. So it remains to show that if y is in level k, then x is in level $k-3$.

$$\sum ix_i = y_1 + 2 + 2(y_2 - 1) + 3(y_3 - 1) = y_1 + 2y_2 + 3y_3 - 3 = k - 3.$$

Thus if there exists a $y \in Y_{i,k}$ then there exists an $x \in X_{i-1,k-3}$ with $x \geq_M y$ as desired.

□

Lemma 5.9. *Let $Y_{i,k}$ and $X_{i-1,k-3}$ be level slices with $i \geq 1$. For any two $y, y' \in Y_{i,k}$ there exist distinct $x, x' \in X_{i-1,k-3}$ with $x \geq_M y$ and $x' \geq_M y'$.*

Again, we warn that the converse of Lemma 5.9 is not true in general. For example in $P'(4,6)$, elements $(4,1,1,0)$ and $(3,3,0,0)$ are in level-slice $L_{0,3}$ but there is only one element in level-slice $L_{1,6}$, namely $(3,1,1,1)$. So again Lemma 5.10 below will provide a "close enough" converse that will suffice for our purposes.

Proof. If $y_0 - y_1 \leq 6$ and $y'_0 - y'_1 \leq 6$ or if $y_3 = y'_3 \in \{1,2\}$, then $x = f(y) \neq g(y') = x'$, or if $y_0 - y_1 \geq 2$ and $y'_0 - y'_1 \geq 2$, then $x = g(y) \neq g(y') = x'$ by lemma 5.8.

The only remaining possibility for distinct y and y' is if $y_3 = y'_3 \geq 3$ and $7 \geq y_0 - y_1 > 6$ and $y'_0 - y'_1 \leq 1$. Since we have assumed that y, y' are within the same level-slice, then by lemma 5.2 since $y_0 - y_1 = 7$ and $y'_0 - y'_1 \leq 1$ then $y'_0 - y'_1 = 1$. Thus it remains to show that without loss of generality, if $y_0 - y_1 = 7$, $y'_0 - y'_1 = 1$ and $y_3 = y'_3 \geq 3$, then $g(y) = x = (y_0, y_1 + 2, y_2 - 1, y_3 - 1)$, and $x' = f(y') = (y'_0 + 1, y'_1, y'_2, y'_3 - 1)$ are in fact distinct. By lemma 5.8 It is clear that x, x' are both in slice $i-1$ and level $k-3$. Moreover,

$$x_0 - x_1 = y_0 - (y_1 + 2) = 7 - 2 = 5$$

$$x'_0 - x'_1 = y'_0 + 1 - y'_1 = 1 + 1 = 2$$

So $x \neq x'$ as desired. □

Again, in general the converses of lemmas 5.8 and 5.9 are not true, but we can say something about a hybrid converse of these lemmas, that is if there are distinct $x, x' \in X_{i-1,k-3}$ then there exists a $y \in Y_{i,k}$ with $x \geq_M y$.

Lemma 5.10. *Let $Y_{i,k}$ and $X_{i-1,k-3}$ be level slices with $i \geq 1$. For distinct $x, x' \in X_{i-1,k-3}$ with $x_0 - x_1 = x'_0 - x'_1 + 3$, then there exists a $y \in Y_{i,k}$ such that $f^{-1}(x) = y = g^{-1}(x')$. Moreover if $x_3 = x'_3 = 0$ or if $3 \leq x_0 - x_1 \leq 8$ then y is not removed.*

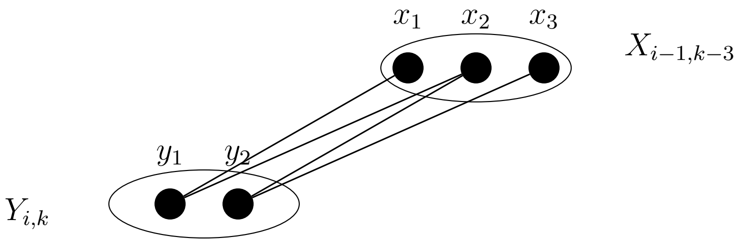


Figure 5.5: Edges from $X_{i-1,k-3}$ to $Y_{i,k}$

Proof. Let $x, x' \in X_{i-1,k-3}$ with $x'_0 - x'_1 + 3 = x_0 - x_1 \geq 0$. Since $x'_0 - x'_1 + 3 = x_0 - x_1 \geq 0$ then by lemma 5.2 we have that $x' = (x_0 - 1, x_1 + 2, x_2 - 1, x_3)$, thus $x_2 - 1 \geq x_3 \geq 0$. Now consider $y = (x_0 - 1, x_1, x_2, x_3 + 1)$. Since $x_0 - x_1 \geq 3$ and $x_2 \geq x_3 + 1$ then $y \in P'(n, m)$. Moreover if $x_3 = 0$ then $y_3 = 1$ and so $y \in R$ or if $3 \leq x_0 - x_1 \leq 8$ then $2 \leq y_0 - y_1 \leq 7$ and so again we have that $y \in R$. By definition of y we have that $y = f^{-1}(x)$ and by definition of x' we see that $y = g^{-1}(x')$. □

Observation 5.11. *Putting lemmas 5.8, 5.9, and 5.10 together with the structures from figures 5.2 and 5.3 we have that the very top-most non-empty level-slices are also the left-most non-empty level-slices within the given level k.*

Proof. If the top-most non-empty level-slices $L_{i,k}$ were not left-most within their level, k, this implies there is something in level-slice $L_{i+1,k}$ and by lemma 5.8 this would mean there is something in level-slice $L_{i,k-3}$ which cannot happen since $L_{i,k}$ is the top-most non-empty level-slice. □

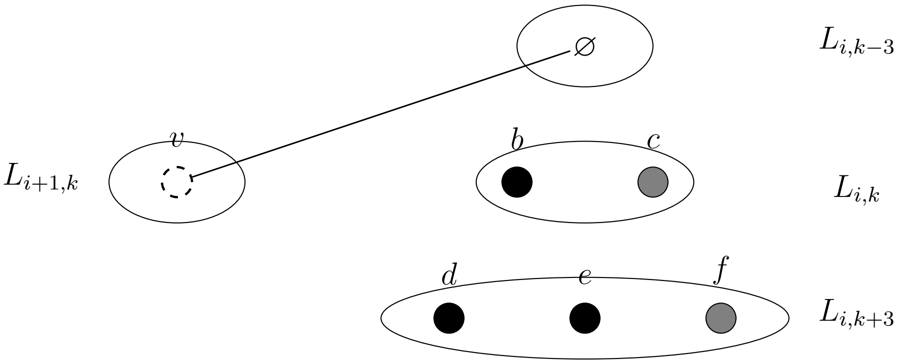


Figure 5.6: The top-most, non-empty, level-slices are also left-most

Similarly putting together lemma 5.10 with the structures from figures 5.2 and 5.3 we have that the first maximal level-slices which have exactly two elements in their top-most non-empty level-slices are not left most.

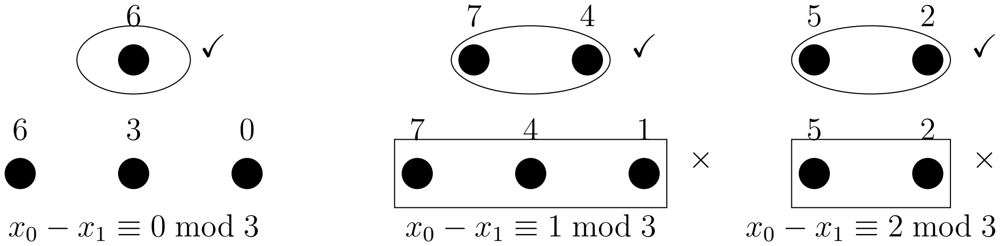


Figure 5.7: Left-most so far when m is Even

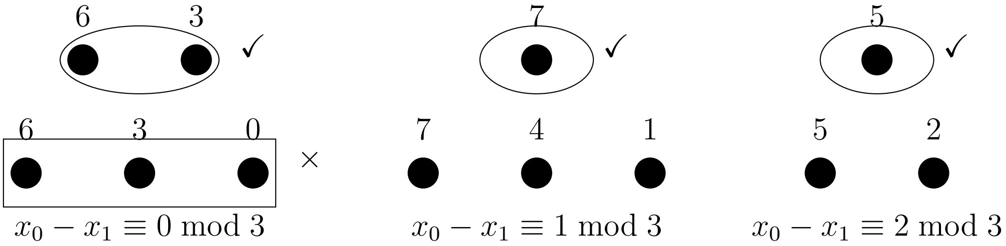


Figure 5.8: Left-most so far when m is Odd

This leaves three level-slices still to be determined whether they are leftmost or not namely, when m is even and $x_0 - x_1 \equiv 0 \bmod 3$ or the two odd cases where $x_0 - x_1 \equiv 1$ or $2 \bmod 3$.

Lemma 5.12. *For $i \geq 2$, let a be the only element in $L_{i,k}$ and let b and c be the only elements in $L_{i,k+3}$, then there exists a $d \in L_{i+1,k+3}$ with $d \leq_M a$.*

Proof. Suppose we are looking at the 1-2 pair. And consider the following structure.

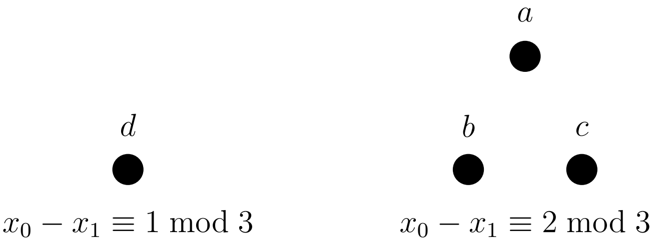


Figure 5.9: The 2 of 1-2 is not leftmost

From lemma 5.7 we have that $a_0 - a_1 = 5$ and $a_2 = a_3 = i$. We also have that $b = (a_0 - 1, a_1 - 1, a_2 + 2, a_3)$ and $c = (a_0 - 2, a_1 + 1, a_2 + 1, a_3)$. We now claim that $d = (b_0 + 1, b_1 - 1, b_2 - 1, b_3 + 1)$ is in $L_{i+1,k+3}$ with $d_0 - d_1 \leq 7$, and $d \leq_M a$. We have that $d_3 = b_3 + 1 = a_3 + 1 = i + 1$ and that $d_1 + 2d_2 + 3d_3 = b_1 - 1 + 2(b_2 - 1) + 3(b_3 + 1) = b_1 + 2b_2 + 3b_3 - 1 - 2 + 3 = k + 3$ since $b \in L_{i,k+3}$. And we have that $d_0 - d_1 = b_0 + 1 - (b_1 - 1) = a_0 - 1 + 1 - (a_1 - 1 - 1) = a_0 - a_1 + 2 = 7$. Thus it remains to show that $d \leq_M a$.

$$d_0 = b_0 + 1 = a_0 - 1 + 1 = a_0 \leq a_0$$
$$d_0 + d_1 = a_0 + b_1 - 1 = a_0 + a_1 - 2 \leq a_0 + a_1$$
$$d_0 + d_1 + d_2 = a_0 + a_1 - 2 + b_2 - 1 = a_0 + a_1 + a_2 - 1 \leq a_0 + a_1 + a_2$$
$$d_0 + d_1 + d_2 + d_3 = a_0 + a_1 + a_2 - 1 + b_3 + 1 = a_0 + a_1 + a_2 + a_3 \leq a_0 + a_1 + a_2 + a_3$$

Thus $d \leq_M a$ and $d \in L_{i+1,k+3}$ and is not removed. Thus b and c are not left-most in their level. □

Lemma 5.13. *For $i \geq 2$, let a be the only element in $L_{i,k}$ and $b, c, d \in L_{i,k+3}$ then there does not exist an $f \in L_{i+1,k+3}$ with $f \leq_M a$.*

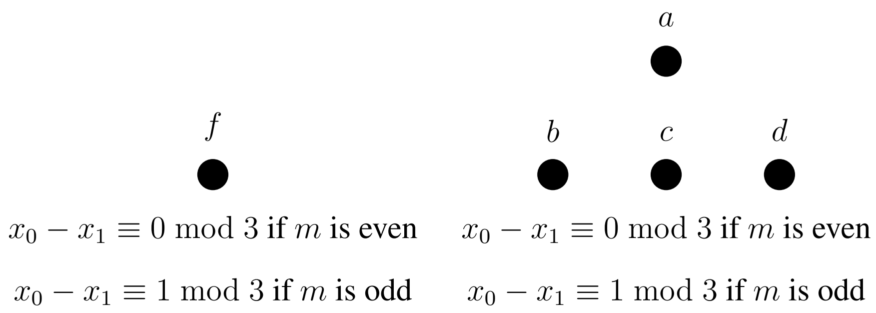


Figure 5.10: The 3 of the 1-3 is leftmost

Proof. From lemma 5.7 we know that $a_0 - a_1 \in \{6, 7\}$, and that $a_2 = a_3 = i$ and by lemma 5.8 if there exists an f in $L_{i+1,k+1}$ then it must be of the form $d = a_0, a_1 - 2, a_2 + 1, a_3 + 1$, but this would imply that $d_0 - d_1 = a_0 - (a_1 - 2) \in \{8, 9\}$. Which implies that if d exists it is removed. Thus b, c, d are left-most in their level. □

So from the structures in figures 5.7 and 5.8 as well as lemmas 5.12 and 5.10 we can see what is to the left of the structures from figures 5.7 and 5.8 as shown in figures 5.11 and 5.12.

