On Chemotaxis Model with Linear and Porous Medium Diffusion, Logistic Source and Consumption on $\mathbb{R}^N$

by

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Abstract

This dissertation is devoted to the study of chemotaxis systems with both linear diffusion and porous-medium-type diffusion, logistic source terms, and consumption of a chemical substance on $\mathbb{R}^N$. Chemotaxis systems are mathematical models describing the aggregation of cells driven by their directed movement in response to gradients of chemicals in their environment, which may act as attractants or repellents.

In the first part of this dissertation, we investigate a chemotaxis model with linear diffusion. We study fundamental problems such as the local and global existence of classical solutions with nonnegative initial data, which may be integrable or non-integrable. Under suitable smallness assumptions on the product of the initial chemical concentration and the chemotactic sensitivity, we prove the existence of a unique global classical solution. For non-integrable initial data, we develop a novel weighted energy method to establish global existence and boundedness. By introducing carefully chosen cut-off functions, we localize L^p-estimates uniformly in space. This approach extends known results for bounded domains and is applicable to other chemotaxis systems. We also study the stability of strictly positive solutions and the spreading behavior of solutions with compactly supported initial data. We show that the chemical does not, in general, hinder the spreading of the species, and it does not accelerate the spreading speed when the initial chemical concentration decays spatially or in the chemorepellent case with small sensitivity. Numerical simulations further reveal a phase transition in the sensitivity χ: when the chemical is initially uniformly distributed in space, acceleration occurs only when χ exceeds a critical positive value.

In the second part, we study the local and global existence of weak solutions for the porous-medium diffusion case. For general bounded, possibly non-integrable initial data, we prove the existence of global weak solutions that remain uniformly bounded for all time. The proof is based on local L^p estimates, uniform in time, obtained through a new continuity-type argument combined with Moser iteration to derive L^∞ bounds. We also investigate regularity and prove uniqueness of weak solutions for sufficiently smooth initial data under suitable conditions on the diffusion exponent.

Artificial Intelligence (AI) Use Disclosure Statement

ChatGPT was used only for language assistance in the preparation of this dissertation. In particular, it was used to check grammar, improve sentence clarity, and refine wording. All mathematical results, proofs, and original research contributions are my own.

Digital Accessibility Disclosure Statement

In the preparation of this dissertation, the following digital accessibility tools were used to ensure this document complies with federal requirements: Adobe Acrobat accessibility tools, screen-reader compatibility checks, tagged PDF tools in LaTeX, and manual review of headings, figures, tables, captions, and alternative text. The author acknowledges full responsibility for the intellectual content of this work and has made a good faith effort to comply with digital accessibility requirements in publishing, wherein the nature of the content does not significantly change in order to do so. Furthermore, all content has been reviewed and revised to meet these requirements prior to final publication.

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Notation

Let $N \geq 1$.

- Let
$$B(r;x) := \{y \in \mathbb{R}^N : |x-y| < r\}, \qquad \text{and} \qquad B_r := B(r;0).$$
- For any $t > 0$, let
$$\Omega_t := [0,t] \times \mathbb{R}^N.$$
- By $C_1 \lesssim C_2$, we mean that there exists a constant $C > 0$ such that
$$C_1 \leq C\, C_2.$$
  Throughout this dissertation, C may depend on the parameters in the model unless otherwise specified. Whenever a constant is independent of certain quantities, we will state this explicitly.
- We define
$$C^b_{\text{unif}}(\mathbb{R}^N) := \left\{ u \in C(\mathbb{R}^N) \,\middle|\, u \text{ is uniformly continuous on } \mathbb{R}^N \text{ and } \sup_{x \in \mathbb{R}^N} |u(x)| < \infty \right\},$$
  equipped with the norm
$$\|u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} := \|u\|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|.$$
- We define
$$C^{m,b}_{\text{unif}}(\mathbb{R}^N) := \left\{ u \in C^b_{\text{unif}}(\mathbb{R}^N) \,\middle|\, \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \in C^b_{\text{unif}}(\mathbb{R}^N),\ k = 1, 2, \dots, m,\ 1 \leq i_1, i_2, \dots, i_k \leq N \right\}$$
  equipped with the norm
$$\|u\|_{C^{m,b}_{\text{unif}}} := \|u\|_\infty + \sum_{k=1}^{m} \sum_{1 \leq i_1, i_2, \dots, i_k \leq N} \left\| \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \right\|_\infty,$$
  where $m \in \mathbb{N}$.

- For given $0 < \nu < 1$ and $m \geq 1$, let

  $$C^{\nu,b}_{\text{unif}}(\mathbb{R}^N) := \left\{ u \in C^b_{\text{unif}}(\mathbb{R}^N) \;\middle|\; \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\nu} < \infty \right\},$$

  with norm

  $$\|u\|_{\infty,\nu} := \sup_{x\in\mathbb{R}^N} |u(x)| + \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\nu},$$

  and

  $$C^{m,\nu,b}_{\text{unif}}(\mathbb{R}^N) := \left\{ u \in C^{m,b}_{\text{unif}}(\mathbb{R}^N) \;\middle|\; \frac{\partial^m u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}} \in C^{\nu,b}_{\text{unif}}(\mathbb{R}^N),\ 1 \leq i_1, i_2, \dots, i_m \leq N \right\},$$

  with norm

  $$\|u\|_{m,\nu,b} := \|u\|_{C^{m,b}_{\text{unif}}(\mathbb{R}^N)} + \sum_{1\leq i_1,i_2,\dots,i_m\leq N} \left\| \frac{\partial^m u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}} \right\|_{C^{\nu,b}_{\text{unif}}(\mathbb{R}^N)}.$$

- For $0 < \theta < 1$, let

  $$C^\theta\Big((t_1,t_2), C^{\nu,b}_{\text{unif}}(\mathbb{R}^N)\Big) := \Big\{ u(\cdot) \in C\Big((t_1,t_2), C^{\nu,b}_{\text{unif}}(\mathbb{R}^N)\Big) \;\Big|\; u(t) \text{ is locally Hölder continuous in } t \text{ with exponent } \theta \Big\}.$$

Chapter 1

Introduction

1.1 Overview

Many microorganisms, including *Escherichia coli*, exhibit coordinated movement at the collective level by adjusting their motion in response to environmental stimuli. One of the most important examples of such behavior is **chemotaxis**, in which cells move along gradients of a chemical signal in their surrounding medium. Chemotaxis plays a fundamental role in a variety of biological processes, including bacterial aggregation, immune cell migration, and angiogenesis during embryonic development and tumor growth.

The recognition that signal-driven cell movement can generate complex spatial and temporal patterns led to significant developments in mathematical biology, particularly following the pioneering work of Keller and Segel in the 1970s [34, 35]. Their system of partial differential equations (PDEs) provided a mathematical framework for describing chemotactic aggregation. A general form of a Keller-Segel-type chemotaxis model for one species and one chemical signal is given by

$$\begin{cases} \partial_t u = \nabla \cdot \Big(D(u,v)\nabla u - \chi(u,v)\nabla v \Big) + f(u,v), & x \in \Omega, \\ \tau \partial_t v = \Delta v + g(u,v) - h(u,v)v, & x \in \Omega. \end{cases} \tag{1.1.1}$$

In (1.1.1), the unknown function $u = u(t,x)$ denotes the population density, while $v = v(t,x)$ represents the concentration of the chemical signal. In addition to the standard diffusive flux, directed movement toward regions of higher chemical concentration produces a chemotactic flux, $J_{\text{chemo}} = \chi(u,v)\,\nabla v$, where $\chi(u,v)$ is the chemotactic sensitivity to the chemical gradient. The coefficient $D(u,v)$ describes the diffusivity of the cells, and $f(u,v)$ models cell growth and death, whereas $g(u,v)$ and $h(u,v)$ represent the production and degradation of the chemical signal, respectively. The parameter τ is associated with the diffusion rate of the chemical substance, biologically, $\tau = 0$ indicates that the chemical substance diffuses much faster than the cells. Finally, the spatial domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ may be bounded or unbounded. For bounded domains, the system is typically studied with homogeneous Neumann boundary conditions.

System (1.1.1) has become one of the most widely studied models in mathematical biology for various choices of the functions D, χ, f, g, and h. This is due both to its flexibility in describing important biological phenomena, such as aggregation, pattern formation, and population stabilization, and to the rich mathematical challenges it presents. Among the central problems in the study of general chemotaxis systems are the global existence and uniqueness of solutions, the prevention or occurrence of finite-time blow-up, boundedness versus unboundedness, the formation of spatial patterns, the stability of equilibria and the influence of chemotactic sensitivity on population density. A fundamental objective is therefore to determine how the interplay among diffusion, chemotactic movement, reaction terms, and chemical dynamics shapes these qualitative properties.

There is a substantial body of work on various special cases of (1.1.1). A particularly important class corresponds to the choices $D(u,v) = mu^{m-1}$ with $m \geq 1$, $\chi(u,v) = \chi u$ with $\chi \in \mathbb{R}$, and $f(u,v) = au - bu^2$, representing density-dependent diffusion, linear chemotactic sensitivity, and logistic growth, respectively. When the chemical signal is produced by the cells at rate μ and decays at rate λ, this leads to the well-studied chemotaxis system with linear signal production:

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \Omega, \\ \tau v_t = \Delta v + \mu u - \lambda v, \quad x \in \Omega, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1.2}$$

and

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad x \in \Omega, \\ 0 = \Delta v + \mu u - \lambda v, \quad x \in \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1.3}$$

Here, $\chi > 0$ corresponds to the case in which the species is attracted toward regions of higher chemical concentration (positive taxis), whereas $\chi < 0$ corresponds to the case in which the species moves away from such regions (negative taxis). Moreover, (1.1.3) may be viewed as the parabolic-elliptic reduction of (1.1.2), obtained by setting $\tau = 0$. Without the logistic source i.e. when $a = b = 0$, systems such as (1.1.2) and (1.1.3) are often called *minimal chemotaxis systems.*

In addition, the case $m = 1$ corresponds to linear diffusion, where one is typically interested in the existence of global classical solutions, while the case $m > 1$, the diffusion becomes degenerate of porous medium type, for which the natural question is often the existence of global weak solutions. Porous medium-type diffusion in chemotaxis models is motivated by the fact that cell migration in biological tissues is often better described by *nonlinear* diffusion, especially in densely populated regions where cell crowding and mechanical stress become significant [64]. In this setting, degenerate diffusion plays an important role in capturing key biological features that linear diffusion cannot reproduce. In particular, the invasion of migrating cell populations is often characterized by a distinct moving boundary (or sharp front). Such behavior is not consistent with linear diffusion, which typically produces smooth profiles that become immediately positive everywhere. Let us present some result on the existence and uniqueness of solution to (1.1.2) and (1.1.3).

We first recall some results for the regular diffusion case $m = 1$, in bounded smooth domains $\Omega \subset \mathbb{R}^N$. For the fully parabolic problem (1.1.2) supplemented with homogenous Neumann boundary condition

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega. \tag{1.1.4}$$

- If $a = b = 0$, then finite-time blow-up does not occur when $N = 1$, whereas blow-up may occur when $N \geq 2$; see, for example, [25, 48].
- If $a, b > 0$, then finite-time blow-up does not occur when $N = 1, 2$; see [51, 50]. Thus, in low dimensions, the logistic source prevents finite-time blow-up in a certain sense.
- Assume that $a, b > 0$ and $\tau = 1$. It is known that finite-time blow-up does not occur provided

  $$b > \frac{N|\chi|\mu}{4}, \tag{1.1.5}$$

  see [29, 74].
- More generally, assume that $a, b > 0$ and $\tau > 0$. It was proved in [29, Theorem 1.1] (see also [79, Theorem 2.2] for the case $\tau = 1$) that finite-time blow-up does not occur provided

  $$b > \inf_{\gamma > \max\{1, \frac{N}{2}\}} \left(\frac{\gamma - 1}{\gamma} \left(\widetilde{C}_{\gamma+1,N} \right)^{\frac{1}{\gamma+1}} \right) |\chi|\mu,$$

  where $\widetilde{C}_{\gamma+1,N}$ is a positive constant arising from the maximal regularity theory for the corresponding parabolic equation. $\widetilde{C}_{\gamma+1,N} = 0$ when $N = 1, 2$ meaning finite-time blow-up does not occur when $N = 1, 2$.

For the parabolic–elliptic problem (1.1.3) + (1.1.4), the following results are known:

- If $a = b = 0$, then finite-time blow-up does not occur when $N = 1$, while it may occur when $N \geq 2$; see [24, 30, 48].

- Assume that $a, b > 0$, $\lambda, \mu > 0$. If

  $$b > \frac{(N-2)_+}{N}\chi\mu,$$

  then classical solutions of (1.1.3) with positive initial data exist globally and remain bounded; see [69, Theorem 2.5]. In particular, global existence is automatic when $N = 1, 2$. It remains open whether finite-time blow-up can be ruled out for every $b > 0$.

For the porous-medium diffusion case $m > 1$, several results are available in both the parabolic–parabolic ((1.1.2) with $\tau = 1$) and parabolic–elliptic (1.1.3) in bounded smooth domain; see [7, 8, 44, 66, 72]. In the parabolic–parabolic case, [66] proved the existence of a nonnegative weak solution when $m > 2-\frac{2}{N}$, and also for $m \leq 2-\frac{2}{N}$ if the logistic coefficient b is sufficiently large. This condition was later improved in [77] where the author established existence of globally bounded weak solutions for $m > \frac{2N}{N+2}$. On the other hand, in the absence of the logistic source ($a = b = 0$), finite-time blow-up may occur when $m \leq 2 - \frac{2}{N}$; see [75]. For the parabolic–elliptic case existence of a globally bounded weak solution was established in [72] under the condition $m > 2-\frac{2}{N}$, with the large-b assumption again covering the regime $m \leq 2 - \frac{2}{N}$, whereas [8] showed finite-time blow-up in the logistic-free case when $m < 2 - \frac{2}{N}$. Altogether, these results indicate that sufficiently strong logistic damping, together with stronger diffusion, can rule out blow-up and guarantee global boundedness.

There are fewer results for these models in the whole space $\mathbb{R}^N$. In the regular diffusion case $m = 1$, and $a, b > 0$, global existence for (1.1.2) on $\mathbb{R}^N$ was studied in [58], where the authors proved existence of a unique globally bounded classical solutions under same conditions (1.1.5) of the bounded-domain. They further investigated spreading properties in [59]. See [55] for a study on global existence for (1.1.2) on $\mathbb{R}^N$ with $m = 1$. See also [56, 57] for a recent study of (1.1.2) and (1.1.3) on compact metric graph.

For the porous medium case $m > 1$, on the whole space $\mathbb{R}^N$ the theory is more delicate. Consider when $a = b = 0$ and initials are integrable and bounded.

- When $\tau = 1$: In [61], the author proved existence of global weak solution for $m \geq 2$. This condition was latter improved Ishida and Yakota in [28], where the established global existence of solution for $m > 2-2/N$. Miura and Sugiyama [47, Theorem 2.3(1)] established the uniqueness of nonnegative weak solutions that are Hölder continuous up to the initial time for integrable initial data $u_0 \in C^{\alpha}(\mathbb{R}^N)$ and $v_0 \in C^{2+\alpha}(\mathbb{R}^N)$. Their argument is based on a vanishing-viscosity duality method.

- When $\tau = 0$: In [61], the author established existence of globally bounded weak solution when $m > 2 - 2/N$ and for initials that are small enough in some sense when $1 < m \leq 2 - 2/N$. In [60], the author construct initial data for which solutions blow up in finite time when $1 < m < 2 - 2/N$ and $N \geq 3$. The works [33, 39, 63] used the L^1-contraction principle to establish uniqueness of weak solutions with integrable initial data, under additional regularity assumptions on u, $\partial_t u$, and ∇u.

This dissertation is devoted to the study of system (1.1.1) with density-dependent diffusion $D(u, v) = mu^{m-1}$ for $m \geq 1$, linear sensitivity $\chi(u, v) = \chi u$ with $\chi \in \mathbb{R}$, logistic growth $f(u, v) = au - bu^2$, no chemical signal production $g(u, v) = 0$, and chemical consumption $h(u, v) = u$, posed on $\mathbb{R}^N$. This leads to the following system:

$$\begin{cases} u_t = m\nabla \cdot (u^{m-1}\nabla u) - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - uv, & x \in \mathbb{R}^N,\ t > 0, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.1.6}$$

For the linear diffusion case $m = 1$, we studied (1.1.6) with initial data satisfying

$$u_0(\cdot) \in C^b_{\mathrm{unif}}(\mathbb{R}^N), \quad v_0(\cdot) \in C^{1,b}_{\mathrm{unif}}(\mathbb{R}^N) \qquad \text{or} \qquad u_0(\cdot) \in L^p(\mathbb{R}^N), \quad v_0(\cdot) \in W^{1,p}(\mathbb{R}^N), \tag{1.1.7}$$

for some $p > 1$. For the porous medium type diffusion ($m > 1$), we studied (1.1.6) with initial data satisfying

$$u_0(\cdot) \in L^{\infty}(\mathbb{R}^N), \quad v_0(\cdot) \in W^{1,\infty}(\mathbb{R}^N) \qquad \text{or} \qquad u_0(\cdot) \in L^p(\mathbb{R}^N), \quad v_0(\cdot) \in W^{1,p}(\mathbb{R}^N), \tag{1.1.8}$$

for some $p > 1$. Let us emphasize that our framework allows both u_0 and v_0 to be non-integrable. This level of generality is important, as it lays the foundation for further studies, including the large-time propagation behavior of chemotaxis models in the whole space $\mathbb{R}^N$ (see [17, 22, 53, 54, 55]). There are relatively few studies of (1.1.6) in bounded domains. To the best of our knowledge, there are no results on the global existence or asymptotic behavior of solutions to (1.1.6) in $\mathbb{R}^N$.

In the rest of this chapter, we review existing results on (1.1.6) in bounded domains and then present our results for the linear diffusion case on $\mathbb{R}^N$ in Section 1.2. Section 1.3 is devoted to a review of the porous medium diffusion case in bounded domains, together with our contributions on $\mathbb{R}^N$.

1.2 Linear Diffusion $(m = 1)$

In the first part of this dissertation, we studied the dynamical features of nonnegative classical solutions to the following system:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - uv, & x \in \mathbb{R}^N,\ t > 0, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2.1}$$

with initial data satisfying (1.1.7). For comparison, consider the counterpart of (1.2.1) on a bounded smooth domain Ω, complemented with homogeneous Neumann boundary conditions:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2.2}$$

Assume that $a = b = 0$ and $\tau = 1$, $\chi > 0$. It is proved in [65, Theorem 1.1] that there exists a unique globally defined bounded classical solution of (1.2.2) with nonnegative initial data $(u_0, v_0) \in \left(W^{1,p}(\Omega)\right)^2$ for some $p > N$, provided that

$$0 < \|v_0\|_{L^\infty(\Omega)} \cdot \chi < \frac{1}{6(N + 1)}. \tag{1.2.3}$$

It is also proved in [78, Theorem 4.4] that any globally defined bounded positive classical solution of (1.2.2) converges to $\left(\frac{1}{|\Omega|} \int_\Omega u_0, 0\right)$ as $t \to \infty$ (see also [67] for the global existence and convergence of weak solutions in the case $N \geq 3$). Very recently, Lankeit and Winkler proved in [43] the existence of a global weak solution that becomes classical after some finite time, without any additional condition. It remains open whether this weak solution is unique.

Assume that $a, b > 0$, $\tau = 1$, $\chi > 0$. It was proved in [71, Theorem 3.3] that there exists a unique globally defined bounded classical solution of (1.2.2) with nonnegative initial data $(u_0, v_0) \in \left(W^{1,p}(\Omega)\right)^2$ for some $p > N$, provided that (1.2.3) holds. It was also proved in [42] that there exists a unique globally defined bounded classical solution with nonnegative initial data $(u_0, v_0) \in C(\bar{\Omega}) \times C^1(\bar{\Omega})$, provided that $\chi \|v_0\|_{L^\infty(\Omega)}$ is sufficiently small relative to b (see [42, Theorem 1.1]), and that any positive bounded globally defined classical solution converges to $\left(\frac{a}{b}, 0\right)$ as $t \to \infty$ (see [42, Theorem 1.2]). The reader is also referred to [44] for the global existence of unique classical solutions when $N = 2$.

Besides the additional difficulties caused by the unboundedness of the spatial domain, the lack of a comparison principle for solutions also presents significant challenges. Establishing global boundedness of solutions to (1.2.1) on $\mathbb{R}^N$ for non-integrable initial data is highly nontrivial. Unlike the bounded-domain setting, where L^p-bounds of solutions can be propagated via Gagliardo–Nirenberg inequalities and Moser iteration, solutions on $\mathbb{R}^N$ may fail to belong to $L^p(\mathbb{R}^N)$ for any $1 \le p < \infty$, and even when they do, their norms may not remain uniformly bounded in time. To overcome this difficulty on $\mathbb{R}^N$, we developed a novel weighted energy method to establish local L^p-bounds for the solution for some $p > \max\{1, \frac{N}{2}\}$, and then showed that this implies the global existence and uniform boundedness of the solution.

We call (u, v) a nonnegative classical solution of (1.2.1) on $[0, T) \times \mathbb{R}^N$ if $(u, v) \in C^{1,2}\Big((0, T) \times \mathbb{R}^N\Big)$, with $u(t, x) \ge 0$ and $v(t, x) \ge 0$ for all $(t, x) \in (0, T) \times \mathbb{R}^N$, and if (u, v) satisfies (1.2.1) in the classical sense for all $(t, x) \in [0, T) \times \mathbb{R}^N$. A global classical solution of (1.2.1) is a classical solution on $(0, \infty) \times \mathbb{R}^N$. In the first part of this dissertation, we prove, among other things, the following results:

Local existence. For any nonnegative initial data satisfying (1.1.7), there exists a unique nonnegative local classical solution of (1.2.1) satisfying the prescribed initial condition (see Proposition 3.1.1).

Global existence. If

$$|\chi| \cdot \|v_0\|_\infty < \max\Big\{b \cdot C_N^*,\ D_{\tau,N}^*\Big\},$$

where C_N^* and $D_{\tau,N}^*$ are defined in Theorem 3.2.2, then the classical solution exists for all time and remains bounded. Moreover, $C_N^* = \infty$ when $N = 1, 2$, which means that in one and two dimensions we obtain a unique global bounded classical solution without any smallness condition. Also, $D_{1,N}^* = \sqrt{\frac{2}{N}}$, which is weaker than (1.2.3). Hence [71, Theorem 3.3] is improved.

When a classical solution is globally defined in time and the initial datum is strictly positive, it is important to understand whether the solution remains *uniformly* positive for all future times, that is, whether $\inf_{x \in \mathbb{R}^N} u(t, x)$ stays bounded away from 0 as $t \to \infty$, or whether the solution eventually dies out. Moreover, when a positive entire solution is stable, it is natural to study the asymptotic behavior of solutions arising from *front-like* or *compactly supported* initial data. These questions are closely related to the persistence and stability of the constant steady state $\left(\frac{a}{b}, 0\right)$, as well as to how the chemotactic sensitivity χ influences spreading properties, for instance, the asymptotic spreading speed.

Observe that, in the absence of chemotaxis, that is, when $\chi = 0$, the dynamics of (1.2.1) are governed by the reaction–diffusion equation

$$u_t = \Delta u + u(a - bu), \qquad x \in \mathbb{R}^N. \tag{1.2.4}$$

Equation (1.2.4) is also known as the Fisher–KPP equation, due to the pioneering works of Fisher [12] and Kolmogorov–Petrovsky–Piskunov [40] on traveling waves and take-over properties. It is known that (1.2.4) admits traveling-wave solutions of the form $u(t,x) = \phi(x \cdot \xi - ct)$, with $\xi \in \mathbb{S}^{N-1}$, connecting the steady states $\frac{a}{b}$ and 0, that is, $\phi(-\infty) = \frac{a}{b}$, and $\phi(\infty) = 0$, for every speed $c \geq 2\sqrt{a}$, and that no such traveling-wave solution exists for $c < 2\sqrt{a}$. The minimal wave speed $c^* := 2\sqrt{a}$ is also called the *spreading speed* of (1.2.4) in the following sense: for any bounded $u_0 \in C(\mathbb{R}^N; \mathbb{R}^+)$ with nonempty compact support, the corresponding solution satisfies

$$\lim_{t\to\infty} \sup_{|x| \leq c' t} \left| u(t,x) - \frac{a}{b} \right| = 0 \qquad \forall\, c' < 2\sqrt{a},$$

and

$$\lim_{t\to\infty} \sup_{|x| \geq c'' t} u(t,x) = 0 \qquad \forall\, c'' > 2\sqrt{a}$$

(see [3, 2]). Now, for the chemotaxis system (1.2.1) with front-like initial data, note that if $\phi(x \cdot \xi - ct)$ is a traveling-wave solution of (1.2.4) connecting $\frac{a}{b}$ and 0, then

$$(u(t,x), v(t,x)) = \big(\phi(x \cdot \xi - ct),\, 0\big)$$

is a traveling-wave solution of (1.2.1) connecting $\left(\frac{a}{b}, 0\right)$ and $(0,0)$, since the v-equation admits the trivial solution $v \equiv 0$ and the chemotaxis term vanishes when $\nabla v \equiv 0$. Consequently, we focus on persistence and spreading speeds for compactly supported initial data. Our main results include the following.

Stability of the constant steady state $\left(\frac{a}{b}, 0\right)$. For every globally bounded nonnegative classical solution with strictly positive initial data, the constant steady state $\left(\frac{a}{b}, 0\right)$ is asymptotically stable (see Theorem 3.3.1).

Lower bound on spreading. For every globally bounded nonnegative classical solution arising from compactly supported, nonnegative, and nontrivial initial data u_0, one has

$$\lim_{t\to\infty} \sup_{|x| \leq ct} \left| u(t,x) - \frac{a}{b} \right| = 0, \qquad \lim_{t\to\infty} \sup_{|x| \leq ct} v(t,x) = 0, \qquad \forall\, 0 < c < 2\sqrt{a}.$$

This is Theorem 3.3.2.

This shows that chemotaxis does not slow down the propagation of the biological species when the initial population distribution is nontrivial. Concerning possible *speed-up*, we first prove that the chemical signal does not cause the population to spread infinitely fast. Moreover, if the initial chemical distribution is spatially sparse, for instance, if $v_0 \in L^p(\mathbb{R}^N)$ for some $p \geq 1$ or $v_0 \in C_0(\mathbb{R}^N)$, then it has no lasting effect on the spreading speed of the population. The reason is that the chemical is gradually consumed and becomes negligible in the long-time dynamics.

On the other hand, if v_0 remains significant as $|x| \to \infty$, then one expects that negative chemotaxis should not accelerate spreading, whereas positive chemotaxis may enhance it. This intuition is partially confirmed by the following result: for every globally bounded nonnegative classical solution arising from compactly supported, nontrivial initial data u_0, the statements below hold.

Upper bound on spreading. There exists a constant $c^*_{\text{up}} = c^*_{\text{up}}(u_0, v_0) \geq 2\sqrt{a}$ such that, for every $c > c^*_{\text{up}}$,

$$\lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0, \qquad \lim_{t\to\infty} \sup_{|x|\geq ct} \big|v(t,x) - V(t,x)\big| = 0,$$

where V denotes the solution of the heat equation $\tau V_t = \Delta V$ with initial condition $V(0,x) = v_0(x)$. This is Theorem 3.3.3.

Existence of the spreading speed. If $v_0 \in C_0(\mathbb{R}^N)$ or $v_0 \in L^p(\mathbb{R}^N)$ for some $p \geq 1$, then chemotaxis neither slows down nor speeds up the rate of spread, in the sense that $c^*_{\text{up}} = 2\sqrt{a}$. In addition, if $\tau = 1$, $u_0 \in C^{1,b}_{\text{unif}}(\mathbb{R}^N)$, and $v_0 \in C^{2+\alpha,b}_{\text{unif}}(\mathbb{R}^N)$ for some $\alpha > 0$, and if $1 - v_0$ is nonnegative and compactly supported, then there exists $\chi_0 > 0$ such that for any $-\chi_0 < \chi < 0$, chemotaxis again neither slows down nor speeds up the spreading speed. In particular, the spreading speed remains $2\sqrt{a}$. This is Theorem 3.3.4.

However, the case of chemoattractants ($\chi > 0$) is considerably more subtle. Numerical simulations (see Section 3.3.5) suggest a phase-transition phenomenon: there exists a critical threshold $\chi^* > 0$ such that the spreading speed is unchanged for $\chi < \chi^*$, but increases strictly once $\chi > \chi^*$. Establishing such a threshold rigorously remains a challenging open problem.

1.3 Porous Medium Diffusion ($m > 1$)

The second part of this dissertation is devoted to the study of (1.1.6) in the case $m > 1$,

$$\begin{cases} u_t = m\nabla \cdot (u^{m-1}\nabla u) - \chi\nabla \cdot (u\nabla v) + u(a - bu), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - uv, & x \in \mathbb{R}^N,\ t > 0, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3.1}$$

with initial data satisfying (1.1.8). Here the diffusion is *degenerate* (porous-medium type), and our focus is on the existence and uniqueness of globally bounded weak solutions. There have been only a few studies of the corresponding problem even on bounded smooth domains $\Omega \subset \mathbb{R}^N$

$$\begin{cases} u_t = \Delta u^m - \chi\nabla \cdot (u\nabla v) + u(a - bu), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ (\nabla u^m - \chi u\nabla v) \cdot \nu = 0, \qquad \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(0, x) = u_0(x), \qquad v(0, x) = v_0(x), & x \in \Omega. \end{cases} \tag{1.3.2}$$

Global existence of weak solutions to (1.1.2) has been studied, for example, in [26, 31, 32, 76] for $N = 3$. In particular, it is shown in [31, Theorem 1.1] that, for $N = 3$, any $m > 1$, and any nonnegative initial data $u_0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$, $v_0 \in C^2(\overline{\Omega})$, system (1.1.2) admits a global weak solution that remains uniformly bounded for all time. For $N \geq 3$ in the case $a = b = 0$, the existence of globally bounded weak solutions was established in [73] under the condition

$$m > 2 - \frac{2 + N}{2N}.$$

There are fewer results on uniqueness. For $m > 9/8$ and $a = b = 0$, uniqueness of Hölder-continuous weak solutions to (1.1.2) in dimension $N = 3$ was proved in [4] under suitable regularity and integrability assumptions on the initial data.

Existing results for porous-medium-type chemotaxis models on unbounded domains typically treat only *integrable* initial data; see, for example, [60, 61, 62]. Moreover, global existence results in both bounded and unbounded domains often require restrictions either on the diffusion exponent m or on the spatial dimension N. To the best of our knowledge, we establish for the first time global existence of weak solutions on unbounded domains for *non-integrable* initial data, without any restrictions on m or N, thereby extending and improving several results in the literature. We also prove uniqueness within the class of weak solutions to (1.3.1) that are Hölder continuous up to the initial time, without imposing integrability conditions on the initial data, without restrictions on the spatial dimension, and without requiring any additional regularity of u_t. Among other contributions, we obtain the following results.

Global existence and regularity of weak solutions. For any $m > 1$, there exists a uniformly bounded nonnegative weak solution of (1.1.6) with initial data satisfying (1.1.8), and this solution is locally Hölder continuous; see Theorem 4.1.2 and Remark 4.1.1(4) for the integrable case.

Uniqueness of Hölder-continuous weak solutions. Let $u_0 \in C^{\alpha}(\mathbb{R}^N)$ and $v_0 \in C^{2+\alpha}(\mathbb{R}^N)$. If $m \in (1,3]$, then there exists a unique bounded nonnegative weak solution that is uniformly Hölder continuous up to the initial time; see Theorem 4.1.3.

The results presented in this dissertation appear in [19, 20, 21].

Chapter 2

Preliminary

In this chapter, we present several preliminary lemmas and propositions that will be used throughout the dissertation. These include basic properties of the analytic semigroup generated by $\Delta - I$ on $C^b_{\text{unif}}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, a maximal regularity lemma for parabolic initial-boundary value problems, a useful exponentially decaying function, several lemmas concerning estimate for the gradient of solutions to the second equation in (1.1.6) and certain special Harnack inequalities.

2.1 The Analytic Semigroup Generated by $\Delta - I$ on $C^b_{\text{unif}}(\mathbb{R}^N)$

In this section, we present some basic properties of the analytic semigroup, denoted by $T(t)$, generated by $\Delta - I$ on $X := C^b_{\text{unif}}(\mathbb{R}^N)$. Observe that

$$(T(t)u)(x) = e^{-t}(G(\cdot, t) * u)(x) = \int_{\mathbb{R}^N} e^{-t} G(x-y, t) u(y)\, dy$$

for every $u \in X$, $t > 0$, and $x \in \mathbb{R}^N$, where $G(t, x)$ is the heat kernel defined by

$$G(t, x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}. \tag{2.1.1}$$

Let $X^\alpha = \text{Dom}(A^\alpha)$ be the fractional power spaces associated with $A = I - \Delta$ on X ($\alpha \in [0, \infty)$). We have $X^0 = C^b_{\text{unif}}(\mathbb{R}^N)$ and $X^1 = C^{2,b}_{\text{unif}}(\mathbb{R}^N)$, together with the following continuous embedding:

$$X^\alpha \subset C^{\lfloor\nu\rfloor, \nu - \lfloor\nu\rfloor, b}_{\text{unif}}(\mathbb{R}^N) \quad \text{if} \quad 0 \le \nu < 2\alpha \tag{2.1.2}$$

(see [23, Exercise 9]). Furthermore, for $0 < \delta < 1$ and $\alpha \ge 0$, there exists a constant C_α

such that

$$\|A^\alpha T(t)u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} \le C_\alpha t^{-\alpha} e^{-(1-\delta)t} \|u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} \tag{2.1.3}$$

for every $u \in C^b_{\text{unif}}(\mathbb{R}^N)$ and $t > 0$ (see [23, Theorem 1.4.3]).

Lemma 2.1.1. *([55, Lemma 3.2]) For every $t > 0$, the operator $T(t)\nabla\cdot$ has a unique bounded extension on $\left(C^b_{\text{unif}}(\mathbb{R}^N)\right)^N$ satisfying*

$$\|T(t)\nabla \cdot u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} \le \frac{N}{\sqrt{\pi}} t^{-\frac{1}{2}} e^{-t} \|u\|_{C^b_{\text{unif}}(\mathbb{R}^N)} \quad \forall\, u \in \left(C^b_{\text{unif}}(\mathbb{R}^N)\right)^N, \ \forall\, t > 0. \tag{2.1.4}$$

2.2 The Analytic Semigroup Generated by $\Delta - I$ on $L^p(\mathbb{R}^N)$

In this section, we present some basic properties of the analytic semigroup, denoted by $T_p(t)$, generated by $A_p := \Delta - I$ on $X_p := L^p(\mathbb{R}^N)$ $(p \ge 1)$. We also have

$$(T_p(t)u)(x) = e^{-t}(G(\cdot,t) * u)(x) = \int_{\mathbb{R}^N} e^{-t} G(x-y,t) u(y)\, dy \tag{2.2.1}$$

for every $u \in X_p$, $t > 0$, and $x \in \mathbb{R}^N$, where $G(t,x)$ is the heat kernel defined in (2.1.1). It follows from the L^p-L^q estimates for the convolution product that there exists $C_{p,q} > 0$ $(1 \le p < q \le \infty)$ such that

$$\|T_p(t)u\|_{L^q(\mathbb{R}^N)} \le C_{p,q} t^{-(\frac{1}{p}-\frac{1}{q})\frac{N}{2}} e^{-t} \|u\|_{L^p(\mathbb{R}^N)}, \tag{2.2.2}$$

and

$$\|\nabla T_p(t)u\|_{L^q(\mathbb{R}^N)} \le C_{p,q} t^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})\frac{N}{2}} e^{-t} \|u\|_{L^p(\mathbb{R}^N)}, \tag{2.2.3}$$

for every $u \in L^p(\mathbb{R}^N)$ and $t > 0$. Note that if $u \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $T_p(t)u \in L^\infty(\mathbb{R}^N)$ and

$$\|T_p(t)u\|_{L^\infty(\mathbb{R}^N)} \le e^{-t} \|u\|_{L^\infty(\mathbb{R}^N)}. \tag{2.2.4}$$

Let $X_p^\alpha = \text{Dom}(A_p^\alpha)$ be the fractional power spaces associated with $A_p = I - \Delta$ on X_p $(\alpha \in [0,\infty))$. We have $X_p^0 = X_p = L^p(\mathbb{R}^N)$ and $X_p^1 = W^{2,p}(\mathbb{R}^N)$ $(1 \le p < \infty)$, together with the following continuous embeddings (see [23, Theorem 1.6.1]):

$$X_p^\alpha \subset C^{\lfloor \nu \rfloor, \nu - \lfloor \nu \rfloor, b}_{\text{unif}}(\mathbb{R}^N) \quad \text{if} \quad 0 \le \nu < 2\alpha - \frac{N}{p}, \tag{2.2.5}$$

$$X_p^\alpha \subset W^{1,q}(\mathbb{R}^N) \quad \text{if } \alpha > \frac{1}{2} \text{ and } \frac{1}{q} > \frac{1}{p} - \frac{(2\alpha - 1)}{N}, \tag{2.2.6}$$

and

$$X_p^\alpha \subset L^q(\mathbb{R}^N) \quad \text{if } \frac{1}{q} > \frac{1}{p} - \frac{2\alpha}{N}, \quad q \geq p. \tag{2.2.7}$$

Furthermore, for $0 < \delta < 1$ and $\alpha \geq 0$, there exists $C_{\alpha,p,q} > 0$ such that

$$\|A_p^\alpha T_p(t)u\|_{L^q} \leq C_{\alpha,p,q} t^{-\alpha - (\frac{1}{p} - \frac{1}{q})\frac{N}{2}} e^{-(1-\delta)t} \|u\|_{L^p(\mathbb{R}^N)} \quad \text{for} \quad 1 \leq p \leq q < +\infty \tag{2.2.8}$$

for every $u \in L^p(\mathbb{R}^N)$ and $t > 0$ (see [55, (2.12)]).

Lemma 2.2.1. *([55, Lemma 3.1]) Let $p \in [1, \infty)$ and let $\{T_p(t)\}_{t>0}$ be the semigroup in* (2.2.1) *generated by A_p on $L^p(\mathbb{R}^N)$. For every $t > 0$, the operator $T_p(t)\nabla\cdot$ has a unique bounded extension on $\left(L^p(\mathbb{R}^N)\right)^N$ satisfying*

$$\|T_p(t)\nabla \cdot u\|_{L^p(\mathbb{R}^N)} \leq \tilde{C}_p t^{-\frac{1}{2}} e^{-t} \|u\|_{L^p(\mathbb{R}^N)} \quad \forall\, u \in \left(L^p(\mathbb{R}^N)\right)^N, \ \forall\, t > 0, \tag{2.2.9}$$

where $\tilde{C}_p$ depends only on p and N. Furthermore, for every $q \in [p, \infty]$, we have $T_p(t)\nabla \cdot u \in L^q(\mathbb{R}^N)$ with

$$\|T_p(t)\nabla \cdot u\|_{L^q} \leq \tilde{C}_{p,q} t^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{p} - \frac{1}{q})} e^{-t} \|u\|_{L^p(\mathbb{R}^N)} \quad \forall\, u \in \left(L^p(\mathbb{R}^N)\right)^N, \ \forall\, t > 0, \tag{2.2.10}$$

where $\tilde{C}_{p,q}$ is a constant depending only on N, q, and p.

Now we state a maximal regularity lemma for a parabolic initial value problem, which will be useful in controlling terms involving v in the system (1.1.6).

2.3 Maximal Regularity of a Parabolic Equation

In this section, we present a lemma on the maximal regularity for parabolic equations on $\mathbb{R}^N$ to be used in Chapter 3 and Chapter 4 to prove global existence of solutions to (1.1.6).

Lemma 2.3.1. *Let $\gamma > 1$ and let $v_0 \in W^{1,\gamma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. There exists $C_{\gamma,N}$ such that for any $T \in (0, \infty)$, if $g \in L^\gamma((0,T), L^\gamma(\mathbb{R}^N))$ and*

$$v(\cdot,\cdot) \in W^{1,\gamma}((0,T), L^\gamma(\mathbb{R}^N)) \cap L^\gamma((0,T), W^{2,\gamma}(\mathbb{R}^N))$$

solves the following initial value problem,

$$\begin{cases} \tau v_t = \Delta v - v + g, & x \in \mathbb{R}^N,\ 0 < t < T, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.3.1}$$

then

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \left(|v(t,x)|^\gamma + |\nabla v(t,x)|^\gamma + |\Delta v(t,x)|^\gamma\right) dx\, dt \\ &\le C_{\gamma,N} \left[\int_0^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} |g(t,x)|^\gamma \, dx\, dt + T \left(\|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} + \|\nabla v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} \right) \right], \end{aligned} \tag{2.3.2}$$

and for any $t_0 \in (0,T)$,

$$\begin{aligned} &\int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \left(|v(t,x)|^\gamma + |\nabla v(t,x)|^\gamma + |\Delta v(t,x)|^\gamma\right) dx\, dt \\ &\le C_{\gamma,N} \left[\int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} |g(t,x)|^\gamma \, dx\, dt + (T + \tau^\gamma t_0^{1-\gamma}) \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} \right]. \end{aligned} \tag{2.3.3}$$

Here $C_{\gamma,N}$ *is the smallest positive constant such that the inequalities* (2.3.2) *and* (2.3.3) *hold.*

Proof. First, we extend g to the whole of $\mathbb{R}$ by setting

$$\tilde{g}(t,x) = \begin{cases} g(t,x), & \text{for} \quad x \in \mathbb{R}^N,\ t \in (0,T), \\ 0, & \text{for} \quad x \in \mathbb{R}^N,\ t > T. \end{cases}$$

By [46, Theorem 3.1], the initial value problem

$$\begin{cases} \tau \tilde{v}_t = \Delta \tilde{v} - \tilde{v} + \tilde{g}, \quad x \in \mathbb{R}^N,\ 0 < t < \infty, \\ \tilde{v}(0,x) = 0, \quad x \in \mathbb{R}^N, \end{cases}$$

has a unique solution

$$\tilde{v}(\cdot,\cdot) \in W^{1,\gamma}((0,\infty), L^\gamma(\mathbb{R}^N)) \cap L^\gamma((0,\infty), W^{2,\gamma}(\mathbb{R}^N)).$$

Next, let $= \tilde{w}(t,x) := e^{\frac{t}{\tau}}\tilde{v}(t,x)$, which solves

$$\begin{cases} \tau \tilde{w}_t = \Delta \tilde{w} + e^{\frac{t}{\tau}}\tilde{g}, \quad x \in \mathbb{R}^N,\ 0 < t < \infty, \\ \tilde{w}(0,x) = 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{2.3.4}$$

We conclude that (2.3.4) has a unique solution in

$$W^{1,\gamma}((0,\infty), L^\gamma(\mathbb{R}^N)) \cap L^\gamma((0,\infty), W^{2,\gamma}(\mathbb{R}^N)).$$

By the closed graph theorem, there exists $C_{\gamma,\tau} > 0$, independent of T, such that for any $t_0 \in [0,T]$,

$$\int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \Big(|\tilde{v}(t,x)|^\gamma + |\nabla \tilde{v}(t,x)|^\gamma + |\Delta \tilde{v}(t,x)|^\gamma \Big)\, dx\, dt \le C_{\gamma,\tau} \int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} |g(t,x)|^\gamma \, dx\, dt. \tag{2.3.5}$$

Next, let $w := v - \tilde{v}$, which satisfies

$$\begin{cases} \tau w_t = \Delta w - w, \quad x \in \mathbb{R}^N,\ t \in (0,T), \\ w(0,x) = v_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{2.3.6}$$

By classical results for the heat equation,

$$w(t,x) = e^{-\frac{t}{\tau}} (G * v_0)(t/\tau, x),$$

where G is the heat kernel from (2.1.1). Since $|\nabla G(t,x)| \le \frac{|x|}{2t} G(t,x)$ and $v_0 \in W^{1,\gamma}(\mathbb{R}^N)$, there holds

$$\nabla w(t,x) = \int_{\mathbb{R}^N} e^{-\frac{t}{\tau}} G(t/\tau, x-y) \nabla v_0(y)\, dy$$

and

$$\Delta w(t,x) = \int_{\mathbb{R}^N} e^{-\frac{t}{\tau}} \nabla G(t/\tau, x-y) \cdot \nabla v_0(y)\, dy.$$

Then, by Young's convolution inequality, there exists $C_{\gamma,\tau,N} > 0$ such that for all $t > 0$,

$$\|w(t,\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} + \|\nabla w(t,\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} + \|\Delta w(t,\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} \le C_{\gamma,\tau,N} e^{-\frac{\gamma t}{\tau}} \|v_0(\cdot)\|^\gamma_{W^{1,\gamma}(\mathbb{R}^N)}. \tag{2.3.7}$$

This implies that

$$\int_0^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \Big(|w|^\gamma + |\nabla w|^\gamma + |\Delta w|^\gamma \Big)\, dx\, dt \le C_{\gamma,\tau,N} T \|v_0(\cdot)\|^\gamma_{W^{1,\gamma}(\mathbb{R}^N)}.$$

The above estimate and (2.3.5) with $t_0 = 0$ yield the conclusion of (2.3.2).

We next modify the bound on w in (2.3.7) to prove (2.3.3) when $t_0 > 0$. Note that for $t_0 > 0$ and $t \in (t_0, T)$, we can find some absolute constant $C > 0$ such that

$$|\nabla G(t,x)| \le \frac{|x|}{2t} G(t,x) \le \frac{C}{\sqrt{t}} G(2t,x),$$

and

$$|G_t(t,x)| \le \left(\frac{N}{2t} + \frac{|x|^2}{4t^2}\right) G(t,x) \le \frac{CN}{t} G(t,x) + \frac{C}{t} G(2t,x).$$

Then, by Young's convolution inequality, it is not hard to see that for all $t > 0$,

$$\begin{aligned} &\|w(t,\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} + \|\nabla w(t,\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} + \|\Delta w(t,\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} \\ &\qquad \le C^\gamma e^{-\frac{\gamma t}{\tau}} \left[\left\| (1 + \frac{\tau}{t}) G\big(\frac{2t}{\tau}, \cdot\big) \right\|^\gamma_{L^1(\mathbb{R}^N)} + \left\| (1 + \frac{N\tau}{t}) G\big(\frac{t}{\tau}, \cdot\big) \right\|^\gamma_{L^1(\mathbb{R}^N)} \right] \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)} \\ &\qquad \le C^\gamma e^{-\frac{\gamma t}{\tau}} \left[1 + \big(\frac{N\tau}{t}\big)^\gamma \right] \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)}. \end{aligned} \tag{2.3.8}$$

Multiplying (2.3.8) by $e^{\frac{\gamma t}{\tau}}$ and integrating over $t \in (t_0, T)$ yield, for some absolute constant $C > 0$,

$$\int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \Big(|w|^\gamma + |\nabla w|^\gamma + |\Delta w|^\gamma \Big)\, dx\, dt \le C^\gamma N^\gamma \Big(T + \tau^\gamma t_0^{1-\gamma} \Big) \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)}.$$

The above and (2.3.5) yield the existence of $C_{\gamma,\tau,N} > 0$ such that

$$\begin{aligned} &\int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \Big(|v(t,x)|^\gamma + |\nabla v(t,x)|^\gamma + |\Delta v(t,x)|^\gamma \Big)\, dx\, dt \\ &\qquad \le C_{\gamma,\tau,N} \int_{t_0}^T \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} |g(t,x)|^\gamma \, dx\, dt + C_{\gamma,\tau,N} (T + \tau^\gamma t_0^{1-\gamma}) \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)}. \end{aligned} \tag{2.3.9}$$

In the following, we always assume that $C_{\gamma,\tau,N}$ is the smallest positive constant such that the inequality (2.3.2) holds. We claim that $C_{\gamma,\tau,N} = C_{\gamma,1,N}$ for all $\tau > 0$. Indeed, by a scaling argument, one can consider $\tilde{v}(t,x) := v(\tau t, x)$, which solves

$$\tilde{v}_t = \Delta \tilde{v} - \tilde{v} + g(\tau t, x), \quad \tilde{v}(0,x) = v_0.$$

Then applying (2.3.9) with $\tau = 1$, and with t_0, T replaced by $\frac{t_0}{\tau}, \frac{T}{\tau}$, respectively, to $\tilde{v}$ yields

$$\begin{aligned}
&\int_{\frac{t_0}{\tau}}^{\frac{T}{\tau}} \int_{\mathbb{R}^N} e^{\gamma \tilde{t}} \Big(|\tilde{v}(\tilde{t}, x)|^\gamma + |\nabla \tilde{v}(\tilde{t}, x)|^\gamma + |\Delta \tilde{v}(\tilde{t}, x)|^\gamma \Big) \, dx \, d\tilde{t} \\
&\qquad \le C_{\gamma,1,N} \int_{\frac{t_0}{\tau}}^{\frac{T}{\tau}} \int_{\mathbb{R}^N} e^{\gamma \tilde{t}} |g(\tau \tilde{t}, x)|^\gamma \, dx \, d\tilde{t} + C_{\gamma,1,N} \Big(\frac{T}{\tau} + \big(\frac{t_0}{\tau} \big)^{1-\gamma} \Big) \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)}.
\end{aligned}$$

This, together with the variable change $\tilde{t} = \frac{t}{\tau}$, implies that

$$\begin{aligned}
&\frac{1}{\tau} \int_{t_0}^{T} \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} \Big(|v(t, x)|^\gamma + |\nabla v(t, x)|^\gamma + |\Delta v(t, x)|^\gamma \Big) \, dx \, dt \\
&\qquad \le \frac{1}{\tau} C_{\gamma,1,N} \int_{t_0}^{T} \int_{\mathbb{R}^N} e^{\frac{\gamma t}{\tau}} |g(t, x)|^\gamma \, dx \, dt + C_{\gamma,1,N} \Big(\frac{T}{\tau} + \big(\frac{t_0}{\tau} \big)^{1-\gamma} \Big) \|v_0(\cdot)\|^\gamma_{L^\gamma(\mathbb{R}^N)}.
\end{aligned}$$

This implies (2.3.3) with $C_{\gamma,N} = C_{\gamma,1,N}$. □

Now we state a useful exponential decay function

2.4 An Exponential Decay Function

In this section, we present a lemma establishing the existence of certain special exponentially decaying functions that will be used in later chapters to handle the possible non-integrability of solutions.

Take a decreasing smooth function f on $\mathbb{R}$ such that

$$f(r) = 1 \quad \text{when } r \le N \qquad \text{and} \qquad f(r) = 2^{-1} e^{N+1-r} \quad \text{when } r \ge N + 1.$$

For each fixed $\kappa \in (0, 1)$ and for some $\gamma \in (0, 1)$, define

$$\psi(x) := \psi_\kappa(x) = f(\gamma \kappa |x|). \tag{2.4.1}$$

Lemma 2.4.1. *There exist dimensional constants $C, \gamma > 0$ such that for any $\kappa \in (0, 1)$, ψ from (2.4.1) satisfies, for all $x \in \mathbb{R}^N$,*

$$0 < \psi(x) \le 1, \quad |\nabla \psi(x)| \le \kappa \, \psi(x), \quad |D^2 \psi(x)| \le \kappa^2 \, \psi(x), \tag{2.4.2}$$

$$\psi(x) \le C \psi(y) \quad \textit{whenever} \quad |x - y| \le \kappa^{-1}, \tag{2.4.3}$$

and

$$\kappa^N \int_{\mathbb{R}^N} \psi(x)\,dx \le C, \qquad \sum_{\kappa z\in\mathbb{Z}^N} \psi(z) \le C. \tag{2.4.4}$$

Proof. The proof of (2.4.2) and (2.4.3) follows by direct computation when $\gamma > 0$ is sufficiently small, depending only on N. Moreover, a direct computation yields

$$\kappa^N \int_{\mathbb{R}^N} \psi(x)\,dx = \kappa^N \int_{\mathbb{R}^N} f(\gamma\kappa|x|)\,dx = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\mathbb{R}^+} f(\gamma r) r^{N-1}\,dr \le C,$$

where $\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ is the surface area of an N-dimensional unit sphere and Γ is the gamma function. Note that the collection of balls $\{B_{N/\kappa}(z) \,|\, \kappa z \in \mathbb{Z}^N\}$ covers the whole space $\mathbb{R}^N$, and by (2.4.3), for any z,

$$\psi(z) \le C^N \psi(x) \quad \text{if } x \in B_{\kappa^{-1}N}(z).$$

Thus, it follows that for some dimensional constant C,

$$\sum_{\kappa z\in\mathbb{Z}^N} \psi(z) \le C\kappa^N \sum_{\kappa z\in\mathbb{Z}^N} \int_{B_{\kappa^{-1}N}(z)} \psi(x)\,dx \le C\kappa^N \int_{\mathbb{R}^N} \psi(x)\,dx.$$

□

Next we present some estimate involving ∇v.

2.5 Estimate for ∇v.

In this section, we present several lemmas related to ∇v in the second equation of (1.1.6). The first proposition is particularly useful for proving global boundedness of solutions to (1.1.6) in both the regular diffusion case ($m = 1$) and the porous medium diffusion case ($m > 1$). More precisely, it asserts that if u is locally bounded in L^p for some $p > N$, then ∇v is uniformly bounded.

Proposition 2.5.1. *Let* $v(t,x) = v(t,x;u_0,v_0)$ *be the solution of the* v*-equation in* (1.2.1) *with* $v_0 \in W^{1,\infty}$*, that is,* v *solves*

$$\begin{cases} \tau v_t = \Delta v - uv, & x\in\mathbb{R}^N,\ t>0,\\ v(0,x) = v_0(x), & x\in\mathbb{R}^N. \end{cases}$$

If for some $p > N$ and $C_1 > 0$,

$$\sup_{t\in[0,T],x_0\in\mathbb{R}^N} \int_{B(x_0,1)} u^p(t,x)\,dx \le C_1, \tag{2.5.1}$$

then there is a constant C depending only on p, N, C_1, and $\|v_0\|_{W^{1,\infty}}$ such that

$$\sup_{t\in[0,T]} \|\nabla v(t,\cdot)\|_\infty < C.$$

In particular, when $m = 1$ in (1.1.6) and

$$(u(t,x), v(t,x)) = (u(t,x;u_0,v_0), v(t,x;u_0,v_0))$$

is the classical solution of (1.1.6) on $(0, T_{\max}(u_0,v_0))$, if (2.5.1) holds on $[0, T_{\max}(u_0,v_0))$, then the conclusion also holds on $[0, T_{\max}(u_0,v_0))$, that is,

$$\sup_{t\in[0,T_{\max})} \|\nabla v(t,\cdot)\|_\infty < \infty. \tag{2.5.2}$$

Proof. The result follows from [41, Theorem 3.1, Chapter V]. For completeness, we provide a more direct proof below.

Let ψ be as in Lemma 2.4.1, with some $\kappa > 0$ to be determined. First, note that

$$\tau\partial_t(v\psi) = \Delta(v\psi) - v\psi + (v\psi - 2\nabla v\cdot\nabla\psi - v\Delta\psi - uv\psi)$$

and that $v_0\psi \in W^{1,p}(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$ for any $p > 1$. Hence, for any $p \le q \le \infty$,

$$\begin{aligned}
\|(\nabla v(t,\cdot))\psi\|_{L^q} &\le \|v(t,\cdot)\nabla\psi\|_{L^q} + \|\nabla(v\psi)\|_{L^q} \\
&\le \underbrace{\|v(t,\cdot)\nabla\psi\|_{L^q}}_{A_1(q)} + \underbrace{\|\nabla e^{(\Delta-I)\frac{t}{\tau}}(v_0\psi)\|_{L^q}}_{A_2(q)} \\
&\quad + \underbrace{\frac{1}{\tau}\int_0^t \left\|\nabla e^{(\Delta-I)\frac{t-s}{\tau}}\Big(v\psi - 2\nabla v\cdot\nabla\psi - v\Delta\psi - \psi uv\Big)\right\|_{L^q} ds}_{A_3(q)}\,.
\end{aligned} \tag{2.5.3}$$

Next, we estimate $A_1(q)$, $A_2(q)$, and $A_3(q)$ for $q = p$ or $q = \infty$. Below, we write C_κ for a constant that may depend on C, p, and κ, while C is independent of κ. Since v is uniformly bounded,

$$A_1(p) \le C_\kappa \quad \text{and} \quad A_1(\infty) \le C\kappa. \tag{2.5.4}$$

Next, it follows from (2.2.2), (2.2.4), and $\|v_0\|_{W^{1,\infty}} \le C$ that

$$A_2(p) \le C\|\nabla(v_0\psi)\|_{L^p} \le C_\kappa \quad \text{and} \quad A_2(\infty) \le C. \tag{2.5.5}$$

Applying (2.2.3) with $q = p$ to $A_3(p)$ yields

$$\begin{aligned}
A_3(p) &\le C\int_0^t e^{-\frac{t-s}{\tau}}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{2}}\|v\psi - 2\nabla v\cdot\nabla\psi - v\Delta\psi - \psi uv\|_{L^p}\,ds\\
&\le C_\kappa + C\int_0^t e^{-\frac{t-s}{\tau}}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{2}}\Big(\|u\psi\|_{L^p} + \|\nabla v\cdot\nabla\psi\|_{L^p}\Big)\,ds\\
&\le C_\kappa + C\sup_{s\in[0,T]}\|u(s,\cdot)\psi\|_{L^p} + C\kappa\int_0^t e^{-\frac{t-s}{\tau}}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{2}}\|(\nabla v(s,\cdot))\psi\|_{L^p}\,ds. \qquad (2.5.6)
\end{aligned}$$

Now, by (2.5.3), (2.5.4), (2.5.5), and (2.5.6), we have

$$\|(\nabla v(t,\cdot))\psi\|_{L^p} \le C_\kappa + C\sup_{s\in[0,T]}\|u(s,\cdot)\psi\|_{L^p} + C\kappa\sup_{0\le s\le T}\|(\nabla v(s,\cdot))\psi\|_{L^p} \quad \forall\, 0\le t\le T.$$

By (2.5.1), and by taking the supremum over $t\in[0,T]$ and choosing κ sufficiently small, depending on C, we obtain

$$\sup_{0\le t\le T}\|(\nabla v(t,\cdot))\psi\|_{L^p} \le C_\kappa. \tag{2.5.7}$$

In the rest of the proof, we fix one such κ and may drop it from the notation of C_κ.

