Problems in Computational Algebraic Geometry: Lefschetz Properties and Toric Varieties

by

Sean Grate

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Approved by

Hal Schenck, Professor and Rosemary Kopel Brown Eminent Scholars Chair
Michael Brown, Associate Professor
Melinda Lanius, Assistant Professor
Luke Oeding, Associate Professor
George Flowers, Dean of the Graduate School

Abstract

The first chapter gives the mathematical background needed for the subsequent chapters. The second chapter consists of the publication [16], a joint work with Hal Schenck, in which the we examine the influence of geometry on the weak Lefschetz property. We show that with a certain configuration of points in projective space, their Artnian reduction does have the weak Lefschetz property. We then generalize this result to a Boij-Söderberg theoretic condition on Betti tables of Artinian algebras.

In the third chapter, the focus shifts to toric varieties and results on the Castelnuovo-Mumford regularity of toric surfaces. Inspired by a result of L'vovsky in 1996, we show that the combinatorics of a projective toric surface can provide a *combinatorial* bound on the Castelnuovo-Mumford regularity. An overview is given on the tools used to approach this problem, and the chapter closes with some open questions.

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Chapter 1

Background

Often in applications, one wishes to understand the solutions to a system of equations. When these equations are polynomials, tools in commutative algebra and algebraic geometry have been developed to describe these solution sets. When the system of polynomial equations consists of *homogeneous* polynomials, the solution sets lie in projective space. Because projective space is covered by affine patches (places where polynomials need not be homogeneous), much work has been put into understanding objects in the projective setting. We refer to texts such as [34] and [12] for overviews on the background presented throughout this chapter.

More formally, let $\mathbb{K}$ be a field, $R = \mathbb{K}[x_0, \dots, x_n]$, and $I \subseteq R$ a homogeneous ideal. For a polynomial $f \in R$, we will use the notation

$$\mathbf{V}(f) \coloneqq \{(a_0, \dots, a_n) \in \mathbb{P}^n : \ f(a_0, \dots, a_n)\}$$

to denote the solution set of f, called the *variety* of f. More generally, if $I \subseteq R$ is an ideal, $\mathbf{V}(I)$ consists of the common zeros of all polynomials in I. This only depends on the generators of I, and by Hilbert's basis theorem, I is finitely generated. Thus, if $I = (f_1, \dots, f_s)$, we may write $\mathbf{V}(I) = \mathbf{V}(f_1, \dots, f_s)$. We sometimes will shorthand this to $\mathbf{V}(I) = X_I$.

Similar to the case for varieties, given $X \subseteq \mathbb{P}^n$, we will use the notation

$$\mathbf{I}(X) = \{f \in R : \ f(x) = 0 \text{ for all } x \in X\}.$$

As before, this becomes shorthanded to $\mathbf{I}(X) = I_X$.

Polynomial functions on $\mathbb{K}^n$ restrict to give polynomial functions on X. Any two functions which differ by an element of I_X define the same function on X. This is encoded through the *coordinate ring* of X, denoted $\mathbb{K}[X] := R/I_X$. When I is homogeneous, R/I_X is the coordinate ring of a projective variety. In many scenarios, understanding the geometry of X amounts to understanding the algebra of R/I_X.

With this machinery, one can frequently pass back and forth between algebra and geometry. To answer questions on the algebra side, passing to the geometric side might provide some intuition, or vice versa. A lot of times, however, a practioner will have a system of equations they want to solve. Thus, given some polynomials, we wish to describe the (geometry of the) common zeros.

1.1 Free resolutions

To first understand the geometry, one must first understand the algebra. One common approach to understanding an ideal I is to construct its minimal graded free resolution. While many of the following definitions and results hold for more general objects, we present only up to the level needed for the results to follow. For now, let $R = \mathbb{K}[x_1, \ldots, x_n]$ and suppose $I \subseteq R$ is a homogenous ideal.

Definition 1.1. Suppose M is a finitely generated R-module. A free resolution of M is an exact complex

$$0 \leftarrow M \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} F_n \leftarrow \ldots$$

where each F_i is a free R-module.

A remarkable theorem of Hilbert says that when we take an R-module M to be finitely generated, the free resolution is finite.

Theorem 1.2 (Hilbert's syzygy theorem)**.** *Suppose M is a finitely generated R-module. Then there exists a graded exact sequence of finitely generated, free R-modules:*

$$0 \leftarrow R/I \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots F_n \leftarrow 0.$$

Because the F_i are finitely generated, the maps in the resolution may be considered as matrices with polynomial entries. We may further require that the graded minimal free resolution of M be *minimal*. This means that all of the entries in the matrices corresponding to the maps in the resolution lie in the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$; equivalently, there are no constants appearing in any of the maps in the resolution.

In this way, the minimal free resolution is unique up to change of basis. For example, the ideal $(x^2 - y^2, x^2 + y^2)$ is the same as the ideal (x^2, y^2). So, the matrices are not unique, but their ranks and degrees are. Thus, we can really say *the* minimal graded free resolution of M without issue. Furthermore, there is no issue in resolving I versus R/I as we can always augment the minimal graded free resolution of R/I as follows:

$$F_\bullet \colon 0 \leftarrow I \leftarrow R/I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow 0.$$

One can construct the resolution by iteratively computing kernels, and so the F_{i+1} become relations (or *syzygies*) among the generators of F_i. The intuition here is that the minimal graded free resolution of R/I acts as an approximation for I with terms further out in the resolution encoding higher-order information about I. This fits nicely into the picture of resolutions by forming some diagramatic triangles:

$$\begin{tikzcd}
 & & & & & \ker d_1 \arrow[dl, dashed] & & & \\
0 & R/I \arrow[l] & F_0 \arrow[l, "d_0"'] & & F_1 \arrow[ll, "d_1"'] \arrow[dl, dashed] & & F_2 \arrow[ll, "d_2"'] \arrow[ul, dashed] & & \cdots \arrow[ll, "d_3"'] \arrow[dl] \\
 & & & \ker d_0 \arrow[ul, dashed] & & & & \ker d_2 \arrow[ul, dashed] &
\end{tikzcd}$$

where each d_i factors through $\ker d_{i-1}$.

1.2 Betti tables

When $M = R/I$ is associated to a projective variety $X \subseteq \mathbb{P}^n$, the minimal free resolution $F_\bullet$ of M encodes important data about X such as its degree and dimension. The graded minimal free resolution

$$F_\bullet \colon 0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_{n+1} \leftarrow 0$$

of M can be partially summarized with a *Betti table*, where the entry $b_{i,j}$ in column i and row j is the equal to the rank of F_i in degree $i+j$. In this way, for an entry $b_{i,j}$, i is the *homological degree* and j is the *internal degree*. We follow the `Macaulay2` [17] convention where zero entries are symbolized by dots.

From the Betti table, two important invariants of M can be read off directly: the *projective dimension* and the *Castelnuovo-Mumford regularity*.

Definition 1.3. Suppose M is a finitely generated $\mathbb{K}[x_0, \dots, x_n]$-module with

$$F_\bullet : 0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \dots \leftarrow F_{n+1} \leftarrow 0$$

its minimal free resolution; write

$$F_i \cong \bigoplus_j R(-i-j)^{\beta_{i,i+j}}.$$

The *projective dimension* of M is the length of the minimal free resolution of M, i.e.,

$$\operatorname{pdim}(M) \coloneqq \max \{i \ : \ F_i \neq 0\}.$$

The *Castelnuovo-Mumford regularity* (or simply, *regularity*) of M is

$$\operatorname{reg}(M) \coloneqq \max_{i,j} \{j \ : \ \beta_{i,i+j} \neq 0\} = \max_{i,j} \{j \ : \ b_{i,j} \neq 0\}.$$

With this setup, the projective dimension is the length of the minimal free resolution of M, and the regularity is the "width". Once the Betti table is produced, $\operatorname{pdim}(M)$ is the index of the right-most nonzero column of the Betti table, and $\operatorname{reg}(M)$ is the index of the bottom-most nonzero row. Regularity measures the complexity of M in terms of its generators and syzygies. An immediate corollary of this definition is that if $\operatorname{reg}(I) = m$, then the minimal generators of I are generated in degrees $\leq m$.

If I is the defining ideal of a projective variety X, we define $\operatorname{reg}(X)$ by $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$. The equality $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$ follows from the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

and analyzing the induced long exact sequence in Tor.

Example 1.4. Let $R = \mathbb{K}[x, y, z]$ and consider the ideal $\mathfrak{m} = (x, y, z)$. Then we can resolve $R/\mathfrak{m} \cong \mathbb{K}$ over R with the resolution

$$F_\bullet \colon 0 \leftarrow \mathbb{K} \leftarrow R \xleftarrow{\begin{bmatrix} x & y & z \end{bmatrix}} R(-1)^3 \xleftarrow{\begin{bmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{bmatrix}} R(-2)^3 \xleftarrow{\begin{bmatrix} z \\ -y \\ x \end{bmatrix}} R(-3) \leftarrow 0.$$

The shifts ensure that the differentials are homogenous, i.e., degree 0. One can check that this is truly a resolution of $\mathbb{K}$ over R, and it is minimal because all of the entries in each differential lie in the maximal ideal $\mathfrak{m}$ of R. We encode shifts in the gradings by $R(-(i+j)) = R(-i-j)$, so that in this notation, we may write the resolution as

$$F_\bullet \colon 0 \leftarrow \mathbb{K} \leftarrow R \xleftarrow{\begin{bmatrix} x & y & z \end{bmatrix}} R(-1-0)^3 \xleftarrow{\begin{bmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{bmatrix}} R(-2-0)^3 \xleftarrow{\begin{bmatrix} z \\ -y \\ x \end{bmatrix}} R(-3-0) \leftarrow 0$$

So, we only have four nonzero graded Betti numbers: $b_{0,0} = 1$, $b_{1,0} = 3$, $b_{2,0} = 3$, and $b_{3,0} = 1$. Thus, the Betti table for $F_\bullet$ is

	0	1	2	3
0	1	3	3	1

From the Betti table, we can read off $\operatorname{pdim}(\mathbb{K}) = 3$ and $\operatorname{reg}(\mathbb{K}) = 0$. This resolution is an instance of the *Koszul complex*.