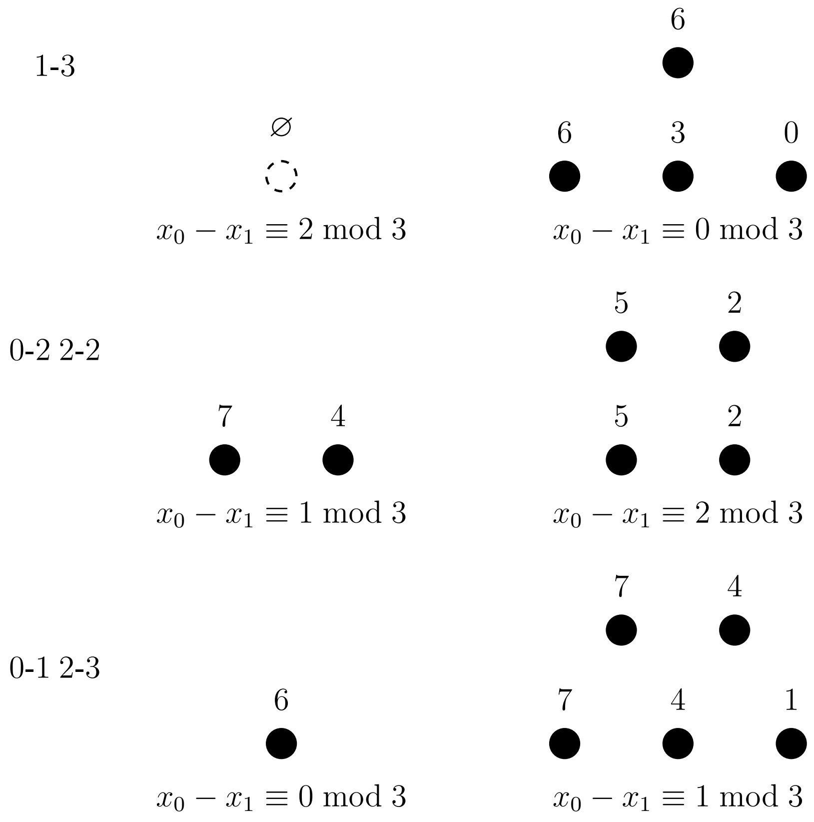


Figure 5.11: Left-most Elements in a Level-Slice Pair when m is Even

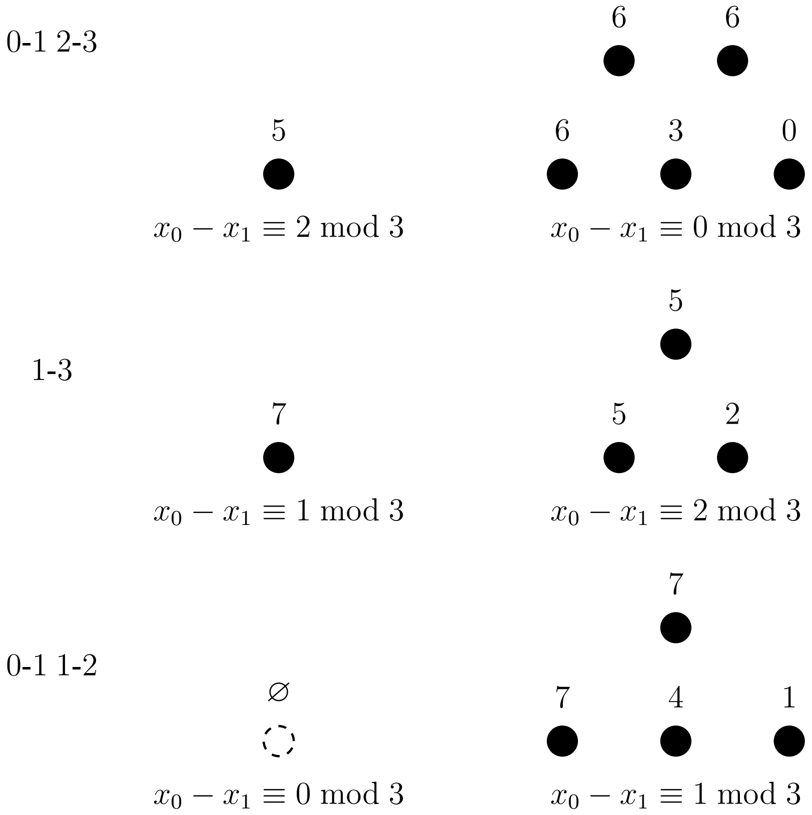


Figure 5.12: Left-most Elements in a Level-Slice Pair when m is Odd

We notice that the bottom of the level-slice pairs look like the top of another level-slice pair. So when we start to stack these level-slice pairs together, like they would be in the entire poset, we see that the structure has a staggered effect. However because the 1-3 structure is entirely left-most then when end up with a “missing structure” that will be between the 1-3 and 0-2 2-2 structure for the even case and between the 1-3 and 2-3 structure in the odd case. Namely we get a 0-2 structure with nothing at all in the top slice and only two things in the bottom slice. We can see this staggered effect in its entirety in figures 5.13 and 5.14.

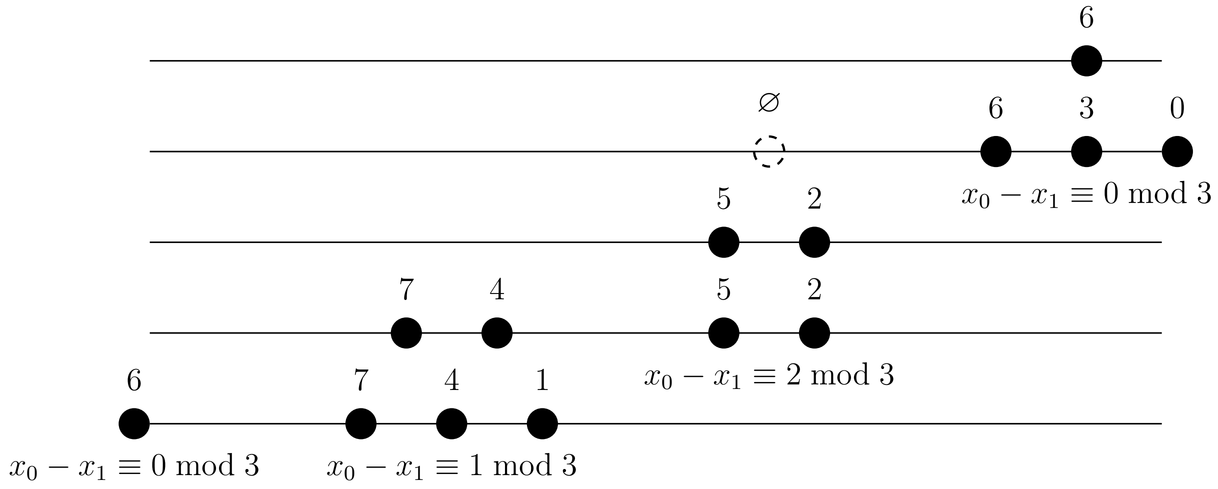


Figure 5.13: Left-most Level-Slice Pairs when m is Even

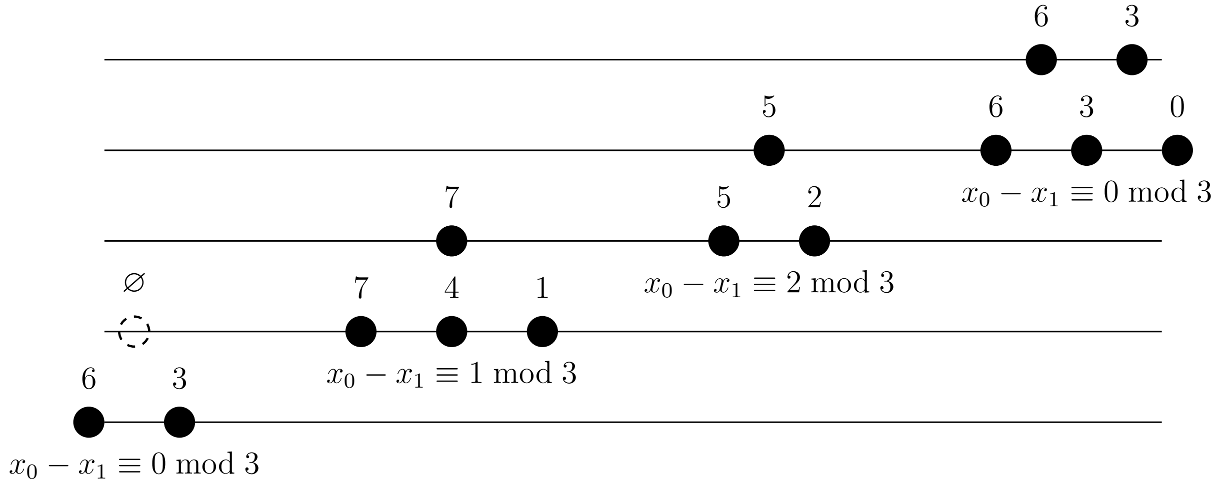


Figure 5.14: Left-most Level-Slice Pairs when m is Odd

Now that we know that the left sides of the level pairs increase from level k to $k+3$, for $i \geq 2$, we now look to the right side of levels k and $k+3$; that is we look at these levels in slices 0 and 1. Because if the induced levels k and $k+3$ when restricted to R have the same size, then somewhere to the right we have to have a decrease of the same size.

Lemma 5.14. *In* $P'(3,m)$, $|L_k| - |L_{k+3}| \leq 1$.

Proof. Suppose there exists $a, b \in L_k$ with $b = a + (-1, 2, -1, 0)$. We claim that there exists a $c \in L_{k+3}$ such that $c \leq_M a$ and $c \leq_M b$ with $c = (a_0 - 2, a_1 + 1, a_2 + 1$. By lemma 5.3 we know that $a_0 - a_1 \geq 3$ and thus c is indeed decreasing. By definition of c we know that $c \leq_M a$, so it remains to show that $c \leq_M b$ and $c \in L_{k+3}$. By Assumption $b = a + (-1, 2, -1, 0) = (a_0 - 1, a_1 + 2, a_2 - 1)$ so again we see that $b \geq_M c$. From the definition of c we see that

$$(a_1 + 1) + 2(a_2 + 1) = a_1 + a_2 + 3 = k + 3$$

thus c is indeed in L_{k+3}. Thus $|L_k| \leq |L_{k+3}| + 1$.

Suppose now that there exists $c, d \in L_{k+3}$ with $d = c + (-1, 2, -1, 0)$. We now claim that there exists an $a \in L_k$ where $a = c_0 + 2, c_1 - 1, c_2 - 1$ and with $a \geq_M c$ and $a \geq_M d$. Since c is decreasing then by definition of a we know that a is also decreasing and since by assumption we know that d exists and is in L_{k+3} then by definition of d we get $d = c + (-1, 2, -1, 0) = (c_0 - 1, c_1 + 2, c_2 - 1)$. Which implies that $c_2 - 1 \geq 0$ and from the existence of c we know that $c_1 \geq c_2$ so $a = (c_0 + 2, c_1 - 1, c_2 - 1)$ has $a_0 \geq a_1 \geq a_2 \geq 0$. So a exists and is decreasing. By the definitions of $a, c,$ and d we have that $a \geq_M c$ and $a \geq_M d$. Thus it remains to show that a is in L_k.

$$(c_1 - 1) + 2(c_2 - 1) = c_1 + 2c_2 - 3 = k + 3 - 3 = k$$

Thus $a \in L_k$, and so $|L_{k+3}| \leq |L_k| + 1$. □

Now that we know that in each of the slices of $P'(4, m)$, $\big||L_k| - |L_{k+3}|\big| \leq 1$ we will try to align the largest level-slice from slices 0 and 1, or as best as we can. From the proof of Theorem 3.9, we know that width of slice 0 is $\lfloor \frac{m}{6} \rfloor + 1$ and moreover we know that when we induct we peel off a single chain so the width will go down by exactly one each time. This means that the maximal levels in each of the slices in $P'(4, m)$ will occur in the same levels as where their respective base cases are in the poset, just shifted by $\lfloor \frac{m}{6} \rfloor * 3$. So depending on what m is modulo 6 we have the following levels in $P'(3, m)$ and thus in slice 0 in P'(4,m) are maximal:

$$\begin{aligned}
m &\equiv 0 \bmod 6 \ \text{ level } \frac{m}{2} \text{ is maximal.}\\
m &\equiv 1 \bmod 6 \ \text{ level } \left\lfloor \frac{m}{2} \right\rfloor \text{ is maximal.}\\
m &\equiv 2 \bmod 6 \ \text{ levels } \frac{m}{2}-1, \frac{m}{2} \text{ are maximal.}\\
m &\equiv 3 \bmod 6 \ \text{ levels } \left\lfloor \frac{m}{2} \right\rfloor -1, \left\lfloor \frac{m}{2} \right\rfloor, \left\lfloor \frac{m}{2} \right\rfloor +2 \text{ are maximal.}\\
m &\equiv 4 \bmod 6 \ \text{ levels } \frac{m}{2}-2, \frac{m}{2}-1, \frac{m}{2}, \frac{m}{2}+1 \text{ are maximal.}\\
m &\equiv 5 \bmod 6 \ \text{ levels } \left\lfloor \frac{m}{2} \right\rfloor -2, \left\lfloor \frac{m}{2} \right\rfloor -1, \left\lfloor \frac{m}{2} \right\rfloor, \left\lfloor \frac{m}{2} \right\rfloor +1, \left\lfloor \frac{m}{2} \right\rfloor +2 \text{ are maximal.}
\end{aligned}$$

Slice 1 is constructed from $P'(3, m-4)$ and so the maximal levels will occur in different places depending on what $m-4$ is congruent to modulo 6 and using $m-4$ rather than m in the above fractions. We then will need to shift this level by 6 to get the correct level for all of $P'(4, m)$.

So for slice 1 when $m \equiv 0 \bmod 6$ then $m-4 \equiv 2 \bmod 6$ and so levels $\frac{m-4}{2}-1+6 = \frac{m}{2}+3$ and $\frac{m-4}{2}+6 = \frac{m}{2}+4$ are maximal level-slices in slice 1. Now these two maximal level-slices in slice 1 do not exactly align with the maximal level-slices from slice 0, but $\frac{m}{2}$ and $\frac{m}{2}+3$ are three levels apart which means that $L_{1,\frac{m}{2}}$ has exactly one less element than maximal and similarly for $L_{0,\frac{m}{2}+3}$. So level $L_{\frac{m}{2}}$ is the first potential maximal level in R with level $L_{\frac{m}{2}+3}$ having the same size. In slice 2, $(\frac{m}{2}+1, \frac{m}{2}-5, 2, 2)$ is the very top-most element in R, which is in level $\frac{m}{2}-5+2(2)+3(2) = \frac{m}{2}+5$. Thus levels $L_{\frac{m}{2}}$ and $L_{\frac{m}{2}+3}$ contain only elements from slices 0 and 1 in R. Now level $L_{\frac{m}{2}+5}$ is not maximal in slice 1 and is not maximal in slice 0 so this level may have the same size as levels $L_{\frac{m}{2}}$ and $L_{\frac{m}{2}+3}$ but will not be bigger. Now level $\frac{m}{2}+6$ contains elements $(\frac{m}{2}+1, \frac{m}{2}-6, 3, 2)$ and $(\frac{m}{2}, \frac{m}{2}-4, 2, 2)$ from slice 2, is exactly 3 below a maximal level in slice 1, and exactly 6 below a maximal level in slice 0 and so level $L_{\frac{m}{2}+6}$ will have the same size as levels $L_{\frac{m}{2}+3}$ and $L_{\frac{m}{2}}$. Now that we have elements in slice 2 the structures from Figures 5.13 and 5.14 show that we get an increase of at most 2 on the left and a decrease of exactly one in slices 0 and 1, provided that we have not gone past level m. So when $m \equiv 0 \bmod 6$ then the maximal levels in R appear every three apart, starting with level

$\frac{m}{2}$. Because $m \equiv 0 \bmod 6$ then $\frac{m}{2} \equiv 0 \bmod 3$ and so will each of the levels three apart going down.

