Combinatorial Methods for Undergraduate Mathematics Curriculum Analysis & Design

by

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Abstract

Student retention is a significant issue in Research in Undergraduate Mathematics Education (RUME), and freshman calculus is notorious for “weeding out” students from STEM majors. Since the late 1980s, various attempts have been made to remedy Calculus courses around the United States. However, the percentage of students earning grades of D, F, W, or I (the DFWI rate) remains relatively high compared to freshman courses in other disciplines. Through a collection of research projects, this dissertation explores the theme of misalignment in various aspects of the university Calculus I curriculum by employing novel applications of combinatorics to mathematics education research. This is an example of Mathematics Discipline-Based Education Research (Math DBER), which is an emerging field that leverages mathematics methodologies in RUME.

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Chapter 1

Introduction

1.1 The STEM Retention Problem

The work discussed in this dissertation is focused on *The STEM Retention Problem*.

> **The STEM Retention Problem** refers to the large number of students (recently estimated at 40-50%) that enter college intending to pursue a STEM degree but end up either completing a non-STEM degree or not completing any degree [30].

For decades, researchers have sought to understand the root causes of this problem. In a 2019 study [20], researchers compiled a list of 24 main reasons why students leave STEM. Nine of these were related to "issues of poor teaching, curricular design, and the negative climate of STEM" [20]. In this dissertation, I focus on three issues under this umbrella.

1. Negative effects of weed-out classes
2. STEM curricular design problems: pace, overload, labs, alignment
3. Conceptual difficulties with one or more STEM subject(s)

A *Large Foundational Course* (LFC) is a university STEM course that satisfies the following criteria: (1) Large: enrollment is typically greater than 100 students per section, (2) Lower Division: enrollment consists primarily of first and second-year students, (3) Established: the course has records going back more than four years, (4) Required: STEM majors must take the course [39].

The *DFWI rate* of a course is the proportion of initially enrolled students that earn a grade of D, F, W, or I at the end of the semester [39]. A subset of LFCs known as *Severe LFCs* satisfy the additional criteria of having a DFWI rate of 20% or higher [39]. In a 2019 study, Calculus and Chemistry were found to have the highest average DFWI rate (20%) of any LFC across multiple institutions [39]. In Auburn's Calculus I course (Math 1610), this rate has ranged from 30-40% in all Fall and Spring semesters since 2018, excluding the semester the university returned to in-person instruction from the COVID-19 Pandemic when the DFWI rate exceeded 50%.

Students call these classes *weed-out classes* and characterize them as having (1) assessments that are misaligned to content and understanding, (2) heavy volume and pace, (3) too much abstractness for an intro class, and (4) a reliance on rote learning and dull lectures, among others [39].

Calculus I is the focus of this dissertation because of its reputation as a weed-out course and its ubiquity across the university STEM curriculum. I approach the STEM Retention Problem from the perspective of a Mathematics Discipline-Based Education Researcher (Math DBER).

1.2 What is Math DBER?

Research in Mathematics Education has a long history, emerging as a discipline at the turn of the 20th century with the founding of the first national organization for Mathematics Educators in the United States (The Central Association of Science and Mathematics Teachers) and accompanying journal (*School Science and Mathematics*) [26]. This initial organization was focused on general mathematics and science education in primary and secondary schools. This was closely followed in 1908 by the founding of the *Mathematics Teacher*, a journal focused on general mathematics education that still exists today [26].

Mathematics Education Research grew into a global field in the late 1960s, with the establishment of international research journals specific to mathematics education (*Educational Studies in Mathematics* and the *Journal of Research in Mathematics Education*) [22].

Research in Undergraduate Mathematics Education (RUME) emerged in the 1980s when mathematicians such as Edward Dubinsky (originally a functional analyst) shifted their attention to undergraduate mathematics education research [29]. Since mathematics education research journals were already well-established, and many were reluctant to publish research on Undergraduate Mathematics Education (UME) topics [29], these first forays into the undergraduate realm borrowed their methodological approaches from existing mathematics education research in order to gain acceptance.

The emergence of RUME coincided with the "Calculus Reform Movement," which highlighted the teaching and learning of university calculus in the eyes of government [35] and professional stakeholders [32]. Furthermore, the establishment of the NSF-funded newsletter, *UME Trends: News and Reports on Undergraduate Mathematics Education*, disseminated research in undergraduate mathematics education freely to mathematics faculty nationwide [29].

After three successful conferences on RUME demonstrated widespread interest in the field [29], the Mathematical Association of America (MAA) formed the Special Interest Group (SIGMAA) on RUME in 2001 [31]. Since this time, the SIGMAA on RUME has hosted an rapidly expanding annual conference and regularly hosted special sessions at both the MAA's Mathfest and the Joint Mathematics Meetings (JMM) [29]. As recently as 2015, RUME expanded globally with the establishment of the International Journal on Research in Undergraduate Mathematics Education [21].

On the other hand, since *Discipline-Based Education Research* (DBER) disciplines (Chemistry, Physics, Geology, etc.) were not omnipresent in primary and secondary school curricula, the various DBER fields originally sought acceptance from their parent scientific disciplines rather than from education researchers. Thus, DBER is an area of scientific research that "investigates learning and teaching in a discipline using a range of methods with deep grounding in the discipline's priorities, worldview, knowledge, and practices. It is informed by and complementary to more general research on human learning and cognition" [34]. The first DBER area, Physics Education Research (PER), emerged in the 1960s and 1970s at the University of California, Berkeley, and the University of Washington and saw its first PhD graduates in the late 1970s [34]. Chemistry Education Research (CER) quickly followed suit, with its first PhDs graduating in the 1990s [34]. Other scientific disciplines' DBER areas have emerged more recently; Astronomy Education Research started graduating PhDs around 2012 [34]. As of 2012, six DBER areas were defined: Physics, Chemistry, Engineering, Biology, Geosciences, and Astronomy [34], with Mathematics notably absent.

In the past decade, faculty positions around the U.S. have been advertised as "Mathematics DBER" [5, 6, 25, 27, 38]. This is largely an administration-driven initiative to complete institutions' collections of STEM disciplines within the DBER umbrella. Math DBER has also been identified in scholarly publications as synonymous with RUME as depicted in Figure 1.1 [19].

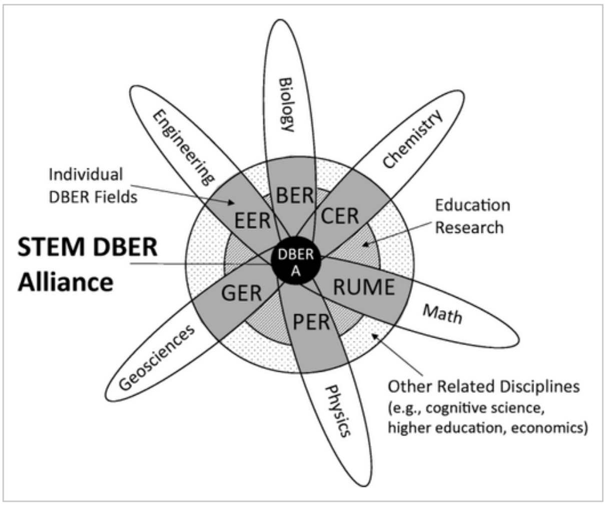


Figure 1.1: The relationship between DBER fields. Math DBER is labeled as "RUME" [19].

However, I argue that this characterization of Math DBER is not entirely accurate, and that RUME and Math DBER are two related but distinct areas of focus. While RUME and Math DBER share interests, topics, and even foundations (recall that RUME was initially investigated by career mathematicians), Math DBER seeks to employ methodologies unique to the mathematics discipline in order to study undergraduate mathematics education phenomena. For someone like me, educated as a theoretical discrete mathematician, this means applying my expert knowledge of combinatorics to investigate the analysis and design of undergraduate mathematics curricula. I am a mathematician, and I find novel ways to use mathematics to study mathematics education (see Figure 1.2).

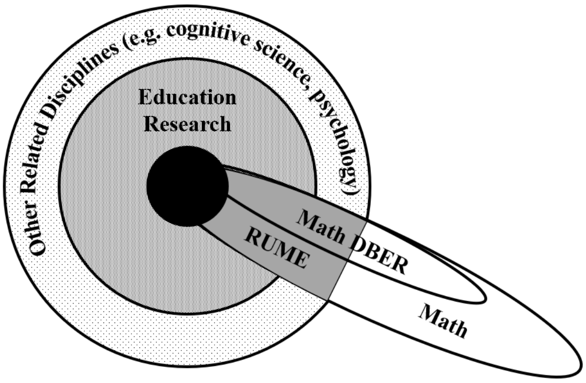


Figure 1.2: The distinction between Math DBER and RUME.

In this dissertation, I demonstrate the distinctive potential of Math DBER by employing various ideas from discrete mathematics to analyze and redesign the university Calculus I curriculum. Since undergraduate mathematics education has been widely understudied from this perspective, it invites the possibility of rich interdisciplinary collaboration between Mathematics and RUME.

Traditionally, there has been a lack of communication between Mathematics and RUME, as demonstrated by the invisibility of undergraduate mathematics education efforts in the political arena [7]. Furthermore, the characterization of mathematics education researchers as “muggles” in university math departments [12] highlights the cultural differences between these two research areas.

One example of a difference between these two fields is how researchers choose to write their publications. In mathematics, it is common for authors to use “royal we”. That is, the pronoun “we” is typically used regardless of the number of authors on the publication. However, in mathematics education, it is common for authors to use a pronoun that delineates the number of authors; if the paper has a single author, “I” is used, and “we” is only used if the paper has multiple authors. Each chapter in this dissertation will use one of these two conventions, depending on the intended audience.

Because of its interdisciplinary nature, the field of Math DBER has the potential to act as an intermediary between RUME and Mathematics. It not only fosters forward communication of mathematics methodologies to RUME through a shared interest in research topics but also conversely communicates ideas from Research in Undergraduate Mathematics Education to Mathematics through the use of methodologies grounded in the mathematics discipline.

In this way, Math DBER offers a bridge between mathematics education and mathematics itself, two disciplines that have stark cultural differences, yet share a common curiosity [12].

1.3 Combinatorial Methods are not Common in Math Education Research

A *combinatorial method* is a research methodology that leverages definitions, theorems, and ideas from the branch of mathematics called Combinatorics. Since many of the methods in this dissertation specifically rely on graphs, digraphs, and networks, it is useful to study a subset of combinatorial methods known as graph-theoretic methods. An abbreviated version of this subsection has been published as a solo-authored poster proposal in a set of peer-reviewed conference proceedings [15]. This subsection is written in the math education convention; "I" is used to signify a single contributor.

Leveraging the definition of a graph [10], a *graph-theoretic method* is a research method that makes use of a set of objects and the relationships between them [15]. To assess the prevalence of graph-theoretic methods in mathematics education research, I conducted a metasummary, a type of qualitative data analysis, on proceedings from prominent mathematics education conferences in North America (SIGMAA-RUME) and Europe (CERME) [15]. Out of the collection of 3,940 articles published in SIGMAA-RUME and CERME conference proceedings from 2015 to 2022, only 80 (2%) were determined to employ graph-theoretic methods. The authors of these 80 articles represented 21 different countries located on five different continents, giving this metasummary a rich international scope.

Though it was found that mathematics education research has employed graph-theoretic methods in the past, the vast majority of these methods did not consider the graphs as mathematical objects. Open coding [37] was used to classify the prominent graph-theoretic methods among the 80 proceedings articles. From this analysis, six themes emerged.

1. **Data Visualization** occurred when researchers used graphs to visualize results of an analysis that had already been conducted. 5/80 articles (6.25%) were classified as using Data Visualization. See [36] for an example.

2. **Flowcharts** were directed graphs that researchers analyzed without any reference to mathematical digraphs. These were used predominantly to visualize the structure of processes. 1/80 articles (1.25%) utilized Flowcharts. See [2] for an example.

3. **Graphs** were used when researchers specifically referred to the structures as "graphs" rather than "maps" or "networks". Most of these still only considered graphs in a visual capacity, but a very small minority did think about them mathematically. 12/80 articles (15%) used Graphs. See [8] for an example.

4. **Grids** were tables or charts that could be considered as adjacency matrices or tables of a graph, which was not visualized. These went by various context-specific titles. 7/80 articles (8.75%) used Grids. See [23] for an example.

5. **Maps** were the undirected analogue of flowcharts. These were used only in a visual capacity with no reference to mathematical graphs. 38/80 articles (47.5%) used Maps. See [11] for an example.

6. **Network Analysis** articles specifically referenced the methodology as "network analysis". This was predominantly in a Social Network Analysis context. 16/80 articles (20%) used a Network Analysis methodology. See [4] for an example.

Though graph-theoretic methods were determined to be rare in mathematics education research, across the 3,940 SIGMAA-RUME and CERME proceedings articles published from 2015-2022, the word "network" appears 1,861 times, "connection" appears 5,553 times, and "relationship" appears 8,935 times. This suggests that mathematics education researchers subconsciously view their body of research as a graph, even if they do not engage with graph-theoretic research methods.

In this metasummary, I also wanted to determine the most popular topics in mathematics education research for using graph-theoretic methods. To do this, I modeled the topics present among the 80 articles as a graph (see Figure 1.3).

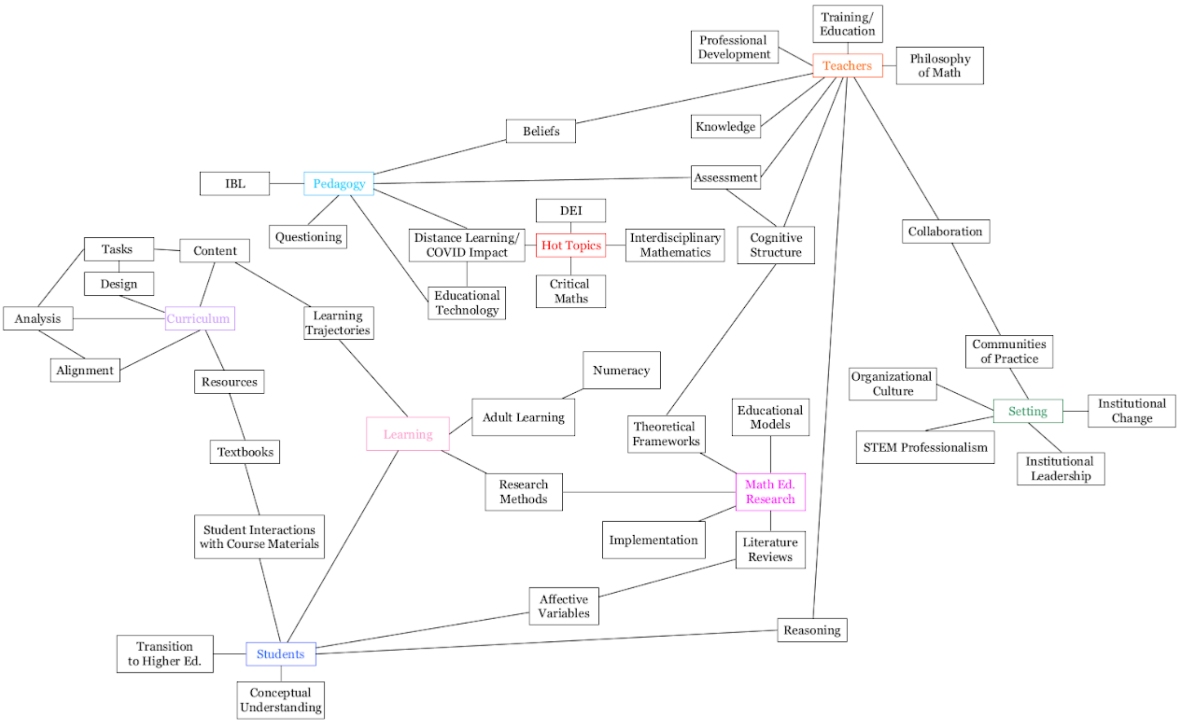


Figure 1.3: Network of research topics using graph-theoretic methods.

Note: The articles analyzed were from SIGMAA-RUME and CERME conference proceedings, 2015-2022.

First, open coding [37] was used to define the vertex set. Then, I drew an edge between two vertices if an article discussed the two topics. Then, I used degree-centrality analysis to determine vertices of highest relative degree (the major themes among the topics). From this analysis, I determined eight areas of focus, which were not mutually exclusive (the entire graph was connected).

1. **Curriculum** including design, content, analysis, and alignment of curricula. See [3] for an example.

2. **Hot Topics** including DEI, Distance Learning & COVID Impact, Critical Maths, and Interdisciplinary Mathematics. See [9] for an example.

3. **Learning** including learning trajectories, adult learning, and student learning. See [24] for an example.

4. **Mathematics Education Research** including Educational models, theoretical frameworks, implementation, literature reviews, and research methods. See [18] for an example.

5. **Pedagogy** including Inquiry-Based Learning, assessment, questioning, teacher beliefs, Educational technology, and COVID Impact. See [1] for an example.

6. **Setting** including communities of practice, institutional change, organizational culture, and STEM professionalism. See [4] for an example.

7. **Students** including students' interactions with course materials, student learning, student conceptual understanding, and transition from secondary to higher education. See [28] for an example.

8. **Teachers** including teachers' training, knowledge, professional development, cognitive structure, and philosophy of math, as well as teacher collaborations and assessment. See [33] for an example.

The wide variety of topics in which graph-theoretic methods have already been used suggests that methodologies in this vein have the potential for a wide variety of applications in mathematics education research.

1.4 Research Goals

This dissertation introduces novel methods for analyzing and redesigning the undergraduate mathematics curriculum that rely on ideas from Discrete Mathematics, or Combinatorics. Though these methods could be generalized to other courses, this dissertation focuses on Calculus I due to its ubiquitous presence in the undergraduate mathematics curriculum and across STEM disciplines.

Since this dissertation's research is interdisciplinary, we define a couple of basic combinatorial concepts that are foundational to understanding the later chapters. A *graph G* is a finite nonempty set V of objects called *vertices* together with a possibly empty set E of 2-element subsets of V called *edges*. A *directed graph*, or *digraph*, D, is a finite nonempty set V of objects called *vertices* together with a possibly empty set E of ordered 2-element subsets of V called *edges* (or *arcs*). A *graph drawing* is a representation of a graph G in $\mathbb{R}^2$ that uses points to represent vertices and lines (or arrows for a digraph) that connect pairs of vertices to represent edges.

1.5 Thesis Structure

This dissertation explores the theme of misalignment in various aspects of the university Calculus I curriculum by employing novel applications of combinatorics to mathematics education research. The remainder of this dissertation is organized into four standalone chapters.

Chapter 2 presents two textbook analysis studies on discipline-specific Calculus. Both of these studies use network analysis to identify sources of misalignment in Calculus textbooks. The first looks at differences between different scientific disciplines' instantiations of Calculus. The second introduces a novel metric for task difficulty to analyze differences between two textbooks meant for Calculus for Life Sciences (or Biocalculus).

Chapter 3 presents three studies focused on Calculus I Derivative Rules. The first two studies present analyses of General Calculus textbooks using various concepts from Graph Theory and Poset Theory. The third study presents new results in Combinatorial Design Theory that apply to constructing Calculus I task sets on the Chain Rule.

Chapter 4 presents a scholarly reflection on adjusting the order of topics in a Calculus I curriculum using Graph Isomorphisms and the subsequent development of a research-backed Calculus I workbook, which was implemented in one coordinated section of Calculus I in Fall 2023 with positive results.

Chapter 5 presents a study about perceptions of the quality of Calculus I homework from the perspective of mathematics faculty and graduate student instructors and the impact of a novel instrument for measuring course alignment on these perceptions.

Some results contained in this dissertation can be found in published and/or accepted peer-reviewed articles [44, 15, 46] as well as works still undergoing peer review [14, 80, 17].

Chapter 2

Textbooks Can Be Analyzed Using Networks

In this chapter, I discuss two projects which are primarily focused on disciplinary variations of Calculus I textbooks. However, theoretically, the methods presented in this chapter can also be applied to general Calculus I textbooks. The results in this chapter are published in short peer-reviewed conference proceedings articles [44, 46] and are under review as part of a peer-reviewed book chapter [45].

2.1 Discipline-Specific and General Calculus Books are Structurally Different

Abbreviated results of this section are published as joint work with Dr. Melinda Lanius in a 4-page peer-reviewed conference proceedings article [44]. I spearheaded the project and contributed the network analysis. Melinda contributed the narrative analysis. This section is written using the math education convention, where "we" indicates that the published version has more than one author.

2.1.1 Introduction

Because life science students are historically outperformed by their peers in calculus [42], many colleges and universities now offer a discipline-specific calculus course, with core calculus concepts motivated by and contextualized within life sciences [49]. Because motivation can improve student engagement [40], a 'biocalculus' theoretically should improve performance outcomes for biology majors. However, preliminary evidence indicates this curriculum is not effective at improving life science students' academic performance [42, 49]. To investigate, we will compare a biocalculus to a business and standard curriculum. Because textbooks are a near-universal course component that mediates teaching [50], we will use textbook analysis to explore structural differences in presenting a core concept of calculus: the derivative. Our research question is: How do discipline-specific calculus textbooks develop the definition of the derivative compared to a general calculus text? Our corpus is *Calculus: Early Transcendentals, eighth ed.* by Stewart [53], which is typically presented to math, engineering, physics, and chemistry majors; *Calculus for Business, Economics, and the Social and Life Sciences, brief tenth ed.* by Hoffmann & Bradley [47]; and *Calculus for Biology and Medicine, fourth ed.* by Neuhauser & Roper [51]. We refer to these texts as CALC, BUSCALC, and BIOCALC, respectively.

2.1.2 Narrative Analysis

To compare how the three texts present the definition of the derivative, we conducted a narrative analysis using the graph framework developed by Weinberg, Wiesner, and Fukawa-Connelly in [58]. In brief, narrative analysis is a technique borrowed from literature studies that considers the sequence of ideas and how earlier items influence and shape the presentation of later concepts. We coded the following key ideas in the presentation of the definition of the derivative:

A. secant lines limiting to a tangent line

B. formula for the slope of a tangent line

C. instantaneous velocity at a point

D. derivative at a point

E. instantaneous rate of change at a point

F. the derivative function

In Figure 2.1, a solid arrow indicates that the tail idea was utilized in presenting the idea at the head of the arrow. A dashed arrow represents an imprecise motivation rather than an explicit linking of ideas.

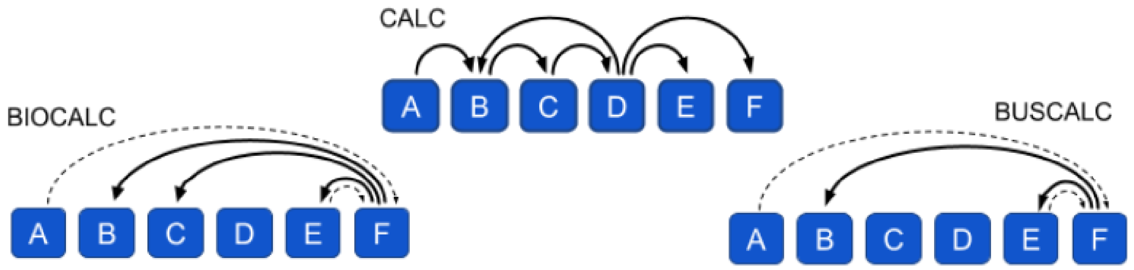


Figure 2.1: Narrative graphs for the definition of the derivative in each text.

The graphs reveal that narratively, BIOCALC and BUSCALC were quite similar, painting a motivating picture before beginning their rigorous discussion with the definition of the derivative function. From this function starting point, the reader is then shown numerous "applications" of the derivative. On the other hand, the CALC text assembles the derivative function by piecing together the point-wise derivative and directly links the concept of slope of a tangent line, instantaneous velocity, and the derivative at a point.

2.1.3 Network Analysis

To analyze the development of the definition of the derivative in each textbook, we constructed a directed graph, or digraph, in yEd (Version 3.22, yWorks GmbH) as follows: the vertices of the digraph are mathematical objects (concepts, definitions, theorems, examples, and exercise types) organized into four groups ordered from left to right: Prior Knowledge (not in the text), Prerequisites (in the text), Section (introducing the limit definition of the derivative), and Exercises (for the section). We drew an arrow from one vertex, x, to another vertex, y, if x is required to learn y.

Figures 2.2, 2.3, and 2.4, show the digraphs for CALC's, BUSCALC's, and BIOCALC's development of the definition of derivative, respectively. For clarity, the vertex corresponding to the definition of the derivative is bolded in black and all arrows from a prior knowledge vertex to an in-text vertex are colored blue.

Visual Inspection

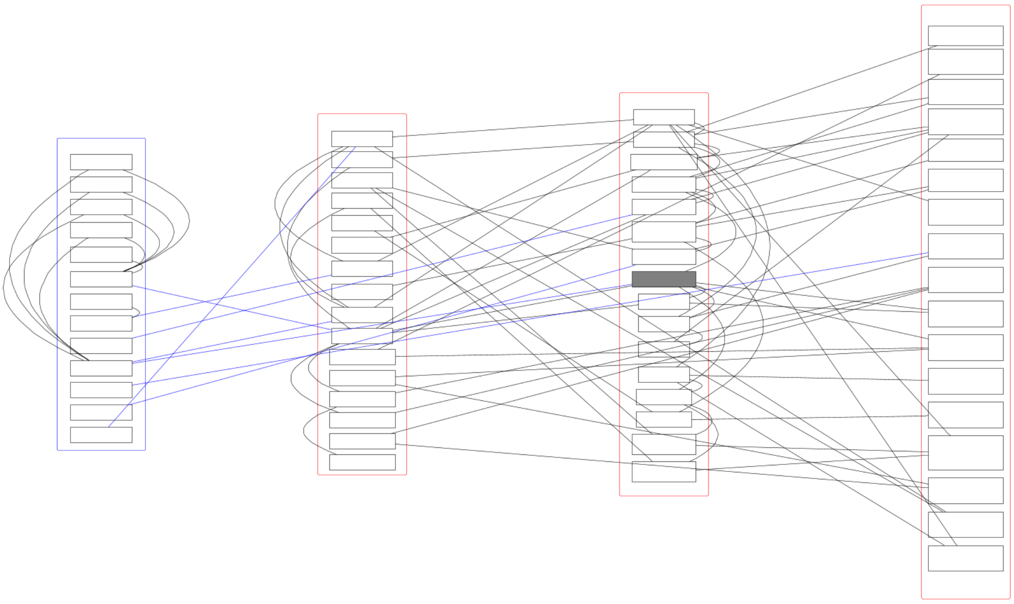


Figure 2.2: Development of the definition of derivative in CALC

Initial visual inspection of the digraphs revealed that CALC required the most prior knowledge not in the text (see Table 2.1). However, BUSCALC required the most prerequisite knowledge in the text. This is because many of the concepts included in BUSCALC were assumed to be prior knowledge in both CALC and BIOCALC. BIOCALC had the least number of concepts presented in the section covering the definition of the derivative as well as the least variety in types of exercises.

Textbook	v(PK)	v(PQ)	v(S)	v(EX)
CALC	13	16	16	17
BUSCALC	8	20	15	12
BIOCALC	10	10	8	8

Table 2.1: Frequency of PK, PQ, S, and EX vertices in each textbook's digraph model.

Since a goal of discipline-specific calculus is to apply calculus concepts to other disciplines, we analyzed each textbook's attention to application problems by computing the proportions of example problems and exercises on applications (App.) in each textbook's section on the definition of the derivative (see Table 2.2).

Each black arrow between categories in the digraph indicates in-text preparation for successive concepts. Thus, a quick measure of how well the text prepares the reader for exercises

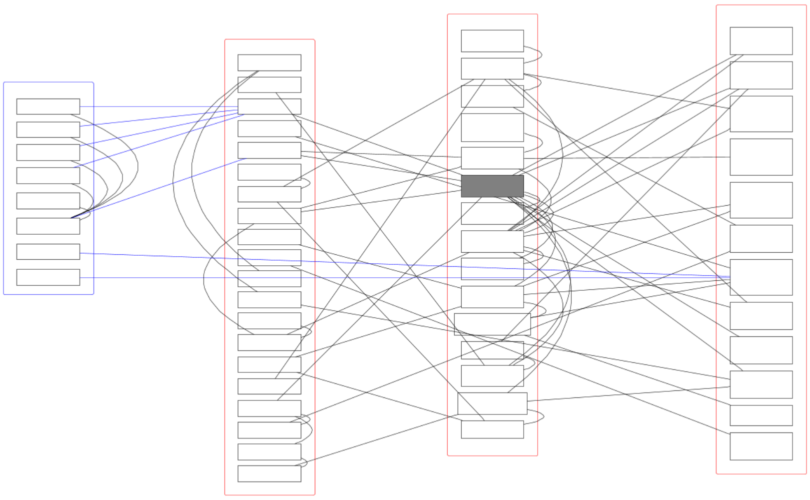


Figure 2.3: Development of the definition of derivative in BUSCALC

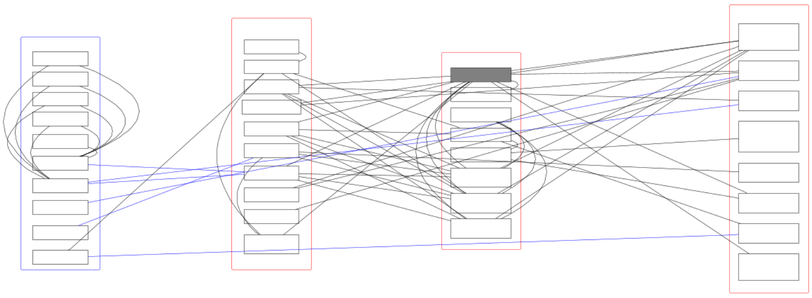


Figure 2.4: Development of the definition of derivative in BIOCALC

relating to the definition of the derivative may be gleaned by counting the number of black arrows pointing to each exercise vertex (the indegree of a vertex). In CALC, the exercise vertex with maximum indegree was "Find the equation of the tangent line to the curve at the given point" with four arrows. In BUSCALC, there was a three-way tie between "Find the rate of change at the given point", "Application word problems", and "Proofs involving differentiation" each with 3 arrows. In BIOCALC, "Find the derivative of a function at the given point" had maximum indegree with six arrows. Furthermore, in BIOCALC, most black arrows clustered at just two exercise vertices, indicating more variation in preparedness for certain exercise types than CALC and BUSCALC.

Centrality Analysis

Following the initial visual inspection, we used two measures of centrality to analyze the digraphs: degree and node betweenness. For both measures, a value of 1 indicates the topic(s)

Textbook	n(App. Exm.)	n(Exm)	% App. Exm.	n(App. Exr.)	n(Exr)	% App. Exr.
CALC	3	7	43%	20	61	33%
BUSCALC	3	7	43%	15	65	23%
BIOCALC	0	3	0%	0	38	0%

Table 2.2: Proportions of application examples and exercises in the section on definition of the derivative

most emphasized in the presentation of the definition of the derivative. Degree centrality measures the number of arrows connected to a vertex (degree of a vertex) relative to the maximum degree in the digraph. In CALC, two topics had degree centrality 1 (“Geometric interpretation of secant & tangent lines” and “Definition of derivative”), whereas in BUSCALC and BIOCALC, “Definition of derivative” was the lone topic with degree centrality 1. To explain node betweenness centrality, imagine that a vertex is a train station. A train station is “between” a pair of cities if we must stop there to travel between the two cities. The number of pairs of cities that a given train station is between is the train station’s “betweenness”. Taking this measure relative to the maximum betweenness among all train stations is node betweenness centrality. All three texts had exactly one topic with node betweenness centrality 1. In CALC, it was “Geometric interpretation of secant & tangent lines”, while in both BUSCALC and BIOCALC, it was “Definition of derivative”. The centrality analysis results suggest that CALC contains two concepts that compete for importance in the development of the definition of derivative, while BUSCALC and BIOCALC focus most on the definition of the derivative itself.

2.1.4 Discussion

Through our narrative and network analysis of general calculus, business calculus, and biocalculus textbooks, we found significant structural differences between the general text and the two discipline-specific texts. The discipline-specific texts emphasized the definition of the derivative most, while the general text emphasized the definition’s geometric roots so much that this competed for importance with the definition itself. More analysis needs to be done to determine whether these competing topics benefit or hinder learning. Furthermore, between the two discipline-specific texts, there were differences in presenting the definition of the derivative, namely in emphasis on application problems and exercise preparedness. The business calculus text both emphasized applications more and better prepared the reader for its exercises concerning the definition of the derivative than its biocalculus counterpart. These structural differences in the texts may explain the lack of improvement in performance outcomes for life science students. Curriculum designers should note when selecting textbooks: the stated aim of a biocalculus course, with core calculus concepts remaining but with life science context, was not supported by the BIOCALC text. Further, our analysis suggests education researchers should be careful to decouple motivation from narrative structure differences when comparing student performance outcomes between calculus and discipline-specific calculus courses.

2.2 Different Biocalculus Books Take Different Approaches to Task Design

After receiving feedback at the *Teaching and Learning of Calculus Across Disciplines* conference on the "exercise preparedness" analysis in Section 2.1, I wanted to unpack the meaning of "exercise preparedness" further. This section presents this extended idea in the context of Biocalculus texts. Abbreviated results from this section are published in a short solo-authored peer-reviewed conference proceedings article [46]. In this section, the math education convention is used, where "I" indicates that the published version has a single author.

2.2.1 Introduction

Researchers have used various approaches to predict the difficulty of mathematical tasks including statistical modeling [48], hand-rubrics [57], and machine learning [56]. By contrast, instructors tend to judge mathematical tasks through a series of intuitive factors [54]. However, because of their lack of experience, novice instructors' intuition is underdeveloped compared to experienced instructors [41], which may affect the reliability of their judgement in task difficulty. The notion of difficulty explored in this section addresses how well a textbook prepares its reader to complete a particular task. For example, providing the reader with a similar fully-worked example gives the reader a source of "mimicry" which classifies a task as an exercise. However, the absence of such examples, which instead requires synthesis of theoretical material (i.e., definitions and theorems), characterizes a task as a problem [52]. This notion of preparedness was explored previously [44] using graph-theoretic methods, but the "exercise preparedness" measurement used did not distinguish to what degree a task could be characterized as an exercise or a problem. The aim of this section is to define and demonstrate the use of a new measurement of task difficulty, the difficulty ratio, which quantifies this distinction. Such a measure may be particularly useful for novice instructors to complement their still-developing intuition.

2.2.2 Theoretical Perspective

Worked Example Effect

The definition of the difficulty ratio given in the next section draws on the theory of Worked Examples from Cognitive Load Theory. A worked example is merely a solution to a task with all the steps shown, usually consisting of a problem statement followed by a procedural solution [55]. In Cognitive Load Theory, evidence of a Worked Example Effect occurs when learners perform better when given worked examples to study [55]. In the definition of the difficulty ratio, there are two types of logical connections measured for a particular task: theoretical and exemplary. Theoretical connections (or an absence of worked examples) imply that the reader must synthesize ideas from the text to solve the task, while exemplary connections imply that the reader may solve a task by mimicking a provided example from the text. Through the lens of Cognitive Load Theory's Worked Example Effect, it is assumed that exemplary connections detract from a task's difficulty, while theoretical connections contribute to a task's difficulty.

Graph Theory

In the analysis that follows, we model textbooks with digraphs (or directed graphs). The definition of a Textbook Structure Map, TSM(T), is as follows.