Example 1.5. Let $R = \mathbb{K}[x, y, z]$ and consider the ideal $I = (x^2, y^2, z^3)$. The minimal free resolution of R/I over R is

$$F_\bullet\colon 0 \leftarrow R/I \leftarrow R \xleftarrow{\begin{bmatrix} x^2 & y^2 & z^3 \end{bmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \end{matrix} \xleftarrow{\begin{bmatrix} -y^2 & -z^3 & 0 \\ x^2 & 0 & -z^3 \\ 0 & x^2 & y^2 \end{bmatrix}} \begin{matrix} R(-4) \\ \oplus \\ R(-5)^2 \end{matrix} \xleftarrow{\begin{bmatrix} z^3 \\ -y^2 \\ x^2 \end{bmatrix}} R(-7) \leftarrow 0$$

Because all of the entries in each differential lie in the maximal ideal $\mathfrak{m}$ of R, this resolution is minimal. To get the Betti table, we may write the resolution as

$$F_\bullet\colon 0 \leftarrow R/I \leftarrow R \xleftarrow{\begin{bmatrix} x^2 & y^2 & z^3 \end{bmatrix}} \begin{matrix} R(-1-1)^2 \\ \oplus \\ R(-1-2) \end{matrix} \xleftarrow{\begin{bmatrix} -y^2 & -z^3 & 0 \\ x^2 & 0 & -z^3 \\ 0 & x^2 & y^2 \end{bmatrix}} \begin{matrix} R(-2-2) \\ \oplus \\ R(-2-3)^2 \end{matrix} \xleftarrow{\begin{bmatrix} z^3 \\ -y^2 \\ x^2 \end{bmatrix}} R(-3-4) \leftarrow 0$$

Thus, the Betti table for $F_\bullet$ is

	0	1	2	3
total	1	3	3	1
0	1	.	.	.
1	.	2	.	.
2	.	1	1	.
3	.	.	2	.
4	.	.	.	1

Sometimes we include the total Betti numbers in the Betti table by placing them in the top row; they are obtained by adding the entries in each column together. From the Betti table, we can read off $\mathrm{pdim}(R/I) = 3$ and $\mathrm{reg}(R/I) = 4$.

Example 1.6. Let $R = \mathbb{K}[x, y, z, w]$ and $I = (xz - y^2, xw - yz, z^2 - yw)$. We can think of I as coming from the 2-minors of the matrix

$$\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}.$$

The minimal free resolution of R/I is

$$F_\bullet \colon 0 \leftarrow R/I \leftarrow R \xleftarrow{\begin{bmatrix} y^2 - xz & yz - xw & z^2 - yw \end{bmatrix}} R(-2)^3 \xleftarrow{\begin{bmatrix} -z & w \\ y & -z \\ -x & y \end{bmatrix}} R(-3)^2 \leftarrow 0$$

and has Betti table

	0	1	2
0	1	.	.
1	.	3	2

From this, we can read off that I has 3 quadric minimal generators and $\operatorname{reg}(R/I) = 1$. This is the *twisted cubic*, an example of a *rational normal curve*, all of which are resolved by the *Eagon-Northcott complex.*

Chapter 2

Betti tables forcing failure of the weak Lefschetz property

2.1 Betti tables for failure of the weak Lefschetz property

The following chapter is taken verbatim from the author's joint work [16]. The other authors have agreed to its inclusion in this thesis.

2.1.1 Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and let $I \subseteq R$ be a homogeneous ideal. We call $A = R/I$ *Artinian* if it is a finite-dimensional $\mathbb{K}$-vector space. We say A has the *weak Lefschetz property* (*WLP*) if multiplication by a general linear form $\cdot\ell\colon A_i \to A_{i+1}$ is full rank (it is either injective or surjective) for all $i \geq 0$. If $\cdot\ell^k\colon A_i \to A_{i+k}$ is full rank for all $i, k \geq 0$, then we say A has the *strong Lefschetz property* (*SLP*). Since A is Artinian, $A_i = 0$ for $i \gg 0$, so there are only finitely many maps to check.

The quintessential problem is to characterize those Artinian algebras which do (or do not) have WLP. Many of the approaches to this problem in the literature tackle this by imposing constraints on the ideal I used in the construction of the Artinian algebra A; see [27] for a nice overview of the field and open problems.

In general, the Hilbert function of A does not determine if an Artinian graded algebra has WLP, but nevertheless the study of the relationship between Hilbert functions and Lefschetz properties has been fruitful. For example, in [35], Stanley constructs an Artinian-Gorenstein algebra with non-unimodal h-vector $(1, 13, 12, 13, 1)$; failure of unimodality forces a concomitant failure of WLP. Further results along these lines appear in [5], which proves existence of monomial Artinian level algebras of codimension $r \geq 3$ also having non-unimodal h-vector.

In this note, we examine a finer invariant than the Hilbert series: the graded Betti numbers of A, and we show that certain Betti tables force failure of WLP in situations where the failure is not guaranteed from behavior of the Hilbert function.

This project began with the goal of probing the weak Lefschetz property from a geometric perspective. Specifically, given a pointset $X \subset \mathbb{P}^n$, does the geometry of the points determine if the associated Artinian reduction has the WLP? Some other natural questions motivate the results of this chapter:

- Is there ever a set of points $Y \subset X$ (and perhaps conditions on $|Y|$ relative to $|X|$) that guarantee failure or success of the WLP?
- Given a configuration of points X, if WLP fails, can we predict the degree in which WLP fails in terms of the geometry of X?
- If a set of points has multiple "subconfigurations" of points—for example, two susbets of points lying on different hypersurfaces $\mathbf{V}(f)$ and $\mathbf{V}(g)$—then how does this structure influence WLP? What if there are conditions on f and g?

We enumerate some of the recent results which are closest in spirit to our work in this note; the Lefschetz literature is vast and so the list is not exhaustive.

- Proposition 5.3 of [30] gives the existence of a reduced, level set of points where the WLP fails at semi-specified degrees via liason theory.
- Corollary 5.4 of [5] shows the existence of pointsets in $\mathbb{P}^n$ with non-unimodal Δh-vector, hence their Artinian reductions fail to have WLP.
- For points on a rational normal curve, Theorem 4.3 of [2] shows that an Artinian Gorenstein algebra related to the points has the SLP.
- A similar result to that above is proved in [2] for points on a conic in $\mathbb{P}^2$.

Unexpected curves also seem to play an important role in the interplay between geometry and Lefschetz properties of pointsets. Theorem 7.5 of [9] shows that the set of points dual to a line arrangement in $\mathbb{P}^2$ has an unexpected curve of a specified degree if and only if the the Artinian algebra formed from the line arrangement fails the SLP in a specified range and degree. Expanding on the influence of unexpected curves on the WLP, the papers [3], [11], and [19] give many more examples of unexpected curves, providing a richer collection of geometries to explore.

Geproci sets provide another approach to understanding how the geometry of a set of points influences WLP. From a geproci set, a dual set of linear forms is defined and one can ask if the ideal generated by certain powers of the linear forms has WLP. In the case that the geproci set is an (a, b) grid, [6] shows that WLP does not hold for most pairs (a, b). Because of this connection between geproci sets and Lefschetz properties, there is much recent work on this topic; see for example [8], [6], [7].

We approach Lefschetz questions from a geometric standpoint: our starting point is a configuration of reduced points $X \subset \mathbb{P}^n$ (with $n \geq 3$), and we then study the Lefschetz properties of the Artinian reduction of I_X by a generic hyperplane. Our aim is to investigate how the geometry of the points manifests itself in the associated Artinian reduction; in particular, if it is possible to detect whether or not the Artinian reduction has WLP.

The pointsets we study do not typically correspond to level algebras. A number of papers such as [35], [28], [29], [26], and [2] have studied the influence of the Hilbert function on Lefschetz properties. A novel aspect of our approach is that the finer data available in the Betti table can distinguish WLP properties for pointsets having the same Hilbert function. In particular, from the Betti table of a pointset $X \subset \mathbb{P}^n$, we describe a simple indicator which is sufficient (but far from necessary) to show that the Artinian reduction of a pointset fails to have WLP.

Throughout this section, X will denote a finite set of distinct points in $\mathbb{P}^n$ (with $n \geq 3$), $S = \mathbb{K}[x_0, \ldots, x_n]$ will be the coordinate ring of $\mathbb{P}^n$, and $\mathbb{K}$ a field. It is well known that the characteristic of $\mathbb{K}$ has a major impact on the WLP; however the results in this chapter are characteristic independent. Let $I_X \subseteq S$ denote the ideal of X, and form the Artinian reduction

A_X by quotienting by a generic linear form L, which by definition misses all the points of X. Hence L is a nonzero divisor on S/I_X. Let J_X denote the Artinian reduction of I_X, so that

$$A(X) = A_X = S/(I_X, L) \simeq R/J_X.$$

We write $\mathrm{B}(X)$ to denote the *Betti table* (or *Betti diagram*) of A_X (or equivalently, of J_X). The entries of $\mathrm{B}(X)$ are given by the ranks and shifts of the summands appearing in the minimal free resolution of J_X (or I_X, since L is a nonzero-divisor the Betti table is unchanged), so $\mathrm{B}(X)$ is a table whose entry in the j^{th} row and i^{th} column is

$$\beta_{i,i+j} = \dim_{\mathbb{K}} \mathrm{Tor}_i^R(R/J_X, \mathbb{K})_{i+j}.$$

2.1.2 All but one point on a hypersurface

Given a pointset $X \subset \mathbb{P}^n$ (with $n \geq 3$), we will show that if nearly all of the points lie on a unique hypersurface, then A_X fails to have WLP. We illustrate this fact first with an example.

Example 2.1. Consider the points $X \subset \mathbb{P}^3$ given as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and let X_f denote the seven points obtained from the first seven columns of the matrix above. The Hilbert functions of the Artinian reductions are $\{1, 2, 3, 1\}$ for X_f and $\{1, 3, 3, 1\}$ for X.