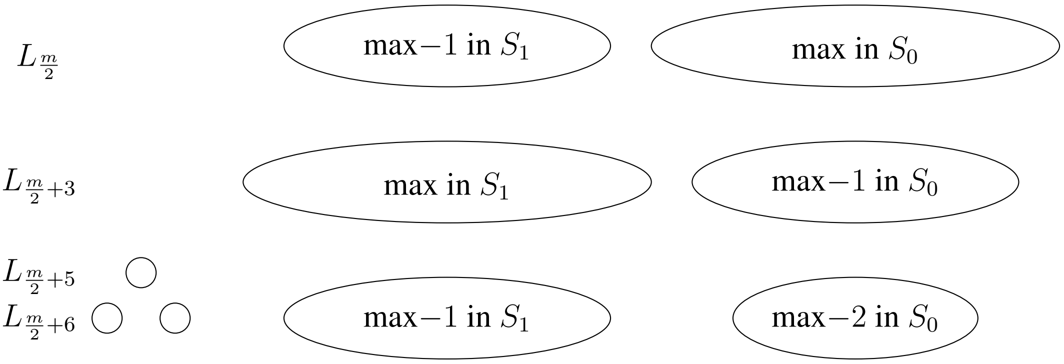


Figure 5.15: First Maximal levels when $m \equiv 0 \bmod 6$

For slice 1 when $m \equiv 1 \bmod 6$ then $m-4 \equiv 3 \bmod 6$ and so levels $\lfloor \frac{m-4}{2} \rfloor - 1 + 6 = \lfloor \frac{m}{2} \rfloor + 3$, $\lfloor \frac{m-4}{2} \rfloor + 6 = \lfloor \frac{m}{2} \rfloor + 4$, and $\lfloor \frac{m-4}{2} \rfloor + 2 + 6 = \lfloor \frac{m}{2} \rfloor + 6$ are maximal level-slices in slice 1. Again these levels slices do not align with the maximal level slices in slice 0, but $\lfloor \frac{m}{2} \rfloor$ and $\lfloor \frac{m}{2} \rfloor + 3$ are three levels apart which means that $L_{1,\lfloor \frac{m}{2} \rfloor}$ has exactly one less element than maximal and similarly for $L_{0,\lfloor \frac{m}{2} \rfloor+3}$. So level $L_{\lfloor \frac{m}{2} \rfloor}$ is the first potential maximal level in R with level $L_{\lfloor \frac{m}{2} \rfloor+3}$ having the same size. Now level $\lfloor \frac{m}{2} \rfloor + 6$ is also maximal in slice 1, but will be two less than maximal in slice 0. In slice 2, $(\lceil \frac{m}{2} \rceil + 1, \lfloor \frac{m}{2} \rfloor - 5, 2, 2)$ is the very top-most element in R, which is in level $\lfloor \frac{m}{2} \rfloor - 5 + 2(2) + 3(2) = \lfloor \frac{m}{2} \rfloor + 5$. Thus levels $L_{\lfloor \frac{m}{2} \rfloor}$ and $L_{\lfloor \frac{m}{2} \rfloor+3}$ contain only elements form slices 0 and 1 in R. Now $L_{\lfloor \frac{m}{2} \rfloor+5}$ is not maximal in slice 0 and is not maximal in slice 1 so this level may have the same size as $L_{\lfloor \frac{m}{2} \rfloor}$ and $L_{\lfloor \frac{m}{2} \rfloor+3}$ but will not be bigger. Now level $L_{\lfloor \frac{m}{2} \rfloor+6}$ contains only one element $(\lceil \frac{m}{2} \rceil, \lfloor \frac{m}{2} \rfloor - 4, 2, 2)$ from slice 2 and this level is exactly three below a maximal level in slice 1 and exactly 6 below a maximal level in slice 0 and so level $L_{\lfloor \frac{m}{2} \rfloor+6}$ will have the same size as levels $L_{\lfloor \frac{m}{2} \rfloor+3}$ and $L_{\lfloor \frac{m}{2} \rfloor}$. Now that we have elements in slice 2 the structures from Figures 5.13 and 5.14 show that we get an increase of at most 2 on the left when we move down by 3 and in slices 0 and 1 we will get a decrease of exactly one in each slice, provided we have not gone past level m. So when $m \equiv 1 \bmod 6$ then $\lfloor \frac{m}{2} \rfloor \equiv 0 \bmod 3$ and so will each of the levels three apart going down.

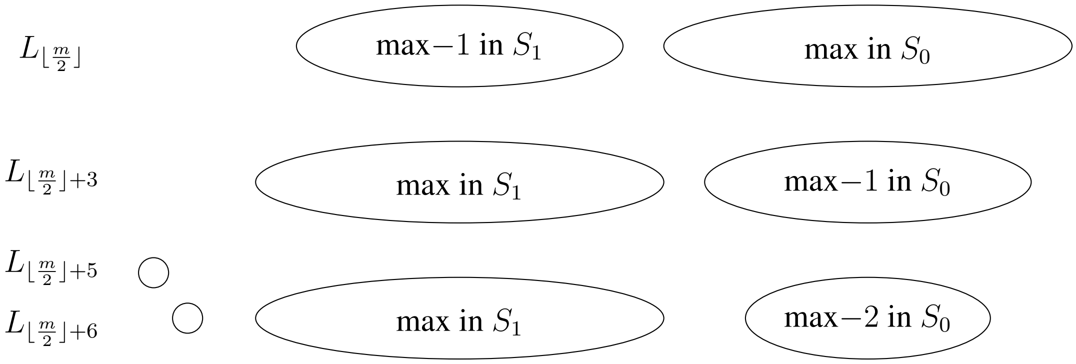


Figure 5.16: First Maximal levels when $m \equiv 1 \bmod 6$

For slice 1 when $m \equiv 2 \bmod 6$ then $m-4 \equiv 4 \bmod 6$ and so levels $\frac{m-4}{2} - 2 + 6 = \frac{m}{2} + 2$, $\frac{m-4}{2} - 1 + 6 = \frac{m}{2} + 3$, $\frac{m-4}{2} + 6 = \frac{m}{2} + 4$, and $\frac{m-4}{2} + 1 + 6 = \frac{m}{2} + 5$ are maximal level-slices in slice 1. Again these levels slices do not align with the maximal level slices in slice 0, but $\frac{m}{2} - 1$ and $\frac{m}{2} + 2$ are three levels apart which means that $L_{1, \frac{m}{2} - 1}$ has exactly one less element than maximal and similarly for $L_{0, \frac{m}{2} + 2}$. So level $L_{\frac{m}{2} - 1}$ is the first potential maximal level in R with level $L_{\frac{m}{2} + 2}$ having the same size. Now level $\frac{m}{2} + 5$ is also maximal in slice 1, but will be two less than maximal in slice 0. In slice 2, $(\frac{m}{2} + 1, \frac{m}{2} - 5, 2, 2)$ is the very top-most element in R, and is in level $\frac{m}{2} - 5 + 2(2) + 3(2) = \frac{m}{2} + 5$. Thus levels $L_{\frac{m}{2} - 1}$ and $L_{\frac{m}{2} + 2}$ contain only elements form slices 0 and 1 in R. Now level $L_{\frac{m}{2}} + 5$ contains only one element $(\frac{m}{2} + 1, \frac{m}{2} - 5, 2, 2)$ and this level is maximal in slice 1 and exactly 3 below a maximal level in slice 0 and so level $L_{\frac{m}{2} + 5}$ will have the same size as levels $L_{\frac{m}{2} + 2}$ and $L_{\frac{m}{2} - 1}$. Now that we have elements in slice 2 the structures from Figures 5.13 and 5.14 show that we get an increase of at most 2 on the left when we move down by 3 and in slices 0 and 1 we will get a decrease of exactly one in each slice, provided we have not gone past level m. So when $m \equiv 2 \bmod 6$ then $\frac{m}{2} - 1 \equiv 0 \bmod 3$ and so will each of the levels three apart going down.

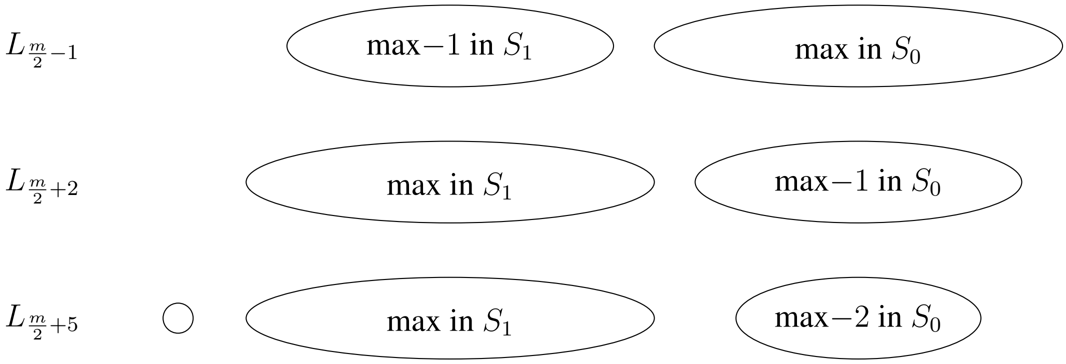


Figure 5.17: First Maximal levels when $m \equiv 2 \bmod 6$

For slice 1 when $m \equiv 3 \bmod 6$ then $m-4 \equiv 5 \bmod 6$ and so levels $\lfloor \frac{m-4}{2} \rfloor - 2 + 6 = \lfloor \frac{m}{2} \rfloor + 2$, $\lfloor \frac{m-4}{2} \rfloor - 1 + 6 = \lfloor \frac{m}{2} \rfloor + 3$, $\lfloor \frac{m-4}{2} \rfloor + 6 = \lfloor \frac{m}{2} \rfloor + 4$, $\lfloor \frac{m-4}{2} \rfloor + 1 + 6 = \lfloor \frac{m}{2} \rfloor + 5$, and $\lfloor \frac{m-4}{2} \rfloor + 2 + 6 = \lfloor \frac{m}{2} \rfloor + 6$ are maximal level-slices in slice 1. In this case we do actually have a level which is maximal in both slice 0 and slice 1, that is level $L_{\lfloor \frac{m}{2} \rfloor + 2}$. So this level is maximal in slices 0 and 1. When we look at level $L_{\lfloor \frac{m}{2} \rfloor + 5}$, it will still be maximal in slice 1 and will be one less than maximal in slice 0. In slice 2, $(\lceil \frac{m}{2} \rceil + 1, \lfloor \frac{m}{2} \rfloor - 5, 2, 2)$ is the very top-most element in R, which is in level $\lfloor \frac{m}{2} \rfloor - 5 + 2(2) + 3(2) = \lfloor \frac{m}{2} \rfloor + 5$; moreover this is the only element in this level-slice. Thus level $L_{\lfloor \frac{m}{2} \rfloor + 2}$ contains only elements form slices 0 and 1 in R. So with the addition of this element we now have that level $L_{\lfloor \frac{m}{2} \rfloor + 5}$ is the same size as level $L_{\lfloor \frac{m}{2} \rfloor + 2}$. Now that we have elements in slice 2 the structures from Figures 5.13 and 5.14 show that we get an increase of at most 2 on the left when we move down by 3 and in slices 0 and 1 we will get a decrease of exactly one in each slice, provided we have not gone past level m. So when $m \equiv 3 \bmod 6$ then $\lfloor \frac{m}{2} \rfloor + 2 \equiv 0 \bmod 3$ and so will each of the levels three apart going down.

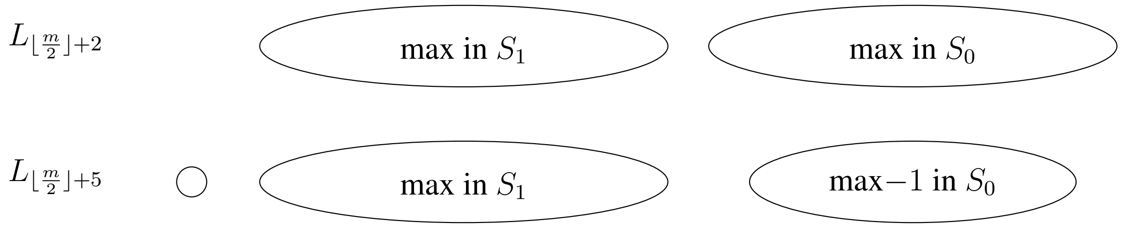


Figure 5.18: First Maximal levels when $m \equiv 3 \bmod 6$

For slice 1 when $m \equiv 4 \bmod 6$ then $m-4 \equiv 0 \bmod 6$ and so level $\frac{m-4}{2}+6 = \frac{m}{2}+4$ is the only maximal level-slice in slice 1. Again this level-slice does not align with the maximal level slices in slice 0, but $\frac{m}{2}+1$ and $\frac{m}{2}+4$ are three levels apart which means that $L_{1,\frac{m}{2}+1}$ has exactly one less element than maximal and similarly for $L_{0,\frac{m}{2}+4}$. So level $L_{\frac{m}{2}+1}$ is the first potential maximal level in R with level $L_{\frac{m}{2}+4}$ having the same size. In slice 2, $(\frac{m}{2}+1, \frac{m}{2}-5, 2, 2)$ is the very top-most element in R, and is in level $\frac{m}{2} - 5 + 2(2) + 3(2) = \frac{m}{2} + 5$. Thus levels $L_{\frac{m}{2}+1}$ and $L_{\frac{m}{2}+4}$ contain only elements form slices 0 and 1 in R. Now level $L_{\frac{m}{2}} + 5$ contains only one element $(\frac{m}{2} + 1, \frac{m}{2} - 5, 2, 2)$ and this level is not maximal in slice 1 and exactly 3 below a non-maximal level in slice 0 and so level $L_{\frac{m}{2}+5}$ be not be a maximal level slice. However level $L_{\frac{m}{2}+7}$ contains two elements $(\frac{m}{2}, \frac{m}{2} - 5, 3, 2)$ and $(\frac{m}{2} - 1, \frac{m}{2} - 3, 2, 2)$ and this level is exactly three below a maximal level in slice 1 and exactly 6 below a maximal level in slice 0 so level $L_{\frac{m}{2}+7}$ will have the same size as levels $L_{\frac{m}{2}+4}$ and $L_{\frac{m}{2}}$. Now that we have elements in slice 2 the structures from Figures 5.13 and 5.14 show that we get an increase of at most 2 on the left when we move down by 3 and in slices 0 and 1 we will get a decrease of exactly one in each slice, provided we have not gone past level m. So when $m \equiv 4 \bmod 6$ then $\frac{m}{2} + 1 \equiv 0 \bmod 3$ and so will each of the levels three apart going down.

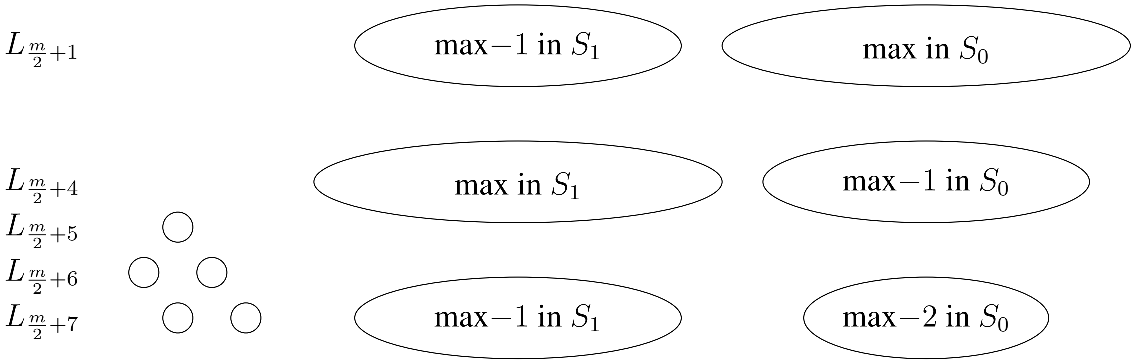


Figure 5.19: First Maximal levels when $m \equiv 4 \bmod 6$

Finally for slice 1 when $m \equiv 5 \bmod 6$ then $m-4 \equiv 1 \bmod 6$ and so levels $\lfloor \frac{m-4}{2} \rfloor + 6 = \lfloor \frac{m}{2} \rfloor + 4$ is the only maximal level-slice in slice 1. Again this level-slice does not align with the maximal level slices in slice 0, but $\lfloor \frac{m}{2} \rfloor + 1$ and $\lfloor \frac{m}{2} \rfloor + 4$ are three levels apart which means that $L_{1,\lfloor \frac{m}{2} \rfloor+1}$ has exactly one less element than maximal and similarly for $L_{0,\lfloor \frac{m}{2} \rfloor+4}$. So level $L_{\lfloor \frac{m}{2} \rfloor+1}$ is the first potential maximal level in R with level $L_{\lfloor \frac{m}{2} \rfloor+4}$ having the same size. In slice 2, $(\lceil \frac{m}{2} \rceil + 1, \lfloor \frac{m}{2} \rfloor - 6, 2, 2)$ is the very top-most element in R, which is in level $\lfloor \frac{m}{2} \rfloor - 5 + 2(2) + 3(2) = \lfloor \frac{m}{2} \rfloor + 5$. Thus levels $L_{\lfloor \frac{m}{2} \rfloor+1}$ and $L_{\lfloor \frac{m}{2} \rfloor+4}$ contain only elements form slices 0 and 1 in R. Now $L_{\lfloor \frac{m}{2} \rfloor+5}$ is not maximal in slice 0 and is not maximal in slice 1 so this level may have the same size as $L_{\lfloor \frac{m}{2} \rfloor+1}$ and $L_{\lfloor \frac{m}{2} \rfloor+4}$ but will not be bigger. Now level $L_{\lfloor \frac{m}{2} \rfloor+7}$ contains exactly two elements $(\lceil \frac{m}{2} \rceil, \lfloor \frac{m}{2} \rfloor - 5, 3, 2)$ and $(\lceil \frac{m}{2} \rceil - 1, \lfloor \frac{m}{2} \rfloor - 3, 2, 2)$ from slice 2 and this level is exactly three below a maximal level in slice 1 and exactly 6 below a maximal level in slice 0 and so level $L_{\lfloor \frac{m}{2} \rfloor+7}$ will have the same size as levels $L_{\lfloor \frac{m}{2} \rfloor+4}$ and $L_{\lfloor \frac{m}{2} \rfloor+1}$. Now that we have elements in slice 2 the structures from Figures 5.13 and 5.14 show that we get an increase of at most 2 on the left when we move down by 3 and in slices 0 and 1 we will get a decrease of exactly one in each slice, provided we have not gone past level m. So when $m \equiv 5 \bmod 6$ then $\lfloor \frac{m}{2} \rfloor + 1 \equiv 0 \bmod 3$ and so will each of the levels three apart going down.

