Orientations with forbidden out-degree and total-coloring extensions

by

Owen Henderschedt

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 8, 2026

Keywords: Orientations, Degree Restrictions, Total-coloring, Coloring Extensions

Approved by

Jessica McDonald, Chair, Professor of Mathematics and Statistics
Joseph Briggs, Assistant Professor of Mathematics and Statistics
Songling Shan, Assistant Professor of Mathematics and Statistics
Zach Walsh, Assistant Professor of Mathematics and Statistics
Ash Abebe, Interim Dean of the Graduate School

Abstract

This thesis studies two topics in graph theory: graph orientations with forbidden out-degrees and extensions of total-colorings.

An orientation of a graph assigns a direction to each edge, thereby prescribing for each vertex the number of edges directed away from it, called its out-degree. A natural question is to determine when it is possible to orient a graph so that each vertex avoids a prescribed set of forbidden out-degrees. More formally, given a graph G and a function $F : V(G) \to 2^{\mathbb{N}}$, an F-avoiding orientation is one in which no vertex v has out-degree in its forbidden set $F(v)$. Akbari et al. conjectured that when $|F(v)| < \frac{1}{2}\deg_G(v)$ for all v, then G admits an F-avoiding orientation. This conjecture remains open even when G is regular and all vertices share the same forbidden list. We extend known results on this problem by verifying the conjecture when G is 5- and 6-regular, as well as confirming the general conjecture in several additional cases, including for sparse graph classes and structured families of forbidden sets.

The second part of the thesis focuses on coloring extension problems, which ask whether a given partial coloring can be extended to a coloring of the entire graph without introducing new colors. A total-coloring of a graph assigns colors to both its vertices and edges so that no adjacent or incident elements receive the same color. A central open problem in this area, the Total Coloring Conjecture, asserts that every graph admits a total-coloring using at most $\Delta + 2$ colors, where Δ is the maximum degree.

We initiate a systematic study of total-coloring extensions by formulating new conjectures inspired by the literature on vertex- and edge-coloring extensions. Our conjectures strengthen the Total-Coloring Conjecture, and thus natural candidate graph classes for their study are those for which the Total-Coloring Conjecture is known to hold. To this end, we establish extension results for planar graphs with large maximum degree. We also prove total-coloring extension results for several other families of graphs.

Artificial Intelligence (AI) Use Disclosure Statement

In the preparation of this dissertation, no Artificial Intelligence (AI) tools were used.

Digital Accessibility Disclosure Statement

In the preparation of this dissertation, the following digital accessibility tools were used to ensure this document complies with federal requirements: PAC and Adobe Acrobat. The author acknowledges full responsibility for the intellectual content of this work and has made a good faith effort to comply with digital accessibility requirements in publishing, wherein the nature of the content does not significantly change in order to do so. Furthermore, all content has been reviewed and revised to meet these requirements prior to final publication.

Acknowledgments

I would first like to thank Dr. Jessica McDonald for her endless support throughout my time at Auburn. Without her guidance, mentorship, and encouragement, none of this would have been possible.

I am also grateful to Dr. Joseph Briggs, Dr. Songling Shan, and Dr. Zach Walsh for their collaboration, insightful feedback, and many valuable mathematical discussions. It has been such a joy to learn from you all during my PhD.

I would like to thank Dr. András Bezdek for his listening ear and his endless supply of interesting mathematical stories and problems, which made many conversations both enjoyable and inspiring.

I would also like to thank the late Dr. Dean Hoffman, whose love for mathematics was contagious and will forever have an impact on me.

I am thankful for my colleagues Dr. Jared DeLeo, Dr. Sean Grate, and Dr. Haile Gilroy for their friendship, support, and their constant willingness to talk about math.

Finally, I want to thank my family—Katherine, my parents, and my sisters. I quite literally could not have done this without you.

Table of Contents

Abstract 2
Artificial Intelligence (AI) Use Disclosure Statement 3
Digital Accessibility Disclosure Statement 4
Acknowledgments 5
List of Figures 8
Chapter 1 Introduction 9
1.1 Introduction to Orientations with Forbidden Out-degrees 9
1.2 Introduction to Total-Coloring Extensions 13
1.2.1 What is a coloring extension? 14
1.2.2 Our total-coloring extension conjectures 15
1.2.3 Planar graphs with large maximum degree 17
1.2.4 Other graph classes 20
1.3 Organization of Thesis 21
Chapter 2 Orientations with Forbidden Out-degrees 23
2.1 Minimum counterexamples and special graph classes 23
2.2 Two results about specialized lists 28
2.3 The lasso and proofs of Theorems 1.2 and 1.5 30
Chapter 3 Total-Coloring Extensions in Planar Graphs with Large Maximum Degree . 37
3.1 Tight examples for total-coloring extensions 37
3.2 Extensions from distance-3 cliques 39
3.3 Extension from subgraphs with bounded maximum degree 51
3.4 Extensions from matchings with no distance condition 56
Chapter 4 Total-Coloring Extensions in Some Other Families of Graphs 62
4.1 Path, Cycles and Trees 62
4.2 Ladders and Prisms 66

Chapter 5 Conclusions and Future Work 77
Bibliography 79

List of Figures

2.1 An out-lasso and a flipped out-lasso. 31

3.1 Example 3.1 with $k = 3$ showing that the color palette $[2\Delta + 1]$ cannot be reduced in size for guaranteeing an extension of a total-precoloring to total-colorings. . . 38

3.2 Example 3.2 with $t = 4$ which shows a totally precolored distance-2 matching with color palette $[\Delta + 2]$ which cannot extend to a total-$(\Delta + 2)$-coloring 38

3.3 Example 3.3 showing a set of distance-3 cliques in a graph G, precolored with color palette $[\Delta + 3]$, that does not extend to a total-$(\Delta + 3)$-coloring of G. . . . 39

3.4 All possible poor configurations are depicted (with black edges) along with their corresponding cliques (with red edges and vertices). Vertex degrees are indicated for all the clique vertices, and h indicates the degree of the high vertex, which exists in each case. 44

3.5 An example of a face f which is the helpful face to 6 different poor configurations. 46

3.6 Needy faces with respect to v of type 1 (left) and type 2 (right). Precolored vertices and edges indicated in red. 58

4.1 How to extend a precolored distance-3 matching in C_n to a total-4-coloring. . . . 63

4.2 The unique total-3-coloring of the path P_6 65

4.3 Two examples $\varphi : V(C_4) \cup E(C_4) \to [5]$ which are both good with respect to v_1v_2. The left gives a total-5-coloring whereas the right does not. 67

4.4 An illustration of the first extension step in Theorem 1.19. 76

Chapter 1

Introduction

A finite simple graph G is an ordered pair (V, E) where $V = V(G)$ is a finite set of *vertices* and $E = E(G)$ is a set of two-element subsets of V, called *edges*. In this thesis all graphs are assumed to be finite and simple unless otherwise stated. We refer the reader to [1] for terms not defined here.

This thesis studies two topics in graph theory: graph orientations with forbidden out-degrees and extensions of total-colorings. The following two sections introduce these topics and highlight our main contributions towards them in this thesis. Section 1.3 gives an overview of how the remainder of the thesis is organized.

1.1 Introduction to Orientations with Forbidden Out-degrees

Given a graph G, an *orientation* D of G is an assignment of a direction to each edge $e \in E(G)$. If u is directed toward v, we denote the corresponding directed edge by uv. Given a vertex $v \in V(G)$ and an orientation D, the *out-degree* (*in-degree*) of v, denoted $\deg_D^+(v)$ ($\deg_D^-(v)$), is the number of edges directed away from (toward) v. The *maximum out-degree* of D is $\Delta^+(D) = \max\{\deg_D^+(v) : v \in V(G)\}$, and the *minimum out-degree* is defined analogously. A foundational question in studying graph orientations is the following: Given some restrictions on the desired out-degrees (in-degrees) of the vertices of G, when does an orientation exist? In particular, one may ask how small the maximum out-degree can be over all orientations of G. This motivates the definition

$$\Delta^+(G) = \min\{\Delta^+(D) : D \text{ is an orientation of } G\}.$$

The parameter $\Delta^-(G)$ is defined analogously. One of the first well-celebrated results answering this question is due to Hakimi in 1965. The *maximum average degree* of G is

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|},$$

where the maximum is taken over all subgraphs H of G with at least one vertex.

Theorem 1.1 (Hakimi [2]). Let G be a graph. Then $\Delta^+(G) = \left\lceil \frac{1}{2}\text{mad}(G) \right\rceil$.

A recent generalization of Hakimi's foundational result replaces global bounds on maximum out-degree with vertex-specific lists of allowed or forbidden out-degrees. Given a function $F : V(G) \to 2^{\mathbb{N}}$ assigning to each vertex a set of forbidden values, we say that an orientation D is F-*avoiding* if $\deg_D^+(v) \notin F(v)$ for all $v \in V(G)$. In 2020, Akbari et al. conjectured that if each forbidden list is sufficiently small relative to the *degree* of its vertex (the number of edges incident to a vertex), then G admits an F-avoiding orientation. More precisely, they conjectured the following.

Conjecture 1.1 (Akbari, Dalirrooyfard, Ehsani, Ozeki, and Sherkati [3]). Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$. If $|F(v)| < \frac{1}{2}\deg(v)$ for all $v \in V(G)$, then G admits an F-avoiding orientation.

Conjecture 1.1 remains open even in the special case where all vertices share the same forbidden list and G is d-*regular*, that is every vertex in G has degree d.

Conjecture 1.2 (Akbari, Dalirrooyfard, Ehsani, Ozeki, and Sherkati [3]). Let G be a d-regular graph and let $F \subseteq \{0, 1, \dots, d\}$ be a list of forbidden out-degrees. If $|F| < \frac{1}{2}d$, then G admits an F-avoiding orientation, that is, an orientation where no out-degrees are in the forbidden list F.

Conjecture 1.2 is tight if true – in particular, the authors of the conjecture proved in [3] that K_{2k+1} has no F-avoiding orientation for $F = \{k, \dots, 2k-1\}$. In terms of specific d, Conjecture 1.2 is trivial for $d = 1, 2$ (where the forbidden list is empty), and true for $d = 3, 4$ (see Ma and Lu [4]). In this thesis, we verify the next cases of $d = 5, 6$ with the following theorem.

Theorem 1.2. Let G be a d-regular graph with $d \geq 5$, and let $F \subseteq \{0, 1, 2, \dots, d\}$ with $|F| \leq 2$. Then G admits an F-avoiding orientation.

The known tight examples for Conjecture 1.1 are the same as those for Conjecture 1.2. The more general conjecture has been verified for bipartite graphs (Akbari et al. [3]) and for cliques (see Bradshaw et al. [5] where they also use work from Huang et al. [6]). In this thesis, we add to the families known to satisfy Conjecture 1.1 with the following two results. Note that for a graph G, we let $\Delta(G)$ and $\delta(G)$ denote the *maximum and minimum degrees* over all vertices, respectively.

Theorem 1.3. Let G be a 2-degenerate graph and let $F : V(G) \to 2^{\mathbb{N}}$ with $|F(v)| < \frac{1}{2}\deg(v)$ for all $v \in V(G)$. Then G has an F-avoiding orientation.

Theorem 1.4. Let G be a graph, and let $F : V(G) \to 2^{\mathbb{N}}$ where $|F(v)| < \frac{1}{2}\deg(v)$ for all $v \in V(G)$. Suppose that the edges of G can be decomposed into a bipartite graph and a subgraph H with $\Delta(H) \leq 2$ and such that every vertex $v \in V(H)$ with $\deg_H(v) = 2$ has $\deg_G(v) \equiv 0 \pmod 2$. Then G admits an F-avoiding orientation.

Note that by a 2-degenerate graph, we mean a graph where every subgraph contains a vertex of degree at most 2. Hence the class of 2-degenerate graphs includes all 1-degenerate graphs (i.e. all forests). It also includes all planar graphs of girth at least 6. (To see this, note that for a planar graph G of girth g, Euler's formula gives $(\mathrm{mad}(G) - 2)(g - 2) < 4$).

There has been more success with approximations to Conjecture 1.1 than verifications for special families. In their initial paper, Akbari et al.[3] showed that replacing $\frac{1}{2}$ with $\frac{1}{4}$ results in a true statement. Recently Bradshaw, Chen, Ma, Mohar, and Wu [5] improved this upper bound to $\lfloor \frac{1}{3}\deg_G(v) \rfloor$ using the combinatorial Nullstellensatz of Alon and Tarsi [7][8].

Another approach to Conjecture 1.1 has been to consider special sorts of lists. To this end, let G be a graph with forbidden lists $F : V(G) \to 2^{\mathbb{N}}$. For every vertex $v \in V(G)$ we can partition $\{0, \dots, \deg(v)\}$ into maximal intervals in $F(v)$, which we call *holes*, and maximal intervals not in $F(v)$, which we call *homes*. Ma and Lu [4] proved that given a graph G and function $F : V(G) \to 2^{\mathbb{N}}$, if every hole has size at most one then G admits an F-avoiding orientation. Note that lists of this sort may satisfy $|F(v)| \sim \frac{1}{2}\deg_G(v)$ for all v. In this thesis, we take another step in this direction and allow holes of size two, provided that the overall ratio is somewhat worse, and given a condition on the *end-intervals*. The *end-intervals* for a vertex v are those holes or homes containing 0 or $\deg(v)$; note that each vertex has two end-intervals, but they need not be distinct.

Theorem 1.5. Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$ be such that for each $v \in V(G)$: all holes have size at most two; between any pair of distinct holes is a home of size at least 3, and both end-intervals are homes of size at least two. Then G admits an F-avoiding orientation.

Observe that in Theorem 1.5, the lists may satisfy $|F(v)| \sim \frac{2}{5}\deg_G(v)$, beating the $\frac{1}{3}$ bound. Our proof of Theorem 1.5 uses a new tool for modifying orientations called a *lasso*; we will introduce this concept and prove Theorem 1.5 in Section 2.2. In Section 2.2 we prove the following two results about very special forbidden lists that satisfy Conjecture 1.1.

Theorem 1.6. Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$ where $|F(v)| \leq \frac{1}{2}\deg(v)$ for all $v \in V(G)$. If every hole is an end-interval, then G admits an F-avoiding orientation.

Theorem 1.7. Let $k \in \mathbb{Z}^+$. Every graph with minimum degree at least $2k+1$ admits a $\{1, 2, ..., k\}$-avoiding orientation.

We will see that that proof of Theorem 1.6 follows from a classic theorem on orientations due to Frank and Gyarfas [9] (also Hakimi [2]), and Theorem 1.7 follows from a result about maximum directed cuts due to Alon, Bollobás, Gyárfás, Lehel and Scott [10]. Theorem 1.7 is interesting in that it is close to the tight example of Akbari et al. [3] for Conjecture 1.2 (and Conjecture 1.1) that we mentioned above. Note that we will use Theorem 1.7 as part of our proof of Theorem 1.2. We will see that the other needed cases for Theorem 1.2 come from Theorem 1.5 and our new lasso technique.

1.2 Introduction to Total-Coloring Extensions

Given a graph $G = (V, E)$, a *k-coloring* is a coloring of the vertices of G, that is, a function $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \varphi(v)$ for any $uv \in E(G)$. The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G admits a k-coloring.

Vertex coloring has been at the forefront of graph theory for decades. Among its most celebrated achievements is the Four Color Theorem, proved by Appel and Haken in 1977, which gives a sharp bound on the chromatic number of *planar graphs*, that is graphs that can be drawn in the plane without and edge crossings.

Theorem 1.8 (The Four Color Theorem [11, 12]). If G is a planar graph, then $\chi(G) \leq 4$.

A *k-edge-coloring* is a coloring of the edges of G, that is, a function $\varphi : E(G) \to [k]$ such that $\varphi(e) \neq \varphi(f)$ whenever e and f share a common endpoint. The *chromatic index* of G, denoted $\chi'(G)$, is the minimum k such that G admits a k-edge-coloring. Clearly, $\chi'(G) = \chi(L(G))$.

Edge coloring has been equally central to the development of the field. If v is a vertex of maximum degree in G, then any proper edge-coloring must assign distinct colors to the edges incident to v, and hence $\chi'(G) \geq \Delta(G)$. In the 1960s, Vizing proved that no more than one additional color is ever required.

Theorem 1.9 (Vizing's Theorem [13]). Let G be a graph. Then $\chi'(G) \leq \Delta(G) + 1$.

Given a graph G, a *total-k-coloring* is an assignment of the colors $\{1, 2, \dots, k\} = [k]$ to the vertices and edges of G such that the vertices induce a k-coloring, the edges induce a k-edge-coloring, and no edge can receive the same color as either of its endpoints. The *total chromatic number* of a graph G, denoted $\chi_T(G)$, is the smallest k such that there exists a total-k-coloring of G. By considering the edges incident with a vertex of maximum degree $\Delta(G)$, along with the vertex itself, we see that $\chi_T(G) \geq \Delta(G) + 1$. In the 1960's, Behzad [14] and Vizing [15] independently proposed the *Total Coloring Conjecture* which says that $\chi_T(G)$ is at most one more than this lower bound.

Conjecture 1.3 (The Total Coloring Conjecture [14, 15]). For any graph G,

$$\chi_T(G) \leq \Delta(G) + 2.$$

Conjecture 1.3 has been verified for several classes of graphs, and in many of these cases one can further establish extension results. We begin with planar graphs, where substantial progress has been made. Through a series of works (see [16, 17]), the Total Coloring Conjecture has been confirmed for all planar graphs G with $\Delta(G) \neq 6$. In fact, it is known that $\chi_T(G) = \Delta(G) + 1$ for planar graphs with $\Delta(G) \geq 9$ (see [18]), although this is false when $\Delta(G) \leq 3$. Other classes of graphs for which Conjecture 1.3 is known to hold include *subcubic graphs*, which are graphs with maximum degree at most 3, *bipartite graphs*, which are graphs whose vertices can be partitioned into two parts such that every edge has one endpoint in each part, and, more generally, graphs with sufficiently large maximum degree. For much more on the Total Coloring Conjecture, both in the planar case and in general, the interested reader is referred to [19] and [20].

1.2.1 What is a coloring extension?

Once a coloring result is known, it is natural to ask how robust that result is via a *coloring extension problem*. Namely, if we know that a graph G is k-colorable (k-edge-colorable or total-k-colorable) we may ask: *when can a k-coloring (k-edge-coloring or total-k-coloring) of some subgraph H of G be extended to a k-coloring (k-edge-coloring or total-k-coloring) of G?* Extension problems have been studied for both vertex-colorings and edge-colorings, particularly when G is planar. We initiate the study of total-coloring extension problems. All of our results are motivated by work that has been done on vertex-coloring and edge-coloring extension problems, and as well as total and list-total coloring results. For any necessary background on list-coloring, see [21]. Some results on edge-coloring extensions, in particular, will motivate our work on total-coloring extensions, and we present them here.

Edwards, Girão, van den Heuvel, Kang, Puleo, and Sereni [22] conjectured the following edge-coloring extension strengthening of Vizing's Theorem (Theorem 1.9), where the precolored subgraphs are matchings, that is, sets of edges whose endpoints are pairwise distinct. A distance-ℓ matching is a matching in which the endpoints of distinct edges are at distance at least ℓ, where the distance between any two vertices in G is the number of edges in a shortest path between them.

Conjecture 1.4 (Edwards, et. al. [22]). Let G be a graph with maximum degree Δ, and let H be a subgraph of G. If H is a distance-2 matching, then any $(\Delta + 1)$-edge-coloring of H can be extended to G.

To be clear, by 'extending the $(\Delta + 1)$-edge-coloring φ of H to G', we mean that G has a $(\Delta + 1)$-edge-coloring which agrees with φ on the edges of H. Conjecture 1.4 is a strengthening of an older conjecture of Albertson and Moore in [23], and there are other results supporting it by Girão and Kang [24] and Cao, Chen, Qi, and Shan [25] (including for a multigraph version).

Edwards et. al [22] prove Conjecture 1.4 for a variety of graph classes which again will motivate our total-coloring extension results. They prove Conjecture 1.4 for bipartite graphs, subcubic graphs, and planar graphs with large maximum degree. Specifically, they show the following.

Theorem 1.10 (Edwards, et. al. [22]). Let G be a planar graph with maximum degree $\Delta \geq 17$, and let H be a subgraph of G. If H is a distance-2 matching, then any $(\Delta + 1)$-edge-coloring of H can be extended to G.

We now turn to total-coloring extensions and initiate analogous conjectures, giving extension strengthenings of the Total-Coloring Conjecture.

1.2.2 Our total-coloring extension conjectures

Suppose we have a graph G with a subgraph H and φ is a total-k-coloring of H. We want to know whether φ can be *extended* to G, that is, whether there exists a total-k-coloring of G

which agrees with φ on all the vertices and edges of H. Of course, if there are vertices u and v which are adjacent in G but non-adjacent in H and moreover $\varphi(u) = \varphi(v)$, then we cannot hope to extend φ to G. We say that φ is a *total-k-coloring of H in G* if no such vertices u, v exist, that is, if the coloring φ is proper also with respect to G. In general, if φ is a total-k-coloring of H in G, then any uncolored vertex in G sees at most $2\Delta(G)$ colors from φ (all adjacent vertices and incident edges) and any uncolored edge in G sees at most $2\Delta(G)$ colors ($2\Delta(G) - 2$ adjacent edges and 2 endpoints). So we get the following basic observation by coloring greedily.

Observation 1.1. If G is a graph with maximum degree Δ and H is a subgraph of G, then any total-$(2\Delta + 1)$-coloring of H in G can be extended to G.

Without putting any restrictions on H, the $(2\Delta + 1)$ in Observation 1.1 is best possible due to an example we will show in Chapter 2, namely Example 3.1. So we must make restrictions on the precolored subgraph H if we want to lower the number of colors beyond the $(2\Delta + 1)$, and certainly if we want to approach the $(\Delta + 2)$ suggested by the Total Coloring Conjecture. Inspired by Conjecture 1.4 and Theorem 1.10, our starting point for this is matchings.

While a matching is typically defined as a set of edges within a graph whose endpoints are vertex disjoint, we extend this notion here and say that a subgraph H of a graph G is a *matching* if it induces a set of matching edges in H, i.e., if H consists of vertex-disjoint copies of K_2 within G. We say that H is *distance-ℓ* if the distance between any two vertices in distinct components of H is at least ℓ in G. We now preset our total-coloring analog to Conjecture 1.4.

Conjecture 1.5. Let G be a graph with maximum degree Δ and let H be a subgraph of G. If H is a distance-3 matching, then any total-$(\Delta + 2)$-coloring of H can be extended to G.

Note that we do not need to say that the coloring of H is 'in G' in Conjecture 1.5 or Theorem 1.12, since this is implicit from the distance condition. In comparing Conjectures 1.5 and 1.4, observe that while Vizing's Theorem [26] guarantees that all graphs have a $(\Delta + 1)$-edge-coloring, Conjecture 1.5 would be stronger than the Total Coloring Conjecture.