Finally, applying (2.2.3) with $q=\infty$ to $A_3(\infty)$ yields that for $p>N$,

$$\begin{aligned}
A_3(\infty) &\le C\int_0^t e^{-\frac{(t-s)}{\tau}}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{2}-\frac{N}{2p}}\|v\psi - 2\nabla v\cdot\nabla\psi - v\Delta\psi - \psi uv\|_{L^p}\,ds\\
&\le C_\kappa + C\int_0^t e^{-\frac{(t-s)}{\tau}}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{2}-\frac{N}{2p}}\Big(\|u\psi\|_{L^p} + \|\nabla v\cdot\nabla\psi\|_{L^p}\Big)\,ds\\
&\le C_\kappa + C\sup_{s\in[0,T]}\|u(s,\cdot)\psi\|_{L^p} + C\kappa\int_0^t e^{-\frac{(t-s)}{\tau}}\left(\frac{t-s}{\tau}\right)^{-\frac{1}{2}-\frac{N}{2p}}\|(\nabla v(s,\cdot))\psi\|_{L^p}\,ds.
\end{aligned}$$

Since $\int_{B(x_0,1)} u^p(t,x)\,dx \le C_1$ uniformly for all $t\in[0,T]$ and $x_0\in\mathbb{R}^N$ by assumption, this and (2.5.7) imply that

$$A_3(\infty) \le C \quad \text{for some } C = C(C_1). \tag{2.5.8}$$

By (2.5.3), (2.5.4), (2.5.5), and (2.5.8), there holds

$$\|(\nabla v(t,\cdot))\psi(\cdot)\|_{L^\infty} \le C \quad \forall\, t\in[0,T].$$

Replacing $\psi(\cdot)$ by $\psi(\cdot - x_0)$ yields

$$\|(\nabla v(t,\cdot))\psi(\cdot - x_0)\|_{L^\infty} \le C \quad \text{uniformly for all } t \in [0,T] \text{ and } x_0 \in \mathbb{R}^N.$$

This implies that

$$\sup_{0\le t\le T} \|\nabla v(t,\cdot)\|_{L^\infty} \le C,$$

where C depends only on p, N, C_1, and $\|v_0\|_{W^{1,\infty}}$. The proof is complete. $\square$

We now present a lemma that provide estimates for ∇v. This estimate will be used in Chapter 4 to derive local L^p bounds for u in the porous medium diffusion case, specifically in the proof of Theorem 4.1.1(1).

For any given $x_0 \in \mathbb{R}^N$ and $r > 1$, define

$$Z_{r,x_0}(t) := \iint_{\Omega_t} e^{-\frac{r(t-s)}{\tau}} (u+\varepsilon)^r(s,x)\psi^2(x-x_0)\,dx\,ds \quad \text{and} \quad Z_r(t) := \sup_{x_0\in\mathbb{R}^N} Z_{r,x_0}(t).$$

Lemma 2.5.1. *For any fixed $0 < \delta < 1$ and $r' > r > 1$, there exist c_r depending only on r and N, and C_δ depending only on C, r, r' and δ such that for all $\kappa \in (0, c_r)$ we have*

$$\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} |\nabla v|^{2r}\psi^2 \le \delta Z_{r'}(t) + C_\delta\,\kappa^{-N} \quad \forall\, t \in (0,T]. \tag{2.5.9}$$

Proof. First, note that

$$\begin{aligned}\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} |\nabla v|^{2r}\psi^2 &\lesssim \delta \iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} |\nabla v|^{2r'}\psi^2 + C_\delta\kappa^{-N} \\ &\lesssim \delta \iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} |\nabla (v\psi^{\frac{1}{r'}})|^{2r'} + \kappa^{2r'} \iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} v^{2r'}\psi^2 + C_\delta\kappa^{-N}\end{aligned} \tag{2.5.10}$$

Denoting $\psi_1 := \psi^{\frac{1}{r'}}$, since v is bounded by $\|v_0\|_\infty$ and ψ is bounded by 1, we have

$$\|v\psi_1\|_{2r'}^2 \le \|v\psi_1\|_\infty \|v\psi_1\|_{r'} \lesssim \|v\psi_1\|_{r'}. \tag{2.5.11}$$

By the Gagliardo-Nirenberg inequality,

$$\|\nabla(v\psi_1)\|_{2r'} \lesssim \|v\psi_1\|_\infty^{\frac12} \|D^2(v\psi_1)\|_{r'}^{\frac12} \lesssim \|D^2(v\psi_1)\|_{r'}^{\frac12}.$$

By [16, Theorem 9.11],

$$\|\nabla(v\psi_1)\|_{2r'} \lesssim \|D^2(v\psi_1)\|_{r'}^{\frac{1}{2}} \lesssim \|\Delta(v\psi_1)\|_{r'}^{\frac{1}{2}} + \|v\psi_1\|_{r'}^{\frac{1}{2}}. \tag{2.5.12}$$

Next, note that

$$\begin{aligned}\tau(v\psi_1)_t &= \Delta(v\psi_1) + (-2\nabla v\cdot\nabla\psi_1 - v\Delta\psi_1 - uv\psi_1)\\ &= \Delta(v\psi_1) - v\psi_1 + (v\psi_1 - 2\nabla v\cdot\nabla\psi_1 - v\Delta\psi_1 - uv\psi_1).\end{aligned}$$

Lemma 2.3.1 then yields

$$\begin{aligned}&\int_0^t e^{-\frac{r'(t-s)}{\tau}}\left(\|\Delta(v\psi_1)\|_{r'}^{r'} + \|\nabla(v\psi_1)\|_{r'}^{r'} + \|v\psi_1\|_{r'}^{r'}\right)\\ &\le C\int_0^t\int_{\mathbb{R}^N} e^{-\frac{r'(t-s)}{\tau}}|v\psi_1 - 2\nabla v\nabla\psi_1 - v\Delta\psi_1 - uv\psi_1|^{r'}\,dx\,ds + Cte^{-\frac{r't}{\tau}}\|v_0\psi_1\|_{W^{1,r'}}^{r'}\\ &\le C\|\psi\|_1 + C\int_0^t\int_{\mathbb{R}^N} e^{-\frac{r'(t-s)}{\tau}}\left[\kappa^{r'}|\nabla(v\psi_1)|^{r'} + |u\psi_1|^{r'}\right]dx\,ds,\end{aligned}$$

where C only depends on r' and N, and is independent of T. Therefore, if κ is sufficiently small depending only on r' and N, by property (2.4.4) in Lemma 2.4.1, we get

$$\int_0^t e^{-\frac{r'(t-s)}{\tau}}\left(\|\Delta(v\psi_1)\|_{r'}^{r'} + \|v\psi_1\|_{r'}^{r'}\right) \lesssim \kappa^{-N} + \int_0^t\int_{\mathbb{R}^N} e^{-\frac{r'(t-s)}{\tau}}|u\psi_1|^{r'},$$

which, combining with (2.5.10), (2.5.11) and (2.5.12), implies that

$$\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}}|\nabla v|^{2r}\psi^2 \lesssim C_\delta\kappa^{-N} + \delta\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}}u^{r'}\psi. \tag{2.5.13}$$

In the following, we derive a connection between $Z_{r'}(t)$ and $\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}}u^{r'}\psi$. Recall that the collection of balls $\{B_{N/\kappa}(z)\,|\,\kappa z\in\mathbb{Z}^N\}$ covers the whole space. Hence, by property (2.4.3) of Lemma 2.4.1, and using that $\psi(x-z)=1$ when $x\in B_{N/\kappa}(z)$ by the definition of

ψ, we have

$$\begin{aligned}
\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} u^{r'}\psi &\lesssim \sum_{\kappa z\in\mathbb{Z}^N}\psi(z)\int_0^t e^{-\frac{r'(t-s)}{\tau}}\int_{B_{N/\kappa}(z)} u^{r'}(s,x)\,dx\,ds\\
&\lesssim \sum_{\kappa z\in\mathbb{Z}^N}\psi(z)\int_0^t e^{-\frac{r'(t-s)}{\tau}}\int_{B_{N/\kappa}(z)} u^{r'}(s,x)\psi^2(x-z)\,dx\,ds\\
&\lesssim \sup_{x_0\in\mathbb{R}^N}\iint_{\Omega_t} e^{-\frac{r'(t-s)}{\tau}} u^{r'}(s,x)\psi^2(x-x_0)\,dx\,ds\\
&\lesssim Z_{r'}(t).
\end{aligned}$$

The lemma is thus proved by combinging this inequality with (2.5.13). $\square$

The following lemma concerns the regularity of classical solutions to the linear diffusion problem (1.2.1) and will be used repeatedly in the analysis of its asymptotic properties. Although it follows essentially from Section 3.1, a detailed proof is provided in the appendix of [21].

Lemma 2.5.2. ($C^{1+\theta,2+\nu}$-boundedness) *Suppose that $u_0\in X^+$ and $v_0\in X_1^+$, and $(u(t,x;u_0,v_0),v(t,x;u_0,$ is a globally defined bounded solution of (3.0.1). Then for any given $0<\nu\ll 1$, $0<\theta\ll 1$, and $t_0>0$, there is $C>0$ depending on the system parameters, $\|v_0\|_{X_1}$, $\sup_{t\geq 0}\|u(t,\cdot;u_0,v_0)\|_\infty$, ν, θ and t_0 such that*

$$\|w\|_{C^{\theta,\nu}_{\rm unif}([t_0,\infty)\times\mathbb{R}^N)}\leq C, \tag{2.5.14}$$

where $w(t,x)$ is any one of the following functions: $u(t,x;u_0,v_0)$, $\partial_{x_i}u(t,x;u_0,v_0)$, $\partial_t u(t,x;u_0,v_0)$, $\partial^2_{x_ix_j}u(t,x;u_0,v_0)$, $v(t,x;u_0,v_0)$, $\partial_{x_i}v(t,x;u_0,v_0)$, $\partial_t v(t,x;u_0,v_0)$, or $\partial^2_{x_ix_j}v(t,x;u_0,v_0)$ for $1\leq i,j\leq N$.

Moreover, if $u_0\in X_1^+$ and $v_0\in C^{2+\alpha,b}_{\rm unif}(\mathbb{R}^N)$ for some $\alpha>0$, then (2.5.14) holds for $t_0=0$ with w being $v(t,x;u_0,v_0)$, $\partial_{x_i}v(t,x;u_0,v_0)$, $\partial_t v(t,x;u_0,v_0)$, or $\partial^2_{x_ix_j}v(t,x;u_0,v_0)$ for $1\leq i,j\leq N$.

2.6 Special Harnack inequality

In this section, we present two lemmas that will be used frequently in Chapter 3, Section 3.3, in the study of the asymptotic behavior of classical solutions to the linear diffusion problem (1.2.1). The first lemma is a local-in-time Harnack inequality, which can be viewed as a slight generalization of [6, Theorem 1.2] and [18, Lemma 2.2], both of which concern advection Fisher–KPP equations. Unless otherwise stated, the constants C may depend on the parameters a, b, σ, and τ, as well as on the dimension N.

Lemma 2.6.1. *For given $\sigma > 0$, $u_0 \in X^+$ and $v_0 \in X_1^+$, assume that $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ is a globally defined bounded classical solution of* (1.2.1)*. Then for any $s_0 \geq 0$, $R > 0$, and $p \in (1,\infty)$, there exists a constant $C = C(s_0, R, p, \chi, \|v_0\|_{X_1}, \|u\|_\infty)$ such that if $t \geq 1$, $s \in [0, s_0]$, and $|x-y| \leq R$, then*

$$u(t,x;u_0,v_0) \leq C u^{\frac{1}{p}}(t+s, y; u_0, v_0) \tag{2.6.1}$$

and

$$|\nabla u(t,x;u_0,v_0)| \leq C u^{\frac{1}{p}}(t,y;u_0,v_0). \tag{2.6.2}$$

The constant C is uniform as $\chi \to 0$.

The proof is similar to the one of [18, Lemma 2.2] and can be found in the Appendix of [21]. The following estimates on v can be obtained as a corollary of Lemma 2.6.1.

Lemma 2.6.2. *Let $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ be as in Lemma 2.6.1. For any $\varepsilon > 0$ and $p \in (1,\infty)$, there are $T(\varepsilon, \chi, \|v_0\|_{X_1}) \geq 1$ and $C = C(\varepsilon, p, \chi, \|v_0\|_{X_1}, \|u\|_\infty) > 0$ such that*

$$|\nabla v(t,x;u_0,v_0)| \leq \varepsilon + C u^{\frac{1}{p}}(t,x;u_0,v_0), \quad \forall\, t \geq T,\ x \in \mathbb{R}^N \tag{2.6.3}$$

and

$$|\Delta v(t,x;u_0,v_0)| \leq \varepsilon + C u^{\frac{1}{p}}(t,x;u_0,v_0), \quad \forall\, t \geq T,\ x \in \mathbb{R}^N. \tag{2.6.4}$$

The constants T, C are uniform as $\chi \to 0$.

Proof. For simplicity of notations, we drop u_0, v_0 from the notations of $u(t,x;u_0,v_0)$, $v(t,x;u_0,v_0)$.

First, note that, for any $t > t_1 \geq 0$ and $x \in \mathbb{R}^N$, we have

$$\begin{aligned} v(t,x) = &\left(\frac{\tau}{4\pi(t-t_1)}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4(t-t_1)}} v(t_1,y)\,dy \\ &- \frac{1}{\tau}\int_{t_1}^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds \\ = &\left(\frac{\tau}{4\pi(t-t_1)}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4(t-t_1)}} v(t_1,y)\,dy \\ &- \frac{1}{\tau}\int_{t_1}^t \left(\frac{\tau}{\pi}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\tau|z|^2} u(s, x+2\sqrt{t-s}\,z)\, v(s, x+2\sqrt{t-s}\,z)\,dz\,ds. \end{aligned} \tag{2.6.5}$$

Since $u, v \geq 0$, this with $t_1 = 0$ implies that $\|v\|_\infty \leq \|v_0\|_\infty$. By Proposition 2.5.1, there is $C = C(\|v_0\|_{X_1}, \|u\|_\infty)$ such that

$$v(t,x) + |\nabla v(t,x)| \le C \quad \forall\, t \ge 0,\ x \in \mathbb{R}^N. \tag{2.6.6}$$

By Lemma 2.5.2, there is $C = C(\chi, \|v_0\|_{X_1}, \|u\|_\infty) > 0$ such that

$$|\nabla v(t,x)| + |\Delta v(t,x)| \le C \quad \forall\, t \ge 1,\ x \in \mathbb{R}^N.$$

This yields that for any $\varepsilon > 0$, there is $T(\varepsilon, \chi, \|v_0\|_{X_1}, \|u\|_\infty)$ such that

$$\begin{aligned}
\left|\nabla \frac{\tau^{N/2}}{(4\pi T)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4T}} v(t_1,y)\,dy\right| &= \left|\frac{\tau^{N/2}}{(4\pi T)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4T}} \nabla v(t_1,y)\,dy\right| \\
&\le \varepsilon/3 \quad \forall\, t_1 \ge 1,\ x \in \mathbb{R}^N, \\
\left|\Delta \frac{\tau^{N/2}}{(4\pi T)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4T}} v(t_1,y)\,dy\right| &= \left|\frac{\tau^{N/2}}{(4\pi T)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4T}} \Delta v(t_1,y)\,dy\right| \\
&\le \varepsilon/3 \quad \forall\, t_1 \ge 1,\ x \in \mathbb{R}^N.
\end{aligned} \tag{2.6.7}$$

Moreover, (2.6.6) implies that there is $R = R(\varepsilon, T, \|v_0\|_{X_1}, \|u\|_\infty) > 0$ such that for all $t_1 \le t \le t_1 + T$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
&\left|\int_{t_1}^t \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z|>R} e^{-\tau|z|^2} |\nabla_x u(s, x + 2\sqrt{t-s}\,z)|\, v(s, x + 2\sqrt{t-s}\,z)\,dz\,ds\right| \le \frac{\varepsilon\tau}{3}, \\
&\left|\int_{t_1}^t \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z|>R} e^{-\tau|z|^2} u(s, x + 2\sqrt{t-s}\,z)\, |\nabla_x v(s, x + 2\sqrt{t-s}\,z)|\,dz\,ds\right| \le \frac{\varepsilon\tau}{3}, \\
&\left|\int_{t_1}^t \left(\frac{\tau}{\pi}\right)^{N/2} \frac{1}{\sqrt{t-s}} \int_{|z|>R} e^{-\tau|z|^2} |z \cdot \nabla_x u(s, x + 2\sqrt{t-s}\,z)|\, v(s, x + 2\sqrt{t-s}\,z)\,dz\,ds\right| \le \frac{\varepsilon}{3}, \\
&\left|\int_{t_1}^t \left(\frac{\tau}{\pi}\right)^{N/2} \frac{1}{\sqrt{t-s}} \int_{|z|>R} e^{-\tau|z|^2} u(s, x + 2\sqrt{t-s}\,z)\, |z \cdot \nabla_x v(s, x + 2\sqrt{t-s}\,z)|\,dz\,ds\right| \le \frac{\varepsilon}{3}.
\end{aligned} \tag{2.6.8}$$

Next, note that, for any $t \ge 1+T$, by (2.6.5) with $t_1 = t-T$, we have

$$\begin{aligned}\nabla v(t,x) = {} & \nabla \frac{\tau^{N/2}}{(4\pi T)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4T}} v(t-T,y)\,dy \\ & - \frac{1}{\tau}\int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \int_{\mathbb{R}^N} e^{-\tau|z|^2} \nabla_x u(s,x+2\sqrt{t-s}\,z)\, v(s,x+2\sqrt{t-s}\,z)\,dz\,ds \\ & - \frac{1}{\tau}\int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \int_{\mathbb{R}^N} e^{-\tau|z|^2} u(s,x+2\sqrt{t-s}\,z)\, \nabla_x v(s,x+2\sqrt{t-s}\,z)\,dz\,ds.\end{aligned} \tag{2.6.9}$$

By (2.6.6)–(2.6.9), we have

$$\begin{aligned}|\nabla v(t,x)| \le {} & \varepsilon + \frac{1}{\tau}\int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \int_{|z|\le R} e^{-\tau|z|^2} |\nabla_x u(s,x+2\sqrt{t-s}\,z)|\, v(s,x+2\sqrt{t-s}\,z)\,dz\,ds \\ & + \frac{1}{\tau}\int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \int_{|z|\le R} e^{-\tau|z|^2} u(s,x+2\sqrt{t-s}\,z)\, |\nabla_x v(s,x+2\sqrt{t-s}\,z)|\,dz\,ds \\ \le {} & \varepsilon + \frac{C}{\tau}\int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \int_{|z|\le R} e^{-\tau|z|^2} |\nabla_x u(s,x+2\sqrt{t-s}\,z)|\,dz\,ds \\ & + \frac{C}{\tau}\int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \int_{|z|\le R} e^{-\tau|z|^2} u(s,x+2\sqrt{t-s}\,z)\,dz\,ds.\end{aligned}$$

This, together with $2\sqrt{t-s}|z| \le 2\sqrt{T}\,R$ for $|z| \le R$ and Lemma 2.6.1, implies that there is $C(\varepsilon,p,\chi,T,\|v_0\|_\infty,\|u\|_\infty) > 0$ such that

$$|\nabla v(t,x)| \le \varepsilon + C u^{1/p}(t,x) \quad \forall\, t \ge 1+T.$$

This proves (2.6.3).

Now, note that, for any $t \ge 1+T$, by (2.6.5) with $t_1 = t-T$, (2.6.7), and (2.6.8), we have

$$\begin{aligned}|\Delta v(t,x)| \le {} & \left| \Delta \Big(\frac{\tau}{4\pi T}\Big)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4T}} v(t-T,y)\,dy \right| \\ & + \left| \int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^N} e^{-\tau|z|^2} \left[z\cdot\nabla_x u(s,x+2\sqrt{t-s}\,z)\right] v(s,x+2\sqrt{t-s}\,z)\,dz\,ds \right| \\ & + \left| \int_{t-T}^{t} \Big(\frac{\tau}{\pi}\Big)^{N/2} \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^N} e^{-\tau|z|^2} u(s,x+2\sqrt{t-s}\,z) \left[z\cdot\nabla_x v(s,x+2\sqrt{t-s}\,z)\right] dz\,ds \right|.\end{aligned}$$

This, together with (2.6.6), (2.6.7), (2.6.8) and Lemma 2.6.1 implies that there is $C(\varepsilon,p,\chi,T,\|v_0\|_\infty,\|u\|$

0 such that

$$|\Delta v(t,x)| \leq \varepsilon + Cu^{1/p}(t,x) \quad \forall\, t \geq 1+T,$$

which proves (2.6.4). □

Chapter 3

The Chemotaxis System with Linear Diffusion

In this chapter, we consider the regular diffusion case, i.e.,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \mathbb{R}^N, \ t > 0, \\ \tau v_t = \Delta v - uv, & x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.0.1}$$

with initial data (u_0, v_0) satisfying

$$u_0(\cdot) \in C^b_{\rm unif}(\mathbb{R}^N), \quad v_0(\cdot) \in C^{1,b}_{\rm unif}(\mathbb{R}^N) \qquad \text{or} \qquad u_0(\cdot) \in L^p(\mathbb{R}^N), \quad v_0(\cdot) \in W^{1,p}(\mathbb{R}^N)$$

for $p > 1$. In Section 3.1, we establish local existence and uniqueness of classical solutions. Section 3.2 is devoted to proving that the solution exists for all time and remains bounded under suitable conditions on the initial data. Finally, Section 3.3 is devoted to the study of the asymptotic properties of (3.0.1).

3.1 Local Existence of Classical solutions

In this section, we study the local existence and uniqueness of classical solutions of (3.0.1). We have the following proposition on the local existence of classical solutions of (3.0.1).

Proposition 3.1.1. *(1) For given* $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ *and* $v_0 \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$*, there is* $T_{\max} := T_{\max}(u_0, v_0) \in (0, \infty]$ *such that* (3.0.1) *has a unique classical solution* $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *on* $(0, T_{\max}(u_0,v_0))$ *satisfying*

$$\lim_{t \to 0+} \Big(\|u(t,\cdot;u_0,v_0) - u_0(\cdot)\|_\infty + \|v(t,\cdot;u_0,v_0) - v_0(\cdot)\|_\infty + \|\nabla v(t,\cdot;u_0,v_0) - \nabla v_0(\cdot)\|_\infty \Big) = 0, \tag{3.1.1}$$

$$u(\cdot,\cdot;u_0,v_0) \in C([0,T_{\max}), C^b_{\rm unif}(\mathbb{R}^N)) \cap C^1((0,T_{\max}), C^b_{\rm unif}(\mathbb{R}^N)), \tag{3.1.2}$$

$$v(\cdot,\cdot;u_0,v_0) \in C([0,T_{\max}), C^{1,b}_{\rm unif}(\mathbb{R}^N)) \cap C^1((0,T_{\max}), C^{1,b}_{\rm unif}(\mathbb{R}^N)), \tag{3.1.3}$$

$$u(\cdot,\cdot;u_0,v_0),\ \partial_{x_i}u(\cdot,\cdot;u_0,v_0),\ \partial^2_{x_ix_j}u(\cdot,\cdot;u_0,v_0),\ \partial_t u(\cdot,\cdot;u_0,v_0)\in C^{\theta}((0,T_{\max}),C^{\nu,b}_{\rm unif}(\mathbb{R}^N)), \tag{3.1.4}$$

$$v(\cdot,\cdot;u_0,v_0),\ \partial_{x_i}v(\cdot,\cdot;u_0,v_0),\ \partial^2_{x_ix_j}v(\cdot,\cdot;u_0,v_0),\ \partial_t v(\cdot,\cdot;u_0,v_0)\in C^{\theta}((0,T_{\max}),C^{\nu,b}_{\rm unif}(\mathbb{R}^N)) \tag{3.1.5}$$

for all $i,j=1,2,\cdots,n$, $0<\theta\ll 1$, *and* $0<\nu\ll 1$. *Moreover, if* $T_{\max}(u_0,v_0)<\infty$, *then*

$$\limsup_{t\to T_{\max}(u_0,v_0)-}\Big(\|u(t,\cdot;u_0,v_0)\|_\infty+\|v(t,\cdot;u_0,v_0)\|_\infty+\|\nabla v(t,\cdot;u_0,v_0)\|_\infty\Big)=\infty. \tag{3.1.6}$$

If $u_0(x)\ge 0$ *and* $v_0(x)\ge 0$ *for* $x\in\mathbb{R}^N$, *then* $u(t,x;u_0,v_0)\ge 0$ *and* $v(t,x;u_0,v_0)\ge 0$ *for* $t\in[0,T_{\max}(u_0,v_0))$ *and* $x\in\mathbb{R}^N$.

(2) Suppose that $q>N$ *and* $q\ge 2$. *For given* $u_0\in L^q(\mathbb{R}^N)$ *and* $v_0\in W^{1,q}(\mathbb{R}^N)$, *there is* $T_{\max}(u_0,v_0)\in(0,\infty]$ *such that* (3.0.1) *has a unique classical solution* $(u(t,x;u_0,v_0),v(t,x;u_0,v_0))$ *on* $(0,T_{\max}(u_0,v_0))$ *satisfying*

$$\lim_{t\to 0+}\Big(\|u(t,\cdot;u_0,v_0)-u_0(\cdot)\|_{L^q}+\|v(t,\cdot;u_0,v_0)-v_0(\cdot)\|_{L^q}+\|\nabla v(t,\cdot;u_0,v_0)-\nabla v_0(\cdot)\|_{L^q}\Big)=0, \tag{3.1.7}$$

$$u(\cdot,\cdot;u_0,v_0)\in C([0,T_{\max}),L^q(\mathbb{R}^N))\cap C((0,T_{\max}),C^b_{\rm unif}(\mathbb{R}^N))\cap C^1((0,T_{\max}),C^b_{\rm unif}(\mathbb{R}^N)), \tag{3.1.8}$$

$$v(\cdot,\cdot;u_0,v_0)\in C([0,T_{\max}),W^{1,q}(\mathbb{R}^N))\cap C((0,T_{\max}),C^{1,b}_{\rm unif}(\mathbb{R}^N))\cap C^1((0,T_{\max}),C^{1,b}_{\rm unif}(\mathbb{R}^N)), \tag{3.1.9}$$

and (3.1.4)-(3.1.5). *Moreover, if* $T_{\max}(u_0,v_0)<\infty$, *then*

$$\limsup_{t\to T_{\max}(u_0,v_0)-}\Big(\|u(t,\cdot;u_0,v_0)\|_{L^q}+\|v(t,\cdot;u_0,v_0)\|_{L^q}+\|\nabla v(t,\cdot;u_0,v_0)\|_{L^q}\Big)=\infty. \tag{3.1.10}$$

If $u_0(x)\ge 0$ *and* $v_0(x)\ge 0$ *for* $x\in\mathbb{R}^N$, *then* $u(t,x;u_0,v_0)\ge 0$ *and* $v(t,x;u_0,v_0)\ge 0$ *for* $t\in[0,T_{\max}(u_0,v_0))$ *and* $x\in\mathbb{R}^N$.

Outline of the proof of Proposition 3.1.1. (1) We provide an outline of the proof of Proposition 3.1.1(1) in the following five steps.

Step 1. In this step, we prove that (3.0.1) has a unique mild solution $(u(t,x;u_0,v_0),v(t,x;u_0,v_0))$ on $[0,T]$ satisfying (3.1.1) for some $T>0$, that is,

$$[0,T]\ni t\mapsto (u(t,\cdot;u_0,v_0),v(t,\cdot;u_0,v_0))\in C^b_{\rm unif}(\mathbb{R}^N)\times C^{1,b}_{\rm unif}(\mathbb{R}^N)\quad\text{is continuous,}$$

and

$$\begin{cases} u(t,\cdot;u_0,v_0) = T(t)u_0 - \chi\int_0^t T(t-s)\nabla\cdot(u(s,\cdot;u_0,v_0)\nabla v(s,\cdot;u_0,v_0))\,ds \\ \qquad\qquad\qquad + \int_0^t T(t-s)u(s,\cdot;u_0,v_0)\Big(1+a-bu(s,\cdot;u_0,v_0)\Big)\,ds, \\ v(t,\cdot;u_0,v_0) = T(\frac{t}{\tau})v_0 + \frac{1}{\tau}\int_0^t T(\frac{t-s}{\tau})\Big(1-u(s,\cdot;u_0,v_0)\Big)v(s,\cdot;u_0,v_0)\,ds \end{cases} \tag{3.1.11}$$

for $t\in[0,T]$.

To this end, first, for given $T>0$ and $R>0$, let

$$\mathcal{X}_T = C([0,T],C^b_{\rm unif}(\mathbb{R}^N))\times C([0,T],C^{1,b}_{\rm unif}(\mathbb{R}^N))$$

be equipped with the sup-norm, and let

$$\mathcal{S}_{R,T} = \{(u,v)\in\mathcal{X}_T\,:\,\sup_{s\in[0,T]}\|u(s)\|_{C^b_{\rm unif}(\mathbb{R}^N)}\le R,\ \sup_{s\in[0,T]}\|v(s)\|_{C^{1,b}_{\rm unif}(\mathbb{R}^N)}\le R\}.$$

Then $\mathcal{S}_{R,T}$ is a closed subset of $\mathcal{X}_T$. Define a map on $\mathcal{S}_{R,T}$ as follows:

$$\begin{aligned}\Psi(u,v)(t) &= \begin{pmatrix}\Psi_1(u,v)(t)\\ \Psi_2(u,v)(t)\end{pmatrix}\\ &= \begin{pmatrix} T(t)u_0 - \chi\int_0^t T(t-s)\nabla\cdot(u(s)\nabla v(s))\,ds + \int_0^t T(t-s)u(s)(1+a-bu(s))\,ds \\ T(\frac{t}{\tau})v_0 + \frac{1}{\tau}\int_0^t T(\frac{t-s}{\tau})(1-u(s))v(s)\,ds\end{pmatrix}.\end{aligned}$$

Next, we show that Ψ is a well-defined map from $\mathcal{S}_{R,T}$ to $\mathcal{X}_T$. Moreover, for any given $0<\beta<\frac{1}{2}$ and $(u,v)\in\mathcal{S}_{R,T}$, the maps

$$(0,T]\ni t\mapsto\Psi_1(u,v)(t)\in X^\beta\quad\text{and}\quad(0,T]\ni t\mapsto\Psi_2(u,v)(t)\in X^{2\beta}$$

are locally Hölder continuous.

We then prove that for given

$$R>\max\{\|u_0\|_{C^b_{\rm unif}(\mathbb{R}^N)},\|v_0\|_{C^{1,b}_{\rm unif}(\mathbb{R}^N)}\},$$

there is $T:=T(R)>0$ such that the map Ψ maps $\mathcal{S}_{R,T}$ into itself and is a contraction. Hence Ψ has a unique fixed point $(u(\cdot,\cdot;u_0,v_0),v(\cdot,\cdot;u_0,v_0))\in\mathcal{S}_{R,T}$, and therefore (3.0.1) has a unique mild solution $(u(t,x;u_0,v_0),v(t,x;u_0,v_0))$ on $[0,T]$ satisfying (3.1.1) for some

$T > 0$. Moreover,

$$(0, T] \ni t \mapsto u(t, \cdot; u_0, v_0) \in X^{\beta} \quad \text{and} \quad (0, T] \ni t \mapsto v(t, \cdot; u_0, v_0) \in X^{2\beta}$$

are locally Hölder continuous for any $0 < \beta < \frac{1}{2}$.

Step 2. In this step, by the standard extension argument, we prove that there is $T_{\max} = T_{\max}(u_0, v_0) \in (0, \infty]$ such that (3.0.1) has a unique mild solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ satisfying (3.1.11) for $t \in [0, T_{\max})$, and

$$(0, T_{\max}) \ni t \mapsto u(t, \cdot; u_0, v_0) \in X^{\beta} \quad \text{and} \quad (0, T_{\max}) \ni t \mapsto v(t, \cdot; u_0, v_0) \in X^{2\beta}$$

are locally Hölder continuous for any $0 < \beta < \frac{1}{2}$. Moreover, if $T_{\max} < \infty$, then (3.1.6) holds.

Step 3. In this step, we prove that the mild solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ obtained in the above two steps is a classical solution of (3.0.1) satisfying (3.1.2)-(3.1.5).

To this end, first, fix any $t_1 \in (0, T_{\max}(u_0, v_0))$. By (2.1.2),

$$(-t_1, T_{\max}(u_0, v_0) - t_1) \ni t \mapsto u(t + t_1, \cdot; u_0, v_0) \in C^{\nu, b}_{\text{unif}}(\mathbb{R}^N)$$

is locally Hölder continuous for some $\nu > 0$.

Next, consider the initial value problem

$$\begin{cases} \tau \frac{\partial}{\partial t} \tilde{v} = (\Delta - 1)\tilde{v} + \Big(1 - u(t + t_1, x; u_0, v_0)\Big)\tilde{v}, \quad x \in \mathbb{R}^N, \ 0 < t < T_{\max}(u_0, v_0) - t_1, \\ \tilde{v}(0, x) = v_1(x) := v(t_1, x; u_0, v_0), \quad x \in \mathbb{R}^N. \end{cases} \tag{3.1.12}$$

By [15, Theorem 11 and Theorem 16 in Chapter 1], (3.1.12) has a unique classical solution $\tilde{v}(t, x)$ on $(0, T_{\max} - t_1)$ with $\lim_{t \to 0+} \|\tilde{v}(t, \cdot) - v_1\|_{C^b_{\text{unif}}(\mathbb{R}^N)} = 0$. By a priori interior estimates for parabolic equations (see [15, Theorem 5]), we have

$$\tilde{v}(\cdot, \cdot) \in C^1((0, T_{\max} - t_1), C^{1,b}_{\text{unif}}(\mathbb{R}^N)), \tag{3.1.13}$$

and the mappings

$$t \mapsto \tilde{v}(t, \cdot) \in C^{\nu, b}_{\text{unif}}(\mathbb{R}^N), \quad t \mapsto \frac{\partial \tilde{v}}{\partial x_i}(t, \cdot) \in C^{\nu, b}_{\text{unif}}(\mathbb{R}^N), \tag{3.1.14}$$

$$t \mapsto \frac{\partial^2 \tilde{v}}{\partial x_i \partial x_j}(t, \cdot) \in C^{\nu, b}_{\text{unif}}(\mathbb{R}^N), \quad t \mapsto \frac{\partial \tilde{v}}{\partial t}(t, \cdot) \in C^{\nu, b}_{\text{unif}}(\mathbb{R}^N) \tag{3.1.15}$$

are locally Hölder continuous in $t \in (0, T_{\max} - t_1)$ for $i, j = 1, 2, \cdots, N$ and $0 < \nu \ll 1$. By

[23, Lemma 3.3.2], $\tilde{v} = \tilde{v}(t, \cdot)$ is also a mild solution of (3.1.12) and therefore satisfies the following integral equation:

$$\tilde{v}(t, \cdot) = T\Big(\frac{t}{\tau}\Big)v_1 + \frac{1}{\tau}\int_0^t T\Big(\frac{t-s}{\tau}\Big)\big(1 - u(s+t_1, \cdot; u_0, v_0)\big)\tilde{v}(s, \cdot)\, ds \quad \forall\, t \in [0, T_{\max} - t_1).$$

Note that for any $t \in [0, T_{\max} - t_1)$,

$$v(t+t_1, \cdot; u_0, v_0) = T\Big(\frac{t}{\tau}\Big)v_1 + \frac{1}{\tau}\int_0^t T\Big(\frac{t-s}{\tau}\Big)\big(1 - u(s+t_1, \cdot; u_0, v_0)\big)v(s+t_1, \cdot; u_0, v_0)\, ds.$$

It is not difficult to prove that

$$v(t+t_1, \cdot; u_0, v_0) = \tilde{v}(t, \cdot) \quad \text{for every } t \in [0, T_{\max} - t_1).$$

Hence, by (3.1.14) and (3.1.15), the mappings

$$t \mapsto \nabla v(t+t_1, \cdot; u_0, v_0) \in C^{\nu}_{\text{unif}}(\mathbb{R}^N), \qquad t \mapsto \Delta v(t+t_1, \cdot; u_0, v_0) \in C^{\nu}_{\text{unif}}(\mathbb{R}^N)$$

are locally Hölder continuous in $t \in (-t_1, T_{\max} - t_1)$ for $0 < \nu \ll 1$.

Now, consider the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\tilde{u} = (\Delta - 1)\tilde{u} + F(t, x, \tilde{u}, \nabla\tilde{u}), \quad x \in \mathbb{R}^N,\ 0 < t < T_{\max} - t_1, \\ \tilde{u}(0, x) = u_1(x) := u(t_1, x; u_0, v_0), \quad x \in \mathbb{R}^N, \end{cases} \tag{3.1.16}$$

where

$$\begin{aligned} F(t, x, \tilde{u}, \nabla\tilde{u}) = {} & -\chi\nabla v(t+t_1, x; u_0, v_0)\cdot\nabla\tilde{u} \\ & + \Big(-\chi\Delta v(t+t_1, x; u_0, v_0) + 1 + a - bu(t+t_1, x; u_0, v_0)\Big)\tilde{u}. \end{aligned}$$

By similar arguments as above, (3.1.16) has a unique classical solution $\tilde{u}(t, x)$ on $(0, T_{\max} - t_1)$ with $\lim_{t\to 0+}\|\tilde{u}(t, \cdot) - u_1\|_{C^b_{\text{unif}}(\mathbb{R}^N)} = 0$, and $\tilde{u}(t, \cdot)$ satisfies (3.1.13)-(3.1.15) with $\tilde{v}$ replaced by $\tilde{u}$. Moreover,

$$u(t, \cdot; u_0, v_0) = \tilde{u}(t, \cdot) \quad \text{for } t \in [0, T_{\max} - t_1).$$

Therefore, $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ is a classical solution of (3.0.1) satisfying (3.1.2)-(3.1.5).

Step 4. In this step, by the uniqueness of mild solutions of (3.0.1), we prove that (3.0.1) has a unique classical solution satisfying (3.1.2)-(3.1.5).

Step 5. In this step, by the comparison principle for parabolic equations, we prove that

$$u(t, x; u_0, v_0) \ge 0 \quad \text{and} \quad v(t, x; u_0, v_0) \ge 0 \quad \forall\, t \in [0, T_{\max}(u_0, v_0)),\ x \in \mathbb{R}^N$$

provided that $u_0(x), v_0(x) \ge 0$ for $x \in \mathbb{R}^N$. Proposition (3.1.1)(1) for (3.0.1) is thus proved.

(2) We provide an outline of the proof of Proposition 3.1.1(2) for (3.0.1) in the following five steps.

Step 1$^{'}$. In this step, we prove that (3.0.1) has a unique mild solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ on $[0, T]$ satisfying (3.1.7) for some $T > 0$, that is,

$$[0, T] \ni t \mapsto (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \in L^q(\mathbb{R}^N) \times W^{1,q}(\mathbb{R}^N) \quad \text{is continuous,}$$

and

$$\begin{cases} u(t, \cdot; u_0, v_0) = T_q(t)u_0 - \chi \int_0^t T_q(t-s)\nabla \cdot (u(s, \cdot; u_0, v_0)\nabla v(s, \cdot; u_0, v_0))\, ds \\ \qquad\qquad\qquad + \int_0^t T_q(t-s) u(s, \cdot; u_0, v_0)\Big(1 + a - bu(s, \cdot; u_0, v_0)\Big)\, ds, \\ v(t, \cdot; u_0, v_0) = T_q(\frac{t}{\tau})v_0 + \frac{1}{\tau}\int_0^t T_q(\frac{t-s}{\tau})\Big(1 - u(s, \cdot; u_0, v_0)\Big) v(s, \cdot; u_0, v_0)\, ds \end{cases} \tag{3.1.17}$$

for $t \in [0, T]$.

To this end, as in Step 1 of the outline of the proof of Proposition 3.1.1(1), first, for given $T > 0$ and $R > 0$, let

$$\mathcal{X}_{q,T} = C([0, T]; L^q(\mathbb{R}^N)) \times C([0, T]; W^{1,q}(\mathbb{R}^N))$$

be equipped with the sup-norm, and let

$$\mathcal{S}_{q,R,T} = \{(u, v) \in \mathcal{X}_{q,T} : \sup_{s\in[0,T]} \|u(s)\|_{L^q(\mathbb{R}^N)} \le R,\ \sup_{s\in[0,T]} \|v(s)\|_{W^{1,q}(\mathbb{R}^N)} \le R\}.$$

Define Ψ_q on $\mathcal{S}_{q,R,T}$ by

$$\begin{aligned} \Psi_q(u, v)(t) &= \begin{pmatrix} \Psi_{1,q}(u, v)(t) \\ \Psi_{2,q}(u, v)(t) \end{pmatrix} \\ &= \begin{pmatrix} T_q(t)u_0 - \chi \int_0^t T_q(t-s)\nabla \cdot (u(s)\nabla v(s))\, ds + \int_0^t T_q(t-s)u(s)(1 + a - bu(s))\, ds \\ T_q(\frac{t}{\tau})v_0 + \frac{1}{\tau}\int_0^t T_q(\frac{t-s}{\tau})(1 - u(s))v(s)\, ds \end{pmatrix}. \end{aligned}$$

Next, we show that the map Ψ_q is a well-defined map from $\mathcal{S}_{q,R,T}$ to $\mathcal{X}_{q,T}$. Moreover, for any given $0 < \beta < \frac{1}{2} - \frac{N}{2q}$, $0 < \gamma < \frac{1}{2}$, and $(u, v) \in \mathcal{S}_{q,R,T}$, the maps

$$(0, T] \ni t \mapsto \Psi_1(u, v)(t) \in X^{\beta} \quad \text{and} \quad (0, T] \ni t \mapsto \Psi_2(u, v)(t) \in X^{2\gamma}$$

are locally Hölder continuous.

We then prove that for given

$$R > \max\{\|u_0\|_{L^q(\mathbb{R}^N)}, \|v_0\|_{W^{1,q}(\mathbb{R}^N)}\},$$

there is $T := T(R) > 0$ such that the map Ψ_q maps $\mathcal{S}_{q,R,T}$ into itself and is a contraction. Hence Ψ_q has a unique fixed point $(u(\cdot, \cdot; u_0, v_0), v(\cdot, \cdot; u_0, v_0))$ in $\mathcal{S}_{q,R,T}$, and therefore (3.0.1) has a unique mild solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ on $[0, T]$ satisfying (3.1.7) for some $T > 0$. Moreover, the maps

$$(0, T] \ni t \mapsto u(t, \cdot; u_0, v_0) \in X_q^{\beta} \quad \text{and} \quad (0, T] \ni t \mapsto v(t, \cdot; u_0, v_0) \in X_q^{2\gamma}$$

are locally Hölder continuous for any $0 < \beta < \frac{1}{2} - \frac{N}{2q}$ and $0 < \gamma < \frac{1}{2}$.

Step $2'$. In this step, by the standard extension argument, we prove that there is $T_{\max} = T_{\max}(u_0, v_0) \in (0, \infty]$ such that (3.0.1) has a unique mild solution $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ satisfying (3.1.17) for $t \in [0, T_{\max})$, and the maps

$$(0, T_{\max}) \ni t \mapsto u(t, \cdot; u_0, v_0) \in X_q^{\beta} \quad \text{and} \quad (0, T_{\max}) \ni t \mapsto v(t, \cdot; u_0, v_0) \in X_q^{2\gamma}$$

are locally Hölder continuous for any $0 < \beta < \frac{1}{2} - \frac{N}{2q}$ and $0 < \gamma < \frac{1}{2}$. Moreover, if $T_{\max} < \infty$, then (3.1.10) holds.

Step $3'$. In this step, using the fact that $(0, T_{\max}(u_0, v_0)) \ni t \mapsto v(t, \cdot; u_0, v_0) \in X_q^{\gamma}$ is locally Hölder continuous for any $0 < \gamma < 1$, we prove that

$$(0, T_{\max}(u_0, v_0)) \ni t \mapsto u(t, \cdot; u_0, v_0) \in X_q^{\beta}$$

is locally Hölder continuous for any $0 < \beta < \frac{1}{2}$. Then, by (2.2.5),

$$u(t, \cdot; u_0, v_0) \in C^b_{\text{unif}}(\mathbb{R}^N) \quad \text{and} \quad v(t, \cdot; u_0, v_0) \in C^{1,b}_{\text{unif}}(\mathbb{R}^N)$$

for all $t \in (0, T_{\max}(u_0, v_0))$.

To be a little more specific, by (2.2.5), for any $0 < t_1 < t_2 < T_{\max}(u_0, v_0)$, the map

$$[t_1, t_2] \ni t \mapsto \nabla v(t, \cdot; u_0, v_0) \in C^b_{\text{unif}}(\mathbb{R}^N)$$

is Hölder continuous. Then for any $0<\beta<\frac{1}{2}$ and $0<\delta<1$, there is $C>0$ such that for any $t\in[t_1,t_2]$, we have

$$\begin{aligned}
\|A^\beta u(t,\cdot;u_0,v_0)\|_{L^q} &\le \|A^\beta T_q(t-t_1)u(t_1,\cdot;u_0,v_0)\|_{L^q}\\
&\quad+\chi\int_{t_1}^t \|A^\beta T_q(t-s)\nabla\cdot(u(s,\cdot;u_0,v_0)\nabla v(s,\cdot;u_0,v_0))\|_{L^q}\,ds\\
&\quad+\int_{t_1}^t \|A^\beta T_q(t-s)u(s)\Big(1+a-bu(s)\Big)\|_{L^q}\,ds\\
&\le \|A^\beta T_q(t-t_1)u(t_1,\cdot;u_0,v_0)\|_{L^q}\\
&\quad+C\chi\max_{t_1\le t\le t_2}\|\nabla v(t,\cdot;u_0,v_0)\|_\infty\int_{t_1}^t (t-s)^{-\beta-\frac{1}{2}}e^{-(1-\delta)(t-s)}\|u(s,\cdot;u_0,v_0)\|_{L^q}\,ds\\
&\quad+C\int_{t_1}^t e^{-(1-\delta)(t-s)}\Big((t-s)^{-\beta}\|u(s)\|_{L^q}+(t-s)^{-\beta-\frac{N}{2q}}\|u^2(s)\|_{L^{\frac{q}{2}}}\Big)\,ds.
\end{aligned}$$

This implies that for any $t\in(t_1,t_2)$, $u(t,\cdot;u_0,v_0)\in X_q^\beta$, and hence for any $t\in(0,T_{\max}(u_0,v_0))$, $u(t,\cdot;u_0,v_0)\in X_q^\beta$. Moreover, we can prove that

$$(0,T_{\max}(u_0,v_0))\ni t\mapsto u(t,\cdot;u_0,v_0)\in X_q^\beta$$

is locally Hölder continuous.

Step $4^{'}$. In this step, we prove that (1.2.1) has a unique classical solution satisfying (3.1.7)-(3.1.9) and (3.1.4)-(3.1.5).

In fact, by Step $3^{'}$, for any $t_1\in(0,T_{\max}(u_0,v_0))$, we have

$$u(t_1,\cdot;u_0,v_0)\in C^b_{\rm unif}(\mathbb{R}^N)\quad\text{and}\quad v(t_1,\cdot;u_0,v_0)\in C^{1,b}_{\rm unif}(\mathbb{R}^N).$$

Then by Proposition 3.1.1(1), the mild solution $(u(t,x;u_0,v_0),v(t,x;u_0,v_0))$ obtained above is a classical solution of (3.0.1) satisfying (3.1.8)-(3.1.9) and (3.1.4)-(3.1.5). Note that a classical solution of (3.0.1) satisfying (3.1.7)-(3.1.9) and (3.1.4)-(3.1.5) is also a mild solution of (3.0.1) satisfying (3.1.7). Then, by the uniqueness of mild solutions of (3.0.1), the system (3.0.1) has a unique classical solution satisfying (3.1.7)-(3.1.9) and (3.1.4)-(3.1.5).

Step $5^{'}$. In this step, by the comparison principle for parabolic equations, we prove that

$$u(t,x;u_0,v_0)\ge 0\quad\text{and}\quad v(t,x;u_0,v_0)\ge 0\quad\forall\, t\in[0,T_{\max}(u_0,v_0)),\ x\in\mathbb{R}^N$$

provided $u_0(x)\ge 0$ and $v_0(x)\ge 0$ for all $x\in\mathbb{R}^N$. Proposition 3.1.1(2) is thus proved. □

3.2 Global Existence of a Classical Solution

In this section, we study the global existence of a classical solution of (3.0.1). We begin by stating the main theorems of this section.

Theorem 3.2.1. *Let* $(u(t,x), v(t,x)) := (u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *be the solution of* (3.0.1)*. We denote* $T_{\max} := T_{\max}(u_0, v_0)$*. Assume that there is* $p > \max\{1, \frac{N}{2}\}$ *such that*

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} u^p(t,x)\,dx < \infty. \tag{3.2.1}$$

Then $T_{\max} = \infty$ *and*

$$\limsup_{t\to\infty} \|u(t,\cdot)\|_\infty < \infty.$$

Theorem 3.2.2. *For given* $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ *and* $v_0 \in C^{1,b}_{\text{unif}}(\mathbb{R}^N)$*, or* $u_0 \in L^q(\mathbb{R}^N)$ *and* $v_0 \in W^{1,q}(\mathbb{R}^N)$*, with* $u_0, v_0 \geq 0$*, assume that*

$$|\chi| \cdot \|v_0\|_\infty < \max\Big\{b \cdot C^*_N,\, D^*_{\tau,N}\Big\}, \tag{3.2.2}$$

where

$$C^*_N := \sup_{\gamma>\max\{1,N/2\}} \frac{\gamma}{\gamma-1}\Big(C_{\gamma+1,N}\Big)^{-\frac{1}{\gamma+1}}, \tag{3.2.3}$$

$C_{\gamma+1,N}$ *is given in Lemma 2.3.1, and*

$$D^*_{\tau,N} := \begin{cases} \frac{2}{\tau N^*}\left(2\sqrt{\frac{(\tau^*)^2}{4}+\frac{1}{\tau N^*}}+j|\tau^*|\right)^{-1} & \text{if } \sqrt{\frac{(\tau^*)^2}{4}+\frac{1}{\tau N^*}} > -j|\tau^*|, \\ \frac{2}{\tau N^*}\left(\sqrt{\frac{(\tau^*)^2}{4}+\frac{1}{\tau N^*}}\right)^{-1} & \text{if } \sqrt{\frac{(\tau^*)^2}{4}+\frac{1}{\tau N^*}} \leq -j|\tau^*|. \end{cases} \tag{3.2.4}$$

with $N^* := \max\{1, \frac{N}{2}\}$*,* $\tau^* := \frac{1}{\tau} - 1$*, and* $j := \operatorname{Sign}(\chi\tau^*)$ *denoting the sign of* $\chi\tau^*$*. Then the classical solution* $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *of* (3.0.1) *exists for all* $t > 0$*, and both* $\|u(t,\cdot;u_0,v_0)\|_\infty$ *and* $\|\nabla v(t,\cdot;u_0,v_0)\|_\infty$ *remain bounded as* $t \to \infty$*.*

Theorem 3.2.1 states that a local L^p bound for u implies global existence of the solution. The proof is based on the following two propositions:

1. The first proposition shows that a local L^p bound for u, for some $p > \frac{N}{2}$, yields a local $W^{1,2p}$ bound for v.

2. The second proposition shows, via an interpolation inequality, that a local $W^{1,2p}$ bound for v, again for some $p > \frac{N}{2}$, yields a local L^γ bound for u for some $\gamma > N$. From this

> local L^γ bound, Proposition 2.5.1 gives a uniform bound for ∇v. Finally, combining this uniform gradient bound with a semigroup argument, we obtain the theorem.

On the other hand, Theorem 3.2.2 provides a condition under which the local L^p norm of the solution remains bounded. All local estimates will be derived using the decay function ψ from Lemma 2.4.1.

We now state the two propositions mentioned above, which will be useful in proving Theorem 3.2.1. The first proposition shows that a local L^p bound for u, for some $p > \frac{N}{2}$, yields a local $W^{1,2p}$ bound for v.

Proposition 3.2.1. *Let $(u(t,x), v(t,x)) := (u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ be the solution of (3.0.1). We denote $T_{\max} := T_{\max}(u_0,v_0)$. For given $p \geq 1$, assume that*

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} u^p(t,x)\,dx < \infty \quad \text{and} \quad \sup_{x_0\in\mathbb{R}^N} \int_{B(x_0,1)} |\nabla v_0(x)|^p\,dx < \infty. \tag{3.2.5}$$

Then the following hold.

(1)

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} \Big(v^p(t,x) + |\nabla v(t,x)|^p\Big)\,dx < \infty. \tag{3.2.6}$$

(2) In addition, if $p > \frac{N}{4}$, then

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} v^{2p}(t,x)\,dx < \infty. \tag{3.2.7}$$

(3) In addition, if $p > \frac{N}{2}$ and $\sup_{x_0\in\mathbb{R}^N} \int_{B(x_0,1)} |\nabla v_0(x)|^{2p}\,dx < \infty$, then

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} |\nabla v(t,x)|^{2p}\,dx < \infty. \tag{3.2.8}$$

Proof of Proposition 3.2.1. First of all, we note that, to prove (3.2.6), (3.2.7), and (3.2.8), it suffices to prove

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{\mathbb{R}^N} \Big(v^p(t,x)\psi^p(x-x_0) + |\nabla v|^p(t,x)\psi^p(x-x_0)\Big)\,dx < \infty, \tag{3.2.9}$$

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{\mathbb{R}^N} v^{2p}(t,x)\psi^{2p}(x-x_0)\,dx < \infty, \tag{3.2.10}$$

and

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla v|^{2p}(t,x)\psi^{2p}(x-x_0)\,dx<\infty, \tag{3.2.11}$$

respectively, where ψ is as in Lemma 2.4.1; the proof then follows by estimates similar to those in the proof of Proposition 2.5.1.

First, if (u,v) is the solution of (3.0.1), then by the comparison principle,

$$\|v(t,\cdot)\|_\infty\le\|v_0\|_\infty\quad\forall\, t\in[0,T_{\max}).$$

Hence, for $p\le p'$ and $t\in[0,T_{\max})$,

$$\|v(t,\cdot)\psi\|_{L^{p'}}\le\|v_0\|_\infty\|\psi\|_{L^{p'}}. \tag{3.2.12}$$

Next, we note that

$$\tau\partial_t(v\psi)=\Delta(v\psi)-v\psi+v\psi-2\nabla v\cdot\nabla\psi-v\Delta\psi-\psi uv.$$

Hence

$$v(t,\cdot)\psi=e^{(\Delta-I)\frac{t}{\tau}}v_0\psi+\int_0^t e^{(\Delta-I)\frac{t-s}{\tau}}\Big(v(s,\cdot)\psi-2\nabla v(s,\cdot)\cdot\nabla\psi-v(s,\cdot)\Delta\psi-\psi u(s,\cdot)v(s,\cdot)\Big)\,ds.$$

This gives

$$\begin{aligned}\|(\nabla v(t,\cdot))\psi\|_{L^{p'}}\le{}&\|v(t,\cdot)\nabla\psi\|_{L^{p'}}+\|\nabla e^{(\Delta-I)\frac{t}{\tau}}(v_0\psi)\|_{L^{p'}}\\&+\frac{1}{\tau}\int_0^t\Big\|\nabla e^{(\Delta-I)\frac{t-s}{\tau}}\Big(v\psi-2\nabla v\cdot\nabla\psi-v\Delta\psi-\psi uv\Big)\Big\|_{L^{p'}}\,ds.\end{aligned}$$

Using (2.2.2), (2.2.3), and (2.4.2), we get

$$\begin{aligned}&\|(\nabla v(t,\cdot))\psi\|_{L^{p'}}\le\kappa\|v_0\|_\infty\|\psi\|_{L^{p'}}+C_{p'}e^{-\frac{t}{\tau}}\|\nabla(v_0\psi)\|_{L^{p'}}\\&\quad+2\kappa\frac{C_{p,p'}}{\tau}\sup_{r\in[0,t]}\|(\nabla v(r,\cdot))\psi\|_{L^p}\int_0^t e^{-\frac{t-s}{\tau}}\Big(\frac{t-s}{\tau}\Big)^{-\frac{1}{2}-\left(\frac{1}{p}-\frac{1}{p'}\right)\frac{N}{2}}\,ds\\&\quad+\frac{C_{p,p'}}{\tau}\|v_0\|_\infty\Big((1+\kappa)\|\psi\|_{L^p}+\sup_{r\in[0,t]}\|u(r,\cdot)\psi\|_{L^p}\Big)\int_0^t e^{-\frac{t-s}{\tau}}\Big(\frac{t-s}{\tau}\Big)^{-\frac{1}{2}-\left(\frac{1}{p}-\frac{1}{p'}\right)\frac{N}{2}}\,ds.\end{aligned} \tag{3.2.13}$$

Equation (3.2.7) follows by choosing $p'=p$ in (3.2.13) and (3.2.12), then shifting in space and taking the supremum over $x_0 \in \mathbb{R}^N$. Now, for $p > \frac{N}{2}$ and $p' = 2p$,

$$\int_0^t e^{-\frac{t-s}{\tau}} \Big(\frac{t-s}{\tau}\Big)^{-\frac{1}{2}-\left(\frac{1}{p}-\frac{1}{p'}\right)\frac{N}{2}}\, ds < \infty.$$

Then by the assumption, we get that there are constants $C > 0$ independent of κ and $\tilde{C} > 0$ such that

$$\|\nabla v(t,\cdot)\psi\|_{L^{2p}} \le \tilde{C} + C\kappa \sup_{s\in[0,t]} \|\nabla v(s,\cdot)\psi\|_{L^{2p}} \quad \forall\, t \in [0, T_{\max}). \tag{3.2.14}$$

By choosing $0 < \kappa \ll 1$, it follows that for any $0 \le t < T_{\max}$,

$$\Big(1 - C\kappa\Big) \sup_{t\in[0,T_{\max})} \|\nabla v(t,\cdot)\psi\|_{L^{2p}} \le \tilde{C}.$$

The results then follow after shifting in space and taking the supremum over $x_0 \in \mathbb{R}^N$. $\square$

In the following proposition, we show that the $W^{1,2p}$ estimate of v obtained from the previous proposition can be used to improve the L^p-norm of u.