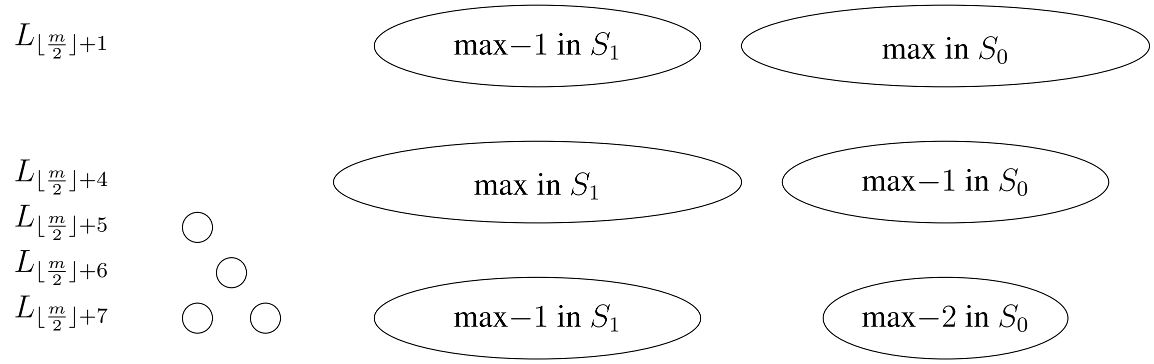


Figure 5.20: First Maximal levels when $m \equiv 5 \bmod 6$

At this point we have where the maximal levels will start and that they appear every three apart as long as there are elements in slice 0, that is when $k \leq_M m$. We also know that these levels are all congruent to $0 \bmod 3$, so the last level in slice 0 which is congruent to $0 \bmod 3$ will be m when $m \equiv 0$ or $3 \bmod 6$, $m-1$ when $m \equiv 1$ or $4 \bmod 6$, and $m-2$ when $m \equiv 2$ or $5 \bmod 6$. We now show where these maximal levels in R stop by looking at what happens in the last couple of level pairs with at least one level having an element in slice 0 which are also congruent to $0 \bmod 3$.

So when $m \equiv 0 \bmod 3$ we compare levels L_{m-3}, L_m, and L_{m+3}. In slice 0, $L_{0,m-3}$ contains exactly two elements $(\frac{m}{3}+2, \frac{m}{3}-1, \frac{m}{3}-1, 0)$ and $(\frac{m}{3}+1, \frac{m}{3}+1, \frac{m}{3}-2, 0)$; $L_{0,m}$ contains only one element, that is $(\frac{m}{3}, \frac{m}{3}, \frac{m}{3}, 0)$ and So $|L_{0,m}| < |L_{0,m-3}|$; and this is the smallest element in slice 0 so $L_{0,m+3} = \varnothing$. Which gives us that $|L_{0,m-3}| > |L_{0,m}| > |L_{0,m+3}$. Comparing $L_{1,m-3}$, $L_{1,m}$, and L_{m+3} we also get that $|L_{1,m}| < |L_{1,m-3}|$, since $L_{1,m}$ contains exactly one element, $(\frac{m}{3}+1, \frac{m}{3}-1, \frac{m}{3}-1, 1)$, and $L_{1,m-3}$ contains exactly two elements, $(\frac{m}{3}+3, \frac{m}{3}-2, \frac{m}{3}-2, 1)$ and $(\frac{m}{3}+2, \frac{m}{3}, \frac{m}{3}-3, 1)$. Since $(\frac{m}{3}, \frac{m}{3}, \frac{m}{3}-1, 1)$ is the smallest element in slice 1 and is in level $\frac{m}{3} + 2\big(\frac{m}{3}-1\big) + 3(1) = m+1$ then $L_{m+3} = \varnothing$. Which gives us that $|L_{1,m-3}| > |L_{1,m}| > |L_{1,m+3}|$. We now compare $L_{2,m-3}$, $L_{2,m}$, and $L_{2,m+3}$. Since $i \geq 2$ we can use the bounds from lemma 5.6 to get that $m + 2i - k = m + 2(2) - m = 4 < 5$ so $L_{2,m}$ is not a maximal level-slice and so certainly neither will $L_{2,m+3}$. But $L_{2,m-3}$ is maximal since

$m + 2i - k = m + 2(2) - (m - 3) = 7$. So we have that $|L_{2,m-3}| > |L_{2,m}| > |L_{2,m+3}|$. So since we can have an increase of at most 2 in the larger slices and have shown a decrease of at least 3 from slices $0, 1$, and 2 then level $|L_{m-3}| > L_m > L_{m+3}$. So level L_m and L_{m+3} are not maximal levels when $m \equiv 0 \bmod 3$.

When $m \equiv 1 \bmod 3$ we compare levels L_{m-4}, L_{m-1}, and L_{m+2}. In slice 0, $L_{0,m-4}$ contains exactly two elements $(\lceil\frac{m}{3}\rceil + 2, \lfloor\frac{m}{3}\rfloor - 1, \lfloor\frac{m}{3}\rfloor - 1, 0)$ and $(\lceil\frac{m}{3}\rceil + 1, \lfloor\frac{m}{3}\rfloor + 1, \lfloor\frac{m}{3}rfloor - 2, 0)$; $L_{0,m-1}$ contains only one element, that is $(\lceil\frac{m}{3}\rceil, \lfloor\frac{m}{3}\rfloor, \lfloor\frac{m}{3}\rfloor, 0)$ and So $|L_{0,m}| < |L_{0,m-3}|$; and this is the smallest element in slice 0 so $L_{0,m+2} = \varnothing$. Which gives us that $|L_{0,m-4}| > |L_{0,m-1}| > |L_{0,m+2}$. Comparing $L_{1,m-4}$, $L_{1,m-1}$, and $L_{1,m+2}$, we get that $L_{1,m-4}$ has exactly 3 elements: $(\lceil\frac{m}{3}\rceil + 3, \lfloor\frac{m}{3}\rfloor - 2, \lfloor\frac{m}{3}\rfloor - 2, 1)$, $(\lceil\frac{m}{3}\rceil + 2, \lfloor\frac{m}{3}\rfloor, \lfloor\frac{m}{3}\rfloor - 3, 1)$, and $(\lceil\frac{m}{3}\rceil + 1, \lfloor\frac{m}{3}\rfloor + 2, \lfloor\frac{m}{3}\rfloor - 4, 1)$. Level-slice $L_{1,m-1}$ has exactly 2 elements: $(\lceil\frac{m}{3}\rceil + 1, \lfloor\frac{m}{3}\rfloor - 1, \lfloor\frac{m}{3}\rfloor - 1, 1)$ and $(\lceil\frac{m}{3}\rceil, \lfloor\frac{m}{3}\rfloor + 1, \lfloor\frac{m}{3}\rfloor - 2, 1)$. And $L_{1,m+2}$ has exactly 1 element $(\lceil\frac{m}{3}\rceil - 1, \lfloor\frac{m}{3}\rfloor, \lfloor\frac{m}{3}\rfloor, 1)$. Which gives us that $|L_{1,m-4}| > |L_{1,m-1}| > |L_{1,m+2}|$. We now compare $L_{2,m-4}$, $L_{2,m-1}$, and $L_{2,m+2}$. Since $i \geq 2$ we can use the bounds from lemma 5.6 to get that $m + 2i - k = m + 2(2) - (m + 2) = 2 < 5$ so $L_{2,m+2}$ is not a maximal level-slice. But $L_{2,m-4}$ is maximal since $m + 2i - k = m + 2(2) - (m - 4) = 8$, and more over contains elements $(\lceil\frac{m}{3}\rceil + 3, \lfloor\frac{m}{3}\rfloor - 1, \lfloor\frac{m}{3}\rfloor - 4, 2)$ and $(\lceil\frac{m}{3}\rceil + 2, \lfloor\frac{m}{3}\rfloor + 1, \lfloor\frac{m}{3}\rfloor - 5, 2)$. These both have $x_0 - x_1 2 \bmod 3$ so our lower bound from lemma 5.6 is 5 so $L_{2,m-1}$ will also be maximal since $m + 2i - k = m + 2(2) - (m - 1) = 5$. So we have that $|L_{2,m-4}| = |L_{2,m-1}| > |L_{2,m+3}|$. So since we can have an increase of at most 2 in the larger slices and have shown a decrease of exactly 2 from L_{m-4} to L_{m-1} then these two levels are equal. But since we have a decrease in slices $0, 1$, and 2 then level $L_{m-1} > L_{m+2}$. So L_{m+2} is not a maximal level when $m \equiv 1 \bmod 3$.

When $m \equiv 2 \bmod 3$ we compare levels L_{m-5}, L_{m-2}, and L_{m+1}. In slice 0, $L_{0,m-5}$ contains exactly two elements $(\lceil\frac{m}{3}\rceil + 3, \lceil\frac{m}{3}\rceil - 2, \lfloor\frac{m}{3}\rfloor - 1, 0)$ and $(\lceil\frac{m}{3}\rceil + 2, \lceil\frac{m}{3}\rceil, \lfloor\frac{m}{3}\rfloor - 2, 0)$; $L_{0,m-2}$ contains only one element, that is $(\lceil\frac{m}{3}\rceil + 1, \lceil\frac{m}{3}\rceil - 1, \lfloor\frac{m}{3}\rfloor, 0)$ and So $|L_{0,m-2}| < |L_{0,m-5}|$; and $(\lceil\frac{m}{3}\rceil, \lceil\frac{m}{3}\rceil, \lfloor\frac{m}{3}\rfloor, 0)$ is the smallest element in slice 0, and is in level $\lceil\frac{m}{3}\rceil + 2\big(\lfloor\frac{m}{3}\rfloor\big) + 3(0) = m - 1$ so $L_{0,m+1} = \varnothing$. Which gives us that $|L_{0,m-5}| > |L_{0,m-2}| > |L_{0,m+1}$. Comparing $L_{1,m-5}$, $L_{1,m-2}$, and $L_{1,m+1}$, we get that $L_{1,m-5}$ has exactly 3 elements: $(\lceil\frac{m}{3}\rceil + 4, \lceil\frac{m}{3}\rceil - 3, \lfloor\frac{m}{3}\rfloor - 2, 1)$,

$(\lceil \frac{m}{3} \rceil + 3, \lceil \frac{m}{3} \rceil - 1, \lfloor \frac{m}{3} \rfloor - 3, 1)$, and $(\lceil \frac{m}{3} \rceil + 2, \lceil \frac{m}{3} \rceil + 1, \lfloor \frac{m}{3} \rfloor - 4, 1)$. Level-slice $L_{1,m-2}$ has exactly 2 elements: $(\lceil \frac{m}{3} \rceil + 2, \lceil \frac{m}{3} \rceil - 2, \lfloor \frac{m}{3} \rfloor - 1, 1)$ and $(\lceil \frac{m}{3} \rceil + 1, \lceil \frac{m}{3} \rceil, \lfloor \frac{m}{3} \rfloor - 2, 1)$. And $L_{1,m+1}$ has exactly 1 element $(\lceil \frac{m}{3} \rceil, \lceil \frac{m}{3} \rceil - 1, \lfloor \frac{m}{3} \rfloor, 1)$. Which gives us that $|L_{1,m-5}| > |L_{1,m-2}| > |L_{1,m+1}|$. We now compare $L_{2,m-5}$, $L_{2,m-2}$, and $L_{2,m+1}$. Since $i \geq 2$ we can use the bounds from lemma 5.6 to get that $m + 2i - k = m + 2(2) - (m + 1) = 3 < 5$ so $L_{2,m+1}$ is not a maximal level-slice. But $L_{2,m-5}$ is maximal since $m + 2i - k = m + 2(2) - (m - 5) = 9$, and more over contains elements $(\lceil \frac{m}{3} \rceil + 4, \lceil \frac{m}{3} \rceil - 2, \lfloor \frac{m}{3} \rfloor - 4, 2)$, $(\lceil \frac{m}{3} \rceil + 3, \lceil \frac{m}{3} \rceil, \lfloor \frac{m}{3} \rfloor - 5, 2)$, and $(\lceil \frac{m}{3} \rceil + 2, \lceil \frac{m}{3} \rceil + 2, \lfloor \frac{m}{3} \rfloor - 6, 2)$. These all have $x_0 - x_1 0 \bmod 3$ so our lower bound from lemma 5.6 is 6 so $L_{2,m-2}$ will also be maximal since $m + 2i - k = m + 2(2) - (m - 2) = 6$. So we have that $|L_{2,m-5}| = |L_{2,m-2}| > |L_{2,m+1}|$. So since we can have an increase of at most 2 in the larger slices and have shown a decrease of exactly 2 from L_{m-5} to L_{m-2} then these two levels are equal. But since we have a decrease in slices $0, 1$, and 2 then level $L_{m-2} > L_{m+1}$. So L_{m+2} is not maximal a level when $m \equiv 2 \bmod 3$.

Once we have no more elements in slice 0, we are going to have at most an increase of 2 on the left and a decrease of at least 2 on the right. So we have the following bounds on maximal levels

- When $m \equiv 0 \bmod 6$: $\frac{m}{2} \leq k \leq m - 3$
- When $m \equiv 1 \bmod 6$: $\lfloor \frac{m}{2} \rfloor \leq k \leq m - 1$
- When $m \equiv 2 \bmod 6$: $\frac{m}{2} - 1 \leq k \leq m - 2$
- When $m \equiv 3 \bmod 6$: $\lfloor \frac{m}{2} \rfloor + 3 \leq k \leq m - 3$
- When $m \equiv 4 \bmod 6$: $\frac{m}{2} + 1 \leq k \leq m - 1$
- When $m \equiv 5 \bmod 6$: $\lfloor \frac{m}{2} \rfloor + 1 \leq k \leq m - 2$,

and whenever $k \equiv 0 \bmod 3$. There may be other maximal levels within theses bounds that have $k \not\equiv 0 \bmod 3$, but they will have the same size as those levels with $k \equiv 0 \bmod 3$.