Let $T = (K, L)$ be a digraph where the set of vertices K is the set of knowledge units presented in a particular textbook, and the set of directed edges L is the set of logical connections between knowledge units (represented with arrows). Then, TSM(T) is a drawing of T such that:

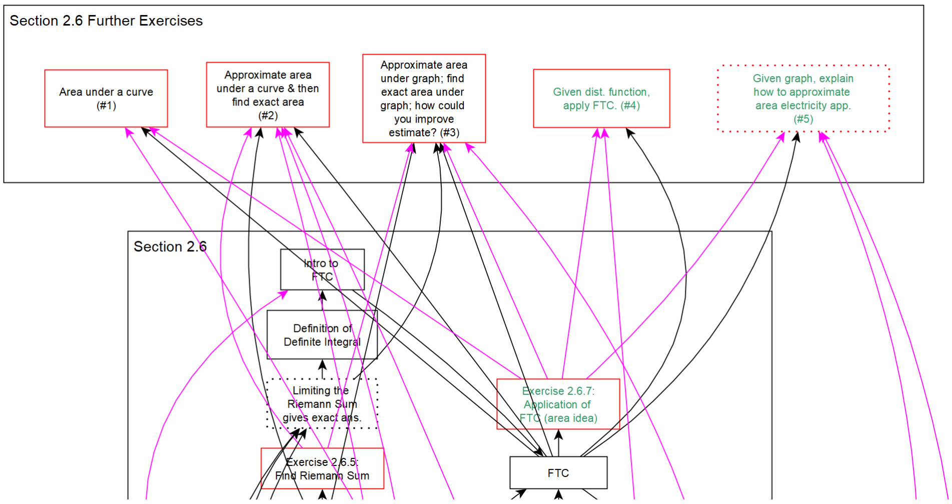


Figure 2.5: A sample from TSM(T2).

1. Knowledge units are positioned: (a) in clusters from bottom to top in order of section, (b) from left to right within a section cluster in order of subsection, and (c) from bottom to top within a subsection in order of appearance.

2. Text color of vertex labels signifies applied knowledge units vs. non-applied knowledge units.

3. Border color of a vertex signifies knowledge unit type: prerequisite, theory, example, task, or required technology use.

4. Border style of a vertex signifies the dominant representation used in the knowledge unit: algebraic, visual, verbal, or numerical.

5. Edge color signifies the type of logic used between two knowledge units: theoretical (one unit requires use of an idea, definition, theorem, etc.) or exemplary (one unit demonstrates how to complete another).

The definition above reflects the vertex and edge types present in this analysis. However, note that this definition is adaptable to meet the level of precision required by the instructor (or researcher). Figure 2 presents a sample from a TSM.

In a TSM, the difficulty ratio of a task vertex p is the ratio of theoretical (black) to exemplary (pink) arrows pointing to p,

$$\operatorname{diff}(p) = \frac{\operatorname{in}_t(p)}{\operatorname{in}_e(p)}$$

with $\operatorname{diff}(p) = \infty$ if $\operatorname{in}_e(p) = 0$. This ratio rates textbook tasks on a scale $0 \leq \operatorname{diff}(p) < \infty$ where values close or equal to 0 indicate exercises that can be completed merely by mimicking provided example solutions (e.g., tasks that are less difficult from a logical standpoint). For example, the task vertex $p =$ “Area under a curve (#1)” in the top left corner of Figure 2.5 has $\operatorname{diff}(p) = 0.5$ because one black arrow (theoretical) and two pink arrows (exemplary) point into p.

2.2.3 Analysis of Tasks

For this analysis, two Biocalculus textbooks from different publishers were studied: (T1) *Calculus for Biology and Medicine 4th edition* by Neuhauser and Roper [51] and (T2) *Modeling Life: The Mathematics of Biological Systems* by Garfinkel, Shevtsov, and Guo [43]. Biocalculus was chosen specifically because the Auburn University Department of Mathematics & Statistics is currently considering adding a new Biocalculus course. Comparable portions of each textbook concerning integration (Sections 6.1, 6.2, 6.3.1 from T1 and Section 2.6 from T2) were used to draw the corresponding portions of TSMs. Due to space restrictions, it was infeasible to include the entire drawings here, but a sample from T2 is presented in Figure 2.5. After constructing the TSM portions, the difficulty ratio was calculated for all tasks contained in each drawing. Note that one task vertex can represent multiple tasks with the same instruction set. Then, for each book, a histogram was constructed to visualize the distribution of the difficulty ratio among Integration tasks (see Figure 2.6).

Finally, the histograms’ similarity was measured by computing the percentage of overlap between them (68%), meaning they exhibit 32% dissimilarity.

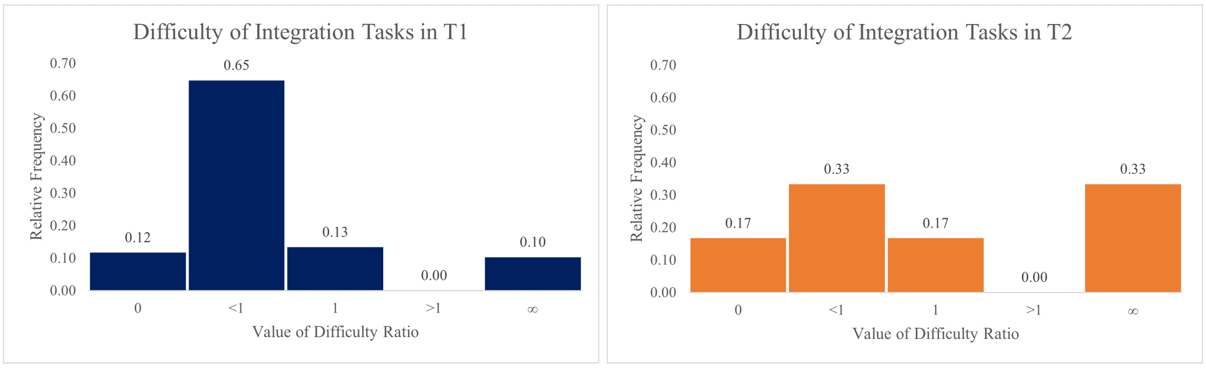


Figure 2.6: Relative Frequency Distributions of the Difficulty Ratio in T1 and T2.

2.2.4 Discussion, Extensions, & Generalizations

Two notes on similarity are (1) most tasks in both books were exercises $0 < \text{diff}(p) < 1$ and (2) there were no tasks with $1 < \text{diff}(p) < \infty$ in both textbooks. It would be interesting to see if these are common trends among mathematics textbooks. The majority of the 32% dissimilarity between the two histograms occurred in the "< 1" bin and the "∞" bin. Tasks with a difficulty ratio less than 1 are exercises that require minimal theoretical knowledge and have plenty of examples to reference. Tasks with difficulty ratio ∞ are problems with no examples to reference. Despite initial observations, the dissimilarity in the histograms becomes most interesting when the textbooks are placed in context.

While both textbooks are meant for use in Biocalculus courses, T1's target audience is specifically Biomathematicians, while T2's target audience is specifically Biologists who are taking math courses. It is interesting to note, then, that T1 contained far more "< 1" exercises and far less "∞" problems than T2, when I would expect the opposite. Since math courses are outside Biologists' area of specialization, I would expect them to need more support in mathematics task-solving than Biomathematicians. This large number of "< 1" exercises in T1 can be attributed to an enormous chunk of definite and indefinite integration exercises in Section 6.2.

The differences between the books' values of the Difficulty Ratio indicate that, on the topic of Integration, T1's approach consists almost entirely of exercises and can be characterized as drill-oriented, procedure-heavy, and calculation-based. While T2's approach also consists of mostly exercises, it appears to focus more on conceptualization and problem novelty than

T1. This hints at a disconnect between the curriculum of undergraduate math courses and the necessary skills for pursuing a career in mathematics, which is what I would expect of Biomathematics students. It would be an interesting future project to see if this hint becomes more defined with a larger sampling of textbooks.

On a final note, the definition of the Difficulty Ratio can be generalized for any number of arrow types an instructor (or researcher) may want to use by dividing the number of arrows pointing into a task that contribute to its difficulty by the number of arrows pointing into a task that detract from its difficulty.

2.3 Conclusion

In this chapter, we showed how digraph models of textbooks can be analyzed using innovative combinatorial network analysis techniques. Specifically, we refined a general measure of exercise preparedness to the difficulty ratio, which took into account different types of "preparedness". Both of these measures depended on the indegree of specific vertices in the digraph model of the text. These methods illuminated subtle differences between textbooks designed for various discipline-specific calculus courses and those intended for the same version of calculus within a particular discipline.

The two textbook analyses presented in this chapter revealed a misalignment between the calculus textbooks' intended audience and the text's structure. Misalignment has been reported as a key inhibitor of student success in entry-level courses. Thus, the approaches to textbook analysis presented in this chapter could profoundly impact mathematics departments' textbook adoption decisions, thereby promoting student success in undergraduate mathematics courses.

Chapter 3

Homework Can Be *Designed* Using Graph Decompositions

In this chapter, I explore the idea of intentional homework design to ensure students are getting maximal benefit from doing their homework, focusing on Calculus I derivative computation tasks. First, I analyze existing Calculus I textbook tasks and identify misalignment between what these tasks require of students and how mathematicians (who typically teach Calculus I courses) believe students should be assessed on such tasks. Then, I present new results in combinatorial designs in order to construct desired task sets.

3.1 Calculus Books *Love* the Power Rule

Abbreviated results from this section are published as a solo-authored poster proposal in peer-reviewed conference proceedings [77]. In this section, the math education convention is used where "I" indicates that this project features contributions from a single author.

3.1.1 Introduction

Despite calls for reforming Calculus instruction to stress conceptual understanding over procedural fluency [75, 101], procedural fluency in the computation of derivatives remains a ubiquitous foundational skill in Calculus I [67]. Throughout this section, I refer to tasks of the form "Find the derivative of [insert function]" as Derivative Computation tasks, abbreviated DC tasks. Furthermore, in tertiary-level mathematics, textbooks often serve as encyclopedic sources of curricular material, which instructors must whittle down into courses [94]. Since it is widely accepted that procedural fluency improves with repeated exposure [76], it may be inferred that the textbook tasks instructors choose to assign to students as homework will directly influence student proficiency, with an unbalanced emphasis among skills potentially leading to unbalanced proficiency. In this paper, I answer the following research question concerning the nature of DC tasks in Calculus I textbooks: What skills are students exposed to (and in what proportions) among DC textbook tasks? The goal of this analysis is to give Calculus instructors cause for reflection on the foundational tasks they assign to students and, in turn, proactive foresight into their students' learning.

3.1.2 Literature Review

A graph-theoretic method is a research method that directly employs the relationships within a set of objects [79]. While attention to the relationships between ideas in mathematics education research is prevalent, graph-theoretic methods are rare, accounting for under 2% of publications in SIGMAA-RUME (North American) and CERME (European) conference proceedings since 2015 [79]. In task analysis, Maps (including variations of concept maps) have been applied to study Linear Algebra [104], but I found nothing that analyzes Calculus I tasks using the notion of graphs. However, Calculus I textbook tasks have been investigated extensively to determine textbooks' alignment to curricula [93] and concerning cognitive categorizations like Bloom's Taxonomy [59]. I introduce a class of directed graphs called Elementary Function Trees to offer a novel approach to Calculus task analysis involving graph-theoretic methods.

3.1.3 Elementary Function Trees

An *elementary function* is a function comprised of a finite collection of field operations (addition, subtraction, division, extracting roots) and simple functions (constant functions, algebraic functions, exponential functions, and the inverses of these types) [70]. Let f be an elementary function. Then, I define the *Elementary Function Tree* of f, EFT(f), as the directed graph (or digraph) whose vertex set is the collection of simple functions and commutative operations contained in f, and for two vertices u and v, there exists a directed edge (arrow) from u to v if v is contained in u. Figure 3.1 presents an example of an EFT. The vertex labels and the dotted arrow representing a reciprocal function will be explained in more detail later.

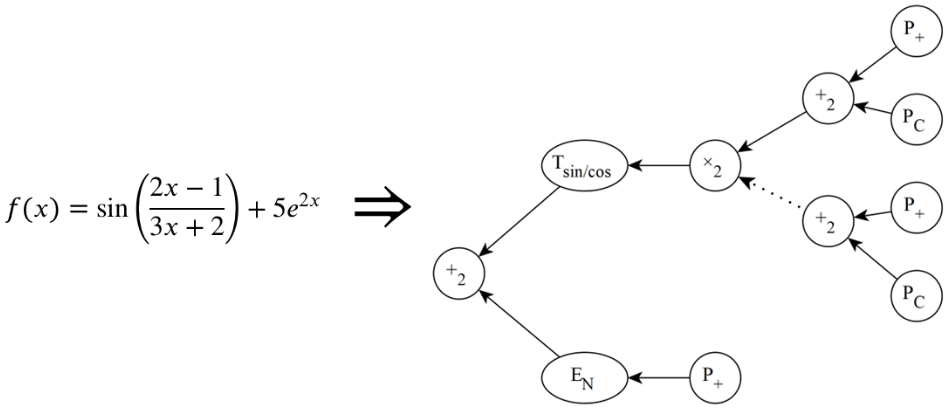


Figure 3.1: An example of an EFT (Analysis Version)

3.1.4 Theoretical Perspective

The basis of my design choices for EFTs lies at the intersection of Harel's theory of teacher knowledge and the psychology of student learning and perception of mathematical content through Schemas and Symbolic Forms. Harel & Lim [83] define teacher knowledge as consisting of three parts: mathematics content, epistemology, and pedagogy. This section focuses on teachers' knowledge of epistemology, which is the understanding of students' learning of mathematics, including psychological principles of learning [83].

Schema Theory

In this section, the underlying assumptions of human cognition are based in the work of Piaget and Bartlett and rely on the notion of schemas – for a formal definition, see [72]. However, one way to think about Schema Theory is to imagine the brain as a computer whose memory stores and organizes information in a file directory. The file directory of a novice learner, such as a Calculus I student, will contain numerous file folders related to Calculus content. For example, in the case of derivative rules, there might be separate file folders for the Power Rule, the Product Rule, the Quotient Rule, and the Chain Rule. Furthermore, the Power Rule folder might be multiple folders, with one for positive exponents, one for negative exponents, and yet another for rational exponents. However, with adequate exposure over time, the information contained in these separate folders gets compressed into one single folder about Derivative Rules in the fashion of an expert learner, such as a Calculus I instructor. The Elementary

Function Trees introduced in this paper consider the structure of elementary functions from a novice learner's (or maximum file folders) point of view, meaning that each simple function type will contain multiple sub-types.

Symbolic Forms

In physics education research, Sherin poses the notion of *Symbolic Forms* [97], which are knowledge elements consisting of a symbol template and conceptual schema. In this case, a derivative rule like "differentiate term-by-term" is associated with the symbol template $\square + \square$. Although Symbolic Forms were initially developed in physics education research, because they involve understanding of mathematical content, this theory has also been applied to mathematics education research and across other disciplines [95]. Symbolic Forms motivate the inclusion of operations of any kind in the vertex set of EFTs. However, I chose to only use commutative operations as vertices because the position of vertices in a digraph is not significant. Noncommutative operations, such as division, are represented in terms of commutative operations, such as multiplication by a reciprocal, as nontrivial sub-structures in the EFT. For example, in Figure 1, the sub-structure with vertex set $\{\times_2, +_2, +_2\}$ and edge set $\{+_2 \to \times_2, +_2 \to \times_2\}$ represents a quotient. For commutative operation vertices, together with noncommutative operation nontrivial sub-structures, I use the collective term *operation structures*.

3.1.5 Analysis

For this analysis, I included the DC tasks in three popular Calculus textbooks in the United States [87, 99, 100]. In addition to their popularity, these books were chosen to represent different publishers and decades. Since "The Odds" is the proverbial textbook task set assigned to students [68, 63], I chose to analyze the textbooks' odd-numbered DC tasks to simulate a student's realistic exposure to DC skills in a Calculus I course. Table 3.2 lists the tasks from each textbook included in this analysis.

For each task, its EFT was drawn. First, open coding (Thornberg & Charmaz, 2014) was used to classify the simple functions of each EFT as follows:

Stewart		Larson & Edwards		Strang & Herman	
§3.1	#3-32 odd	§2.2	#3-24, 39-54 odd	§3.3	#106-117 odd
§3.2	#3-26 odd	§2.3	#1-12, 25-54 odd	§3.5	#175-184 odd
§3.3	#1-16 odd	§2.4	#7-36, 45-66 odd	§3.6	#228-237 odd
§3.4	#7-46 odd	§5.1	#47-76 odd	§3.7	#279-288 odd
§3.5	#49-60 odd	§5.4	#39-60 odd	§3.9	#331-345 odd
§3.6	#2-22 odd	§5.5	#41-62 odd		
		§5.6	#43-62 odd		

Table 3.2: DC tasks analyzed

1. $P_i,\ i \in \{+,-,Q^+,Q^-,C\}$ represents a power function with a positive integer, negative integer, positive non-integer, negative non-integer, or zero exponent, respectively;
2. E_N and E_G represent a natural and general exponential function, respectively;
3. L_N and L_G represent a natural and general logarithmic function, respectively;
4. $T_{sin/cos}$ is sine or cosine, $T_{sec/csc}$ is secant or cosecant, $T_{tan/cot}$ is tangent or cotangent;
5. $IT_{sin/cos}$ is arcsine or arccosine, $IT_{sec/csc}$ is arc-secant or arc-cosecant, $IT_{tan/cot}$ is arctangent or arc-cotangent;
6. *FTC1* represents a function defined as an integral.

The operation structures of each EFT were classified according to the following scheme:

1. a vertex $+_i$ with i a positive integer ≥ 2 represents a sum of i terms,
2. a vertex $\times_i$ with i a positive integer ≥ 2 represents a product of i terms,
3. a sub-structure containing $\times_i$ as the root with two leaves, one connected to the root with a solid arrow and the other connected to the root with a dotted arrow represents a quotient rule,
4. a sub-structure of the form (a) simple function $\rightarrow$ simple function or (b) operation structure $\rightarrow$ simple function represents a chain rule.

Figure 3.1 contains an example of every type of operation structure. In the following analysis, because I represent elementary functions as directed graphs, I categorize DC skills by

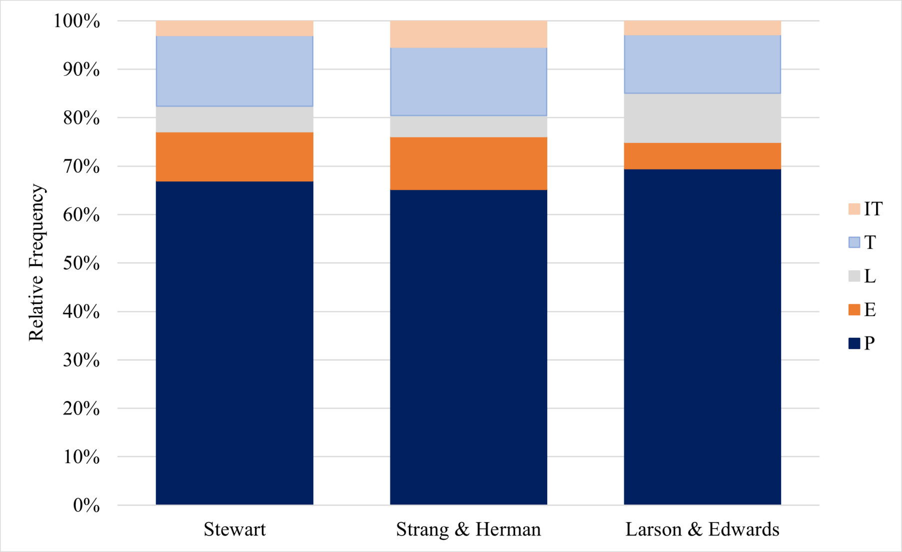


Figure 3.2: Relative frequencies of simple function vertices among EFTs

their "dimension". For example, I consider recalling a fact, such as $f(x) = x^2 \implies f'(x) = 2x$, as a one-dimensional skill because the function $f(x) = x^2$ can be represented by a single vertex. However, a two-dimensional skill is something that can be represented by an edge in an EFT.

To analyze one-dimensional DC skills, I calculated the relative frequencies of simple function vertices for all three textbooks. These are presented in Figure 3.2. In all three texts, power functions were the most common class of simple function vertices (67%, 65%, 69% from left resp.), followed distantly by trigonometric functions (15%, 14%, 12% from left resp.) and exponential functions (10%, 11%, 5% from left resp.).

To analyze two-dimensional DC skills, I superimposed each textbook's set of EFTs into a single digraph (one is given in Figure 3.3) and labeled each edge with its frequency within the textbook. In the mathematics literature, this graph is called the *minimum common supergraph* of a set of graphs [66].

Box-and-whisker plots of the edge frequencies (labels) for each superimposed graph are given in Figure 3.4. The advantage of using box-and-whisker plots in this case is the easy detection of outliers, which represent skills that are practiced significantly more than others. For Stewart's text, ten outliers were detected; for Strang & Herman's text, seven outliers were detected; and for Larson & Edwards' text, four outliers were detected. Figure 3.5 depicts the intersectional relationships among the three sets of outliers.

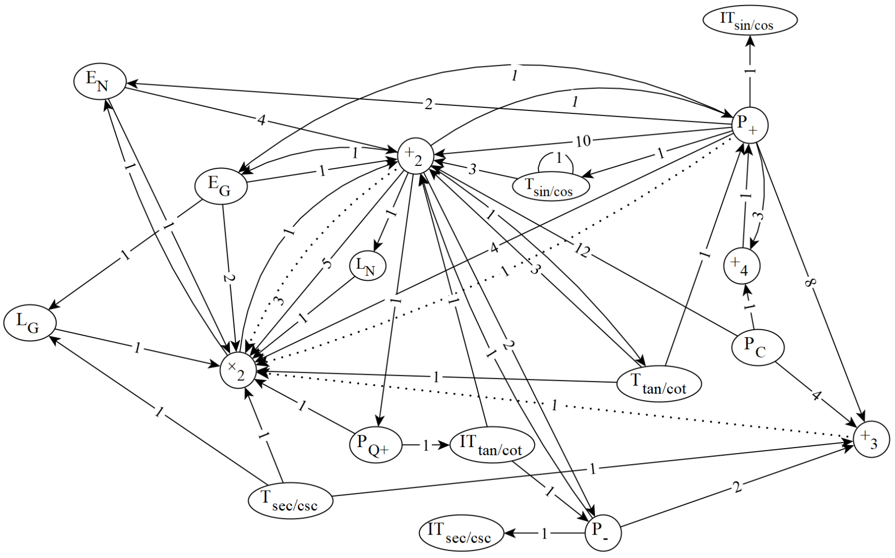


Figure 3.3: Graph of superimposed EFTs (Strang & Herman)

3.1.6 Discussion

In general, my analysis indicates an imbalance among DC skills that a student might encounter during a typical Calculus I experience, highlighting a particular focus on power functions. Theoretically, this could lead to students mastering the overemphasized skills (power functions) but not mastering underemphasized skills (particularly logarithmic and inverse trigonometric functions). This finding agrees with my experience in teaching and tutoring Calculus I – students tend to struggle less with the power rule than any other DC topic. The analysis of the simple function vertices of the EFTs indicates this imbalance clearly. Roughly two-thirds of all simple function vertices involved power functions, and one-third accounted for exponential, trigonometric, logarithmic, and inverse trigonometric functions combined. Furthermore, the analysis of the edges of the EFTs indicates a significant emphasis by all three textbooks specifically on power functions with nonnegative integer exponents. Among the sets of outlier edges, only one is a dotted arrow and only one is a chain rule arrow, indicating a relatively low emphasis on exercises requiring the quotient rule and chain rule that is common to all three texts. With this evidence, it is less surprising that, in my experience, Calculus I students tend to struggle with these two topics.

This observed skewness toward power functions likely stems from the organizational approach the textbooks use for topics related to DC rules. All three textbooks explain the power

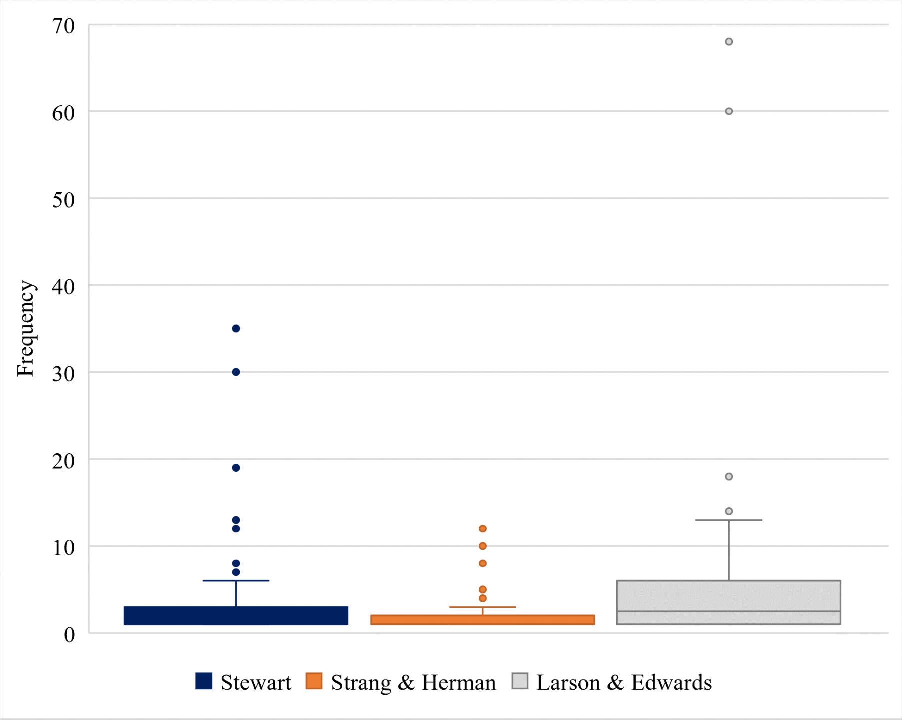


Figure 3.4: Box-and-whisker plots for edges present among EFTs.

rule first and introduce other functions' derivatives concurrently with their theoretical derivations. For example, in Stewart, logarithms are some of the last derivatives discussed because the text introduces this topic with logarithmic differentiation. In Larson & Edwards, logarithms are introduced after integration because the text defines logarithmic functions in terms of integrals. Furthermore, each new topic assumes mastery of those previously discussed. These organizational choices are indicative of an expert approach to presenting mathematics. On the other hand, a novice student would probably organize such information very differently, possibly focusing on facts about derivatives first before building up to a theoretical understanding of how a mathematician arrives at the facts in their final form as well as practicing individual DC rules in isolation before building to combinations of rules.

3.1.7 Future Directions

From a mathematics education research perspective, variations of this approach to task analysis may be feasible for other common classes of skills-based tasks involving elementary functions. For example, Factoring, Limits, and Integration tasks all seem like viable candidates. In addition, it might be interesting to see how instructors' reflection on tasks using EFTs affects

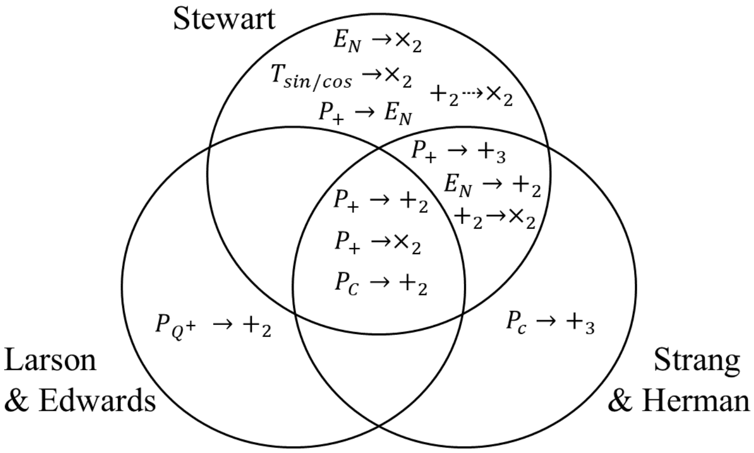


Figure 3.5: Intersections of sets of outlier edges.

students' perception of Calculus course alignment. Would it make students feel better prepared for assessments? Tangentially, would restructuring the presentation of DC topics to reflect a novice learner's preference rather than a mathematician's preference result in better outcomes for Calculus I students? In other words, can some information be rearranged in a Calculus course so that it perhaps lacks the logical flow of a published paper, but somehow makes more sense to someone who doesn't have the experience of an expert? From an interdisciplinary perspective, EFTs offer a promising bridge between theoretical discrete mathematics research and mathematics education research. For example, might it be possible to use existing theoretical discrete mathematics results to resolve the imbalance of DC skills among textbook tasks? In either case, EFTs offer a variety of interesting future research trajectories.

3.2 Derivative Computations are More Complicated Than We Think

Results from this section are under peer-review as part of a solo-authored book chapter [78]. The math education convention is used where "I" indicates that this project features contributions from a single author.

3.2.1 Introduction

In this study, I refine the Simple Procedures category of White & Mesa's COF [106] by narrowing the scope of my analysis to tasks in various textbooks with a particular prompt. A derivative computation (DC) task is a task whose prompt is in the form “Find the derivative of the function.” Such tasks have been studied as a way of training students’ procedural flexibility [89]. This study will answer the research question: What is the complexity of DC textbook tasks, and what does this imply about the cognitive orientation of such tasks?

3.2.2 Methodology

The corpus for this study includes the corpus of the previous study with two additional general calculus texts [85, 87, 92, 99, 100]. Since “The Odds” is regarded as the proverbial mathematics homework assignment [63, 68], I analyzed each textbook’s odd-numbered derivative computation tasks to simulate students’ exposure to textbook tasks in a semester using a particular digraph drawing that models the structure of an elementary function. The *elementary function tree* of a function f, EFT(f), is a digraph, T, such that $V(T)$ is the collection of simple functions and commutative operations in f, and $E(T)$ is the set of composition relations among the vertices, with reciprocal functions represented by dotted arrows (see Figure 3.6). In graph drawing theory, the *area* of a graph drawing is the area of the smallest bounding box containing the drawing [74]. Since I am focused on directed trees, the area is easily computed using the tree’s height and width. The *height*, h, of a directed tree is the length of the longest directed path contained in the tree [73]. The *width*, w, of a directed tree is the maximum number of vertices in any level of the tree [88]. Then, the area of a directed tree can be expressed as $(h+1)w$. Computing the area of an EFT allows quantification of the complexity of a DC task.

3.2.3 Results

The height of an EFT approximates how many applications of Chain Rule one needs to consider when evaluating the derivative of an elementary function. The mean heights of EFTs in the textbooks were, from least to greatest: Hoffmann & Bradley (1.63), Larson & Edwards (1.75), Strang & Herman (1.76), Stewart (1.79), and Neuhauser & Roper (1.83).

The width of an EFT indicates the maximum number of derivative rules one must consider at any one level of an elementary function when evaluating its derivative. The mean widths of

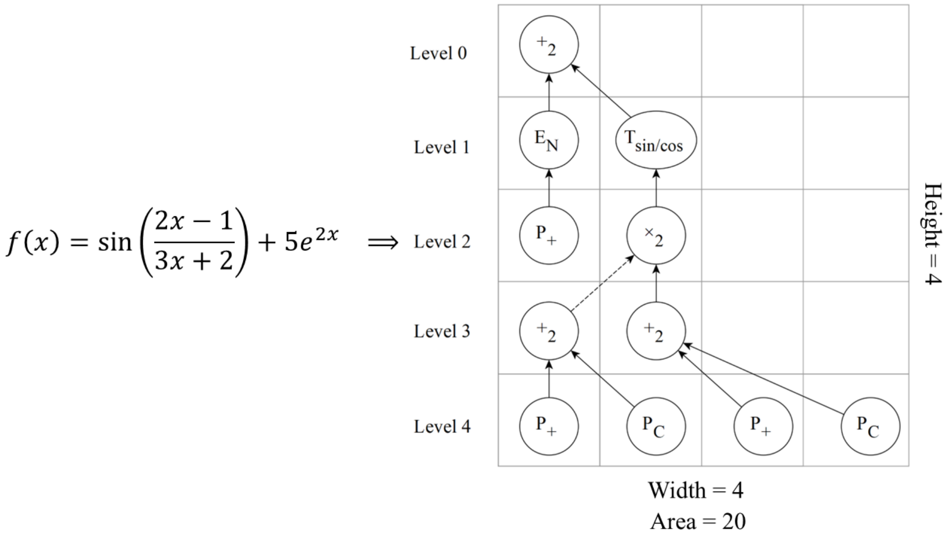


Figure 3.6: The height, width, and area of an EFT.

Note: Vertices correspond to functions and commutative operations – for example, vertices labeled “P” indicate a power function. Edges correspond to compositions – for example, the vertex labeled “+” has two incoming arrows representing the sum’s terms: a sine function and a natural exponential function.

EFTs in the textbooks were, from least to greatest: Larson & Edwards (2.12), Stewart (2.28), Neuhauser & Roper (2.33), Strang & Herman (2.34), and Hoffmann & Bradley (2.55).

The area of an EFT combines the height and the width to capture the complexity of evaluating an elementary function’s derivative. The mean areas of EFTs in the textbooks were, from least to greatest: Larson & Edwards (6.24), Strang & Herman (6.62), Stewart (6.75), Neuhauser & Roper (6.91), Hoffmann & Bradley (7.12). These results are summarized visually in Figure 3.7. Furthermore, relative frequency distributions were constructed for each textbook’s EFTs’ areas (see Figure 3.8).

3.2.4 Discussion

The results of this study demonstrated that a single textbook provides some variety in the complexity of DC tasks but that students’ potential exposure to DC tasks is relatively standardized across textbooks. While all the textbooks studied included DC tasks with a relatively large area, surprisingly, the discipline-specific textbooks had the largest mean area, implying that these books contain some of the most difficult DC tasks. Though DC tasks are characterized in the literature in lower levels of White & Mesa’s [106], Bloom’s [64], and Smith & Stein’s [98]

Figure 3.7: Mean areas of various calculus textbooks' DC tasks (drawn to scale).

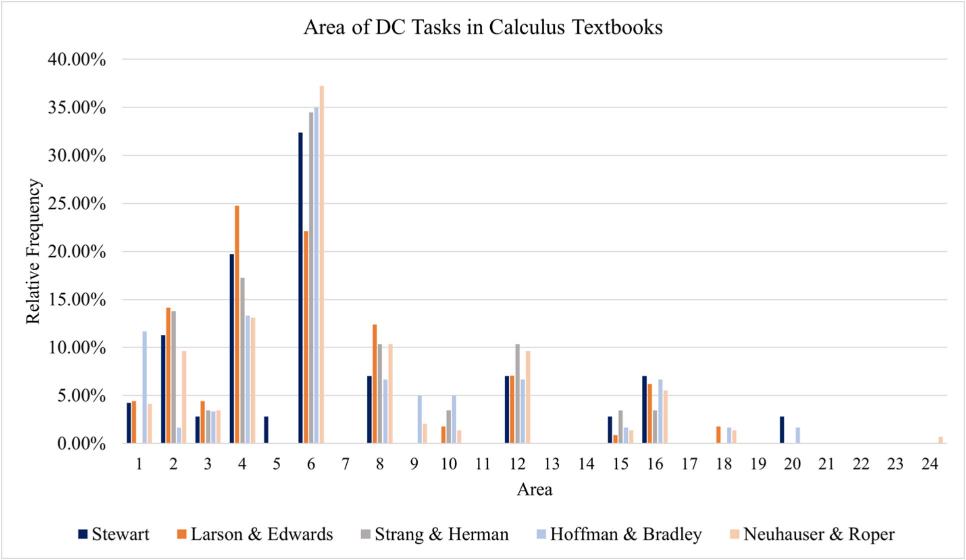


Figure 3.8: Relative frequency distributions of the area of EFTs drawn from various Calculus textbooks' odd-numbered DC tasks.

cognition taxonomies [61, 86], this study's unique methodology demonstrated that, while some of these tasks are simple, many require deeper cognition and offer a way to develop students' procedural flexibility [89].

3.3 "Designed" Derivative Computation Tasks Balance the Load

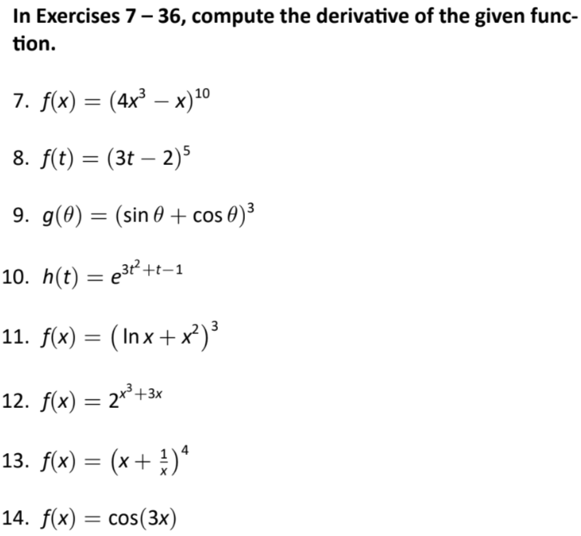
The results in this section have been submitted to a combinatorics journal for peer review and have been released online as a preprint [80]. This section uses the mathematics convention where "we" is used regardless of the number of contributors. However, this project was the product of a single author.