The distance condition in Conjecture 1.5 is best possible by another example given in Section 2, namely Example 3.2, which presents a graph G and a precolored distance-2 matching that does not extend to a total-$(\Delta(G)+2)$-coloring. Moreover, in Conjecture 1.5 we cannot, in general, reduce the color palette to $\Delta(G)+1$, since not all graphs satisfy $\chi_T(G) = \Delta(G)+1$. More broadly, we will see that $\Delta(G)+2$ in Conjecture 1.4 cannot always be replaced by $\chi_T(G)$; however, this replacement is possible for certain graph classes, as we will see shortly (see Chapter 4 for further discussion).

Since a distance-2 matching is a natural barrier to Conjecture 1.5 (according to the afore-mentioned Example 3.2), it is natural to ask what color palette is required when no distance condition is imposed on the precolored matching. We conjecture that only one additional color suffices.

Conjecture 1.6. Let G be a graph with maximum degree Δ and let H be a subgraph of G. If H is a matching, then any total-$(\Delta+3)$-coloring of H in G can be extended to G.

We point out that $(\Delta+3)$ cannot be replaced by $(\chi_T(G)+1)$, and this is demonstrated by the afore-mentioned Example 3.2.

Conjecture 1.5 is a strengthening of the Total-Coloring Conjecture (TCC), and thus natural graph class candidates for proving Conjecture 1.5 are classes of graphs in which the TCC is known to hold. Our contributions toward our conjectures Conjecture 1.5 and Conjecture 1.6 are divided into two chapters of this thesis (Chapter 3 and Chapter 4), and we introduce these results here. These include results for planar graphs with large maximum degree, as well as for a variety of other graph classes.

1.2.3 Planar graphs with large maximum degree

Inspired by Theorem 1.10, we aim to prove Conjecture 1.5 and Conjecture 1.6 for planar graphs with sufficiently large maximum degree. We achieve this in the following results.

Theorem 1.11. Let G be a planar graph with maximum degree $\Delta \geq 28$ and let H be a subgraph of G. If H is a matching, then any total-$(\Delta+3)$-coloring of H in G can be extended to G.

Theorem 1.12. Let G be a planar graph with maximum degree $\Delta \geq 27$ and let H be a subgraph of G. If H is a distance-3 matching, then any total-$(\Delta + 1)$-coloring of H can be extended to G.

We remind the reader that we do not need to say that the coloring of H is 'in G' in Theorem 1.12, since this is implicit from the distance condition. We will actually get Theorem 1.12 as a corollary of another result in this thesis, which we will discuss towards the end of this subsection.

The next result in this thesis on extending colorings in planar graphs does not require any particular structure on the subgraph H, but instead assumes that H has bounded maximum degree. Note we have no requirement on the maximum degree of G.

Theorem 1.13. Let G be a planar graph with maximum degree Δ and let H be a subgraph of G. If H has maximum degree d, then any total-$(\Delta + d + 6)$-coloring of H in G can be extended to G.

Theorem 1.13 is a total-coloring analog of the following edge-coloring extension result.

Theorem 1.14 (Harrelson, McDonald, Puleo [27]). Let G be a planar graph with maximum degree Δ and let H be a subgraph of G. If H has maximum degree d, then any $(\Delta + d + 4)$-edge-coloring of H can be extended to G.

Example 3.1 shows that the $(\Delta + d + 6)$ in Theorem 1.13 cannot be lowered beyond $(\Delta + d + 2)$; in that example $d = \Delta - 1$ and a total-$(\Delta + d + 1)$-coloring of H in G cannot be extended to G. Harrelson, McDonald, and Puleo have a similar example in [27] which shows that $(\Delta + d)$ would be best-possible for Theorem 1.14; in fact they prove this strongest result whenever $\Delta \geq 16 + d$. By making a similar assumption on Δ, it may be possible to get a total-coloring analog with $(\Delta + d + 2)$, but we leave this to future work. In fact, our main purpose in including Theorem 1.13 in this thesis is to apply it with $d = 1$ as a base case in our proof for Theorem 1.11 (there, the value $(\Delta + 7)$ is reduced to $(\Delta + 3)$ at the cost of requiring Δ to be sufficiently large).

While the literature on extending edge-colorings focuses on extending from (sufficiently distanced) matchings, the literature on extending vertex-colorings is rich with results for extending from (sufficiently distanced) *cliques*, that is, subgraphs containing all possible edges between their vertices. A variety of such results may be found in [28, 29, 23, 30, 31]; for our purposes we will highlight just one: Albertson [28] proved that in any planar graph G, if a set of distance-3 vertices are 6-colored, then this can always be extended to G. To be clear, given a subgraph H of a graph G, we say that H is a *set of distance-ℓ cliques* if H is a set of vertex-disjoint cliques (which may be of varying sizes) and the distance between any two vertices in distinct components of H is at least ℓ. In order to give more context for our next result of this thesis, which is about extending total-colorings from a set of distance-3 cliques, we need to first discuss total-choosability.

All total-coloring extension problems can be rephrased as list-total-coloring problems. We say that a graph G is *totally-L-choosable* if G can be totally colored so that each $x \in V(G) \cup E(G)$ is assigned a color from a list $L(x)$. If G is totally-L-choosable for all L where $|L(x)| = k$ for each $x \in V(G) \cup E(G)$, then we say that G is *totally-k-choosable*. Suppose we have a graph G and subgraph H, and we want to extend a total-k-coloring φ of H in G. For each $x \in V(G) \cup E(G) \setminus (V(H) \cup E(H))$ we define $L(x)$ as the subset of $[k]$ remaining after removing any color incident or adjacent to x under φ. If we also set $L(x) = \{\varphi(x)\}$ for all $x \in V(H) \cup E(H)$, then G is totally-L-choosable if and only if φ can be extended to G. Moreover, for *any* way we extend L by assigning lists $L(x)$ for $x \in V(H) \cup E(H)$, if we get that G is totally-L-choosable, then it implies that φ can be extended to G. This is because the color assigned to some $x \in V(H) \cup E(H)$ can be removed and replaced with $\varphi(x)$ without causing any conflicts.

Every planar graph G with maximum degree $\Delta \geq 12$ is known to be total-$(\Delta+1)$-choosable by Borodin, Kostochka, and Woodall [32]. Suppose that H is a set of disjoint cliques in G which has been totally colored by φ. If H is a distance-3 matching, then any vertex or edge in G sees at most two different colors of φ. If H is more generally a set of distance-3 cliques, then any vertex or edge in G sees at most $\omega(H)$ colors (and $\omega(G) \leq 4$ since G is planar). So, using the approach described in the above paragraph, Borodin and Kostochka's total choosability result implies the following.

Theorem 1.15 (Borodin, Kostochka, and Woodall [32]). Let G be a planar graph with maximum degree $\Delta \geq 12$ and let H be a subgraph of G.

1. If H is a distance-3 matching, then any total-$(\Delta + 3)$-coloring of H can be extended to G.

2. If H is a set of distance-3 cliques, then any total-$(\Delta + 5)$-coloring of H can be extended to G.

We can compare Theorem 1.15(1) to Conjecture 1.6 and Theorem 1.11 where we have the same number of colors but no distance condition; we can also compare it to Conjecture 1.5 and Theorem 1.12 where we have the same distance condition but we are using one or even two fewer colors, respectively.

In this thesis, we lower the number of colors in Theorem 1.15(2) from $(\Delta + 5)$ all the way down to $(\Delta + 1)$, as follows.

Theorem 1.16. Let G be a planar graph with maximum degree $\Delta \geq 27$, and let H be a subgraph of G. If H is a set of distance-3 cliques, then any $(\Delta + 1)$-total-coloring of H can be extended to G.

We now see that Theorem 1.16 immediately implies Theorem 1.12. Example 3.2 again shows that the distance-3 condition in Theorem 1.16 cannot be lowered to distance-2. The $\Delta \geq 27$ condition in Theorem 1.16 is surely not sharp, but it cannot be eliminated completely. In particular, not all planar graphs with maximum degree $\Delta \leq 3$ are total-$(\Delta+1)$-colorable. If one is concerned with the bound on Δ, we have in fact proved slightly stronger versions of Theorem 1.11 and Theorem 1.16 than those stated in this introduction. In particular, we show that the bound on Δ can be improved when additional colors are allowed. We also allow precolored vertices in Theorem 1.11 at no additional cost. However, even when more colors are permitted, the condition on Δ cannot be completely removed. Example 3.3 (presented in Section 3.1) shows that we must have at least $\Delta \geq 5$ when using $\Delta + 3$ colors; in fact, in this example the cliques are at distance 4.

1.2.4 Other graph classes

Motivated by Edwards et. al proving Conjecture 1.4 for bipartite and subcubic graphs, we prove Conjecture 1.5 for several structured families within these classes. The subclass of bipartite graphs which we validate Conjecture 1.5 for include paths, cycles, and trees. A *path*, denoted P_n, is the graph with vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for all $i \in [n-1]$. A *cycle*, denoted C_n, is the graph obtained from P_n by adding the edge $v_1 v_n$. A *tree* is a connected graph which is acyclic (contains no cycle as a subgraph).

Theorem 1.17. Let $n \geq 1$ and let M be a distance-3 matching in P_n. Then any total-4-coloring of M can be extended to P_n.

Corollary 1.1. Let $n \geq 3$ and let M be a distance-3 matching in C_n. Then any total-4-coloring of M can be extended to C_n.

Theorem 1.18. Let T be a tree with maximum degree Δ, and let M be a distance-3 matching in T. Then any total-$(\Delta+2)$-coloring of M can be extended to T.

Next, regarding subcubic graphs, we prove Conjecture 1.5 for ladders and prisms. A *ladder*, denoted $P_n \square K_2$, is the graph obtained by taking two copies of a path on n vertices, say $x_1 \cdots x_n$ and $y_1 \cdots y_n$, and adding the edges $x_i y_i$ for all $i \in [n]$. A *prism*, denoted $C_n \square K_2$, is the graph obtained by taking two copies of a cycle on n vertices, say $x_1 \cdots x_n x_1$ and $y_1 \cdots y_n y_1$, and adding the edges $x_i y_i$ for all $i \in [n]$. Note that both ladders and prisms are subcubic. We prove the following.

Theorem 1.19. Let $n \geq 3$ and let $G = P_n \square K_2$ or $C_n \square K_2$. Let M be a distance-3 matching in G. Then any total-5-coloring of M extends to a total-5-coloring of G.

1.3 Organization of Thesis

The remainder of this thesis is divided into four main chapters.

Chapter 2 is devoted to orientations with forbidden out-degrees, the results of which were published in [33]. In Section 2.1, we prove Theorem 1.3 by analyzing minimal counterexamples, and in the process establish a collection of structural properties that any smallest counterexample to Conjecture 1.1 must satisfy. These arguments also allow us to strengthen a result of [3] for bipartite graphs, yielding Theorem 1.4.

We then introduce a new tool for modifying orientations, called a *lasso*, which is used to prove Theorem 1.5. The definition of a lasso and the proof of this theorem are given in Section 2.2. We conclude this chapter by proving Theorem 1.6 and Theorem 1.7. In Section 2.3, we establish additional results for regular graphs.

Chapter 3 focuses on total-coloring extension results for planar graphs with large maximum degree; the results in this chapter appear in [34]. In Section 3.1, we present Example 3.1, Example 3.2, and Example 3.3. We then prove Theorem 1.16 in Section 3.2, Theorem 1.13 in Section 3.3, and Theorem 1.11 in Section 3.4.

Next, Chapter 4 establishes extension results for additional graph families. In particular, we prove Theorem 1.17, Corollary 1.1 and Theorem 1.18 in Section 4.1, and then prove Theorem 1.19 in Section 4.2.

Finally, Chapter 5 provides a conclusion and discusses directions for future work.

Chapter 2

Orientations with Forbidden Out-degrees

In this chapter, we study orientations of graphs with forbidden out-degrees. We begin by establishing structural properties of minimal counterexamples, which will serve as a foundation for the results that follow. Using these observations, we then expand the classes of graphs for which Conjecture 1.1 is known to hold.

2.1 Minimum counterexamples and special graph classes

We prove the following properties about any minimum counterexample to Conjecture 1.1.

Lemma 2.1. Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$ with $|F(v)| < \frac{1}{2}\deg_G(v)$ for all $v \in V(G)$. Suppose that G does not admit an F-avoiding orientation, and suppose that $|E(G)|$ is minimum subject to this. Then:

(a) all vertices of even degree in G form an independent set;

(b) for each even-degree vertex $v \in V(G)$ and neighbor $u \in N_G(v)$, we have $0, \deg_G(u) \notin F(u)$; and

(c) for each adjacent pair $u, v \in V(G)$, then either $0 \notin F(u)$ or $\deg_G(v) \notin F(v)$

Moreover, if $|E(G)| + |V(G)|$ is minimum, then $\delta(G) \geq 3$.

Proof. Define the following sets of vertices:

$$A_1 = B_1 = \{v \in V(G) : v \text{ has even degree}\};$$

$$A_2 = \{v \in V(G) : \deg_G(v) \in F(v)\}; \quad B_2 = \{v \in V(G) : 0 \in F(v)\};$$

$$\mathcal{A} = A_1 \cup A_2; \quad \text{and} \quad \mathcal{B} = B_1 \cup B_2.$$

Proving (a) means proving that there is no edge between A_1 and B_1, proving (b) means proving there is no edge between A_1 and B_2 and no edge between B_1 and A_2, and proving (c) means proving there is no edge between A_2 and B_2. We will handle all these cases together by supposing, for a contradiction, that there exists an edge e joining $u \in \mathcal{A}$ with $v \in \mathcal{B}$.

Let $G' = G - e$. We define $F' : V(G') \to 2^{\mathbb{N}}$ as follows:

$$F'(w) = \begin{cases} F(w) & \text{if } w \notin \{u, v\} \\ F(w) & \text{if } w = u \text{ and } u \in A_1 \cap A_2^c \\ F(w) \setminus \{\deg_G(w)\} & \text{if } w = u \text{ and } u \in A_2 \\ \{i - 1 : i \in F(v), i \geq 1\} & \text{if } w = v. \end{cases}$$

We will show that

$$|F'(w)| < \frac{1}{2} \deg_{G'}(w) \quad \text{for all } w \in V(G'). \tag{2.1}$$

To this end, note that for a vertex $w \notin \{u, v\}$, $\deg'_G(w) = \deg_G(w)$ and $F'(w) = F(w)$ so condition (1) is trivially satisfied for such w. If $w = u$ and $u \in A_2$, then $\deg'_G(w) = \deg_G(w) - 1$ and $|F'(w)| = |F(w)| - 1$. Satisfying (1) therefore amounts to $|F(w)| - 1 < \frac{1}{2}\big(\deg_G(w) - 1\big)$, or equivalently, $|F(w)| < \frac{1}{2}\deg_G(w) + \frac{1}{2}$, which we know by assumption. If $w = u$ and $u \in A_1$, then $F'(w) = F(w)$ while $\deg_{G'}(w) < \deg_G(w)$, but since $\deg_G(w)$ is even, (1) follows from $|F(w)| < \frac{1}{2}\deg_G(w)$. Finally, consider that $w = v$, where we know that $\deg'_G(w) < \deg_G(w)$. If $0 \in F(v)$ then $|F'(v)| = |F(v)| - 1$, and the same computation we did in the A_2-case verifies (1). Otherwise, since $v \in \mathcal{B}$, we know that $v \in B_1$ and hence that v has even degree is G, so we get (1) analogously to the A_1-case.

Since our $F' : V(G') \to 2^{\mathbb{N}}$ satisfies (1) and since G is edge-minimal, we get an F'-avoiding orientation D' of G'. Define an orientation D of G from D' by orienting e from v to u.

Then $\deg^+_D(w) = \deg^+_{D'}(w) \notin F'(w) = F(w)$ for any $w \notin \{u, v\}$. Note that $\deg^+_D(u) = \deg^+_{D'}(u)$ since v points towards u. If $u \in A_1$ then $F'(u) = F(u)$ so $\deg^+_D(u) \notin F(u)$; if $u \in A_2$ then $F'(u) = F(u) \setminus \{\deg_G(u)\}$ and $\deg^+_D(u) \neq \deg_G(u)$, so we have $\deg^+_D(u) \notin F(u)$. Finally consider v: since $\deg^+_D(v) - 1 = \deg^+_{D'}(v) \notin \{i - 1 : i \in F(v), i \geq 1\}$, we have $\deg^+_D(v) \notin F(v)$. Thus, D is an F-avoiding orientation of G, contradiction. This completes our proof of (a), (b), and (c).

In order to prove the final statement of our theorem, suppose for a contradiction that there exists $v_0 \in V(G)$ with $\deg_G(v_0) \leq 2$. Since $|F(v_0)| < \frac{1}{2}(2)$, $F(v_0) = \emptyset$. We cannot have $\deg_G(v_0) = 0$, as isolates cannot satisfy this strict inequality, so $\deg_G(v_0) \in \{1, 2\}$. We divide our proof into two cases.

Case 1: *Every $w \in N_G(v_0)$ has $\deg_G(w) \in F(w)$.*

Note that in this case, we know that $\deg_G(v_0) = 1$, by part (b) above; let u_0 be the lone neighbor of v_0 in G. Define $G' = G - v_0$ and define forbidden lists F' for G' by : $F'(w) = F(w)$ if $w \neq u_0$, and $F'(u_0) = F(u_0) \setminus \{\deg_G(u_0)\}$. Clearly $|F'(w)| = |F(w)| < \frac{1}{2}\deg_G(w) = \frac{1}{2}\deg_{G'}(w)$ for all $w \neq u_0$. For u_0 we have

$$|F'(u_0)| = |F(u_0)| - 1 < \frac{1}{2}\deg_G(u_0) - 1 = \frac{1}{2}(\deg_{G'}(u_0) + 1) - 1 < \frac{1}{2}\deg_{G'}(u_0).$$

So by minimality, there exists an orientation D' of G' which is F'-avoiding. Define an orientation D from D' by directing the additional edge from v_0 to u_0. This means that $\deg^+_D(w) = \deg^+_{D'}(w)$ for all $w \in V(G) \setminus \{v_0\}$. Recall that $F(v_0) = \emptyset$, so D trivially satisfies the F-avoiding condition for v_0. Since $F'(w) = F(w)$ for all $w \neq u_0$, the F-avoiding condition is satisfied for all such w. For u_0 there is one extra forbidden value in $F(u_0)$ as compared to $F'(u_0)$, but this extra value is $\deg_G(u_0)$, which is certainly not the out-degree of u_0 in D since the edge between v_0 and u_0 points into u_0. Hence D is an F-avoiding orientation of G, contradiction.

Case 2: *There exists some* $u \in N_G(v_0)$ *with* $\deg_G(u) \notin F(u)$*.*

In particular, the assumption of this case implies that there exists some $\alpha \in F(u)$ such that $\alpha + 1 \notin F(u)$. Let $\tilde{G}$ be the graph obtained from G by deleting one edge, between u and v_0, and also deleting v_0 if $\deg_G(v_0) = 1$. We define the following forbidden lists $\tilde{F}$ of $\tilde{G}$.

$$\tilde{F}(w) = \begin{cases} F(w) & \text{if } w \in V(G), w \neq u \\ \\ F(w) \setminus \{\alpha\} & \text{if } w = u \end{cases}$$

For any $w \notin \{u, v_0\}$ we have $|\tilde{F}(w)| = |F(w)| < \frac{1}{2}\deg_G(w) = \frac{1}{2}\deg_{G'}(w)$. For u we get

$$|\tilde{F}(u)| = |F(u)| - 1 < \frac{1}{2}\deg_G(u) - 1 = \frac{1}{2}(\deg_{\tilde{G}}(u) + 1) - 1 < \frac{1}{2}\deg_{\tilde{G}}(u).$$

If $v_0 \in V(\tilde{G})$, then v_0 is not isolated so $|F(v_0)| = |\tilde{F}(v_0)| = 0$ implies that $|\tilde{F}(v_0)| < \frac{1}{2}\deg_{\tilde{G}}(v_0)$.

By minimality, there exists an exists an $\tilde{F}$-avoiding orientation $\tilde{D}$ of $\tilde{G}$. We now obtain an orientation D from $\tilde{D}$ by orienting our deleted edge as follows. If $\deg^+_{\tilde{D}}(u) = \alpha$, orient the edge from u to v_0. Otherwise orient from v_0 to u. Clearly $\deg^+_D(w) = \deg^+_{\tilde{D}}(w) \notin \tilde{F}(w) = F(w)$ for all $w \neq \{u, v_0\}$. Since $F(v_0) = \varnothing$, in order to show that D is an F-avoiding orientation of G (and get our desired contradiction) it remains only to check that D is F-avoiding at u. If $\deg^+_{\tilde{D}}(u) = \alpha$, then $\deg^+_D(u) = \alpha + 1 \notin F(u)$. On the other hand, if $\deg^+_{\tilde{D}}(u) \neq \alpha$ then $\deg^+_D(u) = \deg^+_{\tilde{D}}(u)$. Since $\tilde{F}(u) \subset F(u)$, we also get that $\deg^+_D(u) \notin F(u)$ in this situation. Hence D is an F-avoiding orientation of G, contradiction. □

Our work in Lemma 2.1, and in particular the last sentence of the lemma statement, immediately implies that Conjecture 1.1 holds for 2-degenerate graphs.

Theorem (1.3). Let G be a 2-degenerate graph and let $F : V(G) \to 2^{\mathbb{N}}$ with $F(v) < \frac{1}{2}\deg_G(v)$ for all $v \in V(G)$. Then G has an F-avoiding orientation.

Proof. Suppose not, and let G be a counterexample where $|V(G)| + |E(G)|$ is smallest. Since G has a vertex of degree at most 2, then G cannot be a minimum counterexample by Lemma

2.1. $\square$

It is tempting to want to say that if G is a graph where all even-degree vertices form an independent set (or G does not have any even-degree vertices), then G satisfies the assumptions of Conjecture 1.1, but this does not follow from Lemma 2.1 because such a class is not closed under taking subgraphs. On the other hand, we can use arguments similar to those above to expand further the class of graphs which are known to satisfy Conjecture 1.1. Here we restate and prove Theorem 1.4.

Theorem (1.4). Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$ where $|F(v)| < \frac{1}{2}\deg_G(v)$ for all $v \in V(G)$. Suppose that G can be decomposed into a bipartite graph and a subgraph H with $\Delta(H) \leq 2$ and such that every vertex $v \in V(H)$ with $\deg_H(v) = 2$ has $\deg_G(v) \equiv 0 \pmod 2$. Then G admits an F-avoiding orientation.

Proof. We may assume without loss of generality that H is spanning, since adding vertices of degree zero to H has no effect on our assumptions. If all vertices have degree 0 or 2 in H, then H is a collection of cycles and isolates, and by consistently orienting the cycles we get an orientation D_H of H with $\deg^+_{D_H}(v) \in \{0,1\}$ for all $v \in V(H)$. If H has a vertex of odd-degree, then by adding one dummy vertex and joining it to every odd-degree vertex of H, we get a collection of cycles (and isolates) H' – by consistently orienting the cycles in H' and then deleting the dummy vertex, we again get an orientation D_H of H with $\deg^+_{D_H}(v) \in \{0,1\}$ for all $v \in V(H)$. It remains now to orient every edge in $E(G) \setminus E(H)$, and show that our overall orientation is F-avoiding.