Proposition 3.2.2. *Let* $(u(t,x), v(t,x)) := (u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *be the solution of* (3.0.1). *We denote* $T_{\max} := T_{\max}(u_0,v_0)$. *Assume that there is* $p > \max\{1, \frac{N}{2}\}$ *such that*

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} u^p(t,x)\,dx < \infty \quad \text{and} \quad \sup_{x_0\in\mathbb{R}^N} \int_{B(x_0,1)} (u_0^{2p}(x) + |\nabla v_0(x)|^{2p})\,dx < \infty. \tag{3.2.15}$$

Then there is $\gamma > N$ *such that*

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N} \int_{B(x_0,1)} u^\gamma(t,x)\,dx < \infty. \tag{3.2.16}$$

Proof of Proposition 3.2.2. First of all, note that if $p > N$, then there is nothing to prove. When $N = 1$, the condition $p > \max\{1, \frac{N}{2}\}$ implies that $p > 1 = N$. Therefore, in the following, we assume that

$$N \ge 2 \quad \text{and} \quad \frac{N}{2} < p \le N.$$

Let ψ be as in Lemma 2.4.1. Then $u_0 \in L^\gamma(B(x_0,1))$ for any $x_0 \in \mathbb{R}^N$ and $\gamma > N$, and to prove (3.2.16), it suffices to prove

$$\sup_{t\in[0,T_{\max}),x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N} u^\gamma(t,x;u_0,v_0)\psi^2(x-x_0)\,dx<\infty \tag{3.2.17}$$

for some $\gamma>N$. If no confusion occurs, for given $x_0\in\mathbb{R}^N$, we put

$$\int_{\mathbb{R}^N} u^\gamma\psi^2\,dx=\int_{\mathbb{R}^N} u^\gamma(t,x;u_0,v_0)\psi^2(x-x_0)dx.$$

Fix $\gamma\in\Big(\max\{N,\frac{2p(p-1)}{N}\},2p\Big)$. Multiplying the u-equation by $u^{\gamma-1}\psi^2$ and integrating both sides gives

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}\int_{\mathbb{R}^N}u^\gamma\psi^2&=\int_{\mathbb{R}^N}u^{\gamma-1}\psi^2u_t\\&=\int_{\mathbb{R}^N}u^{\gamma-1}\psi^2\Delta u-\int_{\mathbb{R}^N}u^{\gamma-1}\psi^2\nabla\cdot(\chi u\nabla v)+a\int_{\mathbb{R}^N}u^\gamma\psi^2-b\int_{\mathbb{R}^N}u^{\gamma+1}\psi^2\\&=\underbrace{-(\gamma-1)\int_{\mathbb{R}^N}u^{\gamma-2}\psi^2|\nabla u|^2}_{I_1}\underbrace{-\int_{\mathbb{R}^N}u^{\gamma-1}\nabla\psi^2\cdot\nabla u}_{I_2}+\underbrace{\chi(\gamma-1)\int_{\mathbb{R}^N}u^{\gamma-1}\psi^2\nabla u\cdot\nabla v}_{I_3}\\&\quad+\underbrace{\chi\int_{\mathbb{R}^N}u^\gamma\nabla v\cdot\nabla\psi^2}_{I_4}+\underbrace{a\int_{\mathbb{R}^N}u^\gamma\psi^2-b\int_{\mathbb{R}^N}u^{\gamma+1}\psi^2}_{I_5}.\end{aligned}$$

Now we estimate each term in the above identity. In the following, C denotes a universal constant independent of κ and t, but it may depend on other parameters such as χ, a, and b. By Young's inequality and (2.4.2) in Lemma 2.4.1, we have the following estimate for $0<\kappa,\delta<1$:

$$\begin{aligned}I_2\le\kappa\int_{\mathbb{R}^N}u^{\gamma-1}|\nabla u|\psi^2&\le\frac{\gamma-1}{4}\int_{\mathbb{R}^N}u^{\gamma-2}|\nabla u|^2\psi^2+C\kappa^2\int_{\mathbb{R}^N}u^\gamma\psi^2\\&\le\frac{\gamma-1}{4}I_1+\delta\int_{\mathbb{R}^N}u^{\gamma+1}\psi^2+C_\delta\kappa^{-N},\end{aligned}\tag{3.2.18}$$

where we used (2.4.4) in the last inequality. Similarly,

$$
\begin{aligned}
I_3 &\le \frac{\gamma-1}{4}\int_{\mathbb{R}^N} u^{\gamma-2}|\nabla u|^2\psi^2 + C\int_{\mathbb{R}^N} u^{\gamma}|\nabla v|^2\psi^2, && (3.2.19)\\
I_4 &\le 2|\chi|\kappa\int_{\mathbb{R}^N} u^{\gamma}|\nabla v|\psi^2\\
&\le C\kappa\int_{\mathbb{R}^N} u^{\gamma}|\nabla v|^2\psi^2 + C\kappa\int_{\mathbb{R}^N} u^{\gamma}\psi^2\\
&\le C\kappa\int_{\mathbb{R}^N} u^{\gamma}|\nabla v|^2\psi^2 + C\kappa\int_{\mathbb{R}^N} u^{\gamma+1}\psi^2 + C\kappa^{-N}. && (3.2.20)
\end{aligned}
$$

Again by Young's inequality and (2.4.4),

$$
I_5 + \int_{\mathbb{R}^N} u^{\gamma}\psi^2 \le -(b-\delta)\int_{\mathbb{R}^N} u^{\gamma+1}\psi^2 + C_\delta\kappa^{-N}. \tag{3.2.21}
$$

Combining (3.2.18)-(3.2.21) gives

$$
\begin{aligned}
\frac{1}{\gamma}\frac{d}{dt}\int_{\mathbb{R}^N} u^{\gamma}\psi^2 + \int_{\mathbb{R}^N} u^{\gamma}\psi^2 \le -\frac{\gamma-1}{2}\int_{\mathbb{R}^N} u^{\gamma-2}|\nabla u|^2\psi^2 - (b-2\delta-C\kappa)\int_{\mathbb{R}^N} u^{\gamma+1}\psi^2\\
+ C\int_{\mathbb{R}^N} u^{\gamma}|\nabla v|^2\psi^2 + C_\delta\kappa^{-N}. \qquad (3.2.22)
\end{aligned}
$$

By Hölder's inequality, we have

$$
\int_{\mathbb{R}^N} u^{\gamma}|\nabla v|^2\psi^2 \le \Big(\int_{\mathbb{R}^N} u^{\frac{\gamma p}{p-1}}\psi^{\frac{\alpha p}{p-1}}\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N} |\nabla v|^{2p}\psi^{\beta p}\Big)^{\frac{1}{p}}, \tag{3.2.23}
$$

for any $\alpha, \beta > 0$ with $\alpha+\beta = 2$. In the following, we estimate $\Big(\int_{\mathbb{R}^N} u^{\frac{\gamma p}{p-1}}\psi^{\frac{\alpha p}{p-1}}\Big)^{\frac{p-1}{p}}$. Note that the collection of balls $\{B_{N/\kappa}(z)\,|\,\kappa z\in\mathbb{Z}^N\}$ covers the whole space. Then using (2.4.3), we have

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^N} u^{\frac{\gamma p}{p-1}}\psi^{\frac{\alpha p}{p-1}}\Big)^{\frac{p-1}{p}} &\lesssim \sum_{\kappa z\in\mathbb{Z}}\Big(\int_{B_{N/\kappa}(z)} u^{\frac{\gamma p}{p-1}}\psi^{\frac{\alpha p}{p-1}}(x)\,dx\Big)^{\frac{p-1}{p}}\\
&\lesssim \sum_{\kappa z\in\mathbb{Z}}\psi^{\alpha}(z)\big(\int_{B_{N/\kappa}(z)} u^{\frac{\gamma p}{p-1}}(x)\,dx\big)^{\frac{p-1}{p}}.
\end{aligned}
$$

Let $R_\kappa := B_{N/\kappa}(z)$. Now we estimate $\Big(\int_{R_\kappa} u^{\frac{\gamma p}{p-1}} \Big)^{\frac{p-1}{p}}$. To this end, let

$$\theta := \frac{N\gamma - Np + N}{N\gamma - Np + 2p}.$$

Note that $\gamma > N \geq p$. Thus, it can be verified directly that

$$\frac{p-1}{2p} = \theta\Big(\frac{1}{2} - \frac{1}{N}\Big) + (1-\theta)\frac{\gamma}{2p} \quad \text{and} \quad \frac{p-1}{p} < \theta < 1.$$

By the Gagliardo–Nirenberg inequality (see [49, Theorem 1]), and a change of variable, there is $C > 0$ independent of κ such that

$$\Big(\int_{R_\kappa} u^{\frac{\gamma p}{p-1}} \, dx \Big)^{\frac{p-1}{p}} = \| u^{\frac{\gamma}{2}} \|^2_{L^{\frac{2p}{p-1}}(R_\kappa)} \leq C \Big(\int_{R_\kappa} \big|\nabla(u^{\gamma/2})\big|^2 \, dx \Big)^{\theta} \Big(\int_{R_\kappa} u^p \Big)^{\frac{(1-\theta)\gamma}{p}} + C \Big(\int_{R_\kappa} u^p \Big)^{\gamma/p}.$$

This together with the assumption (3.2.5) implies that there is $C_2 > 0$ such that

$$\Big(\int_{R_\kappa} u^{\frac{\gamma p}{p-1}} \, dx \Big)^{\frac{p-1}{p}} \leq C_2 \Big(\int_{R_\kappa} \big|\nabla(u^{\gamma/2})\big|^2 \, dx \Big)^{\theta} + C_2.$$

Take $\alpha = \frac{\theta+2}{\theta+1} > 1$. Then $\beta = \frac{\theta}{\theta+1}$ and $\alpha + \beta = 2$. Using that $\psi(x-z) = 1$ when $x \in B_{N/\kappa}(z) = R_\kappa$, the above inequality together with Young's inequality gives

$$\begin{aligned}
\Big(\int_{\mathbb{R}^N} u^{\frac{\gamma p}{p-1}} \psi^{\frac{\alpha p}{p-1}} \Big)^{\frac{p-1}{p}} &\lesssim \sum_{\kappa z \in \mathbb{Z}} \psi^\alpha(z) \Big[\Big(\int_{R_\kappa} |\nabla(u^{\gamma/2})|^2 \psi^2(x-z) \, dx \Big)^{\theta} + C_2 \Big] \\
&\lesssim \delta \sup_{x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} |\nabla(u^{\gamma/2})|^2 \psi^2(x-x_0) \, dx + C_\delta \Big[\sum_{\kappa z \in \mathbb{Z}} \psi^\alpha(z) \Big]^{\frac{1}{1-\theta}} + C_2 \sum_{\kappa z \in \mathbb{Z}} \psi^\alpha(z) \\
&\lesssim \delta \sup_{x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} |\nabla(u^{\gamma/2})|^2 \psi^2(x-x_0) \, dx + C_\delta. \qquad (3.2.24)
\end{aligned}$$

The last inequality holds since $\alpha > 1$ and properties (2.4.2), (2.4.4) in Lemma 2.4.1. Since $u(t,\cdot) \in L^p_{loc}$ for $\frac{N}{2} < p < N$, by Proposition 3.2.1 (3), we can find a constant $C > 0$ such that

$$\Big(\int_{\mathbb{R}^N} |\nabla v|^{2p} \psi^{\beta p} \Big)^{\frac{1}{p}} \leq C.$$

Hence, using this together with (3.2.24) and (3.2.23) gives

$$\begin{aligned}C\int_{\mathbb{R}^N} u^\gamma|\nabla v|^2\psi^2 &\lesssim \Big[\delta \sup_{x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla(u^{\gamma/2})|^2\psi^2(x-x_0)\,dx + C_\delta\Big]\Big(\int_{\mathbb{R}^N}|\nabla v|^{2p}\psi^{\beta p}\Big)^{\frac{1}{p}}\\ &\lesssim \delta \sup_{x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla(u^{\gamma/2})|^2\psi^2(x-x_0)\,dx + C_\delta. \qquad (3.2.25)\end{aligned}$$

Note that

$$\int_{\mathbb{R}^N} u^{\gamma-2}|\nabla u|^2\psi^2 = \frac{4}{\gamma^2}\int_{\mathbb{R}^N}|\nabla(u^{\gamma/2})|^2\psi^2.$$

By (3.2.22), (3.2.23), and (3.2.25), we have

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}\int_{\mathbb{R}^N} u^\gamma\psi^2 + \int_{\mathbb{R}^N} u^\gamma\psi^2 &\lesssim -\frac{2(\gamma-1)}{\gamma^2}\int_{\mathbb{R}^N}\psi^2|\nabla(u^{\frac{\gamma}{2}})|^2 + \delta\sup_{x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla(u^{\gamma/2})|^2\psi^2(x-x_0)\,dx\\ &\quad -(b-2\delta-C\kappa)\int_{\mathbb{R}^N}u^{\gamma+1}\psi^2 + C_\delta\kappa^{-N}. \qquad (3.2.26)\end{aligned}$$

Multiplying both sides by $e^{\gamma t}$ and then integrating in time gives

$$\begin{aligned}\int_{\mathbb{R}^N} u^\gamma\psi^2(x-x_0)\,dx &\lesssim \int_{\mathbb{R}^N}u(0,x)\psi^2(x-x_0)\,dx - \frac{2(\gamma-1)}{\gamma}\iint_{\Omega_t}e^{-\gamma(t-s)}\psi^2(x-x_0)|\nabla(u^{\frac{\gamma}{2}})|^2\,dx\,ds\\ &\quad + \delta\gamma\sup_{x_0\in\mathbb{R}^N}\iint_{\Omega_t}e^{-\gamma(t-s)}|\nabla(u^{\gamma/2})|^2\psi^2(x-x_0)\,dx\,ds + C_\delta\kappa^{-N}\\ &\quad -(b-2\delta-C\kappa)\gamma\iint_{\Omega_t}e^{-\gamma(t-s)}u^{\gamma+1}\psi^2(x-x_0)\,dx\,ds.\end{aligned}$$

Taking the supremum over $x_0\in\mathbb{R}^N$, together with the regularity of the initial data, gives

$$\begin{aligned}\sup_{x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}u^\gamma(t,x)\psi^2(x-x_0)\,dx &\lesssim -\Big(\frac{2(\gamma-1)}{\gamma}-\delta\gamma\Big)\sup_{x_0\in\mathbb{R}^N}\iint_{\Omega_t}e^{-\gamma(t-s)}|\nabla u^{\frac{\gamma}{2}}|^2\psi^2(x-x_0)\\ &\quad -(b-2\delta-C\kappa)\gamma\sup_{x_0\in\mathbb{R}^N}\iint_{\Omega_t}e^{-\gamma(t-s)}u^{\gamma+1}\psi^2(x-x_0) + C_\delta\kappa^{-N}.\end{aligned}$$

Choose $0<\kappa<\delta\ll 1$ such that

$$\frac{2(\gamma-1)}{\gamma^2}-\delta>0 \quad\text{and}\quad b-2\delta-C\kappa>0.$$

Then

$$\sup_{t\in[0,T_{\max}(u_0,v_0)),x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}u^\gamma(t,x;u_0,v_0)\psi^2(x-x_0)\,dx<\infty,$$

that is, (3.2.17) holds, and hence (3.2.16) holds. □

The above proposition together with Proposition 2.5.1 implies that ∇v is uniformly bounded. Now we combines these propositions to establish that Theorem 3.2.1.

Proof of Theorem 3.2.1. By Proposition 3.1.1, we have

$$u(t,\cdot) \in C^b_{\rm unif}(\mathbb{R}^N) \quad \text{and} \quad v(t,\cdot) \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$$

for any $t \in (0, T_{\max})$. Without loss of generality, we assume that $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ and $v_0 \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$.

Next, by (3.2.1), Propositions 3.2.1-3.2.2, and Proposition 2.5.1,

$$\sup_{t\in[0,T_{\max})} \|\nabla v(t,\cdot)\|_\infty < \infty. \tag{3.2.27}$$

Recall that

$$\begin{aligned} u(t,\cdot) = e^{(\Delta-I)t}u_0 - \chi \int_0^t e^{(\Delta-I)(t-s)}\nabla\cdot(u(s,\cdot)\nabla v(s,\cdot))\,ds \\ + \int_0^t e^{(\Delta-I)(t-s)}u(s,\cdot)(1+a-bu(s,\cdot))\,ds \end{aligned} \tag{3.2.28}$$

for $t \in [0, T_{\max})$. Note that there is $M > 0$ such that

$$u(s,x)(1+a-bu(s,x)) \le M \quad \forall\, s\in[0,T_{\max}),\ x\in\mathbb{R}^N.$$

Hence,

$$0 \le u(t,\cdot) \le e^{(\Delta-I)t}u_0 - \chi\int_0^t e^{(\Delta-I)(t-s)}\nabla\cdot(u(s,\cdot)\nabla v(s,\cdot))\,ds + \int_0^t e^{(\Delta-I)(t-s)}M\,ds$$

and then

$$\begin{aligned} \|u(t,\cdot)\|_\infty \le \underbrace{\|e^{(\Delta-I)t}u_0\|_\infty}_{I_0(t)} + \underbrace{|\chi|\int_0^t \|e^{(\Delta-I)(t-s)}\nabla\cdot(u(s,\cdot)\nabla v(s,\cdot))\|_\infty\,ds}_{I_1(t)} \\ + \underbrace{\int_0^t \|e^{(\Delta-I)(t-s)}M\|_\infty\,ds}_{I_2(t)}. \end{aligned} \tag{3.2.29}$$

It is immediate that

$$I_0(t) \le \|u_0\|_\infty, \quad I_2(t) \le M \quad \forall\, t \in [0, T_{\max}).$$

As for $I_1(t)$, by Lemma 2.1.1, we have

$$I_1(t) \le \frac{N|\chi|}{\sqrt{\pi}} \sup_{r \in [0, T_{\max})} \|\nabla v(r,\cdot)\|_\infty \int_0^t e^{-(t-s)}(t-s)^{-\frac{1}{2}} \|u(s,\cdot)\|_\infty\, ds \quad \forall\, t \in [0, T_{\max}).$$

It then follows from (3.2.29) and (3.2.27) that

$$\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty + M + C\int_0^t e^{-(t-s)}(t-s)^{-\frac{1}{2}} \|u(s,\cdot)\|_\infty\, ds \tag{3.2.30}$$

for all $t \in [0, T_{\max})$. This, together with the generalized Gronwall inequality, implies that

$$\|u(t,\cdot)\|_\infty \le C\Big(\|u_0\|_\infty + M\Big)\int_0^t e^{-(t-s)}(t-s)^{-\frac{1}{2}}\, ds \quad \forall\, t \in [0, T_{\max}),$$

which yields

$$\sup_{t \in [0, T_{\max})} \|u(t,\cdot)\|_\infty < \infty$$

and then $T_{\max} = \infty$. The theorem is thus proved. □

We now present the proof of Theorem 3.2.2, which provides a sufficient condition ensuring a local L^p bound for u. By the previous theorem, this in turn implies global existence of the solution. First of all, for given $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ and $v_0 \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$, or $u_0 \in L^p(\mathbb{R}^N)$ and $v_0 \in W^{1,p}(\mathbb{R}^N)$, Proposition 3.1.1 implies that for any $t \in (0, T_{\max}(u_0, v_0))$, $u(t,\cdot; u_0, v_0) \in C^b_{\rm unif}(\mathbb{R}^N)$ and $v(t,\cdot; u_0, v_0) \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$. It therefore suffices to prove the theorem in the case where $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ and $v_0 \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$.

Next, assume that $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ and $v_0 \in C^{1,b}_{\rm unif}(\mathbb{R}^N)$. By Theorem 3.2.2, it suffices to prove that there exists some $p > \max\{1, \frac{N}{2}\}$ such that

$$\sup_{t \in [0, T_{\max}(u_0, v_0)), x_0 \in \mathbb{R}^N} \int_{B(x_0,1)} u^p(t,x; u_0, v_0)\, dx < \infty \tag{3.2.31}$$

provided that

$$|\chi| \cdot \|v_0\|_\infty < b \cdot \sup_{\gamma > \max\{1, N/2\}} \frac{\gamma}{\gamma - 1}\Big(C_{\gamma+1,N}\Big)^{-\frac{1}{\gamma+1}} \tag{3.2.32}$$

or

$$|\chi| \cdot \|v_0\|_\infty < D^*_{\tau,N}. \tag{3.2.33}$$

Let ψ be as in Lemma 2.4.1 and satisfy (2.4.2) for some $\kappa > 0$. To prove (3.2.31), it suffices to prove

$$\sup_{t\in[0,T_{\max}(u_0,v_0)),x_0\in\mathbb{R}^N} \int_{\mathbb{R}^N} u^p(t,x;u_0,v_0)\psi(x-x_0)\,dx < \infty. \tag{3.2.34}$$

If no confusion occurs, for given $x_0 \in \mathbb{R}^N$, we write

$$\int_{\mathbb{R}^N} u^p\psi\,dx = \int_{\mathbb{R}^N} u^p(t,x;u_0,v_0)\psi(x-x_0)\,dx.$$

We prove (3.2.34) in two steps. The first step covers the case where (3.2.32) holds, while the second step deals with the case where (3.2.33) holds.

Step 1: *Proof of Theorem 3.2.2 when* (3.2.32) *holds.* Note that (3.2.32) is equivalent to

$$b > \Big(\inf_{\gamma>\max\{1,\frac{N}{2}\}} \frac{\gamma-1}{\gamma}\big(C_{\gamma+1,N}\big)^{\frac{1}{\gamma+1}}\Big)|\chi|\|v_0\|_\infty.$$

Hence, we assume this and prove (3.2.34).

First, by the u equation and Young's inequality, for any $p>1$ we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} u^p\psi &= -(p-1)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\psi - \int_{\mathbb{R}^N} u^{p-1}\nabla u\cdot\nabla\psi \\
&\quad + \chi(p-1)\int_{\mathbb{R}^N} u^{p-1}\nabla u\cdot(\nabla v)\psi + \chi\int_{\mathbb{R}^N} u^p\nabla v\cdot\nabla\psi + a\int_{\mathbb{R}^N} u^p\psi - b\int_{\mathbb{R}^N} u^{p+1}\psi \\
&\le -(p-1)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\psi + \frac{\kappa}{2}\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\psi + \frac{\kappa}{2}\int_{\mathbb{R}^N} u^p\psi \\
&\quad + \frac{\chi(p-1)}{p}\int_{\mathbb{R}^N}\nabla(u^p)\cdot(\nabla v)\psi + \frac{|\chi|\kappa p}{p+1}\int_{\mathbb{R}^N} u^{p+1}\psi + \frac{|\chi|\kappa}{p+1}\int_{\mathbb{R}^N}|\nabla v|^{p+1}\psi \\
&\quad + a\int_{\mathbb{R}^N} u^p\psi - b\int_{\mathbb{R}^N} u^{p+1}\psi \\
&= -\Big(p-1-\frac{\kappa}{2}\Big)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\psi - \frac{\chi(p-1)}{p}\int_{\mathbb{R}^N} u^p(\Delta v)\psi - \frac{\chi(p-1)}{p}\int_{\mathbb{R}^N} u^p\nabla v\cdot\nabla\psi \\
&\quad + \frac{|\chi|\kappa}{p+1}\int_{\mathbb{R}^N}|\nabla v|^{p+1}\psi + \Big(a+\frac{\kappa}{2}\Big)\int_{\mathbb{R}^N} u^p\psi - \Big(b-\frac{|\chi|\kappa p}{p+1}\Big)\int_{\mathbb{R}^N} u^{p+1}\psi \\
&\le -\Big(p-1-\frac{\kappa}{2}\Big)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\psi + \frac{|\chi|(p-1)}{p}\int_{\mathbb{R}^N} u^p|\Delta v|\psi - \frac{p+1}{\tau p}\int_{\mathbb{R}^N} u^p\psi \\
&\quad + \Big(\frac{|\chi|\kappa}{p+1} + \frac{|\chi|\kappa(p-1)}{p(p+1)}\Big)\int_{\mathbb{R}^N}|\nabla v|^{p+1}\psi + \Big(a+\frac{\kappa}{2}+\frac{p+1}{\tau p}\Big)\int_{\mathbb{R}^N} u^p\psi \\
&\quad - \Big(b - \frac{|\chi|\kappa p}{p+1} - \frac{|\chi|\kappa(p-1)}{p+1}\Big)\int_{\mathbb{R}^N} u^{p+1}\psi.
\end{aligned}$$

Let $r>0$ be determined later. By Young's inequality again, we have

$$\frac{|\chi|(p-1)}{p}\int_{\mathbb{R}^N} u^p|\Delta v|\psi \le r\int_{\mathbb{R}^N} u^{p+1}\psi + \underbrace{\frac{1}{p}\Big(\frac{p-1}{p+1}\Big)^{p+1}}_{A_p} r^{-p}|\chi|^{p+1}\int_{\mathbb{R}^N}|\Delta v|^{p+1}\psi,$$

and for any $\delta>0$,

$$\int_{\mathbb{R}^N}\Big[a+\frac{\kappa}{2}+\frac{p+1}{\tau p}\Big]u^p\psi \le \delta\int_{\mathbb{R}^N} u^{p+1}\psi + \underbrace{\frac{1}{p+1}\Big[\frac{p+1}{p}\delta\Big]^{-p}\Big[a+\frac{\kappa}{2}+\frac{p+1}{\tau p}\Big]^{p+1}\int_{\mathbb{R}^N}\psi}_{C_\delta\kappa^{-N}}.$$

We then have

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} u^p\psi + \frac{p+1}{\tau p}\int_{\mathbb{R}^N} u^p\psi \le -\Big(p-1-\frac{\kappa}{2}\Big)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\psi + A_p r^{-p}|\chi|^{p+1}\int_{\mathbb{R}^N}|\Delta v|^{p+1}\psi \\
+\left(\frac{|\chi|\kappa}{p+1}+\frac{|\chi|\kappa(p-1)}{p(p+1)}\right)\int_{\mathbb{R}^N}|\nabla v|^{p+1}\psi \\
-\Big(b-\frac{|\chi|\kappa p}{p+1}-\frac{|\chi|\kappa(p-1)}{p+1}-r-\delta\Big)\int_{\mathbb{R}^N} u^{p+1}\psi + C_\delta\kappa^{-N}.
\end{aligned}
$$

Here the term $\frac{p+1}{\tau p}\int_{\mathbb{R}^N} u^p\psi\,dx$ on the left-hand side will be used to obtain uniform-in-time bounds. This implies that for any $0 < t_0 < t < T_{\max}$,

$$
\int_{\mathbb{R}^N} u^p(t,\cdot)\psi\,dx \le e^{\frac{(p+1)(t_0-t)}{\tau}}\int_{\mathbb{R}^N} u^p(t_0,\cdot)\psi\,dx + p\int_{t_0}^t e^{\frac{(p+1)(s-t)}{\tau}} f(s)\,ds, \tag{3.2.35}
$$

where

$$
\begin{aligned}
f(s) := A_p r^{-p}|\chi|^{p+1}\int_{\mathbb{R}^N}|\Delta v(s,\cdot)|^{p+1}\psi\,dx + \left(\frac{|\chi|\kappa}{p+1}+\frac{|\chi|\kappa(p-1)}{p(p+1)}\right)\int_{\mathbb{R}^N}|\nabla v(s,\cdot)|^{p+1}\psi\,dx \\
-\left(b-\frac{|\chi|\kappa p}{p+1}-\frac{|\chi|\kappa(p-1)}{p+1}-r-\delta\right)\int_{\mathbb{R}^N} u^{p+1}(s,\cdot)\psi\,dx + C_\delta\kappa^{-N}.
\end{aligned} \tag{3.2.36}
$$

Let $\psi_1 := \psi^{\frac{1}{p+1}}$. Then $v\psi_1$ solves

$$
\tau(v\psi_1)_t = \Delta(v\psi_1) - v\psi_1 + (v\psi_1 - 2\nabla v\cdot\nabla\psi_1 + v\Delta\psi_1 - uv\psi_1)\,.
$$

By Lemma 2.3.1 with $\gamma := p+1$, we have

$$
\begin{aligned}
&\int_{t_0}^t e^{\frac{(p+1)s}{\tau}}\int_{\mathbb{R}^N}\Big((v(s,x)\psi_1^{p+1} + |\nabla(v(s,x)\psi_1|^{p+1} + |\Delta(v(s,x)\psi_1|^{p+1}\Big)\,dx\,ds \\
&\quad\le C_{p+1,N}\int_{t_0}^t\int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}}\Big(v\psi_1 - 2\nabla v\cdot\nabla\psi_1 + v\Delta\psi_1 - uv\psi_1\Big)^{p+1}(s,x)\,dx\,ds \\
&\quad+ C_{p+1,N}(t+\tau^{p+1}t_0^{-p})\|v_0(\cdot)\psi_1\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}.
\end{aligned} \tag{3.2.37}
$$

Note that

$$
\begin{aligned}
\int_{\mathbb{R}^N}(|\nabla v|\psi_1)^{p+1} &\le \int_{\mathbb{R}^N}\Big(|\nabla(v\psi_1)| + v|\nabla\psi_1|\Big)^{p+1} \\
&\le \int_{\mathbb{R}^N}\Big(|\nabla(v\psi_1)| + \tfrac{\kappa}{p+1}v\psi_1\Big)^{p+1},
\end{aligned}
$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta v\psi_1|^{p+1} &\le \int_{\mathbb{R}^N} \Big(|\Delta(v\psi_1)| + 2|\nabla v||\nabla\psi_1| + v|\Delta\psi_1|\Big)^{p+1} \\ &\le \int_{\mathbb{R}^N} \Big[|\Delta(v\psi_1)| + \frac{2\kappa}{p+1}|\nabla v|\psi_1 + \Big(\frac{p\kappa^2}{(p+1)^2} + \frac{\kappa}{p+1}\Big)v\psi_1\Big]^{p+1}.\end{aligned}$$

Then for any $\delta > 0$, by choosing $0 < \kappa \ll 1$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} (|\nabla v|\psi_1)^{p+1} \le (1+\delta)\int_{\mathbb{R}^N} |\nabla(v\psi_1)|^{p+1} + \frac{\delta}{2}\int_{\mathbb{R}^N} (v\psi_1)^{p+1}, \\ &\int_{\mathbb{R}^N} (|\Delta v|\psi_1)^{p+1} \le (1+\delta)\int_{\mathbb{R}^N} |\Delta(v\psi_1)|^{p+1} + \delta\int_{\mathbb{R}^N} (|\nabla v|\psi_1)^{p+1} + \frac{\delta}{2}\int_{\mathbb{R}^N} (v\psi_1)^{p+1}, \quad (3.2.38)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(v\psi_1 - 2\nabla v\cdot\nabla\psi_1 + v\Delta\psi_1 - uv\psi_1\Big)^{p+1}(s,x)\,dx \\ &\quad\le (1+\delta)\int_{\mathbb{R}^N} (v\psi_1)^{p+1} + \delta\int_{\mathbb{R}^N} (|\nabla v|\psi_1)^{p+1} + (\|v_0\|_\infty^{p+1} + \delta)\int_{\mathbb{R}^N} (u\psi_1)^{p+1}. \quad (3.2.39)\end{aligned}$$

Using (3.2.38) yields

$$\begin{aligned}&\int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}}\Big((v\psi_1)^{p+1} + (|\nabla v|\psi_1)^{p+1} + (|\Delta v|\psi_1)^{p+1}\Big)(s,x)\,dx\,ds \\ &\quad\le (1+\delta)^2\int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}}\Big((v\psi_1)^{p+1} + |\nabla(v\psi_1)|^{p+1} + |\Delta(v\psi_1)|^{p+1}\Big)(s,x)\,dx\,ds.\end{aligned}$$

By (3.2.37) and (3.2.39), the above is

$$\begin{aligned}&\le (1+\delta)^2 C_{p+1,N}\int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}}\Big(v\psi_1 - 2\nabla v\cdot\nabla\psi_1 + v\Delta\psi_1 - uv\psi_1\Big)^{p+1}(s,x)\,dx\,ds \\ &\qquad + (1+\delta)^2 C_{p+1,N}(t + \tau^{p+1}t_0^{-p})\|v_0(\cdot)\psi_1\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\ &\le (1+\delta)^2 C_{p+1,N}\int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}}\left((1+\delta)(v\psi_1)^{p+1} + \delta(|\nabla v|\psi_1)^{p+1} + (\|v_0\|_\infty^{p+1} + \delta)(u\psi_1)^{p+1}\right)dx\,ds \\ &\qquad + (1+\delta)^2 C_{p+1,N}(t + \tau^{p+1}t_0^{-p})\|v_0(\cdot)\psi_1\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}.\end{aligned}$$

Recall that $\psi_1^{p+1} = \psi$. This implies that

$$\begin{aligned}
&\int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} \left(|\nabla v|^{p+1}\psi + |\Delta v|^{p+1}\psi \right) dx\, ds \\
&\quad \leq (1+\delta)^2 C_{p+1,N} \int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} \Big(\delta |\nabla v|^{p+1}\psi + (\|v_0\|_\infty^{p+1} + \delta) u^{p+1}\psi \Big) dx\, ds + C_1^*(t),
\end{aligned}$$

where

$$\begin{aligned}
C_1^*(t) :=& (1+\delta)^3 C_{p+1,N} \|v_0\|_\infty^{p+1} \int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} \psi\, dx\, ds \\
&+ (1+\delta)^2 C_{p+1,N} (t + \tau^{p+1} t_0^{-p}) \|v_0(\cdot)\psi_1\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.
\end{aligned} \tag{3.2.40}$$

Now by taking δ sufficiently small so that $(1+\delta)^2 \delta C_{p+1,N} \leq \frac{1}{2}$, we obtain

$$\int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} |\nabla v|^{p+1}\psi\, dx\, ds \leq 2C_2^* \int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} u^{p+1}\psi\, dx\, ds + 2C_1^*(t),$$

where

$$C_2^* := (1+\delta)^2 C_{p+1,N} (\|v_0\|_\infty^{p+1} + \delta).$$

We also get

$$\int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} |\Delta v|^{p+1}\psi\, dx\, ds \leq C_2^* \int_{t_0}^{t} \int_{\mathbb{R}^N} e^{\frac{(p+1)s}{\tau}} u^{p+1}\psi\, dx\, ds + C_1^*(t). \tag{3.2.41}$$

Choose $0 < \kappa \ll 1$ such that

$$\frac{|\chi|\kappa}{p+1} + \frac{|\chi|\kappa(p-1)}{p(p+1)} < \delta \quad \text{and} \quad \frac{|\chi|\kappa p}{p+1} + \frac{|\chi|\kappa(p-1)}{p+1} < \delta.$$

Then it follows from (3.2.35), (3.2.36), and (3.2.41) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} u^p(t,x)\psi\,dx \\
&\le e^{\frac{(p+1)(t_0-t)}{\tau}} \int_{\mathbb{R}^N} u^p(t_0,x)\psi\,dx + pA_p r^{-p}|\chi|^{p+1} \int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)(s-t)}{\tau}} |\Delta v|^{p+1}\psi\,dx\,ds \\
&\quad + p\delta \int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)(s-t)}{\tau}} |\nabla v|^{p+1}\psi\,dx\,ds - p\Big(b-r-2\delta\Big) \int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)(s-t)}{\tau}} u^{p+1}\psi\,dx\,ds \\
&\quad + C_\delta \kappa^{-N} p \int_{t_0}^t e^{\frac{(p+1)(s-t)}{\tau}}\,ds \\
&\le \int_{\mathbb{R}^N} u^p(t_0,\cdot)\psi\,dx + \left(pA_p r^{-p}|\chi|^{p+1} + 2p\delta\right) \left[C_2^* \int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)(s-t)}{\tau}} u^{p+1}\psi\,dx\,ds + e^{-\frac{(p+1)t}{\tau}} C_1^*(t) \right] \\
&\quad - p\Big(b-r-2\delta\Big) \int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)(s-t)}{\tau}} u^{p+1}\psi\,dx\,ds + \frac{C_\delta \kappa^{-N} p\tau}{p+1} \\
&\le -p\Big(b-r-4\delta - A_p r^{-p}|\chi|^{p+1} C_2^*\Big) \int_{t_0}^t \int_{\mathbb{R}^N} e^{\frac{(p+1)(s-t)}{\tau}} u^{p+1}\psi\,dx\,ds + C_3^*(t),
\end{aligned} \tag{3.2.42}$$

where

$$C_3^*(t) := \frac{C_\delta \kappa^{-N} p\tau}{p+1} + \left(pA_p r^{-p}|\chi|^{p+1} + 2p\delta\right) e^{-\frac{(p+1)t}{\tau}} C_1^*(t) + \left\| u(t_0,\cdot)\psi^{1/p} \right\|_{L^p}^p .$$

A direct computation yields $C_1^*(t) \le Ce^{\frac{(p+1)t}{\tau}} + C(t + \tau^{p+1} t_0^{-p})$ by (3.2.40), for some $C > 0$. It is then clear that $C_3^*(t)$ is uniformly bounded for all $t \ge t_0$, depending on $r, t_0, p, \tau, \chi, \delta, u(t_0)$, and $\|v_0\|_\infty$.

Now note that

$$\left| C_2^* - C_{p+1,N} \|v_0\|_\infty^{p+1} \right| = O(\delta),$$

and

$$\min_{r>0} \left(A_p r^{-p} |\chi|^{p+1} C_{p+1,N} \|v_0\|_\infty^{p+1} + r \right) = \left(1 + \frac{1}{p}\right) A_p^{\frac{1}{p+1}} \left(p\, C_{p+1,N}\right)^{\frac{1}{p+1}} |\chi|\, \|v_0\|_\infty .$$

So, in view of the condition

$$b > \left[\inf_{p > \max\{1, \frac{N}{2}\}} \left(1 + \frac{1}{p}\right) A_p^{\frac{1}{p+1}} \left(p\, C_{p+1,N}\right)^{\frac{1}{p+1}} \right] |\chi| \|v_0\|_\infty ,$$

there are $r > 0$ and $p > \max\{1, \frac{N}{2}\}$ such that

$$b > r + A_p r^{-p} |\chi|^{p+1} C_{p+1,N} \|v_0\|_\infty^{p+1}.$$

Fix such $r > 0$ and $p > \max\{1, \frac{N}{2}\}$. Then by selecting $\delta \ll 1$, we obtain

$$b - r - 4\delta - A_p r^{-p} |\chi|^{p+1} C_2^* \geq 0.$$

Then by (3.2.42),

$$\sup_{t \in [t_0, T_{\max}), x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} u^p(t,x) \psi(x - x_0)\, dx < \infty.$$

This implies that (3.2.34) holds provided that (3.2.32) holds.

Step 2. *In this step, we assume that* $|\chi| \cdot \|v_0\|_\infty < D^*_{\tau,N}$ *and prove* (3.2.34).

To this end, define $\varphi(s) := e^{\sigma s^2}$ for $0 \leq s \leq \|v_0\|_\infty$, where $\sigma > 0$ is to be determined later. Recall that $N^* = \max\{1, \frac{N}{2}\}$, $\tau^* = \frac{1}{\tau} - 1$, and $j = \mathrm{Sign}(\chi \tau^*)$. Due to the condition, we can choose a $p > \max\{1, \frac{N}{2}\}$ such that

$$|\chi| \cdot \|v_0\|_\infty < \begin{cases} \frac{2}{\tau p (2\alpha + j|\tau^*|)} & \text{if} \quad \alpha > -j|\tau^*|, \\ \\ \frac{2}{\tau p \alpha} & \text{if} \quad \alpha \leq -j|\tau^*|, \end{cases} \tag{3.2.43}$$

where $\alpha := \sqrt{\frac{(\tau^*)^2}{4} + \frac{1}{\tau p}}$.

Also let ψ be as in Lemma 2.4.1 and satisfy (2.4.2) for some $\kappa > 0$. To prove (3.2.34), it suffices to prove

$$\sup_{t \in [0, T_{\max}(u_0, v_0)), x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} u^p(t, x; u_0, v_0) \varphi(v(t, x; u_0, v_0)) \psi(x - x_0)\, dx < \infty. \tag{3.2.44}$$

We now prove (3.2.44). If no confusion occurs, for given $x_0 \in \mathbb{R}^N$, we write

$$\int_{\mathbb{R}^N} u^p \varphi(v) \psi\, dx = \int_{\mathbb{R}^N} u^p(t, x; u_0, v_0) \varphi(v(t, x; u_0, v_0)) \psi(x - x_0)\, dx.$$

Observe that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} u^p\varphi(v)\psi\\
&=\int_{\mathbb{R}^N} u_t u^{p-1}\varphi(v)\psi+\frac{1}{p}\int_{\mathbb{R}^N} u^p\varphi'(v)\psi v_t\\
&=\int_{\mathbb{R}^N}(\Delta u-\chi\nabla\cdot(u\nabla v)+au-bu^2)u^{p-1}\varphi(v)\psi+\frac{1}{\tau p}\int_{\mathbb{R}^N} u^p\varphi'(v)(\Delta v-uv)\psi\\
&=-(p-1)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\varphi(v)\psi-\int_{\mathbb{R}^N}\varphi'(v)u^{p-1}\nabla u\cdot(\nabla v)\psi-\int_{\mathbb{R}^N} u^{p-1}\nabla u\cdot(\nabla\psi)\varphi(v)\\
&\quad+\chi(p-1)\int_{\mathbb{R}^N} u^{p-1}\varphi(v)\nabla u\cdot(\nabla v)\psi+\chi\int_{\mathbb{R}^N} u^p|\nabla v|^2\varphi'(v)\psi+\chi\int_{\mathbb{R}^n} u^p\varphi(v)\nabla v\cdot\nabla\psi\\
&\quad+\int_{\mathbb{R}^N}(au^p-bu^{p+1})\varphi(v)\psi-\frac{1}{\tau p}\int_{\mathbb{R}^N} u^p\varphi''(v)|\nabla v|^2\psi-\frac{1}{\tau}\int_{\mathbb{R}^N} u^{p-1}\varphi'(v)\nabla u\cdot(\nabla v)\psi\\
&\quad-\frac{1}{\tau p}\int_{\mathbb{R}^N} u^p\varphi'(v)\nabla v\cdot\nabla\psi-\frac{1}{\tau p}\int_{\mathbb{R}^N} u^{p+1}v\varphi'(v)\psi.
\end{aligned}
$$

Since $v\ge 0$, $\varphi'(s)=2\sigma s\varphi(s)\ge 0$ for all $s\ge 0$, and $\psi>0$, we have

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} u^p\varphi(v)\psi\\
&\le-(p-1)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\varphi(v)\psi-\int_{\mathbb{R}^N}\Big(2\sigma v+\frac{2\sigma}{\tau}v-\chi(p-1)\Big)\varphi(v)u^{p-1}\nabla u\cdot(\nabla v)\psi\\
&\quad+\chi\int_{\mathbb{R}^N} u^p|\nabla v|^2\varphi'(v)\psi+\int_{\mathbb{R}^N}(au^p-bu^{p+1})\varphi(v)\psi-\frac{1}{\tau p}\int_{\mathbb{R}^N} u^p\varphi''(v)|\nabla v|^2\psi\\
&\quad-\int_{\mathbb{R}^N} u^{p-1}\varphi(v)\nabla u\cdot\nabla\psi+\int_{\mathbb{R}^N}\Big(\chi-\frac{2\sigma v}{\tau p}\Big)u^p\varphi(v)\nabla v\cdot\nabla\psi.
\end{aligned}
$$

By Young's inequality, we have for any $\delta,\delta'\in(0,1)$,

$$
\begin{aligned}
&-\int_{\mathbb{R}^N}\Big(2\sigma v+\frac{2\sigma}{\tau}v-\chi(p-1)\Big)\varphi(v)u^{p-1}\nabla u\cdot(\nabla v)\psi\\
&\qquad\le(1-\delta)(p-1)\int_{\mathbb{R}^N} u^{p-2}\varphi(v)|\nabla u|^2\psi+\int_{\mathbb{R}^N}\frac{\Big(2\sigma v+\frac{2\sigma}{\tau}v-\chi(p-1)\Big)^2}{4(p-1)(1-\delta)}u^p\varphi(v)|\nabla v|^2\psi,
\end{aligned}
$$

$$
-\int_{\mathbb{R}^N} u^{p-1}\varphi(v)\nabla u\cdot\nabla\psi\le\delta(p-1)\int_{\mathbb{R}^N} u^{p-2}|\nabla u|^2\varphi(v)\psi+\frac{\kappa^2}{4\delta(p-1)}\int_{\mathbb{R}^N} u^p\varphi(v)\psi,
$$

and

$$\int_{\mathbb{R}^N}\Big(\chi-\frac{2\sigma v}{\tau p}\Big)u^p\varphi(v)\nabla v\cdot\nabla\psi\le\delta'\int_{\mathbb{R}^N}u^p|\nabla v|^2\varphi(v)\psi+\frac{\kappa^2}{4\delta'}\int_{\mathbb{R}^N}\Big(\chi-\frac{2\sigma v}{\tau p}\Big)^2u^p\varphi(v)\psi.$$

Then we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N}u^p\varphi(v)\psi\\
&\le\int_{\mathbb{R}^N}\Big[\frac{\Big(2\sigma v+\frac{2\sigma}{\tau}v-\chi(p-1)\Big)^2}{4(p-1)(1-\delta)}+\delta'\Big]u^p\varphi(v)|\nabla v|^2\psi\\
&\quad+\chi\int_{\mathbb{R}^N}u^p|\nabla v|^2\varphi'(v)\psi-\frac{1}{\tau p}\int_{\mathbb{R}^N}u^p\varphi''(v)|\nabla v|^2\psi+\int_{\mathbb{R}^N}(au^p-bu^{p+1})\varphi(v)\psi\\
&\quad+\Big[\frac{\kappa^2}{4\delta(p-1)}+\Big(\chi-\frac{2\sigma v}{\tau p}\Big)^2\frac{\kappa^2}{4\delta'}\Big]\int_{\mathbb{R}^N}u^p\varphi(v)\psi\\
&=\int_{\mathbb{R}^N}\Big[\frac{\Big(2\sigma v+\frac{2\sigma}{\tau}v-\chi(p-1)\Big)^2}{4(p-1)(1-\delta)}+\delta'+2\sigma\chi v-\frac{1}{\tau p}(2\sigma+4\sigma^2v^2)\Big]u^p\varphi(v)|\nabla v|^2\psi\\
&\quad+\int_{\mathbb{R}^N}(au^p-bu^{p+1})\varphi(v)\psi+\int_{\mathbb{R}^N}\Big[\frac{\kappa^2}{4\delta(p-1)}+\Big(\chi-\frac{2\sigma v}{\tau p}\Big)^2\frac{\kappa^2}{4\delta'}\Big]u^p\varphi(v)\psi. \qquad (3.2.45)
\end{aligned}$$

We claim that, if (3.2.43) holds, then with σ defined in (3.2.48) and (3.2.51) below, we have for all $v\in[0,\|v_0\|_\infty]$,

$$\frac{\Big(2\sigma v+\frac{2\sigma}{\tau}v-\chi(p-1)\Big)^2}{4(p-1)}+2\sigma\chi v-\frac{1}{\tau p}(2\sigma+4\sigma^2v^2)<0. \qquad (3.2.46)$$

Indeed, (3.2.46) is equivalent to

$$A\sigma^2v^2+B<C\sigma+D\sigma v, \qquad (3.2.47)$$

where

$$A:=\frac{(\tau^*)^2}{p-1}+\frac{4}{\tau p(p-1)}>0,\quad B:=\frac{\chi^2(p-1)}{4}>0,\quad C:=\frac{2}{\tau p}>0,\quad D:=-\chi\tau^*.$$

First, assume $D<\sqrt{AB}$. In this case we take

$$\sigma:=\sqrt{B}(2\sqrt{AB}-D)/(C\sqrt{A}). \qquad (3.2.48)$$

In order to have (3.2.47) valid, since it is a convex quadratic form of v, it suffices to have (3.2.47) for $v = \|v_0\|_\infty$ and for $v = 0$. When $v = 0$, it suffices to have

$$B \leq \sqrt{B}(2\sqrt{AB} - D)/\sqrt{A} \quad \text{which is the same as} \quad 2\sqrt{AB} > D,$$

which is certainly true since $D \leq \sqrt{AB}$. When $v = \|v_0\|_\infty$, (3.2.47) becomes

$$\frac{\sqrt{AB}(2\sqrt{AB} - D)}{C}\|v_0\|_\infty^2 - D\|v_0\|_\infty + \frac{C(-\sqrt{AB} + D)}{2\sqrt{AB} - D} < 0. \tag{3.2.49}$$

Note that $AB = |\chi|^2\alpha^2$ and $D = -j|\chi||\tau^*|$ with $j = \text{Sign}(\chi\tau^*)$. Thus (3.2.49) reduces to

$$|\chi|\|v_0\|_\infty < \frac{1}{\tau p}\left(\frac{-j|\tau^*| + \sqrt{|\tau^*|^2 + 4\alpha^2 + 4j\alpha|\tau^*|}}{2\alpha^2 + \alpha j|\tau^*|}\right) = \frac{2}{\tau p(2\alpha + j|\tau^*|)}. \tag{3.2.50}$$

Next, assume that $D \geq \sqrt{AB}$, take $\delta'' > 0$, and

$$\sigma := B/C + \delta'' > \sqrt{B}(2\sqrt{AB} - D)/(C\sqrt{A}). \tag{3.2.51}$$

Therefore (3.2.47) holds when $v = 0$. When $v = \|v_0\|_\infty$, since $D \geq \sqrt{AB}$, after taking δ'' sufficiently small,

$$A\sigma^2\|v_0\|_\infty^2 + B < D\sigma\|v_0\|_\infty + C\sigma \Longleftarrow \|v_0\|_\infty < \frac{C}{\sqrt{AB}} \Longleftrightarrow |\chi|\|v_0\|_\infty < \frac{2}{\tau p\alpha}.$$

This and (3.2.50) reduce to (3.2.43). Overall, we have proved the claim (3.2.46).

Now by choosing $0 < \delta, \delta' \ll 1$ and using (3.2.45), we then have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} u^p\varphi(v)\psi &\leq \int_{\mathbb{R}^N}(au^p - bu^{p+1})\varphi(v)\psi + \int_{\mathbb{R}^N}\left(\frac{\kappa^2}{4\delta(p-1)} + \left(\chi - \frac{2\sigma v}{\tau p}\right)^2\frac{\kappa^2}{4\delta'}\right)u^p\varphi(v)\psi\\ &\leq M\int_{\mathbb{R}^N} u^p\varphi(v)\psi - b\int_{\mathbb{R}^N} u^{p+1}\varphi(v)\psi,\end{aligned} \tag{3.2.52}$$

where

$$M = \max_{0\leq v\leq\|v_0\|_\infty}\left(a + \frac{\kappa^2}{4\delta(p-1)} + \left(\chi - \frac{2\sigma v}{\tau p}\right)^2\frac{\kappa^2}{4\delta'}\right).$$

Note that

$$\begin{aligned}\int_{\mathbb{R}^N} u^p\varphi(v)\psi &= \int_{\mathbb{R}^N} u^p\varphi^{\frac{p}{p+1}}(v)\psi^{\frac{p}{p+1}}\cdot\varphi^{\frac{1}{p+1}}(v)\psi^{\frac{1}{p+1}}\\ &\le \Big(\int_{\mathbb{R}^N} u^{p+1}\varphi(v)\psi\Big)^{\frac{p}{p+1}}\Big(\int_{\mathbb{R}^N}\varphi(v)\psi\Big)^{\frac{1}{p+1}}.\end{aligned}$$

This and $\varphi(v)\le e^{\sigma\|v_0\|_\infty^2}$, with σ defined in (3.2.48) and (3.2.51), imply that

$$-b\int_{\mathbb{R}^N} u^{p+1}\varphi(v)\psi \le -b\Big(\int_{\mathbb{R}^N}\varphi(v)\psi\Big)^{-\frac{1}{p}}\Big(\int_{\mathbb{R}^N} u^p\varphi(v)\psi\Big)^{\frac{p+1}{p}} \le -bK_p\Big(\int_{\mathbb{R}^N} u^p\varphi(v)\psi\Big)^{\frac{p+1}{p}},$$

where

$$K_p := \Big(e^{\sigma\|v_0\|_\infty^2}\int_{\mathbb{R}^N}\psi\Big)^{-\frac{1}{p}}.$$

This together with (3.2.52) implies that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} u^p\varphi(v)\psi \le M\int_{\mathbb{R}^N} u^p\varphi(v)\psi - bK_p\Big(\int_{\mathbb{R}^N} u^p\varphi(v)\psi\Big)^{\frac{p+1}{p}}.$$

By the comparison principle for ODEs, we have

$$\int_{\mathbb{R}^N} u^p\psi \le \int_{\mathbb{R}^N} u^p\varphi(v)\psi \le \max\Big[\int_{\mathbb{R}^N} u_0^p\varphi(v_0)\psi, \Big(\frac{M}{bK_p}\Big)^p\Big].$$

Note that M and K_p are independent of $x_0\in\mathbb{R}^N$. Hence (3.2.44) holds, and the theorem is thus proved. □

Remark 3.2.1. *It follows from Theorem 3.2.2 that the classical solution* $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *to* (3.0.1) *exists globally when* $|\chi|$ *is sufficiently small. Moreover, from its proof, we have that* $\sup_{t\ge 0}\|u(t,\cdot;u_0,v_0)\|_\infty$ *stays uniformly finite for all small* $|\chi|$. *Due to this, it follows that the constant* C *in* (2.5.14) *is uniform for all* $|\chi|$ *sufficiently small as well.*

3.3 Asymptotic Behavior of Solution

This section is devoted to the study of the asymptotic behavior of globally defined bounded classical solutions of (3.0.1) for different kinds of initial data, namely, strictly positive initial data and compactly supported initial data. First, we introduce the following definitions.

Let

$$X := C^b_{\rm unif}(\mathbb{R}^N) \quad\text{and}\quad X_1 := C^{1,b}_{\rm unif}(\mathbb{R}^N).$$

We also set

$$X^+ := \{u \in X \,|\, u \geq 0\}, \quad X_c^+ := \big\{u \in X^+ \,|\, \mathrm{supp}(u) \neq \emptyset,\ \mathrm{supp}(u) \text{ is compact}\big\}.$$

We write $X_1^+ := X_1 \cap X^+$. Finally, we use $C_{\mathrm{unif}}^{2+\alpha,b}(\mathbb{R}^N)$ to denote all $u \in X_1$ such that $\frac{\partial^2 u}{\partial x_i \partial x_j}$ are uniformly bounded and α-Hölder continuous for $i, j = 1, 2, \dots, N$, and use the notation

$$C_0(\mathbb{R}^N) := \left\{ u \in C(\mathbb{R}^N) \,\Big|\, \lim_{|x|\to\infty} u(x) = 0 \right\}.$$

For given $(u_0, v_0) \in X^+ \times X_1^+$, assuming that $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ exists for all $t > 0$, let

$$S_{\mathrm{low}}(u_0,v_0) := \{c \,|\, c > 0,\ \liminf_{t\to\infty} \inf_{|x|\leq ct} u(t,x;u_0,v_0) > 0\}$$

and

$$S_{\mathrm{up}}(u_0,v_0) := \{c \,|\, c > 0,\ \limsup_{t\to\infty} \sup_{|x|\geq ct} u(t,x;u_0,v_0) = 0\}.$$

We define

$$c_{\mathrm{low}}^*(u_0,v_0) := \sup\{c \,|\, c \in S_{\mathrm{low}}(u_0,v_0)\}$$

and

$$c_{\mathrm{up}}^*(u_0,v_0) := \inf\{c \,|\, c \in S_{\mathrm{up}}(u_0,v_0)\},$$

where $c_{\mathrm{low}}^*(u_0,v_0) = 0$ if $S_{\mathrm{low}}(u_0,v_0) = \emptyset$ and $c_{\mathrm{up}}^*(u_0,v_0) = \infty$ if $S_{\mathrm{up}}(u_0,v_0) = \emptyset$. By the definition of $c_{\mathrm{low}}^*(u_0,v_0)$ and $c_{\mathrm{up}}^*(u_0,v_0)$, if $c_{\mathrm{low}}^*(u_0,v_0) > 0$, then

$$\liminf_{t\to\infty} \inf_{|x|\leq c't} u(t,x;u_0,v_0) > 0 \quad \forall\, 0 < c' < c_{\mathrm{low}}^*(u_0,v_0),$$

and if $c_{\mathrm{up}}^*(u_0,v_0) < \infty$, then

$$\lim_{t\to\infty} \sup_{|x|\geq c''t} u(t,x;u_0,v_0) = 0 \quad \forall\, c'' > c_{\mathrm{up}}^*(u_0,v_0).$$

If $c_{\mathrm{low}}^*(u_0,v_0) = c_{\mathrm{up}}^*(u_0,v_0)$, then $c^*(u_0,v_0) := c_{\mathrm{low}}^*(u_0,v_0)$ is called the *spreading speed* of the solution $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$. It is well known that when $\chi = 0$, we have $c_{\mathrm{low}}^*(u_0,v_0) = c_{\mathrm{up}}^*(u_0,v_0) = 2\sqrt{a}$ for any $u_0 \in X_c^+$ and $v_0 \in X_1^+$ (see [3, 2]).

Many interesting questions arise when $\chi \neq 0$. For example, whether $c_{\mathrm{low}}^*(u_0,v_0)$ is positive and $c_{\mathrm{up}}^*(u_0,v_0)$ is finite; whether $c_{\mathrm{low}}^*(u_0,v_0) \geq 2\sqrt{a}$, in other words, whether the chemotaxis does not slow down the population's spreading; whether $c_{\mathrm{up}}^*(u_0,v_0) \leq 2\sqrt{a}$, that is, whether the chemotaxis does not speed up the population's spreading; whether $c_{\mathrm{low}}^*(u_0,v_0) = c_{\mathrm{up}}^*(u_0,v_0)$, that is, the population has a single spreading speed; in the case

$c^*_{\text{low}}(u_0, v_0) > 0$, whether $u(t, x; u_0, v_0)$ converges to $\frac{a}{b}$ in the region $|x| < ct$ for any $0 < c < c^*_{\text{low}}(u_0, v_0)$, which is strongly related to the stability of the constant equilibrium $(\frac{a}{b}, 0)$; and how χ affects the spreading properties; etc. The goal of this section is to answer some of these questions. Now we present our main results of this section

Our first main results in on asymptotic stability of the constant solutions for strictly positive initials

Theorem 3.3.1 (Convergence to constant equilibrium)**.** *Suppose that* $u_0 \in X^+$ *and* $v_0 \in X_1^+$*, and* $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ *is a globally defined bounded classical solution of* (3.0.1)*. If* $\inf_{x \in \mathbb{R}^N} u_0(x) > 0$*, then*

$$\lim_{t\to\infty} u(t, x; u_0, v_0) = \frac{a}{b}, \quad \lim_{t\to\infty} v(t, x; u_0, v_0) = 0 \quad \textit{uniformly in } x \in \mathbb{R}^N.$$

The result implies that there are no other positive stationary solutions $(u(x), v(x))$ of (3.0.1) with $\inf_{x \in \mathbb{R}^N} u(x) > 0$ rather than $(\frac{a}{b}, 0)$.

Our second main result provides a lower bound of the spatial spreading speeds of solutions to (3.0.1) with general non-negative initial data u_0.

Theorem 3.3.2 (Lower bound of spreading speeds)**.** *Suppose that* $u_0 \in X^+$ *and* $v_0 \in X_1^+$*, and* $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ *is a globally defined bounded classical solution of* (3.0.1)*. If* $\text{supp}(u_0) = \text{cl}\{x \in \mathbb{R}^N \mid u_0(x) > 0\}$ *is nonempty, then the following hold.*

(1) $c^*_{\text{low}}(u_0, v_0) \geq 2\sqrt{a}$*, or equivalently,*

$$\liminf_{t\to\infty} \inf_{|x| \leq c't} u(t, x; u_0, v_0) > 0 \quad \forall\, 0 < c' < 2\sqrt{a}. \tag{3.3.1}$$

(2)

$$\lim_{t\to\infty} \sup_{|x| \leq c't} \left| u(t, x; u_0, v_0) - \frac{a}{b} \right| = 0 \quad \forall\, 0 < c' < c^*_{\text{low}}(u_0, v_0), \tag{3.3.2}$$

and

$$\lim_{t\to\infty} \sup_{|x| \leq c't} v(t, x; u_0, v_0) = 0 \quad \forall\, 0 < c' < c^*_{\text{low}}(u_0, v_0). \tag{3.3.3}$$

As it is mentioned in the above, when $\chi = 0$, $c^*_{\text{low}}(u_0, v_0) = 2\sqrt{a}$ for any $u_0 \in X_c^+$ and $v_0 \in X_1^+$ (in this case, $u(t, x; u_0, v_0)$ is independent of v_0). Note that, when $\chi < 0$, the chemical substance is a chemorepellent, and when $\chi > 0$, the chemical substance is a chemoattractant. Theorem 3.3.2 reveals an important biological observation: chemical substance does not slow down the propagation of the biological species with nonzero initial distribution even when the chemical substance is a chemorepellent.

