Path and Cycle Decompositions

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Path and Cycle Decompositions

Chandra Dinavahi

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Path and Cycle Decompositions

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Vita

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Dissertation Abstract

Path and Cycle Decompositions

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A G-design is a partition of edge set of K_v in which each element induces a copy of G. The existence of G-designs with the additional property that they contain no proper subsystems has been previously settled when $G \in \{K_3, K_4 - e\}$ by Rodger and Spicer. In this dissertation, we first solved the problem of G-designs with no subsystems where $G = P_3$, considering the problem for both designs and maximum packings with non-empty leaves. We then completely settled the problem for the general case of P_m-designs which contain no proper subsystems for every value of m and v.

We also solved another problem. A 4-cycle system is said to be diagonally switchable if each 4-cycle can be replaced by another 4-cycle obtained by replacing one pair of non-adjacent edges of the original 4-cycle by its diagonals so that the transformed set of 4-cycles forms another 4-cycle system. The existence of diagonally switchable 4-cycle system of K_v has already been solved [1]. In this paper we give an alternative proof of this result and use the method to prove a new result for $K_v - I$, where I is any one factor of K_v.

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Special thanks to my fellow graduate students and friends who made my stay at Auburn as such a fun.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

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Chapter 1

Introduction

For any two graphs G and H, a G-decomposition of H is an ordered pair $T = (V, D)$, where V is the vertex set of H and D is a partition of the edge set of H, each element of which induces a copy of G. For any graph G and any set L of edges in K_v, a G-packing with a leave L of order v is an ordered pair $T = (V, B)$, where V is the vertex set of K_v and B is a partition of the edge set of $K_v - L$, each element of which induces a copy of G. A G-packing of order v with leave L is said to be maximum if there is no G-packing of order v with leave L' such that $|L'| < |L|$. A proper subsystem of T is an ordered pair $S = (V', B')$ where $V' \subset V$, $B' \subset B$ and (V', B') is a G-design of $K_{v'}$ for $|V| > |V'| = v' > 1$. A G-packing with $L = \emptyset$ is said to be a G-design.

When considering graph decompositions the most natural question is to find the set of values of v for which there exists a decomposition of K_v into edge-disjoint copies of a fixed graph G. This set of values is called as the spectrum of G-decompositions of K_v. This question has been settled for many G, for example, where G is: K_v for $v \in \{3, 4\}$ [10, 12], a star [19], a path [18], any graph with no more than four vertices [3], or a connected graph with no more than five edges [9], a cycle [11, 2, 16]. For an up to date survey see [6].

Another question considered in the literature concerns the existence of G-designs which contain no proper subsystems. That is, for which values of v is it possible to find a G-design (V, C) of order v such that there does not exist a G-design (W, D) where $W \subset V$ and $D \subset C$. Doyen [8] settled this question for Steiner triple systems (that is,when $G = K_3$). Rodger and Spicer solved this problem when $G = K_4 - e$ [15]. The reader may also be interested to note that the related problem for Steiner quadruple systems has been considered, but is still unsolved [14].

The main result in this dissertation is to solve the problem of G-designs with no proper subsystems for the particular case where $G = P_3$, considering the problem for both designs and for maximum packings with non-empty leaves. We then solve the problem for the general case in which $G = P_m$, a simple path with m edges, and $H = K_v$ for every value of m and v.

A second problem is also solved. A C_4-decomposition of G is also known in the literature as a 4-cycle system of order G. A C_4-decomposition of K_v is said to be a 4-cycle system of order v and is denoted by $4CS(v)$. Such a decomposition exists if and only if $v \equiv 1 \pmod 8$ [13]. In this dissertation we consider a class of 4-cycle systems with diagonally switchable property. A set of 4-cycles is said to be diagonally switchable if each 4-cycle can be replaced by another 4-cycle obtained by replacing one pair of non-adjacent edges of the original 4-cycle by its diagonals. In this dissertation we solve the problem of a C_4-decomposition of a graph G with the property of being diagonally switchable, where G is K_v or $K_v - I$. Here, I is any one factor of K_v. A 1-factor I of a complete graph is a spanning one regular subgraph of K_v. Note that in order to have a 1-factor v has to be even.

The decomposition of the complete graph K_v into diagonally switchable 4-cycles has already been solved [1], but we have come up with another construction. This construction not only solves the case for K_v in a more efficient way, but is also powerful enough to easily solve the case for $K_v - I$.

Chapter 2

Maximum Packings of K_v with copies of P_3 which contain no proper subsystems

2.1 Introduction

We now turn our attention to decompositions of K_v with copies of P_3 which contain no proper subsystems. We will consider this problem for both designs and for maximum packings with non-empty leaves. The constructions used here are of interest in their own right, being neat modifications of the Bose construction for Steiner triple systems.