Let $G' = G - E(H)$. We define forbidden lists F' of G' as follows.

$$F'(w) = \begin{cases} F(w) & \text{if } \deg^+_{D_H}(w) = 0 \\ \{i-1 : i \in F(w), i \geq 1\} & \text{if } \deg^+_{D_H}(w) = 1. \end{cases}$$

For any $w \in V(G)$ with $\deg^+_{D_H}(w) = 0$, we have $|F'(w)| = |F(w)| < \frac{1}{2}\deg_G(w) \leq \frac{1}{2}\deg_{G'}(w) + 1$ and hence $|F'(w)| \leq \frac{1}{2}\deg_{G'}(w)$. Now suppose $\deg^+_{D_H}(w) = 1$ and hence $\deg_H(w) \in \{1,2\}$.

If $\deg_H(w) = 1$ then

$$|F'(w)| \leq |F(w)| < \frac{1}{2}\deg_G(w) = \frac{1}{2}(\deg_{G'}(w) + 1)$$

which implies $|F'(w)| \leq \frac{1}{2}\deg_{G'}(w)$. Finally if $\deg_H(w) = 2$, then

$$|F'(w)| \leq |F(w)| < \frac{1}{2}\deg_G(w) = \frac{1}{2}(\deg_{G'}(w) + 2).$$

This implies $|F'(w)| \leq \frac{1}{2}(\deg_{G'}(w) + 1)$ but since $\deg_{G'}(w) \equiv \deg_G(w) \equiv 0 \pmod 2$ in this case, $|F'(w)| \leq \frac{1}{2}\deg_{G'}(w)$. In all cases, $|F'(v)| \leq \frac{1}{2}\deg_{G'}(v)$ for all $v \in V(G')$, and since G' is bipartite, there exists an F'-avoiding orientation D' of G' by the above-mentioned result of Akbari et al. (Theorem 4 in [3]). We claim $D := D' \cup D_H$ is an F-avoiding orientation of G.

Clearly for any $v \in V(G)$ with $\deg^+_{D_H}(v) = 0$ then $\deg^+_{D'}(v) \notin F'(v) = F(v)$. Now consider $v \in V(G)$ with $\deg^+_{D_H}(v) = 1$. We have $\deg^+_D(v) - 1 = \deg^+_{D'}(v) \notin F'(v) = \{i - 1 : i \in F(v), i \geq 1\}$, so we have $\deg^+_D(v) \notin F(v)$. Thus, D is an F-avoiding orientation of G. □

2.2 Two results about specialized lists

The following theorem is a classic result of Frank and Gyárfás [9] on orientations of graphs (which also generalizes prior work of Hakimi [2]) .

Theorem 2.10 (Frank and Gyárfás [9]). Let G be a graph and let $\ell, u : V(G) \to \mathbb{N}$ with $\ell \leq u$. Then G has an orientation D with $\ell(v) \leq \deg^+_D(v) \leq u(v)$ for all $v \in V(G)$ if and only if for all $S \subseteq V(G)$,

$$\ell(S) \leq e[S] + \delta(S) \quad \text{and} \quad e[S] \leq u(S),$$

where $\ell(S) = \sum_{x \in S} \ell(x)$, $u(S) = \sum_{x \in S} u(x)$, $e[S] = |E(G[S])|$, and $\delta(S) = |\{uv \in E(G) : u \in S, v \notin S\}|$.

Theorem 2.10 immediately allows us to prove Theorem 1.6, a special case of Conjecture 1.1, where in fact the strict inequality of the conjecture is replaced by $\leq$. Note that here and in what follows, we use the "hole" and "home" terminology that was introduced in Section 1. We restate the theorem here for convenience.

Theorem (1.6). Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$ where $|F(v)| \le \frac{1}{2}\deg(v)$ for all $v \in V(G)$. If every hole is an end-interval, then G admits an F-avoiding orientation.

Proof. Since every hole is an end-interval, each vertex has a single home interval; for every $v \in V(G)$, let $\ell(v)$ be the smallest value in this home interval and let $u(v)$ be the largest. Then for every $v \in V(G)$, $F(v) = \{0, \dots, \ell(v)-1\} \cup \{u(v)+1, \dots, \deg(v)\}$, with $|F(v)| = \ell(v)+\deg(v)-u(v)$. Since $|F(v)| \le \frac{1}{2}\deg(v)$ and $0 \le \ell(v) \le u(v) \le \deg(v)$, this implies that $\ell(v) \le \frac{1}{2}\deg(v)$ and $u(v) \ge \frac{1}{2}\deg(v)$ for all $v \in V(G)$.

Suppose for contradiction that G does not admit an F-orientation. Then by Theorem 2.10, there exists some $S \subseteq V(G)$ with either

$$\ell(S) > e[S] + \delta(S) \tag{2.2}$$

or

$$u(S) < e[S]. \tag{2.3}$$

First suppose that S realizes (2.2). Since $\ell(v) \le \frac{1}{2}\deg(v)$ for all $v \in V(G)$, we get

$$\begin{aligned}\frac{1}{2}\sum_{x\in S}\deg_G(x) &\ge \ell(S)\\ &> e[S] + \delta(S)\\ &= \frac{1}{2}\left(\sum_{x\in S}\deg_G(x) - \delta(S)\right) + \delta(S)\\ &= \frac{1}{2}\left(\sum_{x\in S}\deg_G(x) + \delta(S)\right).\end{aligned}$$

which is a contradiction. We can argue similarly in the case where S realizes (2.3). Here, since

$u(v) \geq \frac{1}{2}\deg(v)$ for all $v \in V(G)$, we get

$$\frac{1}{2}\sum_{x\in S}\deg_G(v) \leq u(S) < e[S] = \frac{1}{2}\left(\sum_{x\in S}\deg_G(x) - \delta(S)\right),$$

which is also a contradiction. □

We now prove Theorem 1.7, which will be used as a special case in the proof of Theorem 1.2. We restate it here for convenience.

Theorem (1.7). Let $k \in \mathbb{Z}^+$. Every graph with minimum degree at least $2k+1$ admits a $\{1, 2, ..., k\}$-avoiding orientation.

Proof. Let G be a graph with minimum degree at least $2k+1$ and let $F(v) = \{1, 2, ..., k\}$ for all $v \in V(G)$. Let I be a maximal independent set. For any edge incident to a vertex in I, orient the edge so that the out-degree of every vertex in I is 0. Let $G' = G - I$ and let $F'(v) = \{0, 1, ..., k - |N_G(v) \cap I|\}$ for each $v \in V(G')$. Note that it suffices to find an F'-avoiding orientation of G' and the resulting orientation of G will be a $\{1, ..., k\}$-avoiding orientation.

Since I is a maximal independent set, every vertex of G' must have at least one neighbor in I. Then since G has minimum degree at least $2k+1$, for all $v \in V(G')$ we have

$$|F'(v)| = k + 1 - |N_G(v) \cap I| \leq \frac{2k + 1 - |N_G(v) \cap I|}{2} \leq \frac{1}{2}\deg_{G'}(v).$$

Since $F'(v)$ are forbidden lists where every hole is an end-interval, by Theorem 1.6, G' admits an F'-avoiding orientation. □

2.3 The lasso and proofs of Theorems 1.2 and 1.5

A common proof strategy when working with orientations is to find a directed path and then reverse the direction of all of its edges; note that in this modification only the endpoints of the path change their out-degrees. In this section we find another directed subgraph, which we call a *lasso*, which can also be used to alter out-degrees in a controlled way.

Consider the graph obtained from the path $(v_1, \dots, v_k)$ by adding a single edge between v_k and v_i for some $2 \le i \le k-1$. The directed graph obtained by orienting the path consistently from v_1 to v_k and from v_k to v_i is called an *out-lasso*; the directed graph obtained from an out-lasso by reversing the direction of every edge is called an *in-lasso*. In either case, we call the directed graph a *lasso* and denote it by $L = (v_1, \dots v_k; v_i)$; we say that L *starts* at v_1. We *flip* L by reversing the orientation of the edges in $(v_1, \dots, v_i)$ and $v_k v_i$; we call the result a *flipped lasso* $L' = (v_i; v_1, \dots, v_k)$; if L is an out-lasso (in-lasso) we may also call L' a *flipped out-lasso (flipped in-lasso)*. See figure Fig. 2.1 for an example.

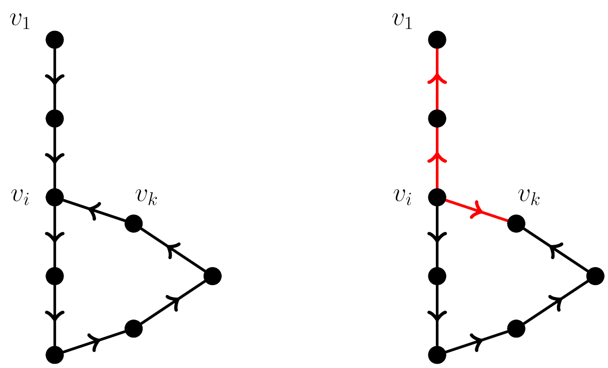


Figure 2.1: An out-lasso and a flipped out-lasso.

Given a graph G and forbidden list $F : V(G) \to 2^{\mathbb{N}}$, for any orientation D of G we define $D_i = \{v \in V(G) : \deg_D^+(v) = i\}$ and $D_F = \{v \in V(G) : \deg_D^+(v) \in F(v)\}$. Thus, an orientation D is F-avoiding if and only if $D_F = \emptyset$. Suppose that we have graph G with orientation D and a lasso subgraph $L = (v_1, \dots v_k; v_i)$ or flipped lasso $L' = (v_i; v_1, \dots v_k)$. For $a, b, c \in \mathbb{Z}$, we say that L or L' is of *type* (a, b, c) (with respect to D) if $v_1 \in D_a$, $v_i \in D_b$, and $v_k \in D_c$.

Suppose that we have found a lasso $L = (v_1, \dots v_k; v_i)$ of type (a, b, c) in a graph G with orientation D. Suppose then that we flip L and obtain the flipped lasso L', with D' the corresponding modified orientation of G. Note that the out-degree of every vertex of G is unchanged from D to D', except for the vertices v_1, v_i, v_k. If L is an out-lasso, then we know that L' is of type $(a-1, b+2, c-1)$ (with respect to D'); if L is an in-lasso, then we know that L' is of type $(a+1, b-2, c+1)$ (with respect to D').

The idea of flipping a lasso is only helpful if we can guarantee the existence of a lasso in the first place. The following lemma helps with that. Here, and in what follows, given a graph G with orientation D, for all $v \in V(G)$ we denote by S_v the set of all vertices reachable from v in D via a directed path. Similarly, we denote by T_v the set of all vertices which can reach v in D via a directed path. A vertex v in an oriented graph is a *source* if all its incident edges point out of v; v is a *sink* if all its incident edges point into v.

Lemma 2.2. Let G be a graph with orientation D and suppose there exists $v \in V(G)$ with $S_v \cap T_v = \{v\}$. If S_v contains no sinks, then there exists an out-lasso starting at v. If T_v contains no sources, then there exists an in-lasso starting at v.

Proof. Both statements of the lemma can be proved in a similar way; we just prove the first. Take a longest directed path P starting at v, say $P = (v = v_1, \dots, v_k)$. Since $v \in S_v$ trivially and S_v contains no sinks, we know that $k \geq 2$. Since $v_k \in S_v$ we know that v_k is not a sink. Since P is however a longest path, we know that v_k has an out-neighbor v_i for some $i \in \{1, \dots, k-1\}$. In fact, $i \geq 2$, since if $i = 1$ then $v_2 \in S_v \cap T_v$. In particular, this implies that $k \geq 3$. But now the path $P + v_k v_i$, gives an out-lasso starting at v. $\square$

We can now prove Theorem 1.5, which we restate for convenience.

Theorem (1.5). Let G be a graph and let $F : V(G) \to 2^{\mathbb{N}}$ be such that for each $v \in V(G)$: all holes have size at most two; between any pair of distinct holes there is a home of size at least 3, and each end-interval is either a hole of size one or a home of size at least 2. Then G admits an F-avoiding orientation.

Proof. For each $v \in V(G)$ we partition $\{0, \dots, \deg(v)\}$ into four sets, one being $F(v)$ and the other three as follows:

$$A(v) = \{i : i \notin F(v), i-1 \in F(v)\} \cap \{0, \dots, \deg(v)\}$$

$$B(v) = \{i : i \notin F(v), i+1 \in F(v)\} \cap \{0, \dots, \deg(v)\}$$

$$X(v) = \{0, \dots, \deg(v)\} \setminus (A(v) \cup B(v) \cup F(v))$$

Note that for any given vertex v, the sets $A(v)$ and $B(v)$ are precisely the integers in homes of v which are adjacent to some hole (*above* and *below* respectively). Moreover, $A(v) \cap B(v) = \varnothing$ since no homes have size 1. Similarly we can think of $X(v)$ as the integers of $\{0, \dots, \deg(v)\}$ which are *far* from (meaning not adjacent to) a hole. For an orientation D of G, recall that $D_F = \{v \in V(G) : \deg_D^+(v) \in F(v)\}$; we now extend this definition so that we may also refer to D_A, D_B, D_X, that is, those vertices $v \in V(G)$ with out-degree in the set $A(v), B(v)$, or $X(v)$, respectively.

Let D be an orientation which minimizes $|D_F|$ and subject to that, maximizes $|D_X|$. If $|D_F| = 0$ then D is an F-avoiding orientation, so suppose for a contradiction that $|D_F| \neq 0$; hence we may choose $v \in D_F$. Note that since all holes of v are of size at most 2, and we cannot have a hole of size two as an end-interval, either $\deg_D^+(v) + 1 \in A(v)$, or $\deg_D^+(v) - 1 \in B(v)$. We shall consider both cases.

Case 1: $\deg_D^+(v) + 1 \in A(v)$. We wish to increase the out-degree of v by one, moving v into a home. To this end, note that there exists $w \in T_v \setminus \{v\}$, since $\deg_D^+(v) + 1 \in A(v)$ implying that $\deg_D^+(v) \neq \deg(v)$. Choose some directed path from w to v in D, and let D' be the orientation obtained from D by reversing the direction of every edge of this path. Note $\deg_{D'}^+(v) = \deg_D^+(v) + 1$ so $v \notin D'_F$ and thus by minimality of $|D_F|$, $w \in D'_F$ and $w \notin D_F$. Since $\deg_{D'}^+(w) = \deg_D^+(w) - 1$, this means that $w \in D_A$. Moreover, $T_v \setminus \{v\} \subseteq D_A$.

Suppose now that there exists some $u \in S_v \cap T_v \setminus \{v\}$. Since $u \in T_v \setminus \{v\}$, we also know that $u \in D_A$. Since $u \in S_v \setminus \{v\}$, we can let D'' be the orientation obtained from D by reversing the direction of each edge in some directed path from v to u. Then $\deg_{D''}^+(v) = \deg_D^+(v) - 1$ and $\deg_{D''}^+(u) = \deg_D^+(u) + 1$. Since $u \in D_A$ this means that $u \in D''_X$ (since all homes between holes have size at least three). We may have $v \in D''_F$ or not, depending on whether the hole is size one or two. But in any case, we have $|D''_F| \leq |D_F|$ and $|D''_X| > |D_X|$, and we obtain a contradiction. Thus, we may assume $S_v \cap T_v = \{v\}$.

Note that $T_v \setminus \{v\}$ contains no sources, since $T_v \setminus \{v\} \subseteq D_A$, and since end-intervals cannot be homes of size one. Thus, by Lemma 2.2 there exists an in-lasso starting at v of type

$(\deg^+_D(v), \deg^+_D(w_1), \deg^+_D(w_2))$ for some $w_1, w_2 \in T_v \setminus \{v\}$ which implies $w_1, w_2 \in D_A$. Let D^* be the orientation obtained from D by flipping the in-lasso. Then the flipped in-lasso is of type $(\deg^+_D(v)+1, \deg^+_D(w_1)-2, \deg^+_D(w_2)+1)$ with respect to D^*. By the assumption of this case, this means that $v \in D^*_A$. Since $w_2 \in D_A$, and each home has size at least 3, we also get that $w_2 \in D^*_X$. We may have $w_1 \in D^*_F$ or not, depending on whether the hole is size one or two. However, in any case, $|D^*_F| \leq |D_F|$ and $|D^*_X| > |D_X|$, and we obtain a contradiction.

Case 2: $\deg^+_D(v)-1 \in B(v)$. We wish to decrease the out-degree of v by one, moving v into a home. To this end, note that there exists $u \in S_v \setminus \{v\}$, since $\deg^+_D(v)-1 \in B(v)$ implying that $\deg^+_D(v) \neq 0$. Choose some directed path from v to u in D, and let D' be the orientation obtained from D by reversing the direction of every edge of this path. Note $\deg^+_{D'}(v) = \deg^+_D(v) - 1$ so $v \notin D'_F$ and thus by minimality of $|D_F|$, $u \in D'_F$ and $u \notin D_F$. Since $\deg^+_{D'}(u) = \deg^+_D(u)+1$, this means that $u \in D_B$. Moreover, $S_v \setminus \{v\} \subseteq D_B$.

Suppose now that there exists some $w \in S_v \cap T_v \setminus \{v\}$. Since $w \in S_v \setminus \{v\}$, we also know that $w \in D_B$. Since $w \in S_v \setminus \{v\}$, we can let D'' be the orientation obtained from D by reversing the direction of each edge in some directed path from v to w. Then $\deg^+_{D''}(v) = \deg^+_D(v) - 1$ and $\deg^+_{D''}(w) = \deg^+_D(w)+1$. Since $w \in D_B$ this means that $w \in D''_X$ (since all holes between homes have size at least three). We may have $v \in D''_F$ or not, depending on whether the hole is size one or two. But in any case, we have $|D''_F| \leq |D_F|$ and $|D''_X| > |D_X|$, and we obtain a contradiction. Thus, we may assume $S_v \cap T_v = \{v\}$.

Note that $S_v \setminus \{v\}$ contains no sinks, since $S_v \setminus \{v\} \subseteq D_B$, and since end-intervals cannot be homes of size one. Thus, by Lemma 2.2 there exists an out-lasso starting at v of type $(\deg^+_D(v), \deg^+_D(u_1), \deg^+_D(u_2))$ for some $u_1, u_2 \in S_v \setminus \{v\}$. Let D^* be the orientation obtained from D by flipping the out-lasso. Then the flipped out-lasso is of type $(\deg^+_D(v)-1, \deg^+_D(u_1)+2, \deg^+_D(u_2)-1)$ with respect to D^*. By the assumption of this case, this means that $v \in D^*_B$. Since $u_2 \in D_B$, and between any two distinct holes is a home of size at least 3, we also get that $u_2 \in D^*_X$. We may have $u_1 \in D^*_F$ or not, depending on whether the hole is size one or two. However, in any case, $|D^*_F| \leq |D_F|$ and $|D^*_X| > |D_X|$, and we obtain a contradiction. $\square$

We can now use Theorem 1.5 (and Theorem 1.7) to validate Conjecture 1.2 for 5- and 6-regular graphs. Our methods work for all cases of the following theorem, although when all holes are of size one, we will just quote the afore-mentioned result of Ma and Lu [4] (namely that Conjecture 1.1 holds when all holes have size 1), for the sake of efficiency. We restate Theorem 1.2 for convenience.

Theorem (1.2). Let G be a d-regular graph with $d \geq 5$, and let $F \subseteq \{0, 1, 2, \dots, d\}$ with $|F| \leq 2$. Then G admits an F-avoiding orientation.

Proof. Let G' be the graph obtained from G by adding a dummy vertex and joining it to every vertex of odd degree in G. There are an even number of these, so G' is Eulerian, and can be decomposed into cycles. By orienting each of these cycles consistently we get an orientation D' of G' where $\deg^+_{D'}(v) = \deg^-_{D'}(v)$ for all $v \in V(G')$. Letting D be the restriction of D' to G we get that $\deg^+_D(v) \in \{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil\}$ for all $v \in V(G)$.

As discussed above, following Ma and Lu [4] we may assume that $F = \{x, x+1\}$ for some $x \in \{0, 1, \dots, d-1\}$. So, if D is not an F-avoiding orientation then $\{x, x+1\} \cap \{\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil\}$ is nonempty.

Suppose first that $x = \lfloor \frac{d}{2} \rfloor$. Then F consists of one hole of size two and two end-homes of sizes $\lfloor \frac{d}{2} \rfloor$ and $\lceil \frac{d}{2} \rceil - 1$, respectively. Since $d \geq 5$ these end-homes are both of size at least two, and hence by Theorem 1.5 G admits an F-avoiding orientation.

Suppose next that $x = \lfloor \frac{d}{2} \rfloor - 1$. Then F consists of one hole of size two and two end-homes of sizes $\lfloor \frac{d}{2} \rfloor - 1$ and $\lceil \frac{d}{2} \rceil$, respectively. If $d \geq 6$ then these end-homes are both of size at least two, and again Theorem 1.5 implies that G admits an F-avoiding orientation. If $d = 5$ then $F = \{1, 2\}$ and we get that G admits an F-avoiding orientation by Theorem 1.7.

We may now assume that $x = \lceil \frac{d}{2} \rceil = \frac{d+1}{2}$ (so d is odd). Then F consists of one hole of size two and two end-homes of sizes $\frac{d+1}{2}$ and $\frac{d-3}{2}$, respectively. If $d \geq 7$ these end-intervals are both of size at least two, and again Theorem 1.5 implies that G admits an F-avoiding orientation. If $d = 5$ then $F = \{3, 4\}$. Apply Theorem 1.7 to get an orientation D^* that is $\{1, 2\}$-avoiding.

Then reverse the direction of every edges in D^*. This new orientation of G is $\{3, 4\}$-avoiding, completing our proof. $\square$

Chapter 3

Total-Coloring Extensions in Planar Graphs with Large Maximum Degree

In this chapter, we present proof of Theorem 1.11, Theorem 1.13, and Theorem 1.16. In all three of these proofs, when we consider an uncolored edge or vertex, we will be counting how many colors are available for it. Such counts are independent of the color palette (eg. $\{1, 2, \dots, k\}$ for a total-k-coloring), and rather just require that each structure starts with a list of k distinct colors. In this way, all our proofs remain valid in the more general setting of *total list-coloring extensions*, in which each vertex and edge is assigned its own list of available colors and a subgraph is precolored from those lists, with the goal of extending the precoloring to the entire graph. In this thesis we have chosen to frame all our results as total coloring extensions, rather than total list-coloring extensions, in order to emphasize readability and present our theorems in what we feel is their most natural setting.

Before presenting the proofs, we give examples showing the tightness in many of the results and our Conjecture 1.5 and Conjecture 1.6.

3.1 Tight examples for total-coloring extensions

Example 3.1. Let G be the tree which is rooted at vertex v with k children $u_1, \dots, u_k$, each themselves with $k-1$ children. Let $W_i = w_1^{(i)}, \dots w_{k-1}^{(i)}$ be the children of u_i. We have $\Delta(G) = k$. We now define a total-precoloring of a subgraph of G as follow. Let $\varphi(u_i) = i$ for all $i \in [k]$, let $\varphi(w_j^{(i)}) = k+1$ for all $i \in [k]$ and $j \in [k-1]$ and let $\varphi(u_i w_j^{(i)}) = i + j \pmod{k}$ (where $0 = k$). Note that φ is a total-k-coloring of k disjoint $K_{1,(k-1)}$'s. Viewing φ as a total-$2k$-coloring (with colors $\{k+1, \dots, 2k\}$ unused) we claim that it does not extend to G. To see this, notice that every edge vu_i and v sees every color in $[k]$ and since vu_i and vu_j are pairwise adjacent for all $i, j \in [k]$, all $k+1$ items must receive a distinct color (see Fig. 3.1).