The following theorem is on upper bound of the spatial spreading speeds of solutions to (3.0.1) with compactly supported u_0.

Theorem 3.3.3 (Upper bound of spreading speeds)**.** *Suppose that* $u_0 \in X_c^+$ *and* $v_0 \in X_1^+$*, and* $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *is a globally defined bounded classical solution of* (3.0.1)*. Then we have*

(1) $c^*_{\rm up}(u_0,v_0) < \infty$*. Moreover, for any* $c'' > c^*_{\rm up}(u_0,v_0)$*, there is* $M > 0$ *such that*

$$u(t,x;u_0,v_0) \le Me^{-\sqrt{a}(|x|-c''t)} \quad \forall\, t>0, \quad x \in \mathbb{R}^N. \tag{3.3.4}$$

(2) There exist $C, \gamma > 0$ *depending on* $c'' - c^*_{\rm up}(u_0,v_0)$ *such that*

$$\sup_{|x|\ge c''t} |v(t,x;u_0,v_0) - V(t,x;v_0)| \le Ce^{-\gamma t} \quad \forall\, t>0,\ c'' > c^*_{\rm up}(u_0,v_0),$$

where $V(t,x) := V(t,x;v_0)$ *is the solution of*

$$\begin{cases} \tau V_t = \Delta V, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ V(0,x) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{3.3.5}$$

Theorem 3.3.3 implies that the chemical substance does not drive the biological species spreads infinitely fast. We point out that the methods developed in the proofs of Theorems 3.3.2 and 3.3.3 can be applied to the study of the asymptotic dynamics of the following modified chemotaxis model for $\sigma > 0$ ((3.0.1) corresponds to the case when $\sigma = 1$),

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu^\sigma), & (t,x) \in [0,\infty) \times \mathbb{R}^N, \\ \tau v_t = \Delta v - uv, & (t,x) \in [0,\infty) \times \mathbb{R}^N. \end{cases} \tag{3.3.6}$$

Theorem 3.3.2 and Theorem 3.3.3 with $\frac{a}{b}$ being replaced by $(\frac{a}{b})^{\frac{1}{\sigma}}$ hold for globally defined bounded positive classical solutions of (3.3.6).

Our last two theorems discuss various sufficient conditions for the existence of spreading speed of globally defined bounded classical solutions of (3.0.1) and (3.3.6).

Theorem 3.3.4 (Existence of spreading speeds)**.** *Suppose that* $u_0 \in X_c^+$ *and* $v_0 \in X_1^+$*, and* $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ *is a globally defined bounded classical solution of* (3.0.1)*.*

(1) If $v_0 \in C_0(\mathbb{R}^N)$ *or* $v_0 \in L^p(\mathbb{R}^N)$ *for some* $p \ge 1$*, then*

$$c^*_{\rm low}(u_0,v_0) = c^*_{\rm up}(u_0,v_0) = 2\sqrt{a}.$$

(2) Further assume $\tau = 1$, and that $u_0 \in X_1^+$, $v_0 \in C_{\rm unif}^{2+\alpha,b}(\mathbb{R}^N)$ for some $\alpha > 0$, and $1 - v_0 \in X_c^+$. Then there exists $\chi_0 > 0$ such that for any $-\chi_0 < \chi < 0$, we have

$$c_{\rm low}^*(u_0, v_0) = c_{\rm up}^*(u_0, v_0) = 2\sqrt{a}.$$

Theorem 3.3.5. *Suppose that $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ is a globally defined bounded classical solution of* (3.3.6)*. Assume that $\tau = 1$, $\sigma \in (0, 1)$, $1 - v_0$, $u_0 \in X_c^+ \cap X_1$, and $v_0 \in C_{\rm unif}^{2+\alpha,b}(\mathbb{R}^N)$ for some $\alpha > 0$. Then there exists $\chi_0 > 0$ such that if $|\chi| < \chi_0$, the conclusion of Theorem 3.3.4(2) holds the same.*

The rest of this section is organized as follows. In Subsection 3.3.1, we study the convergence to the steady state. Subsection 3.3.2 is devoted to the lower bound of spreading and the proof of Theorem 3.3.2. In Subsection 3.3.3, we establish the upper bound of spreading and prove Theorem 3.3.3. Subsection 3.3.4 concerns the existence of spreading, where we prove Theorem 3.3.4 and Theorem 3.3.5. Finally, Subsection 3.3.5 is devoted to numerical simulations on $\mathbb{R}$.

3.3.1 Convergence to the constant equilibrium

In this subsection, we prove the convergence of globally defined bounded solutions with strictly positive initial data to the constant solution $(\frac{a}{b}, 0)$ and prove Theorem 3.3.1.

Proof of Theorem 3.3.1. For any fixed $\varepsilon \in \left(0, \frac{a}{|\chi|}\right)$, by Lemma 2.6.2, there are $T = T(\varepsilon, \|v_0\|_\infty) \geq 1$ and $C = C(\varepsilon, \chi) > 0$ such that for any $t \geq T$, we have

$$u_t \geq \Delta u - \chi \nabla u \cdot \nabla v - |\chi| u(\varepsilon + C u^{1/2}) + au - bu^2.$$

We first claim that

$$\delta_T := \inf_{x \in \mathbb{R}^N} u(T, x; u_0, v_0) > 0.$$

It follows from Proposition 3.1.1 that $u(t, \cdot; u_0, v_0)$ is uniformly continuous in L^∞-norm as $t \to 0$. Since $\inf_{x \in \mathbb{R}^N} u_0(x) > 0$, there is $0 < t_1 < T$ such that

$$\delta_1 := \inf_{x \in \mathbb{R}^N} u(t_1, x; u_0, v_0) > 0.$$

It follows from Lemma 2.5.2 that

$$M := \max\left\{\sup_{t \in [t_1, T],\, x \in \mathbb{R}^N} |\Delta v(t, x; u_0, v_0)|, \sup_{t_1 \leq t \leq T,\, x \in \mathbb{R}^N} u(t, x; u_0, v_0)\right\} < \infty.$$

We then have

$$\begin{cases} u_t \ge \Delta u - \chi \nabla v \cdot \nabla u - |\chi| M u + a u - b M u, & t_1 < t < T, \ x \in \mathbb{R}^N, \\ u(t_1, x) \ge \delta_1, & x \in \mathbb{R}^N. \end{cases}$$

Note that u is a classical solution, and $|\nabla u|$ and $|\nabla v|$ are uniformly bounded and continuous on $[t_1, T] \times \mathbb{R}^N$. Thus, by viewing $\nabla v(t, x; u_0, v_0)$ as a given function, the comparison principle for parabolic equations (see e.g., [52, Proposition 52.10]) yields

$$u(t, x; u_0, v_0) \ge e^{(-|\chi| M + a - bM)(t - t_1)} \delta_1 \quad \forall\, t_1 \le t \le T, \ x \in \mathbb{R}^N.$$

This implies that $\delta_T \ge e^{(-|\chi| M + a - bM)(T - t_1)} \delta_1 > 0$. The claim then follows.

Next, let $\underline{u}(t)$ be the solution to the ODE

$$\begin{cases} \underline{u}_t = -|\chi| \underline{u}(\varepsilon + C \underline{u}^{1/2}) + \underline{u}(a - b\underline{u}), \\ \underline{u}(T) = \delta_T. \end{cases} \tag{3.3.7}$$

Note that $-|\chi|\varepsilon + a > 0$. Hence $\underline{u} = 0$ is an unstable solution of (3.3.7), and there exists $\varepsilon_0 > 0$ such that

$$\underline{u}(t; \delta_T) \ge \varepsilon_0, \qquad \forall\, t > T.$$

Then, by the comparison principle, we have

$$u(t, x; u_0, v_0) \ge \underline{u}(t; \delta_T) \ge \varepsilon_0, \quad \forall\, t \ge T. \tag{3.3.8}$$

This shows that $\tau v_t \le \Delta v - \varepsilon_0 v$ for all $t \ge T$. Thus, by comparing v with $\bar{v}(t)$ which solves the ODE $\tau \bar{v}_t = -\varepsilon_0 \bar{v}$ with $\bar{v}(T) = \|v(T, \cdot; u_0, v_0)\|_\infty \le \|v_0\|_\infty$, we obtain

$$0 \le v(t, x) \le \bar{v}(t) \le \|v_0\|_\infty e^{-\frac{\varepsilon_0}{\tau}(t - T)}, \qquad \forall t \ge T.$$

In particular, we proved

$$\lim_{t \to \infty} v(t, x; u_0, v_0) = 0 \quad \text{uniformly in } x \in \mathbb{R}^N. \tag{3.3.9}$$

Now, we prove that

$$\lim_{t \to \infty} u(t, x; u_0, v_0) = a/b \quad \text{uniformly in } x \in \mathbb{R}^N. \tag{3.3.10}$$

Suppose for contradiction that there exist $\delta_0 > 0$, a sequence $t_n \to \infty$, and $x_n \in \mathbb{R}^N$ such that

$$|u(t_n, x_n; u_0, v_0) - \frac{a}{b}| > \delta_0. \tag{3.3.11}$$

Define

$$u_n(t,x) := u(t+t_n, x+x_n; u_0, v_0) \quad \text{and} \quad v_n(t,x) = v(t+t_n, x+x_n; u_0, v_0), \quad t \geq -t_n.$$

By (3.3.9),

$$\lim_{n\to\infty} v_n(t,x) = 0 \quad \text{locally uniformly in}\, t \in \mathbb{R}, \text{ and uniformly in}\, x \in \mathbb{R}^N. \tag{3.3.12}$$

By Lemma 2.5.2, for any bounded subset $I \subset \mathbb{R}$, there is $n_0 \geq 1$ such that $\{w_n(t,x)\}_{n\geq n_0}$ is uniformly bounded and equi-continuous on $I \times \mathbb{R}^N$, where

$$\begin{aligned} w_n(t,x) = {} & u(t+t_n, x+x_n; u_0, v_0),\ \partial_{x_i} u(t+t_n, x+x_n; u_0, v_0),\ \partial_t u(t+t_n, x+x_n; u_0, v_0), \\ & \partial^2_{x_i x_j} u(t+t_n, x+x_n; u_0, v_0),\ v(t+t_n, x+x_n; u_0, v_0),\ \partial_{x_i} v(t+t_n, x+x_n; u_0, v_0), \\ & \partial_t v(t+t_n, x+x_n; u_0, v_0),\ \partial^2_{x_i x_j} v(t+t_n, x+x_n; u_0, v_0), \end{aligned}$$

for $1 \leq i,j \leq N$. Then, by Arzelà–Ascoli theorem, and (3.3.12), there is a subsequence (u_{n_j}, v_{n_j}) and a smooth function $\tilde{u}(t,x)$ such that

$$\lim_{j\to\infty} u_{n_j}(t,x) = \tilde{u}(t,x), \quad \lim_{j\to\infty} v_{n_j}(t,x) = 0 \quad \forall\, t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{3.3.13}$$

Due to the uniform regularity, $\partial_t u_{n_j}$ converges and the limit is equal to $\partial_t \tilde{u}$. Similarly, and after applying a diagonal argument, we can assume that Du_{n_j}, $D^2 u_{n_j}$, Dv_{n_j} and $D^2 v_{n_j}$ converge to $D\tilde{u}$, $D^2\tilde{u}$, 0 and 0, respectively. Therefore, $\tilde{u}$ satisfies

$$\tilde{u}_t = \Delta \tilde{u} + a\tilde{u} - b\tilde{u}^2, \qquad t \in \mathbb{R},\, x \in \mathbb{R}^N.$$

By (3.3.8) and the global boundedness of $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$, there exists $K > \varepsilon_0$ such that

$$0 < \varepsilon_0 \leq \tilde{u}(t,x) \leq K, \quad \forall\, t \in \mathbb{R},\ x \in \mathbb{R}^N.$$

Set $\underline{u}_0 := \inf_{(t,x)\in\mathbb{R}^{N+1}} \tilde{u}$ and $\overline{u}_0 := \sup_{(t,x)\in\mathbb{R}^{N+1}} \tilde{u}$. For every $t_0 \in \mathbb{R}$, let $\underline{u}(\cdot; t_0)$ and $\overline{u}(\cdot; t_0)$ be the solutions of

$$\begin{cases} \frac{d}{dt}\overline{u} = \overline{u}(a - b\overline{u}), \quad t > t_0, \\ \overline{u}(t_0; t_0) = \overline{u}_0, \end{cases}$$

and

$$\begin{cases} \frac{d}{dt}\underline{u} = \underline{u}(a - b\underline{u}), & t > t_0, \\ \underline{u}(t_0; t_0) = \underline{u}_0, \end{cases}$$

respectively. Then for every $t_0 \in \mathbb{R}$,

$$\lim_{t\to\infty} \overline{u}(t; t_0) = \lim_{t\to\infty} \underline{u}(t; t_0) = \frac{a}{b}. \tag{3.3.14}$$

Since $0 < \underline{u}_0 \le \tilde{u}(t, x) \le \overline{u}_0$ for every $(t, x) \in \mathbb{R}^{N+1}$, we obtain that

$$\underline{u}(t - t_0; 0) = \underline{u}(t; t_0) \le \tilde{u}(t, x) \le \overline{u}(t; t_0) = \overline{u}(t - t_0; 0) \quad \forall\, x \in \mathbb{R}^N,\ t \ge t_0.$$

Taking limit as $t_0 \to -\infty$ on both sides and using (3.3.14) imply

$$\tilde{u}(t, x) \equiv a/b,$$

which contradicts with (3.3.11). Therefore, (3.3.10) holds, and Theorem 3.3.1 is proved. □

3.3.2 Lower bound of spreading speeds and proof of Theorem 3.3.2

In this subsection, we study the lower bounds of spreading speeds and prove Theorem 3.3.2. Throughout this subsection, we fix $(u_0, v_0) \in X^+ \times X_1^+$ and assume that $(u(t, x; u_0, v_0), v(t, x; u_0, v_0))$ is a globally defined, bounded, and nonnegative classical solution of (3.0.1). We will often suppress the dependence of constants on a, b, τ, and N.

The first part of Theorem 3.3.2(1) is proved by a nontrivial modification of the arguments in [59, Theorem 1.2(1)], which studied the chemotaxis system with linear signal production

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + u(a - bu), & x \in \mathbb{R}^N,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \mathbb{R}^N,\ t > 0. \end{cases}$$

In our setting, however, nontrivial modifications are required because of the differences between the two systems. The key idea is to show that, for any given $0 < c' < 2\sqrt{a}$, the quantity $u(t, x + c\xi t; u_0, v_0)$ is bounded away from zero uniformly for $0 < c < c'$, $\xi \in \mathbb{S}^{N-1}$, and $|x| \le l$ for some $l > 0$, whenever $t \gg 1$; see Lemmas 3.3.1, 3.3.2, and 3.3.3. In the proof, we use special Harnack inequalities for bounded solutions of (3.0.1), established in Lemmas 2.6.1 and 2.6.2. In addition, our approach employs the principal eigenvalue and eigenfunction of a suitable linearized operator for u, together with the comparison principle for parabolic equations.

Let us fix any

$$0 < c' < 2\sqrt{a} \quad \text{and} \quad 0 < \delta_0 < \min\left\{1, 2\sqrt{a} - c', (a^{1/4}(2N)^{-1})^N\right\}. \tag{3.3.15}$$

Since $2\sqrt{a} - \delta_0 > c'$, for any $R > 0$,

$$\inf_{|x| \le c't} u(t, x; u_0, v_0) \ge \inf_{-2\sqrt{a}+\delta_0 \le c \le 2\sqrt{a}-\delta_0, \xi \in \mathbb{S}^{N-1}} \left(\inf_{|x| \le R} u(t, x + ct\xi; u_0, v_0) \right).$$

To prove Theorem 3.3.2(1) (i.e. (3.3.1)), it then suffices to prove that there is $R_0 > 0$ such that

$$\liminf_{t \to \infty} \inf_{-2\sqrt{a}+\delta_0 \le c \le 2\sqrt{a}-\delta_0, \xi \in \mathbb{S}^{N-1}} \left(\inf_{|x| \le R_0} u(t, x + ct\xi; u_0, v_0) \right) > 0. \tag{3.3.16}$$

For any $\xi \in \mathbb{S}^{N-1}$ and $c \in \mathbb{R}$, let $\tilde{u}(t, x; \xi, c) := u(t, x + ct\xi; u_0, v_0)$ and $\tilde{v}(t, x; \xi, c) := v(t, x + ct\xi; u_0, v_0)$. Then $(\tilde{u}(t, x; \xi, c), \tilde{v}(t, x; \xi, c))$ satisfies

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} + c\xi \cdot \nabla \tilde{u} - \chi \nabla \cdot (\tilde{u} \nabla \tilde{v}) + \tilde{u}(a - b\tilde{u}), & x \in \mathbb{R}^N, \\ \tau \tilde{v}_t = \Delta \tilde{v} + \tau c\xi \cdot \nabla \tilde{v} - \tilde{u}\tilde{v}, & x \in \mathbb{R}^N. \end{cases} \tag{3.3.17}$$

To prove (3.3.16), it is equivalent to prove that, for any fixed $0 < c' < 2\sqrt{a}$ and $0 < \delta_0 < 2\sqrt{a} - c'$, there is $R_0 > 0$ such that

$$\liminf_{t \to \infty} \inf_{-2\sqrt{a}+\delta_0 \le c \le 2\sqrt{a}-\delta_0, \xi \in \mathbb{S}^{N-1}} \left(\inf_{|x| \le R_0} \tilde{u}(t, x; \xi, c) \right) > 0. \tag{3.3.18}$$

To do so, we first prove some lemmas.

For any $0 < \varepsilon < 1$, let

$$T_\varepsilon := \frac{1}{\varepsilon}, \quad R_\varepsilon := 2\varepsilon^{-(1/2+1/N)} \ge 2. \tag{3.3.19}$$

Our first lemma shows that if supremum of $\tilde{u}$ is small for x in a ball of radius $2R$ and t within a small time interval, then both $\nabla \tilde{v}$ and $\Delta \tilde{v}$ are small for x in a ball of radius R and t within a smaller time interval.

Lemma 3.3.1. *There exists* $\tilde{M} = \tilde{M}(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ *such that for any* $0 < \varepsilon < 1$, $\xi \in \mathbb{S}^{N-1}$, $c \in \mathbb{R}$, *and* $1 < t_1 < t_1 + T_\varepsilon < t_2 \le \infty$, *if*

$$\sup_{x \in B_{2R_\varepsilon}} \tilde{u}(t, x; \xi, c) \le \varepsilon, \quad \forall\, t_1 \le t < t_2, \tag{3.3.20}$$

then

$$\sup_{x\in B_{R_\varepsilon}} |\nabla\tilde{v}(t,x;\xi,c)| \le \tilde{M}\varepsilon^{1/2} \quad \forall\, t_1+T_\varepsilon \le t < t_2, \tag{3.3.21}$$

and

$$\sup_{x\in B_{R_\varepsilon}} |\Delta\tilde{v}(t,x;\xi,c)| \le \tilde{M}\varepsilon^{1/4} \quad \forall\, t_1+T_\varepsilon \le t < t_2. \tag{3.3.22}$$

Proof of Lemma 3.3.1. First, we prove (3.3.21). If no confusion occurs, we may drop ξ, c in the notations of $\tilde{u}(t,x;\xi,c)$ and $\tilde{v}(t,x;\xi,c)$. By the definition of $\tilde{v}(t,x)$, we have for $t>t_1$,

$$\begin{aligned}\tilde{v}(t,x) = v(t,x+ct\xi) = \left(\frac{\tau}{4\pi(t-t_1)}\right)^{N/2}&\int_{\mathbb{R}^N} e^{-\frac{\tau|x+ct\xi-y|^2}{4(t-t_1)}} v(t_1,y)\,dy \\ -\frac{1}{\tau}\int_{t_1}^{t}\left(\frac{\tau}{4\pi(t-s)}\right)^{N/2}&\int_{\mathbb{R}^N} e^{-\frac{\tau|x+ct\xi-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds.\end{aligned}$$

Writing $z=\frac{y-x-ct\xi}{2\sqrt{t-t_1}}$, this implies that

$$\begin{aligned}\nabla\tilde{v}(t,x) = &-\left(\frac{\tau}{\pi}\right)^{N/2}\int_{\mathbb{R}^N}\frac{z\tau}{\sqrt{t-t_1}}e^{-\tau|z|^2}\tilde{v}(t_1,x+2z\sqrt{t-t_1})\,dz \\ &-\int_{t_1}^{t}\left(\frac{\tau}{\pi}\right)^{N/2}\int_{|z|\le\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\tilde{u}(s,x+2z\sqrt{t-s})\tilde{v}(s,x+2z\sqrt{t-s})\,dz\,ds \\ &-\int_{t_1}^{t}\left(\frac{\tau}{\pi}\right)^{N/2}\int_{|z|>\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\tilde{u}(s,x+2z\sqrt{t-s})\tilde{v}(s,x+2z\sqrt{t-s})\,dz\,ds.\end{aligned} \tag{3.3.23}$$

In the following, we estimate each term in (3.3.23).

Note that $\left(\frac{\tau}{\pi}\right)^{N/2}\int_{\mathbb{R}^N}|z|e^{-\tau|z|^2}\,dz \le C_1\tau^{-1/2}$ for some $C_1>0$, which is because

$$\begin{aligned}\left(\frac{\tau}{\pi}\right)^{N/2}\int_{\mathbb{R}^N}|z|e^{-\tau|z|^2}\,dz &= \left(\frac{\tau}{\pi}\right)^{N/2}\int_0^\infty\int_{\partial B(0,r)} re^{-\tau r^2}\,dS(z)\,dr \\ &= \left(\frac{\tau}{\pi}\right)^{N/2}\int_0^\infty \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}r^N e^{-\tau r^2}\,dr \\ &= \left(\frac{\tau}{\pi}\right)^{N/2}\frac{\pi^{\frac{N}{2}}}{\tau^{N/2+1/2}\,\Gamma(\frac{N}{2})}\int_0^\infty R^{\frac{N-1}{2}}e^{-R}\,dR \\ &= \frac{\Gamma(\frac{N+1}{2})}{\tau^{1/2}\Gamma(\frac{N}{2})}.\end{aligned}$$

Thus, for the first integral in (3.3.23) we have for all $t \geq t_1 + T_\varepsilon$ and $x \in \mathbb{R}^N$,

$$\left| \left(\frac{\tau}{\pi}\right)^{N/2} \int_{\mathbb{R}^N} \frac{z\tau}{\sqrt{t-t_1}} e^{-\tau|z|^2} \tilde{v}(t_1, x + 2z\sqrt{t-t_1})\, dz \right| \leq C_1 \tau^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \|v_0\|_\infty. \tag{3.3.24}$$

For the second integral in (3.3.23), using (3.3.20), we have for any $t_1 + T_\varepsilon \leq t \leq \min\{t_1 + 2T_\varepsilon, t_2\}$,

$$\begin{aligned}
&\int_{t_1}^{t} \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z| \leq \frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}} \frac{|z|}{\sqrt{t-s}} e^{-\tau|z|^2} \tilde{u}(s, x + 2\sqrt{t-s}\, z) \tilde{v}(s, x + 2\sqrt{t-s}\, z)\, dz\, ds \\
&\qquad \leq C_1 \tau^{-\frac{1}{2}} \|v_0\|_\infty \Big[\sup_{\substack{s \in (t_1, t), \\ y \in B(x, R_\varepsilon)}} \tilde{u}(s, y) \Big] \int_{t_1}^{t} \frac{1}{\sqrt{t-s}}\, ds \\
&\qquad \leq 2\sqrt{2}\, C_1 \tau^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}} \|v_0\|_\infty \|u\|_\infty \quad \forall\, |x| \leq R_\varepsilon.
\end{aligned} \tag{3.3.25}$$

Then, choose a $C_2 = C_2(N) > 0$ such that $e^{-|\tau^{1/2} z|^2} \leq C_2 |\tau^{1/2} z|^{-2N-1}$ for $|z| \geq \frac{R_\varepsilon}{2\sqrt{T_\varepsilon}} = \varepsilon^{-1/N}$. With this and similarly as before, we obtain for $t_1 + T_\varepsilon < t < \min\{t_1 + 2T_\varepsilon, t_2\}$:

$$\begin{aligned}
&\int_{t_1}^{t} \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z| > \frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}} \frac{|z|}{\sqrt{t-s}} e^{-\tau|z|^2} \tilde{u}(s, x + 2\sqrt{t-s}\, z) \tilde{v}(s, x + 2\sqrt{t-s}\, z)\, dz\, ds \\
&\qquad \leq C_2 \left(\frac{\tau}{\pi}\right)^{N/2} \|u\|_\infty \|v_0\|_\infty \int_{t_1}^{t} \int_{|z| > \frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}} \frac{\tau^{-N-\frac{1}{2}} |z|^{-2N}}{\sqrt{t-s}}\, dz\, ds \\
&\qquad \leq C_2 \tau^{\frac{-1-N}{2}} \|u\|_\infty \|v_0\|_\infty \sqrt{T_\varepsilon} \left(\frac{2\sqrt{T_\varepsilon}}{R_\varepsilon} \right)^N \leq C \varepsilon^{\frac{1}{2}} \tau^{\frac{-1-N}{2}} \|u\|_\infty \|v_0\|_\infty.
\end{aligned} \tag{3.3.26}$$

Combining (3.3.23)–(3.3.26), there is $\tilde{M} = \tilde{M}(N, \tau, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that

$$|\nabla \tilde{v}(t, x)| \leq \tilde{M} \varepsilon^{1/2}, \qquad \forall\, t_1 + T_\varepsilon \leq t \leq \min\{t_1 + 2T_\varepsilon, t_2\},\ |x| \leq R_\varepsilon.$$

Identical argument with $t_3 \in [t_1, t_2)$ in place of t_1 yields the same estimate for $t_3 + T_\varepsilon < t \leq \min\{t_3 + 2T_\varepsilon, t_2\}$. Therefore, we conclude that there is $\tilde{M} = \tilde{M}(N, \tau, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that

$$|\nabla \tilde{v}(t, x)| \leq \tilde{M} \varepsilon^{1/2}, \qquad \forall\, t_1 + T_\varepsilon \leq t \leq t_2,\ |x| \leq R_\varepsilon.$$

(3.3.21) then follows.

Next, we prove (3.3.22). By (3.3.23), we have

$$
\begin{aligned}
\Delta\tilde{v}(t,x) = &-\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{\mathbb{R}^N}\frac{z\tau}{\sqrt{t-t_1}}e^{-\tau|z|^2}\cdot\nabla\tilde{v}(t_1,x+2z\sqrt{t-t_1})\,dz\\
&-\int_{t_1}^{t}\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{|z|\le\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\cdot\nabla\tilde{u}(s,x+2\sqrt{t-s}\,z)\tilde{v}(s,x+2\sqrt{t-s}\,z)\,dz\,ds\\
&-\int_{t_1}^{t}\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{|z|\le\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\tilde{u}(s,x+2\sqrt{t-s}\,z)\cdot\nabla\tilde{v}(s,x+2\sqrt{t-s}\,z)\,dz\,ds\\
&-\int_{t_1}^{t}\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{|z|>\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\cdot\nabla\tilde{u}(s,x+2\sqrt{t-s}\,z)\tilde{v}(s,x+2\sqrt{t-s}\,z)\,dz\,ds\\
&-\int_{t_1}^{t}\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{|z|>\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\tilde{u}(s,x+2\sqrt{t-s}\,z)\cdot\nabla\tilde{v}(s,x+2\sqrt{t-s}\,z)\,dz\,ds
\end{aligned}
\tag{3.3.27}
$$

for all $t \ge t_1$ and $x \in \mathbb{R}^N$. In the following, we estimate each term in (3.3.27).

First, since (3.3.20) and $t_1 \ge 1$, by Lemma 2.6.1 (with s_0, R, p in the lemma being $0, 1, \frac{4}{3}$), there is $C_0 = C_0(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ independent of ε such that

$$
|\nabla\tilde{u}(t,x)| \le C_0\,\tilde{u}(t,x)^{\frac{3}{4}} \le C_0\,\varepsilon^{\frac{3}{4}} \quad \forall\, t_1 \le t < t_2,\ |x| \le 2R_\varepsilon. \tag{3.3.28}
$$

Notice that $\nabla\tilde{v}(t,x)$ is uniformly bounded for $t \ge 1$ by the classical parabolic regularity theory. So, similarly as done in (3.3.24), there is $C = C(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that

$$
\left|\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{\mathbb{R}^N}\frac{z}{\sqrt{t-t_1}}e^{-\tau|z|^2}\cdot\nabla\tilde{v}(t_1,x+2z\sqrt{t-t_1})\,dz\right| \le C\varepsilon^{\frac{1}{2}} \quad \forall\, t \ge t_1+T_\varepsilon,\ x\in\mathbb{R}^N. \tag{3.3.29}
$$

By (3.3.28) and the arguments of (3.3.25), there is $C = C(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that for $t_1 + T_\varepsilon \le t \le \min\{t_2, t_1 + 2T_\varepsilon\}$ and $|x| \le R_\varepsilon$,

$$
\left|\int_{t_1}^{t}\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{|z|\le\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\cdot\nabla\tilde{u}(s,x+2\sqrt{t-s}\,z)\tilde{v}(s,x+2\sqrt{t-s}\,z)\,dz\,ds\right| \le C\varepsilon^{\frac{1}{4}}
$$

and

$$
\left|\int_{t_1}^{t}\Big(\frac{\tau}{\pi}\Big)^{N/2}\int_{|z|\le\frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}}\frac{z}{\sqrt{t-s}}e^{-\tau|z|^2}\tilde{u}(s,x+2\sqrt{t-s}\,z)\cdot\nabla\tilde{v}(s,x+2\sqrt{t-s}\,z)\,dz\,ds\right| \le C\varepsilon^{\frac{1}{2}}.
$$

By (3.3.28) and the arguments of (3.3.26), there is $C = C(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that for $t_1 + T_\varepsilon \le t \le \min\{t_2, t_1 + 2T_\varepsilon\}$ and $|x| \le R_\varepsilon$,

$$\left| \int_{t_1}^{t} \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z| > \frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}} \frac{z}{\sqrt{t-s}} e^{-\tau|z|^2} \cdot \nabla\tilde{u}(s, x + 2\sqrt{t-s}\, z)\tilde{v}(s, x + 2\sqrt{t-s}\, z)\, dz\, ds \right| \le C\varepsilon^{\frac{1}{4}}$$

and

$$\left| \int_{t_1}^{t} \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z| > \frac{R_\varepsilon}{2\sqrt{T_\varepsilon}}} \frac{z}{\sqrt{t-s}} e^{-\tau|z|^2} \tilde{u}(s, x + 2\sqrt{t-s}\, z) \cdot \nabla\tilde{v}(s, x + 2\sqrt{t-s}\, z)\, dz\, ds \right| \le C\varepsilon^{\frac{1}{2}}. \tag{3.3.30}$$

By (3.3.27) and (3.3.29)–(3.3.30), there is $\tilde{M} = \tilde{M}(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that

$$|\Delta\tilde{v}(t,x)| \le \tilde{M}\varepsilon^{1/4}, \qquad \forall\, t_1 + T_\varepsilon \le t \le \min\{t_1 + 2T_\varepsilon, t_2\},\ |x| \le R_\varepsilon.$$

After replacing t_1 by any $t_3 \in [t_1, t_2)$, we can get the estimate for $t_3 + T_\varepsilon < t \le \min\{t_3 + 2T_\varepsilon, t_2\}$. Therefore, we conclude that there exists $\tilde{M} = \tilde{M}(\chi, \|v_0\|_\infty, \|u\|_\infty) > 0$ such that

$$|\Delta\tilde{v}(t,x)| \le \tilde{M}\varepsilon^{1/4}, \qquad \forall\, t_1 + T_\varepsilon \le t \le t_2,\ |x| \le R_\varepsilon.$$

This proves (3.3.22). □

Our second lemma shows that if we can bound $\tilde{u}$ below at some given time t_0 in a ball B_{2R}, then we can bound it below up to some time $t_1 > t_0$ in this ball.

Lemma 3.3.2. *Fix $0 < \varepsilon < 1$. For any $\eta > 0$, there is $\delta_\eta > 0$ such that for any $\xi \in \mathbb{S}^{N-1}$, any $c \in [-2\sqrt{a}, 2\sqrt{a}]$, and any $t_0 \ge 2$, if*

$$\sup_{x \in B_{2R_\varepsilon}} \tilde{u}(t_0, x; \xi, c) \ge \eta,$$

then

$$\inf_{x \in B_{2R_\varepsilon}} \tilde{u}(t, x; \xi, c) \ge \delta_\eta, \quad \forall\, t_0 \le t \le t_0 + T_\varepsilon + 1.$$

Proof of Lemma 3.3.2. Suppose for contradiction that there exist $\eta_0 > 0$, $\xi_n \in \mathbb{S}^{N-1}$, $-2\sqrt{a} \le c_n \le 2\sqrt{a}$, $t_{0n} \ge 2$, $x_n, y_n \in B_{2R}$, and $t_n \in [t_{0n}, t_{0n} + T_\varepsilon + 1]$ such that

$$\lim_{n\to\infty} \tilde{u}(t_{0n}, x_n; \xi_n, c_n) \ge \eta_0, \tag{3.3.31}$$

and

$$\lim_{n\to\infty} \tilde{u}(t_n, y_n; \xi_n, c_n) = 0. \tag{3.3.32}$$

Let $\tilde{u}_n(t,x) = \tilde{u}(t + t_{0n} - 1, x + x_n; \xi_n, c_n)$, $\tilde{v}_n(t,x) = \tilde{v}(t + t_{0n} - 1, x + x_n; \xi_n, c_n)$. Since x_n, y_n and $t_n - t_{0n} + 1$ are bounded sequences, without loss of generality, we may assume that

$$\xi_n \to \xi^*, \quad c_n \to c^*, \quad x_n \to x^*, \quad y_n \to y^*, \quad t_n - t_{0n} + 1 \to t^* \geq 1 \qquad \text{as } n \to \infty,$$

for some $\xi^* \in \mathbb{S}^{N-1}$, $-2\sqrt{a} \leq c^* \leq 2\sqrt{a}$, $x^*, y^* \in B_{2R}$ and $t^* \in [1, T_\varepsilon + 2]$. By Lemma 2.5.2 and Arzelà–Ascoli theorem, after passing to a subsequence, we can assume that there is $(u^*(t,x), v^*(t,x))$ such that

$$(\tilde{u}_n(t,x), \tilde{v}_n(t,x)) \to (u^*(t,x), v^*(t,x)) \quad \text{ as } n \to \infty$$

locally uniformly in $(t,x) \in [0,\infty) \times \mathbb{R}^N$, and (u^*, v^*) is a solution of (3.3.17) with ξ and c being replaced by ξ^* and c^* for $t \geq 0$. By (3.3.31),

$$u^*(1,0) \geq \eta_0.$$

It then follows from Lemma 2.6.1 that $u^*(t^*, \cdot) > 0$ in B_{4R}, which contradicts with $u^*(t^*, y^* - x^*) = 0$ by (3.3.32). The lemma is thus proved. □

To proceed, since $(2\sqrt{a} - \delta_0)^2 + \delta_0\sqrt{a} < 4a$, take $\bar{a} \in (0, a)$ such that

$$4\bar{a} - c^2 \geq \delta_0\sqrt{a} \quad \text{for any} \quad c \in [-2\sqrt{a} + \delta_0, 2\sqrt{a} - \delta_0]. \tag{3.3.33}$$

For $\xi \in \mathbb{S}^{N-1}$, let $\lambda(c, \bar{a})$ be the principal eigenvalue of

$$\begin{cases} \Delta\phi + c\xi \cdot \nabla\phi + \bar{a}\phi = \lambda\phi, & x \in B_{R_0}, \\ \phi(x) = 0, & x \in \partial B_{R_0}, \end{cases}$$

and $\phi(x; \xi, c, \bar{a})$ be the corresponding positive eigenfunction with $\|\phi(\cdot; \xi, c, \bar{a})\|_\infty = 1$. By symmetry, $\lambda(c, \bar{a})$ is independent of ξ. We claim that there are $R_0, \lambda_0 > 0$ such that

$$\lambda(c, \bar{a}) \geq \lambda_0 > 0 \quad \forall\, c \in [-2\sqrt{a} + \delta_0, 2\sqrt{a} - \delta_0],\ \xi \in \mathbb{S}^{N-1}. \tag{3.3.34}$$

The proof for the claim is easy. Indeed, let

$$l_0 = l_0(\delta_0) := 2\pi\sqrt{N}\,(\delta_0\sqrt{a})^{-\frac{1}{2}}, \quad R_0 := \sqrt{N}l_0, \tag{3.3.35}$$

and set

$$D_{l_0} := \{x \in \mathbb{R}^N \mid |x_i| < l_0 \text{ for } i = 1, 2, \cdots N\}.$$

Then it is direct to check that for $\xi \in \mathbb{S}^{N-1}$, we have

$$\tilde{\lambda}(c, \bar{a}) := \bar{a} - \frac{c^2}{4} - \frac{N\pi^2}{4l_0^2} \quad \text{and} \quad \tilde{\phi}(x; \xi, c, \bar{a}) := e^{-\frac{c}{2}\xi\cdot x} \prod_{i=1}^{N} \cos\left(\frac{\pi}{2l_0} x_i\right)$$

satisfy

$$\begin{cases} \Delta\tilde{\phi} + c\xi \cdot \nabla\tilde{\phi} + \bar{a}\tilde{\phi} = \tilde{\lambda}\tilde{\phi}, & x \in D_{l_0}, \\ \tilde{\phi}(x) = 0, & x \in \partial D_{l_0}, \end{cases}$$

and

$$\lambda_0 := \tilde{\lambda}(2\sqrt{a} - \delta_0, \bar{a}) = \min_{-2\sqrt{a}+\delta_0 \le c \le 2\sqrt{a}-\delta_0} \tilde{\lambda}(c, \bar{a}) > 0.$$

Since $D_{l_0} \subseteq B_{R_0}$, the domain monotonicity for the Dirichlet principal eigenvalues yields the claim (3.3.34).

The next lemma shows that if the supremum of $\tilde{u}$ is small on some interval $(t_1, t_2) \subset (2, \infty)$, we can obtain a lower bound for $\tilde{u}$ on that interval.

Lemma 3.3.3. *Recall the notations of* (3.3.19) *and* δ_0 *from* (3.3.15). *There is* $\varepsilon_1 > 0$ *such that for any* $0 < \eta \le \varepsilon_1$*, there is* $\delta_\eta > 0$ *such that for any* $\xi \in \mathbb{S}^{N-1}$*, any* $c \in [-2\sqrt{a} + \delta_0, 2\sqrt{a} - \delta_0]$*, and any* t_1, t_2 *with* $2 \le t_1 < t_2 \le \infty$*, if*

$$\sup_{x \in B_{2R_{\varepsilon_1}}} \tilde{u}(t_1, x; \xi, c) = \eta, \quad \sup_{x \in B_{2R_{\varepsilon_1}}} \tilde{u}(t, x; \xi, c) \le \eta, \qquad \forall\, t_1 < t < t_2, \tag{3.3.36}$$

then

$$\inf_{x \in B_{R_0}} \tilde{u}(t, x; \xi, c) \ge \tilde{\delta}_\eta \qquad \forall\, t_1 < t < t_2.$$

Proof of Lemma 3.3.3. First of all, we give a construction of ε_1. Let $\bar{a}$ from (3.3.33) and set

$$T_0 := \max\left\{1, \lambda_0^{-1} \ln 4\right\}. \tag{3.3.37}$$

Note that $u^*(t,x;\xi,c) := e^{\lambda(c,\bar{a})t}\phi(x;\xi,c,\bar{a})$ is the solution of

$$\begin{cases} u_t = \Delta u + c\xi\cdot\nabla u + \bar{a}u, & x\in B_{R_0},\ t>0,\\ u(t,x) = 0, & x\in \partial B_{R_0},\ t>0,\\ u(0,x) = \phi(x;\xi,c,\bar{a}), & x\in B_{R_0}. \end{cases} \tag{3.3.38}$$

It follows from (3.3.37) and (3.3.34) that

$$u^*(T_0,x;\xi,c) \ge 4\phi(x;\xi,c,\bar{a}) \quad \forall\, x\in B_{R_0},\ c\in[-2\sqrt{a}-\delta_0, 2\sqrt{a}-\delta_0],\ \xi\in\mathbb{S}^{N-1}. \tag{3.3.39}$$

For a given C^1 function $q(t,x)$, consider

$$\begin{cases} u_t = \Delta u + c\xi\cdot\nabla u + \nabla q\cdot\nabla u + \bar{a}u, & x\in B_{R_0},\ t>0,\\ u(t,x) = 0, & x\in \partial B_{R_0},\ t>0,\\ u(0,x) = \phi(x;\xi,c,\bar{a}), & x\in B_{R_0}, \end{cases} \tag{3.3.40}$$

where R_0 is given in (3.3.35). Let $u_q(t,x;\xi,c,\phi)$ be the solution of (3.3.40). We claim that there is $\varepsilon_2>0$ such that for any C^1-function $q(t,x)$, if

$$\|\nabla q\|_\infty := \|\nabla q\|_{C([0,T_0]\times\overline{B}_{R_0})} < |\chi|\tilde{M}\varepsilon_2 \quad \text{with } \tilde{M} \text{ from Lemma 3.3.1,}$$

then

$$u_q(T_0,x;\xi,c,\phi) \ge 2\phi(x;\xi,c,\bar{a}) \quad \forall\, x\in B_{R_0},\ c\in[-2\sqrt{a}+\delta_0, 2\sqrt{a}-\delta_0],\ \xi\in\mathbb{S}^{N-1}. \tag{3.3.41}$$

In fact, recall that u^* is a classical solution of (3.3.38), $u^*(t,x)>0$ for $t>0$ and $x\in B_{R_0}$, and $u^*(t,x)=0$ for $t>0$ and $x\in\partial B_{R_0}$. By [14, Theorem 2] and the continuity of $\frac{\partial u^*(T_0,x,\xi,c)}{\partial\nu}$ in $x\in\partial B_{R_0}$, $|c|\le 2\sqrt{a}-\delta_0$, and $\xi\in\mathbb{S}^{N-1}$, there holds

$$\inf_{x\in\partial B_{R_0},|c|\le 2\sqrt{a}-\delta_0,\xi\in\mathbb{S}^{N-1}} \frac{\partial u^*(T_0,x;\xi,c)}{\partial\nu} < 0 \tag{3.3.42}$$

where $\partial/\partial\nu$ denotes the outer normal derivative. By [23, Theorem 3.4.1],

$$\lim_{\|\nabla q\|_\infty\to 0} \|u_q(T_0,\cdot;\xi,c,\phi) - u^*(T_0,\cdot;\xi,c)\|_{C^1(\bar{B}_{R_0})} = 0 \tag{3.3.43}$$

uniformly in $c \in [-2\sqrt{a} - \delta_0, 2\sqrt{a} + \delta_0]$ and $\xi \in \mathbb{S}^{N-1}$. Note that

$$\frac{\partial u^*(T_0, x; \xi, c)}{\partial \nu} = \frac{1}{R_0} \nabla u^*(t, x; \xi, c) \cdot x, \quad \forall\, x \in \partial B_{R_0}.$$

Thus, by (3.3.42) and (3.3.43), there is $r_1 > 0$ such that

$$\inf_{x \in B_{R_0} \setminus B_{R_0 - r_1}, |c| \le 2\sqrt{a} - \delta_0, \xi \in \mathbb{S}^{N-1}} \nabla u^*(T_0, x; \xi, c) \cdot x < 0. \tag{3.3.44}$$

While, away from the boundary, we have

$$\inf_{x \in B_{R_0 - r_1}, |c| \le 2\sqrt{a} - \delta_0, \xi \in \mathbb{S}^{N-1}} \phi(x; \xi, c, \bar{a}) > 0.$$

Then, by (3.3.39), (3.3.43) and (3.3.44), there is $\varepsilon_2 > 0$ such that for any C^1 function q satisfying that

$$\|\nabla q\|_{C([0,T_0] \times \bar{B}_{R_0})} < |\chi| \tilde{M} \varepsilon_2,$$

we have for all $|c| \le 2\sqrt{a} - \delta_0$ and $\xi \in \mathbb{S}^{N-1}$,

$$u_q(T_0, x; \xi, c, \phi) \ge 2\phi(x; \xi, c, \bar{a}), \quad \text{ in } B_{R_0 - r_1}, \tag{3.3.45}$$

and for all $x \in B_{R_0} \setminus B_{R_0 - r_1}$,

$$-\nabla u_q(T_0, x; \xi, c, \phi) \cdot x \ge -\frac{1}{2} \nabla u^*(T_0, x; c, \xi) \cdot x.$$

This, together with (3.3.39), yields for $x \in B_{R_0} \setminus B_{R_0 - r_1}$,

$$\begin{aligned} u_q(T_0, x; \xi, c, \phi) &= -\left(\frac{R_0}{|x|} - 1\right) \int_0^1 \nabla u_q(T_0, sx + (1-s)R_0 x/|x|; \xi, c, \phi) \cdot x\, ds \\ &\ge -\frac{1}{2}\left(\frac{R_0}{|x|} - 1\right) \int_0^1 \nabla u^*(T_0, sx + (1-s)R_0 x/|x|; \xi, c) \cdot x\, ds \\ &= \frac{1}{2} u^*(T_0, x; \xi, c) \ge 2\phi(x; \xi, c, \bar{a}) \qquad \forall\, |c| \le 2\sqrt{a} - \delta_0,\ \xi \in \mathbb{S}^{N-1}. \end{aligned} \tag{3.3.46}$$

The claim (3.3.41) then follows from (3.3.45) and (3.3.46).

Finally, we let

$$\varepsilon_1 = \min\left\{ \frac{1}{2}, (\varepsilon_2)^2, \left(\frac{a - \bar{a}}{|\chi| \tilde{M} + b}\right)^4, \left(\frac{2}{R_0}\right)^{\frac{2N}{2+N}} \right\}.$$

Recall $R_{\varepsilon_1} = 2\varepsilon_1^{-\frac{2+N}{2N}}$ and so $R_{\varepsilon_1} \geq R_0$.

Next, we prove that the lemma holds with the above ε_1. By the assumption (3.3.36) and Lemma 3.3.2, there is $\delta_1 = \delta_1(\eta) > 0$ such that

$$\inf_{x \in B_{2R_{\varepsilon_1}}} \tilde{u}(t, x; \xi, c) \geq \delta_1, \quad \forall\, t_1 \leq t \leq t_1 + T_{\varepsilon_1} + 1,\ |c| \leq 2\sqrt{a} - \delta_0,\ \xi \in \mathbb{S}^{N-1}.$$

Thus, the proof is finished if $t_2 \leq t_1 + T_{\varepsilon_1} + 1$.

In the following, we assume that $t_2 > t_1 + T_{\varepsilon_1} + 1$. By the assumption and Lemma 3.3.1, we have

$$|\nabla \tilde{v}(t, x; \xi, c)| \leq \tilde{M} \varepsilon_1^{\frac{1}{2}} \quad \text{and} \quad |\Delta \tilde{v}(t, x; \xi, c)| \leq \tilde{M} \varepsilon_1^{\frac{1}{4}} \quad \forall\, t_1 + T_{\varepsilon_1} \leq t < t_2,\ x \in B_{R_{\varepsilon_1}}.$$

By the definition of ε_1, for any $\eta \leq \varepsilon_1$, we have

$$a - |\chi| \tilde{M} \varepsilon_1^{\frac{1}{4}} - b\eta \geq a - \left(|\chi| \tilde{M} + b \right) \varepsilon_1^{\frac{1}{4}} \geq \bar{a}.$$

This implies that

$$\begin{aligned} \tilde{u}_t &\geq \Delta \tilde{u} + c\xi \cdot \nabla \tilde{u} - \chi \nabla \cdot (\tilde{u} \nabla \tilde{v}) + a\tilde{u} - b\eta \tilde{u} \\ &\geq \Delta \tilde{u} + c\xi \cdot \nabla \tilde{u} - \chi \nabla \tilde{u} \cdot \nabla \tilde{v} + \bar{a}\tilde{u}, \quad \forall\, t_1 + T_{\varepsilon_1} + 1 \leq t < t_2,\ x \in B_{R_{\varepsilon_1}}. \end{aligned}$$

Also using that $\tilde{u}(t_1 + T_{\varepsilon_1} + 1, x; \xi, c) \geq \delta_1 \geq \delta_1 u_q(0, x; \xi, c, \phi)$ and that $\tilde{u}$ is non-negative in the whole domain, it follows from the comparison principle that

$$\tilde{u}(t_1 + T_{\varepsilon_1} + 1 + t, x; \xi, c) \geq \delta_1 u_q(t, x; \xi, c, \phi), \quad 0 \leq t < t_2 - t_1 - T_{\varepsilon_1} - 1, \quad x \in B_{R_0},$$

where $q(t, x) = -\chi \tilde{v}(t + t_1 + T_{\varepsilon_1} + 1, x; \xi, c)$.

Let $n_0 \geq 0$ be such that

$$t_1 + T_{\varepsilon_1} + 1 + n_0 T_0 < t_2 \quad \text{and} \quad t_1 + T_{\varepsilon_1} + 1 + (n_0 + 1) T_0 \geq t_2.$$

Since

$$|\nabla q(t, x)| = |\chi| |\nabla \tilde{v}(t, x; \xi, c)| \leq |\chi| \tilde{M} \varepsilon_1^{\frac{1}{2}} \leq |\chi| \tilde{M} \varepsilon_2,$$

by (3.3.41), we get

$$\begin{aligned} \tilde{u}(t_1 + T_{\varepsilon_1} + 1 + kT_0, x; \xi, c) &\geq \delta_1 u_q(kT_0, x; \xi, c, \phi) \\ &\geq 2^k \delta_1 \phi(x; \xi, c, \bar{a}) \quad \forall\, x \in B_{R_0},\ k = 1, 2, \cdots, n_0. \end{aligned}$$

Applying Lemma 3.3.2 implies that there is $0 < \tilde{\delta}_\eta \le \delta_1$ such that for any $-2\sqrt{a}+\delta_0 \le c \le 2\sqrt{a}-\delta_0$, any $\xi \in \mathbb{S}^{N-1}$,

$$\inf_{x\in B_{R_0}} \tilde{u}(t,x;\xi,c) \ge \tilde{\delta}_\eta \qquad \forall\, t_1 \le t < t_2.$$

□

Now, we prove Theorem 3.3.2(1).

Proof of Theorem 3.3.2(1). As it is pointed out in the above, to prove Theorem 3.3.2(1), it suffices to prove (3.3.18).

Let $\varepsilon_0, T_0, R_0, \varepsilon_1, T_{\varepsilon_1}$ be as in the above. Let

$$\delta_* := \inf\left\{\tilde{u}(T_{\varepsilon_1}+1,x;\xi,c)\,|\, x\in B_{R_0},\ \xi\in\mathbb{S}^{N-1},\ |c|\le 2\sqrt{a}-\delta_0\right\}.$$

Since $u_0(x)\ge 0$ has nonempty support, by Lemma 2.6.1, $\delta_*>0$. Let

$$k_* := \inf\{k\in\mathbb{Z}^+\,|\,2^k\delta_*\ge\varepsilon_1\} \quad \text{and} \quad T_* := T_{\varepsilon_1}+1+k_*T_0.$$

We claim that there is a $\delta>0$ independent of c and ξ such that,

$$\inf_{x\in B_{R_0}} \tilde{u}(t,x;\xi,c) \ge \delta \qquad \forall t > T_*. \tag{3.3.47}$$

Case 1: Suppose that for any $t\ge T_{\varepsilon_1}\ge 2$,

$$\sup_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(t,x;\xi,c) \ge \varepsilon_1. \tag{3.3.48}$$

In this case, by applying Lemma 3.3.2 repeatedly, we get

$$\inf_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(t,x;\xi,c) \ge \delta_{\varepsilon_1} \quad \forall\, t\ge T_{\varepsilon_1}.$$

Hence, (3.3.47) holds with $\delta=\delta_{\varepsilon_1}$.

Case 2: Suppose that there exists $t>T_{\varepsilon_1}$ such that (3.3.48) is not true. Then the set $\{t>T_{\varepsilon_1}\,|\,\sup_{x\in B_{2R_{\varepsilon_1}}}\tilde{u}(t,x;\xi,c)<\varepsilon_1\}$ is non-empty, and the set is open by continuity. This means we can write it as union of some disjoint open intervals, i.e.,

$$\left\{t>T_{\varepsilon_1}:\sup_{x\in B_{2R_{\varepsilon_1}}}\tilde{u}(t,x;\xi,c)<\varepsilon_1\right\}=\bigcup_{i\in I}(t_i,s_i), \qquad \text{for some } t_i,s_i\in[T_{\varepsilon_1},\infty),\ t_i<s_i<t_{i+1}.$$

Case 2.1: Suppose that $t_i > T_{\varepsilon_1}$ for all i. Then

$$\sup_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(t_i, x; \xi, c) = \varepsilon_1, \quad \sup_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(t, x; \xi, c) < \varepsilon_1, \ \forall \ t_i < t < s_i.$$

Then by Lemma 3.3.3, there exists a $\tilde{\delta}_{\varepsilon_1} > 0$ independent of ξ and c such that

$$\inf_{x\in B_{R_0}} \tilde{u}(t, x; \xi, c) \geq \tilde{\delta}_{\varepsilon_1} \quad \forall \ t_i \leq t < s_i.$$

For any $t > T_*$ and $t \notin \bigcap_{i\in I}(t_i, s_i)$, we have

$$\sup_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(t, x; \xi, c) \geq \varepsilon_1.$$

Then Lemma 3.3.2 yields

$$\inf_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(s, x; \xi, c) \geq \delta_{\varepsilon_1} \quad \forall\, t \leq s \leq t + T_{\varepsilon_1} + 1.$$

Hence (3.3.47) holds with $\delta = \min\{\delta_{\varepsilon_1}, \tilde{\delta}_{\varepsilon_1}\}$.

Case 2.2: There is i_0 such that $t_{i_0} = T_{\varepsilon_1}$. Note that

$$\sup_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(t, x; \xi, c) < \varepsilon_1, \qquad \forall \ t_{i_0} < t < s_{i_0}. \tag{3.3.49}$$

We claim that $s_{i_0} \leq T_*$. In fact, assuming $s_{i_0} > T_*$, by the definition of δ_*,

$$\inf_{x\in B_{R_0}} \tilde{u}(T_{\varepsilon_1} + 1, x; \xi, c) \geq \delta_*.$$

By comparing $\tilde{u}(T_{\varepsilon_1} + 1 + \cdot, \cdot; \xi, c)$ with $\delta_* u_q(\cdot, \cdot; \xi, c, \phi)$ in $[0, k_* T_0] \times B_{R_0}$, with u_q from Lemma 3.3.3, we get

$$\tilde{u}(T_{\varepsilon_1} + 1 + kT_0, x; \xi, c) \geq 2^k \delta_* \phi(x; \xi, c, \bar{a}) \qquad \forall\, x \in B_{R_0}$$

for $k = 0, 1, 2, \cdots, k_*$. In particular, this implies that

$$\sup_{x\in B_{2R_{\varepsilon_1}}} \tilde{u}(T_*, x; \xi, c) \geq \sup_{x\in B_{R_0}} \tilde{u}(T_{\varepsilon_1} + 1 + kT_0, x; \xi, c) \geq \varepsilon_1,$$

which contradicts with (3.3.49), and hence $s_{i_0} \leq T_*$.

Overall, we conclude that for all $t \geq T_*$,

$$\inf_{\xi \in \mathbb{S}^{N-1},\, |c| \leq 2\sqrt{a}-\delta_0} \left(\inf_{|x| \leq R_0} \tilde{u}(t,x;\xi,c) \right) \geq \min\{\delta_{\varepsilon_1}, \tilde{\delta}_{\varepsilon_1}\}.$$

Theorem 3.3.2(1) is thus proved. $\square$

In the following, we prove Theorem 3.3.2(2).

Proof of Theorem 3.3.2(2). We first prove (3.3.2). Suppose for contradiction that there are $0 < c_1 < c^*_{\rm low}(u_0, v_0)$, $\delta > 0$ and $\{(t_n, x_n)\} \subset \mathbb{R}^{N+1}$ such that $t_n \to \infty$, $|x_n| \leq c_1 t_n$, and

$$\left| u(t_n, x_n) - \frac{a}{b} \right| \geq \delta, \quad \forall\, n \geq 1. \tag{3.3.50}$$

Consider

$$u_n(t,x) = u(t+t_n, x+x_n) \quad \text{and} \quad v_n(t,x) = v(t+t_n, x+x_n) \quad \forall\, t \geq -t_n,\ x \in \mathbb{R}^N.$$

By Lemma 2.5.2, there is a $(\hat{u}, \hat{v}) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^N)$ and a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ such that

$$\lim_{k \to \infty} (u_{n_k}(t,x), v_{n_k}(t,x)) = (\hat{u}(t,x), \hat{v}(t,x)) \quad \text{locally uniformly in } \mathbb{R} \times \mathbb{R}^N.$$

Moreover, $(\hat{u}(t,x), \hat{v}(t,x))$ is an entire solution of (3.0.1).

Choose c' such that $c_1 < c' < c^*_{\rm low}(u_0, v_0)$. Then, for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, let us select k such that $t_{n_k} \geq \frac{|x| - c't}{c' - c_1}$. This yields

$$|x + x_{n_k}| \leq |x| + c_1 t_{n_k} \leq c'(t_{n_k} + t)$$

which then implies that

$$\hat{u}(t,x) = \lim_{k \to \infty} u(t + t_{n_k}, x + x_{n_k}) \geq \liminf_{s \to \infty} \inf_{|y| \leq c's} u(s,y) \quad \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

By (3.3.1),

$$\inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} \hat{u}(t,x) \geq \liminf_{s \to \infty} \inf_{|y| \leq c's} u(s,y) =: \varepsilon_0 > 0. \tag{3.3.51}$$

Similarly to the proof of Proposition 3.3.1, since $(\hat{u}(t,x), \hat{v}(t,x))$ is an entire solution of (3.0.1), we get $\tau \hat{v}_t \leq \Delta \hat{v} - \varepsilon_0 \hat{v}$, which implies

$$0 \leq \hat{v}(t,x) \leq \|\hat{v}(-T,\cdot)\|_\infty e^{-\frac{\varepsilon_0}{\tau}(t+T)} \leq \sup_{s \in \mathbb{R}} \|\hat{v}(s,\cdot)\|_\infty e^{-\frac{\varepsilon_0}{\tau}(t+T)} \qquad \forall t \geq -T.$$

Letting $T \to \infty$ yields

$$\hat{v}(t,x) \equiv 0.$$

Then, $\hat{u}(t,x)$ satisfies

$$\hat{u}_t = \Delta \hat{u} + \hat{u}(a - b\hat{u}), \quad \forall t \in \mathbb{R}.$$

Using (3.3.51) (also see the proof for (3.3.10)), we get

$$\hat{u}(t,x) \equiv \frac{a}{b}.$$

However, this contradicts with (3.3.50). Hence, (3.3.2) holds.