Before proceeding to the constructions, we need some definitions and notation. Let (a, b, c, d) denote the 3-path induced by the edge set $\{\{a, b\}, \{b, c\}, \{c, d\}\}$ (see figure 2.1).

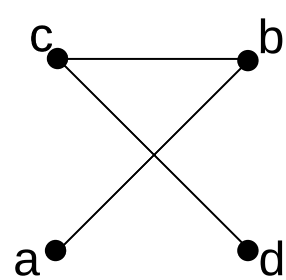

Figure 2.1: Example-1: P_3

Suppose (V, B) is a P_3-packing of order v and suppose (V', B') is a subsystem of (V, B). Notice that if $P_1 = (a, b, c, d) \in B'$ then $\{a, c\} \subset V'$; so if P_2 is the 3-path containing the edge $\{a, c\}$ then any subsystem containing P_1 must also contain all the vertices in P_2. We denote this fact by writing $P_1 \rightarrow P_2$. We also write $P_1 \rightarrow \{a, c\}$ to denote the fact that any subsystem containing P_1 must contain the vertices in $\{a, c\}$ and we will write $\{a, b\} \to P_1$ to note that the edge $\{a, b\}$ is in P_1.

2.2 Constructions

We consider three cases in turn, depending on the congruence of v (mod 3). In all the following constructions, in any ordered pair reduce arithmetic operations modulo k in the first component and modulo 3 in the second component.

2.2.1 Case A: $v = 3k$.

We begin with a construction of a P_3-design.

The $3k$ Construction. Let $V = Z_k \times \{1, 2, 3\}$ and G be a copy of K_{3k} defined on the vertex set V and define a collection B of copies of P_3 as follows (see figure 2.2).

(1) **Type 1:** for each $x \in Z_k$, let

$$P_1(x) = ((x + 1, 2), (x, 1), (x, 3), (x, 2)) \in B.$$

(2) **Type 2:** for $\{x, y\} \subseteq Z_k$, $x \neq y$, let

$$P_2(x, y) = ((x + 1, 2), (y, 1), (x, 1), (y + 1, 2)) \in B.$$

(3) **Type 3:** for $0 \leq x < y \leq k - 1$ and for $3 \leq i \leq 4$, let

$$P_i(x, y) = ((x, i), (y, i - 1), (x, i - 1), (y, i)) \in B.$$

In this case every maximum packing has empty leave, so a P_3-design is required. We now show this is what The $3k$ Construction produces.

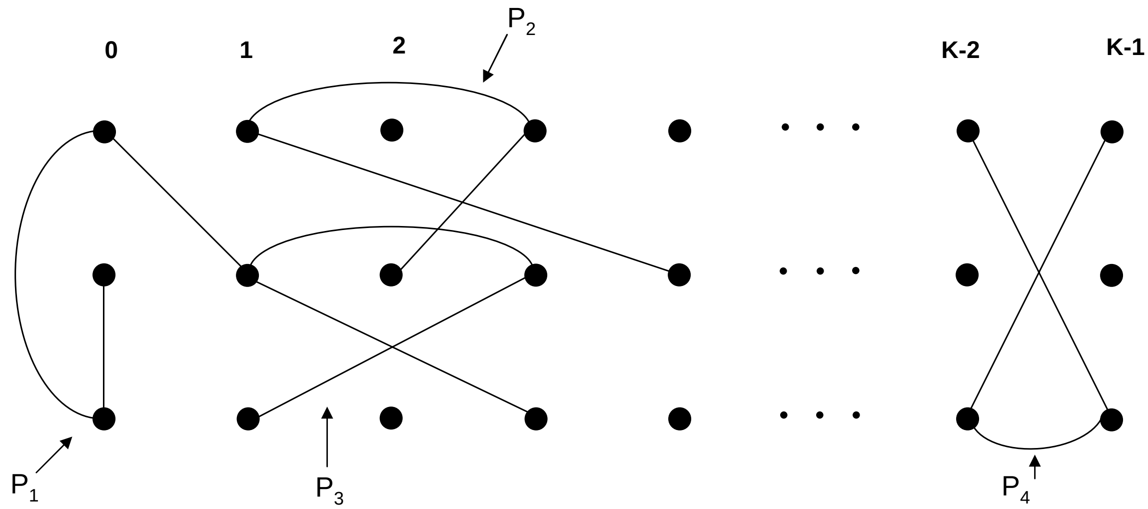

Figure 2.2: The $3k$ Construction

Proposition 2.2.1. *The $3k$ Construction produces a P_3-design of order $3k$.*

Proof. The total number of paths in a P_3-design of order $v = 3k$ is $\binom{v}{2}/3 = k(3k-1)/2$. We begin by counting the number of paths in B.