Now that we have bounds on the maximal levels in R, for our chain decomposition of R to be minimal, each of the elements in a maximal level in R must appear in exactly one chain and each chain must contain exactly one element from a maximal level.

Theorem 5.15. *For $P'(4, m)$, the Hasse Diagram can be decomposed into the Hasse Diagram for $P'(4, m-16)$ and R where R has width*

$$\left\lfloor \frac{m}{6} \right\rfloor + 1 + \left\lfloor \frac{m-4}{6} \right\rfloor + 1 - 1$$

for $m \not\equiv 3 \bmod 6$ and

$$\left\lfloor \frac{m}{6} \right\rfloor + 1 + \left\lfloor \frac{m-4}{6} \right\rfloor + 1$$

for $m \equiv 3 \bmod 6$. Where $\left\lfloor \frac{m}{6} \right\rfloor + 1$ is the width of slice 0, $\left\lfloor \frac{m-4}{6} \right\rfloor + 1$ is the width of slice 1. The -1 is for the misalignment of maximal level sizes whenever $m \not\equiv 3 \bmod 6$ since they align when $m \equiv 3 \bmod 6$ the -1 does not appear.

Proof. Suppose that $m \leq 15$, then in Appendix A we have the chain decompositions for $P'(4, m)$ which have size $\left\lfloor \frac{m}{6} \right\rfloor + 1 + \left\lfloor \frac{m-4}{6} \right\rfloor + 1 - 1$ or $\left\lfloor \frac{m}{6} \right\rfloor + 1 + \left\lfloor \frac{m-4}{6} \right\rfloor + 1 - 1$ if we take the width of slice 1 to be 0 whenever $m < 4$, that is when $S_1 = \varnothing$.

Now assume that $m \geq 16$ and suppose that $P'(4, m)$ is the smallest poset for which the theorem is false. We let R be the set of all elements in $P'(4, m)$ such that $x_0 - x_1 \leq 7$ or $x_3 \leq 1$ or both. We claim that $P'(4, m) \setminus R \cong P'(4, m-16)$ via $f : (x_0, x_1, x_2, x_3) \mapsto (x_0 - 10, x_1 - 2, x_2 - 2, x_3 - 2)$. In particular to check this is well defined and surjective we show that $p = (p_0, p_1, p_2, p_3) \in R$ if and only if $(p_0 - 10, p_1 - 2, p_2 - 2, p_3 - 2) \notin P'(4, m-16)$.

Indeed if $p = (p_0, p_1, p_2, p_3) \in P'(4, m)$ then $p_0 \geq p_1 \geq p_2 \geq p_3 \geq 0$. If $p \in R$, then either $p_3 \leq 1$ or $p_0 - p_1 \leq 7$. If $p_3 \leq 1$, then $(p_0 - 10, p_1 - 2, p_2 - 2, p_3 - 2) \notin P'(4, m-16)$ since $p_3 - 2 \leq -1$. So suppose then that $p_3 \geq 2$ and $p_0 - p_1 \leq 6$. Then $p_0 - 10 \leq (p_1 + 6) - 10 = p_1 - 4 < p_1 - 2$ and thus $(p_0 - 10, p_1 - 2, p_2 - 2p_3 - 2)$ is not decreasing and so $(p_0 - 10, p_1 - 2, p_2 - 2p_3 - 2) \notin P'(4, m-16)$. So $f(P'(4, m) \setminus R) \subseteq P'(4, m-16)$.

Now suppose that $q = (q_0, q_1, q_2, q_3) \in P'(4, m)$ but $(q_0 - 10, q_1 - 2, q_2 - 2, q_3 - 2) \notin P'(4, m-16)$. Thus we have that either $q_3 - 2 < 0$ or $q_0 - 10 < q_1 - 4$. If $q_3 - 2 < 0$ then $q_3 < 2$ which implies that $q_3 \leq 1$ and so $q \in R$. So now suppose that $q_0 - 10 < q_1 - 2$. This gives us that $q_0 - q_1 < 8$ that is $q_0 - q_1 \leq 7$ and thus again we have that $q \in R$. So $f(P'(4, m) \setminus R) \supseteq P'(4, m-16)$. Thus f is a bijection.

Now suppose that $r, s \in P'(4, m) \setminus R$ and let $r \geq_M s$. That is

$$r_0 \geq s_0,\ \ r_0 + r_1 \geq s_0 + s_1,\ \ r_0 + r_1 + r_2 \geq s_0 + s_1 + s_2$$

The sum of all the r_i ands_i will both be equal to m, since $r, s \in P'(4, m)$. Then we have that $f(r) = (r_0-10, r_1-2, r_2-2, r_3-2) \in P'(4, m-16)$ and $f(s) = (s_0-10, s_1-2, s_2-2, s_3-2) \in P'(4, m-16)$. So using the above inequalities we see that

$$r_0-10 \geq s_0-10,\ \ r_0-10+r_1-2 \geq s_0-10+s_1-2,\ \ r_0-10+r_1-2+r_2-2 \geq s_0-10+s_1-2+s_2-2$$

So we have that $f(r) \geq f(s)$. Now suppose that $t, v \in P'(4, m-16)$ and let $t \geq_M v$. That is

$$t_0 \geq v_0,\ \ t_0 + t_1 \geq v_0 + v_1,\ \ t_0 + t_1 + t_2 \geq v_0 + v_1 + v_2.$$

Again the sum of all the t_i and v_i will both be equal to $m - 16$. Then we have that $f^{-1}(t) = (t_0 + 10, t_1 + 2, t_2 + 2, t_3 + 2) \in P'(4, m) \setminus R$ and $f^{-1}(v) = (v_0 + 10, v_1 + 2, v_2 + 2, v_3 + 2) \in P'(4, m) \setminus R$. So using the above inequalities we see that

$$t_0+10 \geq v_0+10,\ \ t_0+10+t_1+2 \geq v_0+10+v_1+2,\ \ t_0+10+t_1+2+t_2+2 \geq v_0+10+v_1+2+v_2+2.$$

So we have that $f^{-1}(t) > f^{-1}(v)$.

So we have that $P'(4, m) \setminus R \cong P'(4, m-16)$. Thus if we can find a minimal chain decomposition of R and ensure that the maximal levels in R align with the maximal levels in $P'(4, m-16)$ we will get a minimal chain decomposition for all of $P'(4, m)$. We have bounded where the maximal levels occur in R and moreover that level L_k will be maximal in R whenever $k \equiv 0 \bmod 3$ and within the established bounds. From the Hasse Diagrams of our base cases we see that a maximal level will occur in level 0 when $m \leq 5$, level 6 whenever $4 \leq m \leq 14$ and level 12 whenever $13 \leq m \leq 15$. And all of these levels are congruent to $0 \bmod 3$. When these are embedded into the poset $P'(4, m+16)$ they will have $(10, 2, 2, 2)$ added and so will be shifted by $2+2(2)+3(2) = 12$ each time they are embedded. The base cases will be embedded

$\lfloor \frac{m}{16} \rfloor$ times and so our maximal levels from the base cases will end up in level $0 + 12 \cdot \lfloor \frac{m}{16} \rfloor$ when $m \leq 5$, level $6 + 12 \cdot \lfloor \frac{m}{16} \rfloor$ whenever $4 \leq m \leq 14$ and level $12 + 12 \cdot \lfloor \frac{m}{16} \rfloor$ whenever $13 \leq m \leq 15$. Which will all fall within the bounds of where maximal levels in R occur and all of these will be congruent to $0 \bmod 3$ so will align with those maximal levels in R. Thus it remains to show that there exists a minimal chain decomposition of R.

For the first part of our chain decomposition, until we reach the first maximal level in R we will follow the chain decomposition as described in the $P'(3, m)$ in slice 0 and the decomposition from $P'(3, m-4)$ in slice 1 adding chains whenever needed, as shown below in figure 5.21.

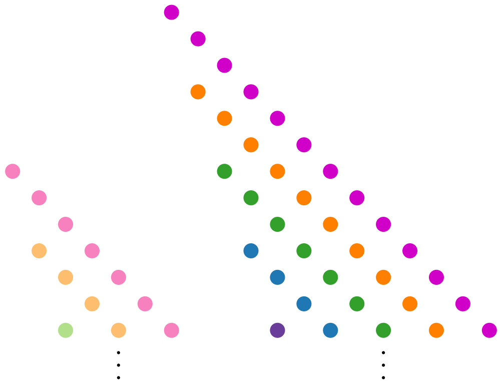


Figure 5.21: Chain Decomposition for the Increasing Parts of S_0 and S_1

Once we reach the first maximal level, but before we reach the structures from figures 5.13 and 5.14, we have the following structures which can be decomposed into minimal chains as shown in figures 5.22, 5.23, 5.24, 5.25, 5.26, and 5.27.

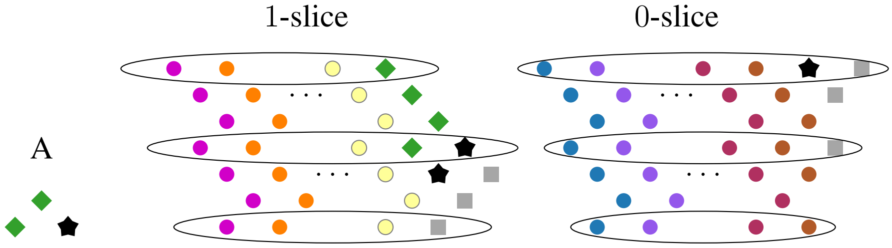


Figure 5.22: Chain decomposition for $m \equiv 0 \bmod 6$ top three maximal levels in R

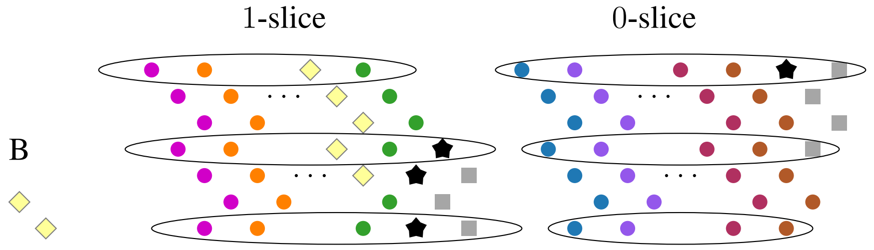


Figure 5.23: Chain decomposition for $m \equiv 1 \bmod 6$ top three maximal levels in R

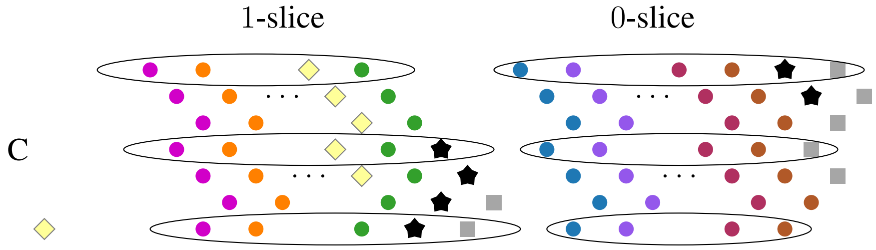


Figure 5.24: Chain decomposition for $m \equiv 2 \bmod 6$ top three maximal levels in R

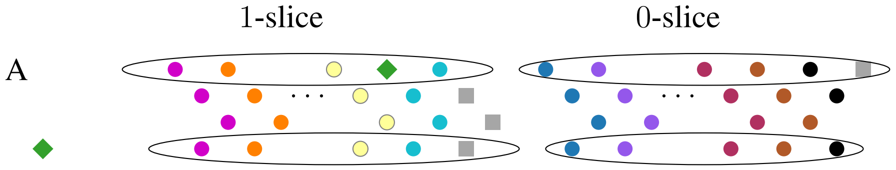


Figure 5.25: Chain decomposition for $m \equiv 3 \bmod 6$ top two maximal levels in R

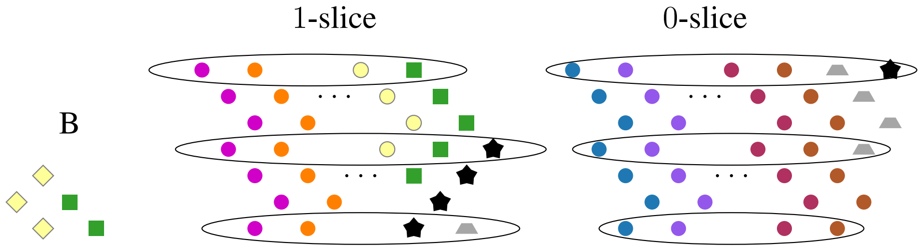


Figure 5.26: Chain decomposition for $m \equiv 4 \bmod 6$ top three maximal levels in R

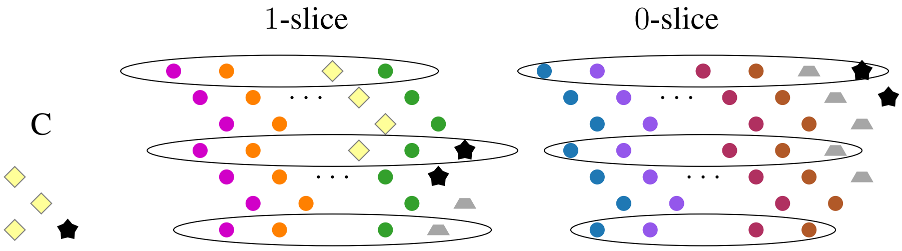


Figure 5.27: Chain decomposition for $m \equiv 5 \bmod 6$ top three maximal levels in R

Once we are decreasing in both slices 0 and 1 we will have the structures from figures 5.13 and 5.14 one one side we will get several possibilities for what our chain decomposition will look like. We will go through one general possibility here in figure 5.28 and the others can be found in Appendix B.

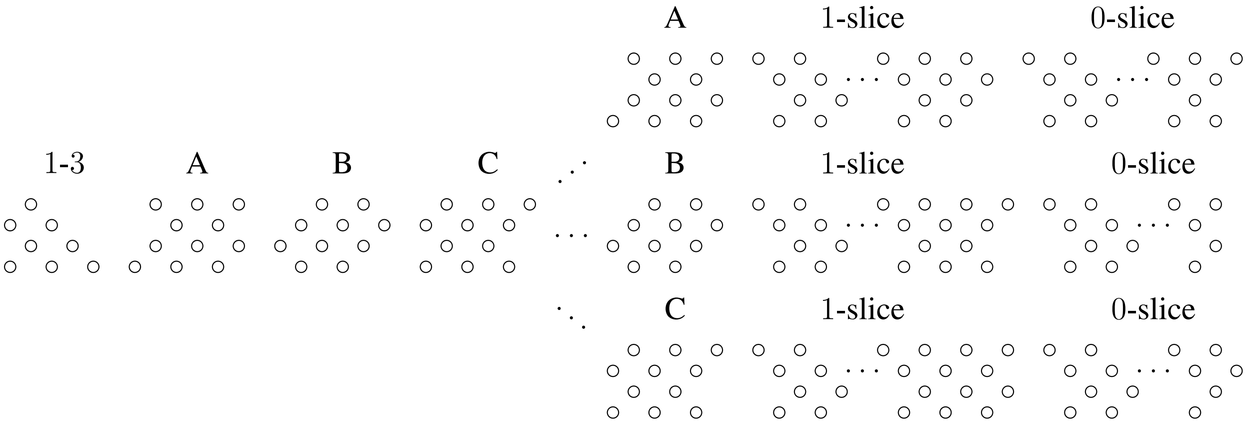


Figure 5.28: 1-3 Even

As we can see from this example we have the increasing 1-3 structure on the left, and the 0 and 1 slices on the right are decreasing. The A, B, and C blocks are those maximal level slices when $i \geq 2$. They will cycle through (A, B, C) until they reach the 1 slice. The 1 slice will still consist of the next block, but since no elements are removed from the 1 slice, then this next block appears in the right of the 1 block. And similarly the 0 slice will consist of whatever block came after the one from the 1 slice. So in figure 5.28, when the cycle ends on an A block then the 1 and 0 slices will include B and C blocks; when the cycle ends on an B block then the 1 and 0 slices will include C and A blocks; when the cycle ends on an C block then the 1 and 0 slices will include A and B blocks. Now these A, B, and C blocks are differentiated based on what the elements in the top and bottom level-slices are congruent to $\mod 3$, and the entire figure is based on even-ness. So we can classify exactly when these levels slice pairs will appear. When $m \equiv 0 \mod 6$ then we know the cycle of (A, B, C) will end on an A , and similarly when $m \equiv 4 \mod 6$ then the cycle ends on a B block, and when $m \equiv 2 \mod 6$ then the cycle ends on a C block.