3.3.1 Introduction

From humble rumblings about arranging schoolgirls, magic squares, and experimental design, the field of Combinatorial Design Theory has evolved into a vast area of inquiry which, over the years, has borrowed and blended ideas from many other research areas [60]. Research in Undergraduate Mathematics Education (RUME) is an analogous field with its own well-established methodologies and research history. Graphs are particularly suited for answering structural questions about real-life scenarios [69], and in recent years, graphs have emerged as approaches to answering various questions in RUME. Although the use of *graph-theoretic methods* in math education research has increased since 2015, this remains a minority approach among education researchers' choices of methodology. Despite this, the notion of research being an interconnected network of relationships is at the heart of many mathematics education research questions [79]. Given this promising connection between discrete mathematics and RUME and the fact that undergraduate mathematics courses, especially freshman courses such as Calculus, feature structured content curricula, it is only natural to wonder if theoretical results from Design Theory, which is both intimately related to Graph Theory and devoted to studying structural phenomena, might apply to RUME.

In mathematics education research, a *task* is a broad term encompassing any unit of questioning assigned to a learner by an instructor. In this section, "task" is preferred over "problem" or "exercise" because education research distinguishes between the latter two depending on the level of possible mimicry of instructor-led examples by the learner [96]. A task is *skills-based* if it requires fluency in narrowly defined procedures, called *skills*. A prime example of a large class of skills-based tasks is Derivative Computation tasks in Calculus I. *Derivative Computation tasks* (or *DC tasks*) are tasks with instructions of the form, "Compute $f'(x)$ of the function $f(x) =$ [elementary function]," as exemplified in Figure 3.9.

Note that in this section, we consider single-variable functions that are neither implicitly-defined functions nor functions defined by exponentiation of non-constant elementary functions.



In Exercises 7 – 36, compute the derivative of the given function.

7. $f(x) = (4x^3 - x)^{10}$

8. $f(t) = (3t - 2)^5$

9. $g(\theta) = (\sin\theta + \cos\theta)^3$

10. $h(t) = e^{3t^2+t-1}$

11. $f(x) = \left(\ln x + x^2\right)^3$

12. $f(x) = 2^{x^3+3x}$

13. $f(x) = \left(x + \frac{1}{x}\right)^4$

14. $f(x) = \cos(3x)$

Figure 3.9: DC Tasks in a Calculus I textbook [84].

A skills-based *task set* is a collection of skills-based tasks, which, in practice, an instructor might assign to students as a homework assignment or quiz. Skills-based task sets are integral to undergraduate (especially freshman) mathematics courses. For Calculus I in particular, approximately 20-25% of the course is devoted to computing derivatives of elementary functions using derivative rules according to a widely accepted Calculus curriculum in the U.S. [91].

A quantitative analysis of Calculus textbooks' DC tasks has revealed that these task sets emphasize the derivatives of power functions more than any other simple function class (e.g., trigonometric functions, exponential functions, logarithmic functions, and inverse trigonometric functions) [77]. However, this finding misaligns with many mathematicians' perspective that Calculus students should be equally prepared for any DC skills they might encounter on assessments. Different types of misalignment in mathematics courses (including this example between instructor expectations and assigned practice), particularly in first-semester Calculus, have been linked to student retention issues in STEM majors [105]. Thus, this section explores the existence of what we call *balanced* DC task sets, or task sets in which each DC skill in a pre-defined list occurs exactly the same number of times. In particular, the Chain Rule is notoriously difficult to teach, as described by this quote from Gordon [81].

> The chain rule is one of the hardest ideas to convey to students in Calculus I. It is difficult to motivate, so that most students do not really see where it comes from;

> it is difficult to express in symbols even after it is developed; and it is awkward to put it into words, so that many students can not remember it and so can not apply it correctly.

So, we focus our attention on constructing task sets that give students practice with this skill.

Within the research area of task design, mathematics education researchers differentiate between types of practice. *Blocked practice* refers to a set of tasks that assesses one particular skill, whereas *mixed practice* refers to a set of tasks that assesses many different skills [82]. However, these two categories of practice are not mutually exclusive and depend on the definition of the skill being assessed. For example, when Calculus students first learn how to take derivatives, they learn some facts like the derivatives of the six trigonometric functions. These facts could constitute blocked or mixed practice depending on how one defines a skill. If we want to assess the skill of "Taking derivatives of trigonometric functions", then recalling these facts is blocked practice. However, if we consider each fact as a skill, then recalling these facts is mixed practice. In this section, we will consider decompositions of a vertex labeled complete directed graph because we aim to construct different types of mixed practice where the skills are the set of vertex labels. Using this approach, every possible ordered pair of skills occurs exactly once in a task set.

The remainder of this section has three main parts. First, we establish the background of the problem, defining an index of notation in Section 3.3.2, preliminary definitions in Section 3.3.3, and known results in Section 3.3.4. Second, in Sections 3.3.5 and 3.3.6 we extend results of Meszka & Skupień [90] on non-Hamiltonian directed path decompositions of the complete directed graph by proving existence results under additional conditions. Third, in Section 3.3.7 we give example Chain Rule task sets using our constructions, and, in Section 3.3.8, we explore possibilities for further research.

3.3.2 Index of Notation

$\mathbb{P}$	positive integers

$\mathrm{id}(v)$	indegree of a vertex v
$\mathrm{od}(v)$	outdegree of a vertex v
$G[E_i]$	the subgraph of G induced by the edge set $E_i \subseteq E$
$\mathscr{D}K_n$	the complete directed graph on n vertices
DCTS(n,t)	a derivative computation task set of order n and size t
BDCTS(n,t)	a balanced derivative computation task set of order n and size t
DPD(G)	a directed path decomposition of a directed graph G
HDPD(n)	a Hamiltonian directed path decomposition of $\mathscr{D}K_n$
NHDPD(n)	a non-Hamiltonian directed path decomposition of $\mathscr{D}K_n$
BNHDPD(n,k)	a balanced non-Hamiltonian directed path decomposition of $\mathscr{D}K_n$ with each vertex appearing k times in the decomposition

3.3.3 Preliminaries

In this section, we introduce terminology from Combinatorial Design Theory necessary for the results in Sections 3.3.5 and 3.3.6.

Definition 3.1 (Graph Decomposition). A *decomposition D* of a (directed) graph $G = (V,E)$ is a collection $\{H_1, H_2, \ldots, H_t\}$ of nonempty subgraphs such that $H_i = G[E_i]$ for some nonempty subset E_i of $E(G)$, and $\{E_1, E_2, \ldots, E_t\}$ is a partition of $E(G)$.

Figure 3.10 illustrates an example of such a decomposition. For simplicity, we label the vertices $1, \ldots, 5$, represent the edge partition using a different color for each part, and omit single-edge parts from the picture.

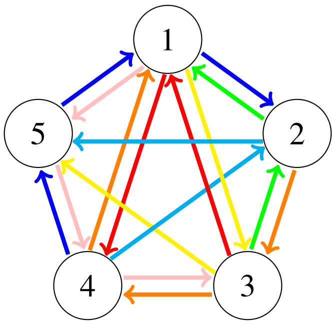


Figure 3.10: A decomposition of $\mathscr{D}K_5$ into directed paths of varying lengths

Many graph decomposition results concern unlabeled graphs. However, when constructing Derivative Computation task sets, it makes sense to consider decompositions of (vertex) labeled graphs so we can specify the simple functions and arithmetic operations present in an elementary function.

Definition 3.2 (Elementary Function Vertex Labeling). An *elementary function vertex labeling* of a graph G is a (not necessarily injective) function $\Lambda : V \to \mathscr{L}$ where $\mathscr{L} = \mathscr{F} \cup \mathscr{O}$ is a set of labels such that:

1. $\mathscr{F}$ is a set of simple function classes, and
2. $\mathscr{O} \subseteq \{+_i \mid i \in \mathbb{P}, i \geq 2\} \cup \{\times_i \mid i \in \mathbb{P}, i \geq 2\} \cup \{\div\}$ is a set of operations where i denotes the number of summands in a sum or factors in a product, respectively.

In the results that follow, we decompose $\mathscr{D}K_n$ into a particular class of directed trees, which we call elementary function trees. Similar representations of functions, called *syntax trees*, are a common data structure in Computer Science.

Definition 3.3 (Construction Version of EFT). An *elementary function tree*, or EFT, is a labeled, rooted in-tree, (V, E, Λ), where

1. V is a collection of simple functions and addition, multiplication, and division operations
2. specified by a labeling Λ, and
3. E is the set of composition relations among the vertex labels (i.e. for $u, v \in V, uv \in E$ iff $\Lambda(v) \circ \Lambda(u)$).

An example of an EFT is given in Figure 3.11. Note that this example is the precise “construction” analog to the example given in Figures 3.1 and 3.6.

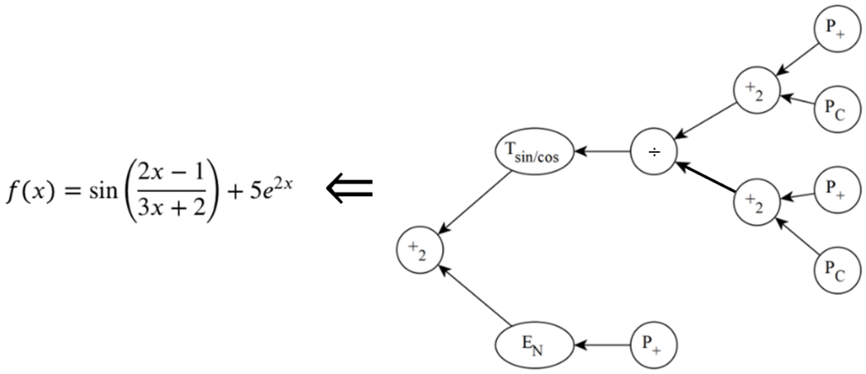


Figure 3.11: An example of an EFT (Construction Version)

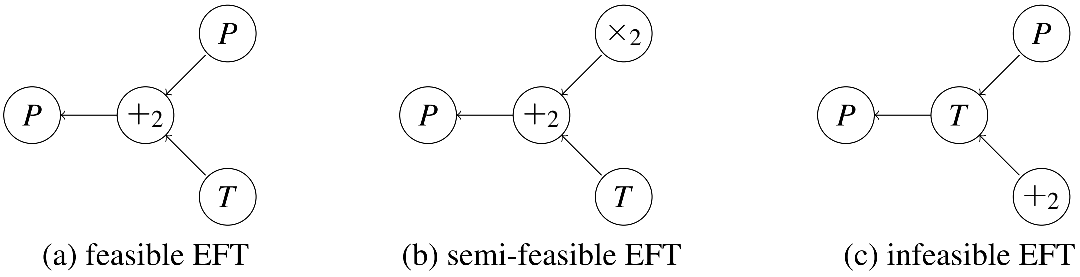
A natural characterization of EFTs exists based on whether they correspond to practical tasks or not.

1. A *feasible* EFT is an EFT that has at least one corresponding elementary function. In other words, if $v \in V$ with $\Lambda(v) \in \mathscr{F}$, then $\mathrm{id}(v) \leq 1$ and $\mathrm{od}(v) \leq 1$, and if $v \in V$ with $\Lambda(v) \in \mathscr{O}$, then $\mathrm{id}(v) = i$ and $\mathrm{od}(v) \leq 1$.

2. A *semi-feasible* EFT is an EFT that can be made feasible by augmenting it with a finite number of additional vertices and arcs. In other words, $\exists\ v \in V$ with $\Lambda(v) \in \mathscr{O}$ such that $\mathrm{id}(v) < i$.

3. An *infeasible* EFT is an EFT that does not have a corresponding elementary function. In other words, either $\exists\ v \in V$ with $\Lambda(v) \in \mathscr{F}$ such that either $\mathrm{id}(v) > 1$ or $\mathrm{od}(v) > 1$, or $\exists\ v \in V$ with $\Lambda(v) \in \mathscr{O}$ such that $\mathrm{od}(v) > 1$.

Figure 3.12 depicts an example of each class of EFTs. The leftmost EFT is feasible because we can find an elementary function it represents; for example, consider the function $f(x) = (x^2 + \sin x)^2$. The middle EFT is semi-feasible because we can make it feasible by adding two simple function vertices pointing into the multiplication vertex. The rightmost EFT is infeasible because a single-variable trigonometric function cannot have two input values. However, note that if we were to consider multivariable functions, these notions of feasible and infeasible would change.



(a) feasible EFT (b) semi-feasible EFT (c) infeasible EFT

Figure 3.12: A feasible, a semi-feasible, and an infeasible EFT.

Note: The labels P, T, $+_2$, and $\times_2$ stand for "power function", "trigonometric function", "sum of two addends", and "product of two factors", respectively.

A feasible EFT is synonymous with a DC task. Therefore, we define a *DC task set*, $\mathscr{T}$, as a collection of feasible EFTs. The *size* of the task set, $t = |\mathscr{T}|$, is the number of tasks (or feasible EFTs) that it contains, whereas the *order* of the task set is $n = |\mathscr{L}| = V(G)$. We denote a DC task set of order n as DCTS(n). If every label appears exactly the same number of times in a task set, then we say the DCTS is *balanced*. We denote a balanced DC task set of order n by BDCTS(n).

3.3.4 Known Results

This section discusses known results on directed path decompositions, DPDs, and how they may be used to construct Derivative Computation Task Sets of order n, or DCTS(n). From a graph-theoretic point of view, the simplest EFTs are directed paths, which correspond to DC tasks that assess the Chain Rule in isolation. Lemma 3.1 allows us to restrict our focus to constructing DPDs with special properties.

Lemma 3.1. *If D is a DPD(G) with $\mathscr{L} = \mathscr{F}$ and $|\mathscr{L}| = |V(G)| = n$, then D is a DCTS(n).*

Proof. Suppose D is a DPD(G). Then, all EFTs in D are directed paths. This means that, for every vertex v contained in any directed path P in D, id(v)≤ 1 and od(v)≤ 1. Since $\mathscr{L} = \mathscr{F}$

($\mathscr{O} = \varnothing$) and all simple functions have exactly one input, all directed paths in D are feasible. Thus, D is a DCTS(n). $\square$

Directed Path Decompositions of $\mathscr{D}K_n$

A natural starting point for studying DPD($\mathscr{D}K_n$) is to require only Hamiltonian paths in the decomposition. The existence problem of HDPD(n) was completely solved by Bosák [65] by synthesizing and extending results of Bermond and Faber [62] and Tillson [103]. However, if we are to assign DCTS to students, Hamiltonian paths are impractical. Each task in a HDPD(n) would require students to practice the Chain Rule $n-1$ times, meaning we would need to construct task sets with relatively small values of n, restricting the variety in simple function classes students would be exposed to in a task set.

Considering the impracticality of Hamiltonian paths, the next natural question is to consider non-Hamiltonian paths, or NHDPD(n). A theorem of Meszka and Skupień [90] completely solves the existence problem of NHDPD(n).

Theorem 3.1 (M. Meszka & Z. Skupień, 2006). *NHDPD(n) with paths of arbitrarily prescribed lengths* ($\leq n-2$) *exist for any positive integer* $n \geq 3$*, provided that the lengths sum up to* $n(n-1)$*, the size of* $\mathscr{D}K_n$*.*

We extend the notion of NHDPD(n) by considering their balanced counterparts.

Definition 3.4. Let $n, k \in \mathbb{P}$. A *balanced non-Hamiltonian directed path decomposition* of $\mathscr{D}K_n$, denoted BNHDPD(n,k), is a NHDPD(n) such that each vertex is contained in exactly k paths in the decomposition.

By Lemma 3.1, all BNHDPD(n,k) are BDCTS(n) where each elementary function occurs k times in the task set.

3.3.5 Necessary Conditions for BNHDPD(n,k)

We now give general necessary conditions for the existence of a BNHDPD(n,k).

Lemma 3.2. *If a BNHDPD(n,k) comprised of x_i non-Hamiltonian directed paths of length i exists, then*

$$\sum_{i=1}^{n-2} i x_i = n(n-1). \tag{3.1}$$

Proof. Since such a decomposition is a non-Hamiltonian directed path decomposition of the complete directed graph, this follows directly from Theorem 3.1. □

The *incidence matrix* of a BNHDPD(n,k), A, is an array whose rows represent the vertices of $\mathscr{D}K_n$, $v_1,\ldots,v_n$, and whose columns represent the EFTs of the task set, $T_1,\ldots,T_{|\mathscr{T}|}$ such that

$$[A] = \begin{cases} 1 \text{ if } v_i \in T_j \\ 0 \text{ if } v_i \notin T_j \end{cases}$$

Lemma 3.3. *If a BNHDPD(n,k) comprised of x_i non-Hamiltonian directed paths of length i exists, then*

$$\sum_{i=1}^{n-2} (i+1) x_i = nk, \tag{3.2}$$

where k is the number of paths incident to each vertex.

Proof. We count the total number of ones in the incidence matrix of a BNHDPD in two different ways. First, count by rows. Since there are n vertices, each incident to k paths, there are nk ones in the incidence matrix. Now, count by columns. Since there are x_i paths of length i and $(i+1)$ vertices in a path of length i, there are $2x_1 + 3x_2 + \cdots + (n-1)x_{n-2}$ ones in the incidence matrix. This gives the desired result. □

Theorem 3.2. *If a BNHDPD(n,k) comprised of x_i non-Hamiltonian directed paths of length i exists, then n divides the number of paths in the decomposition.*

Proof. Substituting (3.1) into (3.2), we arrive at

$$\sum_{i=1}^{n-2} x_i = n(k-(n-1)), \tag{3.3}$$

which is the desired result. □

The next necessary condition concerns the structure of a BNHDPD(n,k) at each vertex of $\mathscr{D}K_n$.

Theorem 3.3. *If a BNHDPD(n,k) comprised of x_i non-Hamiltonian directed paths of length i exists $(i \in \mathbb{P}, \leq n-2)$, then n divides the number of interior vertices among all paths in the decomposition. That is,*

$$n \;\Big|\; \sum_{i=2}^{n-2}(i-1)x_i$$

Proof.

$$\begin{aligned}&\sum_{i=2}^{n-2}(i-1)x_i\\ &=\sum_{i=2}^{n-2} ix_i-\sum_{i=2}^{n-2} x_i\end{aligned} \tag{3.4}$$

Substituting (3.1) and (3) into (4), we arrive at

$$\sum_{i=2}^{n-2}(i-1)x_i = n[n-1-(k-(n-1))],$$

which is the desired result.

□

3.3.6 Sufficient Conditions

In this subsection, we establish sufficiency for $n=5$ and $n=6$. We chose to focus on these cases because they produce a variety of practical, balanced task sets to assign to students. The case $n=4$ results in little possible variety (see Figure 3.13), and only two of these possible task sets meet the divisibility condition for balance. The case $n=7$ is impractical for two reasons. First, paths of length 5 are introduced, meaning each task can require up to 5 successive applications of the chain rule, which we believe is unnecessarily complex for students. Second, though $n=7$ has greater potential variety in tasks, the task sets will be larger than what we would assign to students.

Theorem 3.1 allows us to represent any task set as an nonnegative integer solution to the Diophantine Equation

$$\sum_{i=1}^{n-2} ix_i = n(n-1).$$

Figure 3.13 shows the number of nonnegative integer solutions to this equation for $n = 4,5,6$, categorized by their sum (the size of the task set). This gives us the maximum number of sufficiency cases we will need to construct in each case and hints at the wide variety in possible task sets using our constructions.

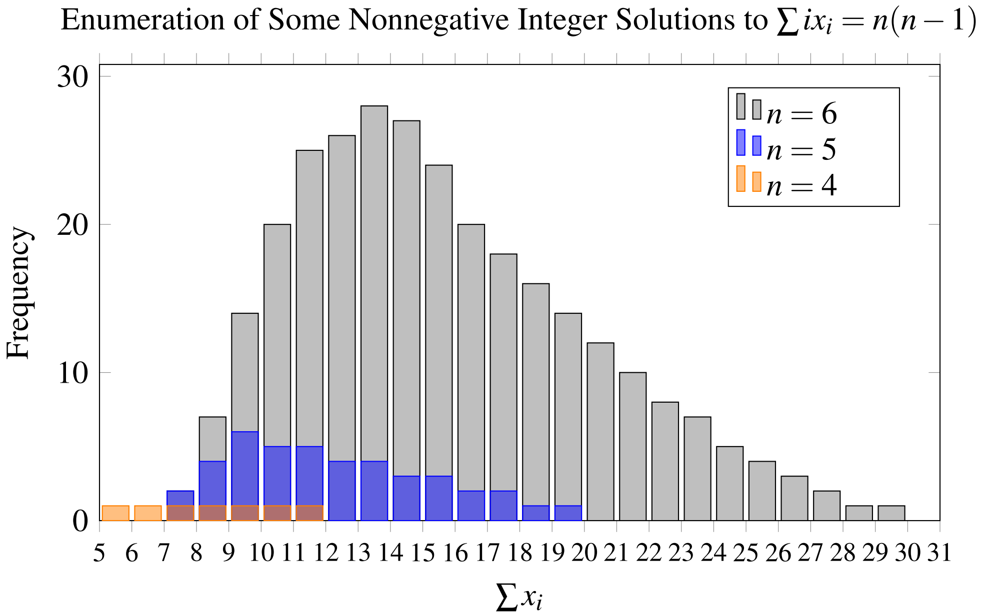


Figure 3.13: Enumeration of some possible DCTS(n,t) to be constructed from NHDPD($\mathscr{D}K_n$).

Before we establish sufficiency, we need the definition for the reverse of a directed path, which appears in many of our constructions.

Definition 3.5. The *reverse* of a directed path $v_1 \to v_2 \to \cdots \to v_m$ is the directed path $v_m \to v_{m-1} \to \cdots \to v_1$.

We also establish one general sufficiency case.

Lemma 3.4. *A BNHDPD*(n,k) *with* $n(n-1)$ *directed paths of length 1 exists for all* $n \in \mathbb{P}$, $n \geq 2$.

Proof. Trivially decompose the arc set of $\mathscr{D}K_n$ into single arcs, resulting in $n(n-1)$ copies of $\mathscr{D}P_2$. Each vertex v appears $2(n-1)$ times in the decomposition ($n-1$ times in which v is the head of an arc and $n-1$ times in which v is the tail of an arc), meaning that the decomposition is balanced. ☐

Sufficiency for BNHDPD$(5,k)$

In this subsection, we establish sufficient conditions for BNHDPD with $n=5$. This is equivalent to constructing BDCTS that contain 5 classes of simple functions and that assess the Chain Rule in isolation. Sample task sets constructed using results in this subsection are given in Section 3.3.7.

To obtain the list of sufficiency cases in the proof of Theorem 3.4, we listed all nonnegative integer solutions to Equation 3.1. Then, we eliminated all solutions that did not meet the additional necessary conditions given in Section 3.3.5.

Theorem 3.4. *A BNHDPD*$(5,k)$ *comprised of* x_1 *copies of* $\mathscr{D}P_2$*,* x_2 *copies of* $\mathscr{D}P_3$*, and* x_3 *copies of* $\mathscr{D}P_4$ *exists for all nonnegative integer solutions of* $x_1+2x_2+3x_3=20$ *such that* $5 \mid x_1+x_2+x_3$.

Proof. Let $1,2,3,4,$ and 5 denote the vertices of $\mathscr{D}K_5$. Then, the following lists of directed paths are BNHDPD$(5,k)$. With each subcase, we give a visual of its construction, not including any reverses or copies of $\mathscr{D}P_2$ to reduce visual clutter.

Case 1: $x_1+x_2+x_3=10$

Subcase 1a. $x_1=0, x_2=10, x_3=0$

Pictured: $1\rightarrow 2\rightarrow 4$,

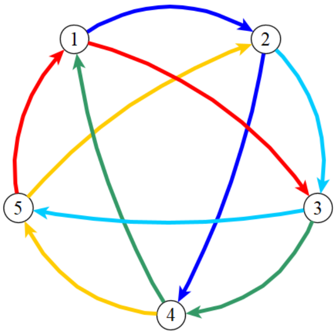
$2\rightarrow 3\rightarrow 5$,

$3\rightarrow 4\rightarrow 1$,

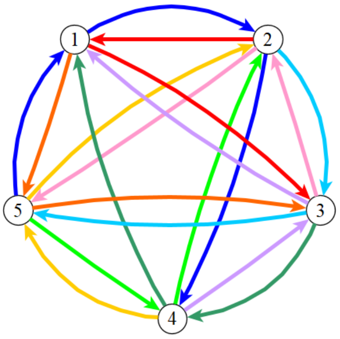
$4\rightarrow 5\rightarrow 2$,

$5\rightarrow 1\rightarrow 3$

Not Pictured: Take reverses of these paths.



Subcase 1b. $x_1 = 1, x_2 = 8, x_3 = 1$



Pictured: $5 \to 1 \to 2 \to 4$,

$2 \to 3 \to 5$, $3 \to 4 \to 1$,

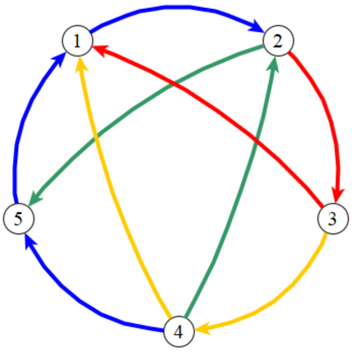
$4 \to 5 \to 2$, $2 \to 1 \to 3$,

$1 \to 5 \to 3$, $5 \to 4 \to 2$,

$4 \to 3 \to 1$, $3 \to 2 \to 5$

Not Pictured: Take remaining arcs as $x_1 = 1$ copies of $\mathscr{D}P_2$.

Subcase 1c. $x_1 = 2, x_2 = 6, x_3 = 2$



Pictured: $4 \to 5 \to 1 \to 2$,

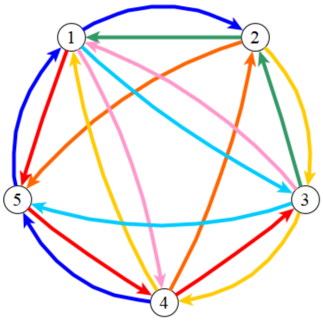
$2 \to 3 \to 1$,

$3 \to 4 \to 1$,

$4 \to 2 \to 5$

Not Pictured: Take reverses of these paths. Take remaining arcs as $x_1 = 2$ copies of $\mathscr{D}P_2$.

Subcase 1d. $x_1 = 3, x_2 = 4, x_3 = 3$



Pictured: $4 \to 5 \to 1 \to 2$,

$1 \to 5 \to 4 \to 3$,

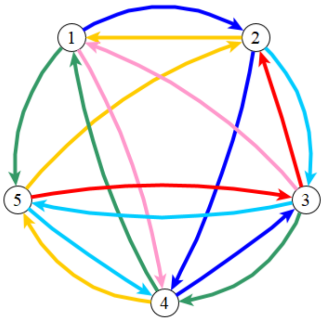
$2 \to 3 \to 4 \to 1$,

$3 \to 2 \to 1$, $3 \to 1 \to 4$,

$4 \to 2 \to 5$, $1 \to 3 \to 5$

Not Pictured: Take remaining arcs as $x_1 = 3$ copies of $\mathscr{D}P_2$.

Subcase 1e. $x_1 = 4, x_2 = 2, x_3 = 4$



Pictured: $1 \to 2 \to 4 \to 3$,

$2 \to 3 \to 5 \to 4$,

$3 \to 4 \to 1 \to 5$,

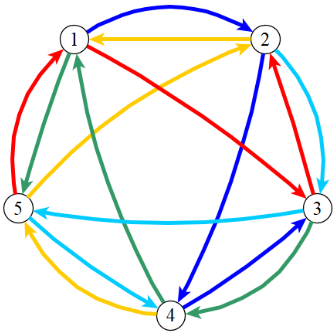
$4 \to 5 \to 2 \to 1$,

$3 \to 1 \to 4$,

$5 \to 3 \to 2$

Not Pictured: Take remaining arcs as $x_1 = 4$ copies of $\mathscr{D}P_2$.

Subcase 1f. $x_1 = 5, x_2 = 0, x_3 = 5$



Pictured: $1 \to 2 \to 4 \to 3$,

$2 \to 3 \to 5 \to 4$,

$3 \to 4 \to 1 \to 5$,

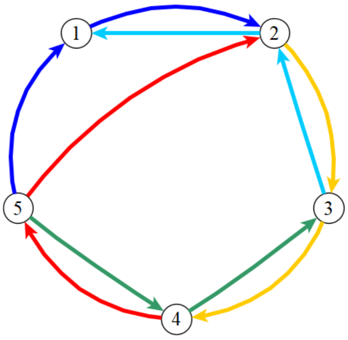
$4 \to 5 \to 2 \to 1$,

$5 \to 1 \to 3 \to 2$

Not Pictured: Take remaining arcs as $x_1 = 5$ copies of $\mathscr{D}P_2$.

Case 2: $x_1 + x_2 + x_3 = 15$

Subcase 2a. $x_1 = 10, x_2 = 5, x_3 = 0$



Pictured: $5 \to 1 \to 2$,

$2 \to 3 \to 4$,

$4 \to 5 \to 2$,

$5 \to 4 \to 3$,

$3 \to 2 \to 1$,

Not Pictured: Take remaining arcs as $x_1 =$ 10 copies of $\mathscr{D}P_2$.

Subcase 2b. $x_1 = 11, x_2 = 3, x_3 = 1$

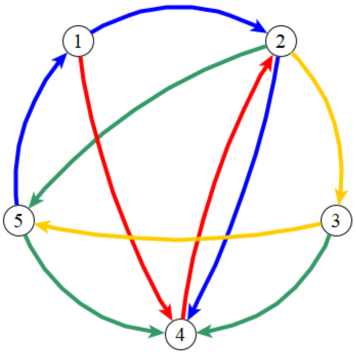
Pictured: $5 \to 1 \to 2 \to 4$,

$2 \to 3 \to 5$,

$1 \to 4 \to 2$,

$2 \to 5 \to 4$

Not Pictured: Take remaining arcs as $x_1 = 11$ copies of $\mathscr{D}P_2$.



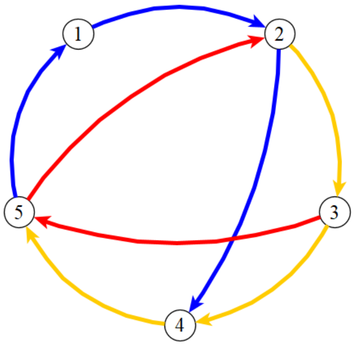
Subcase 2c. $x_1 = 12, x_2 = 1, x_3 = 2$

Pictured: $5 \to 1 \to 2 \to 4$,

$2 \to 3 \to 4 \to 5$,

$3 \to 5 \to 2$

Not Pictured: Take remaining arcs as $x_1 = 12$ copies of $\mathscr{D}P_2$.



Case 3: $x_1 + x_2 + x_3 = 20$. The only solution to this equation meeting the necessary conditions in Section 3.3.5 is $x_1 = 20, x_2 = 0, x_3 = 0$. Lemma 3.4 solves this case.

□

Theorems 3.2 and 3.4 give the complete spectrum for BNHDPD$(5,k)$ since all triples x_1, x_2, x_3 that satisfy Theorem 3.2 also satisfy Theorem 3.3. We summarize this result in Theorem 3.5.

Theorem 3.5. *A BNHDPD$(5,k)$ exists if and only if the following conditions all hold:*

1. $x_1 + 2x_2 + 3x_3 = 20$
2. $5 \mid x_1 + x_2 + x_3$

Sufficiency for BNHDPD$(6,k)$

In this subsection, we establish sufficient conditions for BNHDPD with $n = 6$. This is equivalent to constructing BDCTS that contain 6 classes of simple functions and that assess the Chain Rule in isolation.

To obtain the list of sufficiency cases in the proof of Theorem 3.6, we listed all nonnegative integer solutions to Equation 3.1. Then, we eliminated all solutions that did not meet the additional necessary conditions given in Section 3.3.5.

Theorem 3.6. *A* $BNHDPD(6,k)$ *comprised of* x_1 *copies of* $\mathscr{D}P_2$*,* x_2 *copies of* $\mathscr{D}P_3$*,* x_3 *copies of* $\mathscr{D}P_4$*, and* x_4 *copies of* $\mathscr{D}P_5$ *exists for all nonnegative integer solutions of* $x_1+2x_2+3x_3+4x_4 = 30$ *such that* $6 \mid x_1+x_2+x_3+x_4$ *and* $6 \mid x_2+2x_3+3x_4$.

Proof. Let $1,2,3,4,5$ and 6 denote the vertices of $\mathscr{D}K_6$. Then, the following lists of directed paths are $\text{BNHDPD}(6,k)$. With each subcase, we give a visual of its construction, not including any reverses or copies of $\mathscr{D}P_2$ to reduce visual clutter.

Case 1: $x_1+x_2+x_3+x_4 = 12$

Subcase 1a. $x_1 = 3$, $x_2 = 0$, $x_3 = 9$, $x_4 = 0$

Pictured:

$4 \to 6 \to 1 \to 2$, $6 \to 2 \to 3 \to 4$,

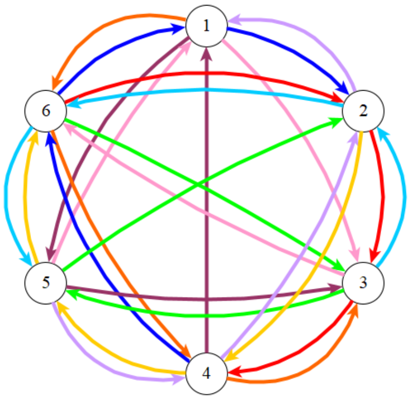
$2 \to 4 \to 5 \to 6$, $3 \to 2 \to 6 \to 5$,

$1 \to 6 \to 4 \to 3$, $5 \to 4 \to 2 \to 1$,

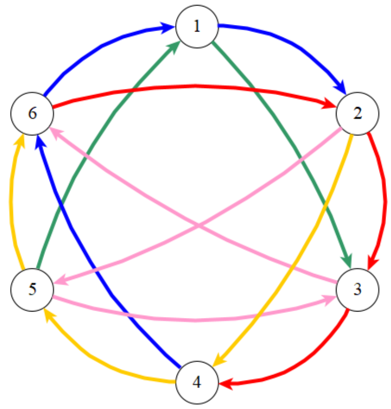
$4 \to 1 \to 5 \to 3$, $5 \to 1 \to 3 \to 6$,

$6 \to 3 \to 5 \to 2$

Not Pictured: Take remaining arcs as $x_1 = 3$ copies of $\mathscr{D}P_2$.



Subcase 1b. $x_1 = 2, x_2 = 2, x_3 = 8, x_4 = 0$



Pictured: $4 \to 6 \to 1 \to 2$,

$6 \to 2 \to 3 \to 4$,

$2 \to 4 \to 5 \to 6$,

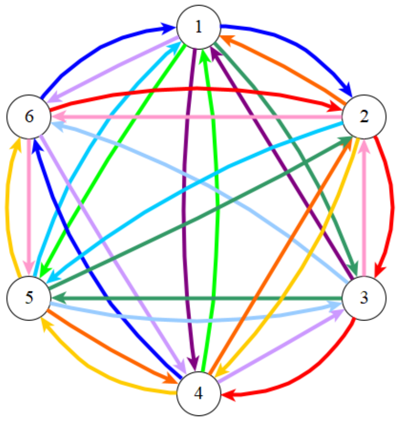
$2 \to 5 \to 3 \to 6$,

$5 \to 1 \to 3$

Not Pictured: Take reverses of these paths.

Take remaining arcs as $x_1 = 2$ copies of $\mathscr{D}P_2$.