Next, we prove (3.3.3). We also prove it by contradiction. Assume for contradiction that there are $0 < c_1 < c^*_{\text{low}}(u_0,v_0)$, $\delta > 0$ and $\{(t_n,x_n)\} \subset \mathbb{R}^{N+1}$ such that $t_n \to \infty$, $|x_n| \le c_1 t_n$, and

$$v(t_n,x_n) \ge \delta, \quad \forall\ n \ge 1. \tag{3.3.52}$$

Let

$$u_n(t,x) = u(t+t_n, x+x_n), \quad \text{and} \quad v_n(t,x) = v(t+t_n, x+x_n) \quad \forall\, t \ge -t_n,\ x \in \mathbb{R}^N.$$

Then, similarly as done in the above, there is a function $(\tilde{u},\tilde{v}) \in C^{2,1}(\mathbb{R} \times \mathbb{R}^N)$ and a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n,v_n)\}$ such that

$$\lim_{k\to\infty} (u_{n_k}(t,x), v_{n_k}(t,x)) = (\tilde{u}(t,x), \tilde{v}(t,x)) \quad \text{locally uniformly in } \mathbb{R} \times \mathbb{R}^N,$$

and, moreover,

$$\tilde{u}(t,x) \equiv \frac{a}{b} \quad \text{and} \quad \tilde{v}(t,x) \equiv 0.$$

However, this contradicts with (3.3.52), and so we proved (3.3.3). □

3.3.3 Upper bound of spreading speeds and proof of Theorem 3.3.3

In this subsection, we provide an upper bound for the spreading speed and prove Theorem 3.3.3. Theorem 3.3.3(1) is proved by the application of special Harnack inequalities for bounded solutions of (3.0.1) established in Lemmas 2.6.1 and 2.6.2. The exponential decay property (3.3.4) for $u(t,x;u_0,v_0)$ and the representation of $v(t,x;u_0,v_0)$ via the Duhamel's principle are the key ingredients in the proof of Theorem 3.3.3(2). Throughout the subsection, we fix $(u_0,v_0) \in X_c^+ \times X_1^+$, and suppose that $(u(t,x;u_0,v_0), v(t,x;u_0,v_0))$ is a globally defined bounded classical solution of (3.0.1).

Proof of Theorem 3.3.3(1). Let us drop u_0, v_0 from the notations of $u(t,x;u_0,v_0)$ and $v(t,x;u_0,v_0)$. We first prove that there is $c_1 > 0$ such that

$$\lim_{t\to\infty} \sup_{|x|\ge c_1 t} u(t,x) = 0. \tag{3.3.53}$$

This will imply that

$$c^*_{\rm up}(u_0,v_0) \le c_1 < \infty.$$

To prove (3.3.53), we first claim that there are $t_0 > 0$ and $M_0 > 0$ such that

$$u(t,x) \le M_0 e^{-\sqrt{a}|x|} \quad \forall\, 0 \le t \le t_0,\ x \in \mathbb{R}^N. \tag{3.3.54}$$

In fact, let $\tilde u(t,x) = e^{\sqrt{a(1+|x|^2)}} u(t,x)$ and $\tilde v(t,x) = v(t,x)$. Writing $h := e^{\sqrt{a(1+|x|^2)}}$, we have

$$\begin{aligned} h\nabla u &= \nabla \tilde u - (h^{-1}\nabla h)\tilde u,\\ h\Delta u &= \Delta\tilde u - \left(h^{-1}\Delta h\right)\tilde u + 2|h^{-1}\nabla h|^2 \tilde u - 2h^{-1}\nabla h\cdot\nabla\tilde u\\ &= \Delta\tilde u + \left(h^{-1}\Delta h\right)\tilde u - 2\nabla(\tilde u h^{-1}\nabla h). \end{aligned}$$

Thus, $(\tilde u(t,x), \tilde v(t,x))$ is a global classical solution of

$$\begin{cases} \tilde u_t = \Delta \tilde u - \chi\nabla\cdot(\tilde u\nabla\tilde v) + \tilde u(a - bh^{-1}\tilde u) & \\ \qquad + \left(h^{-1}\Delta h\right)\tilde u - 2\nabla(\tilde u h^{-1}\nabla h) + \chi \tilde u h^{-1}\nabla h\cdot\nabla\tilde v, & \text{in } (0,\infty)\times\mathbb{R}^N,\\ \tau\tilde v_t = \Delta\tilde v - h^{-1}\tilde u\tilde v, & \text{in } (0,\infty)\times\mathbb{R}^N,\\ \tilde u(0,x) = h(x)u_0(x), \quad \tilde v(0,x) = v_0(x), & x\in\mathbb{R}^N. \end{cases}$$

Note that $\tilde u(0,\cdot)$ has compact support, and h^{-1}, $h^{-1}\nabla h$ and $h^{-1}\Delta h$ are uniformly bounded. By the arguments of local existence of solutions of (3.0.1) in the proof of Proposition 3.1.1, there is $\tilde T_{\max} \in (0,\infty]$ such that the classical solution $(\tilde u, \tilde v)$ is unique in $(0, \tilde T_{\max})$ and it satisfies for any $t_0 \in (0, \tilde T_{\max})$,

$$M_0 = \sup_{t\in[0,t_0], x\in\mathbb{R}^N} \tilde u(t,x) < \infty.$$

This implies that

$$u(t,x) = e^{-\sqrt{a(1+|x|^2)}}\tilde{u}(t,x) \le M_0 e^{-\sqrt{a}|x|}, \quad \forall\, t \in [0,t_0],\ x \in \mathbb{R}^N,$$

which yields that (3.3.54) holds.

Fix a $t_0 > 0$ such that (3.3.54) holds. By Lemma 2.5.2, we have

$$A := \sup_{t\ge t_0} (\|\nabla v(t)\|_\infty + \|\Delta v(t)\|_\infty) < \infty.$$

Then, for some $c_1 > 0$ to be determined and for each $\xi \in \mathbb{S}^{N-1}$, let

$$u_\xi(t,x) := M_0 e^{-\sqrt{a}(x\cdot\xi - c_1 t)}. \tag{3.3.55}$$

It is direct to see that

$$\begin{aligned}
&\partial_t u_\xi - \Delta u_\xi + \chi\nabla\cdot(u_\xi \nabla v) - au_\xi + bu_\xi^2 \\
&\quad \ge c_1\sqrt{a}u_\xi - au_\xi - |\chi|A(\sqrt{a}+1)u_\xi - au_\xi \\
&\quad = (c_1\sqrt{a} - 2a - |\chi|A(\sqrt{a}+1))u_\xi.
\end{aligned}$$

Let us pick $c_1 := 2\sqrt{a} + |\chi|A(1 + 1/\sqrt{a})$, and thus u_ξ is a supersolution to the equation satisfied by u. By (3.3.54),

$$u(t_0,x) \le M_0 e^{-\sqrt{a}x\cdot\xi} \quad \text{for all } \xi \in \mathbb{S}^{N-1}.$$

It then follows from the comparison principle that $u \le u_\xi$ in $[t_0,\infty)\times\mathbb{R}^N$ for all $\xi \in \mathbb{S}^{N-1}$. We obtain

$$u(t,x) \le \min_{\xi\in\mathbb{S}^{N-1}} u_\xi(t,x) = M_1 e^{-\sqrt{a}(|x|-c_1 t)} \quad \forall\, t > t_0,\ x \in \mathbb{R}^N. \tag{3.3.56}$$

This implies (3.3.53) with $c_1 = 2\sqrt{a} + |\chi|A(1+1/\sqrt{a})$.

Next, we prove (3.3.4). Fix a $c_2 > c^*_{\mathrm{up}}(u_0,v_0) \ge c^*_{\mathrm{low}}(u_0,v_0) \ge 2\sqrt{a}$. Let $\delta \in (0,1)$ and $\varepsilon \in (0,\delta)$ to be determined. In view of the definition of $c^*_{\mathrm{up}}(u_0,v_0)$, there is $T_\varepsilon > 0$ such that

$$u(t,x) \le \varepsilon, \quad \forall\ t \ge T_\varepsilon,\ |x| \ge c_2 t.$$

Then, by Lemma 2.6.2, there is $C_\delta > 0$ such that

$$|\nabla v(t,x)|,\ |\Delta v(t,x)| \le \delta + C_\delta u(t,x)^{\frac{1}{2}} \le \delta + C_\delta \varepsilon^{\frac{1}{2}} \quad \forall\, t \ge T_\varepsilon,\ |x| \ge c_2 t. \tag{3.3.57}$$

Let w_ξ be defined similarly as in (3.3.55), that is $w_\xi(t,x) := M_2 e^{-\sqrt{a}(x\cdot\xi - c_2 t)}$, but with M_2 given by

$$M_2 := \max\left\{M_1 \exp(\sqrt{a}T_\varepsilon(c_1 - c_2)), \|u\|_\infty\right\}. \tag{3.3.58}$$

By (3.3.57) and direct computations, in the region $\{t \geq T_\varepsilon, |x| \geq c_2 t\}$, we have

$$\begin{aligned}&\partial_t w_\xi - \Delta w_\xi + \chi\nabla\cdot(w_\xi \nabla v) - a w_\xi + b w_\xi^2 \\ &\qquad \geq \left(c_2\sqrt{a} - 2a - |\chi|(\delta + C_\delta\sqrt{\varepsilon})\left(\sqrt{a}+1\right)\right) w_\xi.\end{aligned}$$

To have the above ≥ 0 (then w_ξ is a supersolution), we need

$$c_2 \geq 2\sqrt{a} + |\chi|(\delta + C_\delta\sqrt{\varepsilon})(1 + 1/\sqrt{a}). \tag{3.3.59}$$

Since $c_2 > \sqrt{2a}$, we can now fix $\delta \in (0,1)$ and then $\varepsilon \in (0,\delta)$ to be sufficiently small such that (3.3.59) holds.

To use the comparison principle to conclude with $w_\xi \geq u$ for all $t \geq T_\varepsilon, |x| \geq c_2 t$, it remains to show that $w_\xi(T_\varepsilon, x) \geq u(T_\varepsilon, x)$ and $|x| \geq c_2 T_\varepsilon$, and $w_\xi(t,x) \geq u(t,x)$ with $t \geq T_\varepsilon$ and $|x| = c_2 t$. Indeed, it follows from (3.3.56) and (3.3.58) that on the bottom boundary,

$$u(T_\varepsilon, x) \leq M_1 e^{\sqrt{a}c_1 T_\varepsilon} e^{-\sqrt{a}|x|} \leq M_2 e^{\sqrt{a}c_2 T_\varepsilon} e^{-\sqrt{a}x\cdot\xi}.$$

On the lateral boundary of $|x| = c_2 t$, we have

$$u(t,x) \leq \|u\|_\infty \leq M_2 \leq w_\xi(t,x).$$

Overall, we can conclude that for all $\xi \in \mathbb{S}^{N-1}$,

$$u(t,x) \leq M_2 e^{-\sqrt{a}(|x| - c_2 t)} \quad \forall\, t \geq T_\varepsilon,\ |x| \geq c_2 t.$$

This, together with (3.3.54) and (3.3.56), finishes the proof of (3.3.4). □

Now we give the proof of Theorem 3.3.3(2)

Proof of Theorem 3.3.3(2). Since V is the solution to (3.3.5), we have

$$v(t,x) = V(t,x) - \frac{1}{\tau}\int_0^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds.$$

Because $u, v \geq 0$, it is direct to see that $v \leq V$, and so to prove the conclusion it suffices to estimate the following from above

$$\int_0^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{\tau|x-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds.$$

Fix $c'' > c^*_{\text{up}}(u_0,v_0)$ and fix x such that $|x| \geq c''t$, and take $c_1 := \frac{1}{2}(c'' + c^*_{\text{up}}(u_0,v_0))$ (so $c'' > c_1 > c^*_{\text{up}}(u_0,v_0)$). We decompose the double integral into two terms

$$\begin{aligned} &\int_0^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{|y|\geq c_1 s} e^{-\frac{\tau|x-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds \\ &\quad + \int_0^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{|y|\leq c_1 s} e^{-\frac{\tau|x-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds := Y^+ + Y^-. \end{aligned} \tag{3.3.60}$$

First, we estimate Y^-. Since u, v are uniformly bounded,

$$\begin{aligned} Y^- &\leq C\left(\frac{\tau}{\pi}\right)^{N/2} \int_0^t (4(t-s))^{-N/2} \int_{|y|\leq c_1 s} e^{-\frac{\tau|x-y|^2}{4(t-s)}}\,dy\,ds \\ &= C\left(\frac{\tau}{\pi}\right)^{N/2} \int_0^t \int_{|z|\leq \frac{c_1 s}{\sqrt{4(t-s)}}} e^{-\tau\left|\frac{x}{\sqrt{4(t-s)}}-z\right|^2}\,dz\,ds. \end{aligned}$$

Using that $|a-z|^2 \geq (|a|-|z|)^2$, $|x| \geq c''t$ and $c'' > c_1$, we get

$$\begin{aligned} \tau^{-N/2} Y^- &\leq C \int_0^t \int_{|z|\leq \frac{c_1 s}{\sqrt{4(t-s)}}} e^{-\tau\left(\frac{|x|}{\sqrt{4(t-s)}}-|z|\right)^2}\,dz\,ds \\ &\leq C \int_0^t \int_{|z|\leq \frac{c_1 s}{\sqrt{4(t-s)}}} e^{-\tau\left(\frac{c''t-c_1 s}{\sqrt{4(t-s)}}\right)^2}\,dz\,ds \\ &\leq C \int_0^t s^N (t-s)^{-\frac{N}{2}} e^{-\tau\frac{\left((c''-c_1)t+c_1(t-s)\right)^2}{4(t-s)}}\,ds \\ &\leq C \int_0^t s^N (t-s)^{-\frac{N}{2}} e^{-\tau\frac{(c''-c_1)^2 t^2}{4(t-s)}}\,ds. \end{aligned}$$

Note that there exist $C, \gamma > 0$ such that for all $s \in (0,t)$,

$$\tau^{-\frac{N}{2}}(t-s)^{-\frac{N}{2}} e^{-\tau\frac{(c''-c_1)^2t^2}{4(t-s)}} \leq Ce^{-\frac{2\tau\gamma t^2}{t-s}} \quad \text{and} \quad (\tau s)^N \leq Ce^{\gamma\tau s} \leq Ce^{\frac{\gamma\tau t^2}{t-s}}.$$

Thus we get

$$Y^- \leq C\int_0^t e^{-\frac{\gamma\tau t^2}{t-s}}\,ds \leq C\int_0^t e^{-\gamma\tau t}\,ds = Cte^{-\gamma\tau t} \tag{3.3.61}$$

which converges to 0 as $t \to \infty$.

Next, we estimate Y^+. By Theorem 3.3.3(1), there exist $C \geq 1$ and $\delta(= \sqrt{a}) > 0$ such that

$$u(s,y) \leq Ce^{-\delta(|y|-(c_1+c^*_{\mathrm{up}})s/2)} \qquad \text{for } |y| \geq c_1 s.$$

Thus, also using that v is uniformly bounded, we obtain

$$\begin{aligned}
Y^+ &= \int_0^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{|y|\geq c_1 s} e^{-\frac{\tau|x-y|^2}{4(t-s)}} u(s,y)v(s,y)\,dy\,ds \\
&\leq C\int_0^t \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} \int_{|y|\geq c_1 s} e^{-\frac{\tau|x-y|^2}{4(t-s)}} e^{-\delta(|y|-(c_1+c^*_{\mathrm{up}})s/2)}\,dy\,ds \\
&\leq C\iint_{(s\geq\frac{t}{2} \text{ or } |y|\geq\frac{c''t}{2})\cap D} \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} e^{-\frac{\tau|x-y|^2}{4(t-s)}} e^{-\delta(|y|-(c_1+c^*_{\mathrm{up}})s/2)}\,dy\,ds \\
&\quad + C\iint_{(s\leq\frac{t}{2} \text{ and } |y|\leq\frac{c''t}{2})\cap D} \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} e^{-\frac{\tau|x-y|^2}{4(t-s)}} e^{-\delta(|y|-(c_1+c^*_{\mathrm{up}})s/2)}\,dy\,ds \\
&=: Y_1^+ + Y_2^+,
\end{aligned}$$

where $D := \{(s,y) \,|\, s \in (0,t), |y| \geq c_1 s\}$. We first show that Y_1^+ is small. If $s \geq \frac{t}{2}$, since $c_1 > c^*_{\mathrm{up}}$, then in D we have for some $\gamma > 0$,

$$e^{-\delta(|y|-(c_1+c^*_{\mathrm{up}})s/2)} \leq e^{-\delta(c_1-c^*_{\mathrm{up}})s/2} \leq e^{-\gamma t}.$$

If $|y| \geq \frac{c''t}{2}$, also using that $|y| \geq c_1 s$ and $c_1 > c^*_{\mathrm{up}}$, there is $\gamma > 0$ such that

$$e^{-\delta(|y|-(c_1+c^*_{\mathrm{up}})s/2)} \leq e^{-\delta\left(|y|-\frac{c_1+c^*_{\mathrm{up}}}{2c_1}|y|\right)} \leq e^{-\gamma t}.$$

Applying these into the definition of Y_1^+ yields

$$\begin{aligned}
Y_1^+ &\leq Ce^{-\gamma t}\iint_{(s\geq\frac{t}{2} \text{ or } |y|\geq\frac{c''t}{2})\cap D} \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} e^{-\frac{\tau|x-y|^2}{4(t-s)}}\,dy\,ds \\
&\leq Ce^{-\gamma t}\int_0^t\int_{\mathbb{R}^N} \left(\frac{\tau}{4\pi(t-s)}\right)^{N/2} e^{-\frac{\tau|x-y|^2}{4(t-s)}}\,dy\,ds \leq Cte^{-\gamma t}.
\end{aligned} \tag{3.3.62}$$

Now we estimate Y_2^+. Recall that $c_1 = \frac{1}{2}(c'' + c^*_{\rm up})$, so

$$\begin{aligned} Y_2^+ &= C\int_0^{t/2}\int_{c_1 s\le |y|\le \frac{c'' t}{2}} \left(\frac{\tau}{4(t-s)}\right)^{N/2} e^{-\frac{\tau|x-y|^2}{4(t-s)}} e^{-\delta(|y|-(c_1+c^*_{\rm up})s/2)}\,dy\,ds \\ &\le C\tau^{\frac{N}{2}}\int_0^{t/2}\int_{\frac{c_1 s}{\sqrt{4(t-s)}}\le |z|\le \frac{c'' t}{2\sqrt{4(t-s)}}} e^{-\tau\left(\frac{|x|}{\sqrt{4(t-s)}}-|z|\right)^2}\,dz\,ds. \end{aligned}$$

Using $|x| \ge c'' t$ and $|z| \le \frac{c'' t}{2\sqrt{4(t-s)}}$ yields $\frac{|x|}{\sqrt{4(t-s)}} - |z| \ge \frac{c'' t}{2\sqrt{4(t-s)}}$. We get that for some $C, \gamma > 0$,

$$\begin{aligned} Y_2^+ &\le C\tau^{\frac{N}{2}}\int_0^{t/2}\int_{\frac{c_1 s}{\sqrt{4(t-s)}}\le |z|\le \frac{c'' t}{2\sqrt{4(t-s)}}} e^{-\tau\left(\frac{c'' t}{2\sqrt{4(t-s)}}\right)^2}\,dz\,ds \\ &\le C\tau^{\frac{N}{2}}\int_0^{t/2}\int_{|z|\le \frac{c''\sqrt{t}}{2\sqrt{2}}} e^{-\tau(c''\sqrt{t}/4)^2}\,dz\,ds \le C\tau^{\frac{N}{2}} t^{1+\frac{N}{2}} e^{-\gamma\tau t}. \end{aligned} \tag{3.3.63}$$

Overall, plugging (3.3.61), (3.3.62) and (3.3.63) into (3.3.60), it follows that

$$\sup_{|x|\ge c' t} |v(t,x) - V(t,x)| \le Cte^{-\gamma\tau t} + Cte^{-\gamma t} + C\tau^{\frac{N}{2}} t^{1+\frac{N}{2}} e^{-\gamma\tau t} \to 0 \quad \text{as } t\to\infty.$$

We conclude the proof. □

3.3.4 Existence of spreading speeds

In this subsection, we prove the existence of spreading speeds under certain natural conditions on v_0 and prove Theorems 3.3.4 and 3.3.5. Throughout this subsection, we assume that (u, v) is a classical solution to (3.3.6) for all time with initial data (u_0, v_0) satisfying the conditions stated in the Theorems. We start by proving some important Lemmas that would be useful in proving Theorems 3.3.4.

Lemma 3.3.4. *Under the assumptions of Theorem 3.3.4, assume that* $v_0 \in C_0(\mathbb{R}^N)$ *or* $v_0 \in L^p(\mathbb{R}^N)$ *for some* $p \ge 1$. *Then*

$$\lim_{t\to\infty}\|v(t,\cdot;u_0,v_0)\|_\infty = 0, \quad \lim_{t\to\infty}\|\nabla v(t,\cdot;u_0,v_0)\|_\infty = 0, \quad \lim_{t\to\infty}\|\Delta v(t,\cdot;u_0,v_0)\|_\infty = 0. \tag{3.3.64}$$

Proof of Lemma 3.3.4. We divide the proof into two steps.

Step 1. In this step, we prove the lemma for the case $v_0 \in C_0(\mathbb{R}^N)$.

First, note that, for any $\varepsilon > 0$, there is $R(\varepsilon) > 0$ such that

$$\left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z|>R(\varepsilon)} e^{-\tau|z|^2} \|v_0\|_\infty \, dz < \varepsilon,$$

and

$$|v_0(y)| < \varepsilon \quad \text{for all} \quad |y| \geq R(\varepsilon).$$

Note also that

$$|x + 2\sqrt{t}z| \geq R(\varepsilon) \quad \text{for all } |x| \geq 2\sqrt{t}R(\varepsilon) + R(\varepsilon),\ |z| \leq R(\varepsilon).$$

Hence, for any $t > 0$ and $|x| > 2\sqrt{t}R(\varepsilon) + R(\varepsilon)$,

$$\begin{aligned} v(t,x;u_0,v_0) &\leq \left(\frac{\tau}{\pi}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\tau|z|^2} v_0(x + 2\sqrt{t}z) \, dz \\ &\leq \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z|>R(\varepsilon)} e^{-\tau|z|^2} \|v_0\|_\infty \, dz + \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z|\leq R(\varepsilon)} e^{-\tau|z|^2} v_0(x + 2\sqrt{t}z) \, dz \\ &\leq \varepsilon + \varepsilon \left(\frac{\tau}{\pi}\right)^{N/2} \int_{|z|\leq R(\varepsilon)} e^{-\tau|z|^2} \, dz. \end{aligned} \tag{3.3.65}$$

Next, by Theorem 3.3.2(2), we have

$$\limsup_{t\to\infty} \sup_{|x|\leq ct} v(t,x;u_0,v_0) = 0 \quad \forall\, 0 < c < c^* := 2\sqrt{a}.$$

Therefore, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ such that

$$2\sqrt{t}R(\varepsilon) + R(\varepsilon) < \sqrt{a}t \quad \forall\, t \geq T(\varepsilon)$$

and

$$v(t,x;u_0,v_0) < \varepsilon \quad \forall\, t > T(\varepsilon),\ |x| \leq \sqrt{a}t. \tag{3.3.66}$$

By (3.3.65)–(3.3.66),

$$v(t,x;u_0,v_0) < \left(1 + \left(\frac{\tau}{\pi}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\tau|z|^2} \, dz\right) \varepsilon \quad \forall\, t > T(\varepsilon).$$

This implies that

$$\lim_{t\to\infty} \|v(t,\cdot;u_0,v_0)\|_\infty = 0.$$

Now, we assume for contradiction that there are $\varepsilon_0 > 0$, $t_n \to \infty$, and $x_n \in \mathbb{R}^N$ such that

$$|\nabla v(t_n, x_n; u_0, v_0)| + |\Delta v(t_n, x_n; u_0, v_0)| \geq \varepsilon_0 \quad \forall\, n \geq 1. \tag{3.3.67}$$

For $t \geq -t_n$ and $x \in \mathbb{R}^N$, consider

$$u_n(t,x) = u(t+t_n, x+x_n; u_0, v_0) \quad \text{and} \quad v_n(t,x) = v(t+t_n, x+x_n; u_0, v_0).$$

By the arguments for (3.3.13) and the statement below it, without loss of generality, we may assume that

$$\nabla v_n(t,x) \to 0, \quad \Delta v_n(t,x) \to 0 \quad \text{as } n \to \infty,$$

locally uniformly in $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. In particular,

$$|\nabla v_n(0,0)| + |\Delta v_n(0,0)| = |\nabla v(t_n, x_n)| + |\Delta v(t_n, x_n)| \to 0 \quad \text{as } n \to \infty,$$

which contradicts with (3.3.67). Hence, we proved (3.3.64) for the case when $v_0 \in C_0(\mathbb{R}^N)$.

Step 2. In this step, we prove the theorem for the case $v_0 \in L^p(\mathbb{R}^N)$ for some $p \geq 1$.

If $v_0 \equiv 0$, nothing needs to be proved. In the following, we assume that $v_0 \not\equiv 0$. Note that

$$\frac{\tau}{p}\frac{d}{dt}\int_{\mathbb{R}^N} v^p(t,x;u_0,v_0)\,dx = -(p-1)\int_{\mathbb{R}^N} v^{p-2}|\nabla v|^2\,dx - \int_{\mathbb{R}^N} uv^p\,dx \leq 0 \quad \forall\, t > 0. \tag{3.3.68}$$

Hence, $\int_{\mathbb{R}^N} v^p(t,x;u_0,v_0)\,dx$ is non-increasing as t increases. We claim that

$$\lim_{t\to\infty} \|v(t,\cdot;u_0,v_0)\|_\infty = 0.$$

In fact, assume this is not true and then there are $\varepsilon_0 > 0$, t_n strictly increasing to ∞, and $x_n \in \mathbb{R}^N$ such that

$$v(t_n, x_n; u_0, v_0) \geq \varepsilon_0. \tag{3.3.69}$$

By the similar arguments of (3.3.13) again, we may assume that there are $u^*(t,x)$ and $v^*(t,x)$ such that

$$\lim_{n\to\infty} u(t+t_n, x+x_n; u_0, v_0) = u^*(t,x), \quad \lim_{n\to\infty} v(t+t_n, x+x_n; u_0, v_0) = v^*(t,x)$$

locally uniformly in $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, and $v^*(t,x)$ satisfies

$$\tau v_t^* = \Delta v^* - u^* v^*, \quad t \in \mathbb{R},\ x \in \mathbb{R}^N.$$

Notice that v is bounded above by the solution to the heat equation with initial data v_0. Then, by the dominated convergence theorem and the monotonicity of $\int_{\mathbb{R}^N} v^p(t,x;u_0,v_0)\,dx$, we have

$$\int_{\mathbb{R}^N} (v^*(t,x))^p\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} v^p(t+t_n,x;u_0,v_0)\,dx \le \int_{\mathbb{R}^N} v_0^p(x)\,dx \quad \forall\, t\in\mathbb{R}. \tag{3.3.70}$$

On the other hand, for any $m \ge 1$, there is $m' \ge m$ such that $t_m + t_n \le t_{n+m'}$ for all $n = 1, 2, \cdots$. Then by (3.3.68) and (3.3.70),

$$\begin{aligned}\int_{\mathbb{R}^N} (v^*(t_m,x))^p\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}^N} v^p(t_m+t_n,x;u_0,v_0)\,dx\\ &\ge \lim_{n\to\infty}\int_{\mathbb{R}^N} v^p(t_{n+m'},x;u_0,v_0)\,dx = \int_{\mathbb{R}^N}(v^*(0,x))^p\,dx.\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^N}(v^*(0,x))^p\,dx \le \int_{\mathbb{R}^N}(v^*(t_m,x))^p\,dx \quad \forall\, m = 1, 2, \cdots. \tag{3.3.71}$$

By (3.3.68) with v and u being replaced by v^* and u^*, respectively, we have that $\int_{\mathbb{R}^N}(v^*(t,x))^p\,dx$ is non-increasing. This, together with (3.3.71), yields that

$$\int_{\mathbb{R}^N}(v^*(t,x))^p\,dx = \int_{\mathbb{R}^N}(v^*(0,x))^p\,dx \quad \forall\, t > 0.$$

By (3.3.68) with v and u being replaced by v^* and u^* again, we find

$$0 = \frac{\tau}{p}\frac{d}{dt}\int_{\mathbb{R}^N}(v^*(t,x))^p\,dx = -(p-1)\int_{\mathbb{R}^N}(v^*)^{p-2}|\nabla v^*|^2\,dx - \int_{\mathbb{R}^N} u^*(v^*)^p\,dx.$$

Thus, we must have $v^*(t,x) \equiv$ constant, and since $\int_{\mathbb{R}^N}(v^*(t,x))^p\,dx < \infty$, we have $v^*(t,x) \equiv 0$. This clearly contradicts with (3.3.69). Hence, the claim holds. By the arguments in Step 1, (3.3.64) holds for the case when $v_0 \in L^p(\mathbb{R}^N)$ for some $p \ge 1$. □

We briefly recall viscosity solutions. We refer readers to [9] for more details. This notion of solutions, as well as the comparison principle, will be one of the main tools we use in proving Theorem 3.3.5.

Consider the following parabolic type equation:

$$u_t + F(t,x,u,\nabla u, D^2u) = 0. \tag{3.3.72}$$

Let $\mathcal{S}^N$ denote the set of $N \times N$ symmetric matrices with the spectral norm. We say that F is uniformly elliptic if there exists $\Lambda > 0$ such that for any positive semi-definite matrix $P \in \mathcal{S}^N$, and any $(t, x, u, p, X) \in [0, \infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$,

$$\Lambda \operatorname{Tr}(P) \leq F(t, x, u, p, X) - F(t, x, u, p, X + P).$$

We assume F to be continuous and uniformly elliptic.

Now we recall the definition of viscosity solutions. Let $\Omega \subseteq \mathbb{R}^N$ be open and $T > 0$.

(i) We say that an upper semicontinuous (resp. lower semicontinuous) function $u : (0, T) \times \Omega \to \mathbb{R}$ is a (viscosity) subsolution (resp. (viscosity) supersolution) to (3.3.72) if the following holds: for any smooth function ϕ on $(0, T) \times \Omega$ such that $u - \phi$ has a local maximum (resp. minimum) at $(t_0, x_0) \in (0, T) \times \Omega$, we have

$$\partial_t \phi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), \nabla \phi(t_0, x_0), D^2 \phi(t_0, x_0)) \leq 0$$

$$\left(\text{resp. } \partial_t \phi(t_0, x_0) + F(t_0, x_0, u(t_0, x_0), \nabla \phi(t_0, x_0), D^2 \phi(t_0, x_0)) \geq 0\right).$$

(ii) We say that a continuous function $u : (0, T) \times \Omega \to \mathbb{R}$ is a (viscosity) solution to (3.3.72) if it is both a subsolution and a supersolution.

It is easy to see that a classical solution is a viscosity solution. For the purpose of the paper, we take

$$F(t, x, u, \nabla u, D^2 u) = -\Delta u + f(t, x) \cdot \nabla u + g(t, x, u) \tag{3.3.73}$$

where f, g are uniformly continuous and bounded functions. It is easy to check that the operator satisfies condition (3.14) in [9]. Consequently, we have the following comparison principle.

Lemma 3.3.5. *Let* $T > 0$ *and let* F *be given in* (3.3.73). *Let*

$$u^+ : (0, T) \times \mathbb{R}^N \to \mathbb{R} \quad \textit{and} \quad u^- : (0, T) \times \mathbb{R}^N \to \mathbb{R}$$

be, respectively, a supersolution and a subsolution to (3.3.72). *If*

$$u^+(0, \cdot) \geq u^-(0, \cdot) \quad \textit{and} \quad \inf_{t \in (0,T)} \liminf_{|x| \to \infty} \Big(u^+(t, x) - u^-(t, x) \Big) \geq 0,$$

then

$$u^+(t, x) \geq u^-(t, x) \quad \textit{for all } (t, x) \in (0, T) \times \mathbb{R}^N.$$

We refer readers to Sections 5D and 8 of [9] for the proof and for more general cases. Next we prove a series of Lemmas that would be used in establishing existence of spreading when $|\chi| \ll 1$ and when v_0 is not small for large $|x|$. We focus on the case that $\tau = 1$, and we consider (3.3.6) with $\sigma \in (0, 1]$. To prove Theorem 3.3.4(2) and Theorem 3.3.5, we make the following change of variable,

$$\zeta := 1 - v.$$

Then (u, ζ) satisfies

$$\begin{cases} u_t = \Delta u + \chi \nabla \cdot (u \nabla \zeta) + u(a - bu^\sigma), & x \in \mathbb{R}^N,\ t > 0, \\ \zeta_t = \Delta \zeta + u(1 - \zeta), & x \in \mathbb{R}^N,\ t > 0 \end{cases}$$

with initial data u_0 and $\zeta_0 := 1 - v_0$. We will estimate $w := \zeta^p / u$ for some $p > 1$.

To this end, we first discuss the inf- and sup-convolution technique. For $T > 0$, suppose $\rho_1, \rho_2 \in C^\infty((0, T) \times \mathbb{R}^N)$ and let $r(t) \in C^\infty((0, T))$ be non-negative. Define

$$\overline{\rho}(t, x) := \sup_{y \in B(x, r(t))} \rho_1(t, y), \qquad \underline{\rho}(t, x) := \inf_{y \in B(x, r(t))} \rho_2(t, y). \tag{3.3.74}$$

Then $\overline{\rho}$ and $\underline{\rho}$ are Lipschitz continuous. Let $y_{1,t} = y_{1,t}(x) \in \overline{B(x, r(t))}$ be such that $\overline{\rho}(\cdot, t) = \rho_1(t, y_{1,t}(\cdot))$. Then the following holds:

$$(\Delta \overline{\rho})(t, x) \geq (\Delta \rho_1)(t, y_{1,t}(x)), \quad (\nabla \overline{\rho})(t, x) = (\nabla \rho_1)(t, y_{1,t}(x)) \tag{3.3.75}$$

and

$$(\partial_t \overline{\rho})(t, x) = (\partial_t \rho_1)(t, y_{1,t}(x)) + r'(t) |\nabla \rho_1|(t, y_{1,t}(x)). \tag{3.3.76}$$

The first inequality in (3.3.75) needs to be understood in the viscosity sense. The proof can be found in Lemmas 5.2, 5.3 [38] and Lemma 5.4 [37] for a more general case. Similarly, assuming $y_{2,t} = y_{2,t}(x) \in \overline{B(x, r(t))}$ to satisfy that $\underline{\rho}(\cdot, t) = \rho_2(t, y_{2,t}(\cdot))$, we have

$$(\Delta \underline{\rho})(t, x) \leq (\Delta \rho_2)(t, y_{2,t}(x)), \quad (\nabla \underline{\rho})(t, x) = (\nabla \rho_2)(t, y_{2,t}(x)),$$

and

$$(\partial_t \underline{\rho})(t, x) = (\partial_t \rho_2)(t, y_{2,t}(x)) - r'(t) |\nabla \rho_2|(t, y_{2,t}(x)).$$

Let us specify the selection of parameters. For given $r_0 > 0$, set ρ_1 to be the unique solution to

$$\partial_t \rho_1 = \Delta \rho_1, \quad \rho_1(0, x) = \inf_{y \in B(x, r_0)} u_0(y), \quad \forall\, x \in \mathbb{R}^N,$$

and ρ_2 the unique solution to

$$\partial_t \rho_2 = \Delta \rho_2, \quad \rho_2(0,x) = \sup_{y \in B(x,r_0)} u_0(y), \quad \forall\, x \in \mathbb{R}^N.$$

Then, let $\overline{\rho}$ and $\underline{\rho}$ be defined in (3.3.74) with $r(t) := r_0(1 - t/(4\beta))$ for some $\beta \in (0,1]$. The construction immediately yields that

$$\overline{\rho}(0,x) = \sup_{y \in B(x,r_0)} \inf_{y' \in B(y,r_0)} u_0(y') \le u_0(x) \le \inf_{y \in B(x,r_0)} \sup_{y' \in B(y,r_0)} u_0(y') = \underline{\rho}(0,x). \quad (3.3.77)$$

It follows from (3.3.75) and (3.3.76) that

$$\overline{\rho}_t \le \Delta \overline{\rho} - (4\beta)^{-1} r_0 |\nabla \overline{\rho}| \quad \text{for} \quad (t,x) \in [0,4\beta) \times \mathbb{R}^N. \quad (3.3.78)$$

Similarly, we have

$$\underline{\rho}_t \ge \Delta \underline{\rho} + (4\beta)^{-1} r_0 |\nabla \underline{\rho}| \quad \text{for} \quad (t,x) \in [0,4\beta) \times \mathbb{R}^N. \quad (3.3.79)$$

Let us comment that here and below, inequalities involving derivatives of sup- or inf-convolutions are understood in the viscosity sense. So (3.3.78)–(3.3.77) and the comparison principle yield that $\overline{\rho} \le \underline{\rho}$ in $[0,4\beta) \times \mathbb{R}^N$.

By the assumption, if $r_0 > 0$ is sufficiently small depending on u_0, we have that $\rho_i(0,x) \not\equiv 0$ with $i = 1,2$. We fix one such $r_0 \in (0,1)$. We claim that for any $p > 1$ there exists $C = C(p) > 0$ such that

$$\underline{\rho}(t,x)^p \le C \overline{\rho}(t,x) \quad \forall\, (t,x) \in [\beta, 2\beta] \times \mathbb{R}^N. \quad (3.3.80)$$

Let $R \ge 1$ be such that $u_0(\cdot)$ is supported inside B_R. Then $\rho_i(0,\cdot)$ with $i = 1,2$ are supported in B_{R+1}. Note that

$$\rho_i(t,x) := (4\pi t)^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \rho_i(0,y)\, dy,$$

and when $|x| \ge LR$ for $L \ge 4$ and $|y| \le R+1$, we have

$$e^{-\frac{|x|^2(1+2/L)^2}{4t}} \le e^{-\frac{|x-y|^2}{4t}} \le e^{-\frac{|x|^2(1-2/L)^2}{4t}}.$$

This implies that for $t \in [\beta, 2\beta]$, we have

$$\rho_1(t,x) \geq (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2(1+2/L)^2}{4t}} \int_{|y|\leq R+1} \rho_1(0,y)\, dy$$

and

$$\rho_2(t,x)^p \leq (4\pi t)^{-\frac{Np}{2}} e^{-\frac{p|x|^2(1-2/L)^2}{4t}} \Big(\int_{|y|\leq R+1} \rho_2(0,y)\, dy \Big)^p.$$

Thus, by picking $L \gg 1$ such that $p \geq \left(\frac{L+2}{L-2}\right)^2$, we get for all $|x| \geq LR$ and $t \in [\beta, 2\beta]$ that

$$\begin{aligned} \rho_2(t,x)^p &\leq C e^{-\frac{|x|^2(1+2/L)^2}{4t}} \Big(\int_{|y|\leq R+1} \rho_2(0,y)\, dy \Big)^p \\ &\leq C e^{-\frac{|x|^2(1+2/L)^2}{4t}} \int_{|y|\leq R+1} \rho_1(0,y)\, dy \leq C \inf_{y\in B(x,r)} \rho_1(t,x). \end{aligned}$$

If $|x| \leq LR$, since ρ_i are strictly positive, the same holds with possibly a larger C in the compact set $[\beta, 2\beta] \times B(0, LR)$. Overall, we can find $C > 0$ such that for $t \in [\beta, 2\beta]$,

$$\underline{\rho}(t,x)^p = \inf_{y\in B(x,r(t))} \rho_2(t,y)^p \leq C \sup_{y\in B(x,r(t))} \rho_1(t,y) = C\overline{\rho}(t,x)$$

which yields (3.3.80).

Similarly, since $\zeta(0,\cdot)$ is compactly supported, the same argument yields that

$$\hat{\zeta}(t,x)^p \leq C\overline{\rho}(t,x) \quad \text{ for } (t,x) \in [\beta, 2\beta] \times \mathbb{R}^N, \tag{3.3.81}$$

where $\hat{\zeta}$ is the unique solution to the heat equation with initial data ζ_0.

Below we use $\overline{\rho}$ and $\underline{\rho}$ to show that $\zeta^p \lesssim u$ in a positive finite time interval. In the proof, we need to estimate u from above. So, as a by-product, we also obtain that $u^p \lesssim \zeta$ in the short time.

Since $u_0 \in X_1^+$ and $v_0 \in X_1^+ \cap C^{2+\alpha,b}_{\text{unif}}(\mathbb{R}^N)$, by Lemma 2.5.2 and Remark 3.2.1, there exist $\chi_1 \in (0,1)$ and $A = A(\|v_0\|_{X_1}, \|u_0\|_\infty) > 0$ such that as long as $|\chi| \leq \chi_1$, we have

$$\sup_{t\geq 0} \Big(\|\nabla \zeta(t,\cdot)\|_\infty + \|\Delta \zeta(t,\cdot)\|_\infty \Big) = \sup_{t\geq 0} \Big(\|\nabla v(t,\cdot)\|_\infty + \|\Delta v(t,\cdot)\|_\infty \Big) \leq A. \tag{3.3.82}$$

Lemma 3.3.6. *Assume $|\chi| \leq \chi_1$. There exists $\beta \in (0,1]$ such that for any $p > 1$ we can find $L \geq 4$ such that for all $(t,x) \in [\beta, 2\beta] \times \mathbb{R}^N$ we have*

$$\zeta(t,x)^p \leq Lu(t,x) \quad \textit{and} \quad u(t,x)^p \leq L\zeta(t,x).$$

Proof of Lemma 3.3.6. Take A and r_0 as above, and let

$$\beta = \min\{1, r_0/(4A)\}. \tag{3.3.83}$$

Let $\overline{\rho}$ and $\underline{\rho}$ be defined as above in $[0, 4\beta) \times \mathbb{R}^N$. Let $\underline{\varphi}(t,x) := e^{Mt}\underline{\rho}(t,x)$ for some $M > 0$ to be determined. Then, by (3.3.79),

$$\underline{\varphi}_t - \Delta\underline{\varphi} \geq M\underline{\varphi} + (4\beta)^{-1}r_0|\nabla\underline{\varphi}|.$$

It follows that

$$\underline{\varphi}_t - \Delta\underline{\varphi} - \chi\nabla\cdot(\underline{\varphi}\nabla\zeta) - \underline{\varphi}(a - b\underline{\varphi}^\sigma) \geq M\underline{\varphi} + ((4\beta)^{-1}r_0 - A|\chi|)|\nabla\underline{\varphi}| - A\chi\underline{\varphi} - a\underline{\varphi},$$

and so, using (3.3.83) and taking $M \geq A + a \geq A|\chi| + a$, $\underline{\varphi}$ is a supersolution to the equation satisfied by u. Also recall (3.3.77), the comparison principle yields $\underline{\varphi} \geq u$.

On the other hand, let $\overline{\varphi}(t,x) := e^{-Mt}\overline{\rho}(t,x)$ and we view $-(a - bu^\sigma)$ as a function of (t,x) that is bounded from above by $C > 0$. Using (3.3.78), we have

$$\overline{\varphi}_t - \Delta\overline{\varphi} - \chi\nabla\cdot(\overline{\varphi}\nabla\zeta) - \overline{\varphi}(a - bu^\sigma) \leq -M\overline{\varphi} - ((4\beta)^{-1}r_0 - A|\chi|)|\nabla\overline{\varphi}| + A|\chi|\overline{\varphi} + C\overline{\varphi} \leq 0,$$

after further assuming $M \geq A + C \geq A|\chi| + C$. Then $\overline{\varphi}$ is a subsolution to the above linear equation and the comparison principle yields $\overline{\varphi} \leq u$. Overall, we obtain for all $t \in [0, 2\beta]$,

$$e^{-2M}\overline{\rho}(t,x) \leq u(t,x) \leq e^{2M}\underline{\rho}(t,x). \tag{3.3.84}$$

Now we estimate ζ. Let us start with the upper bound. Recall that $\hat{\zeta}$ is defined as the solution to the heat equation with initial data ζ_0. We claim that $\zeta \leq \hat{\zeta} + te^{2M}\underline{\rho} =: \underline{\zeta}$ in $[0, 2\beta] \times \mathbb{R}^N$. This is because $\zeta(0,\cdot) = \hat{\zeta}(0,\cdot) = \underline{\zeta}(0,\cdot)$ and, by (3.3.79) and (3.3.84),

$$\underline{\zeta}_t - \Delta\underline{\zeta} \geq e^{2M}\underline{\rho} \geq u,$$

and ζ satisfies $\zeta_t - \Delta\zeta = u(1-\zeta) \leq u$.

For the lower bound, take

$$\overline{\zeta}(t,x) := \alpha t\overline{\rho}(t,x) \qquad \text{with } \alpha \in (0, 2^{-1}e^{-2M}).$$

Then $\overline{\zeta}(0,\cdot) \equiv 0 \leq \zeta_0$. Let us fix $\alpha > 0$ to be sufficiently small such that $\overline{\zeta}(t,\cdot) \leq \frac{1}{2}$ for

$t \in [0, 2\beta]$. Within the time interval, by (3.3.78), (3.3.84) and $\alpha < e^{-2M}/2$, we get

$$\overline{\zeta}_t - \Delta\overline{\zeta} \leq \alpha\overline{\rho} \leq 2^{-1}u \leq u(1-\overline{\zeta}).$$

Therefore, the comparison principle yields $\zeta(t,\cdot) \geq \overline{\zeta}$ for $(t,x) \in [0,2\beta] \times \mathbb{R}^N$. Overall, we obtain for $(t,x) \in [0,2\beta] \times \mathbb{R}^N$,

$$t\alpha\overline{\rho}(t,x) \leq \zeta(t,x) \leq \hat{\zeta}(t,x) + te^{2M}\underline{\rho}(t,x),$$

which, combining with (3.3.80) and (3.3.81), yields

$$C^{-1}\overline{\rho}(t,x)^p \leq \zeta(t,x)^p \leq C\overline{\rho}(t,x) \quad \forall (t,x) \in [\beta, 2\beta] \times \mathbb{R}^N, \text{ for some } C > 1.$$

Finally, since (3.3.80) and (3.3.84) imply

$$C^{-1}\overline{\rho}(t,x)^p \leq u(t,x)^p \leq C\overline{\rho}(t,x) \quad \forall (t,x) \in [\beta, 2\beta] \times \mathbb{R}^N, \text{ for some } C > 1,$$

the conclusion follows immediately. $\square$

Below we prove that $\zeta^p \lesssim u$ for all $t \geq \beta$.

Lemma 3.3.7. *Let* $\beta, L = L(\beta, p)$ *from Lemma 3.3.6 with some* $p > 1$*, and let* χ_1, A *from* (3.3.82)*. If* $|\chi| \leq \min\{\frac{a}{2A}, \chi_1\}$ *and* $-\chi \leq p-1$*, then for all* $(t,x) \in [\beta, \infty) \times \mathbb{R}^N$*,*

$$\zeta(t,x)^p \leq Mu(t,x) \quad \text{where } M := \max\left\{L,\, 4p/a,\, 2(4b/a)^{1/\sigma}\right\}. \tag{3.3.85}$$

Proof of Lemma 3.3.7. Let us consider $w := \frac{\zeta^p}{u}$, which is well-defined and $\leq L$ at least for $t \in [\beta, 2\beta]$ by Lemma 3.3.6. Since the solutions are smooth for all positive times, it suffices to prove a priori estimate that w stays uniformly bounded for all $t \geq \beta$.

By direct computation,

$$\begin{aligned}
\nabla w &= \nabla\left(\frac{\zeta^p}{u}\right) = \frac{u\nabla(\zeta^p) - \zeta^p\nabla u}{u^2},\\
\Delta w &= \Delta\left(\frac{\zeta^p}{u}\right) = \frac{u\Delta(\zeta^p) - \zeta^p\Delta u}{u^2} - \frac{2\nabla u(u\nabla(\zeta^p) - \zeta^p\nabla u)}{u^3}\\
&= \frac{pu\zeta^{p-1}\Delta\zeta - \zeta^p\Delta u}{u^2} + \frac{p(p-1)w|\nabla\zeta|^2}{\zeta^2} - \frac{2\nabla u\cdot\nabla w}{u},\\
\frac{\zeta^p\nabla\cdot(u\nabla\zeta)}{u^2} &= \frac{\zeta^p\Delta\zeta}{u} + \frac{\zeta^p\nabla u\cdot\nabla\zeta - u\nabla(\zeta^p)\cdot\nabla\zeta}{u^2} + \frac{u\nabla(\zeta^p)\cdot\nabla\zeta}{u^2}\\
&= w\Delta\zeta - \nabla w\cdot\nabla\zeta + \frac{pw|\nabla\zeta|^2}{\zeta},
\end{aligned}$$

and

$$\frac{\zeta^p(a - bu^\sigma) - pu\zeta^{p-1}(1-\zeta)}{u} = aw - bu^\sigma w - p\zeta^{p-1}(1-\zeta).$$

Also by the equations, we have

$$\begin{aligned}
w_t &= \frac{\zeta_t^p u - \zeta^p u_t}{u^2} = \frac{pu\zeta^{p-1}(\Delta\zeta + u(1-\zeta)) - \zeta^p(\Delta u + \chi\nabla\cdot(u\nabla\zeta) + u(a - bu^\sigma))}{u^2}\\
&= \frac{pu\zeta^{p-1}\Delta\zeta - \zeta^p\Delta u}{u^2} - \frac{\chi\zeta^p\nabla\cdot(u\nabla\zeta)}{u^2} - \frac{\zeta^p(a - bu^\sigma) - p\zeta^{p-1}u(1-\zeta)}{u}.
\end{aligned}$$

Putting these together, we obtain that w satisfies

$$\begin{aligned}
w_t = \Delta w &+ \frac{2\nabla u\cdot\nabla w}{u} - \frac{p(p-1)w|\nabla\zeta|^2}{\zeta^2} - \chi\left(w\Delta\zeta - \nabla\zeta\cdot\nabla w + \frac{pw|\nabla\zeta|^2}{\zeta}\right)\\
&- (a - bu^\sigma)w + p\zeta^{p-1}(1-\zeta).
\end{aligned}$$

Due to $|\chi\Delta\zeta| \le \frac{a}{2}$ and $-\chi \le p-1$ by the assumption, and $\zeta\in[0,1]$, we get

$$w_t \le \Delta w + \frac{2\nabla u\cdot\nabla w}{u} - \chi\nabla\zeta\cdot\nabla w - \left(\frac{a}{2} - bu^\sigma\right)w + p. \tag{3.3.86}$$

Note that if $bu^\sigma \ge \frac{a}{4}$, we have $u \ge \left(\frac{a}{4b}\right)^{1/\sigma} =: c_*$ and $w = \frac{\zeta^p}{u} \le \frac{1}{c_*}$, and otherwise, we have $\left(\frac{a}{2} - bu^\sigma\right) \ge \frac{a}{4}$. Therefore,

$$\left(\frac{a}{2} - bu^\sigma\right)w \ge \frac{aw}{4} - bu^\sigma w\Psi(w),$$

where Ψ is Lipschitz continuous and satisfies $\Psi(w) \in [\mathbf{1}_{\{w\le 1/c_*\}}, \mathbf{1}_{\{w\le 2/c_*\}}]$ with $\mathbf{1}$ denoting the characteristic function. We deduce from (3.3.86) that

$$w_t \leq \Delta w + \frac{2\nabla u \cdot \nabla w}{u} - \chi \nabla\zeta \cdot \nabla w - \frac{aw}{4} + bu^\sigma w \Psi(w) + p.$$

Recall that $w(\beta, \cdot) \leq L$ by Lemma 3.3.6 and u is uniformly finite. We can compare w with the solution to the following ODE ($z = z(t)$)

$$\frac{d}{dt} z = -\frac{az}{4} + b\|u\|_\infty^\sigma z \Psi(z) + p, \quad z(\beta) = L,$$

to get for all $x \in \mathbb{R}^N$ and $t \geq \beta$,

$$w(t,x) \leq z(t) \leq \max\{L, 4p/a, 2/c_*\} = M.$$

which implies (3.3.85). $\square$

After obtaining the estimate $\zeta^p \leq Mu$, we are able to bound $|\nabla\zeta|$ and $|D^2\zeta|$ in terms of u.

Lemma 3.3.8. *Assume $|\chi| \leq \chi_1$. For any $p' > 1$, there exists $C = C(p') > 0$ such that the following holds. For all $(t,x) \in [1,\infty) \times \mathbb{R}^N$ we have*

$$|\nabla u(t,x)| \leq C u(t,x)^{\frac{1}{p'}}.$$

Under the assumptions of Lemma 3.3.7 and for p, M from the lemma, we have for all $(t,x) \in [3,\infty) \times \mathbb{R}^N$,

$$|\nabla\zeta(t,x)|, \quad |D^2\zeta(t,x)| \leq CM^{\frac{1}{p}} u(t,x)^{\frac{1}{p'p}}.$$

Proof of Lemma 3.3.8. Since $|\chi| \leq \chi_1$, the first claim follows from Lemma 2.6.1.

The estimate for $|\nabla\zeta|$ follows similarly. Indeed, by the equation of ζ, the local L^q-parabolic estimates (see for e.g., [45, Theorem 7.22]) yield for $t \geq 2$ and any $x \in \mathbb{R}^N$,

$$\begin{aligned}\|D^2\zeta\|_{L^{N+3}([t-\frac{1}{2},t]\times B(x,\frac{1}{2}))} &\leq C\left(\|u(1-\zeta)\|_{L^{N+3}([t-1,t]\times B(x,1))} + \|\zeta\|_{L^{N+3}([t-1,t]\times B(x,1))}\right)\\ &\leq C\left(\|u\|_{L^{N+3}([t-1,t]\times B(x,1))} + \|\zeta\|_{L^{N+3}([t-1,t]\times B(x,1))}\right).\end{aligned}$$

The anisotropic Sobolev embedding ([11, Lemma A3]) yields for any $(t,x) \in [2,\infty) \times \mathbb{R}^N$,

$$|\nabla\zeta(t,x)| \leq C\left(\|u\|_{L^{N+3}([t-1,t]\times B(x,1))} + \|\zeta\|_{L^{N+3}([t-1,t]\times B(x,1))}\right). \tag{3.3.87}$$

Since $t \geq 2$, applying (3.3.85) and Lemma 2.6.1 (with $R = 1, s_0 \in [0,1]$ and $p = p'$), we get for some $C = C(p')$,

$$|\nabla\zeta(t,x)| \leq C\left(\|u\|_{L^{N+3}([t-1,t]\times B(x,1))} + M^{\frac{1}{p}}\|u^{\frac{1}{p}}\|_{L^{N+3}([t-1,t]\times B(x,1))}\right) \leq CM^{\frac{1}{p}}u(t,x)^{\frac{1}{p'p}}.$$

For the last claim, note that $q := \nabla\zeta$ satisfies

$$q_t - \Delta q + uq = (\nabla u)(1-\zeta).$$

So Theorem 7.22 in [45], (3.3.85) and (3.3.87) yield for $t \in [3,\infty)$,

$$\begin{aligned}\|D^2 q\|_{L^{N+3}([t-\frac{1}{2},t]\times B(x,\frac{1}{2}))} &\leq C\left(\|\nabla u\|_{L^{N+3}([t-1,t]\times B(x,1))} + \|q\|_{L^{N+3}([t-1,t]\times B(x,1))}\right)\\ &\leq C\Big(\|\nabla u\|_{L^{N+3}([t-1,t]\times B(x,1))} + \|u\|_{L^{N+3}([t-2,t]\times B(x,2))} + M^{\frac{1}{p}}\|u^{\frac{1}{p}}\|_{L^{N+3}([t-2,t]\times B(x,2))}\end{aligned}$$

Again by the anisotropic Sobolev embedding and Lemma 2.6.1, there exists $C = C(p')$ such that for $t \in [3,\infty)$,

$$|D^2\zeta(t,x)| \leq C\|D^2 q\|_{L^{N+3}([t-\frac{1}{2},t]\times B(x,\frac{1}{2}))} \leq CM^{\frac{1}{p}}u(t,x)^{\frac{1}{p'p}}.$$

□

We now prove Theorem 3.3.4(1) using Lemma 3.3.4.

Proof of Theorem 3.3.4(1). First, by Lemma 3.3.4 for any fixed $\varepsilon > 0$, there exists a $T_\varepsilon > 1$ such that

$$|\nabla v(t,x;u_0,v_0)| \leq \varepsilon,\ |\Delta v(t,x;u_0,v_0)| \leq \varepsilon \qquad \forall\, t \geq T_\varepsilon,\ x \in \mathbb{R}^N. \tag{3.3.88}$$

Next, take $c := 2\sqrt{a} + |\chi|\varepsilon(1 + 1/\sqrt{a})$. The rest of the proof is similar to the one of Theorem 3.3.3(1). Indeed, let $c_1 = 2\sqrt{a} + |\chi|A(1 + 1/\sqrt{a})$ and M_1 from (3.3.55), and for any $\xi \in \mathbb{S}^{N-1}$, define

$$u_\xi(t,x) := M_1 e^{-\sqrt{a}(x\cdot\xi - c(t-T_\varepsilon) - c_1 T_\varepsilon)}.$$

Then, by (3.3.88), for all $t \geq T_\varepsilon$ and $x \in \mathbb{R}^N$,

$$\partial_t u_\xi = c\sqrt{a}u_\xi = (2a + |\chi|\varepsilon(\sqrt{a}+1))u_\xi \geq \Delta u_\xi - \chi\nabla\cdot(u_\xi\nabla v) + au_\xi - bu_\xi^2.$$

At $t = T_\varepsilon$, using (3.3.56), we have

$$u(T_\varepsilon, x; u_0, v_0) \le M_1 e^{-\sqrt{a}(x\cdot\xi - c_1 T_\varepsilon)} = u_\xi(T_\varepsilon, x) \qquad \forall\, x \in \mathbb{R}^N \text{ and } \xi \in \mathbb{S}^{N-1}.$$

Thus it follows from the comparison principle that

$$u(t, x; u_0, v_0) \le u_\xi(t, x) \qquad \forall\, x \in \mathbb{R}^N,\ t \ge T_\varepsilon \text{ and } \xi \in \mathbb{S}^{N-1}.$$

This implies that

$$c^*_{\text{up}}(u_0, v_0) \le 2\sqrt{a} + |\chi|\varepsilon(1 + 1/\sqrt{a}) \quad \forall\, \varepsilon > 0.$$

Passing $\varepsilon \to 0$ and applying Theorem 3.3.2(1) show

$$c^*_{\text{low}}(u_0, v_0) = c^*_{\text{up}}(u_0, v_0) = 2\sqrt{a}.$$

□

Next, we prove Theorem 3.3.4(2) and we recall that here $\sigma = 1$ and $\chi < 0$. We would use Lemma 3.3.6–Lemma 3.3.8 to establish Theorem 3.3.4(2) and Theorem 3.3.5

Proof of Theorem 3.3.4(2). First, let

$$w = u - \frac{\chi}{2}|\nabla\zeta|^2.$$

Note that

$$\nabla\zeta \cdot \nabla(\Delta\zeta) = \frac{1}{2}\Delta|\nabla\zeta|^2 - |D^2\zeta|^2.$$

Then w satisfies

$$w_t = \Delta w + \chi u\Delta\zeta + \chi|D^2\zeta|^2 + \chi\zeta\nabla u\cdot\nabla\zeta + \chi u|\nabla\zeta|^2 + u(a - bu).$$

Write $\tilde{\chi} = -\chi > 0$, and by Young's inequality we get

$$\begin{aligned} w_t &= \Delta w - \tilde{\chi}u\Delta\zeta - \tilde{\chi}|D^2\zeta|^2 - \tilde{\chi}\zeta\nabla u\cdot\nabla\zeta - \tilde{\chi}u|\nabla\zeta|^2 + u(a - bu) \\ &\le \Delta w + \frac{N\tilde{\chi}}{4}u^2 - \tilde{\chi}\zeta\nabla u\cdot\nabla\zeta - \tilde{\chi}u|\nabla\zeta|^2 + aw - \frac{a\tilde{\chi}}{2}|\nabla\zeta|^2 - bu^2 \\ &\le \Delta w + \frac{N\tilde{\chi}}{4}u^2 + \frac{\tilde{\chi}}{2a}\zeta^2|\nabla u|^2 - \tilde{\chi}u|\nabla\zeta|^2 + aw - bu^2. \end{aligned}$$

By the assumptions in Theorem 3.3.4(2) and Lemmas 3.3.7 and 3.3.8, there are $C > 0$

and $\chi_2 \in (0, \chi_1)$ such that if $|\chi| \leq \chi_2$,

$$|\nabla u| \leq Cu^{\frac{1}{2}} \quad \text{and} \quad \zeta \leq Cu^{\frac{1}{2}} \quad \text{in } [1, \infty) \times \mathbb{R}^N.$$

With these, w satisfies

$$w_t \leq \Delta w + \frac{N\tilde{\chi}}{4}u^2 + \frac{C\tilde{\chi}}{2a}u^2 + aw - bu^2.$$

which implies that if $\tilde{\chi} \leq \chi_3 := \frac{2ab}{aN+2C}$, then

$$w_t \leq \Delta w + aw - \frac{b}{2}w^2.$$

This shows that the spreading speed of u is $c^* = 2\sqrt{a}$ when $0 < -\chi \leq \chi_0 := \min\{\chi_2, \chi_3\}$. □

Finally, we prove Theorem 3.3.5. Here we consider the case of $\sigma \in (0, 1)$, while allowing $\chi > 0$.