The number of Type 1 paths is clearly k. For $2 \leq i \leq 4$, $P_i(x, y)$ contains $k(k-1)/2$ choices for x and y. Therefore $|B| = k + 3k(k-1)/2 = k(3k-1)/2$ as required.

Now it remains to show that each edge e in $E(K_{3k})$ occurs in some P_3 in B. If $e = \{(x, i), (y, i)\}$ with $x, y \in Z_k$, $x \neq y$ and $1 \leq i \leq 3$, then e occurs in $P_{i+1}(x, y)$. If $e = \{(x, i), (x, j)\}$where $1 \leq i < j \leq 3$ then e occurs in $P_2(x-1, x)$ if $i = 1$ and $j = 2$ and otherwise it occurs in $P_1(x)$. If $e = \{(x, i), (y, i+1)\}$ where $1 \leq i \leq 3$ and $\{x, y\} \subseteq Z_k$, $x \neq y$, then e occurs in

(i) $P_{i+1}(x, y)$ if $2 \leq i \leq 3$,

(ii) $P_2(x, y-1)$ if $i = 1$ and $y \neq x+1$, and

(iii) $P_1(x)$ otherwise.

□

Proposition 2.2.2. *The P_3-design (V, B) of order $v = 3k$ constructed using The $3k$ Construction contains no proper subsystems.*

Proof. Suppose (V', B') is a subsystem of (V, B). We consider each possible copy of P_3 in B' by considering three cases in turn, eventually showing that in fact $V' = V$.

Case 1: The subsystem contains $P_1(x)$ for some $x \in Z_k$. Notice that

$P_1(x) \to \{(x, 2), (x+1, 2)\} \to P_3(x, x+1) \to \{(x+1, 2), (x+1, 3)\} \to P_1(x+1)$. Therefore, if $P_1(x) \in B'$ for some $x \in Z_k$ then $P_1(x) \in B'$ for all $x \in Z_k$. Since $\bigcup_{x \in Z_k} V(P_1(x)) = V$ it follows that $V' = V$.

Case 2: The subsystem contains $P_i(x, y)$ for some $i \in \{3, 4\}$ and some $x, y \in Z_k$ with $x < y$. Then $P_i(x, y) \to \{(y, i-1), (y, i)\} \to P_1(y)$. So by Case 1, $V' = V$.

Case 3: The subsystem contains $P_2(x, y)$ for some $x, y \in Z_k$ with $x < y$. Then $P_2(x, y) \to \{(x+1, 2), (y+1, 2)\} \to P_3(x+1, y+1)$. So by Case 2, $V' = V$.

Therefore since $V' \neq \emptyset$, $V' = V$. So (V, B) contains no proper subsystems. □

2.2.2 Case B: $v = 3k + 1$.

We begin with a construction of a P_3-design.

The $3k + 1$ Construction. Let $V = \{\{\infty\} \cup (Z_k \times \{1, 2, 3\})\}$ and G be a copy of K_{3k+1} defined on the vertex set V and define a collection B of copies of P_3 as follows (see figure 2.3).

(1) **Type 1:** for each $x \in Z_k$, let

$$P_{1,1}(x) = (\infty, (x, 2), (x, 3), (x, 1))$$

$$P_{1,2}(x) = ((x, 3), \infty, (x, 1), (x+1, 2)).$$

$P_1(x) = \{P_{1,i}(x) \mid 1 \leq i \leq 2\}$ and $P_1(x) \subseteq B$.

(2) **Type 2:** for $\{x, y\} \subseteq Z_k$, $x \neq y$, let

$$P_2(x, y) = ((x+1, 2), (y, 1), (x, 1), (y+1, 2)) \in B.$$

(3) **Type 3:** for $0 \leq x < y \leq k-1$ and for $3 \leq i \leq 4$, let

$$P_i(x, y) = ((x, i), (y, i-1), (x, i-1), (y, i)) \in B.$$

In this case every maximum packing has empty leave, so a P_3-design is required. The following result shows this is what The $3k+1$ Construction produces.