For the Betti table of A_X, we have

$$\mathrm{B}(A_X) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & 3 & 3 & 1 \\ 2 & . & 3 & 4 & 1 \\ 3 & . & . & 1 & 1 \end{array}.$$

Let q be the point given by the last column of the matrix above. We constructed X so that all of the points in X_f lie on the plane $x_3 = 0$ and q lies off of it. In this way, we have that

$$(I_{X_f})_1 = (x_3), \text{ and } I_{\{q\}} = (x_0, x_1, x_2).$$

Thus

$$(I_X)_2 = (I_{X_f})_1 \cap (I_{\{q\}})_1 = (x_3) \cap (x_0, x_1, x_2) = (x_0x_3, x_1x_3, x_2x_3).$$

Now we consider the Artinian reduction A_X of I_X by a general linear form

$$L = \sum_{i=0}^{3} a_i x_i.$$

We then have

$$(I_X, L)_2 = (x_0x_3, x_1x_3, x_2x_3, x_3L) = (x_0x_3, x_1x_3, x_2x_3, x_3^2).$$

It is easy to see that multiplication by a generic linear form of A will not have full rank, as x_3 is in the kernel of the multiplication map, hence A_X does not have WLP. This is the motivating example for Theorem 2.2.

Theorem 2.2. *Let $X_f \subset \mathbb{P}^n$ ($n \geq 3$) be a set of distinct points lying on a hypersurface $\mathbf{V}(f)$ for some polynomial $f \in R$ with $\deg(f) = d$, such that $(I_{X_f})_d = (f)$ and $(I_{X_f})_{<d} = 0$. Choose $q \notin \mathbf{V}(f)$ such that if $X = X_f \cup \{q\}$, then $(I_X)_d = 0$ and $\dim_{\mathbb{K}}((I_X)_{d+1}) = n$. Then*

the Artinian reduction A_X does not have WLP. In particular, if l is a general linear form, then $A_d \xrightarrow{\cdot l} A_{d+1}$ does not have full rank.

Proof. By changing coordinates, we may assume that $q = [1 : 0 : \ldots : 0]$ and that $V(x_0) \cap X = \emptyset$. We then have $(I_X)_{\leq d} = (f)$ and $I_{\{q\}} = (x_1, \ldots, x_n)$. So then

$$(I_X)_{d+1} = (f) \cap (x_1, \ldots, x_n) = (x_1 f, \ldots, x_n f).$$

It is clear that $x_i f \in (I_X)_{d+1}$ for $1 \leq i \leq n$, and we know the reverse containment $(x_1 f, \ldots, x_n f) \subseteq (I_X)_{d+1}$ since $\dim_{\mathbb{K}}(I_X)_{d+1} = n$ by assumption. By our choice of coordinates above, the linear form $L = x_0$ is a non-zero divisor, and $A_X = R/J_X$ is the Artinian reduction of X. Let $l = \sum_{i=1}^{n} a_i x_i \in R$ be a general linear form; abusing notation we write f for the reduction of f modulo x_0. Then

$$l \cdot f = \left(\sum_{i=1}^{n} a_i x_i \right) \cdot f = \sum_{i=1}^{n} a_i x_i \cdot f = 0.$$

Thus $A_d \xrightarrow{\cdot l} A_{d+1}$ has nontrivial kernel, and is not injective. Another way to see this is to extend $\{f\}$ to a basis $\{f, g_1, g_2, \ldots, g_s\}$ of A_d, and fix a basis $\{g_1', \ldots, g_t'\}$ of A_{d+1}. The map $A_d \xrightarrow{\cdot l} A_{d+1}$ can then be represented by the matrix

$$\begin{array}{c c} & \begin{array}{ccccc} f & g_1 & g_2 & \cdots & g_s \end{array} \\ \begin{array}{c} g_1' \\ g_2' \\ g_3' \\ \vdots \\ g_{t'} \end{array} & \begin{pmatrix} 0 & . & . & \cdots & . \\ 0 & . & . & \cdots & . \\ 0 & . & . & \cdots & . \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & . & . & \cdots & . \end{pmatrix} \end{array}$$

Thus $A_d \xrightarrow{\cdot l} A_{d+1}$ is not injective since it has a zero column. The map cannot be surjective since $\dim_{\mathbb{K}}(A_d) = \binom{n-1+d}{n-1} < \binom{n+d}{n-1} - n = \dim_{\mathbb{K}}(A_{d+1})$. Note that this inquality is indeed strict since we assumed $n \geq 3$. Therefore $A_d \xrightarrow{\cdot l} A_{d+1}$ does not have full rank, and A does not have WLP. $\square$

The hypotheses of Theorem 2.2 ensure that the points X_f lie on a *unique* hypersurface and that q lies off of $V(f)$. The number of points in $\mathbb{P}^n$ needed to ensure the hypotheses of Theorem 2.2 is one less than the number of degree d hypersufaces, so $|X_f| = \binom{n+d}{d} - 1$ points suffice (if chosen generically) to fix $\mathbf{V}(f)$, and

$$\dim_{\mathbb{K}}((I_{X_f})_d) = 1, \text{ and } \dim_{\mathbb{K}}((I_{X_f})_{<d}) = 0.$$

Now suppose $q \notin V(f)$ and let $\tilde{X} = X_f \cup \{q\}$. Then $\dim_{\mathbb{K}}((I_{\tilde{X}})_d) = 0$ by the choice of q. We can then choose $m := \binom{n+d}{d+1} - n$ distinct generic points on $\mathbf{V}(f) \backslash X_f$ to get $\dim_{\mathbb{K}}((I_{\tilde{X}})_{d+1}) = n$. To see this, note that $\dim_{\mathbb{K}}(R_{d+1}) = \binom{n+d+1}{d+1}$ and solve the equation $\dim_{\mathbb{K}}(R_{d+1}) - (|\tilde{X}| + m) = n$, i.e. the equation

$$\binom{n+d+1}{d+1} - \left(\binom{n+d}{d} + m\right) = n,$$

for m and use the identity $\binom{n+d+1}{d+1} = \binom{n+d}{d} + \binom{n+d}{d+1}$. Let $\hat{X}$ be the set of m extra points chosen, and set $X = \tilde{X} \cup \hat{X}$. Then $\dim_{\mathbb{K}}(I_X)_{d+1} = n$, as desired.

2.1.3 Maximal Koszul tails force failure of the WLP

Definition 2.3. We say a Betti table B has an (n,d)-*Kosul tail* if it has an upper-left principal block of the form

$$\begin{array}{c|cccccccc} & 0 & 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ \hline 0 & 1 & . & . & . & \dots & . & . & . \\ 1 & . & . & . & . & \dots & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d-1 & . & . & . & . & \dots & . & . & . \\ d & . & n & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} & n & 1 \end{array}.$$

If B has an (n,d)-Koszul tail and is the Betti table for an Artnian ring $\mathbb{K}[x_1, \dots, x_n]/I$, then we say B has a *maximal* (n,d)-Koszul tail.

Example 2.4. Consider the set of points $X_f \subset \mathbb{P}^4$ lying on a unique hypersurface $\mathbf{V}(f)$ with $\deg(f) = 3$ (this will consist of 34 points plus 31 extra points). Take 5 points X_Q lying off of $\mathbf{V}(f)$, but on a codimension 3 linear space. Let $X := X_f \cup X_Q$. Then the Betti table of A_X is

$$\mathrm{B}(A_X) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & . & . & . & . \\ 1 & . & . & . & . & . \\ 2 & . & . & . & . & . \\ 3 & . & 3 & 3 & 1 & . \\ 4 & . & 44 & 111 & 90 & 20 \\ 5 & . & . & . & . & 3 \end{array}.$$

So the Betti table of A_X has a $(3,3)$-Koszul tail, and in this case, A_X has WLP. However, if we A_X by another generic linear form $L' \in (A_X)_1$ to obtain $A_{X,L'} = A_X/(L')$, then

$$\mathrm{B}(A_{X,L'}) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & . & . & . \\ 2 & . & . & . & . \\ 3 & . & 3 & 3 & 1 \\ 4 & . & 15 & 27 & 12 \end{array}.$$

The Betti table of $A_{X,L'}$ has a maximal $(3,3)$-Koszul tail, and $A_{X,L'}$ fails WLP from degree 3 to degree 4. This suggests that an Artinian ring whose Betti table has a maximal (n,d)-Koszul tail fails WLP, and motivates Theorem 2.5. The example above then follows from Corollary 2.7.

Theorem 2.5. *An Artinian algebra $A = \mathbb{K}[x_1,\ldots,x_n]/I$ whose Betti table has a maximal (n,d)-maximal Koszul tail does not have the WLP; the map $A_d \xrightarrow{\ell} A_{d+1}$ is not full rank.*

Proof. By reducing to the argument used in Theorem 2.2, it suffices to show that a maximal (n,d)-Koszul tail forces

$$I_{d+1} = f \cdot (x_1, \ldots, x_n) \text{ for some } f \in \mathbb{K}[x_1, \ldots, x_n]_d.$$

Write the last differential in the resolution as d_n. For A Artinian, the maximal Koszul tail hypothesis is equivalent to having the degree $n+d$ component of d_n–that is, the last map in the top strand–equal to

$$\psi = [x_1, \ldots, x_n]^T.$$

To see this, note that the cokernel of d_n^T is $\mathrm{Ext}^n(R/I, R)$, and the cokernel of ψ^T is a direct summand of $\mathrm{Ext}^n(R/I, R)$–this is because the only part of d_n^T whose target is $R(n+d)$ is exactly ψ^T, and we know the only associated prime of $\mathrm{Ext}^n(R/I, R)$ is the maximal ideal. But now we may dualize again, and since R/I is Artinian,

$$\mathrm{Ext}^n(\mathrm{Ext}^n(R/I, R), R) \simeq R/I.$$

In particular, dualizing the resolution of R/I gives a free resolution of $\mathrm{Ext}^n(R/I, R)$, and so the top nonzero row of the Betti table is a resolution of the cokernel of $[x_1, \ldots, x_n]$, which is the Koszul complex on $[x_1, \ldots, x_n]$. Therefore in degree $d+1$ we have that $I_{d+1} = f \cdot (x_1, \ldots, x_n)$, and we conclude as in the proof of Theorem 2.2. $\square$