Subcase 1c. $x_1 = 1, x_2 = 4, x_3 = 7, x_4 = 0$



Pictured:

$4 \to 6 \to 1 \to 2, 6 \to 2 \to 3 \to 4$,

$2 \to 4 \to 5 \to 6, 5 \to 4 \to 2 \to 1$,

$1 \to 6 \to 4 \to 3, 3 \to 2 \to 6 \to 5$,

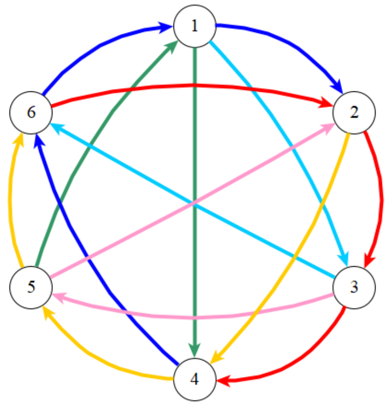
$1 \to 3 \to 5 \to 2$,

$2 \to 5 \to 1, 4 \to 1 \to 5$,

$5 \to 3 \to 6, 3 \to 1 \to 4$,

Not Pictured: Take remaining arcs as $x_1 = 1$ copies of $\mathscr{D}P_2$.

Subcase 1d. $x_1 = 0, x_2 = 6, x_3 = 6, x_4 = 0$



Pictured: $4 \to 6 \to 1 \to 2$,

$6 \to 2 \to 3 \to 4$,

$2 \to 4 \to 5 \to 6$,

$1 \to 3 \to 6$,

$3 \to 5 \to 2$,

$5 \to 1 \to 4$

Not Pictured: Take reverses of these paths.

Subcase 1e. $x_1 = 4, x_2 = 1, x_3 = 4, x_4 = 3$

Pictured: $1 \to 2 \to 4 \to 5 \to 3$,

$2 \to 3 \to 5 \to 6 \to 4$,

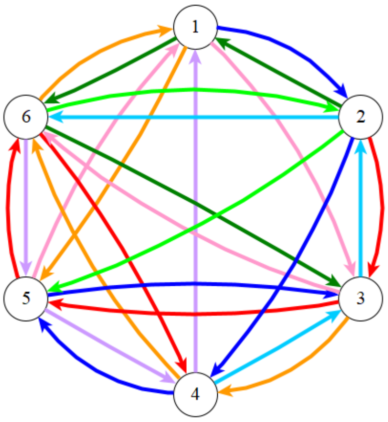
$3 \to 4 \to 6 \to 1 \to 5$,

$2 \to 1 \to 6 \to 3$, $4 \to 3 \to 2 \to 6$,

$6 \to 5 \to 4 \to 1$, $5 \to 1 \to 3 \to 6$,

$6 \to 2 \to 5$

Not Pictured: Take remaining arcs as $x_1 = 4$ copies of $\mathscr{D}P_2$.



Subcase 1f. $x_1 = 3, x_2 = 3, x_3 = 3, x_4 = 3$

Pictured: $1 \to 2 \to 4 \to 5 \to 3$,

$2 \to 3 \to 5 \to 6 \to 4$,

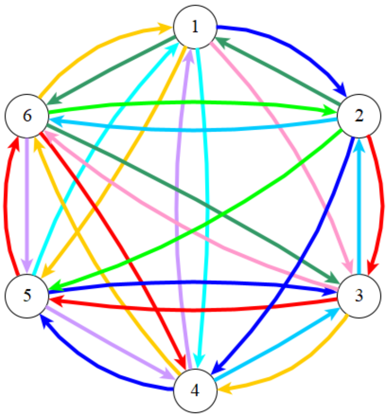
$3 \to 4 \to 6 \to 1 \to 5$,

$2 \to 1 \to 6 \to 3$, $4 \to 3 \to 2 \to 6$,

$6 \to 5 \to 4 \to 1$,

$6 \to 2 \to 5$, $1 \to 3 \to 6$, $5 \to 1 \to 4$

Not Pictured: Take remaining arcs as $x_1 = 3$ copies of $\mathscr{D}P_2$.



Subcase 1g. $x_1 = 2, x_2 = 5, x_3 = 2, x_4 = 3$

Pictured: $1 \to 2 \to 4 \to 5 \to 3$,

$2 \to 3 \to 5 \to 6 \to 4$,

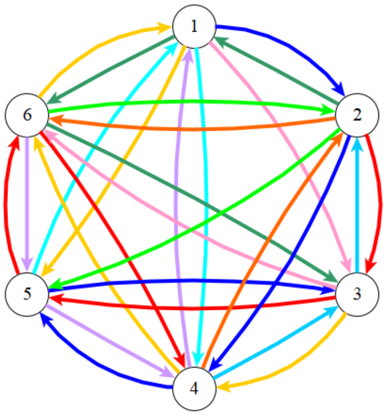
$3 \to 4 \to 6 \to 1 \to 5$,

$2 \to 1 \to 6 \to 3$, $6 \to 5 \to 4 \to 1$,

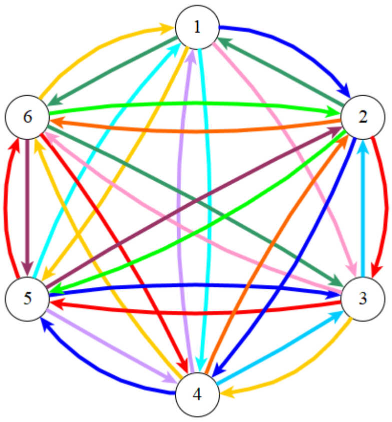
$6 \to 2 \to 5$, $1 \to 3 \to 6$, $5 \to 1 \to 4$,

$4 \to 3 \to 2$, $4 \to 2 \to 6$

Not Pictured: Take remaining arcs as $x_1 = 2$ copies of $\mathscr{D}P_2$.



Subcase 1h. $x_1 = 1, x_2 = 7, x_3 = 1, x_4 = 3$



Pictured: $1 \to 2 \to 4 \to 5 \to 3$,

$2 \to 3 \to 5 \to 6 \to 4$,

$3 \to 4 \to 6 \to 1 \to 5$,

$2 \to 1 \to 6 \to 3$,

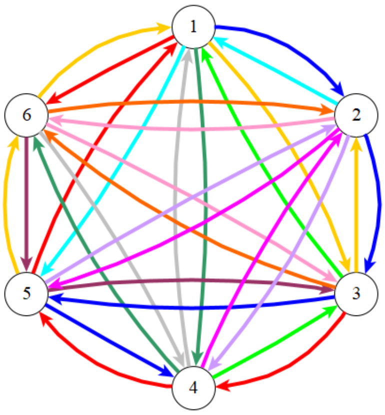
$6 \to 2 \to 5$, $1 \to 3 \to 6$, $5 \to 1 \to 4$,

$4 \to 3 \to 2$, $4 \to 2 \to 6$, $5 \to 4 \to 1$,

$6 \to 5 \to 2$

Not Pictured: Take remaining arcs as $x_1 = 1$ copies of $\mathscr{D}P_2$.

Subcase 1i. $x_1 = 0, x_2 = 9, x_3 = 0, x_4 = 3$



Pictured: $1 \to 2 \to 3 \to 5 \to 4$,

$3 \to 4 \to 5 \to 1 \to 6$,

$5 \to 6 \to 1 \to 3 \to 2$,

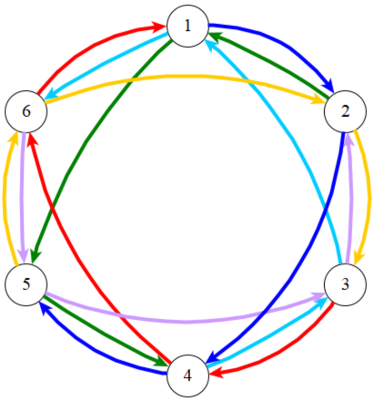
$2 \to 1 \to 5$, $6 \to 5 \to 3$, $4 \to 3 \to 1$,

$5 \to 2 \to 4$, $4 \to 2 \to 5$, $3 \to 6 \to 2$,

$2 \to 6 \to 3$, $1 \to 4 \to 6$, $6 \to 4 \to 1$

Case 2: $x_1 + x_2 + x_3 + x_4 = 18$

Subcase 2a. $x_1 = 12, x_2 = 0, x_3 = 6, x_4 = 0$



Pictured:

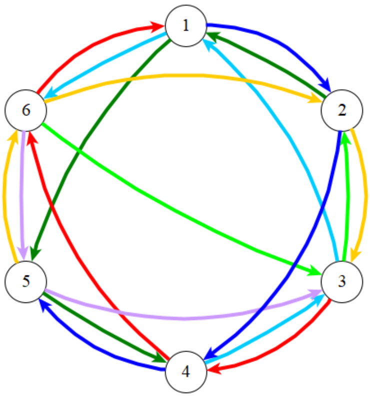
$1 \to 2 \to 4 \to 5$, $2 \to 1 \to 5 \to 4$,

$5 \to 6 \to 2 \to 3$, $6 \to 5 \to 3 \to 2$,

$3 \to 4 \to 6 \to 1$, $4 \to 3 \to 1 \to 6$

Not Pictured: Take remaining arcs as $x_1 = 12$ copies of $\mathscr{D}P_2$.

Subcase 2b. $x_1 = 11, x_2 = 2, x_3 = 5, x_4 = 0$



Pictured:

$1 \to 2 \to 4 \to 5, 2 \to 1 \to 5 \to 4,$

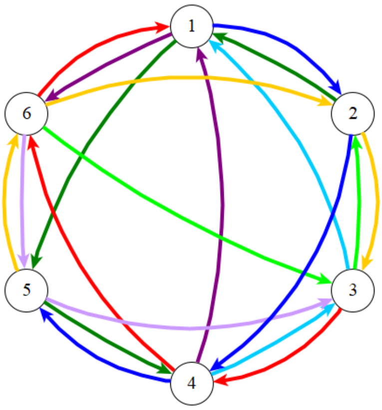
$3 \to 4 \to 6 \to 1, 4 \to 3 \to 1 \to 6$

$5 \to 6 \to 2 \to 3,$

$6 \to 5 \to 3, 6 \to 3 \to 2$

Not Pictured: Take remaining arcs as $x_1 =$ 11 copies of $\mathscr{D}P_2$.

Subcase 2c. $x_1 = 10, x_2 = 4, x_3 = 4, x_4 = 0$



Pictured: $1 \to 2 \to 4 \to 5, 2 \to 1 \to 5 \to 4,$

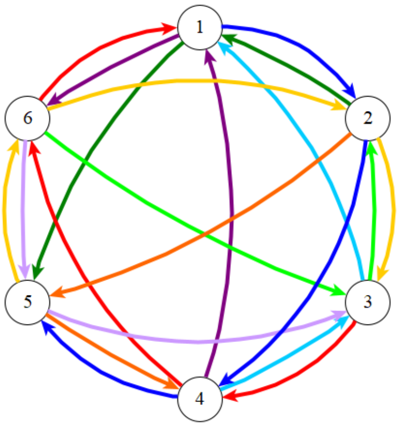
$3 \to 4 \to 6 \to 1, 5 \to 6 \to 2 \to 3,$

$6 \to 5 \to 3, 6 \to 3 \to 2,$

$4 \to 3 \to 1, 4 \to 1 \to 6$

Not Pictured: Take remaining arcs as $x_1 =$ 10 copies of $\mathscr{D}P_2$.

Subcase 2d. $x_1 = 9, x_2 = 6, x_3 = 3, x_4 = 0$



Pictured:

$3 \to 4 \to 6 \to 1, 5 \to 6 \to 2 \to 3,$

$1 \to 2 \to 4 \to 5,$

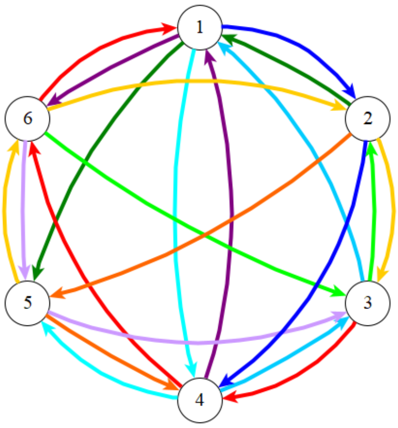
$6 \to 5 \to 3, 6 \to 3 \to 2,$

$4 \to 3 \to 1, 4 \to 1 \to 6,$

$2 \to 1 \to 5, 2 \to 5 \to 4$

Not Pictured: Take remaining arcs as $x_1 = 9$ copies of $\mathscr{D}P_2$.

Subcase 2e. $x_1 = 8, x_2 = 8, x_3 = 2, x_4 = 0$



Pictured:

$3 \to 4 \to 6 \to 1, 5 \to 6 \to 2 \to 3,$

$6 \to 5 \to 3, 6 \to 3 \to 2,$

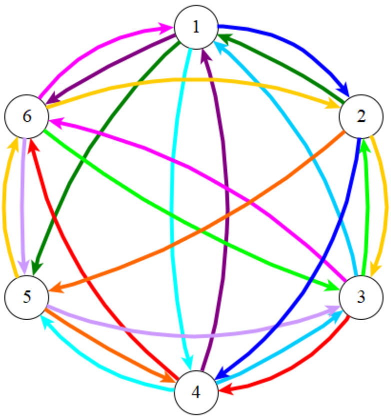
$4 \to 3 \to 1, 4 \to 1 \to 6,$

$2 \to 1 \to 5, 2 \to 5 \to 4,$

$1 \to 2 \to 4, 1 \to 4 \to 5$

Not Pictured: Take remaining arcs as $x_1 = 8$ copies of $\mathscr{D}P_2$.

Subcase 2f. $x_1 = 7, x_2 = 10, x_3 = 1, x_4 = 0$



Pictured: $5 \to 6 \to 2 \to 3,$

$6 \to 5 \to 3, 6 \to 3 \to 2,$

$4 \to 3 \to 1, 4 \to 1 \to 6,$

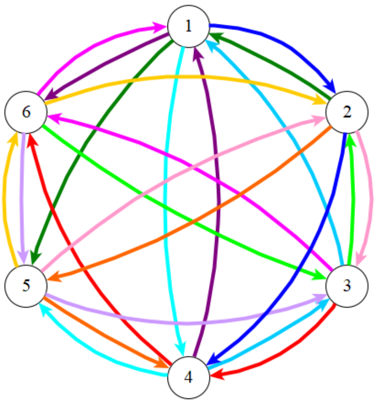
$2 \to 1 \to 5, 2 \to 5 \to 4,$

$1 \to 2 \to 4, 1 \to 4 \to 5,$

$3 \to 4 \to 6, 3 \to 6 \to 1$

Not Pictured: Take remaining arcs as $x_1 = 7$ copies of $\mathscr{D}P_2$.

Subcase 2g. $x_1 = 6, x_2 = 12, x_3 = 0, x_4 = 0$



Pictured: $6 \to 5 \to 3, 6 \to 3 \to 2,$

$4 \to 3 \to 1, 4 \to 1 \to 6,$

$2 \to 1 \to 5, 2 \to 5 \to 4,$

$1 \to 2 \to 4, 1 \to 4 \to 5,$

$3 \to 4 \to 6, 3 \to 6 \to 1,$

$5 \to 6 \to 2, 5 \to 2 \to 3$

Not Pictured: Take remaining arcs as $x_1 = 6$ copies of $\mathscr{D}P_2$.

Case 3: $x_1 + x_2 + x_3 + x_4 = 24$

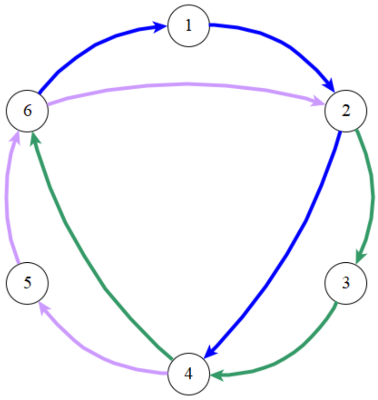
Subcase 3a. $x_1 = 21, x_2 = 0, x_3 = 3, x_4 = 0$

Pictured: $6 \to 1 \to 2 \to 4$,

$2 \to 3 \to 4 \to 6$,

$4 \to 5 \to 6 \to 2$

Not Pictured: Take remaining arcs as $x_1 =$ 21 copies of $\mathscr{D}P_2$.



Subcase 3b. $x_1 = 20, x_2 = 2, x_3 = 2, x_4 = 0$

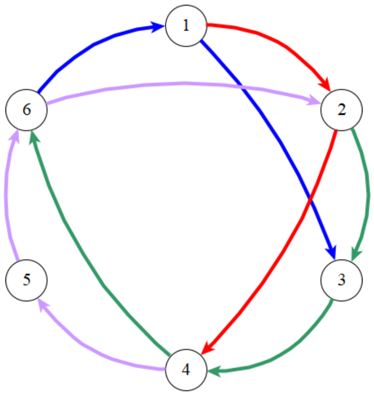
Pictured: $2 \to 3 \to 4 \to 6$,

$4 \to 5 \to 6 \to 2$,

$6 \to 1 \to 3$,

$1 \to 2 \to 4$

Not Pictured: Take remaining arcs as $x_1 =$ 20 copies of $\mathscr{D}P_2$.



Subcase 3c. $x_1 = 19, x_2 = 4, x_3 = 1, x_4 = 0$

Pictured: $4 \to 5 \to 6 \to 2$,

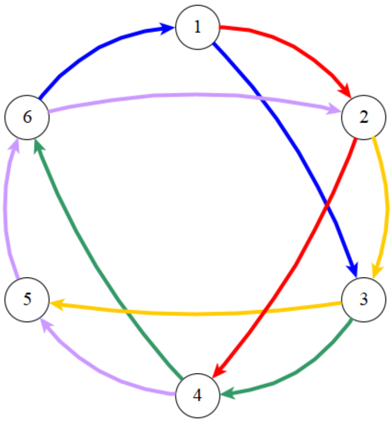
$1 \to 2 \to 4$,

$2 \to 3 \to 5$,

$3 \to 4 \to 6$,

$6 \to 1 \to 3$

Not Pictured: Take remaining arcs as $x_1 =$ 19 copies of $\mathscr{D}P_2$.



Subcase 3d. $x_1 = 18, x_2 = 6, x_3 = 0, x_4 = 0$

Pictured: $1 \to 2 \to 4$,

$2 \to 3 \to 5$,

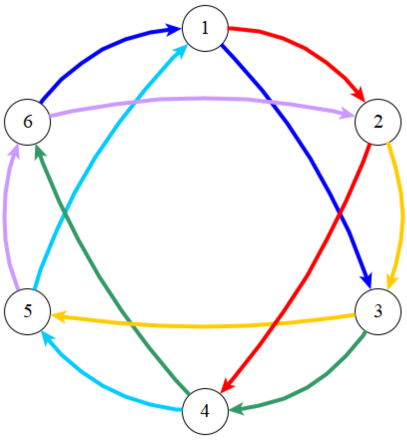
$3 \to 4 \to 6$,

$4 \to 5 \to 1$,

$5 \to 6 \to 2$,

$6 \to 1 \to 3$

Not Pictured: Take remaining arcs as $x_1 =$ 18 copies of $\mathscr{D}P_2$.



Case 4: $x_1 + x_2 + x_3 + x_4 = 30$. The only solution to this equation meeting the necessary conditions in Section 3.3.5 is $x_1 = 30, x_2 = 0, x_3 = 0, x_4 = 0$. Lemma 3.4 solves this case.

□

Theorems 3.2, 3.3, and 3.6 give the complete spectrum for BNHDPD$(6,k)$. We summarize this result in Theorem 3.7.

Theorem 3.7. *A BNHDPD$(6,k)$ exists if and only if the following conditions all hold:*

1. $x_1 + 2x_2 + 3x_3 + 4x_4 = 30$
2. $6 \mid x_1 + x_2 + x_3 + x_4$
3. $6 \mid x_2 + 2x_3 + 3x_4$

3.3.7 Sample *Designed* Task Sets

The task sets given in this section are constructed using Subcase 1d of the proof of Theorem 3.4. First, we show a task set where the skills are derivatives of specific functions.

Task Set 1. Let $1 \mapsto x^2$, $2 \mapsto \sin x$, $3 \mapsto \ln x$, $4 \mapsto e^x$, $5 \mapsto \arctan x$. Then, the balanced task set is

1. $e^{\sin x}$
2. $\sin(\arctan x)$

3. $\ln(\arctan x)$
4. $\arctan(\ln(x^2))$
5. $\arctan(\sin(e^x))$
6. $e^{(\ln x)^2}$
7. $(\sin(\ln x))^2$
8. $(e^{\ln(\sin x)})^2$
9. $\ln(e^{\arctan(x^2)})$
10. $\sin((\arctan(e^x))^2)$

Next, we show a task set where the skills are derivatives of simple function classes and values within each class are randomly chosen.

Task Set 2. Let $1 \in \{x^n \mid 2 \leq n \leq 10\}$, $2 \in \{\sin x, \cos x, \tan x, \sec x, \csc x, \cot x\}$, $3 \in \{\ln x, \log_a x \text{ where } 2 \leq a \leq 10\}$, $4 \in \{e^x, a^x \text{ where } 2 \leq a \leq 10\}$, $5 \in \{\arcsin x, \arccos x, \arctan x\}$. To randomly choose values for each task, the elements of each set defined above were converted to integer values, which were then chosen using the following Excel 2020 formulas.

1. **Power functions** were chosen using `=RANDBETWEEN(2,10)`, where the value of this formula corresponds to the exponent of the power function.

2. **Trigonometric functions** were chosen using `=RANDBETWEEN(1,6)`, where the value of this function corresponds to one of the six trigonometric functions as ordered above.

3. **Exponential and logarithmic functions** were chosen using a two-step formula. First, `=RANDBETWEEN(0,1)` chose between a natural exponential or logarithm and a general exponential or logarithm. This was followed by `=IF(`[cell containing 0 or 1]`=1, RANDBETWEEN(2,10),0)` to choose the base of a general exponential or logarithm.

4. **Inverse trigonometric functions** were chosen using `=RANDBETWEEN(1,3)`, where value of this function corresponds to one of the three inverse trigonometric functions as ordered above.

The first balanced task set produced by this code was

1. $7^{\sec x}$
2. $\csc(\arccos x)$
3. $\log_8(\arccos x)$
4. $\arctan(\ln(x^7))$

5. $\arcsin(\csc(e^x))$
6. $6^{(\log_7 x)^8}$
7. $(\csc(\log_5 x))^5$
8. $(e^{\ln(\csc x)})^4$
9. $\ln(3^{\arcsin(x^{10})})$
10. $\csc((\arccos(e^x))^{10})$

3.3.8 Conclusion

In this section, we established existence theorems for balanced non-Hamiltonian directed path decompositions of the complete directed graph. This extends prior work by Meszka & Skupień [90] by requiring that the decomposition be balanced. Furthermore, these results are applicable to the construction of task sets in Calculus I that assess the Chain Rule due to a correspondence between labeled directed paths and composite functions.

Future Directions

A natural future avenue to explore is the existence of balanced (or near-balanced) task sets assessing additional derivative computation skills: sums, products, and quotients of simple functions of one variable. Since these are binary operations, we will need to decompose the complete directed graph into subgraphs containing directed stars. Preliminary mathematical results by Colbourn, Hoffman, and Rodger [71] are established for the existence of directed star decompositions of the complete directed graph.

Alternatively, we could consider a different type of practice in the task set. Rather than constructing mixed practice task sets, as we did here with the complete directed graph, we can consider the complete directed multigraph with index λ for task sets with a hybrid mixed-block practice. Meszka & Skupień's work on non-Hamiltonian directed paths [90] and Colbourn, Hoffman, & Rodger's work on directed stars [71] contain analogous results regarding decompositions of the complete directed multigraph with index λ.

3.4 Closing Remarks

In this chapter, we developed and implemented a novel method (elementary function trees) for analyzing derivative computation tasks to discover a source of misalignment between mathematics instructor expectations and the practice available to students through textbook tasks; mathematicians expect students to be able to solve a novel task based on an equal mastery of skills through practice, but the practice that textbooks provide focuses on the power rule significantly more than any other derivative computation skill.

We also demonstrated a misalignment between mathematics education researchers classification of derivative computation tasks as "simple procedures" and their inherent complexity revealed by modeling them as elementary function trees.

Finally, we provided a solution to the first source of misalignment (unbalanced practice) by proving new existence results for a specific type of combinatorial design and using the constructions in the proofs to create balanced Chain Rule homework assignments.

The applications of this chapter demonstrate that while Combinatorial Designs are a highly abstract area of discrete mathematics, they have the potential for meaningful applications in undergraduate mathematics education. Since such applications are, as of now, widely understudied, it invites the possibility of rich interdisciplinary collaboration, uniting two fields, Combinatorial Designs and Undergraduate Mathematics Education, in novel ways.

Chapter 4

A Scholarly Reflection on Using Digraphs to Redesign a Curriculum

Students call weed-out classes “hard”, but it has been found that all student-identified characteristics of weed-out courses are due to systemic parameters rather than being intrinsic to the subject matter [139]. This brought to mind teachers in my past who were talented at creating the illusion of lower content volume and slower pace during class. By the end of the semester, I always realized I learned a lot more than I realized during each class. This motivated what I call *The Curriculum Isomorphism Problem*, inspired by graph isomorphisms and some basic notions of posets. This chapter is written using the math education convention where “I” is used to indicate a single contributor; I anticipate that in the future, this project will be aimed at a math education audience.

4.1 The ~~Graph~~ Curriculum Isomorphism Problem

Two graphs G and H are *isomorphic* if there exists a bijective function $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H [10]. In other words, visually, two graphs are isomorphic if I can change the position of the vertices of one graph to make the other graph; for example, see the caterpillar and the butterfly in Figure 4.1. Studying isomorphic graphs naturally leads to the Graph Isomorphism Problem.

> **The Graph Isomorphism Problem** is the problem of determining whether two graphs are isomorphic.

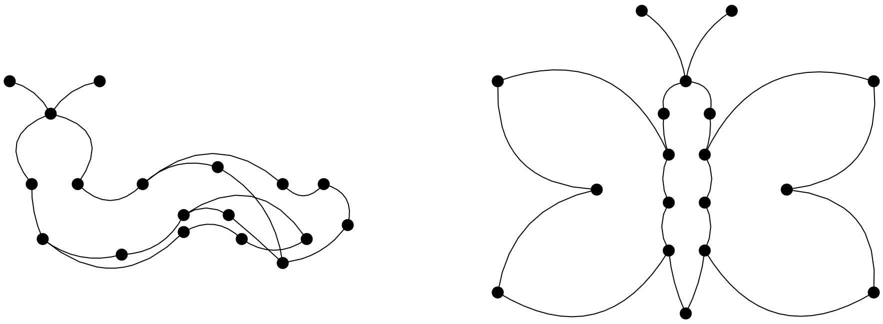


Figure 4.1: A caterpillar and a butterfly that are isomorphic.

In Graph Theory and Computer Science, an open question is whether the Graph Isomorphism Problem is solvable in polynomial time. However, the Graph Isomorphism Problem has been solved for some specific classes of graphs [107, 113, 116, 120].

Although the Graph Isomorphism Problem inspired the Curriculum Isomorphism Problem, fortunately, the Curriculum Isomorphism problem is a little easier. Rather than finding an isomorphism between two graphs (or lack thereof), the goal is to find a "more digestible" graph that is isomorphic to the original. To unpack what I mean by a "more digestible" graph, I introduce a special type of graph called a Course Structure Map.

A *Course Structure Map* is a drawing of a digraph, $D = (V, E)$, such that the following conditions hold:

1. V is the set of topics covered in the course.
2. $E = \{(u, v) \mid u, v \in V$, topic u is required to learn topic v in the textbook$\}$
3. Define the *lecture rank function*, $\mathscr{L} : V \to \mathbb{P}$ by $v \mapsto l$, where l is the day of the course in which the topic is covered. The vertices are positioned in decreasing order of their lecture rank value from top to bottom.
4. Define the *test function*, $\mathscr{T} : V \to \mathbb{P}$ by $v \mapsto t$, where t is the test on which topic v appears. Define $T_t = \{v \in V \mid \mathscr{T}(v) = t\}$ for $t = 1, \ldots, n$, where n is the total number of tests in the semester (excluding the final exam). Assign each $T_1, \ldots, T_n$ a unique color $c_1, \ldots, c_n$.

5. Define the *chapter function*, $\mathscr{C} : V \to \mathbb{P}$ by $v \mapsto k$, where k is the chapter of the textbook in which topic v first appears. Define $C_k = \{v \in V \mid \mathscr{C}(v) = k\}$ for $k = 1, \ldots, m$, where m is the total number of chapters covered in the semester. Assign each $C_1, \ldots, C_m$ a unique shape $s_1, \ldots, s_m$.

This definition assumes that a course is taught from a textbook and assesses students through exams. This decision was made to reflect how large lower-division mathematics courses are typically taught in universities. However, note that this definition could be adapted to model mathematics courses taught using an alternative pedagogy.

The vertex positioning condition (Condition 3) makes a Course Structure Map reminiscent of the Hasse Diagram of a poset, with a slight difference. A *partially ordered set*, or *poset*, is an ordered pair $(X, \leq)$, where X is a set and $\leq$ is a *partial ordering* of the elements of X - a homogeneous relation that is reflexive, antisymmetric, and transitive. For a course structure map, X is the set of course topics, and the partial ordering is the order in which they are taught. Since, in a course, no two topics can be taught simultaneously, a course structure map is a total order, meaning that any two elements are comparable. A *Hasse diagram* of a poset is a drawing in which points represent the elements of X, and a line segment is drawn upward from a vertex x to another vertex y if *y covers x*, that is, whenever $x \neq y$, $x \leq y$ and there is no z distinct from x and y with $x \leq z \leq y$. In the case of a course structure map, though we have a total order, we want to position vertices at the same vertical height that will be taught during the same lecture.

The course structure map for Auburn's Calculus I was constructed using the list of required textbook sections provided to instructors teaching Calculus I, ordered via pacing guides. The left digraph in Figure 4.2 depicts this Course Structure Map.

According to Cognitive Load Theory, the *temporal split-attention effect* is the increase in cognitive load that occurs when two related topics are spaced further apart in time [137]. If related topics are positioned closer together temporally, it becomes easier to see their relationships. This is why courses are naturally organized into content units. To determine if the topics in the course could be reordered to reduce the temporal split-attention effect for students, I rearranged the vertices so that

1. Topics remained in a logical order. That is, no arrows in the drawing had a negative slope unless the reason for initially drawing the arrow was due to something in the textbook that an instructor typically would not discuss during class. For example, our textbook presents the “derivative of logarithmic functions” after “implicit differentiation” because it discusses the derivation of the logarithm’s derivative using implicit differentiation. Since this derivation is not something Calculus I instructors typically spend class time discussing, placing the “derivative of logarithmic functions” before implicit differentiation does not disrupt the logical flow of the course.

2. All arrows were made as short as possible (i.e., decreasing the temporal split-attention effect) without compromising the practicality of the course. For example, some content needs to be taught on all non-exam lecture days, so the rearranged map must have a prescribed height. Futhermore, if too many topics are taught during one lecture, then it would be impossible to discuss them adequately, restricting the width of the rearranged map.

This background knowledge allows us to state the Curriculum Isomorphism Problem now.

> **The Curriculum Isomorphism Problem** is the problem of finding a course structure map (graph) isomorphic to the original that reduces the overall temporal split-attention effect of the course.

Figure 4.2 (Right) shows the map that resulted from rearranging the initial map in this way. Comparing the two drawings visually reveals some notable differences. At the start of the course (located at the bottom of the map), the drawing on the left is significantly more cluttered than the one on the right. During this part of the semester, freshmen are still trying to adjust to the demands of college. Therefore, the less cluttered drawing on the right may be more beneficial for them during this transition period.

Moving to the middle of the maps, the right drawing is wider than the left drawing. This occurred due to shifting various derivative topics for the same reason as the logarithm function described previously. Since proofs and formula derivations are largely avoided by Calculus I instructors, this calls for introducing derivative rules with a list of “basic facts” (see Figure 4.3).

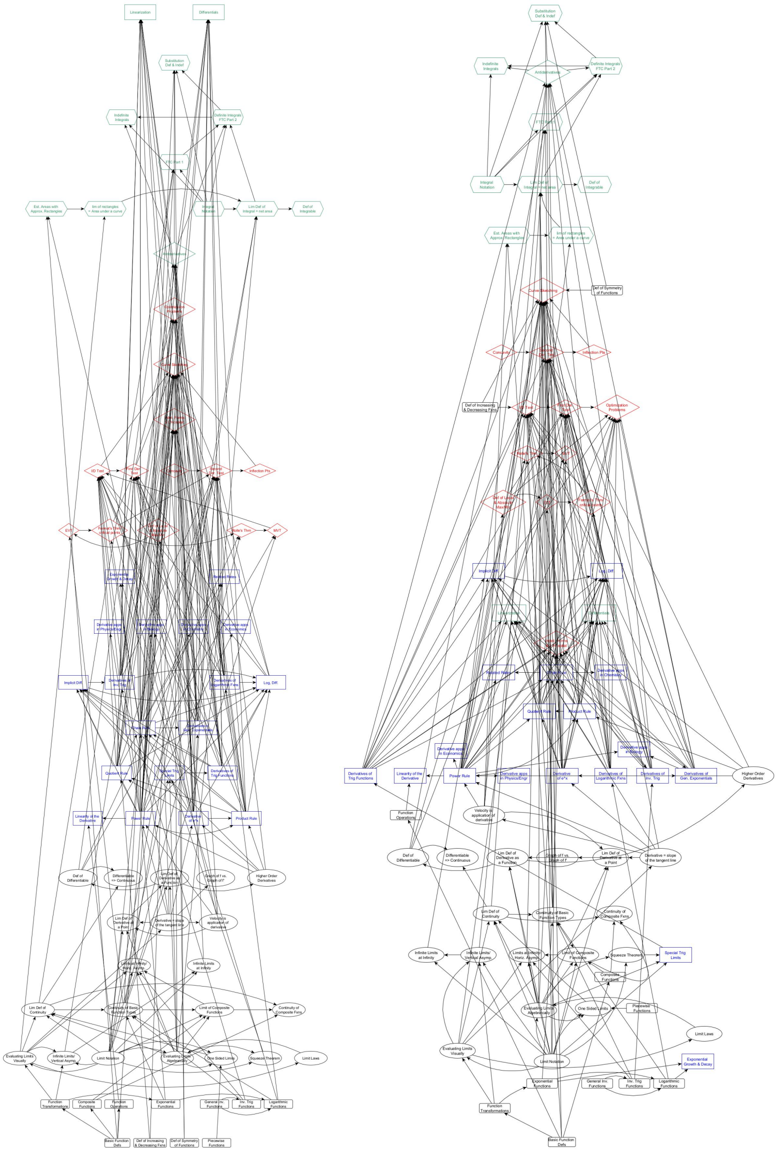
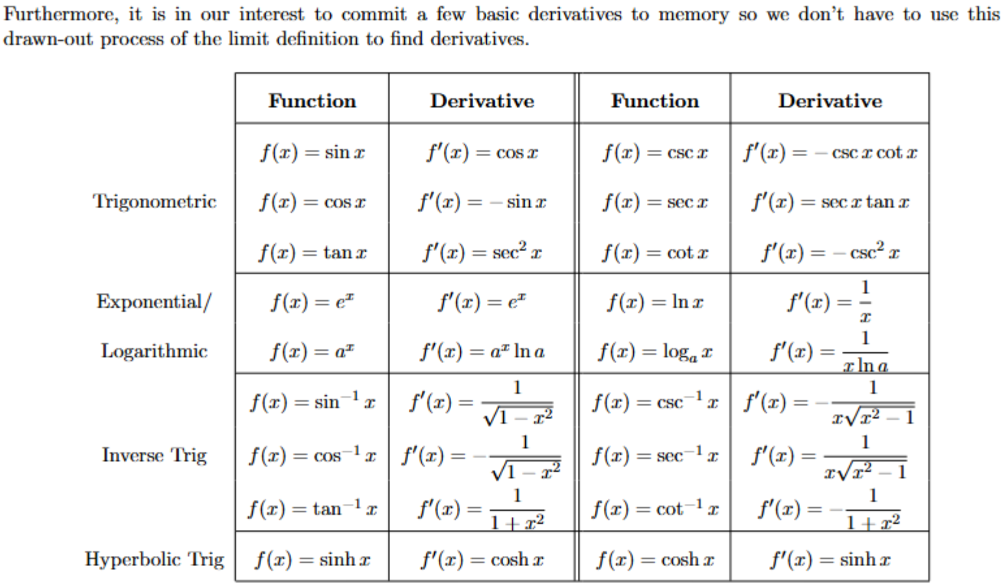


Figure 4.2: (Left) Auburn's Calculus I list of topics via the textbook, and (Right) The same course untangled a little.