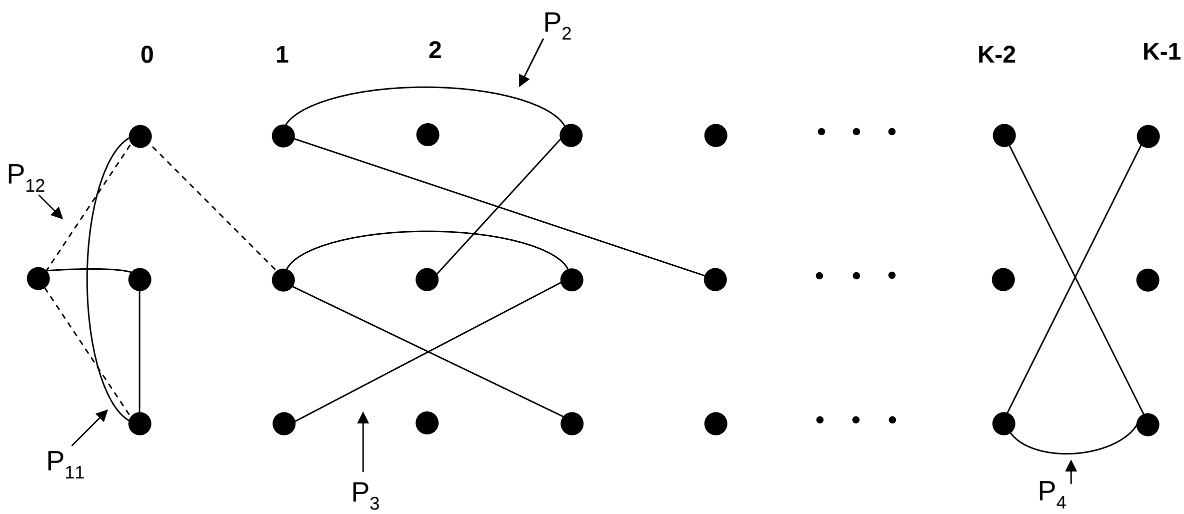

Figure 2.3: The $3k+1$ Construction

For those familiar with modifications of the Bose construction for Steiner triple systems to produce P_3-designs, one would expect to see subsystems on the vertex set $\{\{\infty\} \cup (Z_k \times \{1, 2, 3\})\}$. As the following shows, such subsystems are destroyed by a bijection mapping the edge$\{(x, 1), (y, 2)\}$ to $\{(x, 1), (y+1)\}$ (see$P_{1,2}$). This has the dual effect of keeping paths intact and creates no subsystems in the process, as the following shows.

Proposition 2.2.3. *The* $3k+1$ *Construction produces a* P_3*-design of order* $3k+1$.

Proof. The total number of paths in a P_3-design of order $3k+1$ is $\binom{v}{2}/3 = k(3k+1)/2$. We begin by counting the number of paths in B.

The number of Type 1 paths is clearly $2k$. For $2 \leq i \leq 4$, $P_i(x, y)$ contains $k(k-1)/2$ choices for x and y. Therefore $|B| = 2k + 3k(k-1)/2 = k(3k+1)/2$ as required.

Now it remains to show that each edge e in $E(K_{3k+1})$ occurs in some P_3 in B. If $e = \{(x,i),(y,i)\}$ with $x,y \in Z_k$, $x \neq y$ and $1 \leq i \leq 3$, then e occurs in $P_{i+1}(x,y)$. If $e = \{(x,i),(x,j)\}$ where $1 \leq i < j \leq 3$ then e occurs in $P_2(x-1,x)$ if $i = 1$ and $j = 2$ and otherwise it occurs in $P_{1,1}(x)$. If $e = \{\infty,(x,i)\}$ where$1 \leq i \leq 3$ then e occurs in $P_1(x)$. If $e = \{(x,i),(y,i+1)\}$ where $1 \leq i \leq 3$ and $\{x,y\} \subseteq Z_k$, $x \neq y$, then e occurs in

(i) $P_{i+1}(x,y)$ if $2 \leq i \leq 3$,

(ii) $P_2(x,y-1)$ if $i = 1$ and $y \neq x+1$, and

(iii) $P_{1,2}(x)$ otherwise.

□

Proposition 2.2.4. *The P_3-design (V,B) of order $v = 3k+1$ constructed using The $3k+1$ Construction contains no proper subsystems.*

Proof. Suppose (V',B') is a subsystem of (V,B). We consider each possible copy of P_3 in B' by considering four cases in turn, eventually showing that in fact $V' = V$.

Case 1: The subsystem contains $P_{1,1}(x)$ for some $x \in Z_k$. Notice that

$P_{1,1}(x) \to \{(\infty,(x,1)\} \to P_{1,2}(x) \to \{\infty,(x+1,2))\} \to P_{1,1}(x+1)$. Therefore, if $P_{1,1}(x) \in B'$ for some $x \in Z_k$ then $P_{1,1}(x) \in B'$ for all $x \in Z_k$. Since $\bigcup_{x\in Z_k} V(P_{1,1}(x)) = V$, it follows that $V' = V$.