Now that we know what all of the instances of starting with a 1-3 structure are, and can see that the top and bottom levels are the same size, then we can begin to decompose this part of the poset into chains. We do this in figure 5.29.

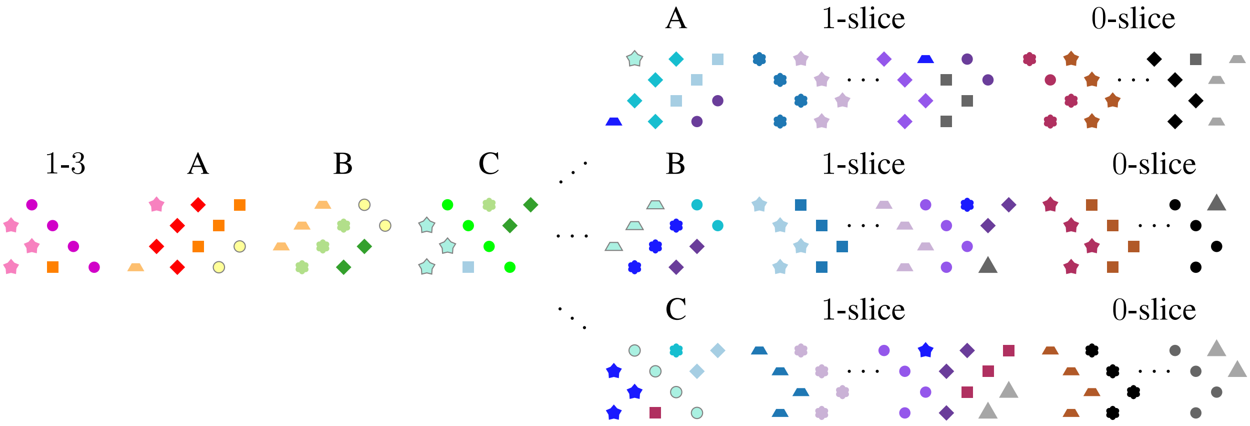


Figure 5.29: 1-3 Even

In this partial chain decomposition, if you look closely at the C block, you will see the same structure as you do in the 1-3 block. So if there are no repetitions of the (A, B, C) blocks then the connections to the 1-slice will remain the same as those connections from the C block. And since the 1-3 block necessarily increases by exactly two, then we have to have that the 1 and 0 slices both decrease by exactly 1. This is also the case with all of the examples in the appendix where there are no (A, B, C) repetitions between the increasing partial block(s). That is the structures from figures 5.13 and 5.14 are “partial” A, B, or C blocks and appear in their respective full blocks.

Thus a chain decomposition of these levels will only use as many chains as there are elements in the top and bottom levels. We then proceed, starting with the bottom of this chain decomposition, to the next structure which would be the 0-2 3-3 structure. We decompose these elements as described in figure B.6. Once we reach a level outside the bounds for maximal levels we will still have an increase of at most 2 in the partial blocks on the left and potentially complete A, B, C blocks but we will eventually reach an A, B, C block in slice 2 which is decreasing, thus partial, and then either decreasing as well in slices 1 and 0 or those slices are empty. To decompose these level-pairs, we will treat these level-pairs as if they were complete and so decompose as described in the figures from Appendix B. We will certainly not use anymore chains than in a maximal level, and so will continue to be a minimal chain decomposition. Now there may be a way to decompose these level-pairs into fewer chains than the Appendix B figures give, but we will not use more chains than what appears in a maximal level in R. Thus we will get a minimal chain decomposition of R where the maximal levels of $P'(4, m-16)$ align with maximal levels in R, giving us a minimal chain decomposition of $P'(4, m)$. $\square$

Chapter 6

Further Directions

In this dissertation we gave an explicit chain decomposition for $P(n, m)$ for $n \leq 4$. In Chapter 4 we compared our results to that of Lindström, West, and Riess, who gave symmetric chain decompositions of a different yet isomorphic poset. We also discussed in this chapter results from O'Hara, who proved the unimodality of the Gaussian Polynomials using a poset chain decomposition. O'Hara's poset consisted of identical elements to Lindström, West, and Riess, but used a different ordering. So while there exists a bijection between the elements of our posets and O'Hara's, the orderings are fundamentally different. So the existence of a chain decomposition for $P(n, m)$ under the majorization ordering remains an open problem.

Conjecture 6.1 (4.22, restated)**.** *$P(n, m)$ under the majorization ordering is Sperner for all $n, m \geq 0$, and moreover admits a symmetric chain decomposition.*

Also in this dissertation we gave explicit chain decompositions for $P'(n, m)$ when $n \leq 2$ and inductive arguments for $n \in \{3, 4\}$, thus establishing that $P'(n, m)$ is Sperner-like. While it seems from the results of O'Hara that $P(n, m)$ is in fact Sperner in general, unfortunately this is not true in general for $P'(n, m)$. In $P'(5, 10)$ we have an example of an antichain whose largest induced level from $P(5, 10)$ has size 3, but has an antichain of size 4. There are two levels which are largest, $L_6 = \{(7, 1, 1, 1, 0), (6, 2, 2, 0, 0), (5, 4, 1, 0, 0)\}$ and $L_9 = \{(5, 2, 2, 1, 0), (4, 4, 1, 1, 0), (4, 3, 3, 0, 0)\}$. We see that $(6, 1, 1, 1, 1) \in L_{10}$ is incomparable with all elements from L_9. Thus $L_9 \cup (6, 1, 1, 1, 1)$ is an antichain of size 4 which is larger than a largest level as we see in figure 6.1.

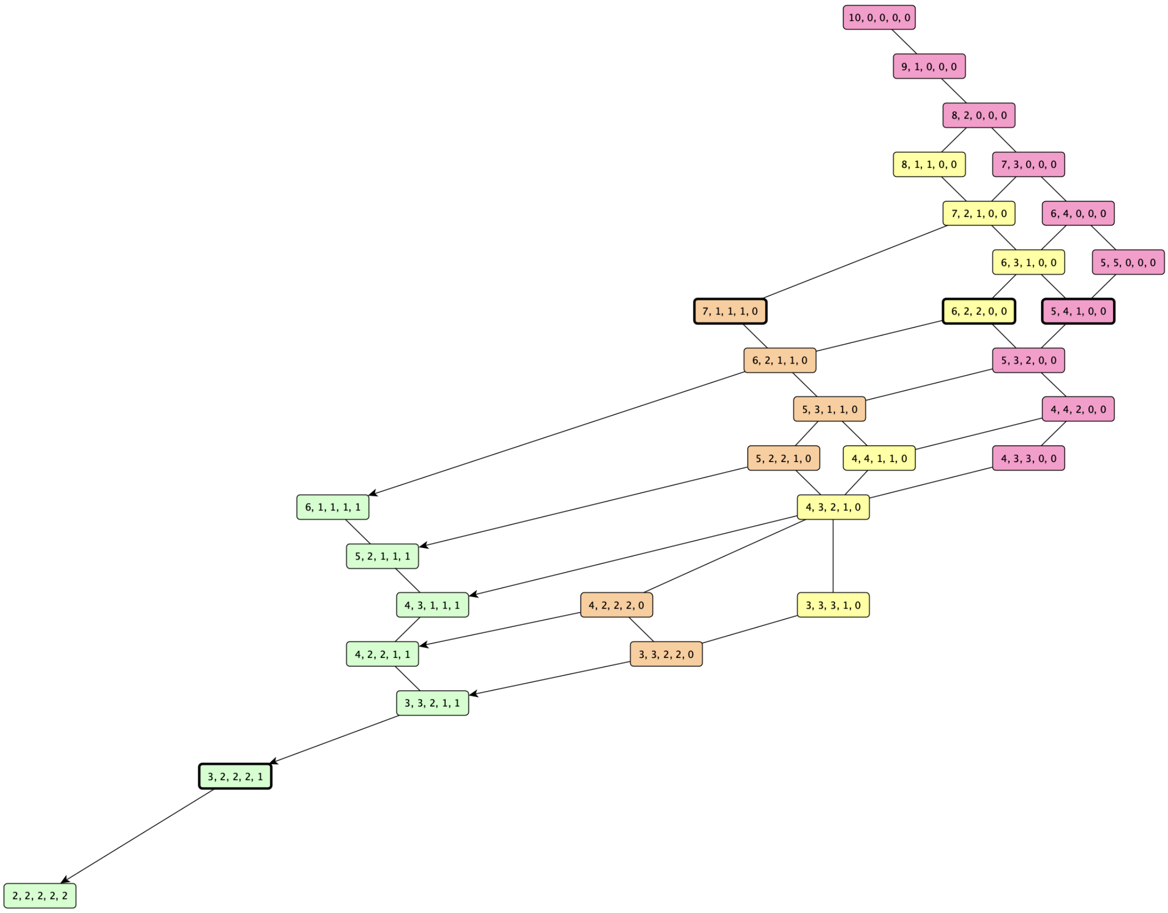


Figure 6.1: $P'(5,10)$ shows $P'(n,m)$ is not Sperner-like in general

Which leads us to the following questions:

Question 6.2. *Is there another way to define a variant of a rank function for $P'(n,m)$ with respect to which it remains Sperner-like?*

Question 6.3. *Can one explicitly describe a chain decomposition witnessing the width of $P'(n,m)$ in general?*