For students, this is reminiscent of how they learn most computational topics in primary and secondary school. First, they are introduced to a new concept, such as addition, through a verbal definition (like "smooshing" two groups of things) and possibly explore the topic through the manipulation of physical objects (like base-10 blocks). After they understand the concept, they memorize some facts (like $2+4=6$) to aid in the efficiency of more complex calculations.

Another benefit of this approach is that it allows for more variety in derivative computation tasks earlier in the course. In the traditional order of topics (Power Rule and Exponentials →



Furthermore, it is in our interest to commit a few basic derivatives to memory so we don't have to use this drawn-out process of the limit definition to find derivatives.

	Function	**Derivative**	**Function**	**Derivative**
Trigonometric	$f(x) = \sin x$	$f'(x) = \cos x$	$f(x) = \csc x$	$f'(x) = -\csc x \cot x$
	$f(x) = \cos x$	$f'(x) = -\sin x$	$f(x) = \sec x$	$f'(x) = \sec x \tan x$
	$f(x) = \tan x$	$f'(x) = \sec^2 x$	$f(x) = \cot x$	$f'(x) = -\csc^2 x$
Exponential/	$f(x) = e^x$	$f'(x) = e^x$	$f(x) = \ln x$	$f'(x) = \frac{1}{x}$
Logarithmic	$f(x) = a^x$	$f'(x) = a^x \ln a$	$f(x) = \log_a x$	$f'(x) = \frac{1}{x \ln a}$
Inverse Trig	$f(x) = \sin^{-1} x$	$f'(x) = \frac{1}{\sqrt{1-x^2}}$	$f(x) = \csc^{-1} x$	$f'(x) = -\frac{1}{x\sqrt{x^2-1}}$
	$f(x) = \cos^{-1} x$	$f'(x) = -\frac{1}{\sqrt{1-x^2}}$	$f(x) = \sec^{-1} x$	$f'(x) = \frac{1}{x\sqrt{x^2-1}}$
	$f(x) = \tan^{-1} x$	$f'(x) = \frac{1}{1+x^2}$	$f(x) = \cot^{-1} x$	$f'(x) = -\frac{1}{1+x^2}$
Hyperbolic Trig	$f(x) = \sinh x$	$f'(x) = \cosh x$	$f(x) = \cosh x$	$f'(x) = \sinh x$

Figure 4.3: Rearranging topics in Calculus I calls for students to begin learning derivative rules by memorizing some "basic facts".

Product and Quotient rule → Trigonometric Derivatives → Chain Rule), there is little variety in possible introductory tasks that isolate one particular rule. For example, with the product and quotient rules, many problems can be simplified using polynomial arithmetic, allowing students to circumvent practicing with these complex formulas. Furthermore, students' initial exposure to the Chain Rule only includes power, exponential, and trigonometric functions, which may make it more challenging to recognize instances of Chain Rule involving other function types (such as logarithms). As an instructor, I aim to use derivative computations to develop problem-solving skills. Greater variety earlier in the unit forces students to attempt more novel problem types.

While formal research needs to be done to ascertain differences in students' learning between approaches to teaching Derivative Rules, I have received positive feedback from students after implementing this approach to derivative computations for three semesters in Calculus I.

4.2 *The War Eagle Calculus I Workbook*

In Fall 2023, Dr. Melinda Lanius served as instructor of record for one section of Calculus I with an initial enrollment of approximately 250 students. Three other Graduate Student Instructors (GSIs) and I were each assigned as recitation leaders to two 32-person recitation sections.

The class met in a 2-2 lecture-recitation format. Lectures met for 75 minutes twice per week, and recitations met for 50 minutes twice per week. We taught the course using *The War Eagle Calculus Workbook*, which consisted of 21 two-day lessons divided into four units. The goal of using the workbook was to present Calculus I content in an efficient way that was aligned to existing coordination guidelines. At the end of the semester, students in this section significantly outperformed the other sections on the common final exam (our section's average score was $\approx 72\%$) and had a significantly lower DFWI rate than the other sections (our section's DFWI rate was $\approx 12\%$).

In October 2023, after securing IRB approval, we asked students for consent for their data to be used for research purposes. 99 of our students agreed to participate. We will use this data to answer future research questions about students' note-taking habits, homework habits, review habits, and metacognition.

The remainder of this section describes the research that inspired the initial design choices in *The War Eagle Calculus I Workbook* and course pedagogical choices with embedded samples from the workbook to illustrate each topic. A full sample lesson from the first edition of *The War Eagle Calculus I Workbook* is included in Appendix A.

4.2.1 Deep Learning

With recent advances in technology, a current trend in mathematics education is to de-emphasize routine computation and instead emphasize "Deep Learning" in mathematics courses. McPhail [127] defines deep learning as "the ability to see relationships between epistemic parts and wholes of a subject". In other words, deep learning is the ability to make connections between seemingly independent facts, allowing a particular subject to be seen as a complex network of facts and their relationships. Coming from a Graph Theory background, I find this idea fascinating, mostly because as mathematicians, this is precisely what we do – we search for connections between things in the form of theorems, trying to unify disparate branches that make up the wild mathematical frontier. If this is how we as researchers view mathematics, it is reasonable to want to give our students a taste of it.

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[1] I want to acknowledge my dad, Brendon Gilroy, for naming the workbook.

Meyers and Nulty [128] list three characteristics of students adopting a deep approach to learning: (1) wanting to develop their own understanding of the material, (2) an ability to apply ideas to new situations, and (3) a highly developed integration of knowledge. However, they also note that not all students are able to do this naturally; many need teachers' guidance. On this note, they list five principles of course materials designed to promote deep learning: (1) authentic, real-world, and relevant, (2) constructive, sequential, and interlinked, (3) require progressively higher order cognitive processes, (4) aligned to the desired learning outcomes, and (5) provide challenge, interest, and motivation to learn.

In creating the workbook, the number-one goal was to promote a deeper understanding of Calculus that will set students up for success in future math courses. The remainder of the topics addressed in this section are interwoven with Meyers and Nulty's [128] five principles in mind.

4.2.2 Content Objectives

Currently, many colleges and universities are transitioning to an alternative grading system called Mastery (or Standards) Based Grading. Elsinger and Lewis [114] define a Standards Based Grading system as one in which: (1) At the beginning of the semester, students are provided with a clear list of course objectives that they are expected to know by the end, (2) Final course grades are based primarily on how many standards a student masters throughout the semester, (3) Students have multiple opportunities to reassess mastery of any objective, and (4) Attempts are graded using pass/fail, with the best score counting toward the final grade. Today, there are concerns about overinflation of grades and that grades do not represent the actual knowledge a student has gained from a course [111]. Mastery-based grading seems like the perfect solution to the grading problem and has been successfully implemented with positive feedback from students at some institutions [114, 143]. With current institutional constraints and limited resources, I felt that transitioning entirely to such a system was a poor choice. However, this may be a future avenue of exploration.

For now, the workbook satisfies the first constraint of mastery-based grading. That is, the content of the course is based on a series of learning goals (or objectives) that the students will be expected to achieve throughout the course. Currently, there is no known such list for Auburn's Calculus I course – merely a list of sections of the textbook that must be covered. Elsinger and Lewis [114] credit Kate Owens with separating learning objectives into "big questions" that students must answer. I took this idea further by addressing "little questions" within these big questions. Finally, I specified base, relational, and procedural outcomes within these little questions, such as knowledge of definitions, theorems, and problems/applications. This structure may be visualized as a pyramid (see Figure 4.4).

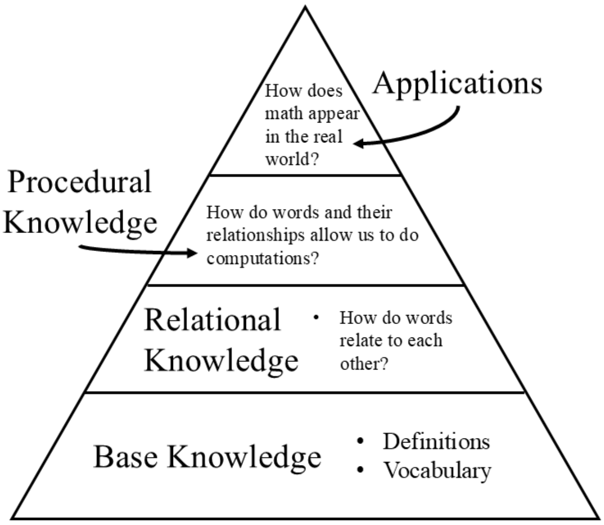


Figure 4.4: Little Question content objective types, visualized as a pyramid.

We include here the comprehensive list of content objectives from the workbook.

Comprehensive List of Content Objectives

At the end of this course, you should be able to answer the following questions and complete the tasks listed under each question.

BQ1: How do we find the slope of a line using only one point?

LQ1: What is the problem with finding the slope of a line using only one point, and how do we start to fix this?

i. Define slope, secant line, tangent line, average rate of change, limit, average velocity.
ii. Relate secant line, average rate of change, and average velocity through slope.
iii. Relate slopes of secant and tangent lines through limits.
iv. Use the slope formula to evaluate the slope of a secant line/average rate of change over an interval/average velocity over an interval.
v. Solve problems about applications of slopes of secant lines to real-world scenarios.
vi. Evaluate limits from a graph.

LQ2: What are some other kinds of limits, and why are they necessary?

i. Define one-sided limit, infinite limit, jump discontinuity, infinite discontinuity, vertical asymptote, domain, continuity (intuitively), limit at infinity, horizontal asymptote
ii. Relate one-sided limit, jump discontinuity through piecewise functions.
iii. Relate infinite limit, vertical asymptote, domain through specific rational functions.
iv. Relate one-sided limit, infinite limit, jump discontinuity, vertical asymptote, domain through rational functions, the natural log function, and the tangent function.
v. Relate limit at infinity, horizontal asymptote through rational functions, the natural exponential function, and inverse tangent function.
vi. Evaluate one-sided limits, infinite limits/vertical asymptotes from a graph and a table of values.
vii. Investigate real-world scenarios involving one-sided limits, infinite limits, and limits at infinity.

LQ3: How do we evaluate limits algebraically?

  i. Define strategies for evaluating limits algebraically including: direct substitution, various algebraic simplifications, verbal argument, algebraic manipulation to use known limits.
  ii. Identify an appropriate strategy to evaluate a limit algebraically based on features of the problem.
  iii. Apply an appropriate strategy to evaluate a limit algebraically.

LQ4: How can limits help us to classify functions as continuous?

  i. Define continuity in terms of limits.
  ii. Define types of discontinuities in terms of limits.
  iii. Identify common classes of continuous functions.
  iv. Apply the limit definition of continuity to various functions to determine intervals of continuity.
  v. Apply continuous and discontinuous functions to real-life scenarios.

LQ5: Why is it useful to classify functions as continuous?

  i. Define the IVT.
  ii. Explore why the conditions of the IVT are necessary.
  iii. Apply the IVT in abstract mathematical and real-life scenarios.

LQ6: What is the derivative, and why is it useful?

  i. Define the derivative in terms of limits.
  ii. Relate the limit definition of the derivative to the concept of limiting the slope of a secant line.
  iii. Relate the graphs of a function and its derivative.
  iv. Apply differentiable functions to real-life scenarios.
  v. Use tangent lines to approximate functions.
  vi. Relate differentiable and continuous functions.

BQ2: How can we make finding derivatives easier?

LQ1: How can we find the derivatives of basic functions?

  i. Define the derivatives of trigonometric functions, exponential functions, logarithmic functions, inverse trigonometric functions, and hyperbolic sine and cosine.
  ii. Define the linearity properties of the derivative.
  iii. Use the definitions of basic derivatives and linearity of the derivative to find derivatives of sums and constant multiples of basic functions and piecewise functions containing basic function parts.

LQ2: What functions' derivatives can be calculated using the power rule, and how do we find these derivatives?

  i. Define the power rule.
  ii. Give examples of functions whose derivatives require using the power rule.
  iii. Identify functions whose derivatives require the power rule.
  iv. Find derivatives of functions that require the power rule.
  v. Apply derivatives involving both the power rule and basic functions to real-life scenarios.

LQ3: What functions' derivatives can be calculated using the product & quotient rules, and how do we find these derivatives?

  i. Define the product and quotient rules.
  ii. Give examples of functions whose derivatives require using the product and quotient rules.
  iii. Identify functions whose derivatives require the product/quotient rules.
  iv. Find derivatives of functions that require the product/quotient rules.
  v. (Cumulative) Determine efficient strategies for solving derivative problems with rules learned so far.

LQ4: What functions' derivatives can be calculated using the chain rule, and how do we find these derivatives?

  i. Define the chain rule.
  ii. Give examples of functions whose derivatives require using the chain rule.
  iii. Identify functions whose derivatives require the chain rule.
  iv. Find derivatives of functions that require the chain rule.
  v. (Cumulative) Determine efficient strategies for solving derivative problems with rules learned so far.

LQ5: What is implicit differentiation, and why is this strategy necessary for finding derivatives?

  i. Define implicitly defined functions, implicit differentiation.
  ii. Relate implicit differentiation to the Chain Rule.
  iii. Find derivatives using implicit differentiation.
  iv. Apply implicit differentiation to real-world scenarios (related rates questions).

LQ6: What is logarithmic differentiation, and how can this make finding complicated derivatives easier?

  i. Define logarithmic differentiation.
  ii. Relate logarithmic differentiation to the Chain Rule and implicit differentiation.
  iii. Find derivatives using logarithmic differentiation.

BQ5: What is the point of finding derivatives?

LQ1: How can knowing about derivatives make some of those algebra limit-solving strategies we learned about easier?

  i. Define L'Hospital's Rule
  ii. Determine whether limits can be solved using L'Hospital's Rule or not.
  iii. Solve limit problems using L'Hospital's Rule

    iv. Analyze a problem to determine if applying L'Hospital's Rule is more efficient than previously learned algebraic methods.

LQ2: What is the Mean Value Theorem, and how does it relate to things we've learned already?

    i. Define the Mean Value Theorem
    ii. Explore why the conditions of the MVT are necessary.
    iii. Apply the MVT in abstract mathematical and real-life scenarios.

LQ3: What is the first derivative specifically used for?

    i. Define local/global Maximum, local/global Minimum, optimization, increasing, decreasing.
    ii. Define Rolle's Theorem.
    iii. Optimize functions.
    iv. Apply optimization to real-life scenarios.

LQ4: What do the first and second derivatives tell us about a function's graph?

    i. Define the First Derivative Test.
    ii. Define Concavity.
    iii. Relate Concavity & Optimization.
    iv. Generate Examples of graphs exhibiting combinations of concavity and increasing/decreasing criteria.

LQ5: How can we figure out what a graph looks like just from the equation and what we know about limits and derivatives?

    i. Analyze a function for: domain, symmetry, asymptotes, increasing/decreasing, extrema, concavity, inflection points.
    ii. Graph a function given results of an analysis.

BQ3: How do we find the areas of weird shapes, and then make it easier?

LQ1: How is the integral defined conceptually?

   i. Define the left endpoint, right endpoint, and midpoint rules.
   ii. Define the integral as the limit of a Riemann Sum.
   iii. Relate integration, area, and accumulation.
   iv. Estimate/evaluate integrals using Riemann Sums, geometric interpretations.

LQ2: How are integrals related to derivatives?

   i. Define FTC.
   ii. Use FTC Part 1 to evaluate derivatives of functions defined in terms of integrals.

LQ3: How do we think about doing basic derivatives in reverse?

   i. Define antiderivatives of basic functions, reverse power rule.
   ii. Evaluate definite and indefinite integrals involving basic functions and/or polynomials.

LQ4: How do we think about doing chain rule in reverse?

   i. Define substitution method for integration.
   ii. Evaluate definite and indefinite integrals using substitution.

Furthermore, daily quizzes in recitation employed a variant of the pass/fail constraint of mastery-based grading. Vocabulary questions were not graded for partial credit.

4.2.3 Student Note-Taking Practices

Nearly four decades of research show that students, especially freshmen students, are not very good at taking notes [125, 108, 122, 118]. However, Locke [125] cites a quantitative study by Crawford from 1925, indicating that students' ability to take notes has been affecting their course grades for a lot longer than we might think. Various studies have shown that note-taking is a multi-faceted issue. Firstly, as novice learners, students are unable to discern what the teacher views as important to the lecture, relying mostly on visual cues to write information [125, 108, 118]. Secondly, note-taking requires three simultaneous high-cognitive-load activities: listening, writing, and processing information meaningfully. This places a high stress on the note-taker's working memory, so what notes are written down and the method students use to take notes greatly affects their overall learning [108, 123, 122], Thirdly, note-taking only facilitates learning if students review their notes after taking them [121], and the way students review their notes also has an impact on their overall learning [135, 119]. For the remainder of this section, I will expand on research related to these three aspects of student note-taking and how they are addressed in the workbook.

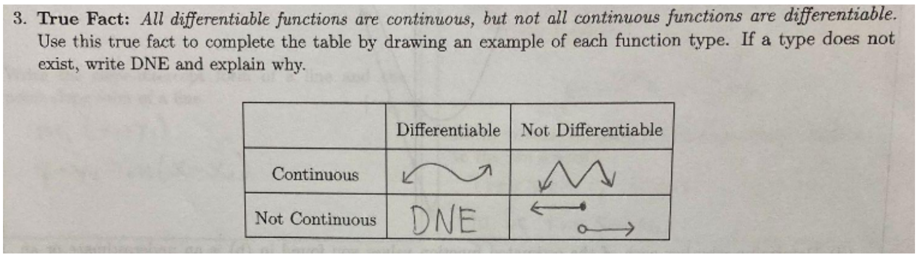
Note-Taking Devices

The current method of note-taking in Auburn's Calculus I course is for students to either take their own notes of the lecture, also called the traditional method, or to use teacher-created outlines of the notes, also called the outline method. Anecdotally, many students' choice of which note-taking strategy they use is dependent on availability of free printing locations on campus. Multiple studies have shown that the outline method of taking notes is far superior to students generating their own notes from scratch [123, 122]. Kiewra [122] also introduces a third note-taking device, the matrix method, which, when used correctly, may be superior to both other methods. The matrix method note-taking strategy involves writing concepts in matrix form and recording notes in each cell concerning relationships between topics corresponding to the cell. The main drawback of the matrix method is students' unfamiliarity with it. Thus, during an in-person lecture, students may be unable to discern relationships between concepts in real time.

As a compromise, the workbook includes outline-style lecture notes with embedded matrix-style pre-recitation, extra practice, and unit review questions so that students can analyze conceptual relations as a lecture-review exercise instead of figuring them out in real-time during a lecture (e.g., see Figure 4.5).

Using Graphic Organizers

Out of the initial matrix method of note-taking, attention grew to the use of graphic organizers in note-taking [135, 119]. A matrix is a form of a graphic organizer, but, more generally, graphic organizers tend to visually connect concepts in a tree shape before proceeding to matrix



3. **True Fact:** *All differentiable functions are continuous, but not all continuous functions are differentiable.* Use this true fact to complete the table by drawing an example of each function type. If a type does not exist, write DNE and explain why.

	Differentiable	Not Differentiable
Continuous		
Not Continuous	DNE	

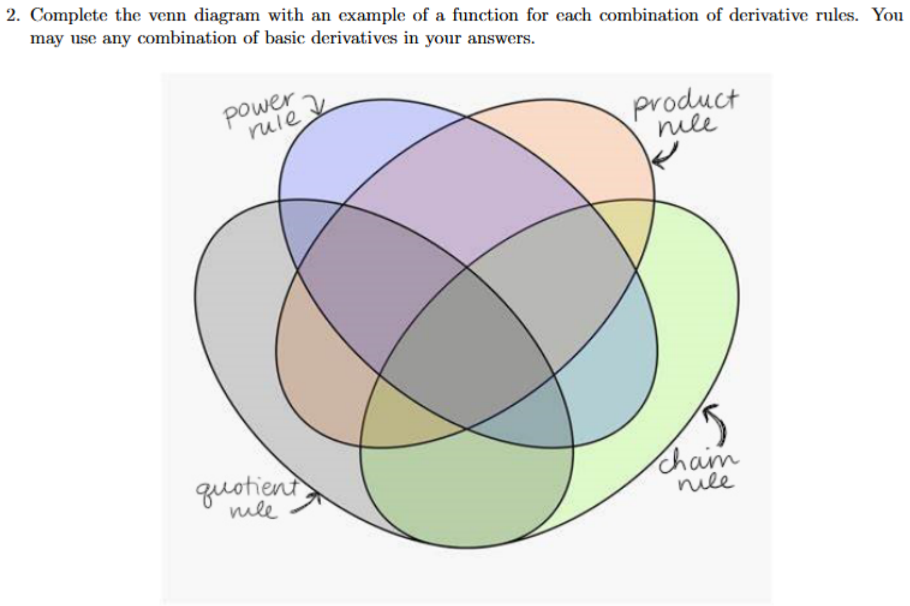
Figure 4.5: A matrix-style pre-recitation task from *Lesson 6: The Derivative*.

form notes under each sub-concept. For examples of a graphic organizer's structure, see Robinson & Kiewra's [135] study and Katayama & Robinson's [119] study. In Robinson & Kiewra's [135] study, graphic organizers were found to aid factual and relational learning, application, and integrated writing when paired with a text. Their belief is that graphic organizers' format and not their content is what facilitated learning. The ability of students to see underlying structure in a topic was invaluable to their ability to write about and apply concepts from the text. In Katayama & Robinson's [119] study, while there was no difference in factual retention between students using graphic organizers and students using text only, however, students studying graphic organizers performed significantly better on a test involving applications of content. Furthermore, students completing partial notes, where roughly 50% of the notes were provided, performed better than the skeletal notes and complete notes groups in both the graphic organizer and text-only categories. They characterize partial notes as a balance between the two high-cognitive-load tasks involved in note-taking: writing and processing of information.

Mathematics courses, especially Calculus I, require not only base knowledge of terms and theorems, but the ability to apply these concepts to problems. Considering the above findings, the workbook includes graphic organizer tasks (e.g., see Figure 4.6).

Note-Taking Cues

A note-taking cue is an event in the classroom setting that prompts students to write information down as notes. One of the first comments on note-taking cues is in Locke's [125] study. He noticed that students wrote down significantly more information when it was written on the board as opposed to when it was spoken only. He estimated that writing down information on the board increased its likelihood of being written by students by almost 37%. Although Locke



2. Complete the venn diagram with an example of a function for each combination of derivative rules. You may use any combination of basic derivatives in your answers.

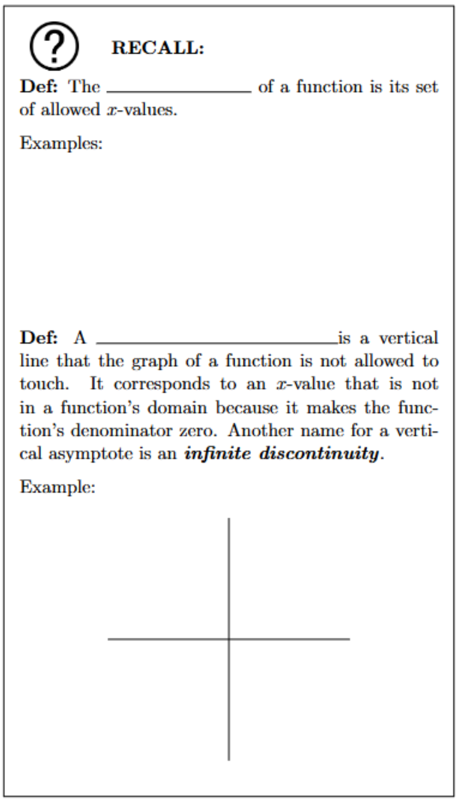
Figure 4.6: A graphic organizer task from the workbook's Unit 2 Review.

commented on note-taking cues, it was not the sole focus of his study. Roughly a decade later, Baker & Lombardi's [108] study commented again on the disparity between visual and other note-taking cues, reporting that all participants recorded all of what was written on the board but only 27% of other information that the investigators deemed important. Huxham's [118] study primarily analyzed differences in note-taking cues, reporting that information on slides (equivalent to blackboard writing) was the most perceived note-taking cue by students, with 98% of participants recording at least some information contained on slides and a significant majority copying slides verbatim. By contrast, 40% of notebooks contained no information presented verbally, suggesting that students are not expecting the lecturer's discussion to be of importance.

To address this in the workbook, lecture notes include printed recall questions about information from previous math courses and printed leading questions to aid in relational knowledge that an instructor might traditionally only present in verbal form (e.g., see Figures 4.7 4.8).

4.2.4 Student Metacognitive Knowledge

Flavell [115] defines metacognitive knowledge as an individual's knowledge of their own learning. Metacognition in relation to mathematics has been studied extensively [132, 133, 134, 136,



RECALL:

Def: The ____________ of a function is its set of allowed x-values.

Examples:

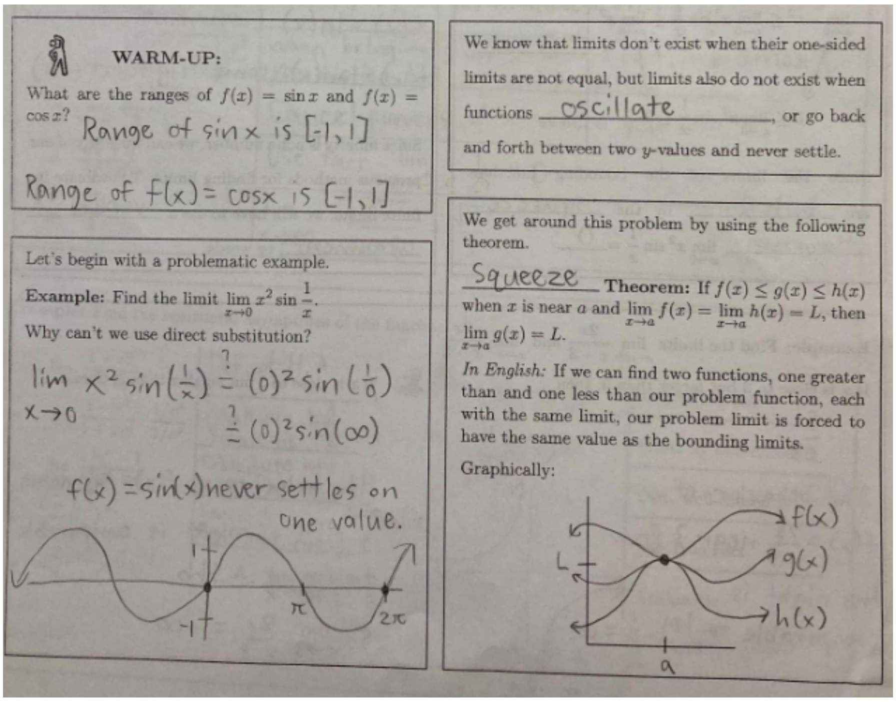
Def: A ____________________ is a vertical line that the graph of a function is not allowed to touch. It corresponds to an x-value that is not in a function's domain because it makes the function's denominator zero. Another name for a vertical asymptote is an ***infinite discontinuity***.

Example:

Figure 4.7: An example of using recall questions in lecture notes from *Lesson 4: Evaluating Other Kinds of Limits Algebraically* to motivate infinite limits.

140]. Yildirim and Ersozlu [140] showed a positive relationship between students' metacognitive knowledge and their ability to solve more challenging, non-routine mathematics problems. This indicates that if we expect students to solve challenging problems, we must also help develop their metacognition. Radmehr and Drake [134] demonstrated a significant difference in metacognition between upper-level secondary school students and first-year university students regarding integral calculus. If this is the case, our Calculus I students are much closer to upper-level high school students than students who have completed Calculus to the level of advanced integration, further supporting the need to aid them in developing their metacognitive knowledge.

To develop student metacognitive awareness, the workbook includes a "Check Your Understanding" activity at the end of each lesson's lecture notes asking students to answer the "little questions" described in the course objectives section (e.g., see Figure 4.9). On pre-recitation assignments, students are provided "How'd it go?" prompts as a space to reflect on



WARM-UP:

What are the ranges of $f(x) = \sin x$ and $f(x) = \cos x$? Range of sin x is [-1, 1]

Range of f(x) = cosx is [-1, 1]

Let's begin with a problematic example.

Example: Find the limit $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

Why can't we use direct substitution?

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \overset{?}{=} (0)^2 \sin\left(\frac{1}{0}\right)$$

$$\overset{?}{=} (0)^2 \sin(\infty)$$

f(x) = sin(x) never settles on one value.

We know that limits don't exist when their one-sided limits are not equal, but limits also do not exist when functions oscillate, or go back and forth between two y-values and never settle.

We get around this problem by using the following theorem.

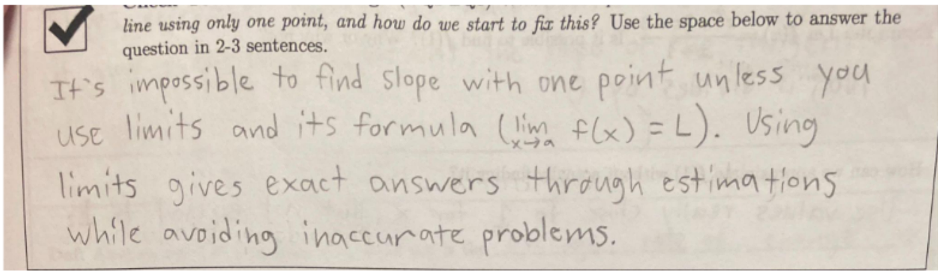
Squeeze **Theorem:** If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

In English: If we can find two functions, one greater than and one less than our problem function, each with the same limit, our problem limit is forced to have the same value as the bounding limits.

Graphically:

Figure 4.8: An example of using leading questions in lecture notes from *Lesson 4: Evaluating Other Kinds of Limits Algebraically* to motivate the Squeeze Theorem.

the difficulty of the assignment and questions they have for the next class period (e.g., see Figure 4.10). In Units 3 and 4, we included a self-regulation exercise at the beginning of each recitation, in which students self-rated where they should begin working for the class period (see Figure 4.11).

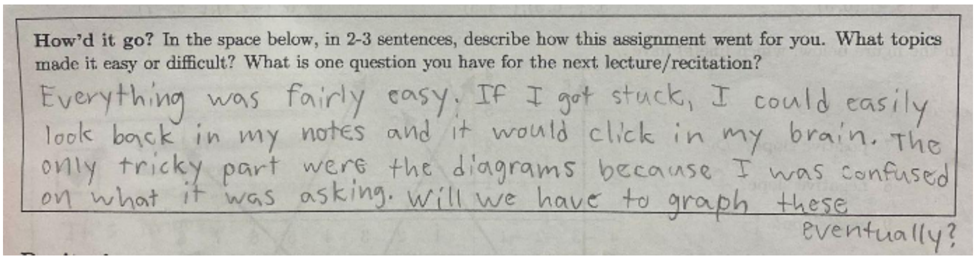


☑ *line using only one point, and how do we start to fix this?* Use the space below to answer the question in 2-3 sentences.

It's impossible to find slope with one point unless you use limits and its formula ($\lim_{x \to a} f(x) = L$). Using limits gives exact answers through estimations while avoiding inaccurate problems.

Figure 4.9: A student's response to the "Check Your Understanding" prompt from *Lesson 1: The Problem With Slopes*.

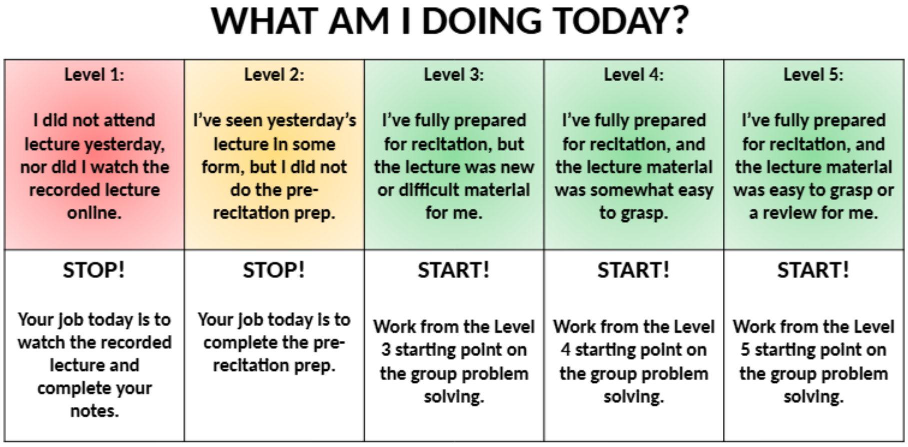
Note: This lesson's little question is "What is the problem with finding the slope of a line using only one point, and how do we start to fix this?"



How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?

Everything was fairly easy. If I got stuck, I could easily look back in my notes and it would click in my brain. The only tricky part were the diagrams because I was confused on what it was asking. Will we have to graph these eventually?

Figure 4.10: A student's response to a "How'd It Go?" prompt from *Lesson 1: The Problem With Slopes*.



WHAT AM I DOING TODAY?

Level 1:	Level 2:	Level 3:	Level 4:	Level 5:
I did not attend lecture yesterday, nor did I watch the recorded lecture online.	I've seen yesterday's lecture in some form, but I did not do the pre-recitation prep.	I've fully prepared for recitation, but the lecture was new or difficult material for me.	I've fully prepared for recitation, and the lecture material was somewhat easy to grasp.	I've fully prepared for recitation, and the lecture material was easy to grasp or a review for me.
STOP! Your job today is to watch the recorded lecture and complete your notes.	STOP! Your job today is to complete the pre-recitation prep.	START! Work from the Level 3 starting point on the group problem solving.	START! Work from the Level 4 starting point on the group problem solving.	START! Work from the Level 5 starting point on the group problem solving.

Figure 4.11: "What Am I Doing Today?" self-regulation exercise included before recitation problems in Units 3 and 4 of the workbook.

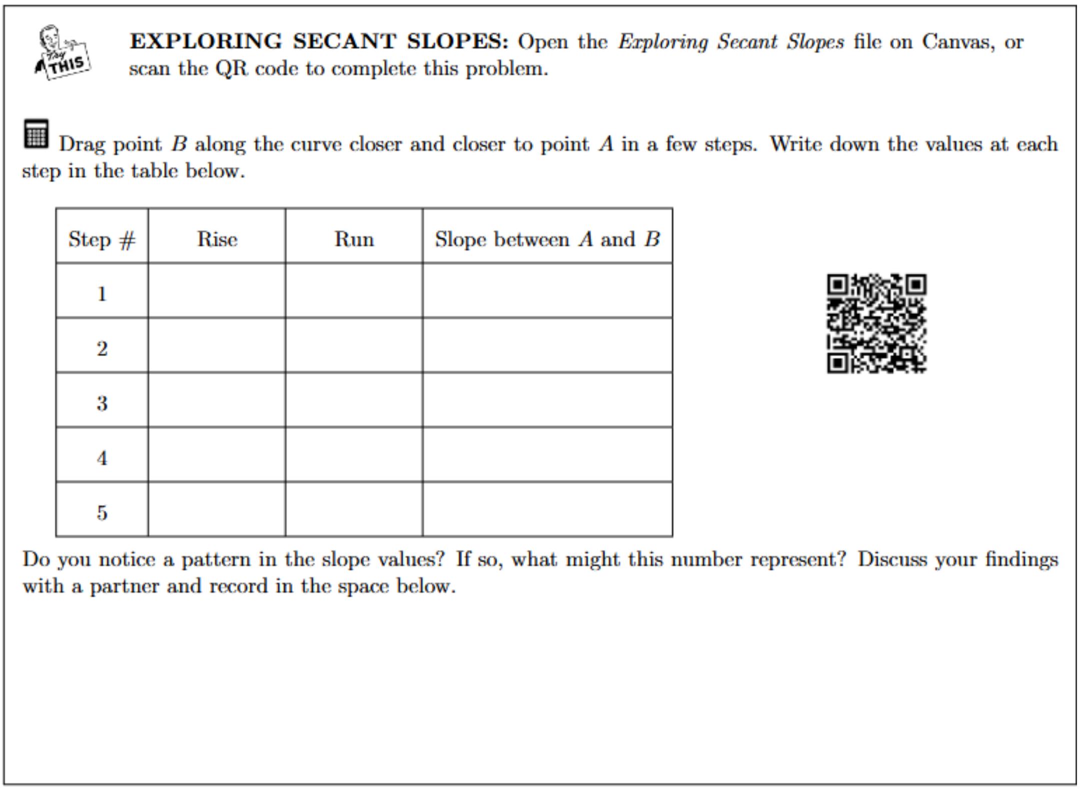
4.2.5 Classroom Technology Use

Since the invention of smartphones, educators have debated regulating their use in classrooms. Lang [124] characterizes classroom technology policies according to four basic types: (1) the Laissez-Faire Approach, (2) The Total Ban, (3) Student-Generated Policies, and (4) The Context-Specific Policy. Researchers have shown that consistently enforcing classroom technology policies (no matter the approach) is important yet challenging [109]. For our 250-person section of Calculus I, enforcing a Total-Ban policy or creating a Student-Generated policy was logistically impossible due to the sheer number of students compared to instructors. Furthermore, the potential for our students to disengage in a 250-person lecture was quite high; research has shown that large class size reduces student motivation and the development of cognitive skills in the classroom [141]. However, due to institutional constraints, it was impossible to offer Calculus I sections with smaller enrollment (hence the two days of 32-person recitations per week). Thus, we chose to leverage technology to integrate short interactive windows into lectures, achieved via GeoGebra applets that were embedded in the printed workbook via QR codes.