Case2: The subsystem contains $P_{1,2}(x)$ for some $x \in Z_k$. Notice that

$P_{1,2}(x) \to \{(x,1),(x,3)\} \to P_{1,1}(x)$. So by Case 1, $V' = V$.

Case 3: The subsystem contains $P_i(x,y)$ for some $i \in \{3,4\}$ and some $x,y \in Z_k$ with $x < y$. Then $P_i(x,y) \to \{(x,i-1),(x,i)\} \to P_{1,1}(x)$. So by Case 1, $V' = V$.

Case 4: The subsystem contains $P_2(x,y)$ for some $x,y \in Z_k$ with $x < y$. Then $P_2(x,y) \to \{(x+1,2),(y+1,2)\} \to P_3(x+1,y+1)$. So by Case 3, $V' = V$.

Since $V' \neq \emptyset$, $V' = V$. So (V, B) contains no proper subsystems. □

2.2.3 Case C: $v = 3k + 2$.

We begin with a construction of a P_3-design.

The $3k+2$ Construction. Let $V = \{\{\infty_1, \infty_2\} \cup (Z_k \times \{1, 2, 3\})\}$ and G be a copy of K_{3k+2} defined on the vertex set V and define a collection B of copies of P_3 as follows.

(1) **Type 1:** for each $x \in Z_k$, let

$$P_{1,1}(x) = (\infty_2, (x, 3), \infty_1, (x, 1))$$

$$P_{1,2}(x) = ((x, 2), \infty_2, (x, 1), (x + 1, 2))$$

$$P_{1,3}(x) = (\infty_1, (x, 2), (x, 3), (x, 1)).$$

$P_1(x) = \{P_{1,i}(x) \mid 1 \leq i \leq 3\}$ and $P_1(x) \subseteq B$ (see figure 2.4).

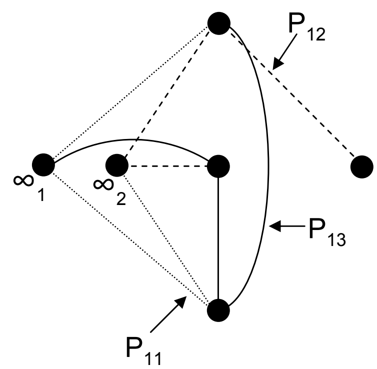

Figure 2.4: $P_1(x)$

(2) **Type 2:** for $\{x, y\} \subseteq Z_k$, $x \neq y$, let

$$P_2(x, y) = ((x + 1, 2), (y, 1), (x, 1), (y + 1, 2)) \in B.$$

(3) **Type 3:** for $0 \leq x < y \leq k - 1$ and for $3 \leq i \leq 4$, let

$$P_i(x, y) = ((x, i), (y, i - 1), (x, (i - 1), (y, i))) \in B.$$

In this case every maximum packing has leave L of size 1. The following result shows this is what The $3k+2$ Construction produces.

Proposition 2.2.5. *The $3k+2$ Construction produces a P_3-packing of order $3k+2$ with a leave $L = \{\infty_1, \infty_2\}$ of size 1.*

Proof. The total number of paths in a maximum P_3-packing of order $3k+2$ with leave of size 1 is $\binom{v}{2}/3 - 1 = 3(k^2+k)/2$. We begin by counting the number of paths in B.

The number of Type 1 paths is clearly $3k$. For $2 \le i \le 4$, in defining $P_i(x,y)$ contains $k(k-1)/2$ choices for x and y. Therefore $|B| = 3k + 3k(k-1)/2 = 3(k^2+k)/2$ as required.

Now it remains to show that each edge e in $E(K_{3k+2}) - \{\infty_1, \infty_2\}$ occurs in some P_3 in B. If $e = \{(x,i),(y,i)\}$ with $x, y \in Z_k$, $x \neq y$ and $1 \le i \le 3$, then e occurs in $P_{i+1}(x,y)$. If $e = \{\infty_1, (x,i)\}$ or $\{\infty_2, (x,i)\}$ with $x \in Z_k$, $1 \le i \le 3$, then e occurs in $P_1(x)$. If $e = \{(x,i),(x,j)\}$ with $x \in Z_k$, $1 \le i \le j \le 3$ and $i \neq 1, j \neq 2$ then e occurs in $P_{1,3}(x)$ otherwise occurs in $P_2(x-1,x)$. If $e = \{(x,i),(y,i+1)\}$ where $1 \le i \le 3$ and $\{x,y\} \subseteq Z_k$, $x \neq y$ then e occurs in

(i) $P_{i+1}(x,y)$ if $2 \le i \le 3$,

(ii) $P_2(x, y-1)$ if $i = 1$ and $y \neq x+1$, and

(iii) $P_{1,2}(x)$ otherwise.