Proof of Theorem 3.3.5. Since $\sigma < 1$, it follows from the assumptions in Theorem 3.3.5 and Lemma 3.3.8 that there exist $C > 0$ and $\chi_2 \in (0, \chi_1)$ such that as long as $|\chi| \leq \chi_2$,

$$|\nabla u|, \quad |\nabla \zeta| \leq Cu^{\frac{1+\sigma}{2}} \quad \text{and} \quad |D^2\zeta| \leq Cu^{\sigma} \quad \text{in } [3, \infty) \times \mathbb{R}^N.$$

Hence, from the equation, we get for $t \geq 3$,

$$\begin{aligned} u_t &= \Delta u + \chi \nabla u \cdot \nabla \zeta + \chi u \Delta \zeta + u(a - bu^{\sigma}) \\ &\leq \Delta u + au - bu^{1+\sigma} + C|\chi|u^{1+\sigma} \leq \Delta u + au, \end{aligned}$$

provided that $|\chi| \leq \chi_0 := \min\{\chi_2, \frac{b}{C}\}$. This implies that the spreading speed of u is $c^* = 2\sqrt{a}$. □

3.3.5 Numerical simulations and biological indications

In this subsection, we present the numerical experiments that explore the influence of chemotaxis on the spread of the biological species in (3.0.1) when $N = 1$ and $a = b = 1$. The equation becomes

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + u(1-u), & x \in \mathbb{R}, \\ \tau v_t = v_{xx} - uv, & x \in \mathbb{R}. \end{cases}$$

For given $u_0 \in X^+$ and $v_0 \in X_1^+$, in order to see the behavior of $u(t,x;u_0,v_0)$ near (t,ct), we consider

$$(\tilde{u}(t,x),\tilde{v}(t,x)) := (u(t,x+ct;u_0,v_0), v(t,x+ct;u_0,v_0)),$$

which solves

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + c\tilde{u}_x - \chi(\tilde{u}\tilde{v}_x)_x + \tilde{u}(1-\tilde{u}), & x \in \mathbb{R}, \\ \tau\tilde{v}_t = \tilde{v}_{xx} + c\tau\tilde{v}_x - \tilde{u}\tilde{v}, & x \in \mathbb{R}, \\ \tilde{u}(0,x) = u_0(x),\ \tilde{v}(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.3.89}$$

For the numerical simulations, we use the following cut-off system of (3.3.89) on $(-L,L)$,

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + c\tilde{u}_x - \chi(\tilde{u}\tilde{v}_x)_x + \tilde{u}(1-\tilde{u}), & x \in (-L,L), \\ \tau\tilde{v}_t = \tilde{v}_{xx} + c\tau\tilde{v}_x - \tilde{u}\tilde{v}, & x \in (-L,L), \\ \tilde{u}(0,x) = u_0(x),\ \tilde{v}(0,x) = v_0(x), & x \in (-L,L). \end{cases} \tag{3.3.90}$$

complemented with the following boundary conditions:

$$\tilde{u}(t,\pm L) = 0, \qquad \frac{\partial \tilde{v}}{\partial x}(t,\pm L) = 0. \tag{3.3.91}$$

If $\chi = 0$, it suffices to solve for the following Fisher-KPP equation with convection,

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + c\tilde{u}_x + \tilde{u}(1-\tilde{u}), \quad -L < x < L, \\ \tilde{u}(0,x) = u_0,\ \tilde{u}(t,-L) = \tilde{u}(t,L) = 0. \end{cases} \tag{3.3.92}$$

Following from the arguments of [13, Theorem 2.2], we have the following dichotomy about the asymptotic dynamics of (3.3.92): for fixed $c \in \mathbb{R}$ and $L > 0$, either $\tilde{u}(t,x) \to 0$ as $t \to \infty$ uniformly in $x \in [-L,L]$ for any $u_0 \in C([-L,L])$ and $u_0 > 0$, or (3.3.92) has a unique positive stationary solution $u^*(x)$ and $u(t,x) \to u^*(x)$ as $t \to \infty$ uniformly for all $x \in [-L,L]$ and $u_0 \in C([-L,L])$ with $u_0 > 0$. The former occurs when $\lambda(c,L) \le 0$ and the latter occurs when $\lambda(c,L) > 0$, where $\lambda(c,L)$ is the principal eigenvalue of

$$\begin{cases} \tilde{u}_{xx} + c\tilde{u}_x + \tilde{u} = \lambda u, \quad -L < x < L, \\ \tilde{u}(-L) = \tilde{u}(L) = 0. \end{cases}$$

One can check that if $c > 2$ and $L \gg 1$, we have $\lambda(c,L) < 0$; and when $0 < c < 2$, we have $\lambda(c,L) > 0$ if $L \gg 1$. Thus, this confirms that the spreading speed for Fisher-KPP is $2\sqrt{a}$, which is 2 here.

When $\chi \neq 0$, if for some $c > 2$ and sufficiently large L we have $\tilde{u}(t,x) \not\to 0$ as $t \to \infty$, then we may conclude that chemotaxis speeds up the spreading of the species. Section 3.3.4 already shows that no speed-up occurs when the initial chemical concentration is not present everywhere. Therefore, we focus on the case where the chemical is initially distributed throughout the domain. To this end, we choose the following initial functions u_0 and v_0:

$$u_0(x) = \begin{cases} 0, & x \leq -1, \\ e^{\frac{1}{x^2-1}}, & x \in (-1,1), \\ 0, & x \geq 1, \end{cases} \qquad v_0(x) = 1, \tag{3.3.93}$$

and take $L = 20$. We compute the numerical solution of (3.3.90)+(3.3.91) with this initials (3.3.93) using the finite difference method. All the numerical simulations were implemented using the Python programming language.

We start by defining the scheme for (3.3.90) and (3.3.91) as follows: We divide the space interval $[-L, L]$ into M subintervals with equal length and divide the time interval $[0, T]$ into N subintervals with equal length. Then the space step size is $h = \frac{2L}{M}$ and the time step size is $\tau^* = \frac{T}{N}$. For simplicity, we denote the approximate value of $\tilde{u}(t_j, x_i), \tilde{v}(t_j, x_i)$ by $\tilde{u}(j,i), \tilde{v}(j,i)$ respectively, with $t_j = (j-1)\tau^*$, $1 \leq j \leq N+1$ and $x_i = -L + (i-1)h$, $1 \leq i \leq M+1$.

Using the central approximation for the spatial derivatives $\tilde{v}_{xx}(t_j, x_i), \tilde{v}_x(t_j, x_i)$ and $\tilde{u}_{xx}(t_j, x_i), \tilde{u}_x(t_j, x_i)$:

$$\begin{aligned} \tilde{v}_x(t_j, x_i) &\approx \frac{\tilde{v}(j, i+1) - \tilde{v}(j, i-1)}{2h}, \\ \tilde{v}_{xx}(t_j, x_i) &\approx \frac{\tilde{v}(j, i-1) - 2\tilde{v}(j,i) + \tilde{v}(j, i+1)}{h^2}, \\ \tilde{u}_x(t_j, x_i) &\approx \frac{\tilde{u}(j, i+1) - \tilde{u}(j, i-1)}{2h}, \\ \tilde{u}_{xx}(t_j, x_i) &\approx \frac{\tilde{u}(j, i-1) - 2\tilde{u}(j,i) + \tilde{u}(j, i+1)}{h^2}. \end{aligned}$$

The forward approximation of the time derivative yields

$$\begin{aligned} \tilde{v}_t(t_j, x_i) &\approx \frac{\tilde{v}(j+1, i) - \tilde{v}(j,i)}{\tau^*}, \\ \tilde{u}_t(t_j, x_i) &\approx \frac{\tilde{u}(j+1, i) - \tilde{u}(j,i)}{\tau^*}. \end{aligned}$$

By the boundary conditions in (3.3.91), we set

$$\tilde{u}(j,1)=\tilde{u}(j,M+1)=0. \tag{3.3.94}$$

Using the forward approximation for the Neumann boundary condition at $-L$ and backward approximation at L, we get

$$\frac{\partial\tilde{v}}{\partial x}(t_j,x_1)\approx\frac{\tilde{v}(j,2)-\tilde{v}(j,1)}{h}=0,\qquad \frac{\partial\tilde{v}}{\partial x}(t_j,x_{M+1})\approx\frac{\tilde{v}(j,M+1)-\tilde{v}(j,M)}{h}=0,$$

and hence, we set

$$\tilde{v}(j,2)=\tilde{v}(j,1),\quad \tilde{v}(j,M+1)=\tilde{v}(j,M). \tag{3.3.95}$$

The equation (3.3.90) can thus be discretized as: for $1\le j\le N$, $2\le i\le M$,

$$\begin{aligned}\frac{\tilde{u}(j+1,i)-\tilde{u}(j,i)}{\tau^*}=&\frac{\tilde{u}(j,i-1)-2\tilde{u}(j,i)+\tilde{u}(j,i+1)}{h^2}\\&+\left(c-\chi\frac{\tilde{v}(j,i+1)-\tilde{v}(j,i-1)}{2h}\right)\frac{\tilde{u}(j,i+1)-\tilde{u}(j,i-1)}{2h}\\&-\chi\tilde{u}(j,i)\frac{\tilde{v}(j,i-1)-2\tilde{v}(j,i)+\tilde{v}(j,i+1)}{h^2}+\tilde{u}(j,i)(1-\tilde{u}(j,i)),\end{aligned}$$

and

$$\begin{aligned}\tau\frac{\tilde{v}(j+1,i)-\tilde{v}(j,i)}{\tau^*}=&\frac{\tilde{v}(j,i-1)-2\tilde{v}(j,i)+\tilde{v}(j,i+1)}{h^2}+c\tau\frac{\tilde{v}(j,i+1)-\tilde{v}(j,i-1)}{2h}\\&-\tilde{u}(j,i)\tilde{v}(j,i).\end{aligned}$$

Simplifying and reordering the two equations, we get for $1\le j\le N$, $2\le i\le M$,

$$\begin{aligned}\tilde{u}(j+1,i)=&\,\tilde{u}(j,i)\left[1+\tau^*-\frac{2\tau^*}{h^2}-\frac{\tau^*\chi}{h^2}\Big(\tilde{v}(j,i-1)-2\tilde{v}(j,i)+\tilde{v}(j,i+1)\Big)\right]\\&+\tilde{u}(j,i-1)\left[\frac{\tau^*}{h^2}-\frac{\tau^*}{2h}\left(c-\chi\frac{\tilde{v}(j,i+1)-\tilde{v}(j,i-1)}{2h}\right)\right]\\&+\tilde{u}(j,i+1)\left[\frac{\tau^*}{h^2}+\frac{\tau^*}{2h}\left(c-\chi\frac{\tilde{v}(j,i+1)-\tilde{v}(j,i-1)}{2h}\right)\right]-\tau^*\tilde{u}(j,i)^2,\end{aligned} \tag{3.3.96}$$

and

$$\tilde{v}(j+1,i) = \tilde{v}(j,i)\left(1 - \frac{\tau^*}{\tau}\tilde{u}(j,i) - \frac{2\tau^*}{\tau h^2}\right) + \tilde{v}(j,i-1)\left(\frac{\tau^*}{\tau h^2} - \frac{c\tau^*}{2h}\right) + \tilde{v}(j,i+1)\left(\frac{\tau^*}{\tau h^2} + \frac{c\tau^*}{2h}\right). \tag{3.3.97}$$

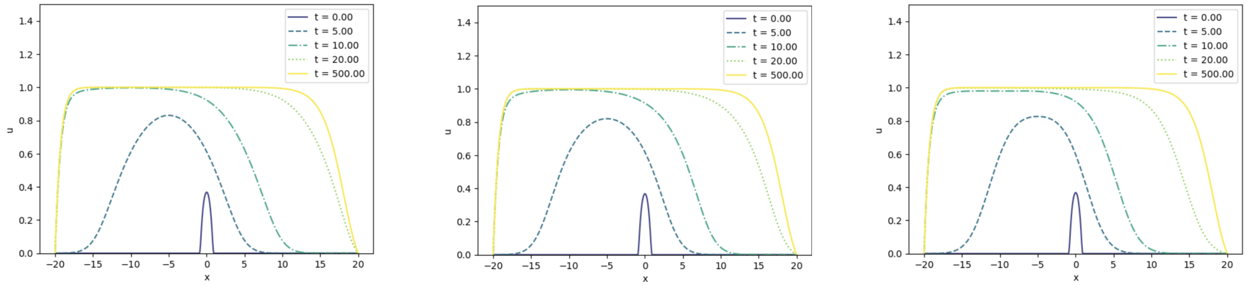
We use (3.3.94), (3.3.95), (3.3.96) and (3.3.97) in implementing the scheme in Python. We apply the same space step size $h = 0.1$ and the same time step size $\tau^* = 0.002$. In all the simulations we take $L = 20$ and $T = 500$.

We run the simulations for $\tau = 0.5$, 1, and 4, and present two-dimensional plots of the solution u. Since $\tilde{u}$ and $\tilde{v}$ exhibit opposite asymptotic behavior - namely, when $\tilde{u}(t,x)$ remains positive, $\tilde{v}(t,x) \to 0$ as $t \to \infty$, whereas when $\tilde{u}(t,x) \to 0$, $\tilde{v}(t,x)$ remains positive as $t \to \infty$, we do not include the graphs of the v-component, since its behavior can be inferred from that of u.

For each figure in the next three subsections, panels (a), (b), and (c) correspond to $\tau = 0.5$, $\tau = 1$, and $\tau = 4$, respectively. Each plot shows the solution at times $t = 0$, 5, 10, 20, and 500. We present the results for $c = 1$, 2.01, and 3, and in each case for $\chi = 1.9$, 5, and 10 in the following subsections.

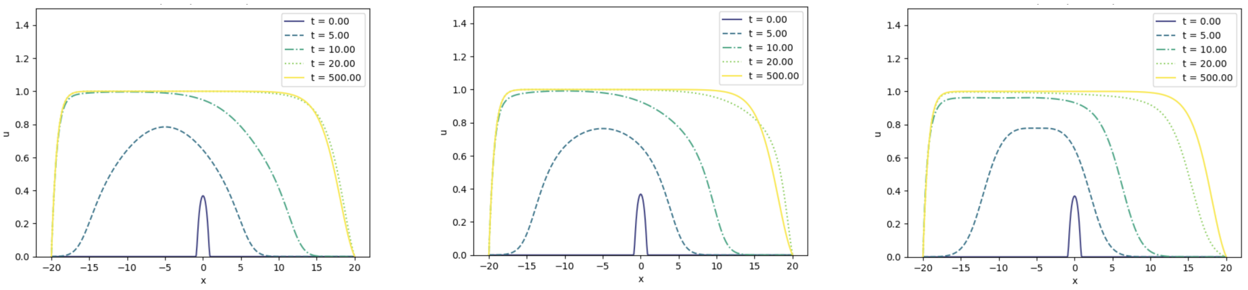
Results of the simulations for $c = 1$

For $c = 1$, the solution remains positive for all values of τ and for each of the values $\chi = 1.9$, 5, and 10. This is consistent with Theorem 3.3.2. The corresponding simulation results are shown below.



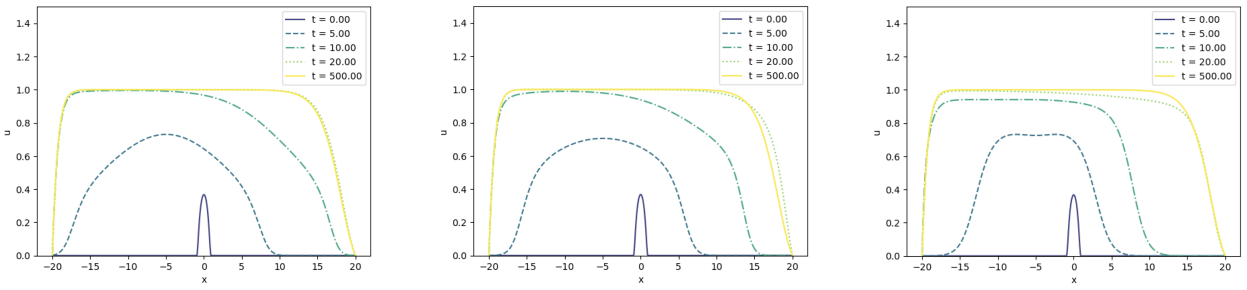
(a) $\tau = 0.5, \chi = 1.9$ (b) $\tau = 1, \chi = 1.9$ (c) $\tau = 4, \chi = 1.9$

Figure 3.1: Two-dimensional plots of the numerical solution u for $\chi = 1.9$ and $c = 1$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at selected times.



(a) $\tau = 0.5,\ \chi = 5$ (b) $\tau = 1,\ \chi = 5$ (c) $\tau = 4,\ \chi = 5$

Figure 3.2: Two-dimensional plots of the numerical solution u for $c = 1$ with $\chi = 5$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at selected times.



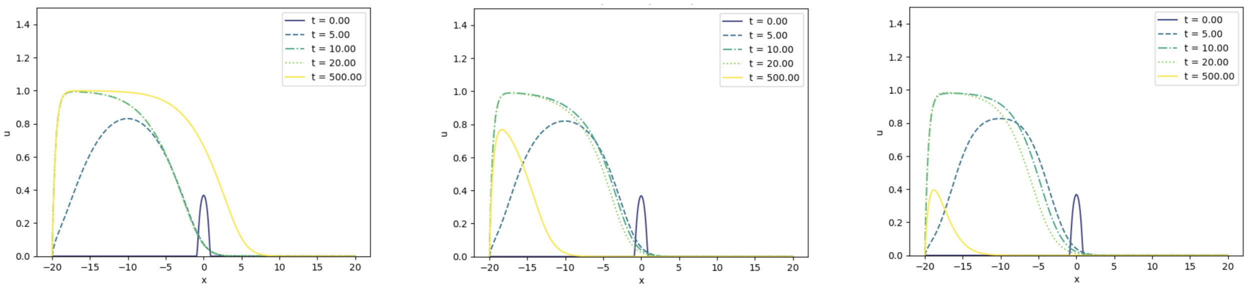
(a) $\tau = 0.5,\ \chi = 10$ (b) $\tau = 1,\ \chi = 10$ (c) $\tau = 4,\ \chi = 10$

Figure 3.3: Two-dimensional plots of the numerical solution u for $c = 1$ with $\chi = 10$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at selected times.

Results of the simulations for $c = 2.01$

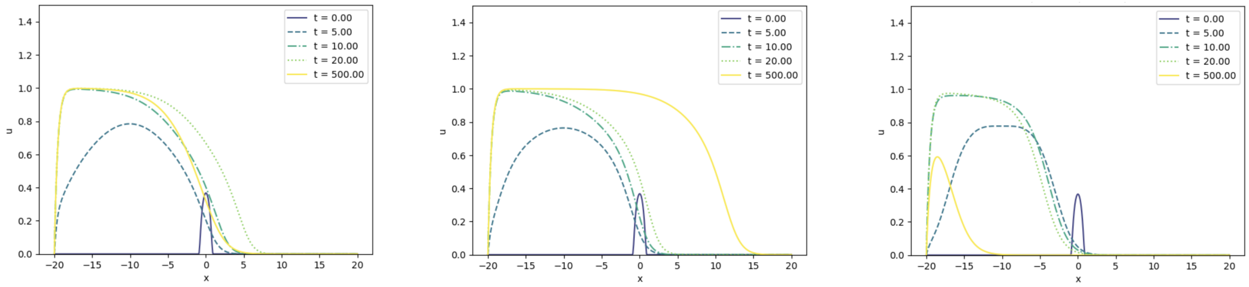
The simulation results indicate that the diffusion rate of v influences the spreading speed. More precisely, as τ increases, the solution u tends to approach zero more rapidly, and the interval of spreading speeds becomes smaller. In addition, for larger values of χ, $\tilde{u}(t, x)$ remains positive as $t \to \infty$ for all values of $\tau = 0.5$, 1, and 4, suggesting the possibility of speed-up in the large-χ regime; see, for example, Figure 3.6. In contrast, when $\chi = 1.9$, $\tilde{u}(t, x)$ tends to zero for $\tau = 1$ and 4, indicating that speed-up may fail to occur when χ is small and $\tau \geq 1$.

Furthermore, when $\tau = 1$, panel (b) in Figures 3.4–3.6 suggests the existence of a critical value χ^*, with $\chi^* \leq 1.9$, such that chemotaxis does not appear to accelerate the spreading rate for $\chi < \chi^*$, whereas it appears to enhance the spreading for $\chi > \chi^*$.



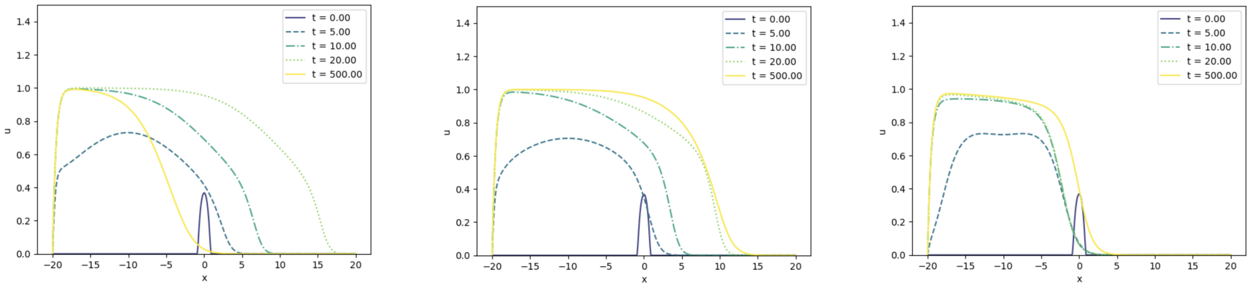
(a) $\tau = 0.5,\ \chi = 1.9$ (b) $\tau = 1,\ \chi = 1.9$ (c) $\tau = 4,\ \chi = 1.9$

Figure 3.4: Two-dimensional plots of the numerical solution u for $c = 2.01$ with $\chi = 1.9$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at times $t = 0$, 5, 20, and 500.



(a) $\tau = 0.5,\ \chi = 5$ (b) $\tau = 1,\ \chi = 5$ (c) $\tau = 4,\ \chi = 5$

Figure 3.5: Two-dimensional plots of the numerical solution u for $c = 2.01$ with $\chi = 5$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at times $t = 0$, 5, 20, and 500.



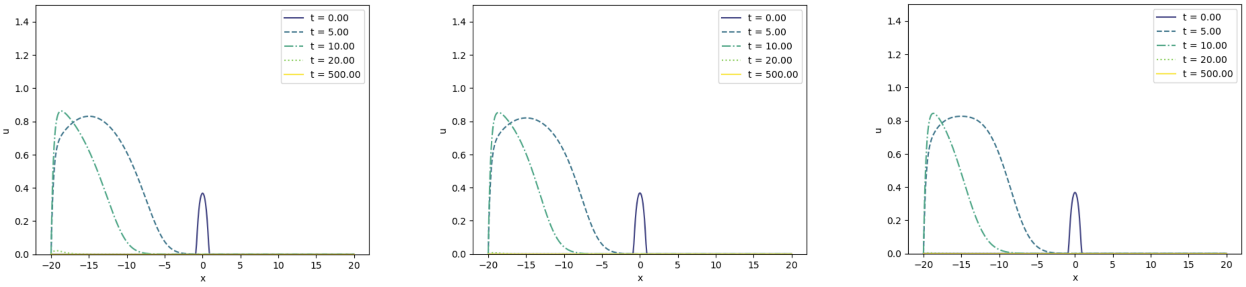
(a) $\tau = 0.5,\ \chi = 10$ (b) $\tau = 1,\ \chi = 10$ (c) $\tau = 4,\ \chi = 10$

Figure 3.6: Two-dimensional plots of the numerical solution u for $c = 2.01$ with $\chi = 10$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at times $t = 0$, 5, 20, and 500.

Results of the simulations for $c = 3$

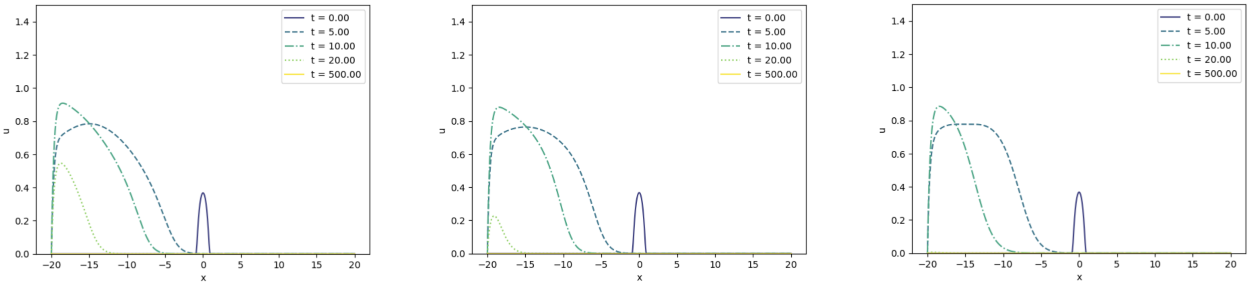
We carry out the simulations for $c = 3$ with $\tau = 0.5$, 1, and 4. In all cases, the results show that the solution approaches zero as time becomes large, for each of the three values of τ

and each of the values $\chi = 1.9$, 5, and 10. This also supports the fact that $c^*_{\mathrm{up}} < \infty$.



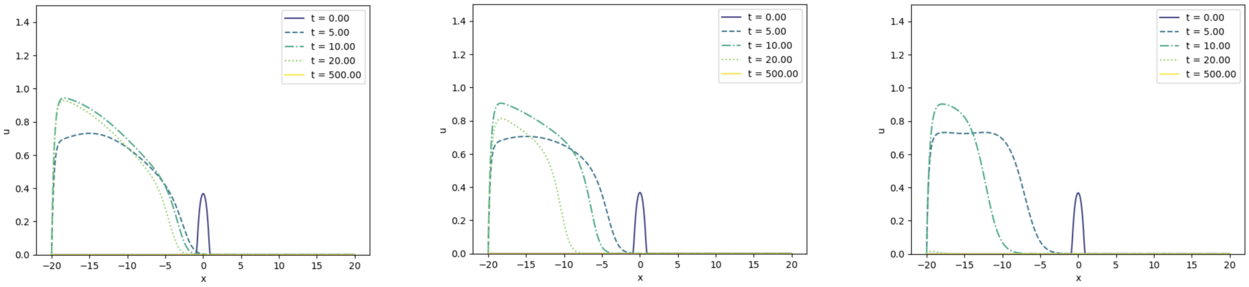
(a) $\tau = 0.5,\ \chi = 1.9$ (b) $\tau = 1,\ \chi = 1.9$ (c) $\tau = 4,\ \chi = 1.9$

Figure 3.7: Two-dimensional plots of the numerical solution u for $c = 3$ with $\chi = 1.9$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at selected times.



(a) $\tau = 0.5,\ \chi = 5$ (b) $\tau = 1,\ \chi = 5$ (c) $\tau = 4,\ \chi = 5$

Figure 3.8: Two-dimensional plots of the numerical solution u for $c = 3$ with $\chi = 5$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at selected times.



(a) $\tau = 0.5,\ \chi = 10$ (b) $\tau = 1,\ \chi = 10$ (c) $\tau = 4,\ \chi = 10$

Figure 3.9: Two-dimensional plots of the numerical solution u for $c = 3$ with $\chi = 10$, shown for $\tau = 0.5$, $\tau = 1$, and $\tau = 4$ at selected times.

See section 6 in [21] for more simulation results. We make the following remark on the experiments

Remark 3.3.1. *In the numerical experiments above, we used the same spatial step size* $h = 0.1$ *and the same time step size* $\tau^* = 0.002$*, since they satisfy the numerical stability condition* $\tau^*/h^2 < 0.5$*. We do not provide an accuracy analysis of the simulations in this paper. However, to assess the reliability of the numerical results, we also used different values of* h *and* τ^* *to simulate the spreading speed of the chemotaxis system on* $\mathbb{R}$*. More precisely, we repeated the above experiments with* $h = 0.1$ *and* $\tau^* = 0.002, 0.004$*, and with* $\tau^* = 0.002$ *and* $h = 0.2$*. The observed outcomes were similar for the different values of* h *and* τ^**.*

Chapter 4

The Chemotaxis System with Porous Medium Diffusion

In this chapter we studied the porous medium diffusion case. Specifically, we study the following system:

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \mathbb{R}^N, \quad t > 0, \\ \tau v_t = \Delta v - uv, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{4.0.1}$$

Here, $m > 1$ and $\tau > 0$. Since the diffusion coefficient mu^{m-1} is degenerate, our main interest is in the existence and uniqueness of globally bounded weak solutions to (4.0.1). We consider (4.0.1) with initial data satisfying

$$u_0(\cdot) \in L^\infty(\mathbb{R}^N), \quad v_0(\cdot) \in W^{1,\infty}(\mathbb{R}^N).$$

(See Remark 4.1.1(4) for the case where the initial data satisfy $u_0(\cdot) \in L^p(\mathbb{R}^N)$ and $v_0(\cdot) \in W^{1,p}(\mathbb{R}^N)$ for some $p > 1$.) We first establish the existence of global weak solutions to (4.0.1). A standard approach in the literature (see, for example, [70]) is to approximate the porous medium equation by a family of nondegenerate parabolic equations and then pass to the limit. For $\varepsilon > 0$, our strategy is to first prove the global existence of classical solutions to the perturbed system

$$\begin{cases} u_t = m \nabla \cdot \Big((\varepsilon + u)^{m-1} \nabla u \Big) - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \mathbb{R}^N, \quad t > 0, \\ \tau v_t = \Delta v - uv, & x \in \mathbb{R}^N, \quad t > 0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.0.2}$$

and then pass to the limit as $\varepsilon \to 0$ to obtain a global weak solution of the degenerate problem (4.0.1). We refer to (4.0.2) as the perturbed problem associated with (4.0.1).

The study of (4.0.1) with non-integrable initial data is highly nontrivial. To explain the difficulty, let us recall the standard energy argument for deriving L^p bounds in the case where the domain Ω is bounded. In that setting, one can multiply the first equation in the perturbed problem (4.0.2) by u^p and integrate over Ω to obtain

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega u^{p+1} \le -mp\int_\Omega u^{p+m-2}|\nabla u|^2 + |\chi|\int_\Omega u^p|\nabla u||\nabla v| + a\int_\Omega u^{p+1} - b\int_\Omega u^{p+2}.$$

From this, one can derive that for some $C > 0$ independent of u and v,

$$\begin{aligned} &\int_\Omega u^{p+1}(t,x) + \frac{mp}{2}\int_0^t\int_\Omega u^{p+m-2}|\nabla u|^2 + \frac{b}{2}\int_0^t\int_\Omega u^{p+2}(s,x) \\ &\le \int_\Omega u^{p+1}(0,x) + C\int_0^t\int_\Omega |\nabla v|^{\frac{2(p+2)}{m}} + C\int_0^t\int_\Omega u(s,x). \end{aligned} \tag{4.0.3}$$

After treating the v-equation carefully, the right-hand side can be bounded by the left-hand side, together with the space-time L^1-norm of u and a constant. This then yields the desired L^p estimate for u, and the proof proceeds by means of a Grönwall-type inequality. We refer the reader to [31] for more details.

In our setting, however, the solutions may be non-integrable over the whole space, so one cannot directly work with global L^p norms. This makes it necessary to localize the problem in space. A natural first attempt is to localize the equation by multiplying the u-equation by $u^p\psi$, where ψ is a suitable cut-off function, as we did in the linear diffusion case in Chapter 3. However, this produces an additional term of the form

$$\iint_{(0,T)\times\mathbb{R}^N} u^{p+m-1}|\nabla u|\,|\nabla\psi|\,dx\,dt,$$

which can be estimated by

$$\iint_{(0,T)\times\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi\,dx\,dt + \iint_{(0,T)\times\mathbb{R}^N} u^{p+m}\frac{|\nabla\psi|^2}{\psi}\,dx\,dt.$$

Although the second term involves only zeroth-order derivatives of u, the exponent of u increases with m. In particular, when $m > 2$, this exponent is at least $p+2$. As a result, it is difficult to control this term directly using only the diffusion term and the logistic term on the left-hand side of (4.0.3). This difficulty does not arise in Chapter 3 (Linear diffusion case $m = 1$). Consequently, the classical argument based on Grönwall-type inequalities no longer applies in the present setting. To overcome this difficulty, we use the decay function ψ from Lemma 2.4.1 to derive an inequality of the form (4.2.19). This inequality plays a crucial role in the analysis, as it allows us to fully exploit all the favorable effects in the problem, namely the diffusion term, the logistic term, and the regularity of the initial data; see Subsection 4.2.2. The overall strategy is close in spirit to a continuity-type argument and, to the best of our knowledge, is new in the study of chemotaxis models.

Our second main contribution in this chapter is to show that the resulting weak solution is in fact unique under a suitable condition on m, provided that the initial data satisfy $u_0 \in C^\alpha$ and $v_0 \in C^{2+\alpha}$. The proof of uniqueness is based on a duality argument, inspired by the approaches in [36, Theorem 3.4] and [70, Theorem 6.5], and suitably adapted to the present chemotaxis–porous medium setting. In addition, to handle the possible non-integrability of the initial data, we again make essential use of the exponentially decaying function ψ.

In the next Section 4.1, we introduce definition of weak solution and state our main result. Section 4.2, is devoted to establishing local L^p-estimates for the perturbed problem and proving part (1) of Theorem 4.1.1. In Section 4.3, we complete the proof of Theorem 4.1.1 by establishing part (2). Section 4.4 is concerned with the global well-posedness of weak and classical solutions to system (4.0.2), where we establish Proposition 4.1.1 and Theorem 4.1.2. Finally, Section 4.5 is devoted to proving the uniqueness of weak solutions, as stated in Theorem 4.1.3.

4.1 Definitions and Main Results

In this section, we introduce the definitions of weak solutions to (4.0.2) for $\varepsilon \geq 0$ and classical solutions to (4.0.2) for $\varepsilon > 0$, and we state the main results of this part.

Definition 4.1.1. *Let $m > 1$ and $T > 0$, and let $u_0 \in L^\infty(\mathbb{R}^N)$ and $v_0 \in W^{1,\infty}(\mathbb{R}^N)$. A pair (u, v) of non-negative functions defined in $[0,T) \times \mathbb{R}^N$ is called a weak solution of* (4.0.2) *on $[0,T)$ if*

(1) $u \in L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^N))$, $(\varepsilon + u)^{m-1}\nabla u \in L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^N))$;

(2) $v \in L^\infty(0,T; H^1_{\text{loc}}(\mathbb{R}^N))$, $u\nabla v \in L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^N))$;

(3) For any continuously differentiable function φ with compact support in $[0,T) \times \mathbb{R}^N$, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} u\varphi_t \, dx\, dt + \int_{\mathbb{R}^N} u_0(x)\varphi(0,x)\, dx \\
&\qquad = \int_0^T \int_{\mathbb{R}^N} \left(m(\varepsilon + u)^{m-1}\nabla u \cdot \nabla\varphi - \chi u \nabla v \cdot \nabla \varphi - au\varphi + bu^2\varphi\right) dx\, dt
\end{aligned}$$

and

$$\int_0^T \int_{\mathbb{R}^N} \tau v \varphi_t \, dx \, dt + \int_{\mathbb{R}^N} \tau v_0(x) \varphi(0,x) \, dx = \int_0^T \int_{\mathbb{R}^N} \left[\nabla v \cdot \nabla \varphi + uv\varphi \right] \, dx \, dt.$$

In the following, we say $v \in C^{1+\alpha/2,2+\alpha}$ if v is $C^{1+\alpha/2}$ in time and $C^{2+\alpha}$ in space; and $u \in C^{\alpha}([0,T) \times \mathbb{R}^N) \cap C^{1+\alpha/2,2+\alpha}((0,T) \times \mathbb{R}^N)$ if u is Hölder continuous in both time and space up to the initial time, and u is $C^{1+\alpha/2}$ in time and $C^{2+\alpha}$ in space for positive time.

Definition 4.1.2. *Let u_0 be uniformly $C^{1+\alpha}$ and v_0 be uniformly $C^{2+\alpha}$. A pair (u,v) of non-negative functions defined in $[0,T) \times \mathbb{R}^N$ is called a classical solution of* (4.0.2) *with $\varepsilon > 0$ on $[0,T)$ if*

(1) v is uniformly $C^{1+\alpha/2,2+\alpha}$ and u is uniformly in $C^{\alpha}([0,T) \times \mathbb{R}^N) \cap C^{1+\alpha/2,2+\alpha}((0,T) \times \mathbb{R}^N)$;

(2) $u(0,\cdot) = u_0$, $v(0,\cdot) = v_0$, and (4.0.2) *is satisfied in the classical sense in $(0,T) \times \mathbb{R}^N$.*

Our first main result is on a priori estimates of classical solutions of (4.0.2), which are stated in the following theorem. We denote $\Omega_T := [0,T] \times \mathbb{R}^N$.

Theorem 4.1.1 (A priori estimate). *Let $\varepsilon \in (0,1)$, $a, b, \tau > 0$, $\chi \in \mathbb{R}$, $m > 1$ and $T > 0$, and let $u_0 \in L^\infty(\mathbb{R}^N)$ and $v_0 \in W^{1,\infty}(\mathbb{R}^N)$ such that $u_0, v_0 \geq 0$. Suppose that $(u_\varepsilon, v_\varepsilon)$ is a weak solution to* (4.0.2) *with initial data (u_0, v_0), and they satisfy the equation in the classical sense for positive times. Then*

(1) (L^p_{loc} a priori estimate). For any $p \geq m$ and $t \in [0,T]$,

$$u_\varepsilon(t,\cdot) \in L^{p+1}_{\mathrm{loc}}(\mathbb{R}^N),\ e^{-\frac{(p+2)T}{2\tau}} (\varepsilon + u_\varepsilon)^{\frac{p+m-2}{2}} \nabla u_\varepsilon \in L^2_{\mathrm{loc}}(\Omega_T),\ v_\varepsilon \in L^\infty(\Omega_T) \cap W^{1,\infty}(\Omega_T)$$

with a bound depending only on $m, |\chi|, a, b, \tau, N, p$, $\|u_0\|_\infty$, $\|v_0\|_{W^{1,\infty}}$ and the diameter of the local spatial domain (but independent of T).

(2) (L^∞ a priori estimate) There exists C depending only on $m, |\chi|, a, b, N, p$, $\|u_0\|_\infty$, $\|v_0\|_{W^{1,\infty}}$ (but independent of T) such that

$$\|\nabla v_\varepsilon\|_{L^\infty(\Omega_T)}, \quad \|u_\varepsilon\|_{L^\infty(\Omega_T)} \leq C.$$

(3) *(Hölder continuity) Let $C > 0$ from (2). Then there exists $\alpha \in (0,1)$ depending only on m, C, and for any $\tau' > 0$ there exists $C' > 0$ depending only on $m, |\chi|, a, b, N, \tau, \tau'$ and C (independent of ε) such that*

$$\|u_\varepsilon\|_{C^\alpha((\tau',\infty)\times\mathbb{R}^N)} \le C'.$$

If, in addition, $u_\varepsilon(0,\cdot)$ is Hölder continuous on $\mathbb{R}^N$, then $u_\varepsilon(\cdot,\cdot)$ is Hölder continuous on $(0,\infty)\times\mathbb{R}^N$ with a bound depending also on the Hölder norm of $u_\varepsilon(0,\cdot)$.

As a corollary of these a priori estimates, we obtain the following proposition on the global existence and boundedness of classical solutions of the perturbed problem (4.0.2) with $\varepsilon \in (0,1)$.

Proposition 4.1.1 (Classical solutions of equation (3.2))**.** *Under the assumptions of Theorem 4.1.1, further assume that u_0 is uniformly $C^{1+\alpha}$, and v_0 is uniformly $C^{2+\alpha}$. Then for each $\varepsilon \in (0,1)$, there exists a unique global classical solution $(u_\varepsilon, v_\varepsilon)$ of (4.0.2) with initial condition u_0, v_0. Moreover, the regularity properties presented in Theorem 4.1.1 hold the same.*

Next, we state our main results concerning the global existence and boundedness of weak solutions to (4.0.1).

Theorem 4.1.2 (Weak solutions of (4.0.1))**.** *Under the assumptions of Theorem 4.1.1, there is a non-negative weak solution (u,v) of (4.0.1) with initial data (u_0, v_0) such that the regularity properties presented in Theorem 4.1.1 hold the same for (u,v). In addition, if u_0 is uniformly Hölder continuous, then u is uniformly Hölder continuous in $[0,\infty)\times\mathbb{R}^N$. Furthermore, if $v_0 \in C^{2,\alpha}$, then v is uniformly $C^{1+\alpha/2,2+\alpha}$ in $[0,\infty)\times\mathbb{R}^N$.*

Our final main result is about the uniqueness of weak solutions to (4.0.1).

Theorem 4.1.3 (Uniqueness of weak solutions of (4.0.1))**.** *Let $a, b > 0$, $u_0 \in C^\alpha(\mathbb{R}^N)$, and $v_0 \in C^{2+\alpha}(\mathbb{R}^N)$. Suppose that there are two weak solutions (u_i, v_i) with $i = 1,2$ to (4.0.1) such that u_i is uniformly Hölder continuous in Ω_T and v_i is uniformly C^2 in Ω_T. Then if $m \in (1,3]$, we have $u_1 = u_2$ and $v_1 = v_2$ in Ω_T.*

Remark 4.1.1. *Several remarks are in order concerning our results.*

(1) *(Bounded domains). Our methods of establishing global solutions extend to bounded domains and, importantly, remove both the dimensional restriction and any condition on m when $a, b > 0$. When $a = b = 0$, the existence of weak solutions to (4.0.1) can still be obtained in a similar manner provided $m > 2$.*

(2) (Hölder continuity). Uniform Hölder continuity estimates for u_ε (the solutions to (1.1.2)*) follow from the regularity theory for degenerate diffusion equations with advection and source terms; see* [5, Theorem 1.3 and Theorem 1.9]*. We also refer the reader to* [27, 36, 39]*.*

(3) (Uniqueness). Although the case $a = b = 0$ is not the focus of this paper, we note that under the same assumptions on solutions as in Theorem 4.1.3, the uniqueness conclusion also holds for $a = b = 0$ and all $m > 1$. The proof requires only minor modifications.

(4) (Integrable data on $\mathbb{R}^N$). In the special case where $u_0 \in L^1 \cap L^\infty$ and $v_0 \in L^1 \cap W^{1,\infty}$, the existence of a global weak solution to (4.0.2) *can be established more easily. In fact, one may simply take the cut-off function to be* 1*, and many of the more delicate estimates become unnecessary. However, on unbounded domains, the potential growth of the L^1 norm over time makes it trickier to obtain uniform-in-time L^∞ bounds. Nevertheless, our argument fully covers this case.*

4.2 L^p_{loc} a Priori Estimate and the Proof of Theorem 4.1.1(1)

In this section, we shall give $L^{p+1}_{\mathrm{loc}}(\mathbb{R}^N)$ a priori estimates of solutions of the perturbed problem (4.0.2) on any finite time interval $[0,T]$, and prove Theorem 4.1.1(1).

Let $0 < \varepsilon < 1$ and $0 < \kappa < 1$, and take p such that

$$p \geq m. \tag{4.2.1}$$

Throughout this section, the constants $c \in (0,1)$ and $C \geq 1$ only depend on $m, \chi, a, b, \tau, N, p$, $\|v_0\|_{W^{1,\infty}}$ and $\|u_0\|_{L^\infty}$, and they may be different at different places, unless otherwise stated. When we say that a constant depends on C, we mean that it might depend on m, χ, a, b, τ, N, $\|v_0\|_{W^{1,\infty}}$ and $\|u_0\|_{L^\infty}$. We emphasize that the constants c, C will always be independent of ε, κ, and T.

Recall that for $t > 0$,

$$\Omega_t := [0,t] \times \mathbb{R}^N.$$

Suppose that $(u(t,x), v(t,x))$ solve the perturbed problem (4.0.2) with $\varepsilon \in (0,1)$ for $t \in (0,T]$ in the classical sense and let ψ be from Lemma 2.4.1 with parameter κ. For any given

$x_0 \in \mathbb{R}^N$ and $r > 1$, recall that

$$Z_{r,x_0}(t) := \iint_{\Omega_t} e^{-\frac{r(t-s)}{\tau}} (u+\varepsilon)^r(s,x)\psi^2(x-x_0)\,dx\,ds \quad \text{and} \quad Z_r(t) := \sup_{x_0\in\mathbb{R}^N} Z_{r,x_0}(t).$$

First we state a useful lemma.

Lemma 4.2.1. *Assume that $N \geq 3$ and let $2^* := \frac{2N}{N-2} > 2$ be the Sobolev conjugate exponent of 2. Then there exists a dimensional constant C such that for any $\delta, \kappa > 0$ and $r > 1$ we have*

$$\kappa^2 \int_{\mathbb{R}^N} u^{2r}\psi^2 \leq \delta \|\nabla(u^r\psi)\|_2^2 + C\kappa^{2+2\theta_r}\delta^{-\theta_r}\left[\int_{\mathbb{R}^N} u^{r+1}\psi^{q_r}\,dx\right]^{2/q_r},$$

where $u = u(x)$ is any function such that u^r is locally uniformly finite in $W^{1,2}$-space, and ψ is from Lemma 2.4.1, and

$$q_r := \frac{r+1}{r} \in (1,2) \quad \text{and} \quad \theta_r := \frac{N(r-1)}{2(r+1)} > 0.$$

In fact, one could replace $u\psi$ with a single function $\tilde{u}$ and assume $\tilde{u}^r \in W^{1,2} \cap L^{\frac{r+1}{r}}$, without imposing separate assumptions on u^r and ψ. However, we choose to state the lemma in its current form for future applicability.

Proof of Lemma 4.2.1. Note that $u^r\psi \in W^{1,2}(\mathbb{R}^N)$. Then by the Gagliardo–Nirenberg–Sobolev inequality, there is a dimensional constant $\widetilde{C}$ such that for any $r > 1$,

$$\|u^r\psi\|_{2^*} \leq \widetilde{C}\|\nabla(u^r\psi)\|_2. \tag{4.2.2}$$

By the interpolation inequality of L^p spaces, we have

$$\|u^r\psi\|_2 \leq \|u^r\psi\|_{q_r}^{1-\vartheta}\|u^r\psi\|_{2^*}^{\vartheta}.$$

Here direct computation yields

$$q_r := \frac{r+1}{r} \in (1,2), \quad \vartheta := \frac{2^*(2-q_r)}{2(2^*-q_r)} = \frac{N(r-1)}{r(N+2)-N+2} \in (0,1),$$

and so

$$2 + 2\theta_r := \frac{2}{1-\vartheta} = \frac{r(N+2)-N+2}{r+1} \quad \text{and so} \quad \theta_r = \frac{N(r-1)}{2(r+1)} > 0.$$

Hence, by Young's inequality and (4.2.2), for any δ,

$$\begin{aligned}\kappa^2 \int_{\mathbb{R}^N} u^{2r}\psi^2 &\le \frac{\delta}{\widetilde{C}}\|u^r\psi\|_{2^*}^2 + \widetilde{C}^{\frac{\vartheta}{1-\vartheta}}\kappa^{\frac{2}{1-\vartheta}}\delta^{-\frac{\vartheta}{1-\vartheta}}\|u^r\psi\|_{q_r}^2 \\ &\le \delta\|\nabla(u^r\psi)\|_2^2 + C\kappa^{2+2\theta_r}\delta^{-\theta_r}\left[\int_{\mathbb{R}^N} u^{r+1}\psi^{q_r}\,dx\right]^{2/q_r},\end{aligned}$$

where $C = \widetilde{C}^{\frac{\vartheta}{1-\vartheta}}$. The lemma is thus proved. □

Now, we proceed to prove local L^{p+1} estimate for u. We recall that the constants c and C below might depend on the data (m, N, etc.) and p, but are independent of ε, κ and T. Multiplying the first equation of (4.0.2) by $(u+\varepsilon)^p\,\psi^2$ and integrating over $\mathbb{R}^N$ yields

$$\begin{aligned}&\frac{1}{p+1}\frac{d}{dt}\int_{\mathbb{R}^N}(u+\varepsilon)^{p+1}\psi^2\,dx \\ &= \int_{\mathbb{R}^N}(u+\varepsilon)^p\psi^2\nabla\cdot\left[m(u+\varepsilon)^{m-1}\nabla u - \chi u\nabla v\right] + (u+\varepsilon)^p\psi^2(au-bu^2) \\ &\le -c\int_{\mathbb{R}^N}(u+\varepsilon)^{p+m-2}|\nabla u|^2\psi^2 + C\int_{\mathbb{R}^N}(u+\varepsilon)^{p+m-1}|\nabla u||\nabla\psi^2| \\ &\quad + C\int_{\mathbb{R}^N}(u+\varepsilon)^{p-1}u|\nabla u||\nabla v|\,\psi^2 + C\int_{\mathbb{R}^N}(u+\varepsilon)^p u|\nabla v||\nabla\psi^2| + \int_{\mathbb{R}^N}(u+\varepsilon)^p(au-bu^2)\psi^2 \\ &\le \underbrace{-c\int_{\mathbb{R}^N}(u+\varepsilon)^{m+p-2}|\nabla u|^2\psi^2}_{-I_1} + \underbrace{C\kappa\int_{\mathbb{R}^N}(u+\varepsilon)^{p+m-1}|\nabla u|\psi^2}_{\kappa I_2} \\ &\quad + \underbrace{C\int_{\mathbb{R}^N}(u+\varepsilon)^p|\nabla u||\nabla v|\,\psi^2}_{I_3} + \underbrace{C\kappa\int_{\mathbb{R}^N}(u+\varepsilon)^{p+1}|\nabla v|\psi^2}_{\kappa I_4} + \underbrace{\int_{\mathbb{R}^N}(u+\varepsilon)^p(au-bu^2)\psi^2}_{I_5},\end{aligned} \tag{4.2.3}$$

where, in the second inequality, we used that $|\nabla\psi| \le \kappa\psi$.

To prove Theorem 4.1.1(1), it is essential to provide proper estimates for I_1, I_2, I_3, I_4, and I_5. We point out that I_1 and I_2 depend on m, and we will estimate them differently for the case $1 < m \le 2$ and $m > 2$. In the following, we will first estimate I_3, I_4 and I_5.

First, we estimate $I_3(t)$. For simplicity of notation, let us write

$$u_+ := u + \varepsilon \ge \varepsilon.$$

Since $\nabla u = \nabla u_+$, by Young's inequality, for any $0 < \delta < 1$,

$$\begin{aligned} I_3(t) &\le C \int_{\mathbb{R}^N} u_+^p |\nabla u_+||\nabla v| \psi^2 \, dx \\ &\le \delta \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx + C\delta^{-1} \int_{\mathbb{R}^N} u_+^{p+2-m} |\nabla v|^2 \psi^2 \, dx \\ &\le \delta \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx + \delta \int_{\mathbb{R}^N} u_+^{p+2} \psi^2 \, dx + C_\delta \int_{\mathbb{R}^N} |\nabla v|^{\frac{2(p+2)}{m}} \psi^2 \, dx, \qquad (4.2.4) \end{aligned}$$

where we used that $p + 2 - m > 0$ by (4.2.1).

Next, we consider $\kappa I_4(t)$. By Young's inequality,

$$\kappa I_4(t) \lesssim \kappa \int_{\mathbb{R}^N} u_+^{p+1} |\nabla v| \psi^2 \, dx \lesssim \kappa \int_{\mathbb{R}^N} u_+^{p+2} \psi^2 \, dx + \kappa \int_{\mathbb{R}^N} |\nabla v|^{p+2} \psi^2 \, dx. \qquad (4.2.5)$$

Finally, since $\varepsilon \in (0, 1)$, there exist $c, C > 0$ depending only on a, b and p such that

$$au - bu^2 \le (a + 2b\varepsilon)(u + \varepsilon) - b(u + \varepsilon)^2 \le -\frac{p+2}{\tau(p+1)} u_+ - cu_+^2 + C.$$

By Lemma 2.4.1, it is clear that

$$I_5 \le -\frac{p+2}{\tau(p+1)} \int_{\mathbb{R}^N} u_+^{p+1} \psi^2 \, dx - c \int_{\mathbb{R}^N} u_+^{p+2} \psi^2 \, dx + C\kappa^{-N}. \qquad (4.2.6)$$

Here the term $\frac{p+2}{\tau(p+1)} \int_{\mathbb{R}^N} u_+^{p+1} \psi^2 \, dx$ on the right-hand side will be used to obtain uniform-in-time bounds. Next, we estimate I_2 and prove Theorem 4.1.1(1) for the cases $1 < m \le 2$ and $m > 2$ in Subsections 4.2.1 and 4.2.2, respectively.

4.2.1 Proof of Theorem 4.1.1 (1) for the case $1 < m \le 2$

In this subsection, we consider the case of $1 < m \le 2$, and prove Theorem 4.1.1(1). This case is much simpler than the case when $m > 2$, which is because for $m \le 2$, the logistic term is strong enough to bound the integral of u^{p+m}.

Proof of Theorem 4.1.1(1) for the case $1 < m \le 2$. Let $p \ge m$. We begin by estimating κI_2. Since

$$p + m \le p + 2 \quad \text{for} \quad 1 < m \le 2,$$

we have

$$u_+^{p+m} \psi^2 \lesssim u_+^{p+2} \psi^2 + \psi^2.$$

By Young's inequality, we have for any $\delta \in (0,1)$ and $0 < \kappa < \delta$,

$$\begin{aligned}
\kappa I_2 &= C\kappa \int_{\mathbb{R}^N} u_+^{p+m-1} |\nabla u_+| \psi^2 \\
&\lesssim \delta \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 + \delta^{-1}\kappa^2 \int_{\mathbb{R}^N} u_+^{p+m} \psi^2 \\
&\lesssim \delta \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx + \delta \int_{\mathbb{R}^N} u_+^{p+2} \psi^2 \, dx + C_\delta \kappa^{-N}, \qquad (4.2.7)
\end{aligned}$$

where in the last inequality, we also used Lemma 2.4.1.

After taking δ to be sufficiently small (then $\kappa < \delta$ is also small), by (4.2.3), (4.2.4), (4.2.5), (4.2.6) and (4.2.7), we have

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^N} u_+^{p+1} \psi^2 \, dx \le &-c \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 - c \int_{\mathbb{R}^N} u_+^{p+2} \psi^2 - \frac{p+2}{\tau} \int_{\mathbb{R}^N} u_+^{p+1} \psi^2 \\
&+ C_\delta \int_{\mathbb{R}^N} |\nabla v|^{\frac{2(p+2)}{m}} \psi^2 + \kappa \int_{\mathbb{R}^N} |\nabla v|^{p+2} \psi^2 + C_\delta \kappa^{-N}.
\end{aligned}$$

Multiplying $e^{\frac{(p+2)t}{\tau}}$ to both sides and integrating in time, this implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N} u_+^{p+1}(t,x) \psi^2(x) \, dx + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx \, ds \\
&\quad + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+2} \psi^2 \, dx \, ds \\
&\le \int_{\mathbb{R}^N} u_+^{p+1}(0,x) \psi^2(x) \, dx + C_\delta \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} |\nabla v|^{\frac{2(p+2)}{m}} \psi^2 \, dx \, ds \\
&\quad + \kappa \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} |\nabla v|^{p+2} \psi^2 \, dx \, ds + C_\delta \kappa^{-N}, \qquad (4.2.8)
\end{aligned}$$

where C_δ is independent of t.