Calculus is often called the mathematics of change [138]. Considering the interactive, dynamic potential of dynamic geometry environments (e.g., GeoGebra or Desmos) [126], it is not surprising that the literature on the effectiveness of learning Calculus concepts using dynamic geometry environments has been overwhelmingly positive; Ziatdinov and Valles' [142] literature review gives an overview of current progress. One study found, through questionnaires completed by students, that all except one had a positive view toward using GeoGebra as part of their class [129]. Another study that quantitatively tested the general effectiveness of using GeoGebra in Calculus course found that it positively impacted students' conceptual understanding of derivative applications [130].

In the workbook, *GeoGebra Explorations*, the name we gave to GeoGebra applets with their associated tasks, were embedded in lectures as short (5-10 minute) interactive activities inviting students to interact with the digital material on their devices. Using this approach allowed the students to gain a better conceptual understanding of Calculus topics but did not take up as much time as a full lab-style activity would have. Using this "interactive window" format in lectures has been shown to be successful in enhancing student engagement, recall, and understanding [117].

From the perspective of Cognitive Load Theory, one potential issue with integrating digital activities into a traditional lecture is the introduction of a spatial split-attention effect, which posits that separating two parts of a task spatially (such as text presented separately from a graphic) increases the cognitive load imposed on the learner [137]. To maximize the dissemination of the applets to the students while minimizing the spatial split-attention effect, we included QR code links to the GeoGebra applets in the students' lecture notes. This approach allows for integrating the digital portion of the task with printed note outlines, making it easy for students to access the activity in class or return to the activity outside of class.



EXPLORING SECANT SLOPES: Open the *Exploring Secant Slopes* file on Canvas, or scan the QR code to complete this problem.

▦ Drag point B along the curve closer and closer to point A in a few steps. Write down the values at each step in the table below.

Step #	Rise	Run	Slope between A and B
1			
2			
3			
4			
5			

Do you notice a pattern in the slope values? If so, what might this number represent? Discuss your findings with a partner and record in the space below.

Figure 4.12: An example of a GeoGebra Exploration from *Lesson 1: The Problem With Slopes*.

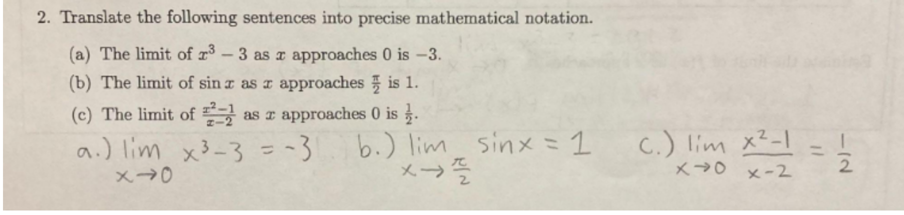
Note: Students investigated the relationship between the slope of secant lines and the slope of the tangent line at a specified point. The corresponding GeoGebra applet may be accessed via https://aub.ie/r4mHdX.

4.2.6 Mathematics Communication Skills

Mathematics is a language, just like English, with its own set of grammatical rules and conventions [131]. Mathematical notation is a shorthand expression of mathematical communication. Before the advent of algebraic symbolism, mathematics was written entirely in words for millennia.

Students entering university mathematics courses have difficulties with understanding mathematical notation, especially when the symbols are unfamiliar or the same symbols are used in different problem contexts [131]. Many symbols used in Calculus are not encountered by students in pre-calculus courses; for example, limit notation, derivative notation, and integral notation are all central to the study of Calculus and yet are not necessary to introduce until students reach Calculus courses.

Pre-recitation tasks emphasized mathematical vocabulary and its connection with mathematical notation to develop students' mathematical written communication skills in the workbook (e.g., see Figure 4.13). Students were expected to append a flashcard deck all semester and quizzed each other at the beginning of each recitation session.



2. Translate the following sentences into precise mathematical notation.

   (a) The limit of $x^3 - 3$ as x approaches 0 is -3.

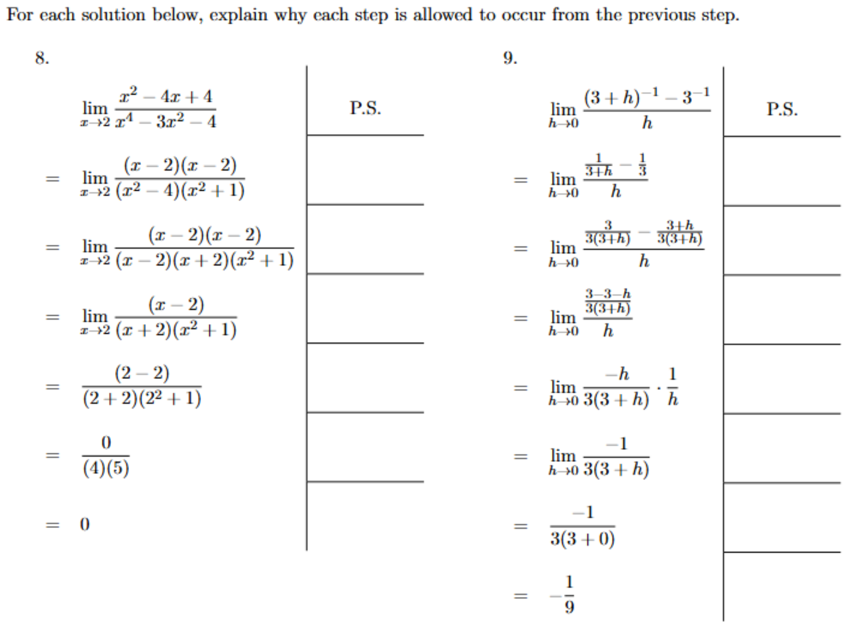
   (b) The limit of $\sin x$ as x approaches $\frac{\pi}{2}$ is 1.

   (c) The limit of $\frac{x^2-1}{x-2}$ as x approaches 0 is $\frac{1}{2}$.

a.) $\lim\limits_{x \to 0} x^3-3 = -3$ b.) $\lim\limits_{x \to \frac{\pi}{2}} \sin x = 1$ c.) $\lim\limits_{x \to 0} \frac{x^2-1}{x-2} = \frac{1}{2}$

Figure 4.13: A pre-recitation notation translation task from *Lesson 1: The Problem With Slopes*.

Furthermore, attention was given to analyzing the structure of problems within lecture notes and task sets (e.g., see Figure 4.14). For example, instead of simply "taking a derivative" of a function for homework, a preliminary question might be to "identify the process of taking the derivative" of a given function.



For each solution below, explain why each step is allowed to occur from the previous step.

8.

$$\begin{aligned} & \lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4} \\ = & \lim_{x\to 2}\frac{(x-2)(x-2)}{(x^2-4)(x^2+1)} \\ = & \lim_{x\to 2}\frac{(x-2)(x-2)}{(x-2)(x+2)(x^2+1)} \\ = & \lim_{x\to 2}\frac{(x-2)}{(x+2)(x^2+1)} \\ = & \frac{(2-2)}{(2+2)(2^2+1)} \\ = & \frac{0}{(4)(5)} \\ = & \ 0 \end{aligned}$$

P.S.

9.

$$\begin{aligned} & \lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h} \\ = & \lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} \\ = & \lim_{h\to 0}\frac{\frac{3}{3(3+h)}-\frac{3+h}{3(3+h)}}{h} \\ = & \lim_{h\to 0}\frac{\frac{3-3-h}{3(3+h)}}{h} \\ = & \lim_{h\to 0}\frac{-h}{3(3+h)}\cdot\frac{1}{h} \\ = & \lim_{h\to 0}\frac{-1}{3(3+h)} \\ = & \frac{-1}{3(3+0)} \\ = & -\frac{1}{9} \end{aligned}$$

P.S.

Figure 4.14: Pre-recitation tasks from *Lesson 3: Evaluating Limits Algebraically* designed to improve students' understanding of mathematical notation and written mathematical communication skills.

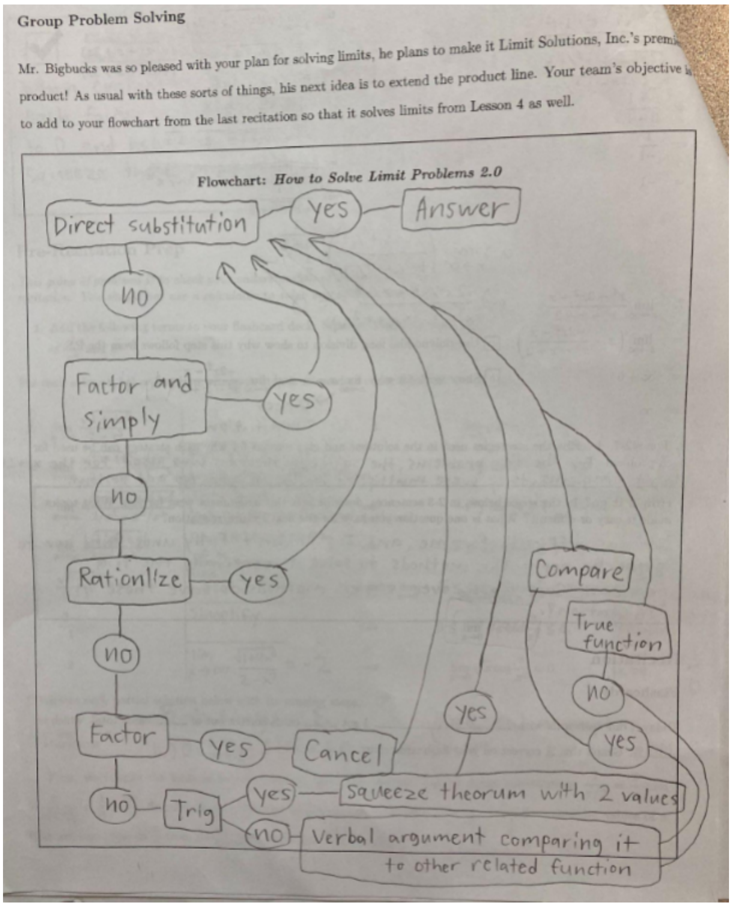
4.2.7 Additional Pedagogical Choices

In addition to the explicit design choices made in the workbook, we highlight a couple of pedagogical choices specific to our section, which, according to research-backed evidence, may have also contributed to the success of our implementation.

Prioritizing Strategic Mindset

Computational problems, such as those in Calculus I, require a solution strategy. Furthermore, Auburn's coordinated approach to Calculus I meant that our students took a final exam that was not created by their instructor of record and was worth a significant proportion of their course grade. Consequently, developing a strategic approach to studying for this exam is crucial to students' success in the course. Thus, we felt that it was essential to embed strategy building into various aspects of the course.

A student with a *strategic mindset* engages in metacognitive strategies while pursuing a goal. This involves identifying actions he or she can do to overcome an obstacle [112]. Throughout the course, we promoted strategic mindset through recitation tasks. In particular, when learning about limit-solving strategies, students created decision trees to describe their strategy for evaluating any limit problem they might encounter (see Figure 4.15). Later in the course, once the students learned about L'Hospital's Rule, their recitation activity asked them to evaluate three limits, once using L'Hospital's Rule and once using a previously learned strategy. Then, students were asked to discuss which strategy they felt was most efficient for each problem (see Figure 4.16).



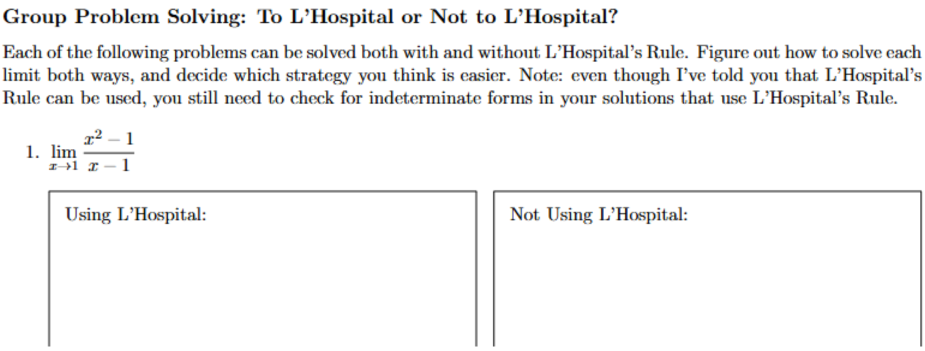
Group Problem Solving

Mr. Bigbucks was so pleased with your plan for solving limits, he plans to make it Limit Solutions, Inc.'s premi product! As usual with these sorts of things, his next idea is to extend the product line. Your team's objective i to add to your flowchart from the last recitation so that it solves limits from Lesson 4 as well.

Flowchart: ***How to Solve Limit Problems 2.0***

Figure 4.15: A recitation task aimed at developing students' strategic mindset from *Lesson 4: Evaluating Other Kinds of Limits Algebraically*. Students created decision trees to describe their strategy for evaluating limits.

Throughout the semester, the "How'd It Go?" prompts, which were originally intended to develop students' metacognitive skills, also provided evidence of students' developing a strategic mindset. As early as Lesson 2, students began alluding to their strategies for solving various tasks in their assignments (e.g., see Figure 4.17).


Group Problem Solving: To L'Hospital or Not to L'Hospital?

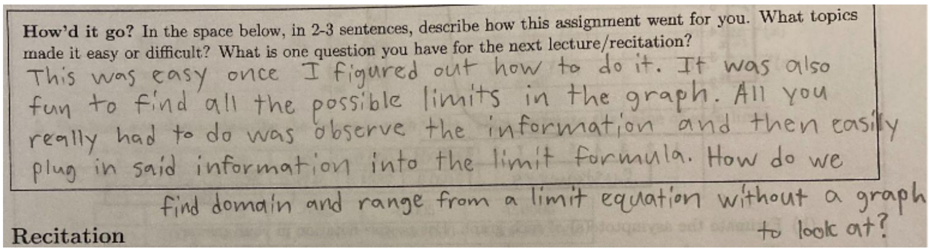
Each of the following problems can be solved both with and without L'Hospital's Rule. Figure out how to solve each limit both ways, and decide which strategy you think is easier. Note: even though I've told you that L'Hospital's Rule can be used, you still need to check for indeterminate forms in your solutions that use L'Hospital's Rule.

1. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Using L'Hospital:	Not Using L'Hospital:

Figure 4.16: A recitation task aimed at developing students' strategic mindset from *Lesson 11: The Magic Limit Shortcut* (L'Hospital's Rule).

Note: Students compared solutions to limits using and not using L'Hospital's Rule, reflecting on which solution they thought was more efficient in each case.


How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?

This was easy once I figured out how to do it. It was also fun to find all the possible limits in the graph. All you really had to do was observe the information and then easily plug in said information into the limit formula. How do we find domain and range from a limit equation without a graph to look at?

Recitation

Figure 4.17: Evidence of a student exhibiting a strategic mindset through a response to a "How'd It Go?" prompt.

From a random sampling of student responses to three lessons' "How'd It Go?" prompts from each of the four content units, we ascertained a net increase in students' writing about their strategic thinking as the semester progressed (see Figure 4.18).

Universal Design

Universal Design is "the design of products and environments to be usable by all people, to the greatest extent possible, without the need for adaptation or specialized design" [110]. Two universal design principles that we applied to our course implementation are (1) Equitable Use and (2) Flexible Use.

Equitable Use refers to the design's utility and marketability to people with diverse abilities [110]. To ensure that our instructional materials were equitably accessible for all our students, we brought printed copies of each lesson's materials to all lectures. We also provided students with index cards to make their flashcard decks.

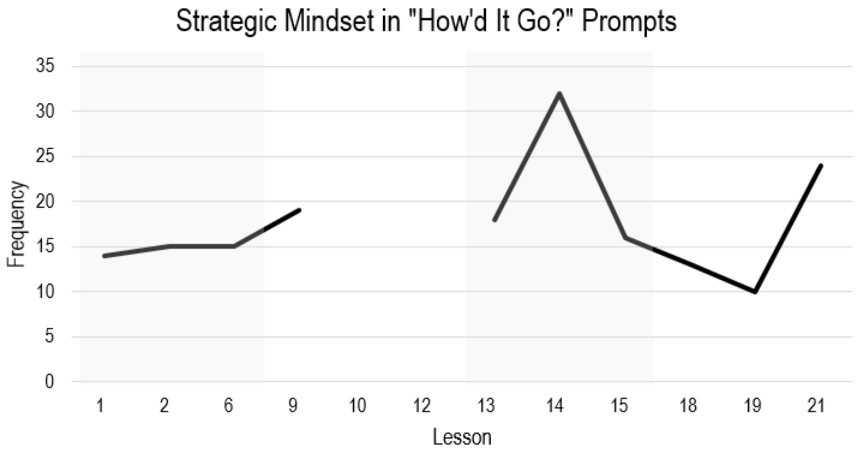


Figure 4.18: Instances of Strategic Mindset in "How'd It Go?" prompt responses.

Note: As the semester progressed, we saw a net increase in responses exhibiting a strategic mindset. Lessons 10 and 12 were randomly chosen from Unit 2 but did not include "How'd It Go?" prompts.

Flexible Use refers to the design's accommodation of a wide range of individual preferences and abilities [110]. To ensure that our instructional materials could be used flexibly, we created video recordings of all the lectures and posted links to these recordings on the class Learning Management System (LMS). We also used the class LMS to store digital copies of all printed materials and solutions to all pre-recitation and recitation tasks so students could access them anywhere and at any time.

4.3 Conclusion

In this chapter, we identified a misalignment between how calculus textbooks order content and how that content is taught in the classroom. We then provided a solution to this misalignment through the curriculum isomorphism problem, which aims to reorder textbook topics in a way that students can theoretically learn more efficiently (from the perspective of reducing cognitive load).

By reordering Calculus I topics using graph isomorphisms and using a workbook designed based on research in education and cognitive psychology, along with a focus on the structure of lecture notes, homework, test reviews, and metacognitive exercises, we significantly reduced the DFWI rate (Drop, Fail, Withdraw, Incomplete) in one section of Auburn University's Calculus I course to approximately 12% (unpublished data). This demonstrates that achieving widespread student success is possible in an exam-heavy, large-enrollment, lecture-recitation formatted course.

This achievement positively addresses the negative perception of Calculus I as a "weed-out" course. Our approach not only covers Calculus content but also emphasizes computational fluency, effective note-taking strategies, and study skills. Ultimately, we aim to enhance our students' abilities in mathematics and improve their overall efficiency as learners.

Chapter 5

Using Graphs to Analyze Homework Promotes Reflection by Coordinated Instructors

This chapter presents a human subjects research study on the efficacy of a novel tool for measuring the alignment of homework assignments to a list of course objectives. The results of this study are joint work with Dr. Melinda Lanius and Sean Grate and have been submitted for peer review to a mathematics education journal. I spearheaded the project, designing the study, developing the instrument, and analyzing the data. Melinda contributed the literature review, situating the study within the broader context of course coordination. Sean developed and refined Python code that examined the distribution of Euclidean distances between horse race sequences of length $n = 4, 5, 6, 7$. Since this project is intended for a math education audience, the use of "we" reflects multiple contributors.

5.1 Introduction

Uniform course components, or course elements that are fixed across all sections of a course, are extremely popular amongst Calculus I courses in Masters or Ph.D. granting mathematics departments in the United States [153]. Unfortunately, at many of these universities, Calculus I is also a notorious 'weed-out' course that operates in such a way that only the highest-performing students pass, contributing to a leaky pipeline where large numbers of STEM majors do not complete a STEM degree; this loss has been partially attributed to a lack of alignment in the content covered between various course components, such as in-class instruction, homework assignments, and assessments [159]. Alignment is the degree to which cohesion exists between course components including learning objectives, in-class materials, homework, and assessments [160]. In this chapter, we explore the role of a uniform course component in instructors' considerations of course alignment.

5.1.1 Relevant Literature & Theoretical Perspectives

In the setting of a large university, sections of calculus are commonly taught by educators with a broad variety of teaching experience and career orientations. Establishing some coordination between the sections is a logical solution to control for variation in the content covered and to promote effective instruction [154]. The role of the course coordinator can be significant in this coordinated system. For example, coordinators can serve as change agents, fostering the widespread implementation of active learning strategies by the instructors they work with [154, 151, 161].

However, there is great variation in how coordinators approach their role. Martinez et al [150] identified two common coordination orientations: Humanistic-Growth and Resource-Managerial. The first mindset is community-based and focuses on the growth of the instructors teaching the coordinated course, while the latter is material-based and focuses more on the logistics of managing the course and course resources. Compared to coordinated chemistry and physics courses, US-based postsecondary mathematics instructors are more likely to experience *controlled coordination* (which we classify as strict Resource-Managerial type coordination), meaning they are mandated to utilize common course elements and are provided no opportunity to give their input on those items or to participate in decision making [144]. The most effective coordination likely requires a balance between the two orientations. For example, institutions that implement uniform course components in conjunction with regular instructor meetings have been found to be more successful concerning both overall passing rates and student persistence in Calculus II [153].

An additional promising strategy to increase the efficacy of coordinated Calculus instruction is the development and use of a set of common learning outcomes within the coordinated community [163, 161]. In their 2015 work [154], Rasmussen & Ellis identified a Calculus

I coordinator who had mapped the uniform homework problems to a set of learning objectives. An instructor working with this particular coordinator found the mapping useful because it communicated what their department thought was important for students to learn and helped them align their in-class instruction to the homework. We consider this an example of a Resource-Managerial approach to coordination. This chapter aims to introduce and evaluate a Humanistic-Growth approach to fostering course alignment through learning objective mapping.

5.1.2 Research Questions

Within our study, we consider the role of common learning objectives in supporting overall course alignment by surveying our participants before and after they have a set of common learning outcomes and they reflect on alignment using a novel instrument called the *Course Alignment Analysis Tool (CAAT)*. CAAT provides a quantitative measurement of alignment, capturing the difference between the learning outcomes that an instructor feels should be prioritised and the learning outcomes most emphasised by an assignment. The goal of CAAT is to provide instructors with a new instrument for reflecting on the design of existing assignments or assessments. In this chapter, we will first explain the graph theoretic underpinnings of CAAT and how to use the instrument. Then, we will answer the following research questions concerning our participants' perceptions of coordinated online homework and CAAT as a reflective tool.

1. How do university mathematics instructors working in a coordinated calculus environment define the 'quality' of uniform homework assignments?
2. Does the use of the CAAT in conjunction with a set of common learning objectives support a Humanistic-Growth coordination style (i.e. did our instructors evolve through participation)?
3. Does CAAT provide evidence of agreement among university mathematics instructors as to the alignment of homework assignments to a common set of objectives?

5.2 Development of CAAT

5.2.1 Backward (Re)Design

The design of CAAT is partly inspired by an instructional design concept called Backward Design. This approach begins with developing a list of learning objectives that students are expected to achieve throughout a course and then constructing the rest of the course around this list of learning objectives, taking care that all course components are aligned [149]. Centers for teaching and learning at universities around the United States have developed tools to aid instructors in creating courses using the Backward Design approach [147, 155, 158, 162]. However, these resources imply that the user is creating their course components from scratch without utilizing any existing resources for assistance. On the other hand, we designed CAAT for use in the Backward *re*-design of mathematics courses, where the alignment of existing course components is analyzed.

5.2.2 The Math Behind the CAAT

Some concepts from Discrete Mathematics are central to the CAAT's design. We explain these concepts in the context of an instructor using the CAAT to analyze an assignment. For reference, a blank copy of the CAAT has been included in Appendix **??**.

A ranking where ties are allowed is called a *weak ordering*. For instance, horse races allow for horses to tie for placement so that two horses might both finish in second place. In voting systems where voters rank candidates by preference, voters can rank multiple candidates the same rank, indicating that they have no preference for one candidate over another. Because an instructor may find two learning objectives equally important, the user begins the CAAT by ranking a list of learning objectives they have deemed should be assessed, using a weak ordering.

Suppose there are n learning objectives an instructor wants to cover in the assignment. The instructor orders the objectives and assigns a 'desired' ranking according to what objectives they hope are emphasized the most. For instance, if $n = 4$, then the instructor might form the ranking $r_{\text{desired}} = (2,1,2,4)$ indicating that the second objective should be emphasized the most, the first and third objectives should receive second priority equally, and the fourth objective should be emphasized the least.

After initially ranking their desired emphasis of each objective, the user completes the Problem-Objective Grid by associating each problem with one or more learning objectives and weighting this association according to the number of times an objective is assessed in a problem. For example, if a problem contains three parts with the same set of instructions for each part, then each objective assessed by the problem would receive a weight of '3'[1].

The Problem-Objective Grid can be viewed as a weighted association matrix of a bipartite multigraph that models the homework assignment in question. An example of a bipartite multigraph, "The Crawfish" (a nod to Haile's Louisiana roots), is shown in Figure 5.1. The Crawfish is bipartite because no two black vertices share an edge and no two white vertices share an edge, and it is a multigraph because multiple edges may exist between pairs of vertices (e.g. the claws).

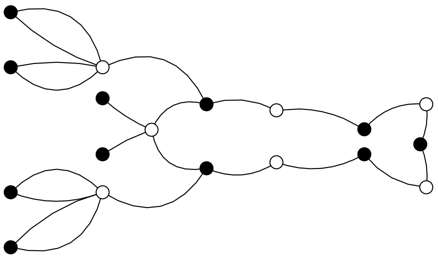


Figure 5.1: An example of a bipartite multigraph, which we call 'The Crawfish'

After the user has completed the Problem-Objective Grid, they sum the columns of the grid. The column sums are then ranked from greatest to least, with the greatest column sum receiving rank 1. Returning to our example with 4 learning objectives, an 'observed' ranking with $n = 4$ might be $r_{\text{observed}} = (4,1,2,3)$ indicating that the second objective appeared the most, and so forth.

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[1] A video of a fully worked example CAAT is available at https://aub.ie/o3BsDL

Lastly, the user compares the observed and desired rankings by computing the standard Euclidean distance[2] (or root squared error) between the two rankings. Keeping with the previous examples, the measured disparity between the two rankings is

$$d(r_{\text{desired}}, r_{\text{observed}}) = \sqrt{(2-4)^2 + (1-1)^2 + (2-2)^2 + (4-3)^2} = \sqrt{5} \approx 2.24.$$

In this way, the instructor can measure how aligned an assignment is to their desired emphasis of learning objectives using a mathematically defined distance, where a smaller distance between ranking vectors indicates better alignment.

5.2.3 The Rating Scale

To help instructors interpret the CAAT's numerical score, we developed a rating scale that translates the numerical alignment score obtained by the CAAT into one of three ratings: 'Excellent', 'Acceptable', or 'Poor'.

To construct the Rating Scale, we first found the distribution of all possible CAAT scores for each number of objectives. To do this, we developed Python code[3] that calculates the Euclidean distance between all pairs of weak orders of length n and outputs the relative frequency distribution of the distances along with quartile values; see Figure 5.2 for $n = 7$ objectives.

The quartile ranges correspond to our final alignment rating categories; the lowest 25% of distances receive a rating of 'Excellent', the distances between the first and second quartiles receive a rating of 'Acceptable', and the highest 50% of distances receive a rating of 'Poor'. Using quartiles to delineate rating categories ensures that each rating category has an equal

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[2] Note that since the rankings are regarded as vectors in $\mathbb{N}^n$, the choice of metric (e.g., standard Euclidean distance) can be changed. Using the L^∞ distance would measure the maximum discrepancy between the ground truth and observed rankings, while the L^1 distance (or absolute error) is less sensitive to large discrepancies. For completeness, the discrepancy from our example would be

$$\begin{aligned} d_1(r_{\text{de}}, r_{\text{ob}}) &= |2-4| + |1-1| + |2-2| + |4-3| = 3, \text{ and} \\ d_\infty(r_{\text{de}}, r_{\text{ob}}) &= \max\{|2-4|, |1-1|, |2-2|, |4-3|\} = 2, \end{aligned}$$

where d_1 is the L^1 distance and d_∞ is the L^∞ distance.

[3] GitHub: https://github.com/TheGrateSalmon/CAAT

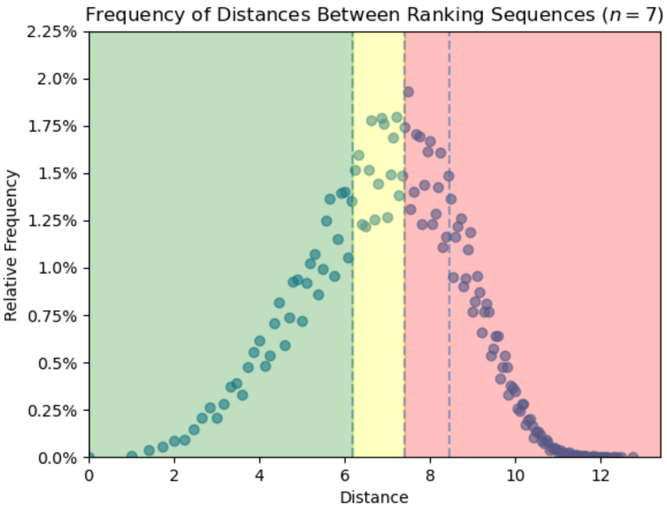

Figure 5.2: The relative frequency of distances between weak orderings on 7 objectives. Note that the dashed lines delimit the quartiles.

probability of being used — in our rating scheme, an assignment has a 50% chance of a positive alignment rating (Excellent/Acceptable) and a 50% chance of a negative alignment rating (Poor).

We were able to construct rating scales for $n \in \{4, 5, 6, 7\}$, after which the number of pairs of weak orders of length n becomes computationally intractable[4]. However, we are currently working on improving the code to handle $n = 8$ and possibly more, which could extend the use of CAAT to analyze course components other than homework (see Section 5.7).

5.3 Study Design

After securing institutional review board approval, we invited all Graduate Teaching Assistants and faculty employed by a large R1 public university's Department of Mathematics & Statistics to participate in our study. 10 people meeting this criteria expressed interest.

The 10 participants in this study consisted of three faculty members, (who we call Lasairfhíona, Siobhán, and Méabh) and seven graduate student instructors or GSIs (who we call Séamus, Alastar, Mícheál, Eoin, Íde, Máire, and Ciarán). Note that these are true-to-gender

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[4] For $n = 7$, there are $\binom{64}{2}$ weak orders to compare, amounting to $1,161,841,311$ distances to compute. Moving to $n = 8$, there are $\binom{128}{2}$ weak orders to compare, amounting to $152,377,145,815$ distances to compute, a substantial increase over $n = 7$.

Irish pseudonyms (a nod to Haile's Irish heritage). The participants reported varying levels of experience with serving as instructor of record, recitation leader, and/or tutor for a Calculus I course; three participants (Alastar, Mícheál, Lasairfhíona) had no experience with Calculus I, three (Séamus, Siobhán, Ciarán) had 1-2 semesters of experience, two (Eoin, Íde) had 3-4 semesters, 1 (Méabh) had 5-6 semesters, and one (Máire) had 7-8 semesters.

Participants completed two phases of data collection. In Phase 1, participants were provided copies of five Calculus I homework assignments and the textbook section(s) corresponding to each assignment. This sample of assignments was chosen to provide variety in length, content, and question formatting. The participants then completed a Pre-CAAT survey that asked them to rate the quality of each assignment on a 5-point Likert scale and explain the rationale behind each of their Likert ratings.

In Phase 2, participants were provided a CAAT with a list of six to seven preselected objectives for each homework assignment. Since the homework assignments were not created using a list of learning objectives, the preselected objectives were adapted from the College Board's AP Calculus AB content objectives [152], which have been thoroughly vetted by a combination of both university educators and high school teachers. After analyzing the five homework assignments using CAAT, the participants completed a Post-CAAT survey with questions that mirrored those on the Pre-CAAT survey and, additionally, asked if the CAAT influenced their Post-CAAT quality ratings of the homework assignments.

5.4 Qualitative Results

We answered Research Questions 1 and 2 using qualitative methods. Subsection 5.4.1 contains our analysis, and Subsection 5.4.2 discusses our answers to Research Questions 1 and 2.

5.4.1 Data Analysis

To analyze participants' free responses to their Likert Scale ratings, we used open coding [157]. During analysis of the Pre-CAAT surveys, the following codes emerged, each containing both a positive and negative sub-code.

Coverage of Topics references the degree to which an assignment includes questions that are discussed in the textbook. Specifically, this code includes broadly stated comments that do not mention specific topics or objectives.

Narrative/Progression is attention to either situating a topic within a broader mathematical context or the ordering of topics or questions on an assignment. For example, in the context of assigning computationally dense problems, Íde noted 'We lose the forest for the trees and in the process break the storyline of the course for tedium and busy work,' which implies that Calculus is a small part of a larger mathematical narrative. Máire specifically focused on the progression of topics from Precalculus to Calculus stating, 'It is unnecessary to mention names of curves (cardioid) because conic sections and polar graphs are not covered in Precalc here.'

Question Format was used for comments referencing either the phrasing of homework questions, the appearance of student answers, or the anticipated 'hacking behaviors' of students. For example, Alastar commented, 'The multiple choice format is even worse then [sic] the regular fill in the box format because it incentivizes not doing any work and getting through the assignment by blind guessing.'

Difficulty manifested through three foci: (1) **Computational Complexity**, (2) **Very procedural, lacking conceptual**, and (3) **Lack of higher-order thinking**. Regarding computational complexity, Íde noted, 'We are asking incredibly tedious questions to the students', and on the procedural nature of the homework, she remarked, 'I think this one is very repetitive, but I would call this section the methodical backbone of calculus.' Finally, regarding the lack of higher-order thinking, Lasairfhíona said, 'No assessment is done that requires a bit higher cognitive thinking.'

Discussion of a specific problem. This code was used for instances in which participants indicated a like or dislike for specific problems by their number. For example, Eoin noted, 'a few of the problems are unnecessarily hard, the biggest culprits being problems #8 and #9.'

Assignment Length. This code was used for instances in which participants commented on the amount of time they anticipated an assignment would take students to complete or the number of questions present on the assignment.

Table 5.1 gives the frequency of each of the homework quality codes, disaggregated by positive and negative subcodes.

No.	Code Title	Freq. (Pre)	Freq. (Post)
1	Coverage of Topics (-)	13	5
2	Narrative/Problem Progression (-)	17	5
3	Question Format (-)	37	18
4	Very Procedural, Lacking Conceptual (-)	13	1
5	Computational Complexity (-)	22	4
6	Lack of higher-level thinking (-)	5	1
7	Disgruntled with one specific problem	26	1
8	Assignment Length (-)	16	10
9	Coverage of Topics (+)	21	13
10	Narrative/Problem Progression (+)	2	2
11	Question Format (+)	0	0
12	Very Procedural, Lacking Conceptual (+)	4	3
13	Computational Complexity (+)	3	0
14	Lack of higher-level thinking (+)	0	0
15	Loved one specific problem	1	0

Table 5.1: Frequencies of homework quality codes Pre- and Post-CAAT.