□

Proposition 2.2.6. *The maximum P_3-packing (V,B) of order $v = 3k+2$ constructed using The $3k+2$ Construction contains no proper subsystems.*

Proof. Suppose (V', B') is a subsystem of (V,B). We consider each possible copy of P_3 in B' by considering five cases in turn, eventually showing that in fact $V' = V$.

Case 1: The subsystem contains $P_{1,1}(x)$ for some $x \in Z_k$. Notice that

$$P_{1,1}(x) \to \{\infty_2, (x,1)\} \to P_{1,2}(x) \to \{\infty_1, (x+1,2)\} \to P_{1,1}(x+1).$$

Therefore, if $P_{1,1}(x) \in B'$ for some $x \in Z_k$ then $P_{1,i}(x) \in B'$ for all $x \in Z_k$ where $1 \leq i \leq 2$. Since $\bigcup_{x \in Z_k} \bigcup_{1 \leq i \leq 2} V(P_{1,i}(x)) = V$, it follows that $V' = V$.

Case 2: The subsystem contains $P_{1,2}(x)$ for some $x \in Z_k$. Notice that $P_{1,2}(x) \to \{(x,2),(x+1,2)\} \to P_3(x,x+1) \to \{\infty_2,(x,3))\} \to P_{1,1}(x)$. So by Case 1, $V' = V$.

Case 3: The subsystem contains $P_{1,3}(x)$ for some $x \in Z_k$. Notice that $P_{1,3}(x) \to \{\infty_1,(x,1))\} \to P_{1,1}(x)$. So by Case 1, $V' = V$.

Case 4: The subsystem contains $P_2(x,y)$ for some $x, y \in Z_k$ with $x < y$. Then $P_2(x,y) \to \{(x,1),(x+1,2)\} \to P_{1,2}(x)$. So by Case 2, $V' = V$.

Case 5: The subsystem contains $P_i(x,y)$ for some $i \in \{3,4\}$ and some $x, y \in Z_k$ with $x < y$. Then $P_i(x,y) \to \{(x,i-1)(x,i)\} \to P_{1,3}(x)$. So by Case 1, $V' = V$.

Therefore since $V' \neq \emptyset$, $V' = V$. So (V,B) contains no proper subsystems. □

2.2.4 Remarks

The constructions described in this section can easily be adapted to construct P_m-designs for odd values of m and specific, and a related construction will produce P_m-designs when m is even. These constructions are likely to produce a framework for constructing P_m-designs with no subsystems for all values of v and m, providing the small values of v can be settled. The results in Chapter 3 abandon this approach since it seems that solving the problem for small values of v can be extended to a method that works for all v.

Chapter 3

Decomposition of a K_v into copies of P_m which contain no proper subsystems

In this chapter we solve the G-design problem with no subsystems for the case $G = P_m$, a simple path with m edges, for every value of m. The existence of P_m-decompositions of K_v was solved by Tarsi [18], by proving the following result.

Theorem 3.1 ([18]). *A necessary and sufficient condition for the existence of a decomposition of a complete multigraph λK_v into edge disjoint simple paths of length m is*

$$\begin{aligned} &v = 1, \text{ or} \\ &\lambda v(v-1) \equiv 0 \ (\text{mod } 2m) \text{ and } v \geq m+1. \end{aligned} \tag{3.1}$$

The approach used in proving the result involves both modifications of Tarsi's constructions and in some cases to come up with a completely new construction to make sure that the P_m-designs have no subsystems. We have created new techniques to check for subsystems in our constructions. These proof techniques can be easily applied to check for subsystems in many G-designs. The following section contains the basic ideas and notation which will be used throughout the rest of the chapter.