Note that rather than pointsets, we can take more general quotients. The key is to pass through a pointset having a maximal Koszul tail, which means having a regular sequence of maximal length. In particular, we get the following:

Corollary 2.6. *If $T = \mathbb{K}[x_0, \ldots, x_n]/I$ is Cohen-Macaulay of dimension m, and T has a maximal $(n-m, d)$-Koszul tail, then the Artinian reduction of T fails the WLP.*

If $T = \mathbb{K}[x_0, \ldots, x_n]/I$ is Cohen-Macaulay of dimension m, and T has a maximal $(n-m, d)$-Koszul tail, then the Artinian reduction of T fails WLP. As Example 2.4 shows, we are not limited to the Cohen-Macaulay situation:

Corollary 2.7. *If $A = \mathbb{K}[x_1, \ldots, x_{m+n}]/I$ is Artinian with an (n,d)-Koszul tail, and there exists a sequence of linearly independent linear forms $I_l = (\ell_1, \ldots, \ell_m)$ such that A/I_L has the same top row of the Betti table as A, then A/I_L fails to have the WLP.*

Example 2.8. Consider the points $X \subset \mathbb{P}^3$ given as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

So $S = \mathbb{K}[x_0, x_1, x_2, x_3]$ and $I_X = (x_0x_2, x_1x_2, x_0x_3, x_1x_3, x_0^2x_1 - x_0x_1^2, x_2^2x_3 - x_2x_3^2)$. The six points lie on the two disjoint lines $\mathbf{V}(x_0, x_1)$ and $\mathbf{V}(x_2, x_3)$. Letting L be a generic linear form, and $A = S/(I_X, L) = R/J_X$, the h-vector of A is $(1, 3, 2)$ so the one-dimensional socle in degree one does not block WLP. The Betti table of A is

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & 4 & 4 & 1 \\ 2 & . & 2 & 4 & 2 \end{array}.$$

A computation with `Macaulay2` verifies that A has WLP. This example highlights that even if there is a single minimal-degree element in the socle of A, this does not imply that A fails WLP. It is the *maximal* (n,d)-Koszul tail that is the key: the map $A_d \xrightarrow{\cdot l} A_{d+1}$ cannot be surjective when $n \geq 3$.

A non-maximal (n, d)-Koszul tail does not determine WLP. In [1], the authors consider the Betti table

	0	1	2	3	4
0	1	.	.	.	.
1	.	2	1	.	.
2	.	5	9	4	.
3	.	4	9	5	.
4	.	.	2	1	.
5	.	.	.	.	1

The following two ideals both yield the Betti table above, but the ideal

$$(x_4^2, x_3x_4, x_3^3, x_2x_3^2 - x_2^2x_4, x_1x_3^2 - x_1x_2x_4 + x_2^2x_4, x_2^2x_3, x_2^3,$$
$$x_1^3x_4 - x_1^2x_2x_4 + x_1x_2^2x_4, x_1^3x_3, x_1^3x_2 - x_1^2x_2^2, x_1^4)$$

has WLP, while the ideal

$$(x_1x_4, x_1^2, x_3x_4^2, x_2x_4^2, x_2^2x_4, x_1x_3^2, x_1x_2^2 - x_3^2x_4, x_3^4, x_2x_3^3 - x_4^4, x_2^2x_3^2, x_2^4)$$

fails to have WLP.

Remark 2.9. All our computations assume we are starting in $\mathbb{P}^n$ with $n \geq 3$. Artinian quotients of the form $\mathbb{K}[x, y]/I$ always have WLP, so our technique does not apply to $(2, d)$-Koszul tails.

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Chapter 3

Combinatorial bounds on the Castelnuovo-Mumford Regularity of Toric Surfaces

3.1 Introduction

Toric varieties are ubitiquous throughout algebraic geometry as they come equipped with additional combinatorial data. When X is a projective toric variety, much effort has been put into understanding the invariants of a X such as its projective dimension, degree, and regularity. The hope is that computing these invariants is made easier by the combinatorics of the toric variety.

For us, a toric variety X is going to come from the lattice points of a lattice polytope $P \subset \mathbb{R}^d$ defined by the convex hull of some lattice points. Given a polytope P, its normal fan Σ defines the toric variety X; see [10] for more details on this construction.

Definition 3.1. The monomial curve with exponents $a_1 \leq \ldots \leq a_{n-1}$ in $\mathbb{P}^n$ is the curve $C \subset \mathbb{P}^n$ of degree $d = a_n$ parameterized by

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^n \quad \text{with} \quad (s,t) \mapsto (s^d, s^{d-a_1}t^{a_1}, \ldots, s^{d-a_{n-1}}t^{a_{n-1}}, t^d).$$

Example 3.2. The twisted cubic is the curve of degree $d = 3$ in $\mathbb{P}^3$ parameterized by

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^n \quad \text{with} \quad (s,t) \mapsto (s^3, s^2t, st^2, t^3).$$

More generally, the monomial curve C in $\mathbb{P}^d$ whose exponents are $1, \ldots, d-1$ of degree d is the curve parameterized by

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^d \quad \text{with} \quad (s,t) \mapsto (s^d, s^{d-1}t, \ldots, st^{d-1}t, t^d);$$

we call C the rational normal curve of degree d. These are the curves from Example 1.6.

Example 3.3. Of course, we need not take consecutive exponents. Consider the curve C in $\mathbb{P}^4$ with exponents $1 \leq 3$ of degree $d = 4$. It is parameterized by

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^3 \quad \text{with} \quad (s^4, s^3t, st^3, t^4).$$

We can think of C as-is, or we may think of C as a projection from the rational normal curve of degree 4. The key insight here is that removing a lattice point corresponds to a projection.

The objects of interest in this chapter are toric surfaces. To obtain one, we can take the convex hull of lattice points lying in $\mathbb{Z}^2$ to define a polytope P, and then consider the toric variety P defines (or any of its projections). When we write the exponents, we will also include a third coordinate to homogenize so that all of the exponents have the same weight. More precisely, for an exponent vector $(a_1, a_2) \in \mathbb{Z}^2$, we will actually consider the exponent vector $(a_1, a_2, N - a_1 - a_2) \in \mathbb{Z}^3$ for some suitably chosen N so that all of the exponents lie on the plane

$$x_1 + x_2 + x_3 = N,$$

where the third coordinate acts as a homogenizing exponent.

Example 3.4. Consider the lattice points given as the columns of the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 \end{pmatrix}.$$

The convex hull of these points is a triangle, shown in Figure 3.1. The image of the map parameterizing this toric variety is given by all monomials of degree 3 in 3 variables.

Suppose P is a lattice polytope defined by the convex hull of a set of points $\mathbf{a}_1, \ldots, \mathbf{a}_n$. We obtain a toric variety X by using all (or some) of the lattice points of P. When computing

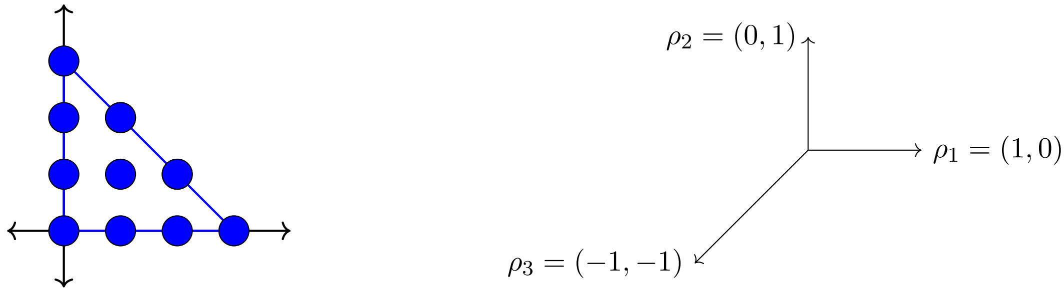


Figure 3.1: The 3-Veronese in 3 variables and its normal fan.

the regularity of X, we can fit coordinate rings into a short exact sequence:

$$0 \longrightarrow R/I \longrightarrow \overline{R/I} \longrightarrow N \longrightarrow 0,$$

where $\overline{R/I}$ is the normalization of R/I. Depending on the perspective (going about computations via resolutions, Tor, or sheaf cohomology), we know

$$\operatorname{reg}(R/I) \leq \max(\operatorname{reg}(\overline{R/I}), \operatorname{reg}(N) + 1)$$

(see, for instance, [12][Corollary 20.19]). So in our case, controlling the regularity of R/I amounts to understanding the regularity of projectively normal toric varieties and controlling the cokernel of the normalization map, $N = \operatorname{coker}(R/I \to \overline{R/I})$.

3.2 Tools for computing Betti numbers

3.2.1 Hochster's formula

Hochster's formula [4, 21] has proven to be useful in studying $\operatorname{reg}(I)$ [15, 20, 22]. Provided enough combintorial structure, this formula allows Betti numbers to be computed from the homology of certain simplicial complexes coming from the chosen lattice points and a fixed multidegree. In effect, if one understands these simplicial complexes well enough, then one understands the Betti numbers of I and hence also $\operatorname{reg}(I)$.

Let $R = \mathbb{K}[x_1, \ldots, x_n]$, $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be an $d \times n$ matrix whose columns are lattice points, and $\mathbf{b} = (b_1, \ldots, b_d)$ be some multidegree. Following [31], define

$$\Delta_{\mathbf{b}} = \left\{ I \subseteq [n] \ : \ \mathbf{b} - \sum_{i \in I} \mathbf{a}_i \text{ lies in } \mathbb{N}A \right\},$$

where $\mathbb{N}A$ denotes the semigroup generated by A, and let I_A be the lattice ideal defined by A. We can directly compute Betti numbers from Hochster's formula.

Theorem 3.5 (Hochster's Formula)**.** *Let I_Δ be a squarefree monomial ideal in variables $X = \{x_1, \ldots, x_n\}$ and let Δ be its Stanley-Reisner complex. Then if $\mathbf{m}$ is a squarefree monomial with support $W = \{x_{i_1}, \ldots, x_{i_j}\} \subseteq X$ with $\deg(\mathbf{m}) = j$, we have*

$$\beta_{i,\mathbf{m}}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \tilde{H}^{j-i-1}(\Delta|_W, \mathbb{K}),$$

where $\Delta|_W$ is the induced subcomplex of Δ on vertices W.