In our analysis of the Post-CAAT surveys, the following additional codes emerged concerning homework quality.

Discussion of a specific objective. Similar to 'Coverage of Topics' before the use of CAAT, this code was used to denote comments that mentioned the material covered by an assignment. However, in 'specific objective' comments, participants used language that specifically referenced an objective listed on the CAATs rather than the more general language denoted by the 'Coverage of Topics' code. For example, before using CAAT, Ciarán commented about one of the assignments, 'I found questions 13 and 27 quite surprising ... I don't recall teaching the normal line in Calculus I ... but it is also not present in the textbook section,' which was coded as ‘Coverage of Topics’. After using CAAT, he said the following about the same assignment

> I’ve now realized that the derivatives homework really tested a ton of things that weren’t really focused on in the textbook, and it almost completely ignored exponential functions except in the context of other problems. I still liked how much it drilled the power rule[5].

Fostering careful reflection. This code was used for responses containing language with contemplative connotations. A few different phrases indicating this code are ‘I’ve now realized...’, ‘the homework did’ or ‘did not give significant attention to...’, and ‘the homework gave a pretty good balance’.

Gut-check refers to responses in which CAAT scores reaffirmed the user’s instinctive opinion about an assignment’s quality.

Complimentary to other measures of homework quality. This code is best characterized by a response from Méabh.

> In my Pre-CAAT assessment ... my primary consideration was the mechanics of completing a homework problem ... The CAAT provided me a complimentary way of thinking about homework quality, one that focused on the overall composition of problems and which objectives were covered.

Table 5.2 gives the frequency of each of the CAAT Influence codes that emerged after participants interacted with the homework assignments using CAAT.

To disaggregate the codes by participant, we constructed a radar chart for each participant’s Pre-CAAT and Post-CAAT survey responses. We colored regions of the radar charts to denote negative subcodes (red), positive subcodes (green), and CAAT Influence codes (yellow). Codes are numbered the same as in Tables 5.1 and 5.2. All 10 participants’ radar charts showed changes from Pre-CAAT to Post-CAAT. We highlight these changes through three participants, Mícheál, Lasairfhíona, and Máire.

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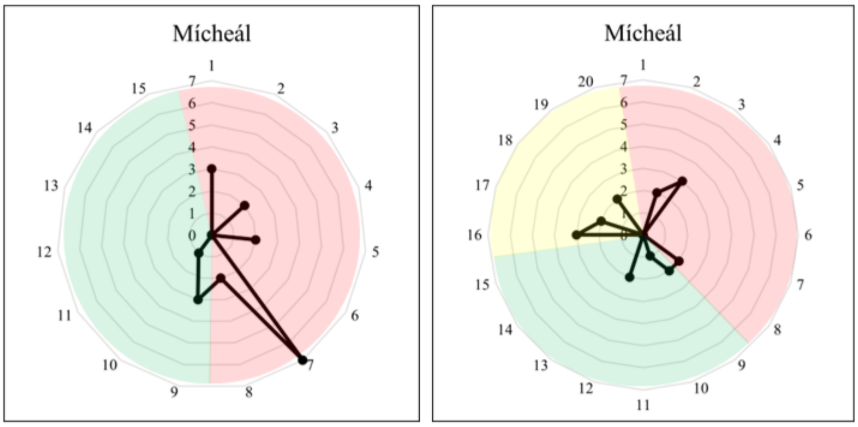
[5] Finding the derivatives of exponential and power functions were listed as objectives on this assignment’s CAAT. Additionally, Ciarán did not mention the power rule in his pre-survey response.

No.	Code Title	Freq. (Post)
16	Coverage of Specific Objective (-)	27
17	Coverage of Specific Objective (+)	9
18	Complementary measure of quality	1
19	Fostered Careful Reflection	28
20	Gut Check	1

Table 5.2: Frequencies of CAAT influence codes Post-CAAT.

Mícheál. Recall that Mícheál was a GSI with no experience as an instructor of record, recitation leader, or tutor for Calculus I. His Pre-CAAT and Post-CAAT radar charts are given in Figure 5.3. In his initial ratings of homework quality, Mícheál focused largely on specific problems that he disliked. However, in his Post-CAAT ratings, his outlook on the homework became slightly more positive (there is 1 point more in the Post-CAAT green region than the Pre-CAAT green region). Furthermore, his Post-CAAT chart contains two more non-zero points than his Pre-CAAT chart, suggesting that overall, his definition of homework quality became more multi-faceted.

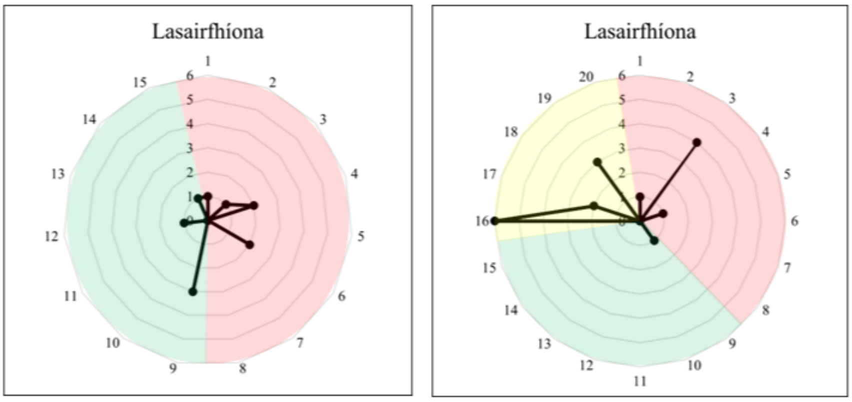


Pre-CAAT

Post-CAAT

Figure 5.3: Mícheál’s Pre-CAAT and Post-CAAT radar charts.

Lasairfhíona. Recall that Lasairfhíona was a faculty member with no Calculus I experience. However, her case was unique because, as a faculty member, she had much more total experience teaching other courses than the GSIs with no Calculus I experience. Her Pre-CAAT and Post-CAAT radar charts are given in Figure 5.4. Unlike Mícheál, Lasairfhíona’s definition

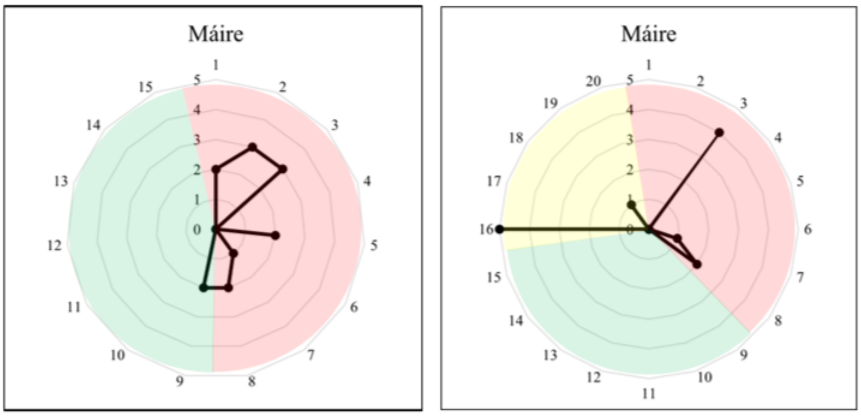

Lasairfhíona
Lasairfhíona

Pre-CAAT
Post-CAAT

Figure 5.4: Lasairfhíona’s Pre-CAAT and Post-CAAT radar charts

of quality did not become increasingly multi-dimensional. Rather, her two charts contained the same number of non-zero points, with some of her negative sentiment shifting to deeper reflection and a heightened focus on objectives. Additionally, her Post-CAAT chart contained far more total instances of codes than her Pre-CAAT chart, suggesting she did experience growth.

Máire. Recall that Máire was a GSI with 7-8 semesters of Calculus I experience, the most of any participant. Her Pre-CAAT and Post-CAAT radar charts are given in Figure 5.5. Given that Máire’s Calculus I experience was on the opposite end of our experience spectrum to Mícheál’s and Lasairfhíona’s experience, it was interesting that her charts were also opposite to the other two. Like Lasairfhíona, Máire’s Pre-CAAT chart showed a preexisting multi-dimensional definition of homework quality. However, in her Post-CAAT chart, her attention narrowly focused on the negative aspects of Question Formatting and specific objectives.


Máire
Máire

Pre-CAAT
Post-CAAT

Figure 5.5: Máire’s Pre-CAAT and Post-CAAT radar charts

5.4.2 Discussion

RQ 1. How do university mathematics instructors working in a coordinated calculus environment define the 'quality' of uniform homework assignments?

Though the CAAT focuses explicitly on alignment, our qualitative analysis of the rationale behind participants' Likert scale quality ratings suggests that instructors working in a coordinated calculus environment have a multi-dimensional definition of homework quality that grows with experience.

Furthermore, by taking the five codes with the highest frequency between Pre-CAAT and Post-CAAT ratings, we have determined that the most important factors instructors consider when defining homework quality are:

1. *Alignment.* Coordinated instructors believe that homework should reflect the content students are exposed to in their textbooks, which should also be reflected in lectures.
2. *Task Difficulty.* Coordinated instructors believe that one aspect of good homework is including both procedural and conceptual tasks that students will find approachable to build their confidence before progressing to more difficult tasks.
3. *Question Format.* Coordinated instructors believe that the way questions are formatted is related to the theme of alignment but in the vein of general mathematical practices rather than content specifics. A prominent issue that emerged among our participants was that online homework in particular does not require students to show work, which is the opposite of what is expected on an exam.
4. *Narrative.* Coordinated instructors view mathematics as a story, and it follows that assignments should tap into this instinct by also telling a story, whether through a gradual increase in task difficulty or by situating assignments within a broader curricular context.
5. *Specific Problems.* Coordinated instructors believe that specific problems they don't like can make or break a good homework assignment.

RQ 2. Does using the CAAT with common learning objectives support a humanistic growth coordination style (i.e., did our instructors evolve through participation)?

In short, yes. Every participant was influenced by CAAT whether they realized it or not. Even Séamus, who significantly disagreed with most other participants and stated that CAAT did not influence him, used the word ‘objectives’ rather than ‘topics’ in his post-CAAT ratings. For the others, the influence was more pronounced; all nine other participants stated that CAAT influenced their post-CAAT ratings, and this was reflected in their radar charts (all participants’ post-CAAT charts contained points in the yellow region).

Furthermore, the influence of CAAT manifested in different ways. More novice Calculus I instructors (such as Micheál) saw growth from their Pre-CAAT to Post-CAAT responses, both in their definition of ‘quality’ and their instances of coded phrases. On the other hand, more experienced Calculus I instructors (such as Máire) saw the opposite effect. From Pre-CAAT to Post-CAAT, they became more narrowly focused on specifics. The evidence of evolution in all participants suggests that using CAAT to onboard instructors to a coordinated setting could support a humanistic growth coordination style in mathematics courses.

5.5 Quantitative Results

To answer Research Question 3, we used Quantitative Methods. Subsection 5.5.1 contains our analysis, and Subsection 5.5.2 discusses our answer to Research Question 3.

5.5.1 Data Analysis

To determine if there was any agreement among participants’ CAAT scores, for each participant $p = 1, \dots, 10$, we defined a CAAT Score Vector

$$\vec{C}_p = \langle s_1, s_2, s_3, s_4, s_5 \rangle$$

where s_i is the CAAT score obtained by participant p for homework assignment $i \in \{1,2,3,4,5\}$. Then, we computed the *(weighted) Jaccard index*, J, between all 45 pairs of participants,

$$J = \frac{\sum_i \min\{x_i, y_i\}}{\sum_i \max\{x_i, y_i\}}$$

where the x_i are the components of the first participant's CAAT score vector, and the y_i are the components of the second participant's CAAT score vector. Values of J range from $[0,1]$ with values closer to 1 indicating more agreement and values closer to 0 indicating less agreement. The Jaccard index has been used previously to validate educational instruments [156]. The five highest and the five lowest Jaccard indices among our participants are presented in Table 5.3.

Participant Pair	Five Highest J	Participant Pair	Five Lowest J
Mícheál, Íde	0.860	Séamus, Máire	0.504
Siobhán, Máire	0.853	Mícheál, Méabh	0.51
Siobhán, Ciarán	0.824	Séamus, Lasairfhíona	0.515
Lasairfhíona, Siobhán	0.793	Séamus, Siobhán	0.515
Lasairfhíona, Ciarán	0.790	Íde, Méabh	0.52

Table 5.3: Participant pairs with the 5 highest and 5 lowest weighted Jaccard indices

To contextualize the weighted Jaccard indices, we estimated a random mean Jaccard index by Monte Carlo method in Microsoft Excel (200 iterations). This yielded a value of 0.622 or 62%, while the mean of our computed Jaccard indices was 0.678 or 68%. Excluding Participant 1, who showed significant disagreement to the other participants (Participant 1 had less than 60% agreement with 5 out of 9 other participants), the mean of our computed Jaccard indices was 0.698 or 70%. This indicates that the 9 participants aside from Participant 1 showed moderate to high agreement in their CAAT scores.

To further explore agreement, we modeled the Jaccard indices indicating high agreement ($\geq 75\%$) with a graph (see Figure 5.6) and looked for sub-cliques (subgraphs with maximum edges between vertices), which could indicate some commonality among a group of participants.

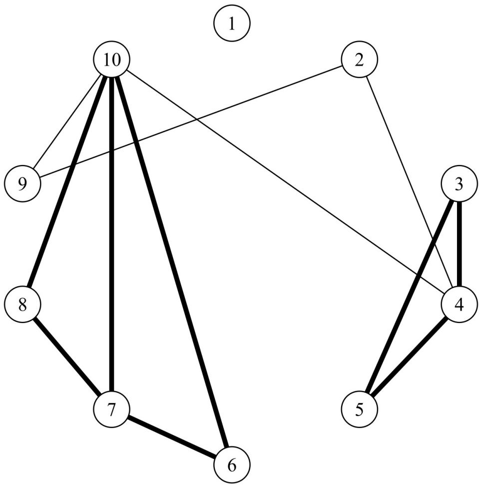


Figure 5.6: Pairs of participants with $J \geq 0.750$.

The largest sub-cliques in the graph were three different triangles: $(3,4,5)$, $(6,7,10)$, and $(7,8,10)$. These are bolded in Figure 5.6. Furthermore, the two triangles involving 7 and 10 are one edge away from forming a 4-clique (the edge between 6 and 8 is missing). This led us to ask if these high-agreement triangles might indicate the existence of common themes among participants' open-ended survey responses.

For the participants in each high-agreement triangle, we used open coding [157] on the participants' initial opinion of online homework (a question on the Pre-CAAT survey) and quality ratings of the homework assignments. For quality ratings, we used the same codes defined in Subsection 5.4.1. However, when coupled with the participants' opinion of online homework, the following themes emerged.

In the first triangle $(3,4,5)$, all three participants' explanations of quality were detail-oriented, focusing on the complexity of procedures among the homework problems. They all referenced specific problem numbers, with one participant even noting that there was a more efficient way to solve one of the problems than was shown by the textbook.

In the second triangle $(6,7,10)$, all three participants focused on a lack of conceptual questions. They didn't all reference specific problem numbers like the first triangle but focused more on the assignments as a whole.

The third triangle $(7,8,10)$ was similar to the second triangle, which may explain why the four participants 6, 7, 8, and 10 nearly formed a 4-clique. The difference was that while all four participants noted a lack of conceptual questions, participants 7, 8, and 10 went further, noting that students both are cheated and cheat their way out of a deep conceptual understanding of the material due to question formatting. Furthermore, all three of these participants focused on an assignment's progression and how the assignment was situated in the overall course narrative as a whole.

The large difference in focus between the first triangle and the other triangles (procedural complexity versus conceptual understanding) may also help explain why these three triangles formed two distinct subgraphs with one edge connecting them in the original graph (the edge between 4 and 10). Participant 10 was the only member of the 'conceptual group' having any high agreement with the 'procedural complexity group'. This may have occurred because Participant 10 also made a few comments about procedural complexity in addition to focusing on conceptual understanding.

5.5.2 Discussion

RQ 3. Does CAAT provide evidence of agreement among university mathematics instructors as to the alignment of homework assignments to a common set of objectives?

In short, yes and no. From our quantitative analysis, most participants were split between a focus on the procedural versus conceptual nature of homework, which could indicate that similar teaching philosophies may be a factor in high-agreement groups. Additionally, most pairs of participants had a non-random agreement. However, instances of significant *disagreement* also existed among the participants. Again, we suspect differences in instructors' teaching philosophies may account for this.

The appearance of both agreement and disagreement provides further evidence for the case of using CAAT as a professional development tool for coordinated course instructional teams. By proactively identifying disagreement amongst team members, a Humanistic-Growth approach to coordination would seek to mediate these differences through group discussion and community-building.

5.6 Study Limitations

This study examined 10 instructors within the same department, five of whom only have experience with their current institution's instantiation of Calculus I. Additionally, this version of Calculus I only involves homework assigned through an online platform.

5.7 Additional Prospective Applications

The implications of our data open up many possible avenues of study including a deeper dive into homework modalities, new approaches to qualitative analysis, and improving our Python code.

Online vs. Paper Homework? Online homework has its own set of unique factors. When interacting with online homework assignments, students use online sources outside their course materials for assistance [145] and are more likely to use solution methods shown in the homework software's help features than those demonstrated in lecture [146]. How might students' interactions with homework and course materials differ for homework that is not assigned online? The results of this study might differ (1) if the study was replicated so that instructors complete study procedures using the online homework platform instead of printed copies of online assignments (giving them a more realistic experience from a student point-of-view) or (2) if the study was replicated using homework assigned from a physical textbook rather than online software.

New Approach to Qualitative Analysis? From our graph-theoretic analysis of participants' CAAT score agreement, there is evidence to suggest that if two instructors have a high agreement in CAAT scores, then they likely focus on similar aspects of a homework assignment's design. Could this method be used more broadly in qualitative data analysis as a preliminary tool for detecting commonalities between study participants?

Improving the Code? If we can improve the Python code to handle larger objective lists, then we could extend the use of CAAT to allow instructors to reflect on the design of other course components that involve larger numbers of objectives such as exams, quizzes, lecture notes, or even recordings of live lectures. However, with the current format of CAAT, improving the Python code may not be feasible. Thus, this goal may require reformatting CAAT so that the score is a probability rather than a distance[6].

5.8 Conclusion

In this chapter, we introduced a novel educational instrument (CAAT) for measuring the alignment of an assignment to a list of content objectives. Through a human-subjects research study, we demonstrated CAAT's potential as a professional development tool for novice instructors, such as Graduate Student Instructors.

This work has practical implications for students in coordinated undergraduate mathematics courses. If the instructional team in such courses is aware of how its members agree and disagree, it can help achieve mutual understanding and compromise, which can positively impact students. Presenting a unified front as instructors prevents possible miscommunication with students.

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[6] The authors would like to acknowledge our colleagues, Jordan Eckert and Padmini Nukala, for this idea.

Chapter 6

Conclusion

In this dissertation, I approached the STEM Retention Problem from the perspective of a Mathematics Discipline-Based Education Researcher with a background in theoretical discrete mathematics. That is, I developed a variety of combinatorial methods for analyzing and redesigning undergraduate mathematics courses. These contributions and their impacts are summarized below.

1. **A New, Enhanced Perspective.** In using graphs, digraphs, and networks to analyze existing curricular materials, such as textbooks, we saw the ability of combinatorial methods to reveal new, macro-level, structural sources of misalignment that are undetectable by traditional education research methods. Additionally, we saw combinatorial methods' ability to complement existing methods, often enhancing conclusions to existing research in a new way. Combinatorial methods can potentially impact critical curricular decisions in mathematics departments, such as textbook adoption, that affect student success.

2. **New Design Theory Results.** Revealing misalignment using combinatorial methods in education research inspired new theoretical results in Combinatorial Designs. These constructions can be used to create calculus homework that aligns more closely with mathematicians' expectations of students on assessments. Although future work will focus on the actual impact of such assignments on student success, theoretically, a greater alignment should affect students positively.

3. **A New Approach to Redesigning Mathematics Courses.** In using combinatorial methods to redesign a calculus course under authentic institutional constraints, we demonstrated that students can experience widespread success in a “weed-out” course. This has broad impacts for institutions similar to Auburn University – schools that offer high-enrollment sections in a lecture-recitation format with exam-heavy grading schemes and teams of multiple instructors per section (often a full-time faculty member with a group of graduate student instructors). Our results show that introductory courses such as calculus do not need to sacrifice rigor to enhance student achievement.

4. **A New Approach to Coordination.** In using combinatorial methods to develop a novel educational instrument, we provided a new way for course coordinators to promote unity among an instructional team. Using our instrument to analyze homework helped to develop novice instructors’ definition of “quality homework” into a multifaceted concept. Our instrument also identified sources of agreement and disagreement among team members, which can help coordinators foster communication between instructors to reach mutual understanding. Enhanced communication promotes alignment between instructors, which may encourage student success, especially in large-enrollment courses where students may encounter multiple instructors, such as a faculty instructor of record and a graduate student recitation leader, in a week.

6.1 Future Directions

Within the realm of Calculus I, I would like to dive deeper into characterizing Calculus I students. The semester I implemented the *War Eagle Calculus I Workbook*, I secured IRB permission to use any work turned in by students during the Fall 2023 semester as research data. 99 students consented to participate in this data collection, which included nearly complete scans of their workbooks (students turned in weekly scans of their workbook for a small participation grade). Since students were not informed of our intention to use their data until after midterm, this massive data set could potentially offer unprecedented authentic insight into students’ conscious and subconscious behaviors as they progressed through an entire Calculus I course. I have only begun to scratch the surface of this goldmine, analyzing student mindsets as manifested through their responses to reflective prompts in collaboration with Devin Hensley [165]. In future work, I would like to deepen this investigation but also analyze other aspects such as note-taking and online homework.

Though this dissertation is focused on Calculus I, in the future, I would like to extend my work to the analysis and design of additional introductory courses (e.g. Precalculus, College Algebra, and Contemporary Math). I have already done some preliminary work analyzing how students are required to use different types of thinking in various Precalculus-Calculus tracks in collaboration with Dr. Isabel Harris [164]. This line of inquiry has made me curious about getting students to think combinatorially about problem solving (e.g. representing a calculus problem as a decision tree). My guess is that mapping their problem strategy may help them develop the gut instinct that experts rely on in the problem solving process.

6.2 Closing Remarks

Recently, I presented my results on constructing balanced Chain Rule task sets (see Section 3.3) at the 56th Southeastern International Conference on Combinatorics, Graph Theory, and Computing in Boca Raton, Florida. It was exciting to see my work spark many interesting discussions about math education with other mathematicians. Though this was just one instance, it demonstrated the potential impact Math DBER could have on the mathematics community. Adopting a mathematician's theory and methodology allows for the translation of mathematics education research into a language that mathematicians find familiar. One day, I hope to see a Math DBER approach heal the rift between math and math education, two fields that are inextricably related through a mutual curiosity about mathematics, merely studied from different perspectives.

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Appendices

Appendix A

Samples from *The War Eagle Calculus I Workbook*

1 The Problem with Slopes

Lecture

By the end of this lesson, you should be able to answer the following little question:

- (BQ1-LQ1) *What is the problem with finding the slope of a line using only one point, and how do we start to fix this?*

By the end of Unit 1 (Lessons 1-6), you should be able to answer the following big question:

- (BQ1) *How do we find the slope of a line using only one point?*

One of the main problems of Calculus is to find the slope of a line given only one point.

WARM-UP:

Calculate the slope between the points $(1, 1)$ and $(-3, 2)$.

What formula calculates slope?

Can you calculate a slope using the formula if all you are given is the point $(1, 1)$? Explain why or why not.

Def: A ______________ curve is a curve with no sharp corners, cusps, or turns. Given a smooth curve, a line between any two points on the curve is called a ______________ ______________. A line that touches the curve at exactly one point is a ______________ ______________.

Example of a smooth curve, secant line, and tangent line:

Try THIS **EXPLORING SECANT SLOPES:** Open the *Exploring Secant Slopes* file on Canvas, or scan the QR code to complete this problem.

Drag point B along the curve closer and closer to point A in a few steps. Write down the values at each step in the table below.

Step #	Rise	Run	Slope between A and B
1			
2			
3			
4			
5			

Do you notice a pattern in the slope values? If so, what might this number represent? Discuss your findings with a partner and record in the space below.

Def: Another name for the slope of a secant line is the ________________ _____ _____ ________________.

Example: Find the average rate of change of $f(x) = x^2 + 2x + 3$ on the given intervals.

(a) from $x = 2$ to $x = 2.1$

(b) from $x = 2$ to $x = 2.01$

(c) Try THIS $[2, 2.001]$

(d) Try THIS $[2, 2.0001]$

Do you notice a pattern in your answers similar to *Exploring Secant Slopes?*

The idea of "approaching" a number like we saw in *Exploring Secant Slopes* and in the previous example is so important, we give it a specific name.

Def: The ________________ of $f(x)$ is the y-value the function approaches (gets really close, but not necessarily equal to) as the x-value approaches a number a.

Notation:

$$\lim_{x \to a} f(x) = L \text{ means "the limit of } f(x) \text{ as } x \text{ approaches } a \text{ is } L\text{".}$$

Limits begin to help us solve the problem of finding slope given only one point because they allow us to find exact answers through estimations while avoiding problems (for example, dividing by 0).

Example: Let $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$. Is it possible to find $f(1)$? Why or why not?

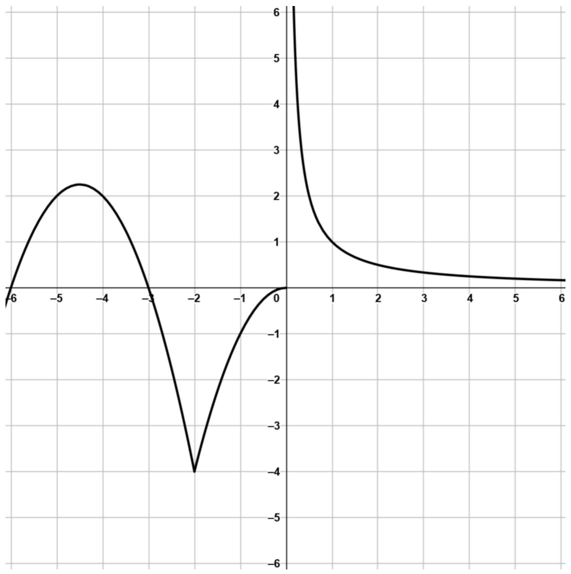
How can we approximate $f(1)$ without actually finding it?

Complete the table below. Round each answer to four decimal places.

x	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$		4.01		4.0001

Estimate the limit of $f(x)$ as x approaches 1:

Example: Decide whether each of the stated limits is **true** or **false** for the graph of $f(x)$ given below. Then, explain your choice.



(a) The limit of $f(x)$ as x approaches -4 is 2. **T F** Because...

(b) $\lim_{x \to -2} f(x) = -4$. **T F** Because...

(c) $\lim_{x \to 1} f(x) = 0$. **T F** Because...

Check Your Understanding! (BQ1-LQ1) *What is the problem with finding the slope of a line using only one point, and how do we start to fix this?* Use the space below to answer the question in 2-3 sentences.

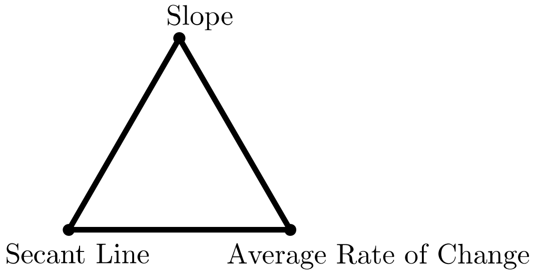
Pre-Recitation Prep

This assignment should be completed for the next recitation session. All problems should be worked on separate paper, which you will be allowed to use for the Prep Check portion of the quiz. You should not use a calculator.

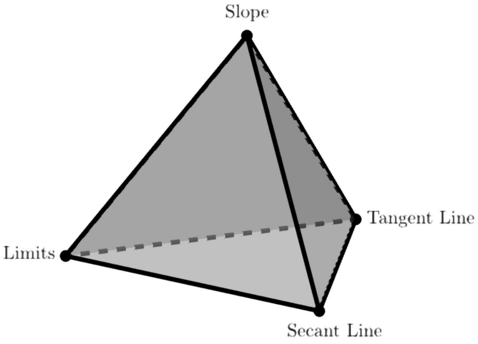
1. Add the following terms to your flashcard deck: slope, secant line, tangent line, limit, average rate of change.
2. Translate the following sentences into precise mathematical notation.
   - (a) The limit of $x^3 - 3$ as x approaches 0 is -3.
   - (b) The limit of $\sin x$ as x approaches $\frac{\pi}{2}$ is 1.
   - (c) The limit of $\frac{x^2-1}{x-2}$ as x approaches 0 is $\frac{1}{2}$.

3. Each diagram below indicates a set of relationships between terms in this lesson. For each diagram, write one sentence that explains the relationships between the terms.

(a)



(b)

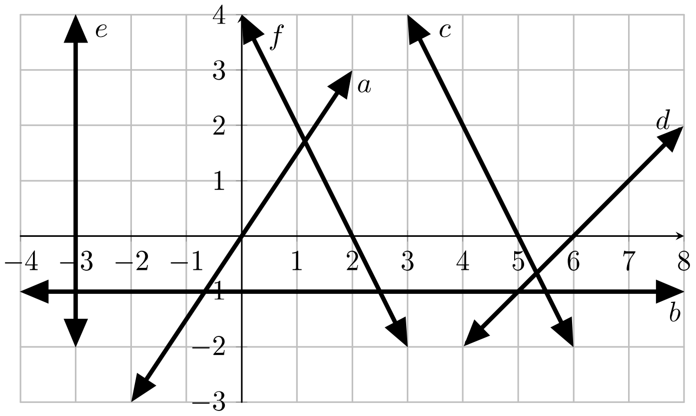


Find the slope of the line through the given points.

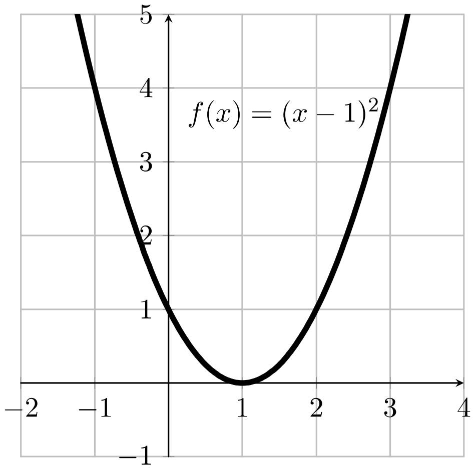
4. $(8, 4), (6, 5)$
5. $(-6, 3), (-4, 5)$
6. $(-2, 7), (-5, -7)$

In the figure below, which line(s) have...

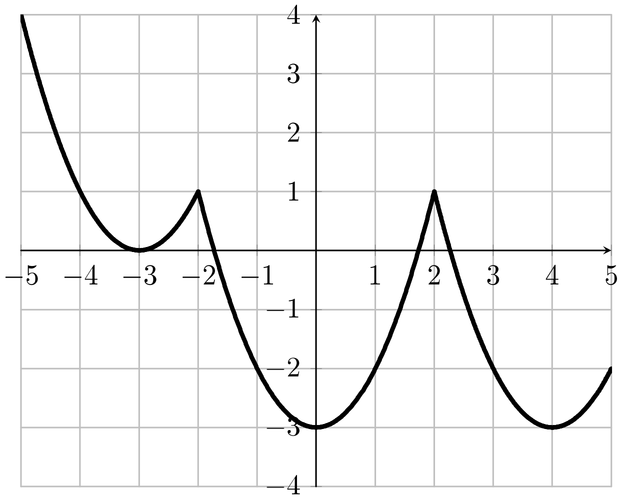
7. positive slope?
8. negative slope?
9. zero slope?
10. no (undefined) slope?



11. Identify as many secant lines as you can using the graph below. Then, find the slopes of the secant lines you identified. Use precise notation when writing your answer(s).



12. Find all the limits you can using the given graph of the function g. Use precise notation when writing your answer(s).



How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?

Recitation

Flashcards

Today, I did flashcards with ______________________. I got _____ cards correct out of _____ cards total, which is _____%. Graph this % correct on your flashcards chart.

Group Problem Solving

Toolbox

- The ***average velocity*** $[v_{avg}]$ of a moving object is the average rate of change of its position $[p(t_2) - p(t_1)]$ with respect to time $[t_2 - t_1]$.

Scenario: *Aubie's Dilemma*

Aubie is running due south down S. College St. 10 minutes into his run, he is located 1 mile south of campus. 18 minutes into his run, he is located 2 miles south of campus. After turning around at the University-Shug Jordan traffic light, Aubie meets his friend, War Eagle. She flies next to him with an average velocity of 8 miles per hour for the entire second half of his 40-minute run. After his run, Aubie gets out of the shower at 7:56 am, but his class starts at 8 am in the ACLC! He knows that his dorm is 0.7 miles from the classroom.

In the scenario *Aubie's Dilemma*, circle all of the time values, box all of the position values, and underline all of the average velocity values.

Figure out what you can from the scenario *Aubie's Dilemma*.

Debate Challenge: Do you think Aubie will make it to class on time if he walks? What if he runs? Why or why not? You may use your phone to research average walking/running paces to defend your answer.

Add the following term to your flashcard deck: average velocity.

Webassign Homework

1.1 #1-4; **2.2** #1, 8, 9; **2.3** #2; **2.7** #5, 9 Lesson 1 Homework Quiz: Wednesday, August 23.

Extra Practice

1. Now that you learned about average velocity in recitation, can you revise the triangle diagram from 3(a) on the Pre-Recitation Prep to be 3D like the diagram in 3(b)? Write one sentence that describes this new 3D diagram.
2. (a) Calculate the average rate of change of the function $f(x) = (x-1)^2$ on the following intervals.
    - i. $[-4.01, -3.99]$
    - ii. $[-2.001, -1.999]$
    - iii. $[-0.02, 0.01]$
    - iv. from $x = 0.9$ to $x = 1.1$
    - v. $[2.999, 3.001]$

   (b) Sketch a graph of $f(x) = (x-1)^2$. Then, plot the points $(-4, a_i), (-2, a_{ii}), (0, a_{iii}), (1, a_{iv}), (3, a_v)$, where the y-coordinates are your answers from part (a) in order.

   (c) What type of graph do the points you plotted in part (b) appear to make? How is this graph related to $f(x) = (x-1)^2$?
3. (a) Graph the piecewise function.

$$f(x) = \begin{cases} -2x - 7, & \text{if } x < -3 \\ -(x+2)^2, & \text{if } -3 \le x \le -1 \\ x^3, & \text{if } -1 < x < 1 \\ (x-2)^2, & \text{if } 1 \le x \le 3 \\ -2x + 7, & \text{if } x > 3 \end{cases}$$

   (b) Find the following limits from your graph. Write each answer as a complete sentence in precise mathematical notation.
    - i. $\lim_{x \to 0} f(x)$
    - ii. $\lim_{x \to -1} f(x)$
    - iii. $\lim_{x \to -3} f(x)$
    - iv. $\lim_{x \to 2} f(x)$
4. Draw an example of one function f that meets the constraints $\lim_{x \to 0} f(x) = 5$, $\lim_{x \to -3} f(x) = -2$, and $\lim_{x \to 4} f(x) = 3$

Evaluate if $x = 4$ and $y = 2$.

5. $x - |y| - |3|$
6. $(9 + xy)^2$
7. $(x^2 y^2)^2$

Find the vertical asymptote(s) of each function.

8. $f(x) = \dfrac{x-1}{x+3}$
9. $f(x) = \dfrac{x-1}{(x-2)(x+5)}$
10. $f(x) = \dfrac{x^2 - 3x + 2}{x^3 - 6x^2 + 5x}$

How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?