3.1 Notation and Basic Ideas

For any G-decomposition $T = (V, C)$, it will be useful to let $E(T)$ denote the edges occurring in $\bigcup_{c \in C} c$; in particular, if $S = (W, D)$ is a subsystem of T, then $E(S) = E(K_{v'})$, where $v' = |W|$. In the following constructions, the set of vertices of K_v will be either $V = Z_v$ or $Z_{v-1} \cup \{\infty\}$. Let $v = |V|$ and $\varepsilon = |E(T)|$ denote the total number of vertices and edges respectively. Let the trail $T = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}\}$ (not all vertices need be distinct) be denoted by $(x_0, x_1, \ldots, x_n)$, and if T is a path P then let the cycle $P + \{x_0, x_n\}$ also be denoted by $(x_0, x_1, \ldots, x_n)$; it will be clear from the context which structure is being used. For each trail $T = (x_0, x_1, \ldots, x_n)$ on the vertex set Z_z or $Z_z \cup \{\infty\}$ (so $z = v$ or $v - 1$ respectively), let $T + i$ be the trail $(x_0 + i, x_1 + i, \ldots, x_n + i)$, where each sum is reduced modulo z if $x_j \neq \infty$, and where $\infty + i$ is defined to be ∞. If $T_1 = (x_0, x_1, \ldots, x_n)$ and $T_2 = (y_0, y_1, \ldots, y_n)$ are two trails with $x_n = y_0$, then denote the concatenation of T_1 and T_2 by $T_1 + T_2 = (x_0, x_1, \ldots, x_n = y_0, y_1, \ldots, y_n)$. If $x \neq \infty$ and $y \neq \infty$ are two elements of Z_z for some $z \in \{v, v-1\}$, then the edge $\{x, y\}$ is said to be of difference k if $k = min\{|y - x|, z - |y - x|\}$. The set of all differences will be denoted by $D_v = \{1, 2, \ldots, \lfloor v/2 \rfloor\}$.

One of the basic ingredients used in the constructions is the trail

$$C(v, k) = (0, k+1, 1, k+2, \ldots, k-1, v-1, k, 0), \tag{3.2}$$

where $k \in D_v$ and $k < (v-3)/2$. Notice that $C(v, k)$ has length $2v$ and contains all the edges of differences k and $k+1$.

For any trail $T = (v_1, v_2, \ldots, v_k)$ and $k \geq m$, let T/m be the set of m-trails $\{(v_i, \ldots, v_{i+m}) \mid i \in \{zm + 1 \mid 0 \leq z \leq \lfloor k - 1/m \rfloor - 1\}\}$. Notice that the edges in T/m partition all but at most the last $m - 1$ of the edges in T. Our aim is to pick T carefully so that each element in T/m is a path. For any trail $T = (v_1, v_2, \ldots, v_k)$, if v_i and v_j are the first occurrences of a and b respectively in T then let $S(T, a, b)$ denote the subtrail $(v_i, v_{i+1}, \ldots, v_j)$ of T. For $x, y \in Z_z$ with $x \neq y$, let $I(x, y)$ be the path $(x, x+1, x+2, \ldots, y)$ consisting entirely of edges of difference 1 reducing the sums modulo z.

In order to prove that a given G-decomposition (V, C) does not have a subsystem (W, D), the argument here is usually based on the observation that if $\{x, y\} \subseteq W$, then there exists a path $c \in C$ containing the edge $\{x, y\}$, implying that $V(c) \subseteq W$. This observation is denoted by $\{x, y\} \to V(c)$. Often a specific vertex $\alpha \in V(c)$ is of specific interest, so we similarly write $\{x, y\} \to \alpha$ to indicate that since $\{x, y\} \subseteq W$ it follows that $\alpha \in W$. A common technique used here to show that a P_m-design has no subsystems when v is even is to focus on the pairs of vertices joined by an edge of difference $v/2$, showing that either the edge $\{u, u + v/2\} \to \{u + 1, u + v/2 + 1\}$ or $\{u, u + v/2\} \to \{u - 1, u + v/2 - 1\}$; in either case we say that the *next* half difference is also in the subsystem.

3.2 Preliminary Results

In order to prove the main result we first make the following useful observations.

Lemma 3.2. *If $m \geq 2v/3$ then every P_m-decomposition of K_v has no subsystems.*

Proof. Suppose $S = (W, D)$ is a subsystem of the P_m-decomposition (V, C) of K_v. Then since D contains a path $|W| \geq m + 1$. Consider an edge $\{x, y\}$, where $x \in W$ and $y \in V - W$. Since S is a subsystem, each edge in the path P that contains the edge $\{x, y\}$ has at least one end in $V - W$. Therefore $| V - W | \geq \lceil m/2 \rceil$. So $|V| = |W| + |V - W| \geq m + 1 + m/2 = 3m/2 + 1 \geq |V| + 1$, a contradiction. Hence the P_m-decomposition contains no subsystems. □

We now prove a lemma that is used regularly in later constructions. It considers the concatenation of various copies of $C(v, k)$ (see Equation 3.2).

Lemma 3.3. *Suppose that $i, j \in D_v$ with $v/2 > j > i$, and that $j - i$ is odd. Let T be the trail formed by the concatenation $C(v,i) + C(v,i+2) + \cdots + C(v,j-1)$. If T contains a cycle C induced by consecutive vertices, then the length of C is at least $2i+1$.*

Proof. We prove the result by showing that if T contains x consecutive edges that form a cycle C then $x \geq 2i+1$.