Example 3.6. Consider the points given by the columns of the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 \end{pmatrix}$$

and consider the multidegrees

$$\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{b}' = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}.$$

Let $R = \mathbb{K}[x_0, \ldots, x_8]$ and let I_A be the lattice ideal defined by A. The Betti table of R/I_A is

	0	1	2	3	4	5	6	7	8
total	1	17	53	91	108	83	37	9	1
0	1	.	.	.	.	.	.	.	.
1	.	17	43	36	8	.	.	.	.
2	.	.	10	55	100	83	37	9	1

The corresponding simplicial complex for $\mathbf{b}$ is shown in Figure 3.2. So, with

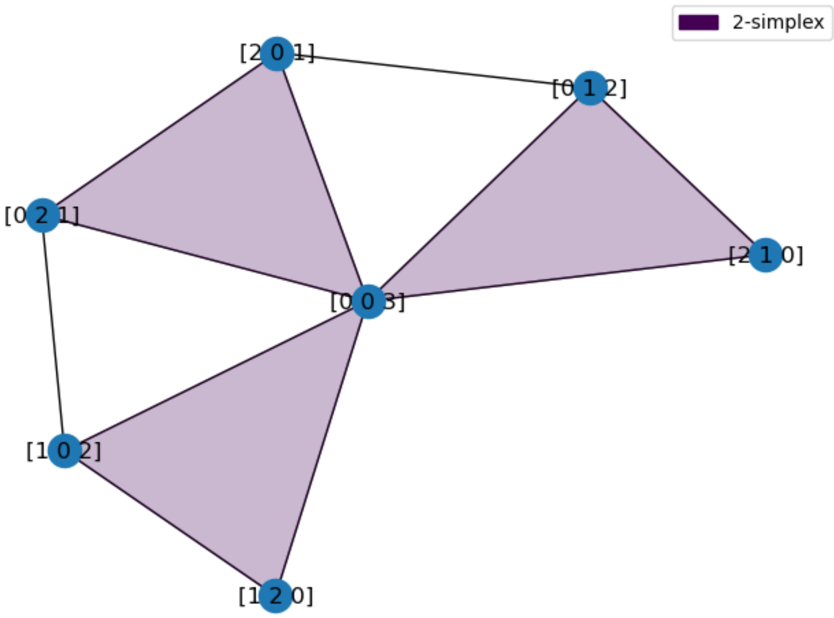


Figure 3.2: The multidegree complex $\Delta_{\mathbf{b}}$.

$$W = \begin{pmatrix} 2 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 \end{pmatrix},$$

and $j = \deg(\mathbf{b}) = 3$, we have

$$\beta_{i+1,\mathbf{b}}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \tilde{H}_{j-i}(\Delta|_W, \mathbb{K}) = \dim_{\mathbb{K}} \tilde{H}_{3-i}(\Delta|_W, \mathbb{K}) = \begin{cases} 0 & \text{if } i = 0, 1, 3 \\ 2 & \text{if } i = 2 \end{cases}.$$

Thus, $\mathbf{b}$ contributes 2 to the Betti number $\beta_{2,3} = b_{2,1}$.

However, this technique is heavily dependent on knowledge of the simplicial complexes for each multidegree. Figure 3.3 still has some structure, but the structure is less obvious compared to Figure 3.2.

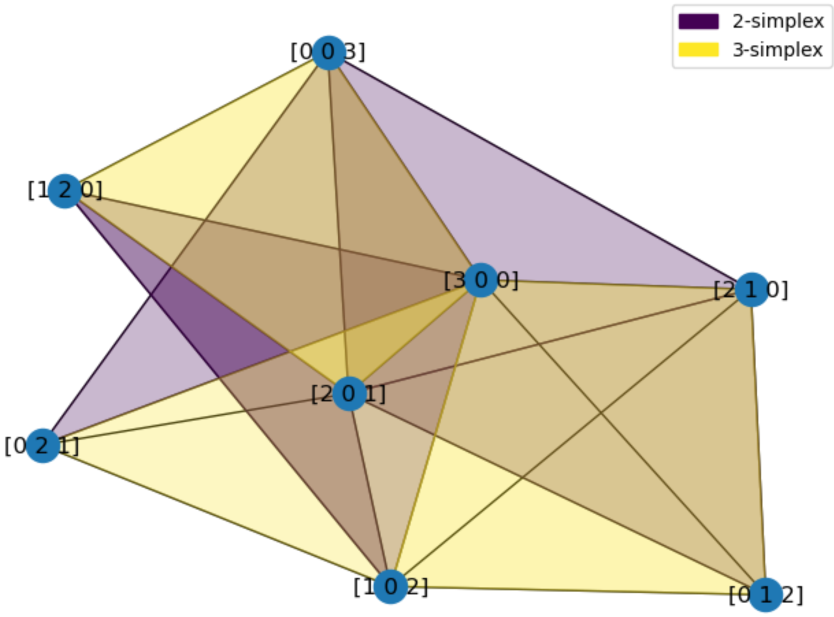


Figure 3.3: The multidegree complex $\Delta_{\mathbf{b}'}$.

One of the main advantages of Hochster's formula is that Betti numbers are upper semi-continuous, i.e., $\beta_{i,j}(I) \leq \beta_{i,j}(\text{in}(I))$, where $\text{in}(I)$ is the *initial ideal* of I. Because $\text{in}(I)$ is a monomial ideal, if it is squarefree, then Stanley-Reisner theory is ripe for use (if it's not, one can polarize to make $\text{in}(I)$ squarefree). Otherwise, Gröbner basis techniques might yield fruitful results. In any case, Hochster's formula can give very powerful combinatorial machinery to understand the Betti numbers of I.

3.2.2 Sheaf cohomology

Sometimes the lens of geometry lends a hand in computing the Betti numbers of R/I. In particular, regularity can be interpreted in terms of sheaf cohomology.

Definition 3.7. A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ is m*-regular* if

$$H^i(\mathscr{F}(m-i)) = 0$$

for all $i > 0$. The *Castelnuovo-Mumford regularity* is

$$\inf\left\{d \,:\, H^i(\mathscr{F}(d-i)) = 0 \text{ for all } i > 0\right\}.$$

Using sheaf cohomology becomes especially useful when exact sequences come into play. The pipeline goes as follows: start with a short exact sequence of sheaves; obtain a long exact sequence in sheaf cohomology; use vanishing theorems to get short exact sequences in sheaf cohomology; analyze the resulting sequence. While a bit of work, this procedure is fairly successful a lot of times. We will see in the proofs of Theorems 3.15 and 3.17 applications of this process.

3.3 Regularity of monomial curves

In 1996, L'vovsky showed that when X is a non-degenerate, projective, monomial curve, its regularity is bounded by the sum of the two largest "gaps" in the associated semigroup [25]. Importantly, this bounds the regularity of I_X using *only* the combinatorial data of the a_i.

Theorem 3.8. *[25] Let $A = (0, a_1, \ldots, a_n)$ be a sequence of non-negative integers such that the g.c.d. of the a_j's equals 1, and let C be the corresponding monomial curve. Then C is m-regular, where*

$$m = \max_{1 \le i < j \le n} \{(a_i - a_{i-1}) + (a_j - a_{j-1})\},$$

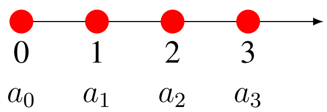
i.e., m is the sum of the two largest gaps in the semigroup generated by A.

This bound is desirable in the sense that it provides a *combinatorial* bound on an *algebraic invariant*, the regularity. Normally, to compute the regularity of a module, a minimal free resolution must be computed, which in turn *usually* involves a Gröbner basis computation. In the worst case, the complexity of a Gröbner basis computation will be doubly exponential in the number of variables. On the other hand, explicitly computing the regularity (or bounding it) can inform Gröbner basis computations so that the computation becomes a linear algebra problem on the vector space of polynomials of a certain finite degree determined by the maximal degree of a minimal generator of I. Thus, this combinatorial bound provides two advantages:

- If one only cares to bound the regularity, then L'vovsky's bound provides a simple way to do this, namely without computing a Gröbner basis.
- The combinatorial bound provides a way to inform Gröbner basis computations of the defining ideal I.

In this setting, we can construct interesting examples of monomial curves by choosing lattice points in the lattice $\mathbb{Z}$ (and then adding another coordinate to homogenize).

Example 3.9. Consider the curve parameterized by the exponents $(0, 1, 2, 3)$.



We then homogenize the exponents to get

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

Thus, the monomial map parameterizing the monomial curve is given by

$$\varphi(s,t) = (\underset{x_0}{s^3}, \quad \underset{x_1}{s^2 t}, \quad \underset{x_2}{st^2}, \quad \underset{x_3}{t^3}).$$

The defining ideal of the curve is

$$I_C = \langle x_2^2 - x_1 x_3, \quad x_1 x_2 - x_0 x_3, \quad x_1^2 - x_0 x_2 \rangle,$$

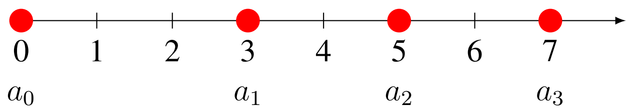
and the coordinate ring has the following minimal free resolution:

$$0 \leftarrow R/I_C \leftarrow R \leftarrow R(-2)^3 \leftarrow R(-3)^2 \leftarrow 0$$

However, if we were to compute the bound using L'vovsky's result, the bound is exactly the regularity we computed.

$$\operatorname{reg}(I_C) = 2 \leq (a_1 - a_0) + (a_2 - a_1) = 1 + 1 = 2.$$

Example 3.10. Consider the curve parameterized by $(0, 3, 5, 7)$.