Example 3.2. Consider the graph G obtained from a star $K_{1,t}$ where each edge is subdivided

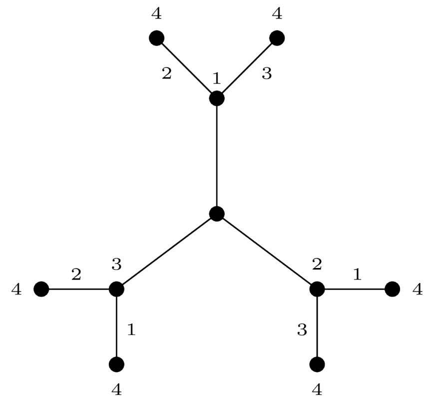


Figure 3.1: Example 3.1 with $k = 3$ showing that the color palette $[2\Delta + 1]$ cannot be reduced in size for guaranteeing an extension of a total-precoloring to total-colorings.

once. Let v be the vertex of degree t with neighbors $x_1, ..., x_t$ of degree 2. Let $y_1, ..., y_t$ be the leaf neighbors of $x_1, ..., x_t$ respectively. Note that $\Delta(G) = t$. Now consider the total-$(t + 2)$-coloring φ of $H = \{x_i y_i : i \in [t]\}$ where $\varphi(x_i) = 1$ for all $i \in [t-1]$ and, $\varphi(x_t) = 2$, $\varphi(x_i y_i) = 2$ for all $i \in [t-1]$ and $\varphi(x_t y_t) = 1$. Finally let $\varphi(y_i) = 3$ for all $i \in [t]$ (note that the coloring of the y_i's can be colored arbitrarily). See Fig. 3.2 for the setup of φ and G. Now notice there are $t + 1$ remaining items (t edges and 1 vertex) to color and they all must be colored with a distinct color. But since 1 and 2 are forbidden for both v and vx_i for all $i \in [t]$, we cannot extend φ to a total-$(t + 2)$-coloring of G.

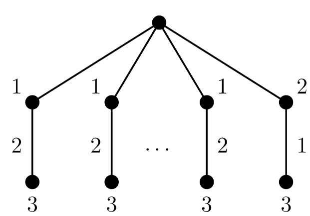


Figure 3.2: Example 3.2 with $t = 4$ which shows a totally precolored distance-2 matching with color palette $[\Delta + 2]$ which cannot extend to a total-$(\Delta + 2)$-coloring

Example 3.3. Let G be the graph obtained by taking $K_{1,4}$ and joining to each leaf, a K_3 (by join we mean add all three edges between the leaf and the K_3). First note that $\Delta(G) = 4$. We precolor the four K_3's identically with colors $[3]$ where $\varphi(v) = \varphi(e)$ iff e is not incident to v (see Fig. 3.3).

We now show that the precoloring φ does not extend to a total-7-coloring of G. Suppose for contradiction that such an extension exists. Then observe, that any leaf of the $K_{1,4}$ along with its incident edges joining the K_3, must receive different colors than [3], say $\{4, 5, 6, 7\}$, since each leaf and incident edge joining the K_3 sees every color in [3]. But this means the edges of the $K_{1,4}$ cannot be colored with $\{4, 5, 6, 7\}$ which is a contradiction since we cannot only use the colors [3] to edge color $K_{1,4}$.

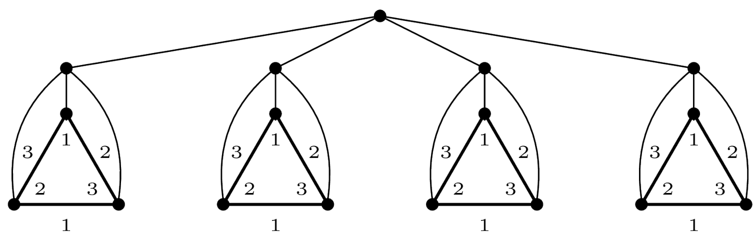


Figure 3.3: Example 3.3 showing a set of distance-3 cliques in a graph G, precolored with color palette $[\Delta + 3]$, that does not extend to a total-$(\Delta + 3)$-coloring of G.

3.2 Extensions from distance-3 cliques

We first introduce some notation and terminology that will be helpful for the rest of the chapter. Given a graph G, we let V_k be the vertices of G with degree k, and we call any $v \in V_k$ a *k-vertex*. We let $V_{[a,b]} = \bigcup_{i=a}^{b} V_i$. Similarly, given a planar embedding of G, we let F_k be the faces of G with length k and we call such $f \in F_k$ a *k-face*.

We prove the following slight strengthening of Theorem 1.16, as discussed in the introduction.

Theorem 3.1. Let $1 \leq t \leq 4$, let G be a planar graph with maximum degree $\Delta \geq 28 - t$, and let H be a subgraph of G. If H is a set of distance-3 cliques, then any $(\Delta + t)$-total-coloring of H can be extended to G.

Proof. Suppose that G is a counterexample for which t is maximum and, subject to that, $|V(G)| + |E(G)|$ is minimum. Then there exists a set H of distance-3 cliques in G such that some total-$(\Delta+t)$-coloring φ_0 of H cannot be extended to G.

We proceed with a series of claims about G and H.

Claim 3.1. If $e \in E(G) \setminus E(H)$, then φ_0 can be extended to $G - e$.

Proof of Claim. Let $G' = G - e$. If $\Delta(G') \geq 28 - t$, then by minimality of G, φ_0 extends to G'. If $\Delta(G') < 28 - t$, then $\Delta(G) = 28 - t$, $\Delta(G') = 28 - (t + 1)$, and we may consider φ_0 as a total-$(\Delta(G') + (t + 1))$-coloring of H. If $t \leq 3$ then by maximality of t, φ_0 extends to total-$(\Delta(G') + (t + 1))$-coloring of G', i.e., a total-$(\Delta + t)$-coloring of G'. If $t = 4$ then we instead apply Theorem 1.15 to get the same conclusion. ⊠

Given $v \in V(G)$, we say v is *high* if $\deg(v) \geq \lceil \frac{\Delta+t}{2} \rceil$ and otherwise v is *low*.

Claim 3.2. Let $uv \in E(G) \setminus E(H)$ such that $u \notin V(H)$ and $\deg(u) \leq \frac{\Delta+t}{2}$. Then $\deg(u) + \deg(v) \geq \Delta + t$. Moreover, every edge in $E(G) \setminus E(H)$ has at least one high endpoint.

Proof of Claim. Note that the second sentence follows from the first, since an edge in $E(G) \setminus E(H)$ cannot have both endpoints in H without violating the distance condition on H. To prove the first sentence, we take u, v as given and consider $G' = G - uv$. By Claim 3.1 we know that φ_0 can be extended to total-$(\Delta+t)$-coloring of G', say φ. We will try to extend φ to G. If $\varphi(u) = \varphi(v)$, then our first step in this extension is to modify φ by recoloring u (which can be considered since $u \notin V(H)$). The recoloring is possible since after removing the color from u, the number of colors seen by u in φ is at most $2(\deg_G(u) - 1) + 1$, which is at most $\Delta + t - 1$ because $\deg(u) \leq \frac{\Delta+t}{2}$. Once we have $\varphi(u) \neq \varphi(v)$, notice that uv sees at most $\deg_G(u) - 1 + \deg_G(v) - 1 + 2 = \deg_G(u) + \deg_G(v)$ colors. If $\deg(u) + \deg(v) < \Delta + t$, then there exists a color for uv and φ_0 can be extended to G, contradiction. Hence $\deg(u) + \deg(v) \geq \Delta + 1$, as desired. ⊠

Claim 3.3. The following hold.

(1) For $t \geq 2$, $V_{[1,t-1]} \subseteq V(H)$.

(2) $V_t \subseteq V(H)$.

(3) If $v \in V_{t+1} \setminus V(H)$, then $N(v) \subseteq V_\Delta$.

Proof of Claim. (1) If $t \geq 2$ and $u \in V_{[1,t-1]} \setminus V(H)$, then any edge uv is in $E(G) \setminus E(H)$. So since $t - 1 \leq \frac{\Delta+t}{2}$, Claim 3.2 tells us that $\deg(v) > \Delta$, contradiction.

(2) Suppose for contradiction there exists $v \in V_t \setminus V(H)$. Let $N(v) = \{u_1, ..., u_t\}$. By Claim 3.2, $u_i \in V_\Delta$ for all $i \in [t]$. Let $G' = G - v$. By Claim 3.1 we know that φ_0 can be extended to total $(\Delta + t)$-coloring of G', say φ. We try to extend φ to G by first coloring $vu_1, vu_2, ..., vu_t$ and finally coloring v. Note that by coloring in this sequence when we come to color edge vu_i the number of forbidden colors is at most $\Delta + i - 1 < \Delta + t$ colors. When we come to v the number of forbidden colors is at most $2\deg(v) = 2t$. Since $t \leq 4$ and $\Delta \geq 24$, we get that $2t < \Delta + t$ and hence that φ can be extended to G, contradiction.

(3) Let $v \in V_{t+1} \setminus V(H)$ and $N(v) = \{u_1, ..., u_{t+1}\}$. By Claim 3.2 we get that $N(v) \subseteq V_{[\Delta-1,\Delta]}$. Suppose for contradiction $u_{t+1} \in V_{\Delta-1}$. By the same argument as in (2), we extend φ_0 to $G - v$; we then attempt to extend to G by first coloring $vu_1, vu_2, ..., vu_t$ and finally coloring v. As in (2), we can color $vu_1, vu_2, ..., vu_t$ greedily. But now we also have at most $\Delta - 1 + t$ colors forbidden when we come to color vu_{t+1}. When we come to v the number of forbidden colors is at most $2\deg_G(v) = 2t + 2 \leq 10$. Since $\Delta \geq 24$, we can indeed extend φ_0 to G, contradiction. $\boxtimes$

Claim 3.4. When $t \in \{1, 2\}$, $|V_{t+1} \setminus V(H)| < |V_\Delta|$.

Proof of Claim. Let $t \in \{1, 2\}$ and let B be the bipartite subgraph G induced by parts $V_{t+1} \setminus V(H)$ and V_Δ. Suppose first that B is acyclic. Then $|E(F)| < |V(F)|$. By Claim 3.3(3), we know that $|E(B)| \geq (t + 1)|V_{t+1} \setminus V(H)| \geq 2|V_{t+1} \setminus V(H)|$. Since $|V(B)| = |V_{t+1} \setminus V(H)| + |V_\Delta|$, this implies that $|V_{t+1} \setminus V(H)| < |V_\Delta|$, as desired.

Suppose now, for a contradiction, that B contains some cycle C. Let $G' = G - E(C)$. We know that φ_0 can be extended to total $(\Delta + t)$-coloring of G', say φ, by Claim 3.1 (extend to all but one edge of C, then remove the colors from the others). We attempt to extend φ to G as follows. We begin by uncoloring each vertex in $V(C) \cap V_{t+1}$ (none of which are in H, by definition of B). The number of colors forbidden for coloring any $uv \in E(G)$ is at most $\deg(u) - 2 + \deg(v) - 2 + 1 \leq \Delta + t - 2$ colors. Thus, there exists two choices for each edge. Since even cycles are 2-edge-choosable, we may color $E(C)$. Now when we go to color some

$v \in V_{t+1} \cap V(C)$ the number of forbidden colors is at most $2(t+1) \leq 6$. Since $\Delta \geq 5, \Delta + t \geq 7$ and thus, we can color v. Hence we have extended φ_0 to G, contradiction. ⊠

We now fix a planar embedding of G for the remainder of the proof. The graph H is made up of vertex-disjoint cliques, and we label these cliques as $H_1, \dots, H_m$. For each H_i, $1 \leq i \leq m$, we define the *configuration of* H_i, denoted $\mathcal{C}(H_i)$, as $G[N[H_i]]$ where for $X \subseteq V(G)$, $N[X] = X \cup N(X)$. Since H is a set of distance-3 cliques, every vertex in G is contained in at most one configuration. If H_i is a k-clique with vertices $v_1, ..., v_k$ and deg$(v_1) \leq ... \leq$ deg(v_k), we say $\mathcal{C}(H_i)$ is a (deg(v_1), ..., deg(v_k))-configuration. We say that a configuration $\mathcal{C}(H_i)$ is *poor* if it contains at most one high vertex. Otherwise we say it is *rich*. As it turns out, there are only 16 poor configurations, each depicted in Fig. 3.4.

Claim 3.5. The only poor configurations are the ones depicted in Fig. 3.4. Moreover, if $\mathcal{C}(H_i)$ is a $(2,2)$-, $(2,3,3)$-, $(3,3,3)$-, $(3,3,4,4)$-, or $(3,4,4,4)$-configuration that is poor, then all the vertices in $V(H_i)$ of degree $|V(H_i)|$ are adjacent to a unique high vertex in $\mathcal{C}(H_i)$.

Proof of Claim. We see from Figure 3.4 that the second sentence follows from the first; we must prove the first sentence of the claim. Let $\mathcal{C}(H_i)$ be a poor configuration, with h the degree of the unique high vertex, if it exists. We start by showing that all of the following properties hold.

(i) The number of high vertices in $V(H_i)$ is at most 1.

(ii) If the number of high vertices in $V(H_i)$ is 1, then all other vertices in $V(H_i)$ must have degree $|V(H_i)| - 1$.

(iii) The degree of any low vertex in $V(H_i)$ is at most $|V(H_i)|$

(iv) All $|V(H_i)|$-vertices in $V(H_i)$ share a common high neighbor.

Property (i) follows from the definition. For (ii), if $v \in V(H_i)$ v low and deg$(v) \geq |V(H_i)|$, then there must exists some $v' \in N(v) \setminus V(H_i)$ which implies that $vv' \notin E(H)$. Thus, by Claim 3.2, v' is high and hence $\mathcal{C}(H_i)$ is rich, contradiction. Similarly, for (iii), if $V(H_i)$ contains a low vertex

v with $\deg(v) \geq |V(H_i)| + 1$, then there must exist two distinct neighbors of v, say v' and v'' with $vv', vv'' \notin E(H)$. Thus, by Claim 3.2 v' and v'' are high and hence $\mathcal{C}(H_i)$ is rich, contradiction. By the same argument again, if $V(H_i)$ contains vertices which have degree $|V(H_i)|$, then they must be adjacent to a shared high vertex as desired in (iv).

First suppose $|V(H_i)| = 1$ with $V(H_i) = \{v\}$. Property (iii) tells us that v must be high (since $G \ncong K_1$). Thus, $\mathcal{C}(H_i)$ is an (h)-configuration, as depicted in the top-left of Figure 3.4.

Next suppose $|V(H_i)| = 2$ with $V(H_i) = \{u, v\}$ and $\deg(u) \leq \deg(v)$. By (i), u must be low and then by (iii), $\deg(u) \leq 2$. If $\deg(u) = 1$, then $\deg(v) \geq 2$ since $G \ncong K_2$. By (iii), this means that when $\deg(u) = 1$, the vertex v is either a 2-vertex or a high vertex, and either way we have a $(1, 2)$- or $(1, h)$-configuration, as depicted in the top center of Figure 3.4. If $\deg(u) = 2$, then by (ii) and (iii), $\deg(v) = 2$ and we have a $(2, 2)$-configuration, as pictured in the top-right of Figure 3.4.

Now suppose $|V(H_i)| = 3$ with $V(H_i) = \{u, v, w\}$ and $2 \leq \deg(u) \leq \deg(v) \leq \deg(w)$. By (i), u and v must be low and hence by (iii) $\deg(u) \leq \deg(v) \leq 3$. If w is high then by (ii), $\deg(u) = \deg(v) = 2$ and if w is low then by (iii), $\deg(w) = 3$ (since $G \ncong K_3$,). Thus we have either a $(2, 2, h)$-, $(2, 2, 3)$-, $(2, 3, 3)$- or a $(3, 3, 3)$-configuration, as pictured in the second row of Figure 3.4.

Finally, suppose $|V(H_i)| = 4$ with $V(H_i) = \{u, v, w, x\}$ and $3 \leq \deg(u) \leq \deg(v) \leq \deg(w) \leq \deg(x)$. By (i), u, v, w are low and hence by (iii), $3 \leq \deg(u) \leq \deg(v) \leq \deg(w) \leq 4$. If x is high, then by (ii), $\deg(u) = \deg(v) = \deg(w) = 3$ and we have a $(3, 3, 3, h)$-configuration as depicted in the bottom row of Figure 3.4. If x is low, then by (iii), and since $G \ncong K_4$, $\deg(x) = 4$. We cannot have a $(4, 4, 4, 4)$-configuration since by (iv), all 4-vertices would share a common neighbor which would mean $G \cong K_5$, contradiction. Thus we have either a $(3, 3, 3, 4)$- $(3, 3, 4, 4)$-, $(3, 4, 4, 4)$-configuration, as depicted in the bottom row of Figure 3.4. $\boxtimes$

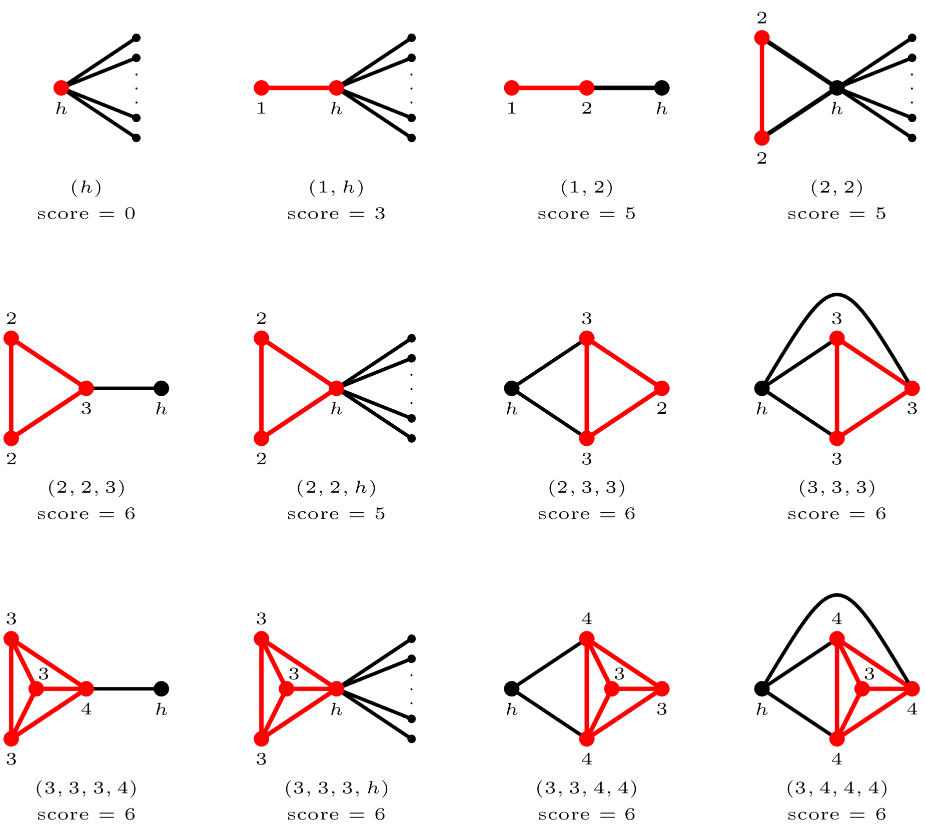


Figure 3.4: All possible poor configurations are depicted (with black edges) along with their corresponding cliques (with red edges and vertices). Vertex degrees are indicated for all the clique vertices, and h indicates the degree of the high vertex, which exists in each case.

We define a *score* for each H_i, $1 \le i \le m$, as follows:

$$s(H_i) = \sum_{v \in V_{[1,3]} \cap V(H_i)} 4 - \deg(v) + |\{f \in F_3 \,:\, f \text{ contains at least one precolored edge from } H_i\}|.$$

Note that the score of each H_i is related to the amount of charge that it will somehow need in our discharging argument, which is still to come. The following claim establishes an upper bound for the score of every H_i.

Claim 3.6. For any H_i, $s(H_i) \le 6$ and moreover if $\mathcal{C}(H_i)$ is poor, then

$$s(H_i) \le \begin{cases} 0 & \text{if } |V(H_i)| = 1 \\ 5 & \text{if } |V(H_i)| = 2 \\ 6 & \text{if } |V(H_i)| \ge 3 \end{cases}$$

Proof of Claim. By Claim 3.5, the moreover statement can be easily verified by checking the configurations in Fig. 3.4. We now show that for any configuration $\mathcal{C}(H_i)$, $s(H_i) \le 6$.

First suppose $|V(H_i)| = 1$ with $V(H_i) = \{v\}$. Then the only thing contributing to $s(H_i)$ is the $4 - \deg(v)$ if $v \in V_{[1,3]}$. Thus, $s(H_i) \le 3$.

Next suppose $|V(H_i)| = 2$ with $V(H_i) = \{u, v\}$. If either u or v is a 1-vertex (note both cannot be) then there is no 3-face incident to the precolored edge uv and hence $s(H_i) \le 3+2 = 5$. If neither u nor v is a 1-vertex, then there are at most two 3-faces incident to the precolored edge uv and thus, $s(H_i) \le 2 + 2 + 2 = 6$.

Next suppose $|V(H_i)| = 3$ with $V(H_i) = \{u, v, w\}$. If two of these vertices are 2-vertices, then the only possible 3-face incident with uv, uw, or vw is the 3-clique itself, and thus, $s(H_i) \le 2 + 2 + 1 + 1 = 6$. If exactly one vertex in $\{u, v, w\}$ is a 2-vertex, say without loss of generality u, then there are at most two 3-faces incident to an edge in uv, uw, vw, namely the 3-clique itself and possibly a 3-face incident with vw. Thus, $s(H_i) \le 2 + 1 + 1 + 2 = 6$. We may now assume no vertex in $\{u, v, w\}$ is a 2-vertex. If all three are 3-vertices, then since $G \ncong K_4$, there are at most three 3-faces incident to the edges uv, uw, vw and thus, $s(H_i) \le 1 + 1 + 1 + 3 = 6$. Finally suppose there is at least one vertex in $\{u, v, w\}$ with degree at least 4, then there are at most four 3-faces incident to an edge in uv, uw, vw, and $s(H_i) \le 1 + 1 + 4 = 6$.

Finally suppose $|V(H_i)| = 4$, with vertices u, v, w, x. Since $G \ncong K_4$, at least one vertex must have degree at least 4. If k vertices in $\{u, v, w, x\}$ are 3-vertices for some $k \in \{0, 1, 2, 3\}$, then there are at most $6 - k$ 3-faces incident to an edge in uv, uw, ux, vw, vx, wx and thus, $s(H_i) \le k + (6 - k) = 6$. ⊠

Consider now a poor configuration $\mathcal{C}(H_i)$ with $|V(H_i)| \geq 2$. We know that the configuration has a high vertex, say v, that it is a cut-vertex in G, and moreover, that $H_i - v$ is a component of $G - v$ (see Figure 3.4). Thus, $V(H_i) \setminus \{v\}$ lies in some unique face f' of $G - (V(H_i) \setminus \{v\})$. We say $f \in F(G)$ is a *helpful face with respect to H_i* if f can be obtained from by extending f' to include at least one edge of H_i. It is clear that every H_i has a unique helpful face, but a face can be helpful with respect to multiple cliques H_i (see Figure 3.5). Even so, in the following claim we show that the length $\ell(f)$ of such a face f is so long that it is still going to be useful in our upcoming discharging argument.