3 Evaluating Limits Algebraically

Lecture

By the end of this lesson, you should be able to answer the following little question:

- (BQ1-LQ3) *How do we evaluate limits algebraically?*

By the end of Unit 1 (Lessons 1-6), you should be able to answer the following big question:

- (BQ1) *How do we find the slope of a line using only one point?*

WARM-UP:

Let $f(x) = 1 + \sqrt{x-7}$. Find $f(16)$.

Factor completely: $9m^3 - 63m^2 + 108m$

Factor completely: $(a+2b)^2 - c^2$

Rationalize the denominator: $\dfrac{1}{2+\sqrt{3}}$

RECALL:

A limit is a ______-value. With this in mind, how might we find the limit of a function like

$$f(x) = \frac{x^2+2x+1}{x-2}, \text{ as } x \to 1$$

We call this method for finding limits

________________ ____________________________.

When can we use direct substitution?

What are some problems that can happen?

What happens if we try to find $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1}$ with direct substitution?

How can we use algebra to remove the problem?

Example:

$\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1}$	Problem Statement
$= \lim\limits_{x \to 1} \dfrac{(x-1)(x+1)}{x-1}$	
$= \lim\limits_{x \to 1} (x+1)$	
$= 1+1$	
$= 2$	

We call this method for finding limits

______________ _____ ______________. On a graph, this is a _____, or a ***removable discontinuity***.

When can we factor and simplify?

Can we factor and simplify the limit $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$?

For this problem, we will need another algebraic strategy: ____________________ _____ ____________________.

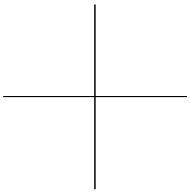
Example: Evaluate $\lim\limits_{u \to 2} \dfrac{\sqrt{4u+1}-3}{u-2}$.

$\lim\limits_{u \to 2} \dfrac{\sqrt{4u+1}-3}{u-2}$	Problem Statement
$= \lim\limits_{u \to 2} \dfrac{\sqrt{4u+1}-3}{u-2} \cdot \dfrac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}$	
$= \lim\limits_{x \to 2} \dfrac{4u+1-9}{(u-2)(\sqrt{4u+1}+3)}$	
$= \lim\limits_{x \to 2} \dfrac{4u-8}{(u-2)(\sqrt{4u+1}+3)}$	
$= \lim\limits_{x \to 2} \dfrac{4(u-2)}{(u-2)(\sqrt{4u+1}+3)}$	
$= \lim\limits_{x \to 2} \dfrac{4}{\sqrt{4u+1}+3}$	
$= \dfrac{4}{\sqrt{4(2)+1}+3}$	
$= \dfrac{4}{12}$	
$= \dfrac{1}{3}$	

When can we rationalize the numerator?

These same strategies apply to one-sided limits. ________________ functions frequently require one-sided limits. One very common piecewise function is the ________________ ________________ function.

Absolute Value Function: $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$



Example: Show that $\lim_{x\to 0}\frac{\lvert x\rvert}{x}$ does not exist.

Proof. We will show that the limit does not exist by showing that...

First,

$\lim_{x\to 0^-}\frac{\lvert x\rvert}{x}$	P.S.
$= \lim_{x\to 0^-}\frac{-x}{x}$	
$= \lim_{x\to 0^-} -1$	
$= -1$	

Next,

$\lim_{x\to 0^+}\frac{\lvert x\rvert}{x}$	P.S.
$= \lim_{x\to 0^+}\frac{x}{x}$	
$= \lim_{x\to 0^+} 1$	
$= 1$	

Since , $\lim_{x\to 0}\frac{\lvert x\rvert}{x}$ does not exist. $\square$

☑ **Check Your Understanding!** (BQ1-LQ3) *How do we evaluate limits algebraically?* Use the space below to answer the question in 2-3 sentences.

Pre-Recitation Prep

This group of problems is to check your understanding of the lecture and make sure you're prepared for the upcoming recitation. You should not use a calculator to solve any of the problems.

1. Add the following terms to your flashcard deck: removable discontinuity, piecewise function, absolute value function (piecewise def.), rationalize.

Factor completely.

2. $r^2 - 6r - 9s^2 + 9$
3. $100 + 4x^2 - 16y^2 - 40x$
4. $(a+b)^2 - (a-c)^2$
5. $m^8 - n^8$
6. $x^2(x+2) - x(x+2) - 12(x+2)$
7. $a^2 + b^2 - c^2 - d^2 - 2ab + 2cd$

For each solution below, explain why each step is allowed to occur from the previous step.

8.

$\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^4-3x^2-4}$	P.S.
$= \lim\limits_{x\to 2} \dfrac{(x-2)(x-2)}{(x^2-4)(x^2+1)}$	
$= \lim\limits_{x\to 2} \dfrac{(x-2)(x-2)}{(x-2)(x+2)(x^2+1)}$	
$= \lim\limits_{x\to 2} \dfrac{(x-2)}{(x+2)(x^2+1)}$	
$= \dfrac{(2-2)}{(2+2)(2^2+1)}$	
$= \dfrac{0}{(4)(5)}$	
$= 0$	

9.

$\lim\limits_{h\to 0} \dfrac{(3+h)^{-1}-3^{-1}}{h}$	P.S.
$= \lim\limits_{h\to 0} \dfrac{\frac{1}{3+h}-\frac{1}{3}}{h}$	
$= \lim\limits_{h\to 0} \dfrac{\frac{3}{3(3+h)}-\frac{3+h}{3(3+h)}}{h}$	
$= \lim\limits_{h\to 0} \dfrac{\frac{3-3-h}{3(3+h)}}{h}$	
$= \lim\limits_{h\to 0} \dfrac{-h}{3(3+h)} \cdot \dfrac{1}{h}$	
$= \lim\limits_{h\to 0} \dfrac{-1}{3(3+h)}$	
$= \dfrac{-1}{3(3+0)}$	
$= -\dfrac{1}{9}$	

Complete each partial solution below with its missing steps.

10.

$$\lim_{h\to 0} \frac{(-5+h)^2-25}{h}$$

$=$

$$= \lim_{h\to 0} \frac{-10h+h^2}{h}$$

$=$

$$= \lim_{h\to 0}(-10+h)$$

$$= -10+0$$

$$= -10$$

11.

$$\lim_{x\to 3^-} (2x+|x-3|)$$

$=$

$$= \lim_{x\to 3^-} (2x-x+3)$$

$=$

$$= 3+3$$

$$= 6$$

12.

$$\lim_{t\to 1} \frac{t^4-1}{t^3-1}$$

$$=$$

$$=$$

$$= \lim_{t\to 1} \frac{(t+1)(t^2+1)}{t^2+t+1}$$

$$=$$

$$= \frac{2\cdot 2}{3}$$

$$= \frac{4}{3}$$

13.

$$\lim_{x\to 25} \frac{5-\sqrt{x}}{25x-x^2}$$

$$=$$

$$= \lim_{x\to 25} \frac{25-x}{(25x-x^2)(5+\sqrt{x})}$$

$$=$$

$$= \lim_{x\to 25} \frac{1}{x(5+\sqrt{x})}$$

$$=$$

$$= \frac{1}{25(5+5)}$$

$$= \frac{1}{250}$$

14. Complete the table with the correct limit strategy and the features of the problem that indicate you should use the particular strategy.

Strategy	**Problem Features**	**Example**
		$\lim_{x\to 5} \frac{x^2-6x+5}{x-5}$
		$\lim_{x\to 16} \frac{4-\sqrt{x}}{16x-x^2}$
Add the fractions, then try direct sub. or factor/simplify		$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right)$

How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?

Recitation

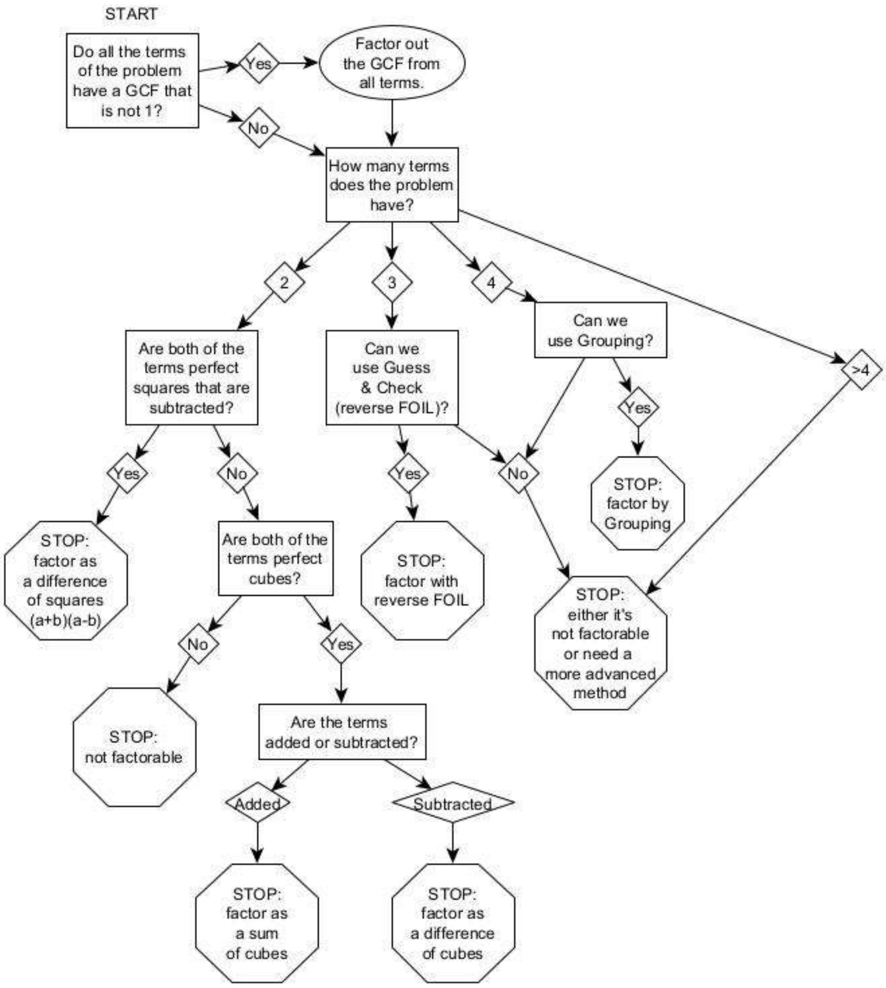
Flashcards

Today, I did flashcards with ______________________. I got _____ cards correct out of _____ cards total, which is _____%. Graph this % correct on your flashcards chart.

Group Problem Solving

Imagine your group is a team of Solution Engineers working for a company called Limit Solutions, Inc. Your boss, Mr. Bigbucks, wants an efficient way to come up with a solution to whatever limit he gives you. After all, he doesn't know how; that's why he makes the big bucks. Your team's objective is to use what you know about solving limits so far (strategies, types of problems, etc.) to create a *flowchart* that solves limits. If you complete the objective, Mr. Bigbucks will be very pleased!

Example of a flowchart that solves factoring problems:



Flowchart: ***How to Solve Limit Problems***

Hint: it's a good idea to always start by seeing if direct substitution works.

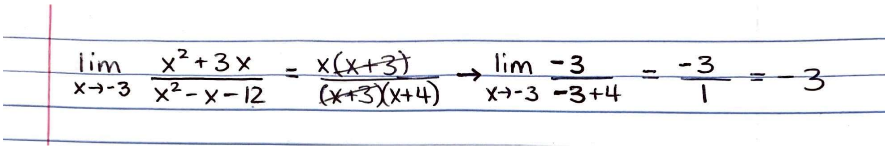
Webassign Homework

1.1 #7, 18-20, **2.3** #1, 3-7 Lesson 3 Homework Quiz: Wednesday, August 30.

Extra Practice

Identify an appropriate strategy for evaluating each limit. Then, evaluate each limit, if it exists, using your strategy. If it does not exist, explain why. Use precise limit notation in your solutions.

1. $\displaystyle\lim_{t\to-3}\frac{t^2-9}{2t^2+7t+3}$
2. $\displaystyle\lim_{x\to-2}\frac{x+2}{x^3+8}$
3. $\displaystyle\lim_{t\to0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$
4. $\displaystyle\lim_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4}$
5. $\displaystyle\lim_{t\to0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$
6. $\displaystyle\lim_{h\to0}\frac{(x+h)^3-x^3}{h}$
7. $\displaystyle\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{|x|}\right)$
8. $\displaystyle\lim_{x\to-6}\frac{2x+12}{|x+6|}$
9. (a) Aubie is also working on limit problems. He turned in the following solution to a problem, but he made 3 mistakes in notation. What are they?



$$\lim_{x\to-3}\frac{x^2+3x}{x^2-x-12}=\frac{x\cancel{(x+3)}}{\cancel{(x+3)}(x+4)}\to\lim_{x\to-3}\frac{-3}{-3+4}=\frac{-3}{1}=-3$$

   (b) Rewrite the solution with all of the mistakes fixed.

10. Look back at #13 on this lesson's Pre-Recitation Prep, and solve the problem using a strategy other than rationalizing the numerator.

Multiply each expression by $\frac{1/x}{1/x}$. Do not simplify.

11. $\dfrac{x^2+1}{x^2-1}$
12. $\dfrac{x^3+2x}{x^2-4}$
13. $\dfrac{x^3+4x^2-x-4}{x^3-x^2+x-1}$

Determine the relationship between each pair of functions: $\geq$ or $\leq$.

14. $f(x)=1$ ___ $g(x)=\sin x$
15. $f(x)=-1$ ___ $g(x)=\cos x$
16. $f(x)=1$ ___ $g(x)=x^2+1$

How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?

4 Evaluating Other Kinds of Limits Algebraically

Lecture

By the end of this lesson, you should be able to answer the following little question:

- (BQ1-LQ3) *How do we evaluate limits algebraically?*

By the end of Unit 1 (Lessons 1-6), you should be able to answer the following big question:

- (BQ1) *How do we find the slope of a line using only one point?*

WARM-UP:

What are the ranges of $f(x) = \sin x$ and $f(x) = \cos x$?

Range of $f(x) = \sin x$ is $[-1, 1]$.

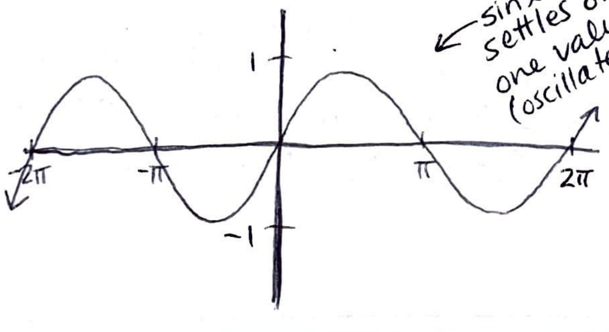
Range of $f(x) = \cos x$ is $[-1, 1]$.

Let's begin with a problematic example.

Example: Find the limit $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

Why can't we use direct substitution?

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0^2 \sin\left(\frac{1}{0}\right)$$
$$= 0^2 \sin(\infty)$$
$$= 0^2 \ ?$$



← sin x never settles on one value (oscillates)

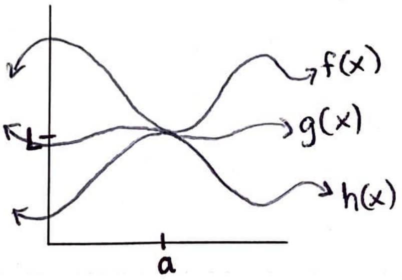
We know that limits don't exist when their one-sided limits are not equal, but limits also do not exist when functions oscillate, or go back and forth between two y-values and never settle.

We get around this problem by using the following theorem.

Squeeze **Theorem:** If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

In English: If we can find two functions, one greater than and one less than our problem function, each with the same limit, our problem limit is forced to have the same value as the bounding limits.

Graphically:



Example: Find the limit $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

We start by bounding the function.

$-1 \le \sin x \le 1$	The range of sin x (A fact we know already)
$\implies \quad -1 \le \sin\frac{1}{x} \le 1$	Still the range of sin x
$\implies \quad -x^2 \le \underbrace{x^2\sin\frac{1}{x}}_{\text{middle matches our problem without limit}} \le x^2$	Multiply through the inequality by x^2

Then, we take the limit.

$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le \lim_{x\to 0} x^2$	Take limit through the inequality now the middle is exactly our problem!
$0 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le 0$	Evaluate limits on either side

Since the limits of the bounding functions are both 0, by the Squeeze Theorem, $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

RECALL:

What two kinds of asymptotes have we learned about, and what type of limit corresponds to each?

① Horizontal Asymptotes → Limits at Infinity → $\lim_{x\to\infty} f(x)$

② Vertical Asymptotes → Infinite Limits → $\lim_{x\to ?} f(x) = \infty$

Think of some types of functions that have vertical asymptotes.

$f(x) = \frac{1}{x}$ (some rational functions)

$f(x) = \tan x$ (some trig functions)

$f(x) = \ln x$ (logarithmic functions)

Since infinity is not a number, we can't use any of our previous methods for finding limits. To evaluate infinite limits, we will have to use a verbal argument, or words.

Example: Find the limits $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$

If x is close to 3 but larger than 3, then...
the denominator is very close to 0, but positive. The numerator is also positive. So, when we divide the numerator by the denominator, we get a very big positive number since the denominator is so small. Therefore, $\lim_{x\to 3^+} \frac{2x}{x-3} = \infty$.

If x is close to 3 but smaller than 3, then...
the denominator is very close to 0, but negative. The numerator is positive though, so when we divide the numerator by the denominator, we get a very big negative number since the denominator is so small. Therefore, $\lim_{x\to 3^-} \frac{2x}{x-3} = -\infty$.

RECALL:

Think of some functions whose horizontal asymptotes we know.

$f(x) = \frac{1}{x} \rightarrow$ HA is $y=0$

$f(x) = e^x \rightarrow$ HA is $y=0$

True Facts:

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0, \text{ where } n \text{ is a positive integer}$$

$$\lim_{x\to\pm\infty} e^{\mp nx} = 0, \text{ where } n \text{ is a positive integer}$$

Our goal with limits at infinity is to manipulate the expression so that we can take advantage of the True Facts.

Example: Find the limit $\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$.

$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$	P.S.
$= \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{1/x}{1/x}$	Multiply the expression by a "well-chosen 1" We choose this according to the denominator's highest power of x
$= \lim_{x\to\infty} \frac{\sqrt{2x^2/x^2+1/x^2}}{3x/x-5/x}$	Multiply the numerator and the denominator. When bringing 1/x under the square root, the 1/x gets squared.
$= \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{3-5/x}$	Simplify each term in the numerator and the denominator.
$= \frac{\sqrt{2+0}}{3-0}$	Use the true fact $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for all positive n.
$= \frac{\sqrt{2}}{3}$	Simplify (arithmetic)

so we can eventually use the True Fact.

Example: Find the horizontal asymptotes of the function $\frac{1-e^x}{1+2e^x}$.

$\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$	Def. of Right HA
$= \lim_{x\to\infty} \frac{1-e^x}{1+2e^x} \cdot \frac{1/e^x}{1/e^x}$	Multiply by well-chosen 1.
$= \lim_{x\to\infty} \frac{1/e^x-1}{1/e^x+2}$	Distribute well-chosen 1
$= \frac{0-1}{0+2}$	Use true fact $\lim_{x\to\infty} \frac{1}{e^x} = 0$
$= -\frac{1}{2}$	simplify (arithmetic)

$\lim_{x\to-\infty} \frac{1-e^x}{1+2e^x}$	Def. of left HA
$= \lim_{x\to-\infty} \frac{1-e^x}{1+2e^x} \cdot ?$	~~Mult. by well-chosen 1~~
$=$	~~Dist. well-chosen 1~~
$= \frac{1-0}{1+2(0)}$	Use True Fact
$= \frac{1}{1} = \boxed{1}$	Arithmetic

don't need these 2 since $\lim_{x\to-\infty} e^x =$

Conclusion: The horizontal asymptotes of $\frac{1-e^x}{1+2e^x}$ are $y=-\frac{1}{2}$ and $y=1$.

☑ **Check Your Understanding!** (BQ1-LQ3) *How do we evaluate limits algebraically?* Use the space below to answer the question in 2-3 sentences.

answers may vary

Lesson 4 Solutions

Sunday, August 20, 2023 10:51 AM

Lesson 4 Pre-Recitation Prep

	P.S
2. $\lim\limits_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$	
$= \lim\limits_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \cdot \frac{1/x^3}{1/x^3}$	Multiply by "well-chosen 1"
$= \lim\limits_{x\to\infty} \frac{\sqrt{1/x^6+4x^6/x^6}}{2/x^3-x^3/x^3}$	Distribute numerator & denominator; $1/x^3$ inside $\sqrt{\ }$ becomes $1/x^6$
$= \lim\limits_{x\to\infty} \frac{\sqrt{1/x^6+4}}{2/x^3-1}$	Simplify each fraction
$= \frac{\sqrt{0+4}}{0-1}$	Use True Fact $\lim\limits_{x\to\infty} \frac{1}{x^n}=0$
$= \frac{2}{-1}$	Arithmetic/ Simplify
$= -2$	Arithmetic/ Simplify

Strategy for #2 is "manipulate to use True Facts". This works because $x\to\infty$.

	P.S.
3. $\lim\limits_{x\to 0} \sqrt{x}\cos\frac{3}{x^2}$	
$-1 \leq \cos x \leq 1$	Range of cos x is $[-1,1]$.
$\implies -1 \leq \cos\frac{3}{x^2} \leq 1$	No matter the argument, range is still $[-1,1]$.
$\implies -\sqrt{x} \leq \sqrt{x}\cos\frac{3}{x^2} \leq \sqrt{x}$	Multiply by $\sqrt{x}$
$\implies \lim\limits_{x\to 0} -\sqrt{x} \leq \lim\limits_{x\to 0} \sqrt{x}\cos\frac{3}{x^2} \leq \lim\limits_{x\to 0} \sqrt{x}$	Apply limit across inequalities (Now middle matches problem)
$\implies 0 \leq \lim\limits_{x\to 0} \sqrt{x}\cos\frac{3}{x^2} \leq 0$	Take limit of bounding functions
$\implies \lim\limits_{x\to 0} \sqrt{x}\cos\frac{3}{x^2} = 0$	By Squeeze Thm, since limits of bounding functions are equal, our problem must be equal to the same value as those limits.

Strategy for #3 is Squeeze Theorem. This works because all other strategies don't work & trig function with power function is good candidate for Squeeze Thm.

4. $\lim\limits_{x\to(\pi/2)^+} \frac{1}{x}\sec x$

First, we rewrite the limit in terms of cosine: $\boxed{a}$. Now, we see that by direct substitution, $\frac{1}{\pi/2} = \frac{2}{\pi}$ and $\frac{1}{\cos(\pi/2)} = \frac{1}{0}$ is $\boxed{b}$. So, there is a vertical asymptote of the function at $x = \frac{\pi}{2}$. Now, if we consider values of x that are very close to $\frac{\pi}{2}$ from the $\boxed{c}$, then the value of cosine is positive, so $\lim\limits_{x\to(\pi/2)^+} \frac{1}{x}\sec x = \boxed{d}$.

$\boxed{a}$ $\lim\limits_{x\to(\frac{\pi}{2})^+} \frac{1}{x\cos x}$

$\boxed{b}$ undefined

$\boxed{c}$ right

$\boxed{d}$ ∞

Strategy for #4 is verbal argument. This works because limit in question is a vertical asymptote.

5. $\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}$

$$= \lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2}\cdot\frac{1/t^2}{1/t^2}$$

$$= \lim_{t\to\infty} \frac{1/t^{3/2}+1}{2/t-1}$$

$$= \frac{0+1}{0-1}$$

$$= \frac{1}{-1}$$

$$= -1$$

Strategy for #5 is "manipulate to use True facts". This works because t→∞.

6. $\lim_{x\to\infty}(\sqrt{9x^2+x}-3x)$

$$= \lim_{x\to\infty}(\sqrt{9x^2+x}-3x)\cdot\frac{(\sqrt{9x^2+x}+3x)}{(\sqrt{9x^2+x}+3x)}$$

$$= \lim_{x\to\infty} \frac{(\cancel{9x^2}+x)-\cancel{9x^2}}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9+1/x}+3}$$

$$= \frac{1}{\sqrt{9+0}+3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}$$

Strategy for #6 is rationalize numerator then manipulate to use True facts. This works because original problem is not a fraction, but rationalizing turns it into a fraction, which we can then multiply by a well-chosen 1 as normal.

7. $\lim_{x\to\infty} \frac{x^4-3x^2+x}{x^3-x+2}$

$$= \lim_{x\to\infty}\left(x+\frac{-2x^2-x}{x^3-x+2}\right)$$

$\boxed{a}$ Use polynomial long division to show why this step follows from the P.S.

$$= \infty + 0$$

$\boxed{b}$ Show work for why $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} \frac{-2x^2-x}{x^3-x+2} = 0$

$$= \infty$$

$\boxed{a}$

$$\begin{array}{r|l} & x + \frac{-2x^2-x}{x^3-x+2} \\ \hline x^3-x+2 & x^4+0x^3-3x^2+x+0 \\ & -(x^4 \qquad -x^2 \; +2x) \\ \hline & \qquad -2x^2-x \end{array}$$

$\boxed{b}$ $\lim_{x\to\infty} x = \infty$ because if we take a really large number for x, then f(x) = x is really large.

$$\lim_{x\to\infty} \frac{-2x^2-x}{x^3-x+2}\cdot\frac{1/x^3}{1/x^3}$$

$$= \lim_{x\to\infty} \frac{-2/x-1/x^2}{1-1/x^2+2/x^3}$$

$$= \frac{0-0}{1-0+0} = 0$$

Strategy for #7 is manipulate to use true facts. Manipulation requires division first because if we try multiplying original problem by a well-chosen 1, the denominator would be 0.

requires division first because if we try multiplying original
by a well-chosen 1, the denominator would be 0.

Lesson 4 Extra Practice

1. $\lim_{x\to 5^+} \frac{x+1}{x-5}$ verbal argument

If we choose x-values really close to, but greater than 5, the denominator will be really close to 0 and positive. The numerator will also be positive. Dividing the two will give a really big positive number since the denominator is so small. Therefore, $\lim_{x\to 5^+} \frac{x+1}{x-5} = \infty$.

2. $\lim_{x\to\infty} \frac{3x-2}{2x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$ Manipulate & Use True Fact

$= \lim_{x\to\infty} \frac{3-{}^2/x}{2+{}^1/x}$

$= \frac{3-0}{2+0}$

$= \boxed{\frac{3}{2}}$

3. $\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right)$ Squeeze Theorem

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$

$\Rightarrow -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) \le \sqrt{x^3+x^2}$

$\Rightarrow \lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) \le \lim_{x\to 0} \sqrt{x^3+x^2}$

$\Rightarrow -\sqrt{0^3+0^2} \le \lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) \le \sqrt{0^3+0^2}$

$\Rightarrow 0 \le \lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) \le 0$

$\Rightarrow \lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) = 0$

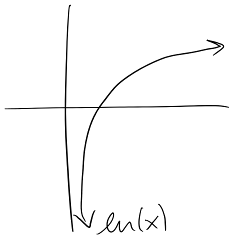
4. $\lim_{t\to\infty} \frac{t-t\sqrt{t}}{2t^{3/2}+3t-5}$ Manipulate & Use True Fact

$= \lim_{t\to\infty} \frac{t-t^{3/2}}{2t^{3/2}+3t-5} \cdot \frac{\frac{1}{t^{3/2}}}{\frac{1}{t^{3/2}}}$

$= \lim_{t\to\infty} \frac{\frac{1}{t^{1/2}}-1}{2+\frac{3}{t^{1/2}}-\frac{5}{t^{3/2}}}$

$= \frac{0-1}{2+0-0} = \boxed{-\frac{1}{2}}$

5. $\lim_{x\to 3^+} \ln(x^2-9)$



$= \ln\left(\lim_{x\to 3^+} x^2-9\right)$

$= \ln(3^2-9)$

$= \ln(0)$ > since ln x has a vertical asymptote at x=0.

$\boxed{= -\infty}$

6. $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$ Squeeze Theorem

7. $\lim_{x\to\pi^-} \cot x$ verbal argument

$= \lim \frac{\cos x}{\ }$

6. $\lim\limits_{x\to 0} x^4\cos\left(\frac{2}{x}\right)$ Squeeze Theorem

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

$$-x^4 \le x^4\cos\left(\frac{2}{x}\right) \le x^4$$

$$\lim_{x\to 0} -x^4 \le \lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) \le \lim_{x\to 0} x^4$$

$$-0^4 \le \lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) \le 0^4$$

$$0 \le \lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) \le 0$$

$\Rightarrow \lim\limits_{x\to 0} x^4\cos\left(\frac{2}{x}\right) = 0$ by the Squeeze Theorem

$x\to\pi^-$

argument

$$= \lim_{x\to\pi^-} \frac{\cos x}{\sin x}$$

If we use x-values less than but really close to π, then sin x is a really small positive number and cos x is a negative number. When we divide a negative number by a really small positive number, we get a really large negative number. Thus, $\lim\limits_{x\to\pi^-} \cot x = -\infty$.

8. $\lim\limits_{x\to-\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} \cdot \frac{e^{3x}}{e^{3x}}$ Manipulate & Use True Facts

$$= \lim_{x\to-\infty} \frac{e^{6x}-1}{e^{6x}+1}$$

$$= \frac{0-1}{0+1}$$

since $\lim\limits_{x\to-\infty} e^{6x} = 0$

$$\boxed{= -1}$$

9. $4x-9 \le f(x) \le x^2-4x+7$

$$\lim_{x\to 4} 4x-9 \le \lim_{x\to 4} f(x) \le \lim_{x\to 4} x^2-4x+7$$

$$4(4)-9 \le \lim_{x\to 4} f(x) \le 4^2-4(4)+7$$

$$7 \le \lim_{x\to 4} f(x) \le 7$$

$\Rightarrow \lim\limits_{x\to 4} f(x) = 7$ by the Squeeze Theorem

10. 0° = 0 rad

30° = $\frac{\pi}{6}$ rad

45° = $\frac{\pi}{4}$ rad

60° = $\frac{\pi}{3}$ rad

90° = $\frac{\pi}{2}$ rad

120° = $\frac{2\pi}{3}$ rad

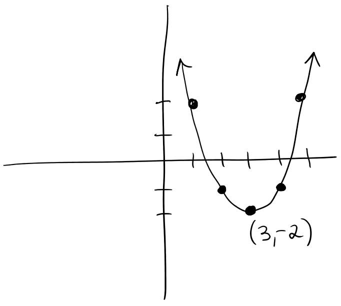
135° = $\frac{3\pi}{4}$ rad

150° = $\frac{5\pi}{6}$ rad

180° = π rad

45 - 4   120 - 3

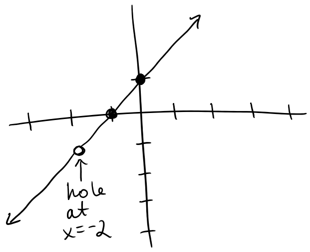
11. $f(x) = (x-3)^2 - 2$



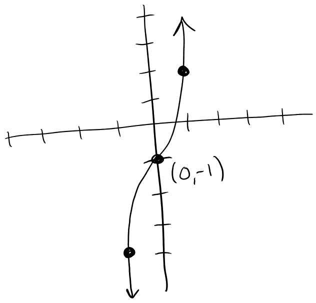
12. $f(x) = \frac{x^2+3x+2}{x+2}$

$= \frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}}$

$= x+1$



13. $f(x) = 3x^3 - 1$



14. $f(x) = 2\sin(3x - \pi) + 1$

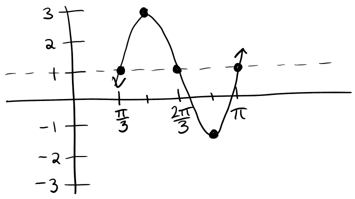
$= 2\sin\left(3\left(x - \frac{\pi}{3}\right)\right) + 1$

Amp = 2

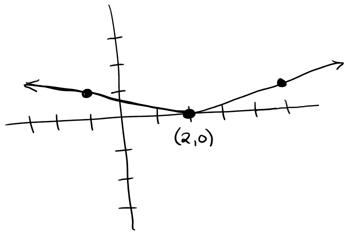
Period = $\frac{2\pi}{3}$

VS = up 1

HS = right $\frac{\pi}{3}$



15. $f(x) = \frac{1}{3}|x-2|$



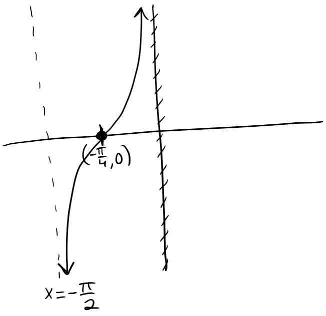
16. $f(x) = \tan\left(2x + \frac{\pi}{2}\right)$

$= \tan\left(2\left(x + \frac{\pi}{4}\right)\right)$

VA: $2\left(x + \frac{\pi}{4}\right) = \frac{\pi}{2}$ $2\left(x + \frac{\pi}{4}\right) = -\frac{\pi}{2}$

$x + \frac{\pi}{4} = \frac{\pi}{4}$ $x + \frac{\pi}{4} = -\frac{\pi}{4}$

$x = 0$ $x = -\frac{\pi}{2}$



Appendix B

The Course Alignment Analysis Tool (CAAT)

Course Alignment Analysis Tool (CAAT)

Note: at any stage in this form, not all boxes may be used due to fewer or more objectives being assessed.

Homework Title: ____

1. **Define Objectives:**

Objectives	Rank
A.	1a.
B.	1b.
C.	1c.
D.	1d.
E.	1e.
F.	1f.
G.	1g.

2. **Affiliate Items:**

Item #	Description	Obj. A	Obj. B	Obj. C	Obj. D	Obj. E	Obj. F	Obj. G
1.								
2.								
3.								
4.								
5.								
6.								
7.								
8.								
9.								
10.								
11.								
12.								
13.								
14.								
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16.								
17.								
18.								
19.								
20.								
21.								
22.								
23.								
24.								
25.								
26.								
27.								
28.								
29.								
30.								

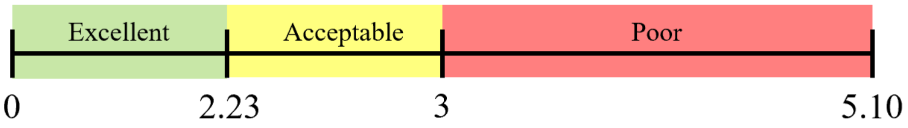
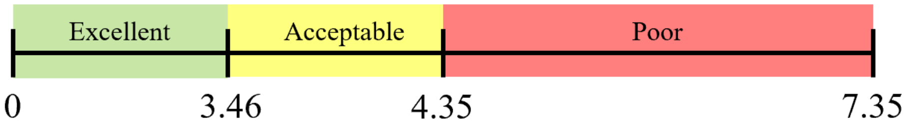
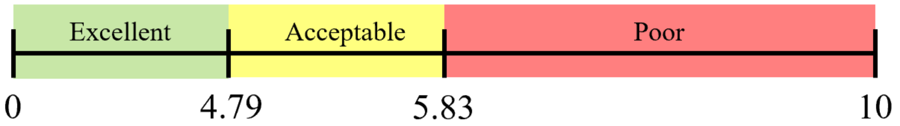
Course Alignment Analysis Tool (CAAT)

3. Calculate Alignment Score:

Column Totals →							
Rank of Column Totals →	3a.	3b.	3c.	3d.	3e.	3f.	3g.
Copy boxes 1a. – 1h. →	1a.	1b.	1c.	1d.	1e.	1f.	1g.
Subtract previous 2 rows and write answers here →	3i.	3j.	3k.	3l.	3m.	3n.	3o.
Square 3i. – 3p. and write answers here →	3q.	3r.	3s.	3t.	3u.	3v.	3w.
Add 3q. – 3x. →	3y.						
Take square root of 3y. This is ALIGNMENT SCORE→							

4. Interpret Alignment Score:

The alignment score measures the difference between what you think the assignment should emphasize and what the assignment actually emphasizes. Scores closer to 0 indicate better alignment, however, maximum possible scores vary by number of objectives. Score ranges for 4, 5, 6, and 7 objectives are presented below.

Number of Objectives	Alignment Score Scale
4	
5	
6	
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