We then homogenize the exponents to get

$$A = \begin{pmatrix} 0 & 3 & 5 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 3 & 5 & 7 \\ 7 & 4 & 2 & 0 \end{pmatrix}$$

Thus, the monomial map parameterizing the monomial curve is given by

$$\varphi(s,t) = (\underset{x_0}{s^7}, \quad \underset{x_1}{s^3t^4}, \quad \underset{x_2}{s^5t^2}, \quad \underset{x_3}{t^7})$$

The defining ideal of the curve is

$$I_C = \langle x_2^2 - x_1x_3, \quad x_1^3x_2 - x_0^2x_3^2, \quad x_1^4 - x_0^2x_2x_3 \rangle$$

and the coordinate ring has the following minimal free resolution:

$$0 \leftarrow R/I_C \leftarrow R \leftarrow \begin{array}{c} R(-2) \\ \oplus \\ R(-4)^2 \end{array} \leftarrow R(-5)^2 \leftarrow 0$$

Using L'vovsky's result, our bound is slightly off from the actual regularity from the cost of avoiding computing the minimal free resolution of R/I_C.

$$\operatorname{reg}(I_C) = 4 \leq (a_1 - a_0) + (a_2 - a_1) = 3 + 2 = 5.$$

3.4 Regularity of toric surfaces

One of the most compelling conjectures toward bounding regularity is the Eisenbud-Goto conjecture which bounds the regularity of I_X in terms of the degree and codimension of X [14]. Under a mild non-degeneracy assumption, the conjecture is true when X is arithmetically Cohen-Macaulay [13][Corollary 4.15]. A celebrated result of Gruson, Lazarsfeld, and Peskine shows it holds for irreducible curves [18]. Pinkham showed that if $X \subseteq \mathbb{P}^r$ is a non-degenerate projective surface of degree d and codimension r, then $\operatorname{reg}(X) \leq d - r + 4$ [33]. Subsequently, Lazarsfeld showed that if additionally X is smooth, then the bound may be improved to $\operatorname{reg}(X) \leq d - r + 3$, and this bound is optimal [24].

More generally, a result attributed to Mumford [32] says that if $X \subseteq \mathbb{P}^r$ is a reduced smooth subscheme and the characteristic is 0, then

$$\operatorname{reg}(X) \leq (\dim(X) + 1)(\deg(X) - 2) + 2.$$

Later, Kwak [23] showed that when $X \subseteq \mathbb{P}^r$ is a smooth variety of either dimension 3 or 4, then

$$\operatorname{reg}(X) \leq \begin{cases} \deg(X) - \operatorname{codim}(X) + 2 & \text{if } \dim(X) = 3 \\ \deg(X) - \operatorname{codim}(X) + 5 & \text{if } \dim(X) = 4 \end{cases}.$$

When $X \subseteq \mathbb{P}^{r-1}$ is taken to be a projective toric variety, Sturmfels [36] has shown that $\operatorname{reg}(X) \leq r \cdot \deg(X) \cdot \operatorname{codim}(X)$. While the Eisenbud-Goto conjecture has been shown to be false in general, it is still open for smooth varieties, and also for toric ideals.

When X is a normal projective toric variety with a very ample line bundle L (so that $L = \mathscr{O}(D)$ for some torus-invariant divisor D), it corresponds to a lattice polytope P. In this setting, the Eisenbud-Goto conjecture may be translated more combinatorially in terms of

k-normality of P as showing

$$k_P \leq \mathrm{Vol}(P) - \#P + \dim P + 1,$$

where k_P is the minimum k such that P is k-normal, $\mathrm{Vol}(P)$ is the (normalized) volume of P, and $\#P$ is the number of lattice points in P. If we take P to be a polygon (so then X is a toric surface), then this becomes

$$k_P \leq \mathrm{Vol}(P) - \#P + 1.$$

Applying Pick's theorem, we get that for toric surfaces, the Eisenbud-Goto conjecture is equivalent to showing

$$k_P \leq \mathrm{Vol}(P) - \#P + 1 = 2(i + \frac{b}{2} - 1) - (i + b) + 1 = i + 1,$$

where i and b denote the number of interior and boundary points of P, respectively.

3.5 Hollow polygons

Our goal is to extend L'vovsky's results on monomial curves to the next setting: toric surfaces. Practically, this now means we are choosing points in the lattice $\mathbb{Z}^2$ (and then adding another coordinate to homogenize). The main issue is to determine the analagous notion of a "gap" as was done for monomial curves.

We examine another interesting class of examples obtained by taking only the integral boundary points of a convex polytope in $\mathbb{Z}^2$. In this way, the polygons have all of their boundary points and none of their interior points. One might expect that since there are a considerable amount of projections happening (removing a lattice point corresponds to a projection), the invariants of these hollow polygons will be ill-behaved. While this is the case some of the time, in many other case, the invariants are still well-controlled so long as there are "enough" lattice points on the boundary. We illustrate these extremes with Example 3.11 having arbitrarily high regulary whereas Theorems 3.15 and 3.17 show hope for control over the regularity.

Example 3.11. We can choose lattice points so that the corresponding toric variety has arbitrarily high regularity, forming a family of "bad boy" examples. Choose the lattice points given as the columns of the matrix

$$\begin{pmatrix} 0 & d & d-1 & d \\ 0 & d-1 & d & d \\ 2d & 1 & 1 & 0 \end{pmatrix}$$

These lattice points are the vertices of a quadrilateral without the interior lattice points. This defines the monomial map $\varphi\colon \mathbb{K}[x_1, x_2, x_3, x_4] \to \mathbb{K}[s, t, u]$ given by

$$(x_1, x_2, x_3, x_4) \mapsto (u^{2d}, us^dt^{d-1}, us^{d-1}t^d, s^dt^d).$$

Then I is generated by a single polynomial of degree $2d$, so $\operatorname{reg}(I) = 2d$ (i.e., $\operatorname{reg}(R/I) = 2d - 1$).

Adding in all of the interior lattice points amounts to taking the integral closure. In this case, we now have the lattice points given as the columns of the matrix

$$\begin{pmatrix} 0 & 1 & 2 & \dots & d-1 & d & d-1 & d \\ 0 & 1 & 2 & \dots & d-1 & d & d & d-1 \\ 2d & 2d-2 & 2d-4 & \dots & 2 & 0 & 1 & 1 \end{pmatrix}.$$

The regularity becomes bounded so that $\operatorname{reg}(\overline{R/I}) = 2$.

A step in the right direction might be analyzing *smooth* lattice polygons (e.g., Examples 3.12 and 3.13). These are polygons in which at every vertex, the vectors emanating along adjacent edges form a basis for $\mathbb{Z}^2$, i.e., they have determinant equal to ± 1 when arranged in a matrix. However, while there is hope in bounding the regularity, it is still not quite as nice as Theorems 3.15 and 3.17.

Example 3.12. Consider the lattice points given as the columns of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & 8 \\ 0 & 4 & 2 & 8 \\ 16 & 10 & 14 & 0 \end{pmatrix},$$

and consider the hollow polygon these define, i.e., take the convex hull and only consider the integral boundary points. Then if I_A is the lattice ideal these points define, we can see that the

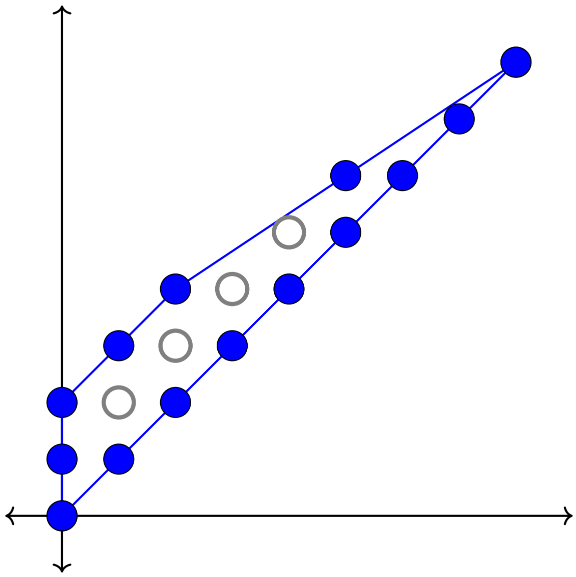


Figure 3.4: A smooth toric surface with $\operatorname{reg}(R/I) = 3$, but with I having two quartic generators.

toric variety of I_A is smooth—the determinant at all the corners is ± 1—and the Betti table of R/I is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
total	1	54	385	1462	3608	6456	8394	7875	5369	2639	913	212	30	2
0	1	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	52	280	730	1128	1050	720	315	80	9	.	.	.	.
2	.	.	81	600	2040	4416	6090	5712	3705	1640	473	80	6	.
3	.	2	24	132	440	990	1584	1848	1584	990	440	132	24	2

So in particular, we have $\operatorname{reg}(R/I_A) = 3$, showing that a *smooth* hollow polygon is not enough to have regularity at most 2.

Example 3.13. Consider the lattice points given as the columns of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 9 & 3 & 0 \\ 11 & 0 & 8 & 9 \end{pmatrix},$$

and consider the hollow polygon these define, i.e., take the convex hull and only consider the integral boundary points. Then if I_A is the lattice ideal these points define, we can see that the

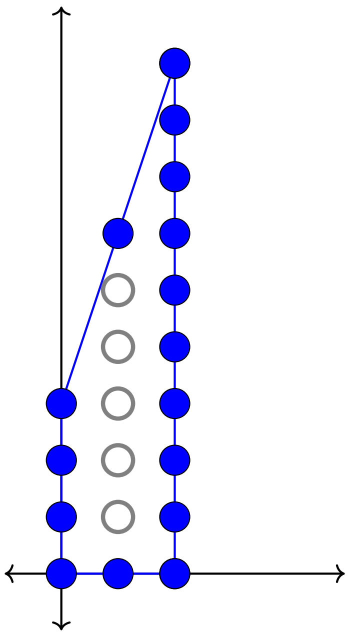


Figure 3.5: A smooth toric surface with $\operatorname{reg}(R/I) = 3$, but with I having no quartic generators.

toric variety of I_A is smooth—the determinant at all the corners is ± 1—and the Betti table of

R/I is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
total	1	74	633	2883	8593	18953	31097	38115	35387	24959	13287	5251	1495	291	35	2
0	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	73	486	1627	3388	4620	4752	3465	1760	594	120	11	.	.	.	.
2	.	1	144	1218	4983	13541	24420	31284	29271	20141	10098	3590	857	123	8	.
3	.	.	3	38	222	792	1925	3366	4356	4224	3069	1650	638	168	27	2

As with Example 3.12, we have $\operatorname{reg}(R/I_A) = 3$, showing another instance where smoothness does not suffice to provide $\operatorname{reg}(R/I) \leq 2$.