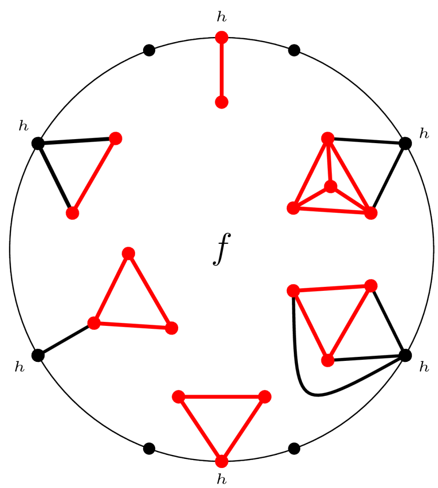


Figure 3.5: An example of a face f which is the helpful face to 6 different poor configurations.

Claim 3.7. Let $f \in F(G)$ and suppose f is the helpful face of $x_2 + x_3$ different poor configurations, with x_2 being the number of these poor-configurations $\mathcal{C}(H_i)$ with $|V(H_i)| = 2$, and x_3 being the number of these poor configurations $\mathcal{C}(H_j)$ with $|V(H_j)| \geq 3$. Then

$$\ell(f) \geq 4(x_2 + x_3)$$

and moreover if $x_2 + x_3 = 1$, then

$$\ell(f) \geq \begin{cases} 5 & \text{if } x_2 = 1 \text{ and } x_3 = 0 \\ 6 & \text{if } x_2 = 0 \text{ and } x_3 = 1. \end{cases}$$

Proof of Claim. Let W be the boundary walk of f. If W does not contain cycle then clearly $\ell(f) \geq 4(x_1 + x_2)$ since each poor configuration has at least 2 edges contained in W which are each counted twice in a face without a cycle. Now suppose W contains an cycle $C = v_1, ..., v_k$. Note that $k \geq \max\{x_2 + x_3, 3\}$.

For each configuration counted by x_3, note there is an additional length of at least 3 in W which is not accounted for in C. So overall, we know that $\ell(f) \geq k + 3x_3$.

For each configuration counted in x_2, there is an additional length of at least 2 in W that not counted by C. However, looking at Figure 3.4 we see that equality is only achieved when the poor configuration is a $(1, h)$-configuration and the high vertex (which is precolored) lies on C (see the top configuration of Fig. 3.5). In this case however, if v_i is the high vertex of the configuration, then since precolored cliques are at a distance of 3, we get that v_{i-1} and v_{i+1} are not precolored and hence not the high vertex of a poor configuration. So in fact C itself must have length at least one longer to compensate for each such configuration, and the configuration still adds at least 3 to $\ell(f)$.

We have now seen that in general, we have $\ell(f) \geq (x_1 + x_2) + 3x_1 + 3x_2 = 4(x_1 + x_2)$. If $x_1 + x_2 = 1$, then $\ell(f) \geq 3 + 2x_2 = 5$ when $x_2 = 1, x_3 = 0$ and $\ell(f) \geq 3 + 3x_2 = 6$ when $x_2 = 0, x_3 = 1$. ⊠

We are now finally ready to define an initial charge μ on $x \in V(G) \cup F(G)$, on every configuration $\mathcal{C}(H_i)$ in G, and on a *global pot* $\mathcal{P}$. Initially, let $\mu(\mathcal{C}(H_i)) = \mu(\mathcal{P}) = 0$ for every configuration in G and let

$$\mu(x)=\begin{cases}\deg(x)-4 & \text{if } x\in V(G)\\ \ell(x)-4 & \text{if } x\in F(G)\end{cases}$$

By Euler's formula we see that

$$\mu(G)=\sum_{H_i\in H}\mu(\mathcal{C}(H_i))+\sum_{v\in V(G)}\mu(v)+\sum_{f\in F(G)}\mu(f)+\mu(\mathcal{P})=4|E(G)|-4|V(G)|-4|F(G)|=-8.$$

The items in $V(G)\cup F(G)$ which receive a negative initial charge are 3-faces, and vertices in $V_{[1,3]}$. We now present the discharging rules, which we perform in order.

Discharging rules

(R1) Each rich configuration $\mathcal{C}(H_i)$ receives $\frac{s(H_i)}{2}$ charge from each high vertex in $\mathcal{C}(T)$.

(R2) Each poor configuration $\mathcal{C}(H_i)$ with $|V(H_i)|=2$ receives charge 1 from its helpful face and receives $s(H_i)-1$ from its high vertex. Each poor configuration $\mathcal{C}(H_i)$ with $|V(H_i)|\geq 3$ receives charge 2 from its helpful face and receives $s(H_i)-2$ from its high vertex.

(R3) Each $v\in V_{[1,3]}\cap V(H)$ receives charge $4-\deg(v)$ from its corresponding configuration.

(R4) Each face $f\in F_3$ with at least one edge in $E(H)$ receives charge 1 from its corresponding configuration.

(R5) Each face $f\in F_3$ with no edges in $E(H)$ receives charge $\frac{1}{2}$ from its incident high vertices.

(R6) When $t\leq 2$, each vertex $u\in V_\Delta$ gives charge $4-\deg(v)$ to $\mathcal{P}$ and each vertex $v\in V_{t+1}\setminus V(H)$ receives charge $4-\deg(v)$ from $\mathcal{P}$.

(R7) When $t=1$, Each vertex $v\in V_3\setminus V(H)$ receives charge $\frac{1}{3}$ from each neighbor.

Let μ' be the final charge after (R1)-(R7) have been applied. We'll prove $\mu' \geq 0$, contradicting $\mu'(G) = \mu(G) = -8$ and completing the proof. The rest of the proof consists of one claim each for $\mathcal{P}$, faces, configurations and vertices, respectively.

Claim 3.8. $\mathcal{P}$ has nonnegative final charge.

Proof of Claim. The only rule affecting $\mathcal{P}$ is (R6). By Claim 3.4, $\mu'(\mathcal{P}) \geq \mu(\mathcal{P}) + |V_{[\Delta-1,\Delta]}| - |V_{t+1} \setminus V(H)| > 0$. ⊠

Claim 3.9. Every face has nonnegative final charge.

Proof of Claim. Let $f \in F(G)$. If f is neither a 3-face, nor is the helpful face for any configuration, then f is not affected by the discharging rules and $\mu'(f) = \mu(f) = \ell(f) - 4 \geq 4 - 4 = 0$.

Now suppose $f \in F_3$. Since any helpful face must have length at least 5 by Claim 3.7, f is only affected by one of (R4), (R5). In the former case, $\mu'(f) = \mu(f) + 1 = 3 - 4 + 1 = 0$. In the latter case, then by Claim 3.2, f must be incident to at least 2 high vertices. Thus, by (R5) $\mu'(f) = \mu(f) + 2 \cdot \frac{1}{2} = 3 - 4 + 1 = 0$.

Finally suppose f is the helpful face for $x_2 + x_3$ poor configurations, with x_2, x_3 as in the statement of Claim 3.7. By (R2), $\mu'(f) = \mu(f) - x_1 - 2x_2 = \ell(f) - 4 - x_1 - 2x_2$. If $x_2 + x_3 \geq 2$, then by Claim 3.7, $\mu'(f) \geq 4x_2 + 4x_3 - 4 - x_2 - 2x_3 = 3x_2 + 2x_3 - 4 \geq 0$. If $x_2 + x_3 = 1$ then by Claim 3.7 $\mu'(f) \geq 5 - 4 - 1 = 0$ when $x_2 = 1, x_3 = 0$ and $\mu'(f) \geq 6 - 4 - 2 = 0$ when $x_2 = 0, x_3 = 1$. In all cases, $\mu'(f) \geq 0$. ⊠

Claim 3.10. Each configuration has nonnegative final charge.

Proof of Claim. Let $\mathcal{C}(H_i)$ be a configuration. By (R1) and (R2) $\mathcal{C}(H_i)$ receives $s(H_i)$ charge. Then by (R3) and (R4) $\mathcal{C}(H_i)$ gives $2|V_2 \cap V(H_i)| + |V_3 \cap V(H_i)| + |\{f \in F_3 : E(f) \cap E(H) \neq \emptyset\}| = s(H_i)$. Thus $\mu'(\mathcal{C}(H_i)) = \mu(\mathcal{C}(H_i)) + s(H_i) - s(H_i) = 0$. ⊠

Claim 3.11. Every vertex has nonnegative final charge.

Proof of Claim. Let $v \in V(G)$. First suppose $v \in V_1$ or $v \in V_{[2,3]} \cap V(H)$. Then by Claim 3.3 $v \in V(H)$, and thus, it is contained in a configuration. By (R3), $\mu'(v) = \mu(v) + 4 - \deg(v) = \deg(v) - 4 + 4 - \deg(v) = 0$. Next suppose $v \in V_{[2,3]} \setminus V(H)$. Then by Claim 3.3 we know $t \leq 2$. If $v \in V_{t+1}$, then by (R6), $\mu'(v) = \mu(v) + 4 - \deg(v) = \deg(v) - 4 + 4 - \deg(v) = 0$. If $v \in V_{t+2}$, then $t = 1$ and $v \in V_3$. By (R7) $\mu'(v) = \mu(v) + \frac{1}{3}\deg(v) = -1 + 1 = 0$. We may now assume $\deg(v) \geq 4$.

If v is low then it is not affected by any of the discharging rules and thus, $\mu'(v) = \mu(v) = \deg(v) - 4 \geq 0$. We may therefore assume that v is high. Note that since $\Delta \geq 28 - t$, $\deg(v) \geq \lceil \frac{\Delta+t}{2} \rceil \geq 14$ which will turn out to be exactly what we need.

Since there are extra rules for vertices in V_Δ when $t = 2$ and for $V_{[\Delta-2,\Delta]}$ when $t = 1$, we take care of those separately later. So suppose v is high but if $t = 2$ then $v \notin V_\Delta$ and if $t = 1$ $v \notin V_{[\Delta-2,\Delta]}$.

Suppose first that v is in a poor configuration $\mathcal{C}(H_i)$. If $|V(H_i)| = 1$, then by (R5), $\mu'(v) \geq \mu(v) - \frac{1}{2}\deg(v) = \frac{1}{2}\deg(v) - 4$. Thus, $\mu'(v) \geq 0$ because $\deg(v) \geq 8$. If $|V(H_i)| = 2$ then v gives charge $s(H_i) - 1$ to its poor configuration by (R2). The only other rule which may affect v is (R5) where it gives charge $\frac{1}{2}$ to any incident 3-face with no edges in $E(H)$. Since v is a cut vertex in G, the number of incident faces to v is at most $\deg(v) - 1$. Moreover, at least one of these faces cannot be a 3-face with no edges in $E(H)$ (since there is at least one face incident to v and H_i). Thus, $\mu'(v) \geq \mu(v) - (s(H_i) - 1 + \frac{1}{2}(\deg(v) - 2))$. By Claim 3.6, $\mu'(v) \geq \deg(v) - 4 - (5 - 1 + \frac{1}{2}(\deg(v) - 2)) = \frac{1}{2}\deg(v) - 7$. Thus, $\mu'(v) \geq 0$ because $\deg(v) \geq 14$. Similarly, if $|V(H_i)| \geq 3$, then by (R2) and (R5), $\mu'(v) \geq \mu(v) - (s(H_i) - 2 + \frac{1}{2}(\deg(v) - 2))$. By Claim 3.6, $\mu'(v) \geq \deg(v) - 4 - (6 - 2 + \frac{1}{2}(\deg(v) - 2)) = \frac{1}{2}\deg(v) - 7$ and hence $\mu'(v) \geq 0$ because $\deg(v) \geq 14$.

Suppose now that v is in a rich configuration or v is in no configuration at all (and again if $t = 2$, then $v \notin V_\Delta$ and if $t = 1$ then $v \notin V_{[\Delta-2,\Delta]}$). Then, by (R1) and (R5), $\mu'(v) \geq \mu(v) - (\frac{s(H_i)}{2} + \frac{1}{2}\deg(v))$. By Claim 3.6, $\mu'(v) \geq \deg(v) - 4 - (3 + \frac{1}{2}\deg(v)) = \frac{1}{2}\deg(v) - 7$. Since $\deg(v) \geq 14$ we get $\mu'(v) \geq 0$, as desired.

Finally we consider the cases when $t = 2$ and V_Δ or $t = 1$ and $v \in V_{[\Delta-2,\Delta]}$. First suppose $t = 2$ and $v \in V_\Delta$. The only difference between the previous counts is that v also gives charge 1 by (R6). Thus, $\mu'(v) \geq \frac{1}{2}\deg(v) - 7 - 1 = \frac{1}{2}\Delta - 8 \geq 0$ since $\Delta \geq 16$. Lastly suppose $t = 1$ and $v \in V_{[\Delta-2,\Delta]}$. If v has two 3-neighbors $u, u' \notin V(H)$, then $uu' \notin E(G)$ and hence there cannot be a 3-face incident to v and u and u'. Thus, if v is adjacent to s vertices in $V_3 \setminus V(H)$ and incident to s' 3-faces which contain no precolored edge, then $s \leq \deg(v), s' \leq \deg(v)$, and $s + s' \leq \frac{3}{2}\deg(v)$. By (R7) v has to give charge $\frac{1}{3}$ to each 3-neighbor in $V_3 \setminus V(H)$ and as mentioned before, v gives charge $\frac{1}{2}$ to each 3-face with no precolored edge. Also, if $v \in V_\Delta$ then by (R6) v gives charge 2 to $\mathcal{P}$ instead of charge 1 seen before. Thus, we can bound the previous count of final charge by subtracting an additional 1 for (R6) if $\deg(v) = \Delta$ and an additional $\frac{1}{3} \cdot (\frac{1}{2}(\deg(v)))$ if $v \in V_{[\Delta-2,\Delta]}$. If $v \in V_{[\Delta-2,\Delta-1]}$ we see that $\mu'(v) \geq \frac{1}{2}\deg(v) - 8 - \frac{1}{3}(\frac{1}{2}\deg(v)) = \frac{1}{3}\deg(v) - 8 = \frac{1}{3}(\Delta - 2) - 8$. thus, $\mu'(v) \geq 0$ since $\Delta \geq 26$. If $v \in V_\Delta$, then we have $\mu'(v) \geq \frac{1}{3}\deg(v) - 9 = \frac{1}{3}\Delta - 9 \geq 0$ since $\Delta \geq 27$. $\boxtimes$

$\square$

3.3 Extension from subgraphs with bounded maximum degree

Before proving Theorem 1.13 in general we prove it when G is planar and bipartite; in fact we'll prove a stronger result in this special case.

Theorem 3.2 (Alon and Tarsi). Every planar bipartite graph is 3-choosable.

Theorem 3.3 (Borodin, Kostochka, Woodall). Let G be a bipartite graph and let L be an edge list assignment on G. If $|L(xy)| \geq \max\{\deg(x), \deg(y)\}$ for every edge $xy \in E(G)$, then G is L-edge-colorable.

Lemma 3.1. Let G be a planar bipartite graph and let L be an total list assignment on G. If $|L(v)| \geq 3$ for all $v \in V(G)$ and $|L(xy)| \geq \max\{\deg(x) + 2, \deg(y) + 2\}$ for every edge $xy \in E(G)$, then G is L-totally-colorable.

Proof. First give an L-coloring of the vertices of G, which is possible by Theorem 3.2. For every edge e, modify $L(e)$ to $L'(e)$ by removing the colors of the incident vertices, if they appear. So $|L'(e)| \geq |L(e)| - 2$. But then for $e = xy$, $|L(xy)| \geq \max\{\deg(x), \deg(y)\}$. Hence by Theorem 3.3, the edges of G can all be L'-colored, giving an L-total-coloring of G. □

Theorem 3.4. Let G be a planar bipartite graph, and let H be a subgraph of G. If H has maximum degree at most d, then any $(\Delta + d + 4)$-total-coloring of H in G can be extended to G.

Proof. Let $G' = G - E(H)$. For each edge e and vertex v in G', let $L'(e), L'(v)$ be obtained from $[\Delta + d + 4]$ by removing all colors used on the vertices and edges of H incident and adjacent to e and v. Let xy be an arbitrary edge of G'. Note that $|L'(xy)| \geq \Delta + d + 4 - \deg_H(x) - \deg_H(y) - 2$. This means that

$$|L'(xy)| \geq \max\{\Delta + 2 - \deg_H(x), \Delta + 2 - \deg_H(y)\} = \max\{\deg_{G'}(x) + 2, \deg_{G'}(y) + 2\}.$$

Now let v be an arbitrary vertex of G' that is not colored under H. Then

$$|L'(v)| \geq \Delta + d + 4 - 2\deg_H(v) \geq 4.$$

Our result now follows by Lemma 3.1. □

We now prove Theorem 1.13 which we restate here for convenience.

Theorem (1.13). Let G be a planar graph with maximum degree Δ and let H be a subgraph of G. If H has maximum degree d, then any $(\Delta + d + 6)$-total-coloring of H in G can be extended to G.

Proof. If $H = \emptyset$ then since planar graphs are 4-colorable and $(\Delta + 1)$-edge-colorable, we trivially get a total-$(\Delta + 5)$-coloring of G.

For any fixed Δ and d, we choose a counterexample (G, H) where the quantity

$$q(G) = 3|E(G)| + |V_{[2,d+5]}|$$

is as small as possible. Throughout the proof, we let φ be the total-$(\Delta + d + 6)$-coloring of H. Our goal is to arrive at a contradiction by extending φ to G.

Claim 3.12. $V_{[2,d+5]} = \varnothing$.

Proof of Claim. Suppose not, and take $v \in V_{[2,d+5]}$. If $v \notin V(H)$, we first color v with any available color and add v to H. Since v sees at most $\deg(v)$ colors, such a choice is possible. Note, adding v to H does not increase the maximum degree of H since no edges are added to H. We may now assume $v \in V(H)$.

We now form G' from G by deleting v, and attaching a leaf v_u to each $u \in N_G(v)$. By construction, $|E(G)| = |E(G')|$ and thus, $q(G') < q(G)$ since the vertex v in $V_{[2,d+5]}$ has been replaced by $\deg(v)$ vertices of degree 1. For any edge $uv \in E(H)$ we assign $\varphi(uv_u) = \varphi(uv)$ and for all $u \in N_G(v)$, we assign $\varphi(v_u) = \varphi(v)$. Now by minimum counterexample, we can extend φ to a total-$(\Delta + d + 6)$-coloring of G'. Since $\varphi(v_u) = \varphi(v)$ for all $u \in N(v)$, we would like to obtain a total-$(\Delta + d + 6)$-coloring of G by merging all such v_u back into one vertex v. Note that the resulting coloring may not be proper if $\varphi(uv_u) = \varphi(u'v_{u'})$ for two $u, u' \in N(v)$. If this occurs, we know at most one of uv or $u'v$ can be in $E(H)$ since H is a proper total-coloring. For any such edge uv which is *not* in $E(H)$, we can recolor uv with a different available color. Since uv sees at most $\deg(u) - 1 + 1 + \deg(v) - 1 + 1 \leq \Delta + d < \Delta + d + 6$ colors, there must be an available color for each such edge. Thus, there exists a total-$(\Delta + d + 6)$-coloring extending H to G, contradicting that G was a counterexample. $\boxtimes$

Claim 3.13. Every vertex v is adjacent to at most d leaves.

Proof of Claim. Let u be a vertex adjacent to a leaf. It suffices to show that $uv \in E(H)$ for any leaf $v \in N(u)$ since then H having maximum degree d implies the desired claim. If $v \notin V(H)$, then we can color v with any color it does not see (which is at most 1 color). Thus, we may assume $v \in V(H)$. If $u \notin V(H)$, then we can color u with any color it does not see (which is at most Δ colors) and color uv with any color it does not see (which is at most $\Delta + 1$ colors). Thus, we may assume that $u, v \in V(H)$ and since H induces a proper vertex coloring, $\varphi(v) \neq \varphi(u)$.

Now obtain G' from G by removing v. Since $E(G') = E(G) - 1$ and $|V_{[2,d+5]}|$ can only increase by at most 1 in G', we see that $q(G') < q(G)$ and by minimum counterexample we get an extension of φ to a total-$(\Delta + d + 6)$-coloring of G'. Since $\varphi(u)$ is precolored with some color distinct from the precoloring of v, we can now add back in v with $\varphi(v)$ and greedily color uv with any available color. Note that this is always possible since uv sees at most $\Delta + 1$ colors and $\Delta + d + 6 > \Delta + 1$. Thus, we have a total-$(\Delta + d + 6)$-coloring extending H to G, contradicting that G was a counterexample. ⊠

For $i \geq 0$, let $\tilde{F}_i$ be the faces of G with exactly i incident vertices of degree at least 3.

Claim 3.14. $\tilde{F}_i = \varnothing$ for $i = 0, 1, 2$.

Proof of Claim. Suppose not for some $i \in \{0, 1, 2\}$. Suppose first that the boundary of f contains no cycle. This means that G is a forest, and f is its one face. In particular, G is bipartite. But then we may apply Theorem 3.4 and we are done. So we may assume that the boundary of f contains a cycle. But then this boundary contains at least three vertices of degree at least three, since $V_2 = \varnothing$ by Claim 3.12, contradiction. ⊠

We now introduce a discharging argument. We first define an initial charge μ of $x \in V(G) \cup F(G)$. Initially let

$$\mu(x) = \begin{cases} 3\deg(x) - 6 & \text{if } x \in V(G) \\ -6 & \text{if } x \in F(G) \end{cases}$$

By Euler's formula we see that

$$\mu(G) := \sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = 6|E(G)| - 6|V(G)| - 6|F(G)| = -12.$$

Note leaves are the only vertices to start with a negative charge. We now present the discharging rules.

Discharging rules

(S1) For each m, every face $f \in \tilde{F}_m$ receives charge $\frac{6}{m}$ from each vertex of degree at least 3 on its boundary.

(S2) Every vertex $v \in V_1$ receives charge 3 from its neighbor.

Let $\mu'(x)$ be the final charge of $x \in V(G) \cup F(G)$ after (S1) and (S2) have been applied, and $\mu'(G)$ be the sum of all these final charges in G. Then $\mu'(G) = \mu(G) = -12$. We complete our proof by showing that $\mu'(G) \geq 0$, which is a contradiction.

First consider a face f. By Claim 3.14, $f \in F_m$ for $m \geq 3$. So by (S1) (the only rule affecting f), $\mu'(f) = \mu(f) + m(\frac{6}{m}) = -6 + 6 = 0$.

It remains only to consider the final charge of an arbitrary vertex v. If $v \in V_1$, then by (S2), we get $\mu'(v) = \mu(v) + 3 = -3 + 3 = 0$. By Claim 3.12, we may now assume that $\deg_G(v) \geq d + 6$.