Next, recall that

$$Z_{p+2}(t) = \sup_{x_0 \in \mathbb{R}^N} \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+2}(s,x) \psi^2(x - x_0) \, dx \, ds.$$

Applying Lemma 2.5.1 with $r = \frac{p+2}{m}$ and $r' = p+2 > r$ (for $m > 1$) yields

$$C_\delta \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} |\nabla v|^{\frac{2(p+2)}{m}} \psi^2 \, dx \, ds \lesssim \delta Z_{p+2}(t) + C_\delta \, \kappa^{-N}. \qquad (4.2.9)$$

Similarly, applying Lemma 2.5.1 with $r = \frac{p+2}{2}$ and $r' = p+2$ yields

$$\kappa \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} |\nabla v|^{p+2} \psi^2 \, dx \, ds \lesssim \kappa \delta Z_{p+2}(t) + C_\delta \, \kappa^{1-N}. \tag{4.2.10}$$

By (4.2.8), (4.2.9), and (4.2.10), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} u_+^{p+1}(t,x)\psi^2(x) \, dx + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx \, ds \\ &\quad + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+2} \psi^2 \, dx \, ds \\ &\lesssim \int_{\mathbb{R}^N} u_+^{p+1}(0,x)\psi^2(x) \, dx + 2\delta Z_{p+2}(t) + C_\delta \, \kappa^{-N}. \end{aligned}$$

Since $u_+(0,x) = u_0(x) + \varepsilon$ ($\varepsilon \in (0,1)$) is uniformly bounded, by Lemma 2.4.1,

$$\int_{\mathbb{R}^N} u_+^{p+1}(0,x)\psi^2(x) \, dx \le C\kappa^{-N}.$$

Then we fix $\delta := c/4$ to get

$$\begin{aligned} &\int_{\mathbb{R}^N} u_+^{p+1}(t,x)\psi^2(x) \, dx + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+2} \psi^2 \, dx \, ds \\ &\quad + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx \, ds \le C\kappa^{-N} + \frac{c}{2} Z_{p+2}(t). \end{aligned} \tag{4.2.11}$$

Now, by shifting the space variable, we can replace $\psi(x)$ by $\psi(x - x_0)$ in (4.2.11), and obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} u_+^{p+1}(t,x)\psi^2(x-x_0) \, dx + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+2} \psi^2(x-x_0) \, dx \, ds \\ &\quad + c \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2(x-x_0) \, dx \, ds \le C\kappa^{-N} + \frac{c}{2} Z_{p+2}(t). \end{aligned}$$

After taking supremum over x_0, this implies that

$$\begin{aligned} &\sup_{x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} u_+^{p+1}(t,x)\psi^2(x-x_0) \, dx + c Z_{p+2}(t) \\ &\quad + c \sup_{x_0 \in \mathbb{R}^N} \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2(x-x_0) \, dx \, ds \le C\kappa^{-N}, \end{aligned}$$

where $0 < \kappa < \frac{c}{4}$, and the constants c, C are independent of κ and t. Hence, recalling

$u_+ = u + \varepsilon$, we get for all $p \geq m$,

$$u(t,\cdot) \in L^{p+1}_{\mathrm{loc}}(\mathbb{R}^N), \quad e^{-\frac{(p+2)T}{2\tau}}(\varepsilon + u)^{\frac{p+m-2}{2}} \nabla u \in L^2_{\mathrm{loc}}(\Omega_T),$$

and

$$\sup_{t\in[0,T],x_0\in\mathbb{R}^N} \int_{\mathbb{R}^N} u^{p+2}(t,x)\psi^2(x-x_0)\,dx \lesssim \kappa^{-N}.$$

The bounds only depend on $m, |\chi|, a, b, \tau, N, p$, $\|u_0\|_\infty$, $\|v_0\|_{W^{1,\infty}}$ and the diameter of the local spatial domain (but independent of T).

Finally, since $p \geq m$ can be arbitrary, picking $p = N+1 \geq 2 \geq m$ yields that $u(t,\cdot)$ is locally uniformly finite in $L^{N+2}(\mathbb{R}^N)$. Then Proposition 2.5.1 implies that

$$\sup_{t\in[0,T],x\in\mathbb{R}^N} |\nabla v(t,x)| \lesssim \kappa^{-N}.$$

This together with the fact that $\|v\|_\infty < \infty$ implies that $v \in L^\infty(\Omega_T) \cap W^{1,\infty}(\Omega_T)$ with a bound independent of T. □

4.2.2 Proof of Theorem 4.1.1 (1) for the case $m > 2$

In this subsection, we consider the case when $m > 2$. Note that the estimate (4.2.7) for κI_2 only holds when $1 < m \leq 2$, and when $m > 2$, we are not able to bound this term by the logistic term and the diffusion term. Therefore, we will estimate both I_1 and κI_2 differently, and a continuity argument will play an important role at the end.

We remark that the argument below actually works for the case of $m \in (1,2]$ as well. We treated the case $m \leq 2$ separately in the previous subsection as it allows a simpler proof.

Proof of Theorem 4.1.1(1) for the case $m > 2$. Let $p \geq m$ and recall that $u_+ = u + \varepsilon$. We start with the following notations: For any given $x_0 \in \mathbb{R}^N$ and $r > 1$, define

$$X_{r,x_0}(t) = \int_{\mathbb{R}^N} u_+^r(t,x)\psi^2(x-x_0)\,dx, \quad X_r(t) = \sup_{x_0\in\mathbb{R}^N} X_{r,x_0}(t),$$

$$\text{and} \quad Y_r(t) := \sup_{s\in[0,t]} X_r(s). \tag{4.2.12}$$

The proof is divided into two cases.

Case 1. We assume $N \geq 3$. First, we will make a better use of the term I_1 as follows:

$$I_1 \gtrsim \int_{\mathbb{R}^N} |\nabla(u_+^{\frac{p+m}{2}})|^2\psi^2 \geq c\int_{\mathbb{R}^N} |\nabla(u_+^{\frac{p+m}{2}}\psi)|^2 - C\kappa^2 \int_{\mathbb{R}^N} u_+^{p+m}\psi^2. \tag{4.2.13}$$

Applying Lemma 4.2.1 with r being replaced by $\frac{p+m}{2}$ (here we need $N \geq 3$), we have

$$\kappa^2 \int_{\mathbb{R}^N} u_+^{p+m} \psi^2 \leq \delta \|\nabla (u_+^{\frac{p+m}{2}} \psi)\|_2^2 + C_\delta \kappa^{2+\theta^*} \left[\int_{\mathbb{R}^N} u_+^{\frac{p+m}{2}+1} \psi^{q^*} \, dx \right]^{2/q^*}, \tag{4.2.14}$$

where

$$q^* := \frac{p+m+2}{p+m} \in (1,2), \quad \textit{and} \quad \theta^* := \frac{N(p+m-2)}{p+m+2} > 0. \tag{4.2.15}$$

Similarly as done before, using the properties of ψ from Lemma 2.4.1, we have

$$\begin{aligned} \int_{\mathbb{R}^N} u_+^{\frac{p+m}{2}+1}(s,x) \psi^{q^*}(x) \, dx &\lesssim \sum_{\kappa z \in \mathbb{Z}^N} \psi^{q^*}(z) \int_{B_{N/\kappa}(z)} u_+^{\frac{p+m}{2}+1}(s,x) \psi^2(x-z) \, dx \\ &\lesssim \sup_{x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} u_+^{\frac{p+m}{2}+1}(s,x) \psi^2(x-x_0) \, dx. \end{aligned}$$

Since $\frac{p+m}{2} + 1 \leq p+1$, by (4.2.1), $u_+^{\frac{p+m}{2}+1} \leq u_+^{p+1} + 1$. By Lemma 2.4.1 again,

$$\begin{aligned} \int_{\mathbb{R}^N} u_+^{\frac{p+m}{2}+1}(s,x) \psi^{q^*}(x) \, dx &\lesssim \sup_{x_0 \in \mathbb{R}^N} \int_{\mathbb{R}^N} u_+^{p+1}(s,x) \psi^2(x-x_0) \, dx + \int_{\mathbb{R}^N} \psi^2(x-x_0) \, dx \\ &\lesssim X_{p+1}(s) + \kappa^{-N}. \end{aligned}$$

It follows from (4.2.14) that

$$\kappa^2 \int_{\mathbb{R}^N} u_+^{p+m} \psi^2 \leq \delta \|\nabla (u_+^{\frac{p+m}{2}} \psi)\|_2^2 + C_\delta \kappa^{2+\theta^*} X_{p+1}(s)^{2/q^*} + C \kappa^{2+\theta^* - 2N/q^*}. \tag{4.2.16}$$

After taking this $\delta > 0$ to be small, (4.2.13) and (4.2.16) yield

$$\frac{1}{2} I_1 \geq c \|\nabla (u_+^{\frac{p+m}{2}} \psi)\|_2^2 - C \kappa^{2+\theta^*} X_{p+1}(s)^{2/q^*} - C \kappa^{2+\theta^* - 2N/q^*}. \tag{4.2.17}$$

Next, we bound I_2. By Young's inequality and (4.2.16), we have for any $\delta \in (0,1)$,

$$\begin{aligned} \kappa I_2 &\leq \delta \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 + C \delta^{-1} \kappa^2 \int_{\mathbb{R}^N} u_+^{p+m} \psi^2 \\ &\leq \delta \int_{\mathbb{R}^N} u_+^{p+m-2} |\nabla u_+|^2 \psi^2 \, dx + \delta \|\nabla (u_+^{\frac{p+m}{2}} \psi)\|_2^2 \\ &\quad + C_\delta \kappa^{2+\theta^*} X_{p+1}(s)^{2/q^*} + C_\delta \kappa^{2+\theta^* - 2N/q^*}. \end{aligned} \tag{4.2.18}$$

By (4.2.3), (4.2.4), (4.2.5), (4.2.6), (4.2.17) and (4.2.18), we have for any $0 < \delta < 1$

sufficiently small depending only on C, any $t \in [0,T]$, and any $0 < \kappa < \delta$, there holds

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} u_+^{p+1}\psi^2\,dx \le -c\int_{\mathbb{R}^N} u_+^{p+m-2}|\nabla u_+|^2\psi^2 - c\int_{\mathbb{R}^N} u_+^{p+2}\psi^2 - \frac{p+2}{\tau}\int_{\mathbb{R}^N} u_+^{p+1}\psi^2 \\ + C_\delta\int_{\mathbb{R}^N} |\nabla v|^{\frac{2(p+2)}{m}}\psi^2 + \kappa\int_{\mathbb{R}^N} |\nabla v|^{p+2}\psi^2 + C\kappa^{-N} \\ + C_\delta\kappa^{2+\theta^*}X_{p+1}(s)^{2/q^*} + C_\delta\kappa^{2+\theta^*-2N/q^*}.\end{aligned}$$

This together with (4.2.9) and (4.2.10) implies that

$$\begin{aligned}&\int_{\mathbb{R}^N} u_+^{p+1}(t,x)\psi^2(x)\,dx + c\iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}}u_+^{p+m-2}|\nabla u_+|^2\psi^2\,dx\,ds + c\iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}}u_+^{p+2}\psi^2\,dx\,ds \\ &\le \int_{\mathbb{R}^N} u_+^{p+1}(0,x)\psi^2\,dx + \delta Z_{p+2}(t) + C_\delta\kappa^{-N} + C_\delta\kappa^{2+\theta^*}Y_{p+1}(t)^{2/q^*} + C_\delta\kappa^{2+\theta^*-2N/q^*},\end{aligned}$$

where the constants c, C_δ are independent of κ and t. Using that $u_+(0,\cdot) = u_0(x) + \varepsilon$ is uniformly bounded, Lemma 2.4.1, and fixing $\delta > 0$ to be sufficiently small, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} u_+^{p+1}(t,x)\psi^2(x)\,dx + \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}}u_+^{p+m-2}|\nabla u_+|^2\psi^2\,dx\,ds + \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}}u_+^{p+2}\psi^2\,dx\,ds \\ &\lesssim \delta Z_{p+2}(t) + \kappa^{2+\theta^*}Y_{p+1}(t)^{2/q^*} + \kappa^{2+\theta^*-2N/q^*} + \kappa^{-N}.\end{aligned}$$

Hence, for $x_0 = 0$, since $Y_{p+1}(\cdot)$ is non-decreasing,

$$\begin{aligned}&X_{p+1,x_0}(t) + \iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}}u_+^{p+m-2}|\nabla u_+|^2\psi^2(x-x_0)\,dx\,ds + Z_{p+2,x_0}(t) \\ &\lesssim \delta Z_{p+2}(t) + \kappa^{2+\theta^*}Y_{p+1}(t_0)^{2/q^*} + \kappa^{2+\theta^*-2N/q^*} + \kappa^{-N}\end{aligned}$$

for $t_0 \in (0,T]$ and $t \in [0,t_0]$. After shifting in space, the same estimate holds for general $x_0 \in \mathbb{R}^N$. Taking supremum in x_0 and $t \in [0,t_0]$ for any $t_0 \in [0,T]$, we get

$$\begin{aligned}&Y_{p+1}(t_0) + Z_{p+2}(t_0) + \sup_{t\in[0,t_0],x_0\in\mathbb{R}^N}\iint_{\Omega_t} e^{-\frac{(p+2)(t-s)}{\tau}}u_+^{p+m-2}|\nabla u_+|^2\psi^2(x-x_0)\,dx\,ds \\ &\lesssim \kappa^{2+\theta^*}Y_{p+1}(t_0)^{2/q^*} + \kappa^{2+\theta^*-2N/q^*} + \kappa^{-N}.\end{aligned}$$

Recall (4.2.15) and then

$$2/q^* = \frac{2(p+m)}{p+m+2} \quad\text{and}\quad 2+\theta^* - 2N/q^* = 2 + \frac{N(p+m-2)}{p+m+2} - \frac{2N(p+m)}{p+m+2} = 2 - N.$$

Therefore, there exists $C_0 > 0$ independent of t_0 such that for all κ sufficiently small,

$$\begin{aligned} &Y_{p+1}(t_0) + \sup_{x_0\in\mathbb{R}^N} \iint_{\Omega_{t_0}} e^{-\frac{(p+2)(t_0-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2(x-x_0)\,dx\,ds \\ &\le C_0 \kappa^{2+\frac{N(p+m-2)}{p+m+2}} \left[Y_{p+1}(t_0)\right]^{\frac{2(p+m)}{p+m+2}} + C_0\kappa^{-N}. \end{aligned} \tag{4.2.19}$$

By further taking C_0 to be large enough if necessary, we can assume

$$Y_{p+1}(0) = X_{p+1}(0) \le C_0\kappa^{-N}.$$

We claim that, if $0 < \kappa < 1$ is sufficiently small,

$$Y_{p+1}(T) \le 2C_0\kappa^{-N}. \tag{4.2.20}$$

In fact, assume for contradiction that there is $t_0 \in [0, T]$ such that

$$Y_{p+1}(t_0) = 2C_0\kappa^{-N}.$$

By (4.2.19),

$$\begin{aligned} 2C_0\kappa^{-N} &\le C_0\kappa^{2+\frac{N(p+m-2)}{p+m+2}} (2C_0\kappa^{-N})^{\frac{2(p+m)}{p+m+2}} + C_0\kappa^{-N} \\ &= C_0 (2C_0)^{\frac{2(p+m)}{p+m+2}} \kappa^{2-N} + C_0\kappa^{-N}. \end{aligned}$$

This implies that

$$1 \le (2C_0)^{\frac{2(p+m)}{p+m+2}} \kappa^2,$$

which is impossible if κ was chosen to be $\frac{1}{2}(2C_0)^{-\frac{p+m}{p+m+2}}$. Therefore, the claim (4.2.20) holds.

By (4.2.19), we also have

$$\sup_{x_0\in\mathbb{R}^N} \iint_{\Omega_T} e^{-\frac{(p+2)(t-s)}{\tau}} u_+^{p+m-2} |\nabla u_+|^2 \psi^2(x-x_0)\,dx\,ds \le C_0(2C_0)^{\frac{2(p+m)}{p+m+2}}\kappa^{2-N} + C_0\kappa^{-N}.$$

The results yield that for any $p \ge m$, $u(t,\cdot)$ is locally uniformly finite in $L^{p+1}(\mathbb{R}^N)$ space for each $t \in [0, T]$, and $e^{-\frac{(p+2)T}{2\tau}} (u+\varepsilon)^{\frac{p+m-2}{2}} \nabla u$ is locally uniformly bounded in $L^2(\Omega_T)$. Furthermore, after picking $p = \max\{m, N\}$, we obtain locally uniform boundedness of $u(t,\cdot)$ in $L^{N+1}(\mathbb{R}^N)$. By Proposition 2.5.1, ∇v is uniformly finite in $L^\infty(\Omega_T)$. Moreover, the bounds of $u(t,\cdot) \in L^{p+1}_{\text{loc}}(\mathbb{R}^N)$ and $v \in L^\infty(\Omega_T) \cap W^{1,\infty}(\Omega_T)$ are independent of T. This proves Theorem 4.1.1(1) for the case when $m > 2$ and $N \ge 3$.

Case 2. $N = 1, 2$. For any $t \geq 0$ and $x \in \mathbb{R}^N$, let $\tilde{x} = (x, x_{N+1}, x_{N+2}) \in \mathbb{R}^{N+2}$ and

$$\tilde{u}(t, \tilde{x}) = u(t, x), \quad \tilde{v}(t, \tilde{x}) = v(t, x).$$

Then $(\tilde{u}(t, \tilde{x}), \tilde{v}(t, \tilde{x}))$ is a solution of (4.0.2) with N being replaced by $N + 2$ with extended initial data $(\tilde{u}_0, \tilde{v}_0)$. The conclusion for the case $N = 1, 2$ thus follows from the case $N \geq 3$. ☐

Remark 4.2.1. *Suppose that ∇v is a given L^∞ vector field, and u solves*

$$u_t = m\nabla \cdot \Big((\varepsilon + u)^{m-1}\nabla u\Big) - \chi\nabla \cdot (u\nabla v) + u(a - bu) \qquad \text{in } [0, T] \times \mathbb{R}^N,$$

with bounded initial data. Then for any $p \geq m$, u is locally uniformly finite in L^p with a bound independent of ε and T, and $e^{-\frac{(p+2)T}{2\tau}}(\varepsilon + u)^{\frac{p+m-2}{2}}\nabla u$ is locally uniformly bounded in $L^2(\Omega_T)$. Moreover, the bounds depend only on $m, |\chi|, a, b, N, p$, $\|u_0\|_\infty$, $\|\nabla v\|_{L^\infty}$ and the diameter of the local spatial domain.

The proof is almost identical to the one of Theorem 4.1.1(1) and it is actually simpler. The only difference is that, in this case, we can replace (2.5.9) *by*

$$\iint_{\Omega_t} |\nabla v|^{2r}\psi^2 \leq C_r\, \kappa^{-N}$$

with C_r depending on r and $\|\nabla v\|_{L^\infty}$.

4.3 L^∞ a Priori Estimate for the Perturbed Problem and Proof of Theorem 4.1.1(2)

In this section, we provide L^∞ a priori estimates (independent of $\varepsilon \in (0, 1)$ and T) for solutions (u, v) of (4.0.2) and prove Theorem 4.1.1(2). It follows from the last part of the proof of Theorem 4.1.1(1) that it suffices to consider the case $N \geq 3$. By Theorem 4.1.1(1), $u(t, \cdot)$ is locally uniformly bounded in L^p for all t, although the bound depends on p. Unlike in Section 3.2 of Chapter 3, there is no integral representation for u as in (3.2.28), and semigroup estimates are therefore not helpful. Here, we apply Moser's iteration method to upgrade this estimate to an L^∞-bound valid for all times.

In the following, we assume $N \geq 3$, and fix $T > 0$ and

$$p_0 := \max\{N + 1, m + 1\}.$$

By Theorem 4.1.1(1), there exists K_0 depending on p_0 (independent of T and ε) such that

$$\sup_{t\in[0,T],x_0\in\mathbb{R}^N} \|u(t,\cdot)\|_{L^{p_0}(B_1(x_0))} \le K_0. \tag{4.3.1}$$

Since $p_0 > N$, by Proposition 2.5.1, there exists K_1 depending only on p_0, N, $\|v_0\|_{W^{1,\infty}}$ and K_0 such that

$$\|\nabla v(t,\cdot)\|_\infty \le K_1 \quad \forall\, 0 \le t \le T.$$

Let ψ be from Lemma 2.4.1 with a fixed parameter $\kappa \in (0,1)$. Multiplying the first equation in (4.0.2) by $u^p\psi^2$ and integrating it over $\mathbb{R}^N$ with $p \ge m$, we get

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_{\mathbb{R}^N} u^{p+1}\psi^2\,dx + \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}\psi^2\,dx \\
&= \int_{\mathbb{R}^N} u^p\psi^2\nabla\cdot\big[m(u+\varepsilon)^{m-1}\nabla u - \chi u\nabla v\big]\,dx + \int_{\mathbb{R}^N}\Big(a+\frac{1}{p+1}\Big)u^{p+1}\psi^2\,dx - b\int_{\mathbb{R}^N} u^{p+2}\psi^2\,dx \\
&= -mp\int_{\mathbb{R}^N} u^{p-1}\psi^2(u+\varepsilon)^{m-1}|\nabla u|^2\,dx + \underbrace{\Big(-m\int_{\mathbb{R}^N} u^p(u+\varepsilon)^{m-1}\nabla u\cdot\nabla\psi^2\,dx\Big)}_{J_1} \\
&\quad + \underbrace{\Big(\chi p\int_{\mathbb{R}^N} u^p\psi^2\nabla v\cdot\nabla u\,dx\Big)}_{J_2} + \underbrace{\Big(\chi\int_{\mathbb{R}^N} u^{p+1}\nabla v\cdot\nabla\psi^2\,dx + \int_{\mathbb{R}^N}\Big(a+\frac{1}{p+1}\Big)u^{p+1}\psi^2\,dx - b\int_{\mathbb{R}^N} u^{p+2}\psi^2\,d}_{J_3}
\end{aligned} \tag{4.3.2}$$

Unlike in the previous Section 4.2, throughout this section, the general constants $c, C, C' \ge 1$ only depend on $m, \chi, a, b, \tau, N, K_0, K_1, \|v_0\|_{W^{1,\infty}}$ and $\|u_0\|_{L^\infty}$, and they are likely to be different from one line to another. Let us emphasize that these general constants can depend on p_0 in this section, but they are absolutely independent of ε, κ, T, and p.

We start by providing fine estimates for J_1, J_2 and J_3. First, using that $|\nabla\psi| \lesssim \kappa\psi$, direct computation yields

$$\begin{aligned}
J_1 &\le \frac{mp}{4}\int_{\mathbb{R}^N} u^{p-1}\psi^2(u+\varepsilon)^{m-1}|\nabla u|^2\,dx + \frac{4\kappa^2 m}{p}\int_{\mathbb{R}^N} u^{p+1}(u+\varepsilon)^{m-1}\psi^2\,dx \\
&\le \frac{mp}{4}\int_{\mathbb{R}^N} u^{p-1}\psi^2(u+\varepsilon)^{m-1}|\nabla u|^2\,dx + \frac{C\kappa^2}{p}\int_{\mathbb{R}^N} (u^{p+m}+1)\psi^2\,dx,
\end{aligned}$$

where in the second inequality we used that

$$u^{p+1}(u+\varepsilon)^{m-1} \le 2^{m-1}u^{p+m} + 2^{m-1}u^{p+1} \le 2^m u^{p+m} + 2^{m-1},$$

and C is independent of ε, κ and p. Then using (2.4.2) and (2.4.4) from Lemma 2.4.1 yields

$$\begin{aligned} J_1 &\leq \frac{mp}{4}\int_{\mathbb{R}^N} u^{p-1}\psi^2(u+\varepsilon)^{m-1}|\nabla u|^2\,dx + \frac{C\kappa^2}{p}\int_{\mathbb{R}^N} u^{p+m}\psi^2\,dx + \frac{C\kappa^{2-N}}{p} \\ &\leq \frac{mp}{4}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + \frac{C\kappa^2}{p}\int_{\mathbb{R}^N} u^{p+m}\psi^2\,dx + \frac{C\kappa^{2-N}}{p}. \end{aligned} \tag{4.3.3}$$

For J_2, since $|\nabla v| \leq C$ and $p \geq m-2$, we have

$$\begin{aligned} J_2 &\leq \frac{mp}{4}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + Cp\int_{\mathbb{R}^N} u^{p+2-m}\psi^2\,dx \\ &\leq \frac{mp}{4}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + Cp\int_{\mathbb{R}^N} u^{p+m}\psi^2\,dx + Cp\kappa^{-N}, \end{aligned} \tag{4.3.4}$$

where we used (2.4.2) and (2.4.4) again. Since $|\nabla v| \leq C$, $m > 1$ and $\kappa < 1$,

$$J_3 \leq C\int_{\mathbb{R}^N} u^{p+1}\psi^2\,dx - b\int_{\mathbb{R}^N} u^{p+2}\psi^2\,dx \leq C\int_{\mathbb{R}^N} u^{p+m}\psi^2\,dx + C\kappa^{-N}. \tag{4.3.5}$$

Let us comment that here we did not use the assumption that $b > 0$. By (4.3.3)–(4.3.5), we have

$$J_1 + J_2 + J_3 \leq \frac{mp}{2}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + C_1 p\int_{\mathbb{R}^N} u^{p+m}\psi^2\,dx + Cp\kappa^{-N}, \tag{4.3.6}$$

where C_1 and C are general constants independent of p and κ.

Next, we provide fine estimates for $\int_{\mathbb{R}^N} u^{p+m}\psi^2$. Let

$$\theta := \frac{N(p+m-2)}{2(p+m+2)} \quad \text{and} \quad q := \frac{p+m+2}{p+m}.$$

It follows from Lemma 4.2.1 with $r = \frac{p+m}{2}$ that there is a dimensional constant $C > 0$ such that

$$\int_{\mathbb{R}^N} u^{p+m}\psi^2 \leq \delta\left\|\nabla(u^{\frac{p+m}{2}}\psi)\right\|_2^2 + C\delta^{-\theta}\left[\int_{\mathbb{R}^N} u^{\frac{p+m+2}{2}}\psi^q\,dx\right]^{2/q}.$$

Using the properties of ψ, direct computation yields

$$\left\|\nabla(u^{\frac{p+m}{2}}\psi)\right\|_2^2 \leq Cp^2\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + C\kappa^2\int_{\mathbb{R}^N} u^{p+m}\psi^2\,dx.$$

We pick $\delta := c/p^2$ and $\kappa := c$ with c sufficiently small depending on C_1 from (4.3.6) to get

$$C_1 p \int_{\mathbb{R}^N} u^{p+m}\psi^2 \le \frac{mp}{4}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + Cp^{1+2\theta}\left[\int_{\mathbb{R}^N} u^{\frac{p+m+2}{2}}\psi^q\,dx\right]^{2/q}.$$

Then, let us bound $\int_{\mathbb{R}^N} u^{\frac{p+m+2}{2}}\psi^q\,dx$. Using Lemma 2.4.1 and that $q \in (1,2)$, there is a dimensional constant $C > 0$ such that for any $t \in [0,T]$ and $x_0 \in \mathbb{R}^N$,

$$\begin{aligned}
\int_{\mathbb{R}^N} u^{\frac{p+m+2}{2}}(t,x)\psi^q(x)\,dx &\le \int_{\mathbb{R}^N} u^{\frac{p+m+2}{2}}(t,x)\psi(x)\,dx \\
&\le C\sum_{\kappa z\in\mathbb{Z}^N}\int_{B_{N/\kappa}(z)} u^{\frac{p+m+2}{2}}(t,x)\psi(x-x_0)\,dx \\
&\le C\sum_{\kappa z\in\mathbb{Z}^N}\psi(z-x_0)\int_{B_{N/\kappa}(z)} u^{\frac{p+m+2}{2}}(t,x)\psi^2(x-z)\,dx \\
&\le CY_{\frac{p+m+2}{2}}(t)\sum_{\kappa z\in\mathbb{Z}^N}\psi(z-x_0) \le CY_{\frac{p+m+2}{2}}(t),
\end{aligned}$$

where we used the notation (4.2.12). This implies that

$$C_1 p\int_{\mathbb{R}^N} u^{p+m}(t,\cdot)\psi^2(\cdot)\,dx \le \frac{mp}{4}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + Cp^{1+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q}. \qquad (4.3.7)$$

It follows from (4.3.6) and (4.3.7) that

$$J_1 + J_2 + J_3 \le \frac{3mp}{4}\int_{\mathbb{R}^N} u^{p+m-2}|\nabla u|^2\psi^2\,dx + Cp^{1+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q} + Cp\kappa^{-N}. \qquad (4.3.8)$$

Now we would use this estimate in the next subsection in order to proof Theorem 4.1.1(2).

4.3.1 Proof of Theorem 4.1.1(2)

The proof of Theorem 4.1.1(2) follows from the following proposition.

Proposition 4.3.1. *Suppose that* ∇v *is a given* L^∞ *vector field, and* u *solves*

$$u_t = m\nabla\cdot\left((\varepsilon+u)^{m-1}\nabla u\right) - \chi\nabla\cdot(u\nabla v) + u(a-bu) \qquad \text{in } [0,T]\times\mathbb{R}^N,$$

with bounded initial data, and u *satisfies* (4.3.1). *Then* u *is uniformly bounded in* $[0,T]\times\mathbb{R}^N$ *with the bound depending on general constants, but independent of* ε *and* T. *Moreover, for any* $M > \|u_0\|_\infty$, *there exists* $T > 0$ *such that* $\|u\|_\infty \le M$ *in* $[0,T]\times\mathbb{R}^N$.

Proof of Proposition 4.3.1. First of all, by (4.3.2) and (4.3.8), there holds

$$\frac{1}{p+1}\frac{d}{dt}\int_{\mathbb{R}^N} u^{p+1}\psi^2\,dx + \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}\psi^2\,dx \le Cp^{1+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q} + Cp\kappa^{-N},$$

where

$$\theta = \frac{N(p+m-2)}{2(p+m+2)} \in (cN, N) \quad \text{for some } c \in (0,1) \text{ and } \quad q = \frac{p+m+2}{p+m}.$$

Multiplying the above inequality on both sides by e^t, and integrating in time give

$$\begin{aligned}
&\int_{\mathbb{R}^N} u^{p+1}(t,x)\psi^2(x)\,dx \\
&\le e^{-t}\int_{\mathbb{R}^N} u^{p+1}(0,x)\psi^2(x)\,dx + \int_0^t e^{-(t-s)}\Big(Cp^{2+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q} + Cp^2\kappa^{-N}\Big)\,ds \\
&\le \int_{\mathbb{R}^N} u^{p+1}(0,x)\psi^2(x)\,dx + Cp^{2+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q} + Cp^2\kappa^{-N},
\end{aligned}$$

where we used that $Y_p(\cdot)$ is non-decreasing in time by its definition. This implies that for $t \in [0,T]$,

$$\int_{\mathbb{R}^N} u^{p+1}(t,x)\psi^2(x)\,dx \le \int_{\mathbb{R}^N} u^{p+1}(0,x)\psi^2(x)\,dx + Cp^{2+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q} + Cp^2\kappa^{-N}.$$

Since $u(0,\cdot)$ is uniformly bounded, by shifting the space variable, we get for any $x_0 \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} u^{p+1}(t,x)\psi^2(x-x_0)\,dx \le \max\left\{Cp^{2+2\theta}Y_{\frac{p+m+2}{2}}(t)^{2/q},\, (C_\kappa)^p\right\},$$

where C_κ depends on general constants and κ. After taking supremum over x_0 and t, we get for $t \in [0,T]$,

$$Y_{p+1}(t) \le \max\left\{Cp^{2+2\theta}\left[Y_{\frac{p+m+2}{2}}(t)\right]^{\frac{2(p+m)}{p+m+2}},\, (C_\kappa)^p\right\}. \tag{4.3.9}$$

Since $\theta \le N$, if denoting

$$W_r(t) := \max\left\{Y_r(t)^{\frac{1}{r}},\, C_\kappa\right\}$$

for some C_κ depending only on general constants and κ, we get

$$\begin{aligned} W_{p+1}(t) &\leq \max\left\{ C^{\frac{1}{p+1}} p^{\frac{2+2\theta}{p+1}} \left[Y_{\frac{p+m+2}{2}}(t) \right]^{\frac{2}{p+m+2}\cdot\frac{p+m}{p+1}}, \, (C_\kappa)^{\frac{p}{p+1}} \right\} \\ &\leq C^{\frac{1}{p+1}} p^{\frac{2+2N}{p+1}} \left[W_{\frac{p+m+2}{2}}(t) \right]^{\frac{p+m}{p+1}} . \end{aligned} \tag{4.3.10}$$

Now, define $r_0 := p_0$ and, iteratively,

$$r_n := 2r_{n-1} - m - 1 \quad \text{for } n \geq 1.$$

It is easy to get

$$r_n = 2^n(p_0 - m - 1) + m + 1.$$

Then (4.3.10) becomes

$$W_{r_n}(t) \leq C^{\frac{1}{r_n}} (r_n)^{\frac{2+2N}{r_n}} \left[W_{r_{n-1}}(t) \right]^{\frac{r_n+m-1}{r_n}} ,$$

and then by iteration,

$$\begin{aligned} W_{r_n}(t) &\leq C^{\frac{1}{r_n} + \frac{1}{r_{n-1}}(1+\frac{m-1}{r_n})} (r_n)^{\frac{2+2N}{r_n}} (r_{n-1})^{\frac{2+2N}{r_{n-1}}(1+\frac{m-1}{r_n})} \left[W_{r_{n-2}}(t) \right]^{(1+\frac{m-1}{r_{n-1}})(1+\frac{m-1}{r_n})} \\ &\leq \ldots \leq \exp\Big(\sum_{j=1}^{n} \frac{\beta_{j+1}}{r_j} \Big) \Big[\prod_{j=1}^{n} (r_j)^{\frac{2+2N}{r_j}\beta_{j+1}} \Big] \left[W_{r_0}(t) \right]^{\beta_1} \end{aligned} \tag{4.3.11}$$

where $\beta_{n+1} := 1$, and for $j = 1, \ldots, n$,

$$\beta_j := (1 + \frac{m-1}{r_n}) \ldots (1 + \frac{m-1}{r_j}).$$

Since $p_0 > m + 1$, $C^{-1}2^n \leq r_n \leq C2^n$ for some $C > 1$. Consequently, $\beta_j \in [1, C]$ for some general constant $C > 1$ independent of j and n, and so

$$\sum_{j=1}^{n} \frac{\beta_{j+1}}{r_j} \leq C,$$

$$\ln \prod_{j=1}^{n} (r_j)^{\frac{2+2N}{r_j}\beta_{j+1}} = \sum_{j=1}^{\infty} \frac{(2+2N)\beta_{j+1}}{r_j} \ln r_j \leq C \sum_{j=1}^{\infty} j2^{-j} < \infty.$$

Hence $\prod_{j=1}^{n} (r_j)^{\frac{2+2N}{r_j}\beta_{j+1}} \leq C$ for some $C > 0$ uniformly for all n. Finally, since $W_{r_0}(t) = W_{p_0}(t) \leq C_\kappa$ by (4.3.1), we obtain from (4.3.11) that there exists C_κ depending only on C

and κ such that

$$W_{r_n}(t) \leq C_\kappa \quad \text{and then} \quad \sup_{x_0 \in \mathbb{R}^N} \left[\int_{B_{N/\kappa}(x_0)} u^{r_n}(t,x)\, dx \right]^{1/r_n} \leq C_\kappa.$$

After passing $n \to \infty$, this implies that there is C independent of ε such that

$$\|u(\cdot,\cdot)\|_{L^\infty([0,T]\times\mathbb{R}^N)} \leq C.$$

Finally, the last claim is clear from the proof. This is because when $T \leq 1$, the constants C and $(C_\kappa)^p$ in (4.3.9) can be replaced by, respectively, $C'T$ and $C'_\kappa(p^2 T + \|u_0\|_\infty^p)$ for some C', C'_κ independent of $T \in (0,1]$. $\square$

In the next subsection, we give the proof of Theorem 4.1.1(3).

4.3.2 Proof of Theorem 4.1.1(3)

In this subsection, we proof uniform Hölder continuity of the perturbed problem.

Proof of Theorem 4.1.1(3). We denote $z(t,x) := (u_\varepsilon(t,x)+\varepsilon)^m$ and $\tilde{V}(t,x) := \nabla v_\varepsilon(t,x)$. Then $z \geq \varepsilon^m$ satisfies the equation

$$(z^{\frac{1}{m}})_t = \Delta z - \chi \nabla \cdot ((z^{\frac{1}{m}} - \varepsilon)\tilde{V}) + a(z^{1/m} - \varepsilon) - b(z^{1/m} - \varepsilon)^2, \quad x \in \mathbb{R}^N,\ t > 0. \quad (4.3.12)$$

Note that by Theorem 4.1.1(2), there exists $C > 0$, independent of ε, such that

$$\|z\|_\infty, \quad \|\tilde{V}\|_\infty \leq C.$$

Hence, it follows from the proof of [5, Theorem 1.3 and Theorem 1.9] that there exist $\alpha = \alpha(m, C) \in (0,1)$, and $C' > 0$ depending only on $m, N, a, b, |\chi|, \tau'$ and C such that for any $x_0 \in \mathbb{R}^N$ and $t_0 \geq 0$,

$$\|z\|_{C^\alpha((t_0+\tau', t_0+2\tau')\times B_1(x_0))} \leq C'.$$

If, in addition, $z(t_0,\cdot)$ is Hölder continuous in $B_2(x_0)$, then $z(\cdot,\cdot)$ is Hölder continuous in $(t_0, t_0+1)\times B_1(x_0)$ with a bound depending only on $m, N, a, b, |\chi|, N$ and the Hölder norm of $z(t_0,\cdot)$.

We note that although the framework in [5] does not exactly cover equation (4.3.12) with $\varepsilon > 0$, since it considers the drift term $\nabla \cdot (z^{1/m}\tilde{V})$ rather than $\nabla \cdot ((z^{1/m} - \varepsilon)\tilde{V})$ (and thus covers the case $\varepsilon = 0$), the conclusions of their theorems remain valid. Indeed, the same proofs apply without modification.

Finally, it is clear that the Hölder continuity property of z yields the Hölder continuity property of u_ε, and the norms are independent of ε.

$\square$

4.4 Global Existence of Weak Solution and Proof of Theorem 4.1.2

In this section, we first study the global well-posedness of weak/classical solutions of (4.0.2) with $\varepsilon > 0$ and prove Proposition 4.1.1 in Subsection 4.4.1. Then we pass $\varepsilon \to 0$ and prove Theorem 4.1.2 in Subsection 4.4.2.

4.4.1 Proof of Proposition 4.1.1

In this subsection, we prove Proposition 4.1.1. We do so by means of the following strategy. First, we establish the existence of a unique weak solution on an initial interval $[0, T]$ via the Banach fixed-point theorem. Next, we extend this solution to a maximal interval $[0, T_{\max})$. Then we show that the solution is a classical solution. Finally, applying the previously derived a priori estimates, we demonstrate that $T_{\max} = \infty$.

Recall that, in the case that $m = 1$ (Chapter 3, Section 3.2), the local existence of a unique classical solution was achieved via semigroup approach. Indeed, we showed, via Banach fixed point theorem, the existence of a unique mild solution satisfying

$$\begin{cases} u(t,\cdot) = e^{(\Delta-I)t}u_0 - \chi\int_0^t e^{(\Delta-I)(t-s)}\nabla\cdot(u\nabla v)\,ds + \int_0^t e^{(\Delta-I)(t-s)}u(s,\cdot)\big(a+1-bu(s,\cdot)\big)\,ds, \\ v(t,\cdot) = e^{(\Delta-I)\frac{t}{\tau}}v_0 + \frac{1}{\tau}\int_0^t e^{(\Delta-I)\frac{(t-s)}{\tau}}v(s,\cdot)\big(1-u(s,\cdot)\big)\,ds, \end{cases}$$

on some small interval $[0, T]$. However, when $m > 1$, a semigroup representation is not available. In the literature on chemotaxis models with nonlinear diffusion, the local well-posedness of solutions to the perturbed problem is typically established using the theory of quasilinear parabolic equations (see, e.g., [1, 41]). However, these arguments are often presented without full details. For completeness, we provide a detailed proof of the local well-posedness of solutions to (4.0.2).

First of all, for given $T > 0$ and $p > m + 1$, let

$$\mathcal{X}^p(T) := \{u \in C([0, T], L^p_{\mathrm{loc}}(\mathbb{R}^N))\},$$

equipped with the norm

$$\|u\|_{\mathcal{X}^p(T)} := \frac{1}{|B_1|} \sup_{0\le t\le T,\, x_0\in\mathbb{R}^N} \|u(t,\cdot)\|_{L^p(B_1(x_0))}.$$

Then $\mathcal{X}^p(T)$ with the norm is a Banach space. Moreover, if $u \in L^\infty$, then $\|u\|_\infty \ge \|u\|_{\mathcal{X}^p(T)}$. For any $M > \|u_0\|_\infty$, let

$$\mathcal{Z}^p(T,M) = \{u \in \mathcal{X}^p(T) \,|\, u(0,\cdot) = u_0,\, u \ge 0,\, \|u\|_{\mathcal{X}^p(T)} \le M\}.$$

It is clear that $\mathcal{Z}^p(T,M)$ is a closed subset of $\mathcal{X}^p(T)$.

The next lemma will be used to define the contraction mapping.

Lemma 4.4.1. *Assume that u_0 is uniformly $C^{1+\alpha}$, and v_0 is uniformly $C^{2+\alpha}$, $M > \|u_0\|_\infty$, $T \ge 1$, and $p > \max\{N, m+1\}$. For $i = 1,2$ and any given $\tilde{u}_i \in \mathcal{Z}^p(T,M)$ such that $\tilde{u}_i$ is uniformly Hölder continuous in space and time, let v_i be the classical solution to*

$$\begin{cases} \tau(v_i)_t = \Delta v_i - \tilde{u}_i(t,x)v_i, & (t,x) \in \Omega_T, \\ v_i(0,x) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

and let u_i be a classical solution to

$$\begin{cases} (u_i)_t = m\nabla\cdot((\varepsilon+u_i)^{m-1}\nabla u_i) - \chi\nabla\cdot(u_i\nabla v_i) + u_i(a-bu_i), & (t,x) \in \Omega_T, \\ u_i(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Then there exists $T_1 \in (0,1]$ depending only on ε, M, C, $\|u_0\|_{C^{1+\alpha}}$ and $\|v_0\|_{C^{2+\alpha}}$ such that

$$\|u_1 - u_2\|_{\mathcal{X}^p(T_1)} \le \frac{1}{2}\|\tilde{u}_1 - \tilde{u}_2\|_{\mathcal{X}^p(T_1)}.$$

Proof of Lemma 4.4.1. Since $\tilde{u}_i$ is C^α, by [41, Theorem 8.1, Chapter V], $v_i(\cdot,\cdot)$ is uniformly $C^{1+\frac{\alpha}{2},2+\alpha}$ in Ω_T with norm depending on the Hölder norm of $\tilde{u}_i$ and $\|v_0\|_{C^{2,\alpha}}$, and then u_i is locally uniformly $C^{1+\frac{\alpha}{2},2+\alpha}$ in the interior of Ω_T. By Proposition 2.5.1, since $\tilde{u}_i \in \mathcal{Z}^p(T,M)$ with $p > N$ is uniformly bounded, there exists K_1 depending only on $\|v_0\|_{W^{1,\infty}}$ and M such that

$$|\nabla v_i(\cdot,\cdot)| \le K_1 \quad \text{in } \Omega_T. \tag{4.4.1}$$

By Proposition 4.3.1, there is $\tilde{K}$ depending only on general constants and K_1 such that

$$\|u_i(\cdot,\cdot)\|_\infty \le \tilde{K} \quad \text{in } \Omega_T. \tag{4.4.2}$$

Let ψ be from Lemma 2.4.1 with parameter $\kappa = 1$. It suffices to bound

$$\int_{\mathbb{R}^N} |u_1(t,x) - u_2(t,x)|^p \psi \, dx.$$

Let $z = u_1 - u_2$, then it satisfies

$$\begin{aligned} z_t = \nabla \cdot \Big[m(u_1+\varepsilon)^{m-1} \nabla z + m \nabla u_2 \Big((u_1+\varepsilon)^{m-1} - (u_2+\varepsilon)^{m-1} \Big) \Big] - \chi \nabla \cdot (z \nabla v_1) \\ + \chi \nabla \cdot (u_2 \nabla (v_2 - v_1)) + az - bz(u_1+u_2) \end{aligned} \tag{4.4.3}$$

in the classical sense. For some $p \geq m+1$ an even integer, multiplying (4.4.3) by $z^{p-1}\psi$ and integrating it over $\mathbb{R}^N$ yields

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} z^p \psi = {} & -m(p-1)\int_{\mathbb{R}^N} z^{p-2}|\nabla z|^2 (u_1+\varepsilon)^{m-1}\psi - m\int_{\mathbb{R}^N} z^{p-1}(u_1+\varepsilon)^{m-1}(\nabla z \cdot \nabla \psi) \\ & - m(p-1)\int_{\mathbb{R}^N} z^{p-2}\Big((u_1+\varepsilon)^{m-1} - (u_2+\varepsilon)^{m-1}\Big)(\nabla z \cdot \nabla u_2)\psi \\ & - m\int_{\mathbb{R}^N} z^{p-1}\Big((u_1+\varepsilon)^{m-1} - (u_2+\varepsilon)^{m-1}\Big)(\nabla u_2 \cdot \nabla \psi) \\ & + \chi(p-1)\int_{\mathbb{R}^N} z^{p-1}(\nabla z \cdot \nabla v_1)\psi + \chi\int_{\mathbb{R}^N} z^p (\nabla v_1 \cdot \nabla \psi) \\ & - \chi(p-1)\int_{\mathbb{R}^N} u_2 z^{p-2}(\nabla z \cdot \nabla(v_2 - v_1))\psi - \chi\int_{\mathbb{R}^N} u_2 z^{p-1}(\nabla(v_2 - v_1)\cdot \nabla \psi) \\ & + \int_{\mathbb{R}^N} (az^p - bz^p(u_1+u_2))\psi. \end{aligned}$$

It follows from (2.5.13) in Lemma 2.5.1 that $D^2 v_i$ is bounded in L^p_{loc} for each $p \geq 1$ (uniformly in space by shifting and locally uniformly in time). Also, recall (4.4.1), (4.4.2), and that $u_i(0,\cdot) = u_0(\cdot)$ is uniformly $C^{1+\alpha}$ and v_i is uniformly bounded in $W^{1,\infty}$. By the second part of [41, Theorem 3.1, Chapter V], we get $|\nabla u_i| \leq K_2$ for all $t \in [0,1]$ for some $K_2 > 0$ depending only on the general constants, ε, M, $\|u_0\|_{C^{1+\alpha}}$ and $\|v_0\|_{C^{2+\alpha}}$. Hence, using these bounds and $|\nabla \psi| \leq \psi$ and $0 \leq u_i \leq M$, we obtain for some $C > 0$ depending only on the general constants, ε, M, p, $\|u_0\|_{C^{1+\alpha}}$ and $\|v_0\|_{C^{2+\alpha}}$ such that for $t \in [0,1]$,

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} z^p\psi \le &-\frac{m(p-1)}{2}\int_{\mathbb{R}^N} z^{p-2}|\nabla z|^2(u_1+\varepsilon)^{m-1}\psi + C\int_{\mathbb{R}^N} z^p\psi \\ &+ C\int_{\mathbb{R}^N} z^{p-2}\frac{|(u_1+\varepsilon)^{m-1}-(u_2+\varepsilon)^{m-1}|^2}{(u_1+\varepsilon)^{m-1}}\psi \\ &+ C\int_{\mathbb{R}^N} z^{p-1}|(u_1+\varepsilon)^{m-1}-(u_2+\varepsilon)^{m-1}|\psi \\ &+ C\int_{\mathbb{R}^N} z^{p-2}|\nabla z||\nabla(v_2-v_1)|\psi + C\int_{\mathbb{R}^N} |z^{p-1}||\nabla(v_2-v_1)|\psi \\ &+ C\int_{\mathbb{R}^N} z^{p-1}|\nabla z|\psi + \int_{\mathbb{R}^N}(az^p - bz^p(u_1+u_2))\psi, \qquad (4.4.4)\end{aligned}$$

where we also applied Young's inequality. Direct computation yields that there is C_ε depending on ε, m, M such that

$$|(u_1+\varepsilon)^{m-1}-(u_2+\varepsilon)^{m-1}| \le C_\varepsilon|u_1-u_2| = C_\varepsilon|z|.$$

Since p is even, z^p and z^{p-2} are non-negative. By Young's inequality again, we have for any $\delta > 0$,

$$\begin{aligned}z^{p-2}|\nabla z||\nabla(v_2-v_1)| &\le \delta z^{p-2}|\nabla z|^2\varepsilon^{m-1} + C_{\delta,\varepsilon}|\nabla(v_2-v_1)|^2 z^{p-2} \\ &\le \delta z^{p-2}|\nabla z|^2(u_1+\varepsilon)^{m-1} + C_{\delta,\varepsilon}|\nabla(v_2-v_1)|^p + C_{\delta,\varepsilon}z^p,\end{aligned}$$

and, similarly,

$$|z|^{p-1}|\nabla(v_2-v_1)| + |z|^{p-1}|\nabla z| \le \delta z^{p-2}|\nabla z|^2(u_1+\varepsilon)^{m-1} + C_{\delta,\varepsilon}z^p + C|\nabla(v_2-v_1)|^p.$$

Fixing $\delta > 0$ to be sufficiently small and plugging these into (4.4.4) yield

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} z^p\psi \le C_\varepsilon\int_{\mathbb{R}^N} z^p\psi + C_\varepsilon\int_{\mathbb{R}^N}|\nabla(v_2-v_1)|^p\psi.$$

This implies that for all $t \in [0,1]$,

$$\|(z\psi_1)(t,\cdot)\|_{L^p}^p \le C_\varepsilon t \sup_{0<s<t}\|(\nabla(v_2-v_1)\psi_1)(s,\cdot)\|_{L^p}^p, \qquad (4.4.5)$$

where $\psi_1 := \psi^{1/p}$ and the constant C_ε only depends on C, ε, M, p, $\|u_0\|_{C^{1+\alpha}}$ and $\|v_0\|_{C^{2+\alpha}}$.

Now, note that $v_i\psi_1$ solves the equation

$$\tau(v_i\psi_1)_t = \Delta(v_i\psi_1) - \big[\tilde{u}_i v_i\psi_1 + 2\nabla v_i \cdot \nabla\psi_1 + v_i\Delta\psi_1\big].$$

Then, denoting $w := (v_2 - v_1)\psi_1$ and recalling T_p from Section 2.2, we get

$$\begin{aligned} w(t,\cdot) = T_p\Big(\frac{t}{\tau}\Big)w_0 + \int_0^t T_p\Big(\frac{t-s}{\tau}\Big)\Big[&w - (\tilde{u}_2 - \tilde{u}_1)v_2\psi_1 - \tilde{u}_1 w \\ &- 2\nabla(v_2 - v_1)\cdot\nabla\psi_1 - (v_2 - v_1)\Delta\psi_1\Big]ds. \end{aligned}$$

Since $|\nabla\psi| \le \psi$, we have $|\nabla\psi_1| \le \psi_1$. In view of (2.2.2) and by Grönwall's inequality, we get for some C and for all $t \in [0,1]$,

$$\begin{aligned} \|w(t,\cdot)\|_{L^p} &\le C\int_0^t \|(\tilde{u}_1 - \tilde{u}_2)(s,\cdot)\psi_1\|_{L^p}\,ds + C\int_0^t \|w(s,\cdot)\|_{L^p}\,ds + C\int_0^t \|\nabla(v_2 - v_1)(s,\cdot)\psi_1\|_{L^p}\,ds \\ &\le Ct\sup_{s\in[0,t]}\Big[\|(\tilde{u}_1 - \tilde{u}_2)(s,\cdot)\psi_1\|_{L^p} + \|w(s,\cdot)\|_{L^p} + \|\nabla(v_2 - v_1)(s,\cdot)\psi_1\|_{L^p}\Big], \end{aligned} \tag{4.4.6}$$

where we also used that $|v_2| \le \|v_0\|_\infty$ and $|\tilde{u}_1| \le M$. Similarly, by (2.2.3), for $t \in [0,1]$ we have

$$\begin{aligned} \|\nabla w(t,\cdot)\|_{L^p} \le{}& C\int_0^t (t-s)^{-\frac{1}{2}}e^{-\frac{p(t-s)}{\tau}}\|(\tilde{u}_1 - \tilde{u}_2)(s,\cdot)\psi_1\|_{L^p}\,ds \\ &+ C\int_0^t (t-s)^{-\frac{1}{2}}e^{-\frac{p(t-s)}{\tau}}\Big[\|w(s,\cdot)\|_{L^p} + \|\nabla(v_2 - v_1)(s,\cdot)\psi_1\|_{L^p}\Big]\,ds \\ \le{}& Ct^{\frac{1}{2}}\sup_{s\in[0,t]}\Big[\|(\tilde{u}_1 - \tilde{u}_2)(s,\cdot)\psi_1\|_{L^p} + \|w(s,\cdot)\|_{L^p} + \|\nabla(v_2 - v_1)(s,\cdot)\psi_1\|_{L^p}\Big]. \end{aligned} \tag{4.4.7}$$

Then (4.4.6) and (4.4.7) yield that for all $t \in (0,1]$ sufficiently small,

$$\|\nabla(v_2 - v_1)(t,\cdot)\psi_1\|_{L^p} \le C(\|w(t,\cdot)\|_{L^p} + \|\nabla w(t,\cdot)\|_{L^p}) \le Ct^{\frac{1}{2}}\sup_{0<s<t}\|(\tilde{u}_1 - \tilde{u}_2)(s,\cdot)\psi_1\|_{L^p}.$$

By (4.4.5), we have

$$\|z(t,\cdot)\psi_1\|_{L^p}^p \le Ct^{1+\frac{p}{2}}\sup_{0<s<t}\|(\tilde{u}_1(s) - \tilde{u}_2(s))\psi_1\|_{L^p}^p.$$

Thus, if $T_1 \le 1$ is sufficiently small, depending on the general constants, ε, M, p, $\|u_0\|_{C^{1+\alpha}}$, and $\|v_0\|_{C^{2+\alpha}}$, then for all $t \in [0, T_1]$ we conclude that

$$\sup_{x_0 \in \mathbb{R}^N} \|(u_1 - u_2)(t, \cdot)\|_{L^p(B_1(x_0))} \le \frac{1}{2} \sup_{\substack{x_0 \in \mathbb{R}^N \\ s \in [0, T_1]}} \|(\tilde{u}_1 - \tilde{u}_2)(s, \cdot)\|_{L^p(B_1(x_0))}.$$

□

It follows from Lemma 4.4.1 that, given $\tilde{u} \in \mathcal{Z}^p(T_1, M)$ with $\tilde{u}$ Hölder continuous, we can uniquely determine a function $u \in \mathcal{Z}^p(T_1, M)$. In the next lemma, we remove the Hölder continuity assumption and conclude that the mapping is a contraction on $\mathcal{Z}^p(T_1, M)$.