3.5.1 Hollow triangles

As a first generalization of [25], we consider lattice points in $\mathbb{Z}^2$ that lie on the boundary of a triangle determined by the coordinate axes and the plane $x + y = k$, for some integer k. We call this a *hollow triangle* of length k, and denote it as $\triangle^k$. For example, the hollow triangle of length 5 is shown in Figure 3.6. Note that the number of points in $\triangle^k$ is $T_{k+1} - T_{k-2} = 3k$.

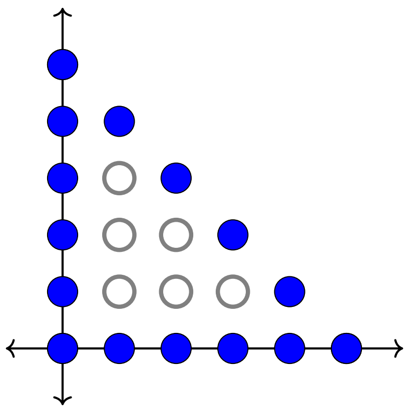


Figure 3.6: The hollow triangle of length 5, $\triangle^5$.

The following lemma allows us to control the cokernel of the normalization map in hopes to get bounds on the regularity of $R/I_{\triangle^k}$.

Lemma 3.14. *For all $d \geq 2$, $(R/I_{\triangle^k})_d = (\overline{R/I_{\triangle^k}})_d$.*

Proof. Since k is fixed, for ease of notation, let $\triangle$ denote $\triangle^k$. Let $\overline{\triangle}$ be the saturation of $\triangle$, and let $\triangle^\circ$ be the interior of $\overline{\triangle}$, i.e. $\triangle^\circ := \overline{\triangle} \setminus \triangle$. To show that $(R/I_\triangle)_2 = (\overline{R/I_\triangle})_2$, we just need to that for any two lattice points $p, q \in \overline{\triangle}$, $p + q = a + b$ for some $a, b \in \triangle$.

If both p and q are in $\triangle$, then we are done. Now assume $p, q \in \triangle^\circ$ and write $p = (p_0, p_1, p_2)$ and $q = (q_0, q_1, q_2)$. We can assume without loss of generality that $q_0 \leq p_0$ and $q_1 \leq p_1$; if not, interchange the roles of the p_i and q_i. Suppose $q_0 \leq q_1$. Then,

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ p_1 + q_1 \\ p_2 + q_2 \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ p_1 - q_0 \\ p_2 \end{pmatrix} + \begin{pmatrix} 0 \\ q_1 + q_0 \\ q_2 \end{pmatrix}.$$

Otherwise, if $q_1 \leq q_0$, then

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_0 + q_0 \\ p_1 + q_1 \\ p_2 + q_2 \end{pmatrix} = \begin{pmatrix} p_0 - q_1 \\ p_1 + q_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} q_0 + q_1 \\ 0 \\ q_2 \end{pmatrix}.$$

In either case, the the coordinates of each point sum to k and the entries are all nonnegative integers. Also, the coordinates are bounded by k; in particular, for the first case,

$$p_0 + q_0 \leq p_0 + q_0 \leq p_0 + q_1 \leq p_0 + p_1 < k,$$

and in the second case,

$$p_1 + q_1 \leq p_1 + q_0 \leq p_1 + p_0 < k.$$

Hence, we have reduced to the case with one boundary point and one interior point, and we are done.

Finally, suppose $p \in \triangle$ and $q \in \triangle^\circ$. Write $p = (p_0, p_1, p_2)$ and $q = (q_0, q_1, q_2)$, and without loss of generality, we may assume $p_0 = 0$. Furthermore, since q is in $\triangle^\circ$, all of its coordinates are positive. Hence, at least one of the following inequalities is true: (1) $p_1 \geq q_1$,

(2) $p_1 \geq q_2$, (3) $p_2 \geq q_1, p_2 \geq q_2$. In the cases that (2) is true or (3) is true, choose the following:

$$\begin{pmatrix} 0 \\ p_1 - q_2 \\ p_2 + q_2 \end{pmatrix} + \begin{pmatrix} q_0 \\ q_1 + q_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ p_1 + q_1 \\ p_2 - q_1 \end{pmatrix} + \begin{pmatrix} q_0 \\ 0 \\ q_2 + q_1 \end{pmatrix},$$

respectively. Note that in either case, the coordinates are truly in $\triangle$. All of the coordinates are nonnegative, and, for instance, in case (2), since $p_2 = k - p_1$, we have

$$p_1 \geq q_2 \iff k - p_2 \geq q_2 \iff k \geq p_2 + q_2.$$

Next, we will show that you can choose points under the conditions of (1), and then by a similar argument, points can also be chosen under the conditions of (4). So, assume (1) holds, and further assume that $q_1 \leq p_2$. Then choose the points

$$\begin{pmatrix} 0 \\ p_1 + q_1 \\ p_2 - q_1 \end{pmatrix} + \begin{pmatrix} q_0 \\ 0 \\ q_1 + q_2 \end{pmatrix}.$$

Both points are in $\triangle$ since the coordinates are nonnegative and the coordinates are less than or equal to k ($q_1 \leq p_2 \implies p_1 + q_1 \leq k$).

Now assume $q_1 > p_2$. First push the boundary point to a corner of $\triangle$ by choosing the points

$$\underbrace{\begin{pmatrix} 0 \\ p_1 + p_2 \\ 0 \end{pmatrix}}_{p_*} + \underbrace{\begin{pmatrix} q_0 \\ q_1 - p_2 \\ q_2 + p_2 \end{pmatrix}}_{q_*}.$$

Move the points to

$$\begin{pmatrix} 0 \\ p_1+p_2 \\ 0 \end{pmatrix} + \begin{pmatrix} q_0 \\ q_1-p_2 \\ q_2+p_2 \end{pmatrix} = \begin{pmatrix} q_0 \\ p_1+p_2-q_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ q_1-p_2+q_0 \\ q_2+p_2 \end{pmatrix}$$
$$= \begin{pmatrix} q_0 \\ k-q_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ k-(q_2+p_2) \\ q_2+p_2 \end{pmatrix}.$$

All of the coordinates are nonnegative since $q_1 > p_2$, and they are bounded by k since

$$p_2 < q_1 \implies q_2 + p_2 < q_2 + q_1 \leq k.$$

In all cases, the chosen points have nonnegative coordinates bounded above by k which also sum to k, have a coordinate which is 0 (i.e., they are both points on the boundary), and the sum of the two points is exactly the sum of the interior and boundary point. Thus $p+q$ is expressible as a sum of lattice points in $\triangle$, and we are done. □

Theorem 3.15. $\operatorname{reg}(\triangle^k) = 2$.

Proof. Fix $k \geq 1$. Since $\triangle^k$ retains the Veronese syzygies contributed by each of its three sides, we know $\operatorname{reg}(\triangle^k) \geq 2$. To show that $\operatorname{reg}(\triangle^k) \leq 2$, we turn to sheaf cohomology.

In a particular twist m, we have a short exact sequence of sheaves

$$0 \to \mathscr{I}_{\triangle^k}(d) \to \mathscr{O}_{\mathbb{P}^{3k-1}}(d) \to \mathscr{O}_{\triangle^k}(d) \to 0.$$

Summing over all twists and taking sheaf cohomology, this induces a long exact sequence

$$0 \to \bigoplus_d H^0(\mathscr{I}_{\triangle^k}(d)) \to \bigoplus_d H^0(\mathscr{O}_{\mathbb{P}^{3k-1}}(d)) \to \bigoplus_d H^0(\mathscr{O}_{\triangle^k}(d)) \to$$
$$\bigoplus_d H^1(\mathscr{I}_{\triangle^k}(d)) \to \bigoplus_d H^1(\mathscr{O}_{\mathbb{P}^{3k-1}}(d)) \to \dots.$$

However,

$$H^i(\mathscr{O}_{\mathbb{P}^{3k-1}}(d)) = \begin{cases} R_d & \text{if } i = 0 \\ S_d & \text{if } i = 3k-1 \\ 0 & \text{if } i \neq 0, 3k-1 \end{cases}$$

for all $d \in \mathbb{Z}$. In particular, $H^1(\mathscr{O}_{\mathbb{P}^{3k-1}}(d)) = 0$, so we have an exact sequence

$$0 \to \bigoplus_d H^0(\mathscr{I}_{\triangle^k}(d)) \to \bigoplus_d H^0(\mathscr{O}_{\mathbb{P}^{3k-1}}(d)) \to \bigoplus_d H^0(\mathscr{O}_{\triangle^k}(d)) \to \bigoplus_d H^1(\mathscr{I}_{\triangle^k}(d)) \to 0.$$

Quotienting by $I_{\triangle^k}$ yields the short exact sequence

$$0 \to R/I_{\triangle^k} \to \overline{R/I_{\triangle^k}} \to N \to 0,$$

where $\overline{R/I_{\triangle^k}}$ denotes the normalization of $R/I_{\triangle^k}$. Combining the short exact sequence and Lemma 3.14, we have N is only generated by degree 1 monomials, i.e., the interior lattice points of $\triangle^k$.

Note that $\operatorname{reg}(\overline{\triangle^k}) = 2$ since it is the Veronese, and $\operatorname{reg}(N) = 1$ since it is resolved by the Koszul complex and is generated by degree 1 monomials. So,

$$\operatorname{reg}(\triangle^k) \leq \max\{\operatorname{reg}(\overline{\triangle^k}), \operatorname{reg}(N) + 1\} = 2.$$

Therefore $\operatorname{reg}(\triangle^k) = 2$, as desired. $\square$

3.5.2 Hollow squares

Next, we consider lattice points in $\mathbb{Z}^2$ that lie on the boundary of a square determined by the coordinate axes with sides of length k, for some integer k. We call this a *hollow square* of length k, and denote it as $\square^k$. For example, the hollow square of length 5 is shown in Figure 3.7. Note that the number of points in $\square^k$ is $(k+1)^2 - (k-1)^2 = 4k$.