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Appendices

Appendix A

Base Case for $P'(4, m)$ Hasse Diagram Chain Decompositions

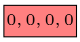

Figure A.1: $m = 0$ base case

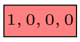

Figure A.2: $m = 1$ base case

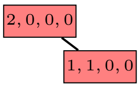

Figure A.3: $m = 2$ base case

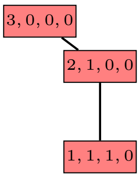

Figure A.4: $m = 3$ base case

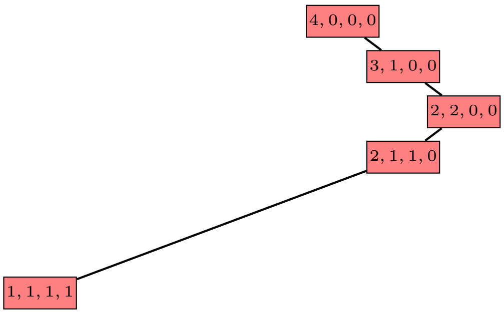

Figure A.5: $m = 4$ base case

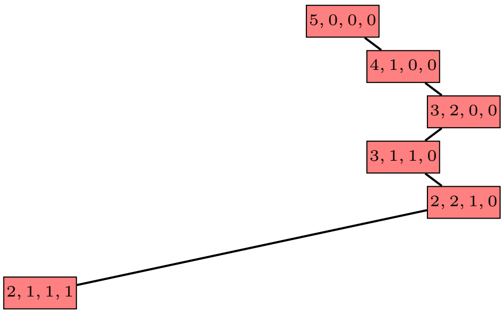

Figure A.6: $m = 5$ base case

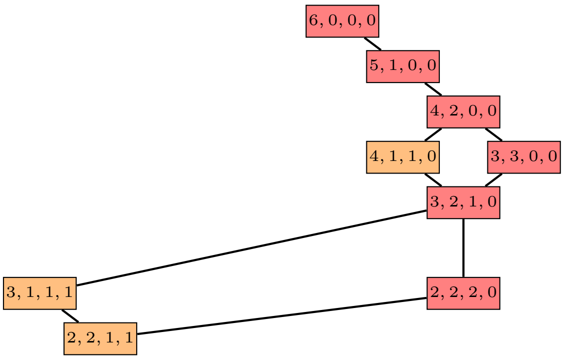

Figure A.7: $m = 6$ base case

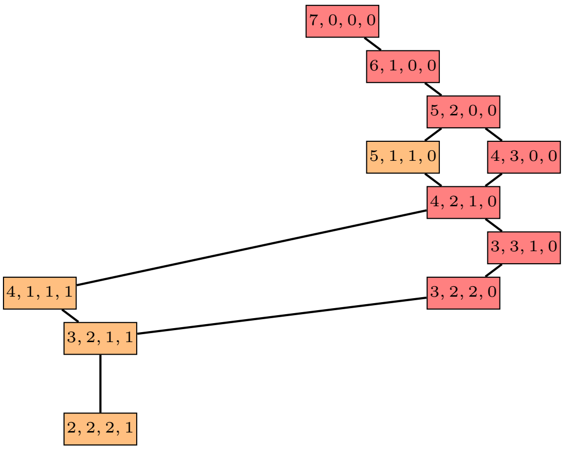

Figure A.8: $m = 7$ base case

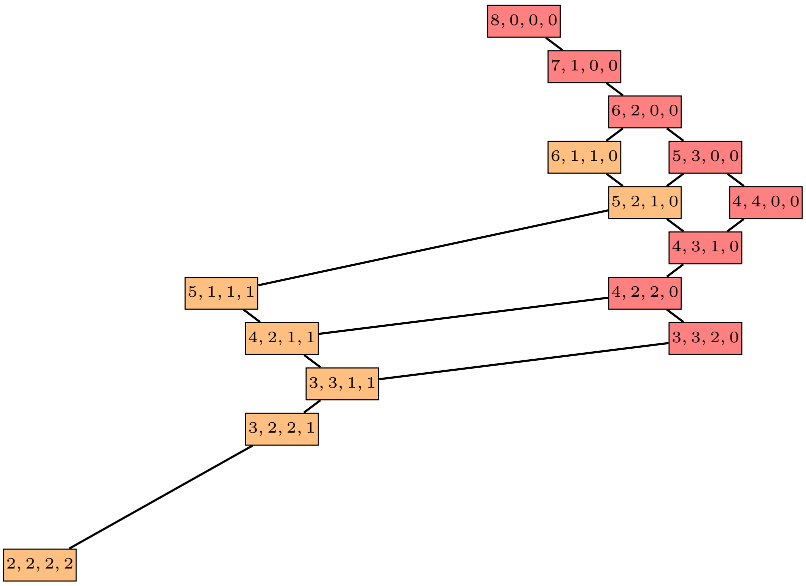

Figure A.9: $m = 8$ base case

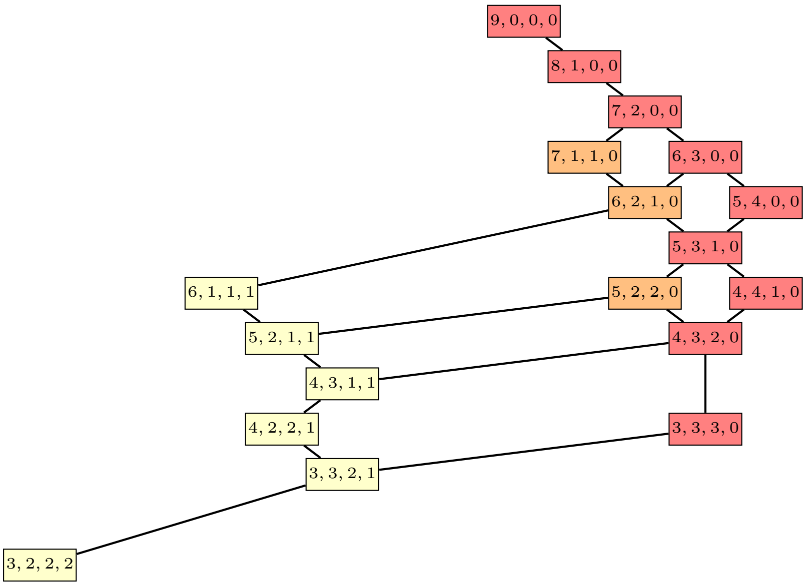

Figure A.10: $m = 9$ base case

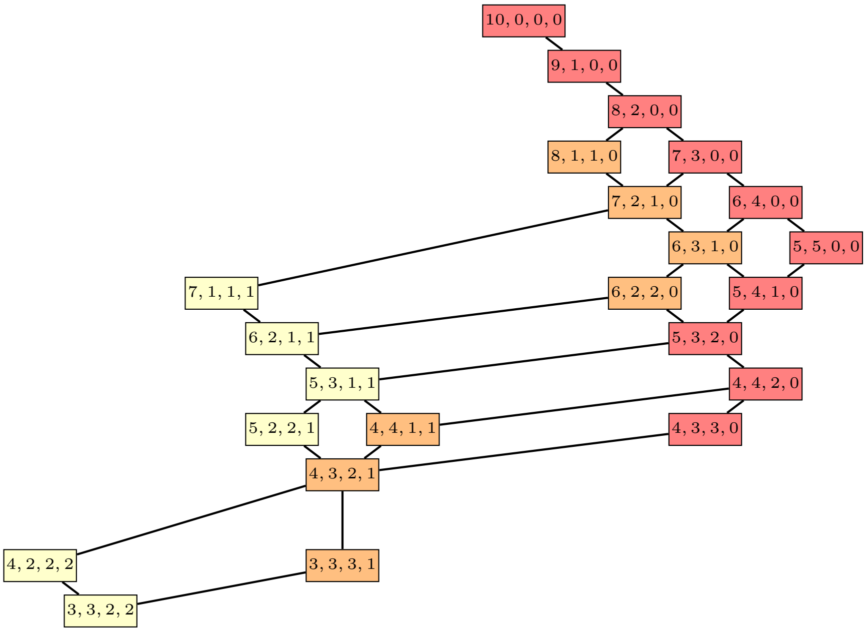

Figure A.11: $m = 10$ base case

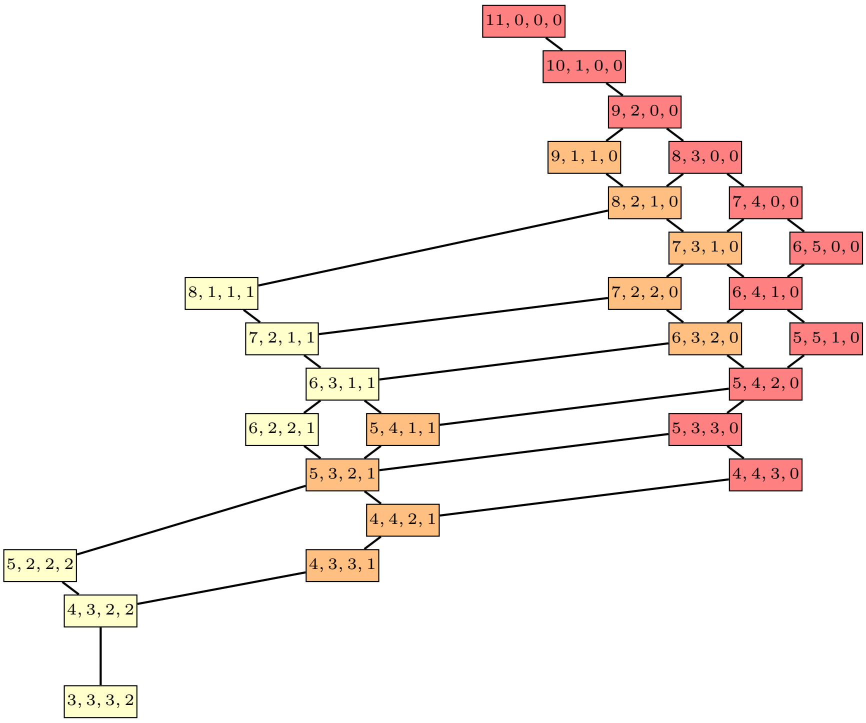

Figure A.12: $m = 11$ base case

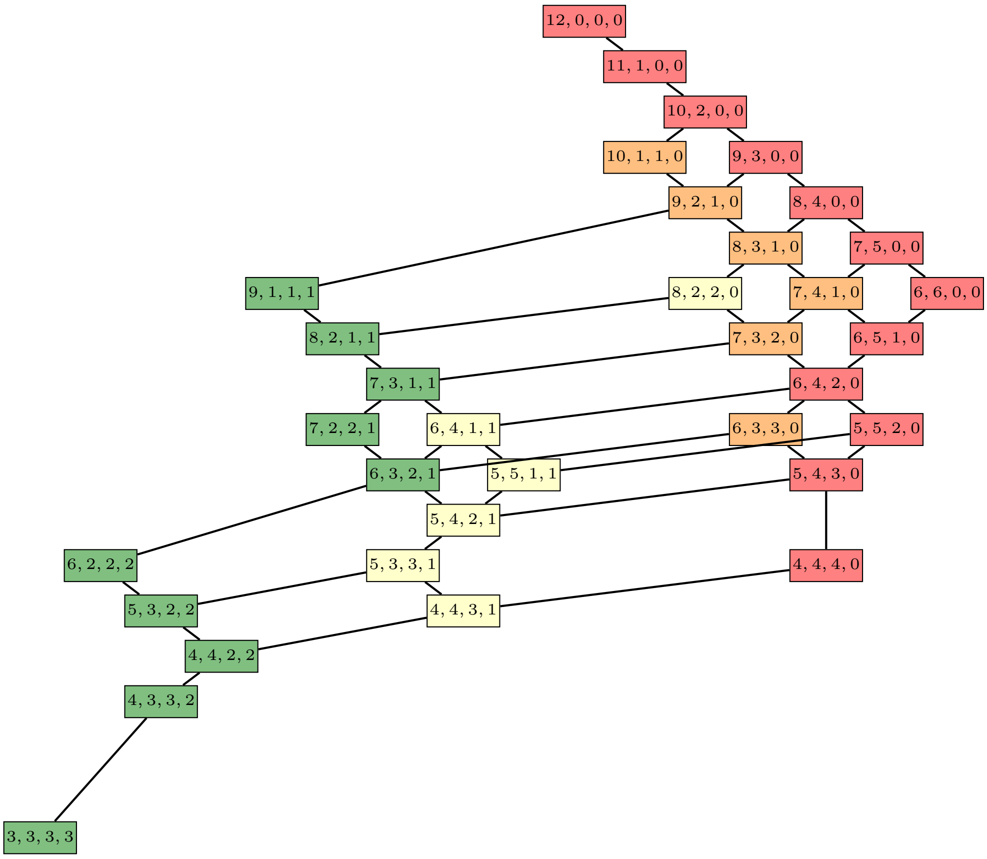

Figure A.13: $m = 12$ base case

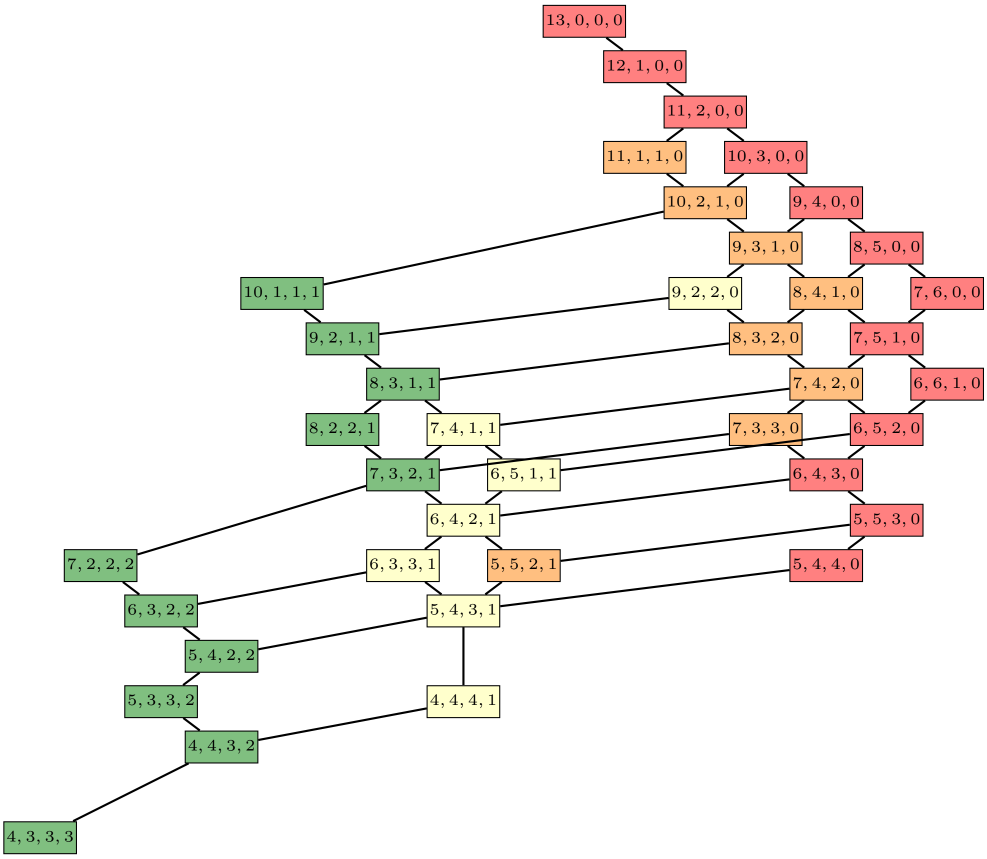

Figure A.14: $m = 13$ base case

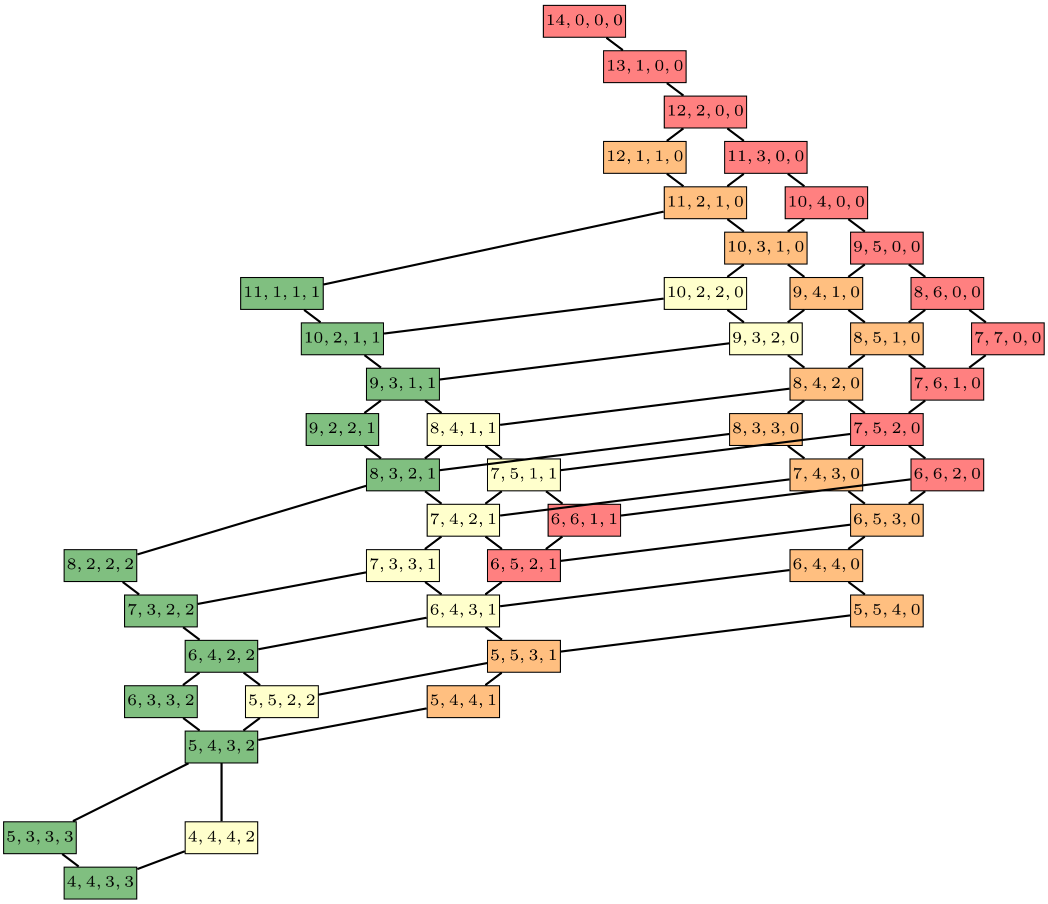

Figure A.15: $m = 14$ base case

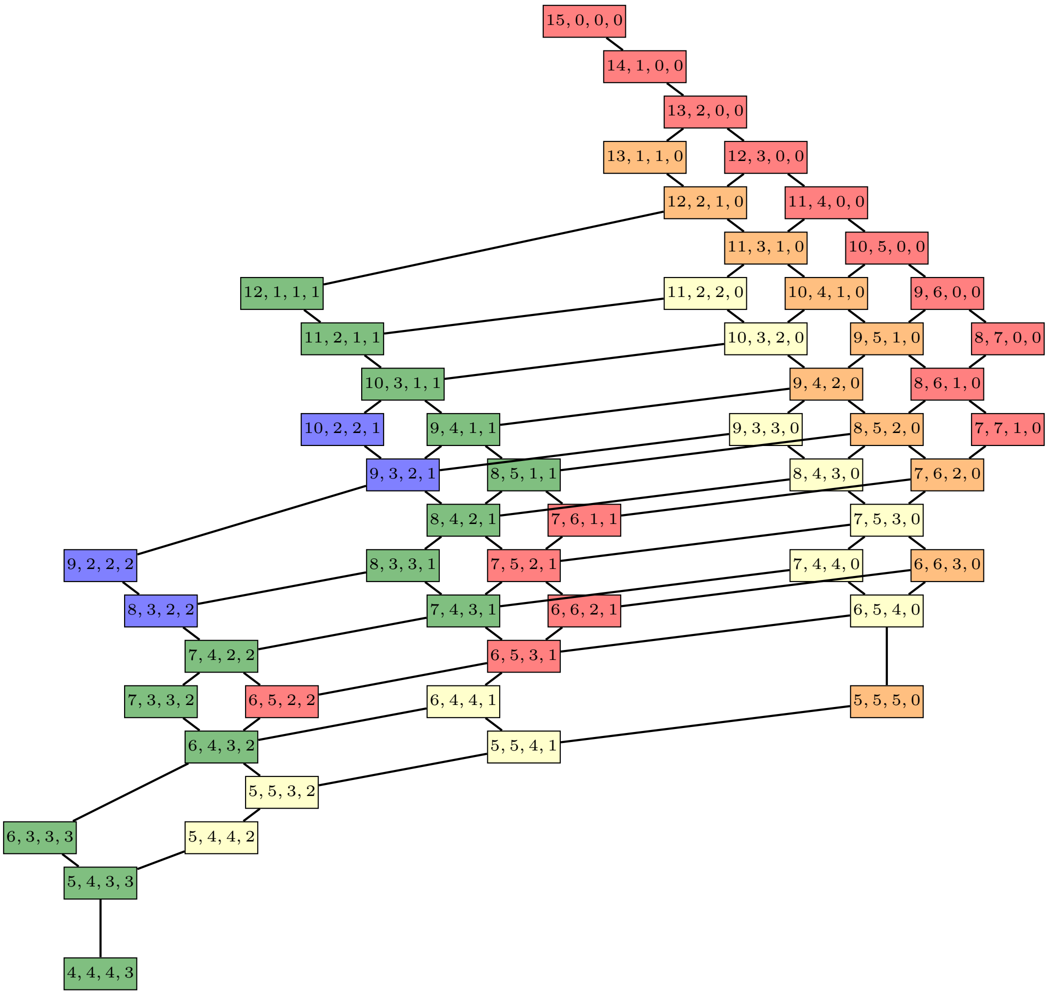

Figure A.16: $m = 15$ base case

Appendix B

Chain Decompositions for Maximal levels in R

In each of the chain decomposition figures, the decompositions are not only based on the left side structure but also the 0 and 1 slice structures on the right. In each of the figures for the odd cases the top sub-case is when $m \equiv 3 \bmod 6$, the middle sub-case is when $m \equiv 1 \bmod 6$ and the bottom sub-case is when $m \equiv 5 \bmod 6$. And similarly in each of the figures for the even cases the top sub-case is when $m \equiv 0 \bmod 6$, the middle sub-case is when $m \equiv 4 \bmod 6$ and the bottom sub-case is when $m \equiv 2 \bmod 6$.