Suppose v lies on the boundary of x distinct faces and is incident to y leaves. We know that x is no more than $\deg_G(v) - y$, so $x + y \leq \deg_G(v)$. We also know that $y \leq d$ by Claim 3.13. By doubling the first inequality and adding the result to the second inequality we get

$$2x + 3y \leq 2\deg_G(v) + d. \tag{3.1}$$

Since $\tilde{F}_i = \varnothing$ for any $0 \leq i \leq 2$ by Claim 3.14, each of the x distinct faces incident to v has at least 3 vertices of degree at least 3 on their boundary. This means that each of these x faces takes charge at most 2 from v, according to (S1). Each of the y leaves incident to v take exactly 3 from v, according to (S2). Hence by inequality (3.1),

$$\mu'(v) = 3\deg_G(v) - 6 - (2x + 3y) \geq \deg_G(v) - 6 - d \geq (d + 6) - d \geq 0. \tag{3.2}$$

Thus, every vertex and face has nonnegative final charge, a contradiction which completes the proof. $\square$

In the case when H is a totally precolored matching ($d = 1$) we get the following corollary of Theorem 1.13

Corollary 3.1. Let G be a planar graph with maximum degree Δ and let H be a subgraph of G. If H is a matching then any total-$(\Delta + 7)$-coloring of H in G can be extended to G.

3.4 Extensions from matchings with no distance condition

We reduce the color palette in Corollary 3.1 from $(\Delta + 7)$ to $(\Delta + 3)$ with the following theorem.

Theorem 3.5. Let $3 \leq t \leq 6$, let G be a planar graph with maximum degree $\Delta \geq 31 - t$, and let H be a subgraph of G. If H is a subgraph with maximum degree one, then any total-$(\Delta + t)$-coloring of H in G can be extended to G.

Note that setting $t = 1$ in the above statement gives a slightly strengthened version of Theorem 1.11 (since here we allow H to also contain isolates).

Proof. Suppose that G is a counterexample for which t is maximum and, subject to that, the quantity $q(G) = 3|E(G)| + |V_{[2,t-1]}|$ is minimum. Then there exists a subgraph H with maximum degree one such that some total-$(\Delta + t)$-coloring φ_0 of H in G cannot be extended to G.

Consider the following claim, whose statements are very similar to claims we have already established within other proofs in this chapter; the previous claims are listed in brackets below. Here, as in our previous work, we define a vertex v to be *high* if $\deg_G(v) \geq \lceil \frac{\Delta+t}{2} \rceil$.

Claim 3.15. The following hold.

1. If $e \in E(G) \setminus E(H)$, then φ_0 can be extended to $G - e$. (Claim 3.1 of Theorem 3.1)

2. Let $uv \in E(G) \setminus E(H)$ such that $u \notin V(H)$ and u is low. Then $\deg(u) + \deg(v) \geq \Delta + t$. Moreover, every edge $uv \in E(G) \setminus E(H)$ has at least one high endpoint. (Claim 3.2 of Theorem 3.1)

3. $V_1 \subseteq V(H)$ and for any $v \in V_{[t,4]}$, $N(v) \subseteq V_{[\Delta-4+t,\Delta]}$ (Claim 3.3(1) and (3) of Theorem 3.1)

4. $V_{[2,t-1]} = \varnothing$ (Claim 3.12 of Theorem 1.13).

To establish Claim 3.15(1), we can use an essentially identical proof to the one indicated in the bracket. To see this, note first that if G' is a proper subgraph of G, then $q(G') < q(G)$ (since any edges removed create at most 2 new vertices in $V_{[2,t-1]}$), so we may apply minimality. When t achieves its upper bound in the other proof, we applied Theorem 1.15, but here we can use Corollary 3.1 instead. To establish Claim 3.15(2), the proof of the bracketed claim works verbatim, except that instead of relying on "Claim 1" there, we instead rely on the analog we just established in Claim 3.15(1). Similarly, the fact that $V_1 \subseteq V(H)$ for Claim 3.15(3) is verbatim from Claim 3.3(1), except we replace "Claim 2" there by our analog in Claim 3.15(2). The second part of Claim 3.15(3) is only relevant when $t \in \{3,4\}$. For $t = 3$ the result follows directly from Claim 3.15(2). For $t = 4$ we follow the same argument as Claim 3.3(3), replacing "Claim 1" and "Claim 2" as above. Finally, to establish Claim 3.15 (4), we look to the proof of the bracketed result, where $\Delta + d + 5$ is replaced by $\Delta + t - 1$ here, and we note that $t - 1 \geq \delta(H) + 1$.

Our next claim is similar to Claim 4 in the proof of Theorem 3.1, but with enough differences that we write a full argument here for clarity.

Claim 3.16. If $t \in \{3,4\}$, then $|V_{[t,4]}| < |V_{[\Delta-4+t,\Delta]}|$

Proof of Claim. Let B be the bipartite graph induced by parts $V_{[t,4]}$ and $V_{[\Delta-4+t,\Delta]}$ in $G - E(H)$. Note that by Claim 3.15(3) all but at most one edge incident to $v \in V_{[t,4]}$ is in $E(B)$. If B is acyclic, then $2|V_{[t,4]}| \leq |E(B)| < |V(B)| = |V_{[t,4]}| + |V_{[\Delta-4+t,\Delta]}|$ as desired.

Now suppose for contradiction that B contains a cycle C. The edges of C are all uncolored, but vertices on C can be precolored. If $v \in V(C)$ and $v \notin V(H)$, then note that v sees at most Δ other colors, since no edge incident to v is precolored (precolored edges have both ends precolored as well). So the precoloring φ_0 of H can be extended to $H' = H \cup V(C)$. Moreover, H' still has maximum degree one, since it is obtained from H by adding isolates. So if we let

$G' = G \setminus E(C)$, we can use Claim 3.15(1) to get a total-$(\Delta + t)$-coloring φ extending φ_0 to G'. It remains then only to color the edges of C.

Let $uv \in E(C)$, with $u \in V_{[t,4]}, v \in V_{[\Delta-4+t,\Delta]}$. If $t = 3$, the edge uv sees at most $1 + \Delta - 2 + 2 = \Delta + 1$ colors in φ (one edge incident to u, $\Delta - 2$ edges incident to v, and the two vertices u, v themselves). Since $\Delta + t = \Delta + 3$ here, there are at least 2 color choices for each such edge, so since even cycles are 2-edge-choosable, we can extend φ to G, contradiction. We can make a similar argument when $t = 4$. In that case, each edge sees at most $2 + \Delta - 2 + 2 = \Delta + 2$ colors in φ. So, since $\Delta + t = \Delta + 4$, there are at least 2 color choices for each such edge, and the 2-edge-choosability of C means we can extend φ to G, contradiction. ⊠

We say a leaf in G is a *high-leaf* (*low-leaf*) if its neighbor is a high (low) vertex. Given a high vertex $v \in V(G) \setminus V(H)$, we say a face f is *needy with respect to v* if v is incident to f and either of the following is true:

1. f is a 3-face and v is its only incident high vertex, or;
2. f is a face with at least two incident low-leaves u, u' with neighbors w, w', respectively, such that w, w' are neighbors of v on f.

We say that a needy face is either *type 1* or *type 2* according to whether it satisfies (1) or (2), respectively See Figure 3.6. Given a high vertex $v \in V(G) \setminus V(H)$, let $\eta(v)$ denote the number of faces that are needy with respect to v.

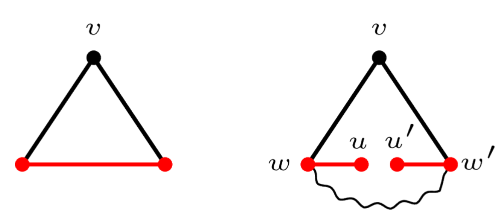


Figure 3.6: Needy faces with respect to v of type 1 (left) and type 2 (right). Precolored vertices and edges indicated in red.

Claim 3.17. Let $v \in V(G) \setminus V(H)$ be a high vertex. Then $\eta(v) \leq \frac{1}{2} \deg(v)$.

Proof of Claim. Suppose for contradiction that $v \in V(G) \setminus V(H)$ and $\eta(v) > \frac{1}{2}\deg(v)$. Then by the pigeonhole principle, there must be two adjacent needy faces with respect to v, say f and f'. If f and f' are of type 2, this would require a neighbor of v to be adjacent to two leaves. By Claim 3.15(3), since $V_1 \subseteq V(H)$, this would contradict H being a matching. So we may assume without loss of generality that f is a 3-face where v is the only high vertex. Let u, v, w be the vertices of f where u, w are both low and u is incident to f'. By Claim 3.15(2), $uw \in E(H)$.

Now consider f'. If f' is of type 2, then u must also have a neighboring leaf z on f' which means $uz \in E(H)$ by Claim 3.15(2), contradicting that H is a matching. If f' is of type 1 where $f' = vuw'$ then by definition u and w' are both low and thus $uw' \in E(H)$ by Claim 3.15(2) again contradicting that H is a matching. In either case we arrive at a contradiction since H is a matching. $\boxtimes$

Claim 3.18. If $f \in F(G)$ is incident with $x \geq 1$ leaves, then $\ell(f) \geq x + 4$.

Proof of Claim. We prove this by induction on x. If $x = 1$, then any face f with one leaf v must have length at least 5 since G is simple. Now suppose a face f has $x \geq 2$ leaves. Let v be a leaf on f and let f' be the face obtained by removing v. By Claim 3.15(4), $V_2 = \varnothing$ so when we remove v from f, we yield a face f' incident with one less leaf. By induction $\ell(f') \geq x - 1 + 4 = x + 3$. By adding v back in we see that $\ell(f) = \ell(f') + 2 = x + 5 \geq x + 4$ as desired. $\boxtimes$

We now define an initial charge μ of $x \in V(G) \cup F(G)$ and to a *global pot* $\mathcal{P}$. Initially let $\mu(\mathcal{P}) = 0$ and let

$$\mu(x) = \begin{cases} \deg(x) - 4 & \text{if } x \in V(G) \\ \ell(x) - 4 & \text{if } x \in F(G) \end{cases}$$

By Euler's formula we see that

$$\mu(G) = \mu(\mathcal{P}) + \sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = 4|E(G)| - 4|V(G)| - 4|F(G)| = -8.$$

We now present the discharging rules.

Discharging rules

(T1) Every high-leaf receives charge 1 from its incident face and receives charge 2 from its neighbor.

(T2) Every low-leaf receives charge 1 from its incident face f, receives charge 1 from its neighbor and receives charge $\frac{1}{2}$ from both of its second neighbors on f.

(T3) Every 3-face receives charge $\frac{1}{y}$ from its y incident high vertices (note $y \geq 1$).

(T4) If $t \in \{3,4\}$, every vertex $v \in V_{[t,4]}$ receives charge 2 from $\mathcal{P}$ and every vertex $v \in V_{[\Delta-4+t,\Delta]}$ gives charge 2 to $\mathcal{P}$.

Let $\mu'(x)$ and $\mu'(\mathcal{P})$ be the final charge of $x \in V(G) \cup F(G)$ and $\mathcal{P}$ after (T1)-(T4) have been applied. We prove via a series of three claims that $\mu' \geq 0$, contradicting $\mu'(G) = \mu(G) = -8$ and completing the proof. Our proof consists of one claim each for $\mathcal{P}$, faces, and vertices, respectively.

Claim 3.19. $\mathcal{P}$ has nonnegative final charge.

Proof of Claim. The only rule affecting $\mathcal{P}$ is (T4). By Claim 3.16, $\mu'(\mathcal{P}) \geq \mu(\mathcal{P}) + 2|V_{[\Delta-4+t,\Delta]}| - 2|V_{[t,4]}| > 0$. ⊠

Claim 3.20. Every face has nonnegative final charge.

Proof of Claim. Let $f \in F(G)$. If $f \in F_3$, then f cannot have any incident leaves. Thus, f is only affected by (T3) and $\mu'(f) = \mu(f) + y \cdot \frac{1}{y} = 0$ where y is the number of high vertices incident to f. We reiterate that $y \geq 1$ since every 3-face can have at most 1 edge in $E(H)$ and hence there is an edge which is not precolored and must be incident to a high vertex. Next suppose $\ell(f) \geq 4$. If f has no incident leaves, f is not affected by any discharging rules and thus, $\mu'(f) = \mu(f) = \ell(f) - 4 \geq 0$. Finally, suppose f is incident to $x \geq 1$ leaves. By (T1) and (T2), $\mu'(f) = \mu(f) - x = \ell(f) - 4 - x$. By Claim 3.18, $\ell(f) \geq 4 + x$ and thus, $\mu'(f) \geq 0$. ⊠

Claim 3.21. Every vertex has nonnegative final charge.

Proof of Claim. Let $v \in V(G)$. First suppose v is a leaf. If v is a high-leaf, by (T1), $\mu'(v) = \mu(v) + 1 + 2 = \deg(v) - 4 + 3 = 0$. If v is a low-leaf, by (T2), $\mu'(v) = \mu(v) + 1 + 1 + 2 \cdot \frac{1}{2} = \deg(v) + 3 = 0$.

Next suppose v is low. Note that since v is low, v does not give any charge to $\mathcal{P}$ in (T4). By (T2), $\mu'(v) \geq \mu(v) - 1 = \deg(v) - 5$ and thus $\mu'(v) \geq 0$ if $\deg(v) \geq 5$. If $2 \leq \deg(v) \leq 4$ then since $V_{[2,t-1]} = \varnothing$ by Claim 3.15(4), $\deg(v) \in \{3, 4\}$ and $t \in \{3, 4\}$. Thus, by (T4), $\mu'(v) \geq \mu(v) + 2 - 1 = \deg(v) - 3 \geq 0$.

Next suppose v is high and v is not affected by (T4). If $v \in V(H)$, then every 3 face which v is incident to, must have at least 2 high vertices incident to it. Thus, by (T1) and (T3) $\mu'(v) \geq \mu(v) - (2 + \frac{1}{2}(\deg(v) - 1)) = \frac{1}{2}\deg(v) - \frac{11}{2}$. Thus, $\mu'(v) \geq 0$ as long as $\deg(v) \geq 11$. Since $\Delta \geq 31 - t$, and v is high, $\deg(v) \geq \lceil \frac{\Delta+t}{2} \rceil \geq 16$. If $v \notin V(H)$, then by (T2) and (T3), v gives a total charge of 1 for any needy fact with respect to v (gives 1 to a needy face of type 1 and $\frac{1}{2}$ to both leaves in a needy face of type 2). For any other incident face, v gives charge at most $\frac{1}{2}$. Thus, by Claim 3.17, we see that $\mu'(v) \geq \mu(v) - (1 \cdot \eta(v) + \frac{1}{2} \cdot (\deg(v) - \eta(v))) \geq \frac{1}{4}\deg(v) - 4$. Hence $\mu'(v) \geq 0$ so long as $\deg(v) \geq 16$. Since v is high and $\Delta \geq 31 - t$, $\lceil \frac{\Delta+t}{2} \rceil \geq 16$.

Finally suppose v is high and is affected by (T4). Then v loses an additional charge of 2 to $\mathcal{P}$. In this case, $t \in \{3, 4\}$ and $\deg(v) \geq \Delta - 1$. If $v \in V(H)$, then by the same computation as the previous paragraph we see that $\mu'(v) \geq \frac{1}{2}\deg(v) - \frac{11}{2} - 2 \geq \frac{1}{2}\Delta - 8 \geq 0$ since $\Delta \geq 25$. Similarly if $v \notin V(H)$, using the same calculation in the previous paragraph, $\mu'(v) \geq \frac{1}{4}\deg(v) - 6 \geq \frac{1}{4}\Delta - \frac{25}{4} \geq 0$ since $\Delta \geq 25$. ⊠

□

Chapter 4

Total-Coloring Extensions in Some Other Families of Graphs

Conjecture 1.5 is a strengthening of the Total-Coloring Conjecture (TCC), and thus natural graph class candidates for proving Conjecture 1.5 are classes of graphs in which the TCC is known to hold. In this section, we investigate several such classes, including paths, cycles, trees, ladders, and prisms.

4.1 Path, Cycles and Trees

We first prove Conjecture 1.5 for paths and cycles, both of which we restate for convenience.

Theorem (1.17). Let $n \geq 1$ and let M be a distance-3 matching in P_n. Then any total-4-coloring of M can be extended to P_n.

Proof. Let $n \geq 1$ and let M be a distance-3 matching in P_n. Let φ_0 be a total-4-coloring of M. Since $P_n - E(M)$ is a disjoint union of paths, and colorings of different components are independent, it suffices to extend the coloring on each component separately. In particular, it is enough to consider a path P_t with $t \geq 6$ and vertices $v_1, \ldots, v_t$ such that $E(M) \cap E(P_t) = \{v_1v_2, v_{t-1}v_t\}$.

We now extend φ_0 to P_t. First, set $\varphi(v_3) = \varphi_0(v_1v_2)$. Since $t \geq 6$, the elements v_3 and v_1v_2 are at distance at least 3, so this coloring preserves properness. Next, we greedily color the remaining uncolored vertices of P_t. At each step, a vertex is adjacent to at most two already colored vertices and no precolored edge (since the endpoints of precolored edges are already colored under φ_0), so at most two colors are forbidden, and a color from [4] is available.

It remains to color the edges $v_2v_3, \ldots, v_{t-2}v_{t-1}$. Observe that each edge in this set is incident with two vertices and possibly adjacent to one precolored edge. In particular, since we colored $\varphi(v_3) = \varphi_0(v_1v_2)$, every such edge sees at most three colors, and all but possibly $v_{t-2}v_{t-1}$ see at most two colors. Therefore, by coloring the edges in the order

$v_{t-2}v_{t-1}, v_{t-3}v_{t-2}, \dots, v_2v_3$, each edge has an available color in $[4]$, and we may complete the coloring greedily.

Thus, φ_0 extends to a total-4-coloring of P_t, and hence to P_n. □

Corollary (1.1). Let $n \geq 3$ and let M be a distance-3 matching in C_n. Then any total-4-coloring of M can be extended to C_n.

Proof. Let $n \geq 3$ and let M be a distance-3 matching in C_n. Let φ_0 be a total-4-coloring of M in C_n. We show that φ_0 extends to a total-4-coloring of C_n. Observe that $C_n - E(M)$ is a disjoint union of paths, say $P_1, \dots, P_{|M|}$. For each path P_i, consider the restriction of φ_0 to the edges of M incident with P_i and their endpoints. This forms a totally-4-colored distance-3 matching relative to P_i. Thus, by Theorem 1.17, this coloring extends to a total-4-coloring of P_i. Since the paths $P_1, \dots, P_{|M|}$ are disjoint, these extensions are compatible. Combining them yields a total-4-coloring of C_n extending φ_0, as desired. □

See Fig. 4.1 for a illustration of how a precolored distance-3 matching in C_n decomposes the graph into uncolored cycles to which we can apply Theorem 1.17.

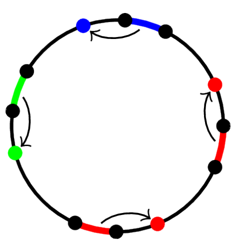


Figure 4.1: How to extend a precolored distance-3 matching in C_n to a total-4-coloring.

Next, we investigate extending total colorings in trees. Let T be a tree with maximum degree Δ. We verify Conjecture 1.5 by showing that any precolored distance-3 matching extends to a total-$(\Delta + 2)$-coloring of T, that is proving Theorem 1.18, which we restate below. By Fig. 3.2, this distance condition is best possible. Since $\chi_T(T) = \Delta + 1$, one might hope to extend a precolored matching to a total-$(\Delta + 1)$-coloring. We show that this is not possible, regardless of any distance condition.

Theorem (1.18). Let T be a tree with maximum degree Δ, and let M be a distance-3 matching in T. Then any total-$(\Delta + 2)$-coloring of M can be extended to T.

Proof. If $\Delta \leq 2$, then T is a path which is proved by Theorem 1.17. Thus we may assume $\Delta \geq 3$, which implies our desired color palette has size at least 5.

We prove this by minimal counterexample with respect to $|V(T)|$. Let T be the smallest such tree with maximum degree Δ, where M is a distance-3 matching in T and φ_0 is a total-$(\Delta+2)$-coloring of M which does not extend to any total-$(\Delta + 2)$-coloring of T.

Suppose there exists an edge $xy \notin M$ with $\deg_T(x) = 1$. Let $T' = T - x$. By minimality, there exists a total-$(\Delta + 2)$-coloring extending M to T'. Next, greedily color xy, which is possible since xy forbids at most Δ colors ($\Delta - 1$ adjacent edges and the color of y). Finally, we can greedily color x, which forbids at most 2 colors. Thus, M is precisely the set of edges incident to a leaf in T.

Let $e_1, e_2 \in M$, both incident to leaves x_1 and x_2 respectively. Moreover, choose e_1 and e_2 such that the distance from x_1 to x_2 is maximized. For $i \in \{1, 2\}$, let $e_i = x_i y_i$ and let P be the shortest (x_1, x_2)-path in T, and let z_i be the other neighbor of y_i on P (note that $z_1 \neq z_2$ since M is distance-3).

We claim that $\deg_T(y_1) = \deg_T(y_2) = \deg_T(z_1) = \deg_T(z_2) = 2$. First consider y_1 and y_2. The arguments are symmetric, so suppose for a contradiction that $\deg_T(y_1) \geq 3$, and let $v \in N(y_1) \setminus \{x_1, z_1\}$. Since $y_1 v \notin M$ and all edges incident to leaves must lie in M, there must exist some edge $e_3 = x_3 y_3 \in M$ with leaf x_3 such that $\mathrm{dist}_T(v, y_3) \geq 2$ and e_3 lies in the component of $T - P$ containing v. But then the distance from e_3 to e_2 is greater than the distance from e_1 to e_2, a contradiction. Now let w_i be the neighbor of z_i on P distinct from y_i. The argument for z_1 and z_2 is identical. Suppose for a contradiction that $\deg_T(z_1) \geq 3$, and let $v \in N(z_1) \setminus \{y_1, w_1\}$. Since $z_1 v \notin M$ and all edges incident to leaves must lie in M, there must exist some edge $e_3 = x_3 y_3 \in M$ with leaf x_3 such that $\mathrm{dist}_T(v, y_3) \geq 1$ and e_3 lies in the component of $T - P$ containing v. But then the distance from e_3 to e_2 is greater than the distance from e_1 to e_2, again a contradiction.

Let $T' = T - \{x_1, y_1, z_1\}$. By minimum counterexample, there exists a total-$(\Delta + 2)$-coloring φ' extending $M \setminus \{x_1y_1\}$ to T'. What remains is to color the edges z_1w_1 and y_1z_1 and the vertices y_1 and z_1. We do this greedily. First color z_1w_1, which is possible since z_1w_1 sees at most Δ colors under φ'. Next color z_1, which is possible since z_1 sees at most 3 colors (the colors of w_1, z_1w_1, and y_1). Finally color y_1z_1, which is possible since y_1z_1 sees at most 4 colors (the colors of x_1y_1, z_1w_1, y_1, and z_1) and $\Delta + 2 \geq 5$. □

Not only do we see from Fig. 3.2 that the distance condition in Theorem 1.18 is tight, but the color palette is also tight for any distance condition. To see this, observe that there is only one total-3-coloring of a path P_n up to relabeling (see Fig. 4.2). Thus, a total-3-coloring of a matching cannot always be extended to P_n, regardless of the distance condition (fix the coloring of the first K_2 of P_n and then alter the final K_2 so that it does not match the coloring in Fig. 4.2). It would be interesting to know whether a tight example exists for trees for all values of Δ.