Lemma 4.4.2. *For any $M > \|u_0\|_\infty$ and $p > N$, there exists $T_1 = T_1(M) \in (0, T]$ such that for any $\tilde{u} \in \mathcal{Z}^p(T, M)$, there exist $u \in \mathcal{Z}^p(T_1, M)$ and a bounded function v such that they are weak solutions to*

$$u_t = m\nabla \cdot ((\varepsilon + u)^{m-1} \nabla u) - \chi \nabla \cdot (u \nabla v) + u(a - bu), \quad u(x, 0) = u_0 \tag{4.4.8}$$

and

$$\tau v_t = \Delta v - \tilde{u} v, \quad v(x, 0) = v_0, \tag{4.4.9}$$

respectively, on $[0, T_1]$. Moreover, there exists a mapping $\mathcal{L} : \mathcal{Z}^p(T_1, M) \to \mathcal{Z}^p(T_1, M)$ such that $\mathcal{L}(\tilde{u}) = u$, where u is a weak solution of (4.4.8) *on $[0, T_1]$, and it is a contraction on $\mathcal{Z}^p(T_1, M)$. Lastly, if $\tilde{u}$ is uniformly Hölder continuous, then u and v are classical solutions to* (4.4.8) *and* (4.4.9), *respectively.*

Proof of Lemma 4.4.2. First of all, we prove that for any $\tilde{u} \in \mathcal{Z}^p(T, M)$, there exist $u \in \mathcal{Z}^p(T_1, M)$ and a bounded function v such that they are weak solutions to (4.4.8) and (4.4.9) on $[0, T_1]$. To this end, we take $\tilde{u}_\delta \in \mathcal{Z}^p(T, M)$ such that $\tilde{u}_\delta$ is uniformly Hölder continuous in space and time, and $\|\tilde{u}_\delta - \tilde{u}\|_{\mathcal{X}^p(T)} \to 0$ as $\delta \to 0$. Then take v_δ and u_δ as v_i and u_i from Lemma 4.4.1, respectively, with $\tilde{u}_\delta$ in place of $\tilde{u}_i$. It follows from (4.4.1) that for some K_1 depending only on $\|v_0\|_{C^1}$, $\|\tilde{u}_\delta\|_{\mathcal{X}^p(T)}$ and M such that

$$|\nabla v_\delta(\cdot, \cdot)| \le K_1 \quad \text{in } \Omega_T.$$

Since $M > \|u_0\|_\infty$, by Remark 4.2.1 and Proposition 4.3.1, there exists T_1 depending on M but independent of δ and ε such that

$$\|u_\delta\|_{L^\infty(\Omega_{T_1})} \le M, \tag{4.4.10}$$

and, for some C independent of $\delta, \varepsilon \in (0,1)$,

$$\sup_{x_0 \in \mathbb{R}^N} \iint_{[0,T_1]\times B_1(x_0)} (\varepsilon + u_\delta)^{2m-2} |\nabla u_\delta|^2 \, dx \, dt \le C.$$

Next, passing $\delta \to 0$ along a subsequence, we can find $v \in L^\infty(0, T_1; W^{1,\infty}(\mathbb{R}^N))$ and $u \in L^\infty(\Omega_{T_1}) \cap L^2((0,T_1); W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N))$ such that

$$\begin{gathered} v_\delta \rightharpoonup v, \quad \nabla v_\delta \rightharpoonup \nabla v, \quad u_\delta \rightharpoonup u \quad \text{ in } L^p_{\mathrm{loc}}(\Omega_{T_1}) \text{ for all } p \ge 1; \\ (\varepsilon + u_\delta)^{m-1} \nabla u_\delta \rightharpoonup (\varepsilon + u)^{m-1} \nabla u \quad \text{ in } L^2_{\mathrm{loc}}(\Omega_{T_1}). \end{gathered} \tag{4.4.11}$$

It is clear that v is a weak solution of (4.4.9) on $[0, T_1]$. We claim that $u \in \mathcal{Z}^p(T_1, M)$ and u is a weak solution of (4.4.8). In fact, by (4.4.10)

$$\|u\|_{L^\infty(\Omega_{T_1})} \le M,$$

and there exists a uniformly bounded vector field g in Ω_{T_1} such that

$$u_\delta \nabla v_\delta \rightharpoonup g \quad \text{ in } L^p_{\mathrm{loc}}(\Omega_{T_1}) \text{ for all } p \ge 1. \tag{4.4.12}$$

We have from equation (4.4.1) that ∇v_δ is uniformly finite independent of δ (and also ε), Theorem 4.1.1(3) yields that u_δ is uniformly Hölder continuous in $[t_1, t_2] \times \mathbb{R}^N$ with $0 < t_1 < t_2 \le T_1$, and the Hölder norm is independent of $\delta, \varepsilon \in (0,1)$. Therefore, after passing $\delta \to 0$ along a subsequence, we actually obtain that $u_\delta \to u$ pointwise locally uniformly in $(0, T_1] \times \mathbb{R}^N$. This and the weak convergence of $\nabla v_\delta \to \nabla v$ yield that

$$g = u \nabla v. \tag{4.4.13}$$

By the u_δ-equation (also see the computations of (4.3.2)), $u_\delta(t, \cdot)$ is continuous in t in the space of $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, by the pointwise convergence of $u_\delta \to u$, the boundedness of u_δ (see (4.4.10)), and the dominated convergence theorem, we have that for each $t \in [0, T_1]$, $u_\delta(t, \cdot) \to u(t, \cdot)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. These together imply

$$u \in \mathcal{X}^p(T_1). \tag{4.4.14}$$

By (4.4.11)–(4.4.14), $u \in \mathcal{Z}^p(T_1, M)$ and u is a weak solution of (4.4.8) on $[0, T_1]$. Without loss of generality, we may assume that T_1 satisfies the conclusion in Lemma 4.4.1. In the following, we show that there exists a mapping $\mathcal{L} : \mathcal{Z}^p(T_1, M) \to \mathcal{Z}^p(T_1, M)$ such that $\mathcal{L}(\tilde{u}) = u$, where u is a weak solution of (4.4.8) on $[0, T_1]$, and it is a contraction on $\mathcal{Z}^p(T_1, M)$. Suppose that along another sequence of $\delta' \to 0$ (or we have different approximations of $\tilde{u}_{\delta'} \to \tilde{u}$), for some $u' \in \mathcal{Z}^p(T_1, M)$ we have $u_{\delta'} \to u'$ in $L^p_{\mathrm{loc}}(\Omega_{T_1})$ and $u_{\delta'}(t, \cdot) \to u'(t, \cdot)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for each $t \in [0, T_1]$. It follows from Lemma 4.4.1 that

$$\|u_{\delta'} - u_\delta\|_{\mathcal{X}^p(T_1)} \leq \frac{1}{2}\|\tilde{u}_{\delta'} - \tilde{u}_\delta\|_{\mathcal{X}^p(T_1)}.$$

Passing δ and δ' to 0 yields that $u = u'$. Hence the weak solution u to (4.4.8) on $[0, T_1]$ obtained via the above approximation process is unique. This allows us to define the mapping $\mathcal{L} : \mathcal{Z}^p(T_1, M) \to \mathcal{Z}^p(T_1, M)$, where $\mathcal{L}(\tilde{u}) = u$ and u is the weak solution of (4.4.8) from the above approximation process. Moreover, Lemma 4.4.1 shows that this mapping $\mathcal{L}$ is a contraction on $\mathcal{Z}^p(T_1, M)$. Finally, if $\tilde{u}$ is uniformly Hölder continuous, we can take $\tilde{u}_\delta = \tilde{u}$. Hence, it follows from the previous proof that $v = v_\delta$ is uniformly $C^{1+\frac{\alpha}{2}, 2+\alpha}$ in Ω_T. The classical parabolic regularity results then yield that $u = u_\delta$ is locally uniformly $C^{1+\frac{\alpha}{2}, 2+\alpha}$ in the interior of Ω_T. So, they are classical solutions. □

Remark 4.4.1. *In the proof of* (4.4.13)*, we invoke Theorem 4.1.1(3) to show that* $u_\delta \to u$ *as* $\delta \to 0$ *along a subsequence almost everywhere (in fact, pointwise) in* $(0, T_1] \times \mathbb{R}^N$*. There is an alternative argument of proving this fact without using Theorem 4.1.1(3). We refer readers to* [31, 61, 68] *for more details. Indeed, by multiplying the* u_δ*-equation by* $u_\delta^m \zeta$ *with* $\zeta \in C_0^\infty(B_1)$*, we get*

$$\frac{1}{m+1}\int_{\mathbb{R}^N} (u_\delta^{m+1})_t \zeta \, dx = -\int_{\mathbb{R}^N} \nabla(u_\delta^m \zeta) \cdot [m(u_\delta + \varepsilon)^{m-1}\nabla u_\delta - \chi u_\delta \nabla v] + u_\delta^m \zeta(a u_\delta - b u_\delta^2).$$

By Young's inequality and Theorem 4.1.1, we obtain

$$\begin{aligned}\left|\int_{B_1} (u_\delta^{m+1})_t \zeta\right| &\leq C\left[1 + \int_{B_1} (u_\delta + \varepsilon)^{2m-2}|\nabla u_\delta(t, x)|^2 \, dx + \|u_\delta\|_{2m}^{2m} + \|u_\delta\|_{m+2}^{m+2}\right]\|\zeta\|_{W_0^{1,\infty}(B_1)}\\ &\leq C\|\zeta\|_{W_0^{1,\infty}(B_1)}.\end{aligned}$$

This implies that $\partial_t(u_\delta^{m+1})$ *restricted to* $(0, T_1) \times B_1$ *is bounded under the norm of*

$$L^1((0, T_1); (W_0^{1,\infty}(B_1))^*) =: L^1((0, T_1); X_1).$$

Since $(\varepsilon + u_\delta)^{m-1}\nabla u_\delta$ *is locally uniformly bounded in* $L^2(\Omega_T)$ *and* u_δ *is uniformly bounded,*

u_δ^{m+1} restricted to $(0,T_1)\times B_1$ is in the space

$$L^2((0,T_1);W^{1,2}(B_1)) =: L^2((0,T_1);X_0).$$

Let $q\in(1,\frac{2N}{N-2})$ and $X := L^q(B_1)$. Then

$$X_0 = W^{1,2}(B_1)\subseteq X\subseteq (L^\infty(B_1))^*\subseteq (W_0^{1,\infty}(B_1))^* = X_1,$$

and the embedding of $X\subseteq X_1$ is continuous. By Rellich–Kondrachov embedding theorem, X_0 is compactly embedded in X. By Aubin–Lions compactness lemma, we have that, along a subsequence of $\delta\to 0$, u_δ^{m+1} converges in the space $L^2((0,T_1);L^q(B_1))$. This yields that $u_\delta\to u$ as $\delta\to 0$ along a subsequence a.e. in $[0,T_1]\times B_1$, which, after shifting, proves the claim.

We now prove Proposition 4.1.1.

Proof of Proposition 4.1.1. First, fix $\varepsilon>0$ and any $M>\|u_0\|_\infty$. Consider the mapping from Lemma 4.4.2:

$$\tilde{u}\in\mathcal{Z}^p(T_1,M)\mapsto\mathcal{L}(\tilde{u}) = u(\cdot,\cdot)\in\mathcal{Z}^p(T_1,M).$$

By Lemmas 4.4.1 and 4.4.2, if $T_1\le 1$ is picked to be sufficiently small, then the mapping is a contraction. By Banach fixed point theorem, there is a unique $u\in\mathcal{Z}^p(T_1,M)$ such that

$$\mathcal{L}(u) = u.$$

By Lemmas 4.4.1 and 4.4.2 again, we obtain a weak solution $(u_\varepsilon,v_\varepsilon)$ to (4.0.2) in the time interval $[0,T_1]$.

Next, we claim that $(u_\varepsilon,v_\varepsilon)$ is a classical solution of (4.0.2) on $(0,T_1]$. In fact, recall that u_0 is uniformly Hölder continuous, and v_0 is uniformly $C^{2+\alpha}$. Let v_ε^δ be uniformly smooth (depending on δ), and for some $p>N$,

$$\nabla v_\varepsilon^\delta\to\nabla v_\varepsilon\quad\text{ as }\delta\to 0\text{ in }L^p_{\text{loc}}(\Omega_{T_1}).$$

Then let u_ε^δ solve (4.4.8) with v_ε^δ in place of v, in the classical sense. Since $\varepsilon>0$, the classical parabolic theory yields that u_ε^δ converges to u_ε as $\delta\to 0$ in $L^p_{\text{loc}}(\Omega_{T_1})$. Since ∇v_ε is uniformly bounded, $\nabla v_\varepsilon^\delta$ can be chosen to be uniformly bounded independently of ε and δ. Hence, Theorem 4.1.1(3) implies that u_ε^δ is uniformly Hölder continuous in Ω_{T_1} with a bound independent of ε and δ, which in turn implies that u_ε is also uniformly Hölder continuous independently of ε. Thus, in view of Definition 4.1.2, the claim follows from the second part of Lemma 4.4.2.

Now, we show that (4.0.2) has a unique classical solution on $(0, T_1]$. To this end, suppose that $(u'_\varepsilon, v'_\varepsilon)$ is another pair of classical solutions of (4.0.2) with initial condition u_0, v_0. Since u_ε and u'_ε are uniformly Hölder continuous in $[0, T_1]$, by the second part of Lemma 4.4.2,

$$\mathcal{L}(u_\varepsilon) = u_\varepsilon \quad \text{and} \quad \mathcal{L}(u'_\varepsilon) = u'_\varepsilon.$$

Therefore, by Lemma 4.4.1, for any $M > \|u_0\|_\infty$, there exists $0 < \tilde{T}_1 \leq T_1$ such that $u_\varepsilon, u'_\varepsilon \in \mathcal{Z}^p(\tilde{T}_1, M)$ and

$$\|u_\varepsilon - u'_\varepsilon\|_{\mathcal{X}^p(\tilde{T}_1)} \leq \frac{1}{2}\|u_\varepsilon - u'_\varepsilon\|_{\mathcal{X}^p(\tilde{T}_1)}.$$

This implies that $u_\varepsilon = u'_\varepsilon$ in $[0, \tilde{T}_1] \times \mathbb{R}^N$. Since both u_ε and u'_ε are uniformly bounded, by iteration, we actually get that $u_\varepsilon = u'_\varepsilon$ in $[0, T_1] \times \mathbb{R}^N$, which proves the uniqueness.

Finally, by standard extension arguments, we can extend the solution $(u_\varepsilon, v_\varepsilon)$ of (4.0.2) on $[0, T_1]$ to a maximal interval $(0, T_{\max})$. Moreover, if $T_{\max} < \infty$, then

$$\limsup_{t \to T_{\max}-} \|u_\varepsilon(t, \cdot)\|_{C^{1+\alpha}} = \infty \quad \text{or} \quad \limsup_{t \to T_{\max}-} \|v_\varepsilon(t, \cdot)\|_{C^{2+\alpha}} = \infty. \tag{4.4.15}$$

However, Theorem 4.1.1 implies

$$\sup_{t \in [0, T_{\max})} \|u_\varepsilon(t, \cdot)\|_{L^\infty} < \infty, \quad \text{and} \quad \sup_{t \in [0, T_{\max})} \|\nabla v_\varepsilon(t, \cdot)\|_{L^\infty} < \infty.$$

By Theorem 4.1.1(3) again, u_ε is uniformly Hölder continuous on $[0, T_{\max})$. In view of Lemma 4.4.2, we have

$$\lim_{t \to T_{\max}-} \left[\|u_\varepsilon(t, \cdot)\|_{C^{1+\alpha}} + \|v_\varepsilon(t, \cdot)\|_{C^{2+\alpha}}\right] < \infty,$$

which implies that (4.4.15) cannot occur. Consequently, $T_{\max} = \infty$, and this completes the proof of Proposition 4.1.1. □

Now we proof Theorem 4.1.2 in the next section.

4.4.2 Proof of Theorem 4.1.2

In this subsection, we prove the existence of a globally bounded weak solution to (4.0.1) by means of approximation through solutions of the perturbed system (4.0.2), together with the a priori estimates in Theorem 4.1.1.

Proof of Theorem 4.1.2. First, let $\eta \in C_c^\infty(\mathbb{R}^N)$ be non-negative with unit total mass. For any $\varepsilon \in (0,1)$, define $\eta_\varepsilon(x) = \varepsilon^{-N}\eta(\varepsilon^{-1}x)$, and let

$$u_{0,\varepsilon} = u_0 * \eta_\varepsilon \quad \text{and} \quad v_{0,\varepsilon} = v_0 * \eta_\varepsilon.$$

Consequently, for any $p \geq 1$, we have

$$\|v_{0,\varepsilon} - v_0\|_{L^p_{\text{loc}}},\ \|\nabla v_{0,\varepsilon} - \nabla v_0\|_{L^p_{\text{loc}}},\ \|u_{0,\varepsilon} - u_0\|_{L^p_{\text{loc}}} \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, $u_{0,\varepsilon} \in L^\infty(\mathbb{R}^N)$, $v_{0,\varepsilon} \in W^{1,\infty}$, and

$$\|u_{0,\varepsilon}\|_\infty \leq \|u_0\|_\infty, \quad \|v_{0,\varepsilon}\|_{W^{1,\infty}} \leq \|v_0\|_{W^{1,\infty}}.$$

Let $(u_\varepsilon, v_\varepsilon)$ be from Proposition 4.1.1 with u_0 and v_0 being replaced by $u_{0,\varepsilon}$ and $v_{0,\varepsilon}$, respectively. For any $T > 0$, by Theorem 4.1.1 and Proposition 4.1.1, there exists C independent of ε and $T > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(\Omega_T)} \leq C, \qquad |\nabla v_\varepsilon(\cdot,\cdot)| \leq C \quad \text{in } \Omega_T. \tag{4.4.16}$$

and a C' independent of ε such that

$$\sup_{x_0 \in \mathbb{R}^N} \iint_{[0,T]\times B_1(x_0)} |\nabla(\varepsilon + u_\varepsilon)^m(t,x)|^2 \, dx\, dt \leq C',$$

Also, we know that v_ε are uniformly bounded. Therefore, after passing $\varepsilon \to 0$ along a subsequence, we can find $v \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^N))$ and $u \in L^\infty(\Omega_T)$ such that

$$\begin{aligned} &v_\varepsilon \rightharpoonup v, \quad \nabla v_\varepsilon \rightharpoonup \nabla v, \quad u_\varepsilon \rightharpoonup u && \text{in } L^p_{\text{loc}}(\Omega_T) \text{ for all } p \geq 1, \\ &(\varepsilon + u_\varepsilon)^{m-1}\nabla u_\varepsilon \rightharpoonup u^{m-1}\nabla u && \text{in } L^2_{\text{loc}}(\Omega_T). \end{aligned} \tag{4.4.17}$$

Moreover, similarly as before, there is a uniformly bounded vector field g in Ω_T such that

$$u_\varepsilon \nabla v_\varepsilon \rightharpoonup g \quad \text{in } L^p_{\text{loc}}(\Omega_T) \text{ for all } p \geq 1.$$

Now, we show that $g = u\nabla v$. As ∇v_ε is uniformly finite independent of ε, Theorem 4.1.1(3) yields that u_ε is uniformly Hölder continuous in $[\tau^*, \infty) \times \mathbb{R}^N$ with fixed $\tau^* > 0$, and the Hölder norm is independent of $\varepsilon \in (0,1)$. This shows that $u_\varepsilon \to u$ pointwise locally uniformly in $(0,\infty) \times \mathbb{R}^N$, and it follows that $g = u\nabla v$. We comment that this fact can also be obtained by the argument in Remark 4.4.1 without invoking Theorem 4.1.1(3). Finally, due to (4.4.16) and (4.4.17), (u, v) is a global weak solution, and they stay uniformly bounded for all time.

The convergence of $(u_\varepsilon, v_\varepsilon)$ to (u, v) in (4.4.17), together with Theorem 4.1.1, implies that the regularity properties stated in the theorem hold equally for (u, v). Moreover, if u_0 is uniformly Hölder continuous, then $u_{0,\varepsilon}$ are uniformly Hölder continuous with bounds independent of ε. In this case, Theorem 4.1.1(3) implies that u_ε, and so u, are uniformly Hölder continuous on $[0, \infty) \times \mathbb{R}^N$, with constants independent of ε. Next, assume further that $v_0 \in C^{2,\alpha}(\mathbb{R}^N)$, so that $v_{0,\varepsilon}$ are uniformly bounded in $C^{2,\alpha}$. Since u_ε is uniformly Hölder continuous, classical parabolic regularity theory implies that v_ε, and hence v, are uniformly bounded in $C^{1+\alpha/2,2+\alpha}$ on the whole domain, with constants independent of $\varepsilon > 0$. □

4.5 Uniqueness of Weak Solutions and Proof of Theorem 4.1.3

In this section, we give the prove of Theorem 4.1.3. In doing so, we apply a duality method similar to that used in [36, Theorem 3.4] and [70, Theorem 6.5]. It should be pointed out that, in those works, the method is applied to a single PDE, whereas our problem (4.0.1) is a coupled system, and so we need to carefully handle the interaction between u and v when applying the duality argument. Moreover, the weak solutions of (4.0.1) considered in this paper may not be integrable on $\mathbb{R}^N$. We need to handle the non-integrability issue.

In the following, we outline our strategy to prove Theorem 4.1.3 in Subsection 4.5.1, present an estimates for the adjoint equation in Subsection 4.5.2, and prove Theorem 4.1.3.

4.5.1 Outline of the proof of Theorem 4.1.3

Let $(u_i, v_i), i = 1, 2$ be two bounded weak solutions of (4.0.1) with initials $u_0 \in C^\alpha(\mathbb{R}^N)$ and $v_0 \in C^{2+\alpha}(\mathbb{R}^N)$ such that for any $T > 0$, $u_i \in C^\alpha([0, T] \times \mathbb{R}^N)$, and $v_i \in C^{1+\alpha/2,2+\alpha}([0, T] \times \mathbb{R}^N)$.

We define

$$\bar{u} := u_2 - u_1, \quad \bar{v} := v_2 - v_1, \quad \text{and} \quad \tilde{u} := u_2 + u_1. \tag{4.5.1}$$

The goal is to show $\bar{u} = 0$ and $\bar{v} = 0$ via a duality method. The key point is to establish the following two inequalities: for a fixed small $T > 0$ and any smooth nonnegative function

$\xi(t,x)$ compactly supported in $[0,T]\times\mathbb{R}^N$, we have

$$\int_0^T\int_{\mathbb{R}^N}\bar{u}\,\xi\,\psi \lesssim \left(\int_0^T\int_{\mathbb{R}^N}|\nabla\bar{v}|^2\psi^2\right)^{\frac{1}{2}}\left(\int_0^T\int_{\mathbb{R}^N}(|\xi|^2+|\nabla\xi|^2)\right)^{\frac{1}{2}}, \tag{4.5.2}$$

and

$$\int_{\mathbb{R}^N}\bar{v}^2(t)\psi^2+\int_0^t\int_{\mathbb{R}^N}|\nabla\bar{v}|^2\psi^2 \lesssim \int_0^t\int_{\mathbb{R}^N}\bar{v}^2\psi^2, \quad \forall\, 0\le t\le T, \tag{4.5.3}$$

where ψ is as in Lemma 2.4.1. Then applying Grönwall's inequality to (4.5.3) implies that $\bar{v}(t,x)=0$ for all $t\in[0,T]$ and $x\in\mathbb{R}^N$, which together with (4.5.2) implies $\bar{u}(t,x)=0$.

In the following, let us roughly describe the duality method. First, set

$$U(t,x):=\bar{u}(t,x)\,\psi(x), \qquad a^*(t,x):=\left(\frac{u_2^m-u_1^m}{u_2-u_1}\right)(t,x). \tag{4.5.4}$$

Using the equation for u, we obtain

$$\begin{aligned} U_t &= \Delta(a^*U)-2\nabla\cdot\left(a^*U\,\frac{\nabla\psi}{\psi}\right)+a^*U\,\frac{\Delta\psi}{\psi} \\ &\quad -\chi\nabla\cdot(U\nabla v_2)+\chi U\,\nabla v_2\cdot\frac{\nabla\psi}{\psi}+aU-b\tilde{u}\,U-\chi\psi\,\nabla\cdot(u_1\nabla\bar{v}). \end{aligned} \tag{4.5.5}$$

Next, let $\varepsilon'>0$, for some $T>0$ and $\xi(t,x)\ge 0$ as above, consider the following adjoint-type equation associated with (4.5.5):

$$\begin{cases} \phi_t+a^{\varepsilon'}\Delta\phi+g^{\varepsilon'}\cdot\nabla\phi+f^{\varepsilon'}\phi+\xi=0, \\ \phi(T,x)=0, \end{cases} \tag{4.5.6}$$

where

$$g^{\varepsilon'}:=2a^{\varepsilon'}\frac{\nabla\psi}{\psi}+\chi\nabla v_2, \qquad f^{\varepsilon'}:=a-b\tilde{u}+a^{\varepsilon'}\frac{\Delta\psi}{\psi}+\chi\nabla v_2\cdot\frac{\nabla\psi}{\psi}. \tag{4.5.7}$$

Here $a^{\varepsilon'}$ is a smooth approximation of a^* with $a^{\varepsilon'}\ge\max\{\varepsilon',a^*\}$; ψ is given in Lemma 2.4.1 with some $\kappa>0$ to be chosen. By the regularity of solutions and smoothness of ψ, both $g^{\varepsilon'}$ and $f^{\varepsilon'}$ are bounded on $[0,T]\times\mathbb{R}^N$. Since ψ and $a^{\varepsilon'}$ are smooth, v_2 is C^2 and $\tilde{u}$ is Hölder continuous; thus, by [41, Theorem 8.1, Chapter V], (4.5.6) admits a classical solution ϕ.

The main idea is to use $\phi\psi$ as a test function in the weak formulation to derive (4.5.2). A main difficulty is controlling the regularity of the auxiliary function ϕ. In particular, the argument relies on estimating the $W^{2,2}$ norm of ϕ in terms of the $W^{1,2}$ norm of ξ, which is crucial for the duality method. This estimate is proved in Lemma 4.5.1, which is also where the condition $m \leq 3$ is required.

Finally, since $\bar{v}\, v_2\, \psi$ can be approximated by smooth compactly supported functions on $[0,T] \times \mathbb{R}^N$, we apply (4.5.2) with $\xi = \bar{v}\, v_2\, \psi$ to obtain (4.5.3).

4.5.2 Estimates for the adjoint equation

Let ξ be a smooth function that is compactly supported in $[0,T] \times \mathbb{R}^N$ and for any $\varepsilon' > 0$, let $a^{\varepsilon'} = a^{\varepsilon'}(t,x)$ be a smooth function satisfying that

$$\max\{\varepsilon', a^*\} \leq a^{\varepsilon'} \leq \max_{i=1,2} \{m(\|u_i\|_\infty + 1)^{m-1}\} =: K \tag{4.5.8}$$

with a^* from (4.5.4). We will select a more specific $a^{\varepsilon'}$ later in the proof of Theorem 4.1.3.

We claim that

$$a^* \geq c^*\, \tilde{u}^{m-1}, \quad \text{where} \quad c^* = \min\left\{1, \frac{m}{2^{m-1}}\right\}. \tag{4.5.9}$$

Indeed, the inequality holds trivially if $u_1 = u_2$. Otherwise, suppose $u_2 > u_1$, and note that

$$a^* = \frac{m}{u_2 - u_1} \int_{u_1}^{u_2} u^{m-1}\, du.$$

If $m \geq 2$, since the map $t \mapsto t^{m-1}$ is convex, and by the Hermite–Hadamard inequality,

$$\frac{1}{u_2 - u_1} \int_{u_1}^{u_2} u^{m-1}\, du \geq \left(\frac{u_1 + u_2}{2}\right)^{m-1} = 2^{1-m} \tilde{u}^{m-1},$$

which yields the claim. If $m \in (1,2]$, for $r := u_1/u_2$, we have

$$\frac{a^*}{(u_1 + u_2)^{m-1}} = \frac{1 - r^m}{1 - r} \frac{1}{(1+r)^{m-1}} \geq 1,$$

as $r \mapsto \frac{1-r^m}{1-r} \frac{1}{(1+r)^{m-1}}$ is an increasing function of $r \in (0,1]$. The claim follows.

Let ψ be as in Lemma 2.4.1 with some parameter $\kappa > 0$, and $\phi^{\varepsilon',\xi}$ solve (4.5.6). We prove the following estimate.

Lemma 4.5.1. *Fix any $\kappa \in (0, \frac{1}{2})$. Assume that m, a, b satisfy that $a, b > 0$ and $1 < m \leq 3$. For any $\delta \in (0, \frac{1}{8})$, there exists $T^* = T^*(\delta) > 0$ such that, for every $T \in (0, T^*]$ and*

$\varepsilon' \in (0,1)$, if $\phi = \phi^{\varepsilon',\xi}$ is the solution to (4.5.6) on $[0,T]\times\mathbb{R}^N$, then the following holds:

$$\int_t^T\int_{\mathbb{R}^N} a^{\varepsilon'}|\Delta\phi^{\varepsilon',\xi}|^2 + \int_t^T\int_{\mathbb{R}^N}|\nabla\phi^{\varepsilon',\xi}|^2 + \int_t^T\int_{\mathbb{R}^N}|\phi^{\varepsilon',\xi}|^2 \le \delta\Big(\int_t^T\int_{\mathbb{R}^N}|\xi|^2 + \int_t^T\int_{\mathbb{R}^N}|\nabla\xi|^2\Big)$$

for every $t \in [0,T]$.

Proof of Lemma 4.5.1. We divide the proof into three steps. In the following, unless otherwise specified, we denote $\phi^{\varepsilon',\xi}$ by ϕ, and $g^{\varepsilon'}$, $f^{\varepsilon'}$ by g, f, respectively.

Step 1. In this step, we prove the following estimate for $\int_{\mathbb{R}^N}\phi(t,x)^2\,dx$:

$$\int_{\mathbb{R}^N}\phi(t,x)^2\,dx \le \delta\int_t^T\int_{\mathbb{R}^N}a^{\varepsilon'}|\Delta\phi|^2 + \delta\int_t^T\int_{\mathbb{R}^N}|\xi|^2 + \|g\|_\infty\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2 + C_\delta\int_t^T\int_{\mathbb{R}^N}|\phi|^2, \tag{4.5.10}$$

where

$$C_\delta := (K+1)/\delta + \|g\|_\infty + 2\|f\|_\infty. \tag{4.5.11}$$

To this end, multiplying (4.5.6) by ϕ and then integrating in space and time, we have

$$\frac{1}{2}\int_t^T\int_{\mathbb{R}^N}\frac{d}{dt}\phi^2\,dx\,ds = -\int_t^T\int_{\mathbb{R}^N}a^{\varepsilon'}\phi\Delta\phi + \phi g\cdot\nabla\phi + f\phi^2 + \phi\xi\,dx\,ds.$$

This together with Young's inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^N}\phi(t,x)^2\,dx &\le \delta\int_t^T\int_{\mathbb{R}^N}a^{\varepsilon'}|\Delta\phi|^2 + \int_t^T\int_{\mathbb{R}^N}\frac{1}{\delta}\|a^{\varepsilon'}\|_\infty|\phi|^2 + \|g\|_\infty\int_t^T\int_{\mathbb{R}^N}(|\nabla\phi|^2+|\phi|^2)\\ &\quad + 2\int_t^T\int_{\mathbb{R}^N}\|f\|_\infty|\phi|^2 + \frac{1}{\delta}\int_t^T\int_{\mathbb{R}^N}|\phi(s)|^2 + \delta\int_t^T\int_{\mathbb{R}^N}\xi^2\\ &\le \delta\int_t^T\int_{\mathbb{R}^N}a^{\varepsilon'}|\Delta\phi|^2 + \delta\int_t^T\int_{\mathbb{R}^N}|\xi|^2 + \|g\|_\infty\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2\\ &\quad + C_\delta\int_t^T\int_{\mathbb{R}^N}|\phi|^2,\end{aligned}$$

where C_δ is given in (4.5.11). Hence (4.5.10) holds.

Step 2. In this step, we establish an estimate for $\int_t^T\int_{\mathbb{R}^N}a^{\varepsilon'}|\Delta\phi|^2\eta$ for some $\eta = \eta(t)$. To be more precise, denote

$$M := \|v_2\|_{C^2} \quad\text{and}\quad C_* := 2C_1M|\chi| + (4K+2)/\delta + 4a + 8|\chi|M + 2\|g\|_\infty + 2,$$

where C_1 is a dimensional constant to be determined. Since u_i are uniformly Hölder continuous, v_i are uniformly C^2 and M is well defined. Also, set $\eta = \eta(t) := 1 + C_* t$. It is clear that

$$\eta_t = C_* \quad \text{and} \quad 1 \le \eta \le 2, \qquad \text{for } t \in [0, 1/C_*]. \tag{4.5.12}$$

We establish the following estimate:

$$\begin{aligned}
&(1-3\delta)\int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} |\Delta\phi|^2 \eta \\
&\le \left(-\frac{C_*}{2} + C_1 M|\chi| + \frac{2K+1}{\delta} + 2a + 4|\chi|M\right) \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 + \delta \int_t^T \int_{\mathbb{R}^N} |\nabla\xi|^2 \\
&\quad + \left(\frac{K}{\delta} + 4|\chi|M + \frac{b^2 \|\tilde{u}\|_\infty^{3-m}}{c^*\delta}\right) \int_t^T \int_{\mathbb{R}^N} |\phi|^2,
\end{aligned} \tag{4.5.13}$$

where $c^* = \min\{1, \frac{m}{2^{m-1}}\}$.

To this end, first, multiplying (4.5.6) by $(\Delta\phi)\eta$ and then integrating in time and space yields that, for any $T \in (0, 1/C_*]$ and $t \in [0, T]$,

$$\int_t^T \int_{\mathbb{R}^N} (\Delta\phi)\eta\phi_t + a^{\varepsilon'} |\Delta\phi|^2 \eta + g \cdot \nabla\phi(\Delta\phi)\eta + f\phi(\Delta\phi)\eta + \xi(\Delta\phi)\eta = 0.$$

This implies that

$$\begin{aligned}
\int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} |\Delta\phi|^2 \eta = &\int_t^T \int_{\mathbb{R}^N} (\nabla\phi \cdot \nabla\phi_t)\eta - \int_t^T \int_{\mathbb{R}^N} g \cdot \nabla\phi(\Delta\phi)\eta \\
&- \int_t^T \int_{\mathbb{R}^N} f\phi(\Delta\phi)\eta + \int_t^T \int_{\mathbb{R}^N} (\nabla\phi \cdot \nabla\xi)\eta.
\end{aligned} \tag{4.5.14}$$

Next, we estimate each term on the right-hand side of (4.5.14). For the first and last terms, using (4.5.12), we have

$$\begin{aligned}
\int_t^T \int_{\mathbb{R}^N} (\nabla\phi \cdot \nabla\phi_t)\eta &= \frac{1}{2} \int_t^T \int_{\mathbb{R}^N} \eta \frac{d}{dt} |\nabla\phi|^2 \\
&= -\frac{1}{2} \int_{\mathbb{R}^N} \eta(t) |\nabla\phi(t,x)|^2 \, dx - \frac{1}{2} \int_t^T \int_{\mathbb{R}^N} \eta_t |\nabla\phi|^2 \\
&\le -\frac{C_*}{2} \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2,
\end{aligned} \tag{4.5.15}$$

and

$$\int_t^T \int_{\mathbb{R}^N} (\nabla\phi \cdot \nabla\xi)\eta \le \delta \int_t^T \int_{\mathbb{R}^N} |\nabla\xi|^2 + \frac{1}{\delta} \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2. \tag{4.5.16}$$

To estimate the second term on the right-hand side of (4.5.14), denote $g_1 := \chi\nabla v_2$ and $g_2 := 2a^{\varepsilon'}\frac{\nabla\psi}{\psi}$, and then $g = g_1 + g_2$. By direct computation,

$$\begin{aligned}
-\int_t^T\int_{\mathbb{R}^N} g_1\cdot\nabla\phi(\Delta\phi)\eta &= \int_t^T\int_{\mathbb{R}^N}\sum_{i,j}\phi_{x_j}(\partial_j g_1^i)\phi_{x_i}\eta + \int_t^T\int_{\mathbb{R}^N}\sum_{i,j} g_1^i\phi_{x_ix_j}\phi_{x_j}\eta\\
&= \int_t^T\int_{\mathbb{R}^N}(\nabla\phi)^T Dg_1(\nabla\phi)\eta + \frac{1}{2}\int_t^T\int_{\mathbb{R}^N}\sum_{i,j} g_1^i\partial_i(\phi_{x_j}^2)\eta\\
&\le \int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2|Dg_1|\eta + \frac{1}{2}\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2|\nabla\cdot g_1|\eta.
\end{aligned}$$

Recall that

$$|Dg_1| = |\chi||D^2v_2| \le |\chi|M.$$

Hence, there is C_1 depending only on N such that

$$-\int_t^T\int_{\mathbb{R}^N} g_1\cdot\nabla\phi(\Delta\phi)\eta \le C_1M|\chi|\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2. \tag{4.5.17}$$

By Young's inequality and using Lemma 2.4.1, $|a^{\varepsilon'}| \le K$, $\kappa < 1$ and $\eta \le 2$, we get

$$\begin{aligned}
-\int_t^T\int_{\mathbb{R}^N} g_2\cdot\nabla\phi(\Delta\phi)\eta &= -\int_t^T\int_{\mathbb{R}^N} 2a^{\varepsilon'}\frac{\nabla\psi}{\psi}\cdot\nabla\phi(\Delta\phi)\eta\\
&\le \delta\int_t^T\int_{\mathbb{R}^N} a^{\varepsilon'}|\Delta\phi|^2\eta + \frac{2K}{\delta}\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2.
\end{aligned}\tag{4.5.18}$$

Combining (4.5.17) and (4.5.18), we have

$$-\int_t^T\int_{\mathbb{R}^N} g\cdot\nabla\phi(\Delta\phi)\eta \le \delta\int_t^T\int_{\mathbb{R}^N} a^{\varepsilon'}|\Delta\phi|^2\eta + \Big(C_1M|\chi| + \frac{2K}{\delta}\Big)\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2. \tag{4.5.19}$$

Now, we estimate the third term on the right-hand side of (4.5.14). Recall that $f = f^{\varepsilon'}$ is defined in (4.5.7). In the case when $m \in (1, 3]$, by Young's inequality, we have

$$\begin{aligned}
-\int_t^T\int_{\mathbb{R}^N}(a - b\tilde{u})\phi(\Delta\phi)\eta &= a\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2\eta + b\int_t^T\int_{\mathbb{R}^N}\tilde{u}\phi(\Delta\phi)\eta\\
&\le 2a\int_t^T\int_{\mathbb{R}^N}|\nabla\phi|^2 + \frac{\delta c^*}{2}\int_t^T\int_{\mathbb{R}^N}\tilde{u}^{m-1}|\Delta\phi|^2\eta + \frac{b^2}{2c^*\delta}\int_t^T\int_{\mathbb{R}^N}\tilde{u}^{3-m}|\phi|^2\eta,
\end{aligned}$$

where $c^* = \min\{1, \frac{m}{2^{m-1}}\}$. It follows from (4.5.9) that $a^{\varepsilon'} \ge \max\{\varepsilon', a^*\} \ge c^*\tilde{u}^{m-1}$. Hence, also using that $\eta \leq 2$,

$$-\int_t^T \int_{\mathbb{R}^N} (a - b\tilde{u})\phi(\Delta\phi)\eta \leq 2a \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 + \delta \int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} |\Delta\phi|^2 + \frac{b^2 \|\tilde{u}\|_\infty^{3-m}}{c^* \delta} \int_t^T \int_{\mathbb{R}^N} |\phi|^2. \tag{4.5.20}$$

By Young's inequality, Lemma 2.4.1, $\eta \leq 2$ and $a^{\varepsilon'} \leq K$, there holds

$$-\int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} \frac{\Delta\psi}{\psi} \phi(\Delta\phi)\eta \leq \delta \int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} |\Delta\phi|^2 \eta + \frac{K}{\delta} \int_t^T \int_{\mathbb{R}^N} |\phi|^2. \tag{4.5.21}$$

By Lemma 2.4.1, $\|v_2\|_{C^2} \leq M$ and $\eta \leq 2$,

$$\begin{aligned} &-\int_t^T \int_{\mathbb{R}^N} \chi \nabla v_2 \cdot \frac{\nabla\psi}{\psi} \phi(\Delta\phi)\eta \\ \leq &\int_t^T \int_{\mathbb{R}^N} \phi \nabla\phi \cdot \nabla \left(\chi \nabla v_2 \cdot \frac{\nabla\psi}{\psi} \right) \eta + |\chi| \|\nabla v_2\|_\infty \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 \eta \\ \leq &\, 4|\chi| M \int_t^T \int_{\mathbb{R}^N} (|\nabla\phi|^2 + |\phi|^2). \end{aligned} \tag{4.5.22}$$

Combining (4.5.20), (4.5.21) and (4.5.22), we get the following estimate for the third term in (4.5.14),

$$\begin{aligned} -\int_t^T \int_{\mathbb{R}^N} f\phi(\Delta\phi)\eta \leq 2\delta \int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} |\Delta\phi|^2 + (2a + 4|\chi|M) \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 \\ + \left(\frac{K}{\delta} + 4|\chi|M + \frac{b^2 \|\tilde{u}\|_\infty^{3-m}}{c^* \delta} \right) \int_t^T \int_{\mathbb{R}^N} |\phi|^2. \end{aligned} \tag{4.5.23}$$

The inequality (4.5.13) then follows from (4.5.14), (4.5.15), (4.5.16), (4.5.19), and (4.5.23).

Step 3. In this step, we prove the inequality stated in the lemma.

First, adding up (4.5.10) and (4.5.13), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\phi(t,x)|^2\,dx + \int_t^T \int_{\mathbb{R}^N} |\phi|^2 + \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 + (1-4\delta)\int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'} |\Delta\phi|^2 \\
&\le \left(-\frac{C_*}{2} + C_1 M|\chi| + \frac{2K+1}{\delta} + 2a + 4|\chi|M + \|g\|_\infty + 1\right)\int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 \\
&\qquad + \left(C_\delta + \frac{K}{\delta} + 4|\chi|M + \frac{b^2\|\tilde{u}\|_\infty^{3-m}}{c^*\delta} + 1\right)\int_t^T \int_{\mathbb{R}^N} |\phi|^2 + \delta\int_t^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2) \\
&\le C'_\delta \int_t^T \int_{\mathbb{R}^N} |\phi|^2 + \delta\int_t^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2), \qquad (4.5.24)
\end{aligned}$$

where, in the last inequality, we used the definition of C_* and

$$C'_\delta := C_\delta + \frac{K}{\delta} + 4|\chi|M + \frac{b^2\|\tilde{u}\|_\infty^{3-m}}{c^*\delta} + 1.$$

Next, fix $\delta \in (0, \frac{1}{8})$, and let

$$T^* := \min\{1/C_*, 1/(2C'_\delta)\}.$$

Assume that $T \in (0, T^*]$. For any $t \in [0,T]$, choose $t_1 \in [t,T]$ such that

$$\int_{\mathbb{R}^N} |\phi(t_1,x)|^2 dx = \max\left\{\int_{\mathbb{R}^N} |\phi(s,x)|^2 dx : s \in [t,T]\right\}.$$

It follows from (4.5.24) with $t = t_1$ and $C'_\delta T \le \frac{1}{2}$ that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\phi(t_1,x)|^2\,dx &\le C'_\delta \int_{t_1}^T \int_{\mathbb{R}^N} |\phi|^2 + \delta\int_{t_1}^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2) \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |\phi(t_1,x)|^2\,dx + \delta\int_{t_1}^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2).
\end{aligned}$$

This implies that for any $t \in [0,T]$,

$$\int_t^T \int_{\mathbb{R}^N} |\phi(t,x)|^2 \le T\int_{\mathbb{R}^N} |\phi(t_1,x)|^2 \le 2\delta T^* \int_t^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2).$$

Using (4.5.24) again, this and $\delta \leq 1/8$ yield

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\phi(t,x)|^2\,dx + \int_t^T \int_{\mathbb{R}^N} |\phi|^2 + \int_t^T \int_{\mathbb{R}^N} |\nabla\phi|^2 + \frac{1}{2}\int_t^T \int_{\mathbb{R}^N} a^{\varepsilon'}|\Delta\phi|^2 \\
&\leq (2\delta T^* C'_\delta + \delta)\int_t^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2) \leq 2\delta \int_t^T \int_{\mathbb{R}^N} (|\xi|^2 + |\nabla\xi|^2).
\end{aligned}$$

This completes the proof of the lemma. □

The next corollary is an immediate consequence of Lemmas 2.4.1 and 4.5.1, combined with the following computation:

$$\begin{aligned}
&|\nabla(\phi\psi)|^2 = |\nabla\phi\,\psi + \phi\nabla\psi|^2 \leq 2\Big(|\nabla\phi|^2\psi^2 + \phi^2\psi^2\Big), \\
&|\Delta(\phi\psi)|^2 = |\Delta\phi\,\psi + 2\nabla\phi\nabla\psi + \phi\Delta\psi|^2 \leq 8(|\Delta\phi|^2 + |\nabla\phi|^2 + |\phi|^2)\psi^2.
\end{aligned}$$

Corollary 4.5.1. *Under the condition of Lemma 4.5.1, for any* δ*, there exists* $T^* > 0$ *such that for all* $T \in [0, T^*]$ *we have*

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} \frac{|\nabla(\phi\psi)|^2}{\psi^2} &\leq \delta \int_0^T \int_{\mathbb{R}^N} \Big(|\xi|^2 + |\nabla\xi|^2\Big), \\
\int_0^T \int_{\mathbb{R}^N} \frac{a^{\varepsilon'}}{\psi^2}\,|\Delta(\phi\psi)|^2 &\leq \delta \int_0^T \int_{\mathbb{R}^N} \Big(|\xi|^2 + |\nabla\xi|^2\Big).
\end{aligned}$$

Now we give the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. Let $(u_i, v_i), i = 1, 2$ be two weak solutions to (4.0.1) with initial data (u_0, v_0) such that u_i are uniformly bounded and Hölder continuous and v_i are uniformly C^2. Then let $\delta \in (0, \frac{1}{8})$ and $T^* = T^*(\delta)$ be as in Lemma 4.5.1. It is sufficient to prove $u_1 = u_2$ and $v_1 = v_2$ for $t \in [0, T^*]$, since the general uniqueness then follows by iteration. As mentioned earlier in Subsection 4.5.1, the key is to establish (4.5.2) and (4.5.3).

Step 1. In this step, we show that (4.5.2) holds up to time T^* (see (4.5.28)). Let us start with choosing $a^{\varepsilon'}$ carefully. For any $\varepsilon' \in (0,1)$, let $a^{\varepsilon'}$ satisfy

$$\max\{\varepsilon', a^*\} \leq a^{\varepsilon'} < K \quad \text{and} \quad \int_0^{T^*} \int_{\mathbb{R}^N} |a^* - a^{\varepsilon'}|^2 \bar{u}^2 \psi^2 \leq C_1 \tau^2, \tag{4.5.25}$$

where

$$C_1 := \int_0^{T^*} \bar{u}^2\psi^2 + 1,$$

and K and $\bar{u}$ are defined in (4.5.8) and (4.5.1), respectively. To see the existence of such $a^{\varepsilon'}$, we first extend the function $a^*(t,x)+\varepsilon'$ to $t\in[0,\infty)$ by

$$a^{\varepsilon',*}(t,x) := a^*(\max\{0,\min\{T^*,t\}\},x)+\varepsilon'.$$

Thus, it is clear that

$$\int_0^{T^*}\int_{\mathbb{R}^N}|a^*-a^{\varepsilon',*}|\bar{u}^2\psi^2 \le (C_1-1){\varepsilon'}^2.$$

Then, let $\eta\in C_c^\infty(\mathbb{R}^{N+1})$ be non-negative with compact support and unit total mass, and set

$$a^{\varepsilon',\epsilon}(t,x) := \frac{1}{\epsilon^{N+1}}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\eta\Big(\frac{1}{\epsilon}(t-s),\frac{1}{\epsilon}(x-y)\Big)a^{\varepsilon',*}(s,y)\,dy\,ds.$$

Since $a^{\varepsilon',*}$ is uniformly continuous on $\mathbb{R}\times\mathbb{R}^N$,

$$\lim_{\epsilon\to 0}a^{\varepsilon',\epsilon}=a^{\varepsilon',*}\quad\text{uniformly on }\mathbb{R}\times\mathbb{R}^N.$$

Hence, after taking $\epsilon>0$ sufficiently small, $a^{\varepsilon'}(t,x):=a^{\varepsilon',\epsilon}$ satisfies the requirement (4.5.25).

Next, let $\phi=\phi^{\varepsilon',\xi}$ be the solution of (4.5.6) with some parameters $\kappa\in(0,\frac{1}{2})$ and $\varepsilon'\in(0,1)$ and $T\in(0,T^*]$. Using $\phi\psi$ as the test function in the equations of u_i for $i=1,2$, and taking the difference of these two equations, we obtain

$$\begin{aligned}-\int_0^T\int_{\mathbb{R}^N}\bar{u}\,\phi_t\psi &= -\int_0^T\int_{\mathbb{R}^N}\nabla(\phi\psi)\cdot\nabla\Big(u_2^m-u_1^m\Big)+\int_0^T\int_{\mathbb{R}^N}\chi u_2\nabla(\phi\psi)\cdot\nabla v_2-\chi u_1\nabla(\phi\psi)\cdot\nabla v_1\\ &\quad+\int_0^T\int_{\mathbb{R}^N}a\bar{u}\,\phi\psi-b(u_2^2-u_1^2)\phi\psi\\ &=\int_0^T\int_{\mathbb{R}^N}\Big[\Delta(\phi\psi)\,a^*\bar{u}+\chi\bar{u}\,\nabla v_2\cdot\nabla(\phi\psi)+\chi u_1\nabla\bar{v}\cdot\nabla(\phi\psi)+(a-b\tilde{u})\bar{u}\,\phi\psi\Big].\end{aligned}\tag{4.5.26}$$

Now, multiplying equation (4.5.6) by ψ gives

$$\begin{aligned}&\phi_t\psi+a^{\varepsilon'}(\Delta\phi)\psi+\chi(\nabla v_2\cdot\nabla\phi)\psi+2a^{\varepsilon'}\nabla\phi\cdot\nabla\psi\\ &\qquad+(a-b\tilde{u})\psi\phi+a^{\varepsilon'}(\Delta\psi)\phi+\chi\nabla v_2\cdot(\nabla\psi)\phi+\xi\psi=0,\end{aligned}$$

which implies that

$$\phi_t\psi+a^{\varepsilon'}\Delta(\phi\psi)+\chi\nabla v_2\cdot\nabla(\phi\psi)+(a-b\tilde{u})\psi\phi+\xi\psi=0.$$

Multiplying this equation by $\bar{u}$ and integrating in space and time yield

$$0 = \int_0^T \int_{\mathbb{R}^N} \bar{u}\,\phi_t\psi + \int_0^T \int_{\mathbb{R}^N} \Big[\Delta(\phi\psi)\,a^{\varepsilon'}\bar{u} + \chi\bar{u}\,\nabla v_2\cdot\nabla(\phi\psi) + (a-b\tilde{u})\bar{u}\,\phi\psi + \bar{u}\xi\psi\Big]. \tag{4.5.27}$$

Finally, subtracting (4.5.26) from (4.5.27), we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} \bar{u}\,\xi\psi &= \int_0^T \int_{\mathbb{R}^N} (a^* - a^{\varepsilon'})\bar{u}\,\Delta(\phi\psi) + \int_0^T \int_{\mathbb{R}^N} \chi u_1 \nabla\bar{v}\cdot\nabla(\phi\psi) \\ &\le \left(\int_0^T \int_{\mathbb{R}^N} \frac{|a^* - a^{\varepsilon'}|^2}{a^{\varepsilon'}}|\bar{u}|^2\psi^2\right)^{1/2} \left(\int_0^T \int_{\mathbb{R}^N} \frac{a^{\varepsilon'}}{\psi^2}|\Delta(\phi\psi)|^2\right)^{1/2} \\ &\quad + |\chi|\|u_1\|_\infty \left(\int_0^T \int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} \frac{|\nabla(\phi\psi)|^2}{\psi^2}\right)^{\frac{1}{2}} \\ &\le (\varepsilon')^{-1/2} \left(\int_0^T \int_{\mathbb{R}^N} |a^* - a^{\varepsilon'}|^2|\bar{u}|^2\psi^2\right)^{1/2} \left(\int_0^T \int_{\mathbb{R}^N} \frac{a^{\varepsilon'}}{\psi^2}|\Delta(\phi\psi)|^2\right)^{1/2} \\ &\quad + |\chi|\|u_1\|_\infty \left(\int_0^T \int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} \frac{|\nabla(\phi\psi)|^2}{\psi^2}\right)^{\frac{1}{2}},\end{aligned}$$

where we also used Hölder's inequality to get the second inequality and the fact that $a^{\varepsilon'} \ge \varepsilon'$ in the last inequality. By Corollary 4.5.1 and (4.5.25), we have for some C depending only on $|\chi|$ and $\|u_1\|_\infty$,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} \bar{u}\,\xi\psi &\le C(KC_1\delta)^{\frac{1}{2}}(\varepsilon')^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} \left(|\xi|^2 + |\nabla\xi|^2\right)\right)^{1/2} \\ &\quad + C\delta^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} \left(|\xi|^2 + |\nabla\xi|^2\right)\right)^{\frac{1}{2}}.\end{aligned}$$

Letting $\varepsilon' \to 0$ yields

$$\int_0^T \int_{\mathbb{R}^N} \bar{u}\,\xi\psi \le C\delta^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2\right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^N} \left(|\xi|^2 + |\nabla\xi|^2\right)\right)^{\frac{1}{2}}. \tag{4.5.28}$$

Step 2. In this step, we prove (4.5.3). First, by the equations satisfied by v_i, we get

$$\tau\bar{v}_t = \Delta\bar{v} - u_1\bar{v} - \bar{u}v_2, \quad \bar{v}(0,x) = 0.$$

Multiplying the above equation by $\bar{v}\psi^2$ and integrating in space and time yield for any $T \in [0, T^*]$,

$$\begin{aligned}
&\frac{\tau}{2}\int_{\mathbb{R}^N} \bar{v}^2(T,x)\psi^2(x)\,dx + \int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2 \\
&\le \int_0^T\int_{\mathbb{R}^N} \bar{v}|\nabla\bar{v}|\,|\nabla(\psi^2)| - \int_0^T\int_{\mathbb{R}^N} u_1\bar{v}^2\psi^2 - \int_0^T\int_{\mathbb{R}^N} \bar{u}\bar{v}v_2\psi^2 \\
&\le \int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2|\nabla\psi|^2 + \int_0^T\int_{\mathbb{R}^N} \bar{v}^2\psi^2 - \int_0^T\int_{\mathbb{R}^N} \bar{u}\bar{v}v_2\psi^2. \qquad (4.5.29)
\end{aligned}$$

Next, since $\bar{v}v_2\psi$ and $\nabla(\bar{v}v_2\psi)$ decay exponentially in space, there is a sequence $\{\xi_n\}$ of smooth, compactly supported functions such that

$$\int_0^T\int_{\mathbb{R}^N} \left|\xi_n + \bar{v}v_2\psi\right|^2 + \left|\nabla\xi_n + \nabla(\bar{v}v_2\psi)\right|^2 \to 0 \qquad \text{as } n \to \infty.$$

Hence, (4.5.28) holds for $\xi = -\bar{v}v_2\psi$, which implies that

$$\begin{aligned}
-\int_0^T\int_{\mathbb{R}^N} \bar{u}\bar{v}v_2\psi^2 &\le C\delta^{\frac{1}{2}}\left(\int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2\right)^{\frac{1}{2}}\left(\int_0^T\int_{\mathbb{R}^N} |\bar{v}v_2\psi|^2 + |\nabla(\bar{v}v_2\psi)|^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{4}\int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2 + C^2\delta\int_0^T\int_{\mathbb{R}^N} \left(|\bar{v}v_2\psi|^2 + |\nabla(\bar{v}v_2\psi)|^2\right).
\end{aligned}$$

Since $|\nabla\psi| \le \kappa\psi$ by Lemma 2.4.1 and $v_2 \in C^2$, there exists C_2 depending only on C and $\|v_2\|_{C^1}$ such that

$$-\int_0^T\int_{\mathbb{R}^N} \bar{u}\bar{v}v_2\psi^2 \le \left(\frac{1}{4} + C_2\delta\right)\int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2 + C_2\delta\int_0^T\int_{\mathbb{R}^N} |\bar{v}|^2\psi^2.$$

Now, by Lemma 2.4.1 again and taking $\delta = \frac{1}{4C_2}$, substituting this inequality into (4.5.29) gives

$$\begin{aligned}
&\frac{\tau}{2}\int_{\mathbb{R}^N} \bar{v}^2(T,x)\psi^2(x)\,dx + \int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2 \\
&\le \left(\kappa^2 + \tfrac{1}{2} + C_2\delta\right)\int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2 + (1 + C_2\delta)\int_0^T\int_{\mathbb{R}^N} \bar{v}^2\psi^2 \\
&\le \int_0^T\int_{\mathbb{R}^N} |\nabla\bar{v}|^2\psi^2 + 2\int_0^T\int_{\mathbb{R}^N} \bar{v}^2\psi^2,
\end{aligned}$$

where, in the second inequality, we used that $\kappa^2 \leq 1/4$ and $C_2\delta = \frac{1}{4}$. We obtain

$$\int_{\mathbb{R}^N} \bar{v}^2(T,x)\psi^2(x)\,dx \leq \frac{4}{\tau}\int_0^T \int_{\mathbb{R}^N} \bar{v}^2\psi^2. \tag{4.5.30}$$

Finally, since (4.5.30) holds for all $T \in [0, T^*]$, Grönwall's inequality and $\bar{v}(0,\cdot) = 0$ yield $\bar{v}(T,\cdot) = 0$ for any $T \in [0, T^*]$. We then have that $\bar{u} = 0$ on $[0, T^*]$ from (4.5.28). Therefore $u_1 = u_2$ and $v_1 = v_2$ on $[0, T^*]$. Uniqueness on any interval $[0, \infty)$ follows by iteration.

□

Chapter 5

Remarks and Future Work

In this chapter, we summarize the main results of this dissertation and discuss current and future directions in the study of chemotaxis systems on $\mathbb{R}^N$.

This dissertation studies a class of chemotaxis models on the whole space $\mathbb{R}^N$ with logistic source, chemical consumption, and either linear or porous-medium diffusion; see (1.1.6). In the first part of this dissertation (Chapter 3), we focused on the linear diffusion case (3.0.1). Specifically, in Section 3.2, we developed tools for proving global existence when the initial data are not necessarily integrable. This setting is substantially more delicate than the bounded-domain case, because standard tools such as global L^p-estimates, Gagliardo–Nirenberg inequalities, and classical Moser iteration do not directly apply when solutions fail to belong to $L^p(\mathbb{R}^N)$. To overcome this difficulty, we developed a new weighted energy method based on spatially localized estimates of the form $\|u(t,\cdot)\psi\|_{L^p(\mathbb{R}^N)}$, where ψ is a carefully chosen cut-off function. This method allows us to localize L^p-control uniformly in space and derive global-in-time bounds without the aid of boundary conditions. Using this approach, we proved the global existence and uniqueness of bounded nonnegative classical solutions for both integrable and non-integrable initial data under a sharp smallness condition involving the chemotactic sensitivity and the size of the initial chemical distribution. Moreover, in low dimensions $N = 1, 2$, we showed that bounded global classical solutions exist without any smallness assumption. It remains open whether global existence can be established without any smallness condition in higher dimensions.

In Section 3.3 of Chapter 3, we studied the asymptotic behavior of solutions for various types of initial data, including strictly positive initial data and compactly supported initial functions. We also investigated the effect of chemotactic sensitivity on the population dynamics. For strictly positive initial data, we proved that the constant steady state $(a/b, 0)$ is asymptotically stable, provided the solution remains globally bounded. In particular, this shows that there are no other positive stationary solutions. For compactly supported initial data, we studied the rate of spatial propagation in Subsection 3.3.2 through Subsection 3.3.4. One might expect that chemorepulsion ($\chi < 0$) slows down spreading, while chemoattraction ($\chi > 0$) accelerates it. Our results show that the situation is subtler. We proved that chemotaxis does not slow down the spreading speed of the population. In particular, when the initial chemical distribution is sufficiently sparse at infinity, for example when $v_0 \in L^p(\mathbb{R}^N)$ for some $p \geq 1$ or $v_0 \in C_0(\mathbb{R}^N)$, chemotaxis neither slows down nor speeds up the asymptotic spreading rate. Thus, in this setting, the spreading speed coincides with the Fisher–KPP speed $2\sqrt{a}$. We also showed that solutions do not spread infinitely fast, and we established upper bounds on the propagation speed for general initial data. When v_0 remains significant at spatial infinity, the influence of chemotaxis becomes more delicate. In the chemorepulsive case $\chi < 0$, we proved that for sufficiently small $|\chi|$, the spreading speed remains unchanged. In contrast, the numerical simulations in Subsection 3.3.5 indicate that for the chemoattractant case $\chi > 0$, there may exist a threshold $\chi^* > 0$ above which the spreading speed increases strictly. This phase transition phenomenon remains open at the theoretical level.

The second part of this dissertation (Chapter 4) focuses on the porous medium diffusion case (4.0.1). We study, for the first time, a chemotaxis system with porous medium diffusion on $\mathbb{R}^N$ for initial data satisfying (1.1.8), which are not necessarily integrable. Motivated by the linear diffusion case, we proved the global existence of bounded weak solutions by using the weighted energy method to show that boundedness of the local L^p-norms for the perturbed problem yields boundedness of the L^∞-norm of the solution through Moser iteration. In particular, we established the existence of globally bounded weak solutions without imposing any smallness condition on the initial data or the chemotactic sensitivity. This shows that porous medium diffusion, which is often more realistic in applications, helps prevent finite-time blow-up. A particularly significant difficulty arises when $m > 2$. In that case, localizing the equation produces an additional term of the form

$$\iint u^{p+m-1} |\nabla u| |\nabla \psi| \, dx \, dt,$$

which leads to a higher-order zero-derivative term whose exponent in u is at least $p + 2$. Classical Grönwall-type arguments are therefore no longer sufficient. We resolved this difficulty by combining the diffusion effect, the logistic damping, and the regularity of the initial data in a continuity framework. This idea is one of the main technical innovations of the dissertation.

From the perspective of porous medium equations, Hölder continuity appears to be the optimal regularity for weak solutions to (4.0.1), since the well-known fundamental solutions (Barenblatt solutions) are only Lipschitz continuous. In Section 4.5, we then established uniqueness in the class of weak solutions that are Hölder continuous up to the initial time, assuming $u_0 \in C^\alpha(\mathbb{R}^N)$ and $v_0 \in C^{2+\alpha}(\mathbb{R}^N)$, for $1 < m \leq 3$, using a duality argument. It remains open whether uniqueness holds for $m > 3$.

This dissertation opens several natural directions for future work. One important direction is the study of the long-time behavior of weak solutions to (4.0.1) when $m > 1$. For strictly positive initial data, we expect that the solution will asymptotically converge to $(a/b, 0)$, since classical solutions exist in this case. For compactly supported initial data, the problem becomes even more interesting because porous-medium diffusion leads to finite-speed propagation and the formation of a free boundary. Without chemotaxis, a key feature of the degenerate Fisher–KPP type problem

$$u_t = \Delta u^m + u(a - bu)$$

is finite-speed propagation: if the initial function is compactly supported, the solution $u(\cdot, t)$ remains compactly supported for all $t > 0$. Consequently, solutions of this equation develop a free boundary separating the positivity region of u from the region where $u = 0$. In [10], the authors proved that for $m > 1$, solutions with compactly supported data propagate outward with a finite speed c^* and that the invasion front exhibits a universal logarithmic delay, while the solution behind the front converges to the equilibrium $u = a/b$. Building on this, we would like to study precisely the large-time behavior of the free boundary in (4.0.1) and investigate the effect of chemotactic sensitivity on the spreading speed.

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