Similar to the hollow triangle case, we can control the cokernel of the normalization map.

Lemma 3.16. *For all $d \geq 2$, $(R/I_{\square^k})_d = (\overline{R/I_{\square^k}})_d$.*

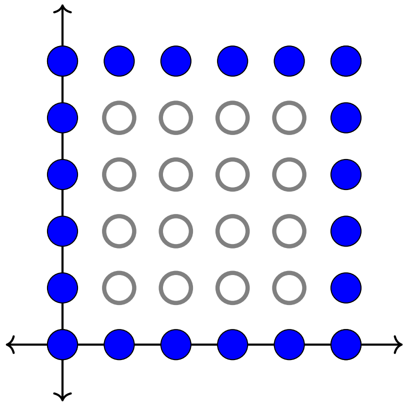


Figure 3.7: The hollow square of length 5, $\square^5$.

Proof. Since k is fixed, for ease of notation, let $\square$ denote $\square^k$. Let $\overline{\square}$ be the saturation of $\square$, and let $\square^\circ$ be the interior of $\overline{\square}$, i.e. $\square^\circ := \overline{\square} \setminus \square$. To show that $(R/I_\square)_2 = (\overline{R/I_\square})_2$, we just need to that for any two lattice points $p, q \in \overline{\square}$, $p + q = a + b$ for some $a, b \in \square$.

If both p and q already lie in $\square$, then we are done.

Write the facets of $\overline{\square}$ as x_1, x_2, x_3, x_4 ordered clockwise with x_1 being the top facet of $\overline{\square}$. Suppose $p = x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}, q = x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4} \in \square^\circ$, and suppose without loss of generality that p lies closest to a facet of $\overline{\square}$, say the facet corresponding to x_1. Then move p to x_1 and q equally in the opposite direction; now $p = x_2^{i_2} x_3^{i_1+i_3} x_4^{i_4}$ and $p = x_1^{i_1+j_1} x_2^{j_2} x_3^{j_3-i_1} x_4^{j_4}$. Since this move was one of minimal distance, all of the exponents in p and q are nonnegative and bounded by k. So, p now lies in $\square$, and if q does as well, then we are done. Otherwise, we have reduced to the case where one point is in $\square^\circ$ and the other is in $\square$.

Now suppose $p \in \square$ and $q \in \square^\circ$. Move p toward the closest vertex of $\square$; if q hits $\square$, then stop and we are done. Else, now p is at a vertex of $\square$ and q still lies $\square^\circ$. Move p in the orthogonal direction of its first move, going along a facet of $\square$, and correspondingly move q in the opposite direction. Since q has strictly less than k spaces to move and p has exactly k spaces available, q will hit $\square$ at some point; stop at this point and we are done.

Hence, in any case, we can rewrite any sum of points in $\overline{\square}$ as a sum of points in $\square$. $\square$

Theorem 3.17. $\operatorname{reg}(\square) = 2$.

Proof. Fix $k \geq 1$. Since $\square^k$ retains the Veronese syzygies contributed by each of its four sides, we know $\operatorname{reg}(\square^k) \geq 2$. To show $\operatorname{reg}(\square^k) \leq 2$, we reduce to the sheaf cohomology computation as in the proof of Theorem 3.15. So, by Lemma 3.16, we have $N = \operatorname{coker}(R/I_{\Box^k} \to \overline{R/I_{\Box^k}})$ is only generated by degree 1 monomials, i.e., the interior lattice points of $\Box^k$.

Note that $\operatorname{reg}(\overline{\Box^k}) = 2$ since it is a rational normal scroll, and $\operatorname{reg}(N) = 1$ since it is resolved by the Koszul complex and is generated by degree 1 monomials. So,

$$\operatorname{reg}(\Box^k) \leq \max\{\operatorname{reg}(\overline{\Box^k}), \operatorname{reg}(N) + 1\} = 2.$$

Therefore $\operatorname{reg}(\Box^k) = 2$, as desired. $\Box$

3.5.3 Hollow polygons

We close with some conjectures based on experimental evidence and heuristics from the topological perspective. For the hollow polygon setup we have had throughout this chapter—where we only take the boundary lattice points of a polygon—we can still associate a normal fan to this polygon as normal. Conjecturally, it seems like the combinatorics of the fan still control the regularity of toric variety. With the duality between the polygon and its fan, the "gaps" manifests as the side lengths of the polygon.

Conjecture 3.18. *Suppose X is a smooth toric surface in the setup of the hollow polygons. Let Σ be its corresponding fan with rays $\rho_1, \dots, \rho_r$. Then the regularity of I_X is bounded by the largest magnitude of a ray of Σ, i.e.,*

$$\operatorname{reg}(R/I_X) \leq \max_{1 \leq i \leq r} \|\rho_i\|_1.$$

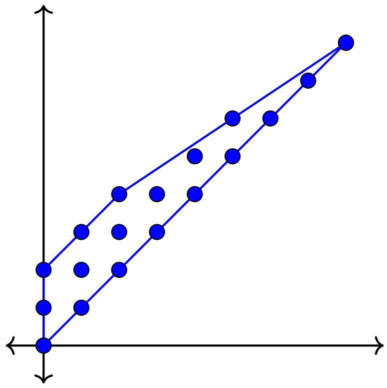
Alternatively, if $v_1, \dots, v_r = v_0$ are the vertices of the corresponding polytope P (ordered clockwise), then

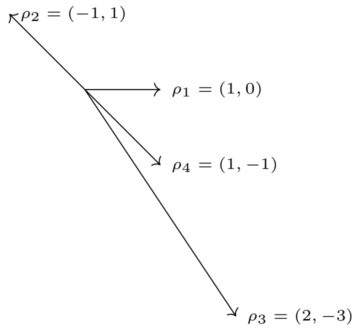
$$\operatorname{reg}(R/I_X) \leq \max_{1 \leq i \leq r} \frac{\|v_i - v_{i-1}\|_1}{\gcd(v_i - v_{i-1})}.$$

Example 3.19. Consider the polygon P with vertices

$$\begin{pmatrix} 0 & 0 & 2 & 8 \\ 0 & 2 & 4 & 8 \end{pmatrix}$$

from before. We know P is smooth and $\operatorname{reg}(I) = 4$. The polygon P and normal fan Σ are pictured below.





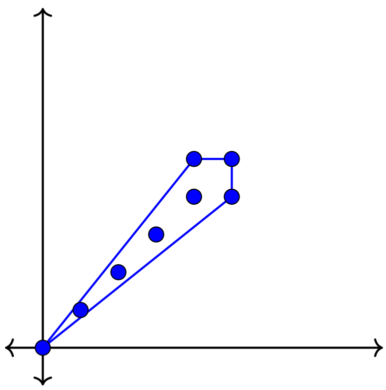
If Conjecture 3.18 were true, this would imply

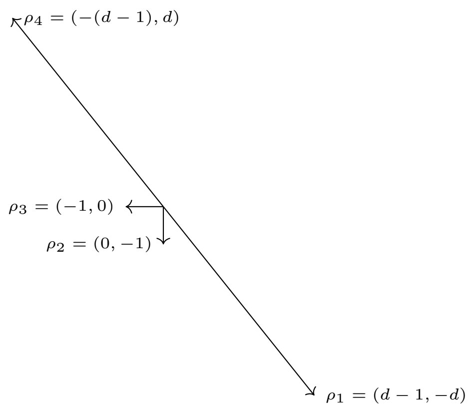
$$3 = \operatorname{reg} R/I_X \leq \max_{\rho_i \in \Sigma(1)} \|\rho_i\|_1 = \max\{1, 5, 2, 2\} = 5.$$

Example 3.20. Consider the "bad boy" polygon P with vertices

$$\begin{pmatrix} 0 & d & d-1 & d \\ 0 & d-1 & d & d \end{pmatrix}.$$

We know that I is generated by a single polynomial of degree $2d$, so $\operatorname{reg}(I) = 2d$. The polygon P and normal fan Σ are pictured below.





Then Conjecture 3.18 would imply

$$2d-1=\operatorname{reg} R/I_X \leq \max_{\rho_i\in\Sigma(1)} \|\rho_i\|_1 = \max\{2d-1,1,1,2d-1\}=2d-1.$$

The conjecture experimentally seems to hold. Since every smooth complete toric surface is obtained via a sequence of blow-ups [10, Theorem 10.4.3], an inductive argument on sheaf cohomology seems like a plausible strategy in proving the conjecture.

In addition to Conjecure 3.18, the general framework provided by hollow polygons still seems promising in constructing examples of non-projectively normal projective toric varieties with well-controlled regularity.

Definition 3.21. A set of lattice points $A\subset\mathbb{Z}^2$ is *polygonal* if A consists of only the integral boundary points of some convex polygon with integral vertices.

Definition 3.22. A set of polygonal lattice points $A\subset\mathbb{Z}^2$ is *primitive* if there does not exist a polygonal subset of lattice points $B\subsetneq \operatorname{Conv}(A)$ such that the $\Sigma_A=\Sigma_B$.

Definition 3.23. For a fan Σ, the *primitive points* of Σ is the set of primitive, polygonal lattice points A such that $\Sigma_A=\Sigma$, and we denote this by $\operatorname{prim}(\Sigma)$.

Conjecture 3.24. *For any set of polygonal points A,* $\operatorname{reg}(I_A)=\operatorname{reg}(I_{\operatorname{prim}(A)})+1$.

This setup encapsulates the hollow triangle and hollow square examples. Moreover, it would explain why dilates of hollow polygons have the same regularity (after enough dilations, as in the case of hollow triangles). If P were to have many symmetries (e.g., if P were centrally symmetric), then the argument used in Lemma 3.16 should generalize as well. Thus it would only remain to be shown that $\overline{R/I_P}$ has low regularity.

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