Figure 4.2: The unique total-3-coloring of the path P_6

Here we take a quick aside to mention some future work. Trees are a special case of bipartite graphs, for which we know the TCC holds trivially (take any Δ-edge-coloring together with a 2-vertex coloring). Thus, it is natural to hope to have a strengthening of the TCC via an extension result. As it turns out, this is very closely related to another celebrated coloring result for bipartite graphs shown by Galvin in 1995 [35].

Theorem 4.1 (Galvin's Theorem [35]). Bipartite graphs are Δ-edge-choosable.

As a corollary of Galvin's Theorem, we can prove a weakening of Conjecture 1.5 by replacing the conjectured $\Delta + 2$ color palette in Conjecture 1.5 with $\Delta + 3$, even when the distance condition is reduced to 2.

Corollary 4.1. Let G be a bipartite graph and let M be a distance-2 matching in G. Then any total-$(\Delta + 3)$-coloring of M can be extended to G.

Proof. Let G be a bipartite graph and let M be a distance-2 matching in G that is total-$(\Delta + 3)$-colored. We first greedily color the vertices of G, which is certainly possible with $\Delta + 3$ colors. Next, observe that each edge sees at most 3 colors: its two endpoints and possibly one adjacent edge in M. Thus, every edge has at least Δ available colors, and by Theorem 4.1, we can extend M to a total-$(\Delta + 3)$-coloring of G. □

4.2 Ladders and Prisms

We now turn to another class of graphs for which the Total-Coloring Conjecture is known to hold, namely subcubic graphs. The The TCC was first verified for subcubic graphs in 1971 by Rosenfeld [36] and independently by Vijayaditya [37] (see [38] for a short proof). Determining whether 4 colors suffices for a total-coloring certain classes of subcubic graphs remains an challenging problem, even for subcubic bipartite graphs.

To systematically investigate total-coloring extension results for subcubic graphs, we restrict our attention to special cases of subcubic graphs with more inherent structure, namely ladders and prisms. We point out that the maximum degree of both graphs is 3 and moreover, the prism is 3-regular.

In order to extend total colorings of matchings to total colorings of ladders and prisms, it will be helpful to control the coloring on some of the induced C_4's contained in these graphs. To this end, we consider the following definition.

Let C be a 4-cycle with $V(C) = \{v_1, v_2, v_3, v_4\}$ and $E(C) = \{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$. We say that a coloring $\varphi : V(C) \cup E(C) \to [5]$ is *good with respect to* v_1v_2 if $\varphi(v_1), \varphi(v_2), \varphi(v_1v_2)$ are pairwise distinct, $\varphi(v_1v_4) \notin \{\varphi(v_1), \varphi(v_1v_2)\}$, $\varphi(v_2v_3) \notin \{\varphi(v_2), \varphi(v_1v_2)\}$, and $\varphi(v_1v_4) \neq \varphi(v_2v_3)$ (see Fig. 4.3). Note that we do not require the coloring φ to be proper, since there is no restriction on $\varphi(v_3)$, $\varphi(v_4)$, or $\varphi(v_3v_4)$.

We first consider a special case: extending a specific precolored matching in L_5.

Lemma 4.1. Let L_5 have vertices $x_1, \dots, x_5$ and $y_1, \dots, y_5$ where the x_i's and y_i's each induce a P_5 and $x_iy_i \in E(L_5)$ for all $i \in [5]$. Let $M = \{x_1, x_2, x_1x_2, y_4, y_5, y_4y_5\}$. Then any total-5-coloring

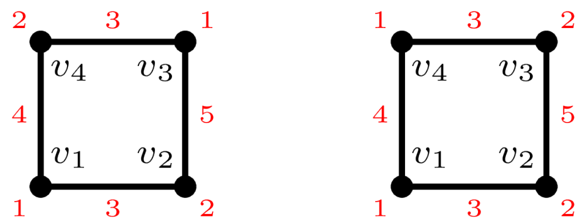


Figure 4.3: Two examples $\varphi : V(C_4) \cup E(C_4) \to [5]$ which are both good with respect to v_1v_2. The left gives a total-5-coloring whereas the right does not.

of M extends to a total-5-coloring of L_5 where $\{x_1, x_2, y_1, y_2\}$ is good with respect to x_1y_1 and $\{x_4, x_5, y_4, y_5\}$ is good with respect to x_5y_5.

Proof. Let φ be a total 5-coloring of M. By relabeling the colors, we may assume that $\varphi(x_1) = 1$, $\varphi(x_1x_2) = 2$, and $\varphi(x_2) = 3$. Next, write $\varphi(y_4) = \alpha$, $\varphi(y_4y_5) = \beta$, and $\varphi(y_5) = \gamma$, where α, β, γ are distinct elements of $[5]$. Let $\{\delta, \varepsilon\} = [5] \setminus \{\alpha, \beta, \gamma\}$.

Case 1. Suppose $(\alpha, \beta, \gamma) \notin \{(4, 5, 2), (5, 1, 3), (5, 1, 4), (5, 2, 4), (5, 4, 2), (4, 1, 2), (5, 1, 2)\}$. We color $C_1 = x_1x_2y_2y_1$ by setting $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 5$, and $\varphi(x_1y_1) = \varphi(x_2y_2) = 4$. We color $C_4 = x_4x_5y_5y_4$ by setting $\varphi(x_4) = \gamma$, $\varphi(x_5) = \alpha$, $\varphi(x_4y_4) = \varphi(x_5y_5) = \delta$, and $\varphi(x_4x_5) = \varepsilon$. Then both C_1 and C_4 are good.

It remains to color

$$x_3, y_3, x_2x_3, x_3x_4, y_2y_3, y_3y_4, x_3y_3.$$

At this point, each of $x_2x_3, x_3x_4, y_2y_3, y_3y_4$ has at least two available colors, each of x_3, y_3 has at least three available colors, and x_3y_3 has all five colors available. To finish the coloring, we prove this by exhaustion in the following claim.

Claim 4.1. Assume $(\alpha, \beta, \gamma) \notin \{(4, 5, 2), (5, 1, 3), (5, 1, 4), (5, 2, 4), (5, 4, 2), (4, 1, 2), (5, 1, 2)\}$. With the coloring of C_1 and C_4 defined above, the elements $x_3, y_3, x_2x_3, x_3x_4, y_2y_3, y_3y_4, x_3y_3$ can be colored so that the coloring remains proper.

Proof of Claim. We let L denote the list of available colors. At this point,

$$L(x_2x_3) = \{1,5\}, \qquad L(x_3x_4) = \{\alpha,\beta\}, \qquad L(y_2y_3) = \{2,3\}, \qquad L(y_3y_4) = \{\gamma,\varepsilon\}.$$

We choose the roles of δ and ε as needed below; this only changes which of the two colors in $[5] \setminus \{\alpha,\beta,\gamma\}$ is assigned to x_4x_5 and which is assigned to the two rungs of C_4.

First suppose $\alpha \neq 5$ and $\gamma \neq 2$. Choose ε arbitrarily. Set $\varphi(x_2x_3) = 5, \varphi(x_3x_4) = \alpha, \varphi(y_2y_3) = 2, \varphi(y_3y_4) = \gamma$. It remains to color x_3y_3, x_3, y_3. We do this according to the following cases:

(α,γ)	$(\varphi(x_3y_3), \varphi(x_3), \varphi(y_3))$
$(1,3)$	$(4,2,5)$
$(1,4)$	$(3,2,5)$
$(1,5)$	$(3,2,4)$
$(2,1)$	$(3,4,5)$
$(2,3)$	$(1,4,5)$
$(2,4)$	$(3,1,5)$
$(2,5)$	$(1,4,3)$
$(3,1)$	$(4,2,5)$
$(3,4)$	$(1,2,5)$
$(3,5)$	$(1,2,4)$
$(4,1)$	$(3,2,5)$
$(4,3)$	$(1,2,5)$
$(4,5)$	$(1,2,3)$

These are all possibilities with $\alpha \neq 5, \gamma \neq 2$, and α,γ distinct.

It remains to consider the triples in Case 1 for which either $\alpha = 5$ or $\gamma = 2$. Since the seven exceptional triples have been excluded, the remaining triples are $(1,3,2), (1,4,2), (1,5,2), (3,4,2), (3,5,2), (4,3,2), (5,2,1), (5,2,3), (5,3,1), (5,3,2), (5,3,4), (5,4,3), (3,1,2)$, and $(5,4,1)$.

For the first twelve of these triples, choose ε as indicated below and color the seven remaining elements as follows.

(α, β, γ)	ε	$(\varphi(x_2x_3), \varphi(x_3x_4), \varphi(y_2y_3), \varphi(y_3y_4), \varphi(x_3y_3), \varphi(x_3), \varphi(y_3))$
$(1, 3, 2)$	5	$(1, 3, 3, 5, 2, 5, 4)$
$(1, 4, 2)$	5	$(1, 4, 3, 5, 2, 5, 4)$
$(1, 5, 2)$	4	$(1, 5, 3, 4, 2, 4, 5)$
$(3, 4, 2)$	5	$(1, 4, 3, 5, 2, 5, 4)$
$(3, 5, 2)$	4	$(1, 5, 3, 4, 2, 4, 5)$
$(4, 3, 2)$	5	$(1, 3, 3, 5, 4, 5, 2)$
$(5, 2, 1)$	4	$(1, 2, 3, 4, 5, 4, 2)$
$(5, 2, 3)$	4	$(1, 2, 3, 4, 5, 4, 2)$
$(5, 3, 1)$	4	$(1, 3, 3, 4, 5, 4, 2)$
$(5, 3, 2)$	4	$(1, 3, 3, 4, 5, 4, 2)$
$(5, 3, 4)$	2	$(1, 3, 3, 2, 5, 2, 4)$
$(5, 4, 3)$	2	$(1, 4, 3, 2, 5, 2, 4)$

Finally, if $(\alpha, \beta, \gamma) = (3, 1, 2)$, choose $\varepsilon = 4$ and set $\varphi(x_2x_3) = 5$, $\varphi(x_3x_4) = 1$, $\varphi(y_2y_3) = 3$, $\varphi(y_3y_4) = 4$, $\varphi(x_3y_3) = 2$, $\varphi(x_3) = 4$, and $\varphi(y_3) = 5$. If $(\alpha, \beta, \gamma) = (5, 4, 1)$, choose $\varepsilon = 2$ and set $\varphi(x_2x_3) = 5$, $\varphi(x_3x_4) = 4$, $\varphi(y_2y_3) = 3$, $\varphi(y_3y_4) = 2$, $\varphi(x_3y_3) = 1$, $\varphi(x_3) = 2$, and $\varphi(y_3) = 4$. Thus in every triple covered by Case 1, the coloring extends over the seven remaining elements. $\boxtimes$

This completes the extension in this case.

Case 2: (α, β, γ) **is one of the exceptional cases.**

Suppose $(\alpha, \beta, \gamma) = (4, 5, 2)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 5$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 4$, $\varphi(x_4) = 2$, $\varphi(x_5) = 4$, $\varphi(x_4y_4) = \varphi(x_5y_5) = 3$, and $\varphi(x_4x_5) = 1$. Finally set $\varphi(x_2x_3) = 1$, $\varphi(x_3x_4) = 5$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 1$, $\varphi(x_3y_3) = 3$, $\varphi(x_3) = 4$, and $\varphi(y_3) = 5$.

Suppose $(\alpha, \beta, \gamma) = (5, 1, 3)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 4$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 5$, $\varphi(x_4) = 3$, $\varphi(x_5) = 5$, $\varphi(x_4y_4) = \varphi(x_5y_5) = 2$, and $\varphi(x_4x_5) = 4$. Finally set $\varphi(x_2x_3) = 4$, $\varphi(x_3x_4) = 1$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 3$, $\varphi(x_3y_3) = 5$, $\varphi(x_3) = 2$, and $\varphi(y_3) = 4$.

Suppose $(\alpha, \beta, \gamma) = (5, 1, 4)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 4$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 5$, $\varphi(x_4) = 4$, $\varphi(x_5) = 5$, $\varphi(x_4y_4) = \varphi(x_5y_5) = 2$, and $\varphi(x_4x_5) = 3$. Finally set $\varphi(x_2x_3) = 4$, $\varphi(x_3x_4) = 1$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 3$, $\varphi(x_3y_3) = 5$, $\varphi(x_3) = 2$, and $\varphi(y_3) = 4$.

Suppose $(\alpha, \beta, \gamma) = (5, 2, 4)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 4$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 5$, $\varphi(x_4) = 4$, $\varphi(x_5) = 5$, $\varphi(x_4y_4) = \varphi(x_5y_5) = 1$, and $\varphi(x_4x_5) = 3$. Finally set $\varphi(x_2x_3) = 4$, $\varphi(x_3x_4) = 2$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 3$, $\varphi(x_3y_3) = 1$, $\varphi(x_3) = 5$, and $\varphi(y_3) = 4$.

Suppose $(\alpha, \beta, \gamma) = (5, 4, 2)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 5$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 4$, $\varphi(x_4) = 2$, $\varphi(x_5) = 5$, $\varphi(x_4y_4) = \varphi(x_5y_5) = 3$, and $\varphi(x_4x_5) = 1$. Finally set $\varphi(x_2x_3) = 1$, $\varphi(x_3x_4) = 4$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 1$, $\varphi(x_3y_3) = 3$, $\varphi(x_3) = 5$, and $\varphi(y_3) = 4$.

Suppose $(\alpha, \beta, \gamma) = (4, 1, 2)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 5$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 4$, $\varphi(x_4) = 3$, $\varphi(x_5) = 4$, $\varphi(x_4y_4) = 2$, $\varphi(x_5y_5) = 3$, and $\varphi(x_4x_5) = 5$. Finally set $\varphi(x_2x_3) = 5$, $\varphi(x_3x_4) = 1$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 3$, $\varphi(x_3y_3) = 4$, $\varphi(x_3) = 2$, and $\varphi(y_3) = 5$.

Suppose $(\alpha, \beta, \gamma) = (5, 1, 2)$. Color $\varphi(y_1) = 3$, $\varphi(y_2) = 1$, $\varphi(y_1y_2) = 4$, $\varphi(x_1y_1) = \varphi(x_2y_2) = 5$, $\varphi(x_4) = 3$, $\varphi(x_5) = 5$, $\varphi(x_4y_4) = 2$, $\varphi(x_5y_5) = 3$, and $\varphi(x_4x_5) = 4$. Finally set $\varphi(x_2x_3) = 4$, $\varphi(x_3x_4) = 1$, $\varphi(y_2y_3) = 2$, $\varphi(y_3y_4) = 3$, $\varphi(x_3y_3) = 5$, $\varphi(x_3) = 2$, and $\varphi(y_3) = 4$.

In each of the above colorings, it is easy to check that the coloring is proper on L_5 and that both C_1 and C_4 are good with respect to x_1y_1 and x_5y_5 respectively. □

Lemma 4.2. Let $n \geq 6$ and let $G = P_n \square K_2$ with vertices $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$, where $x_1, \dots, x_n$ and $y_1, \dots, y_n$ each induce a path and $x_iy_i \in E(G)$ for all $i \in [n]$. For $i \in [n-1]$, let $C_i = G[\{x_i, x_{i+1}, y_i, y_{i+1}\}]$. Let $H = V(C_1) \cup E(C_1) \cup V(C_{n-1}) \cup E(C_{n-1})$, and let $\varphi_0 : H \to [5]$ be a coloring such that φ_0 restricted to C_1 is good with respect to x_2y_2 and φ_0 restricted to C_{n-1} is good with respect to $x_{n-1}y_{n-1}$. Then there exists a coloring $\varphi : V(G) \cup E(G) \to [5]$ extending φ_0 to G such that the only possible violations of φ being a total-5-coloring occur at $x_1, y_1, x_1y_1, x_n, y_n$, and x_ny_n.

Proof. Since coloring elements in $G - (C_1 \cup C_{n-1})$ cannot affect the vertices and edges at the ends of the ladder, namely $x_1, y_1, x_1y_1, x_n, y_n$, and x_ny_n, it suffices to extend φ_0 so that every other vertex and edge of G is properly colored.

We first show how to propagate a good coloring along the ladder so that the problem reduces to the case $n = 6$. Suppose φ is defined on $V(C_i) \cup E(C_i)$, that its restriction to C_i is good with respect to $x_{i+1}y_{i+1}$, and that all currently colored elements are proper except possibly at the two ends of the original ladder. We first set $\varphi(x_{i+2}) = \varphi(y_{i+1})$, $\varphi(y_{i+2}) = \varphi(x_{i+1})$, and $\varphi(x_{i+2}y_{i+2}) = \varphi(x_{i+1}y_{i+1})$. Since $\varphi(x_{i+1})$, $\varphi(y_{i+1})$, and $\varphi(x_{i+1}y_{i+1})$ are pairwise distinct, these three assignments create no conflicts. It remains to color $x_{i+1}x_{i+2}$ and $y_{i+1}y_{i+2}$. Each of these two edges has at least two available colors and thus, we may choose distinct available colors for $x_{i+1}x_{i+2}$ and $y_{i+1}y_{i+2}$. After doing so, the coloring on C_{i+1} is proper and good with respect to $x_{i+2}y_{i+2}$. Repeating this step, we may assume $n = 6$.

Now assume $n = 6$. We consider three cases for the possible color combinations between $\varphi_0(x_1x_2)$ and $\varphi_0(x_5x_6)$ and between $\varphi_0(y_1y_2$ and $\varphi_0(y_5y_6)$.

Case 1: $\varphi_0(x_1x_2) \neq \varphi_0(x_5x_6)$ and $\varphi_0(y_1y_2) \neq \varphi_0(y_5y_6)$. Define $\varphi(x_3) = \varphi_0(x_1x_2)$, $\varphi(y_3) = \varphi_0(y_1y_2)$, $\varphi(x_4) = \varphi_0(x_5x_6)$, $\varphi(y_4) = \varphi_0(y_5y_6)$, $\varphi(x_3y_3) = \varphi_0(x_2y_2)$, and $\varphi(x_4y_4) = \varphi_0(x_5y_5)$. Because φ_0 is good on C_1 and C_5, these assignments preserve properness. At this stage, x_3x_4 and y_3y_4 each see at most four colors, while x_2x_3, y_2y_3, x_4x_5, and y_4y_5 each see at most three colors. We first greedily color x_3x_4 and y_3y_4, and then greedily color the remaining four edges to obtain the desired extension.

Case 2: Exactly one of the equalities $\varphi_0(x_1x_2) = \varphi_0(x_5x_6)$ and $\varphi_0(y_1y_2) = \varphi_0(y_5y_6)$ holds. By symmetry, assume $\varphi_0(x_1x_2) = \varphi_0(x_5x_6)$ and $\varphi_0(y_1y_2) \neq \varphi_0(y_5y_6)$. We first set $\varphi(x_3) = \varphi_0(x_1x_2)$, $\varphi(y_3) = \varphi_0(y_1y_2)$, and $\varphi(x_3y_3) = \varphi_0(x_2y_2)$. Next set $\varphi(y_4) = \varphi_0(y_5y_6)$. As in the previous cases, the edges x_2x_3, y_2y_3, and y_4y_5 can be colored greedily at the end. Thus it remains to color x_3x_4, x_4, x_4x_5, x_4y_4, and y_3y_4.

At this stage, each of x_4, x_4x_5, and y_3y_4 has at least two available colors, x_3x_4 has at least three available colors, and x_4y_4 has at least four available colors. Therefore we may greedily color these elements in the order x_4x_5, x_4, x_3x_4, x_4y_4, and y_3y_4. Finally, we greedily color x_2x_3, y_2y_3, and y_4y_5, completing the extension.

Case 3: $\varphi_0(x_1x_2) = \varphi_0(x_5x_6)$ and $\varphi_0(y_1y_2) = \varphi_0(y_5y_6)$. We first set $\varphi(x_3) = \varphi_0(x_1x_2)$, $\varphi(y_3) = \varphi_0(y_1y_2)$, $\varphi(x_3y_3) = \varphi_0(x_2y_2)$, and $\varphi(x_4y_4) = \varphi_0(x_2y_2)$. We next aim to color x_4, y_4, x_4x_5, and y_4y_5. After that, we will greedily color x_3x_4 and y_3y_4, and finally x_2x_3 and y_2y_3. Indeed, once x_4, y_4, x_4x_5, and y_4y_5 are colored, each of x_3x_4 and y_3y_4 sees at most four colors, since before this step each sees only the two colors on its incident endpoint and rung edge, and coloring x_4, y_4, x_4x_5, and y_4y_5 introduces at most two additional conflicts. Similarly, x_2x_3 and y_2y_3 each see at most four colors when colored last. Thus, it remains only to show that x_4, y_4, x_4x_5, and y_4y_5 can be colored appropriately.

We first show that this can be done greedily except in one special case, namely when $\varphi_0(x_5) = \varphi_0(y_5y_6)$ and $\varphi_0(y_5) = \varphi_0(x_5x_6)$ which will we take care of afterwards. Thus, we may assume without loss of generality that $\varphi_0(x_5) \neq \varphi_0(y_5y_6)$.

Claim 4.2. If $\varphi_0(x_5) \neq \varphi_0(y_5y_6)$, then we can greedily color x_4, y_4, x_4x_5, and y_4y_5 in some order.

Proof of Claim. Since $\varphi_0(x_5) \neq \varphi_0(y_5y_6)$, the five elements x_5, y_5, x_5y_5, x_5x_6, and y_5y_6 must receive at least four colors. Suppose

$$\varphi_0(x_2y_2) \in \{\varphi_0(x_5), \varphi_0(y_5), \varphi_0(x_5y_5), \varphi_0(x_5x_6), \varphi_0(y_5y_6)\}$$

. Note that $\varphi_0(x_2y_2)$ is also the color assigned to both x_3y_3 and x_4y_4. By symmetry, we may assume that x_4x_5 sees at most three colors. Then we may greedily color y_4y_5, y_4, x_4, and x_4x_5 in this order.

Now assume that $\varphi_0(x_2y_2) \notin \{\varphi_0(x_5), \varphi_0(y_5), \varphi_0(x_5y_5), \varphi_0(x_5x_6), \varphi_0(y_5y_6)\}$. Since only five colors are available, the five elements x_5, y_5, x_5y_5, x_5x_6, and y_5y_6 must then use exactly four colors. Because φ_0 restricted to C_5 is good, the only possible repeated color among these five elements is $\varphi_0(y_5) = \varphi_0(x_5x_6)$. Thus we may assume $\varphi_0(y_5) = \varphi_0(x_5x_6)$.

In this case, recolor x_4y_4 with $\varphi_0(x_1x_2)$. Then x_3x_4 can still be greedily colored as discussed earlier, while y_3y_4 now sees three colors and has only two remaining potential conflicts, namely y_4 and y_4y_5. Thus we first color y_3y_4 with $\varphi_0(x_5y_5)$. At this stage, each of x_4, y_4, x_4x_5, and y_4y_5 sees at most three colors. Hence we can greedily color them in the order x_4x_5, x_4, y_4, and y_4y_5. ⊠

It remains to consider the case when $\varphi_0(x_5) = \varphi_0(y_5y_6)$ and $\varphi_0(y_5) = \varphi_0(x_5x_6)$. By a symmetrical argument, we may also assume that $\varphi_0(x_2) = \varphi_0(y_1y_2)$ and $\varphi_0(y_2) = \varphi_0(x_1x_2)$. Hence the elements x_1x_2, x_2, y_1y_2, y_2, x_5, x_5x_6, y_5, and y_5y_6 use exactly two colors, and without loss of generality we may assume these colors are $\{1, 2\}$.