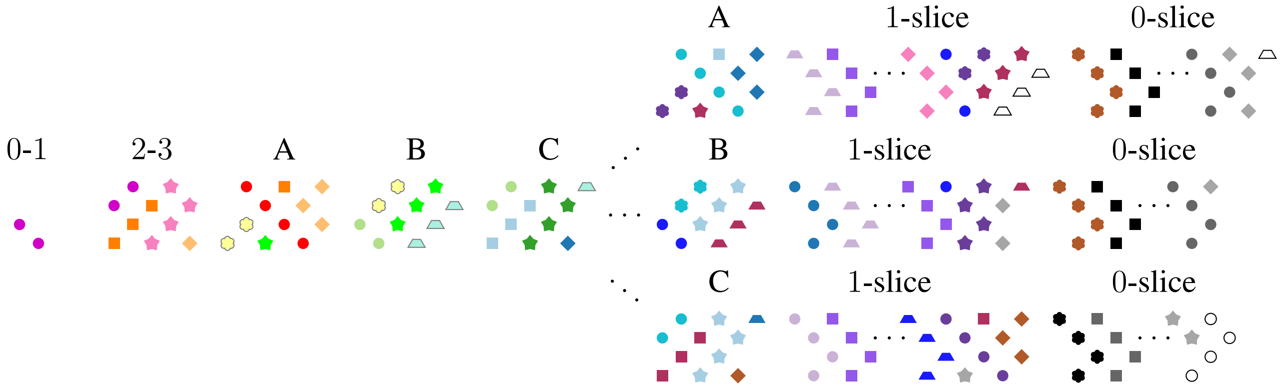

Figure B.1: 0-1 2-3 when m is Odd

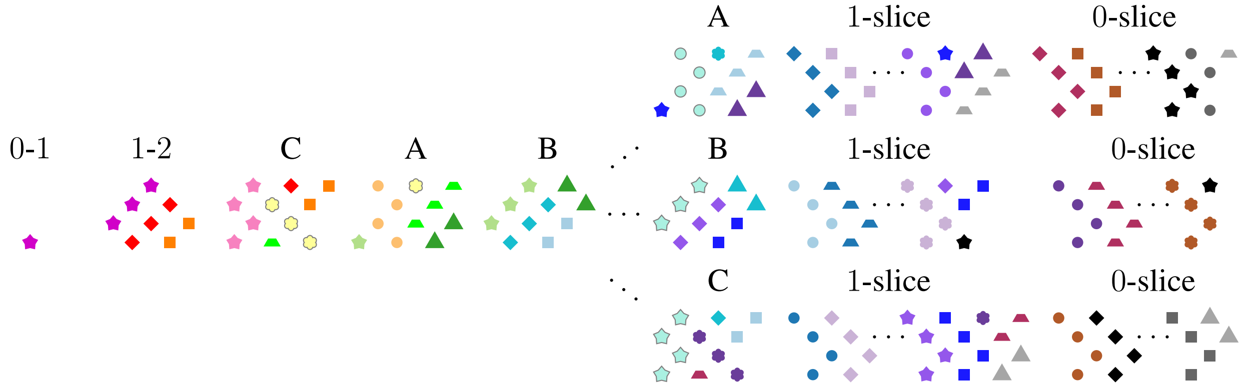

Figure B.2: 0-1 1-2 when m is Odd

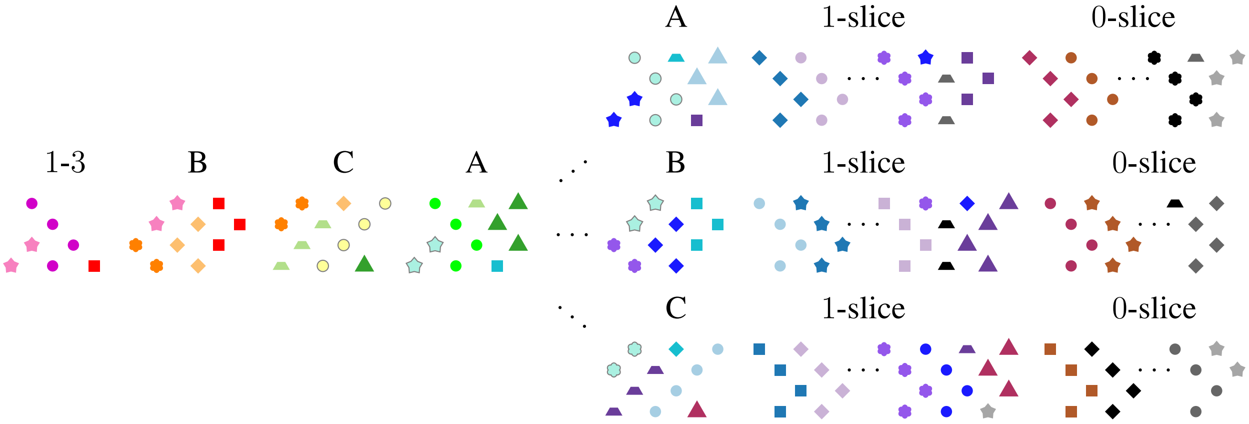

Figure B.3: 1-3 when m is Odd

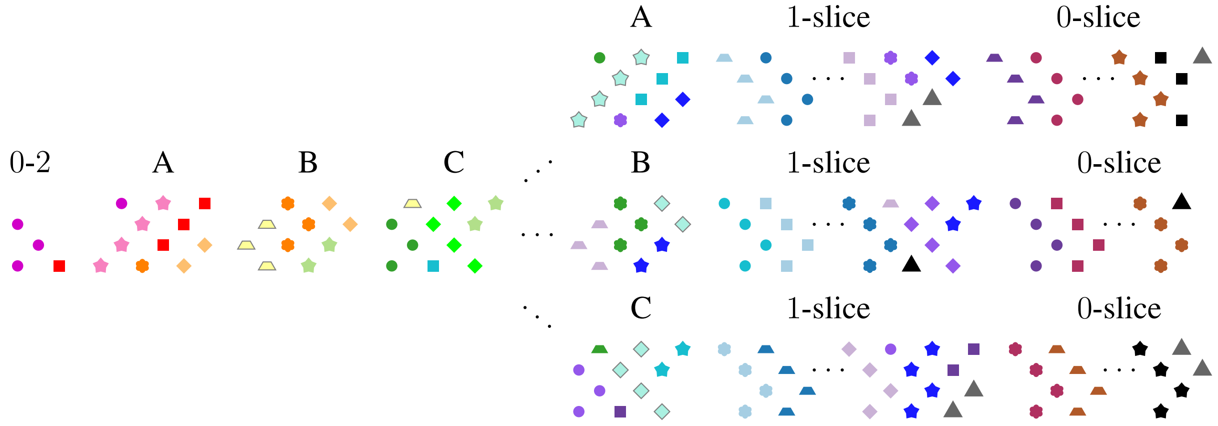

Figure B.4: 0-2 when m is Odd

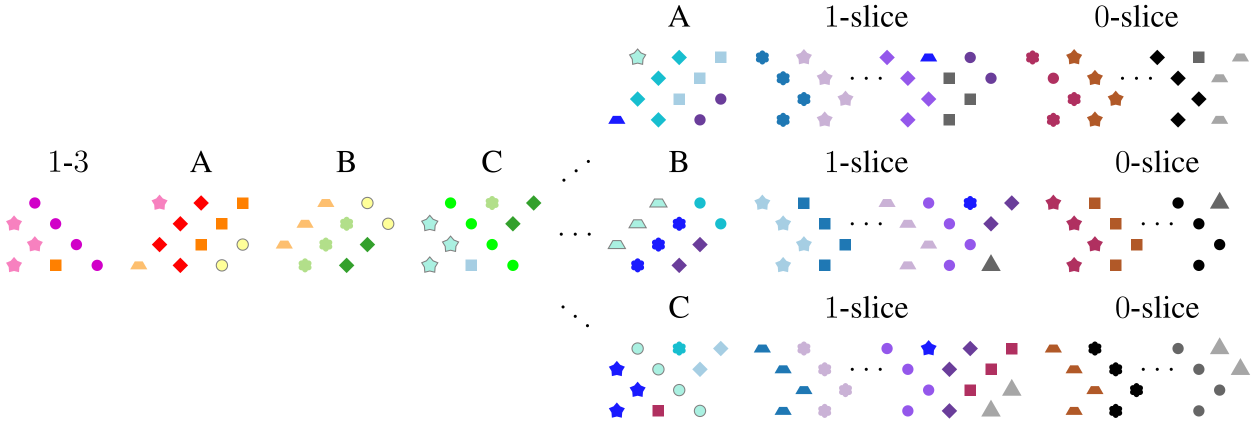

Figure B.5: 1-3 when m is Even

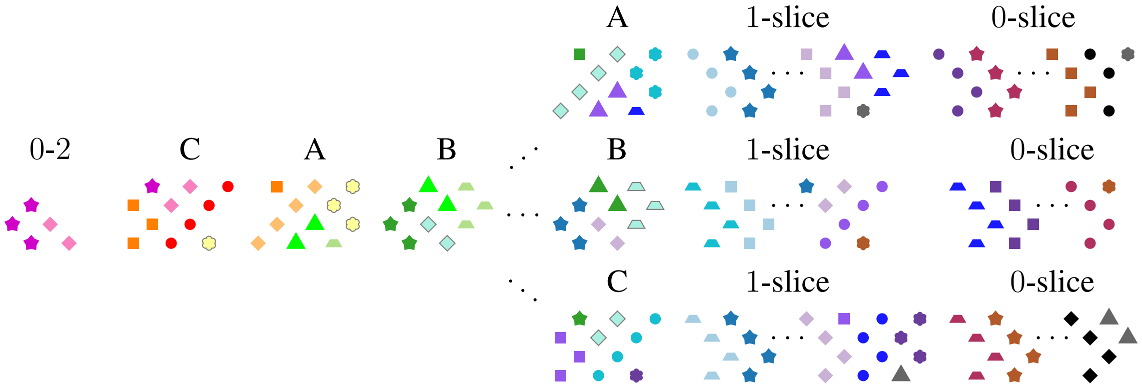

Figure B.6: 0-2 3-3 when m is Even

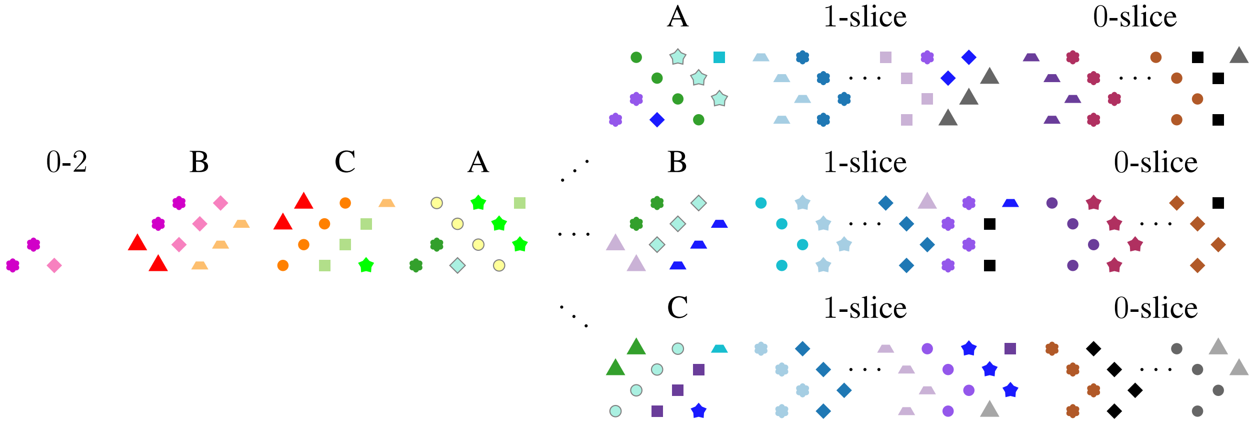

Figure B.7: 0-2 2-2 when m is Even

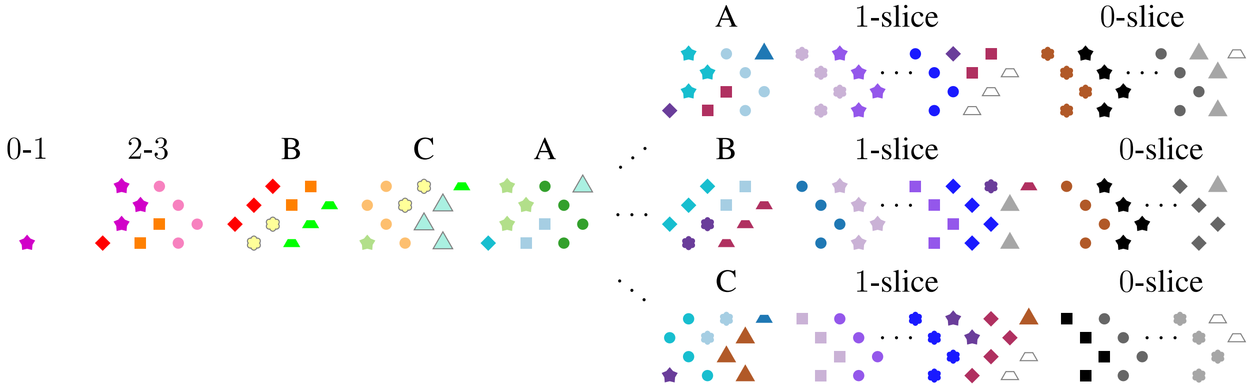

Figure B.8: 0-1 2-3 when m is Even