Claim 4.3. Suppose $\varphi_0(x_1x_2) = \varphi_0(x_5x_6)$ and $\varphi_0(y_1y_2) = \varphi_0(y_5y_6)$, and that the elements $x_1x_2, x_2, y_1y_2, y_2, x_5, x_5x_6, y_5$, and y_5y_6 together use exactly the colors $\{1, 2\}$. Then, without loss of generality, we may assume $\varphi_0(x_1x_2) = \varphi_0(y_2) = \varphi_0(x_5x_6) = \varphi_0(y_5) = 1$ and $\varphi_0(x_2) = \varphi_0(y_1y_2) = \varphi_0(x_5) = \varphi_0(y_5y_6) = 2$. Under these assumptions, φ_0 extends to a coloring of G.

Proof of Claim. Since φ_0 restricted to C_1 and C_5 is good, we have $\varphi_0(x_2y_2), \varphi_0(x_5y_5) \notin \{1, 2\}$. Without loss of generality, let $\varphi_0(x_2y_2) = 3$. If $\varphi_0(x_5y_5) = \varphi_0(x_2y_2)$, then the following coloring extends φ_0 to G: $\varphi(x_3) = \varphi(y_3y_4) = 1, \varphi(y_3) = \varphi(x_3x_4) = 2, \varphi(x_3y_3) = \varphi(x_4y_4) = 3$, $\varphi(x_2x_3) = \varphi(x_4) = \varphi(y_4y_5) = 4$, and $\varphi(y_2y_3) = \varphi(y_4) = \varphi(x_4x_5) = 5$. If $\varphi_0(x_5y_5) \neq \varphi_0(x_2y_2)$, we may assume without loss of generality that $\varphi_0(x_5y_5) = 4$. Then the following coloring extends φ_0 to G: $\varphi(x_3) = \varphi(y_3y_4) = 1, \varphi(y_3) = \varphi(x_3x_4) = 2, \varphi(x_3y_3) = \varphi(x_4) = \varphi(y_4y_5) = 3, \varphi(x_2x_3) = \varphi(y_2y_3) = \varphi(x_4y_4) = 4$, and $\varphi(y_4) = \varphi(x_4x_5) = 5$. ⊠

In all cases, we obtain an extension φ with the desired property. □

We now use Lemma 4.1 and Lemma 4.2 to prove Conjecture 1.5 for both ladders and prisms, that is, to prove Theorem 1.19, which we restate below for convenience.

Theorem (1.19). Let $n \geq 3$ and let $G = C_n \square K_2$. Let M be a distance-3 matching in G. Then any total-5-coloring of M extends to a total-5-coloring of G.

Proof. It suffices to prove the result for prisms, since any ladder can be extended to a prism by connecting its ends with a sufficiently long ladder so as not to violate the distance-3 condition.

Let $n \geq 3$ and let $G = C_n \square K_2$. Partition $V(G)$ as $X \cup Y$, where $X = x_1, \dots, x_n$ and $Y = y_1, \dots, y_n$, such that X and Y each induce a copy of C_n and $x_i y_i \in E(G)$ for all $i \in [n]$. Let M be a distance-3 matching in G, and let φ_0 be a total-5-coloring of M. We will show that there exists a total-5-coloring φ of G extending φ_0.

Every edge of M is of one of the following forms, with indices taken modulo n: $x_i x_{i+1}$, $y_i y_{i+1}$, or $x_i y_i$ for some $i \in [n]$. We say $e \in M$ is a *rung* if $e = x_i y_i$.

We first deal with the only situation in which independently extending non-rung edges to 4-cycles can decrease the distance. Suppose that, for some i, both $x_i x_{i+1}$ and $y_{i+3} y_{i+4}$ belong to M, with indices taken modulo n. Then the subgraph induced by

$$x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, y_i, y_{i+1}, y_{i+2}, y_{i+3}, y_{i+4}$$

is a copy of L_5. By Lemma 4.1, the precoloring on $x_i, x_{i+1}, x_i x_{i+1}, y_{i+3}, y_{i+4}, y_{i+3} y_{i+4}$ extends to this copy of L_5 so that the two end 4-cycles are good with respect to the two boundary rungs. We perform this extension for every such pair.

After doing so, every remaining non-rung edge of M may be extended independently to its corresponding 4-cycle without creating any new conflict with another precolored edge. Thus, for each remaining non-rung edge $x_i x_{i+1}$ or $y_i y_{i+1}$, we color the cycle induced by $x_i, x_{i+1}, y_i, y_{i+1}$ so that the restriction of φ to this cycle is good with respect to both $x_i y_i$ and $x_{i+1} y_{i+1}$, while maintaining that φ is a total-5-coloring on the colored elements. This is always possible. Indeed, if $x_i x_{i+1} \in M$, then after relabeling colors we may assume that $\varphi_0(x_i) = 1$, $\varphi_0(x_i x_{i+1}) = 2$, and $\varphi_0(x_{i+1}) = 3$. We then set $\varphi(y_i) = 3$, $\varphi(y_{i+1}) = 1$, $\varphi(y_i y_{i+1}) = 4$, and $\varphi(x_i y_i) = \varphi(x_{i+1} y_{i+1}) = 5$. The case $y_i y_{i+1} \in M$ is symmetric.

Next, for any rung edge $x_i y_i \in M$, we replace this K_2 with a C_4, say C_i, with vertices x_i', x_i'', y_i', y_i'', where $E(C_i) = \{x_i' x_i'', x_i' y_i', y_i' y_i'', x_i'' y_i''\}$. We define the coloring φ on C_i by setting

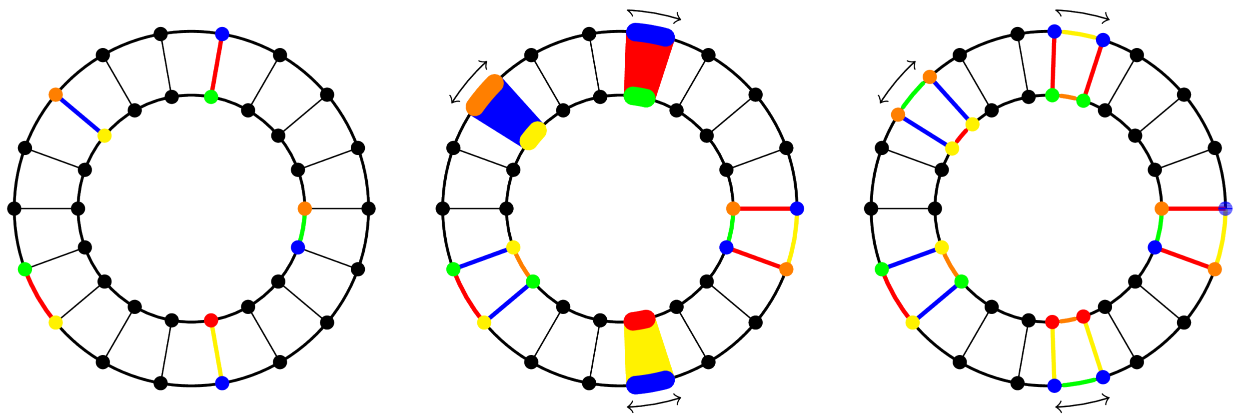
$\varphi(x_i') = \varphi(x_i'') = \varphi_0(x_i)$, $\varphi(y_i') = \varphi(y_i'') = \varphi_0(y_i)$, and $\varphi(x_i'y_i') = \varphi(x_i''y_i'') = \varphi_0(x_iy_i)$. Finally, we color $x_i'x_i''$ and $y_i'y_i''$ with the two distinct colors in $[5] \setminus \{\varphi_0(x_i), \varphi_0(y_i), \varphi_0(x_iy_i)\}$. We note that this coloring of C_i is good with respect to both $x_i'y_i'$ and $x_i''y_i''$. Let G^* be the graph obtained by replacing each rung edge $x_iy_i \in M$ with the corresponding C_i. Once $V(G^*) \cup E(G^*)$ are colored, we recover G by identifying x_i' with x_i'' and y_i' with y_i'' for each i. Since $\varphi(x_i') = \varphi(x_i'')$, $\varphi(y_i') = \varphi(y_i'')$, and $\varphi(x_i'y_i') = \varphi(x_i''y_i'')$, this identification is consistent, and hence the resulting coloring of G is well-defined. See Fig. 4.4 for an illustration of this step.

Notice that G^* consists of uncolored, vertex-disjoint ladders connected by colored C_4's, each of which is good with respect to both edges between X and Y. Order the uncolored ladders as $L_1, \dots, L_t$ according to their cyclic order around the prism, where the choice of L_1 and the direction is arbitrary. For each L_i, let C_i' and C_i'' denote the 4-cycles at the ends of L_i, where $C_i' = C_{i-1}''$ (with indices taken modulo t), and define $\overline{L_i} = L_i \cup C_i' \cup C_i''$.

We first apply Lemma 4.2 to $\overline{L_1}$ to extend φ to L_1. Next, we apply Lemma 4.2 to $\overline{L_2}$, possibly after a modification. If $C_1'' = C_2'$ arises from a rung edge $x_iy_i \in M$, then we modify the colors of $x_i'x_i''$ and $y_i'y_i''$ by setting $\varphi(x_i'x_i'') = \varphi(x_{i-1}x_i')$ and $\varphi(y_i'y_i'') = \varphi(y_{i-1}y_i')$. With this modification, once $\overline{L_1}$ and $\overline{L_2}$ are colored and we identify x_i' with x_i'' and y_i' with y_i'', the two non-rung edges $x_{i-1}x_i$ and x_ix_{i+1} receive distinct colors, and similarly for $y_{i-1}y_i$ and y_iy_{i+1}. Moreover, by the construction of the good C_4's in the initial extension, we have $\varphi(x_{i-1}x_i') \neq \varphi(y_{i-1}y_i')$. Therefore, after modifying C_2', the coloring remains good, and we may safely apply Lemma 4.2.

We continue this process inductively around the prism. Suppose that for some $k \geq 2$, we have extended φ to $\overline{L_1} \cup \cdots \cup \overline{L_k}$, and that for each $i \leq k$, the identification of the corresponding C_4's produces a proper coloring on the vertices and edges of $L_1 \cup \cdots \cup L_k$. To extend to L_{k+1}, we apply Lemma 4.2 to $\overline{L_{k+1}}$, making the same modification to $C_{k+1}' = C_k''$ if it arises from a rung edge in M. By the same argument as above, this modification preserves the good property of the C_4 and ensures that, after identification, the colors on the edges $x_{i-1}x_i$ and x_ix_{i+1} (and similarly on $y_{i-1}y_i$ and y_iy_{i+1}) are distinct. Thus the coloring extends properly to L_{k+1}.

Proceeding in this manner, we extend φ to all ladders $L_1, \dots, L_t$. Finally, after identifying all pairs x_i' with x_i'' and y_i' with y_i'', we obtain a total-5-coloring of G extending φ_0, as desired. $\square$



(i) precolored distance-3 matching

(ii) non-rung edges extend to good C_4's

(iii) rung edges perform a modification

(iv) turn modified rung edges to good C_4's

(v) apply lemma to all ladders

Figure 4.4: An illustration of the first extension step in Theorem 1.19.

Chapter 5

Conclusions and Future Work

In this thesis, we studied orientations with forbidden out-degrees and extensions of total colorings, with a focus on structured graph classes.

In Chapter 2, we extend the known classes of graphs and types of forbidden lists for which Conjecture 1.1 holds,. The conjecture (given by Akbari et al. [3]) asserts that if the number of forbidden out-degrees at each vertex is sufficiently small, namely less than half the degree, then there exists an orientation avoiding all forbidden out-degrees. The graph classes for which we verify Conjecture 1.1 include 2-degenerate graphs in Theorem 1.3 and graphs that can be decomposed into a bipartite graph and a 2-factor in Theorem 1.4. In doing so, we established structural properties of minimal counterexamples to Conjecture 1.1. These results suggest several directions for future work, including extending these methods to broader graph classes such as 3-degenerate graphs, tripartite graphs as well as further refining the structure of minimal counterexamples to Conjecture 1.1.

When the forbidden lists are constant, we extend the best known bounds for d-regular graphs satisfying Conjecture 1.2, a weakening of Conjecture 1.1 for d-regular graphs in which all lists are constant , by verifying the conjecture for $d = 5$ and $d = 6$ in Theorem 1.2. We also obtain additional results for special classes of forbidden lists. Roughly speaking, these include Theorem 1.5, where no three consecutive out-degrees are forbidden, and Theorem 1.6, where the forbidden out-degrees occur at the tail ends of the possible values (from 0 to the degree of a vertex). We also prove Theorem 1.7, which applies to specific lists in graphs with large minimum degree. A natural direction for future research is to determine whether these techniques can be pushed to larger values of d, or adapted to handle more general families of forbidden lists.

The second part of this thesis concerns total-coloring extensions. In Section 1.2, we initiate the study of total-coloring extensions by formulating the conjectures introduced in Section 1.2.2. These conjectures ask when it is possible to extend a totally precolored matching, sometimes under additional distance conditions, to a total-coloring of the entire graph. They are motivated by analogous results in vertex- and edge-coloring. In Chapter 3, we focus on planar graphs with large maximum degree and provide evidence for these conjectures by verifying Conjecture 1.5 and Conjecture 1.6 in Theorem 1.11 and Theorem 1.12. We also establish extension results in Theorem 1.13 when the precolored subgraph has bounded maximum degree. Future work includes extending these results to all planar graphs and relaxing the assumptions on the precolored subgraph.

In Chapter 4, we establish extension theorems beyond planar graphs with large maximum degree and verify Conjecture 1.5 for paths, cycles, trees, and prisms. For a tree T with maximum degree Δ, it is well known that $\chi_T(T) = \Delta + 1$. However, Fig. 4.2 shows that, even for sufficiently far apart precolored matchings, one cannot always extend to a total-$(\Delta + 1)$-coloring. This example occurs when $\Delta = 2$, and it is natural to ask whether the same phenomenon persists for trees with larger maximum degree.

These results for trees, cycles, ladders and prisms provide further evidence toward Conjecture 1.5 and suggest that similar extension properties may hold for broader classes, such as bipartite or subcubic graphs. For a bipartite graph G with maximum degree Δ, we see from Corollary 4.1 (a direct application of Galvin's Theorem) that any distance-2 matching in G extends to a total-$(\Delta + 3)$-coloring, using one additional color beyond the palette in Conjecture 1.5. However, Example 3.2 shows that such a statement cannot hold with a $\Delta + 2$ palette when the distance condition is 2. On the other hand, Conjecture 1.5 suggests that the palette should reduce to $\Delta + 2$ when the distance condition is at least 3. This naturally leads to the question of whether Galvin's theorem can be refined to establish Conjecture 1.5 for bipartite graphs.

Bibliography

[1] D. B. West, *Introduction to Graph Theory*, 2nd ed. Prentice Hall, September 2000.

[2] S. L. Hakimi, "On the degrees of the vertices of a directed graph," *J. Franklin Inst.*, vol. 279, pp. 290–308, 1965. [Online]. Available: https://doi.org/10.1016/0016-0032(65)90340-6

[3] S. Akbari, M. Dalirrooyfard, K. Ehsani, K. Ozeki, and R. Sherkati, "Orientations of graphs avoiding given lists on out-degrees," *J. Graph Theory*, vol. 93, no. 4, pp. 483–502, 2020. [Online]. Available: https://doi-org.spot.lib.auburn.edu/10.1002/jgt.22498

[4] X. Ma and H. Lu, "A characterization on orientations of graphs avoiding given lists on out-degrees," 2023, http://arxiv.org/abs/2310.15650.

[5] P. Bradshaw, Y. Chen, H. Ma, B. Mohar, and H. Wu, "List-avoiding orientations," 2022, http://arxiv.org/abs/2209.09107.

[6] P.-Y. Huang, T.-L. Wong, and X. Zhu, "Application of polynomial method to on-line list colouring of graphs," *European J. Combin.*, vol. 33, no. 5, pp. 872–883, 2012. [Online]. Available: https://doi.org/10.1016/j.ejc.2011.09.020

[7] N. Alon and M. Tarsi, "Colorings and orientations of graphs," *Combinatorica*, vol. 12, no. 2, pp. 125–134, 1992. [Online]. Available: https://doi.org/10.1007/BF01204715

[8] N. Alon, "Combinatorial Nullstellensatz," 1999, vol. 8, no. 1-2, pp. 7–29, recent trends in combinatorics (Mátraháza, 1995).

[9] A. Frank and A. Gyárfás, "How to orient the edges of a graph?" in *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, ser. Colloq. Math. Soc. János Bolyai, vol. 18. North-Holland, Amsterdam-New York, 1978, pp. 353–364.

[10] N. Alon, B. Bollobás, A. Gyárfás, J. Lehel, and A. Scott, "Maximum directed cuts in acyclic digraphs," *J. Graph Theory*, vol. 55, no. 1, pp. 1–13, 2007.

[11] K. Appel and W. Haken, "Every planar map is four colorable. part i: Discharging," *Illinois Journal of Mathematics*, vol. 21, no. 3, pp. 429–490, 1977.

[12] ——, "Every planar map is four colorable. part ii: Reducibility," *Illinois Journal of Mathematics*, vol. 21, no. 3, pp. 491–567, 1977.

[13] V. G. Vizing, "On an estimate of the chromatic class of a p-graph," *Diskret. Analiz*, vol. 3, pp. 25–30, 1964, in Russian.

[14] M. Behzad, "Graphs and Their Chromatic Numbers," *Ph.D. thesis, Michigan State University, East Lansing*, 1965.

[15] V. G. Vizing, "Some unsolved problems in graph theory," *Russ. Math. Surv.*, vol. 23, pp. 125–141, 1968.

[16] A. Kostochka, “The total chromatic number of any multigraph with maximum degree five is at most seven,” *Discrete Mathematics*, vol. 162, no. 1, pp. 199–214, 1996. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0012365X95002866

[17] D. P. Sanders and Y. Zhao, “On total 9-coloring planar graphs of maximum degree seven,” *J. Graph Theory*, vol. 31, no. 1, pp. 67–73, 1999. [Online]. Available: https://doi.org/10.1002/(sici)1097-0118(199905)31:1<67::aid-jgt6>3.0.co;2-c

[18] Ł. Kowalik, J.-S. Sereni, and R. Škrekovski, “Total-coloring of plane graphs with maximum degree nine,” *SIAM J. Discrete Math.*, vol. 22, no. 4, pp. 1462–1479, 2008. [Online]. Available: https://doi.org/10.1137/070688389

[19] J. Geetha, N. Narayanan, and K. Somasundaram, “Total colorings—a survey,” *AKCE Int. J. Graphs Comb.*, vol. 20, no. 3, pp. 339–351, 2023. [Online]. Available: https://doi.org/10.1080/09728600.2023.2187960

[20] T. R. Jensen and B. Toft, *Graph coloring problems*, ser. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995, a Wiley-Interscience Publication.

[21] D. Woodall, *List colourings of graphs*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, 2001, p. 269–301.

[22] K. Edwards, A. Girão, J. van den Heuvel, R. J. Kang, G. J. Puleo, and J.-S. Sereni, “Extension from precoloured sets of edges,” *Electron. J. Combin.*, vol. 25, no. 3, pp. Paper No. 3.1, 28, 2018. [Online]. Available: https://doi.org/10.37236/6303

[23] M. O. Albertson and E. H. Moore, “Extending graph colorings using no extra colors,” *Discrete Math.*, vol. 234, no. 1-3, pp. 125–132, 2001. [Online]. Available: https://doi.org/10.1016/S0012-365X(00)00376-9

[24] A. Girão and R. J. Kang, “A precolouring extension of Vizing’s theorem,” *J. Graph Theory*, vol. 92, no. 3, pp. 255–260, 2019. [Online]. Available: https://doi.org/10.1002/jgt.22451

[25] Y. Cao, G. Chen, G. Jing, X. Qi, and S. Shan, “Precoloring extension of Vizing’s theorem for multigraphs,” *European J. Combin.*, vol. 122, pp. Paper No. 104 037, 17, 2024. [Online]. Available: https://doi.org/10.1016/j.ejc.2024.104037

[26] V. G. Vizing, “On an estimate of the chromatic class of a p-graph,” *Diskret. Analiz*, no. 3, pp. 25–30, 1964.

[27] J. Harrelson, J. McDonald, and G. J. Puleo, “List-edge-colouring planar graphs with precoloured edges,” *European Journal of Combinatorics*, vol. 75, pp. 55–65, 2019. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0195669818301148

[28] M. O. Albertson, “You can’t paint yourself into a corner,” *J. Combin. Theory Ser. B*, vol. 73, no. 2, pp. 189–194, 1998. [Online]. Available: https://doi.org/10.1006/jctb.1998.1827

[29] M. O. Albertson, A. V. Kostochka, and D. B. West, “Precoloring extensions of Brooks’ theorem,” *SIAM J. Discrete Math.*, vol. 18, no. 3, pp. 542–553, 2004/05. [Online]. Available: https://doi.org/10.1137/S0895480103425942

[30] J. P. Hutchinson and E. H. Moore, “Distance constraints in graph color extensions,” *J. Combin. Theory Ser. B*, vol. 97, no. 4, pp. 501–517, 2007. [Online]. Available: https://doi.org/10.1016/j.jctb.2006.08.003

[31] C. Ojima, A. Saito, and K. Sano, “Precoloring extension involving pairs of vertices of small distance,” *Discrete Appl. Math.*, vol. 166, pp. 170–177, 2014. [Online]. Available: https://doi.org/10.1016/j.dam.2013.10.012

[32] O. Borodin, A. Kostochka, and D. Woodall, “List edge and list total colourings of multigraphs,” *Journal of Combinatorial Theory, Series B*, vol. 71, no. 2, pp. 184–204, 1997. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0095895697917809

[33] O. Henderschedt and J. McDonald, “On orientations with forbidden out-degrees,” *Discrete Applied Mathematics*, vol. 387, pp. 116–123, 2026. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0166218X26001265

[34] O. Henderschedt and J. McDonald, “Extending total colorings in planar graphs,” *arXiv preprint arXiv:2509.18940*, 2025.

[35] F. Galvin, “The list chromatic index of a bipartite multigraph,” *Journal of Combinatorial Theory, Series B*, vol. 63, no. 1, pp. 153–158, 1995.

[36] M. Rosenfeld, “On the total colouring of certain graphs,” *Israel Journal of Mathematics*, vol. 9, no. 3, pp. 396–402, 1971.

[37] N. Vijayaditya, “On total chromatic number of a graph,” *Journal of the London Mathematical Society*, vol. 2, no. 3, pp. 405–408, 1971.

[38] Y. Feng and W. Lin, “A concise proof for total coloring of subcubic graphs,” *Information Processing Letters*, vol. 113, no. 18, pp. 664–665, 2013.