Looking at the structure of $C(v,i)$, any cycle consisting only of edges in $C(v,i)$ has length $2i+2$. If C contains edges from both $C(v,l)$ and $C(v,l+2)$ then C must contain precisely the first $2l$ edges in $C(v,l+2)$ together with the edge $\{0,l\}$ in $C(v,l)$, so has length $2l+1$. Since $l \geq i$, we can conclude that the length of the smallest cycle in T is $2i+1$. □

Corollary 3.4. *Suppose that $i, j \in D_v$ with $v/2 > j > i$, and that $j - i$ is odd. Let T be the trail formed by the concatenation $C(v,i) + C(v,i+2) + \cdots + C(v,j-1)$. If $m \leq 2i$ then all trails in T/m are paths.*

Proof. From Lemma 3.3 it follows that each cycle formed by the consecutive vertices in T has length at least $2i+1$. Since $m \leq 2i$, we can conclude that all trails in T/m are paths. □

Next we consider a similar concatenation.

Corollary 3.5. *Suppose that $i, j \in D_v$ with $v/2 > j > i$, that $j - i$ is odd, and that $x \in Z_z$. Let T be the trail formed by the concatenation $I(x,0) + C(v,i) + C(v,i+2) + \cdots + C(v,j-1)$. If*

$$m \leq \begin{cases} \min\{v + x - 2i - 2, 2i\} \text{ when } x \geq i+1, \text{and} \\ \min\{v + x - 1, 2i\} \text{ otherwise} \end{cases}$$

then all trails in T/m are paths.

Proof. Let C be any cycle formed by the consecutive vertices in T. If C consists only of edges in $I(x,0) + C(v,i)$ then C must contain precisely the $v - x$ edges in $I(x,0)$ together with the first $2x-2i-1$ edges of $C(v,i)$ if $x \geq i+1$ and the first $2x$ edges of $C(v,i)$ otherwise. If C contains edges from both $C(v,l)$ and $C(v,l+2)$ then from Lemma 3.3 it follows that the length of the C must be $2i+1$.

Thus we can conclude that whenever

$$m < \begin{cases} \min\{v+x-2i-1, 2i+1\} \text{ when } x \geq i+1, \textit{and} \\ \min\{v+x, 2i+1\} \text{ otherwise }, \end{cases}$$

all trails in T/m are paths.

□

3.3 The Main Result

Now we state and prove the main theorem.

Theorem 3.6. *Let $m \geq 3$. There exists a P_m-decomposition (V,C) of K_v containing no subsystems if and only if either*

$$v = 1, \text{ or } m \text{ divides } \binom{v}{2} \text{ and } v \geq m+1. \tag{3.3}$$

Proof. The necessary condition follows from two observations that if K_v contains at least one edge (so $v > 1$) then C must contain at least one path and so $|V| \geq m+1$; and since each of the $\binom{v}{2}$ edges in K_v occurs in exactly one path and each path contains exactly m edges.

In order to prove the sufficiency we now consider two cases depending on whether v is odd or even, each case considering various subcases in turn. In view of Theorem 3.1 and Lemma 3.2, if $m \geq 2v/3$ then P_m-decompositions exist and clearly have no subsystems; so we can assume that $m < 2v/3$. In particular, since $v \geq m+1 \geq 4$ it follows that $m \leq v-2$.

Case A: v is odd.

Let $C_0 = (v_0, v_1, v_2, \ldots, v_v)$ be a hamiltonian cycle defined by

$$v_i = \begin{cases} \infty \text{ if } i \in \{0, v\}, \text{ and} \\ (-1)^i \lceil (i-1)/2 \rceil \text{ otherwise,} \end{cases}$$

where each sum is reduced modulo v (see figure 3.1). Let $C_i = C_0 + i$ for each $i \in Z_{v-1}$. Then clearly $C_i = C_{i+(v-1)/2}$ for $i \in Z_{(v-1)/2}$. Also note that $\{C_i \mid i \in Z_{(v-1)/2}\}$ is the standard hamiltonian decomposition of K_v. Form an Euler tour $(e_1, e_2, \ldots, e_\varepsilon)$ by the concatenation $C_0 + C_1 + \cdots + C_{(v-3)/2}$ (see figure 3.2). For each $i \in Z_{v(v-1)/2m}$, let π_i be the trail induced by $\{e_{im+1}, e_{im+2}, \ldots, e_{(i+1)m}\}$ (see figure 3.3). Then $(V, C) = (Z_{v-1} \cup \{\infty\}, \{\pi_i \mid i \in Z_{v(v-1)/2m}\})$ is a P_m-decomposition of K_v. Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition of K_v. Now we consider various possibilities, arriving at the contradiction $W = V$ in each case.

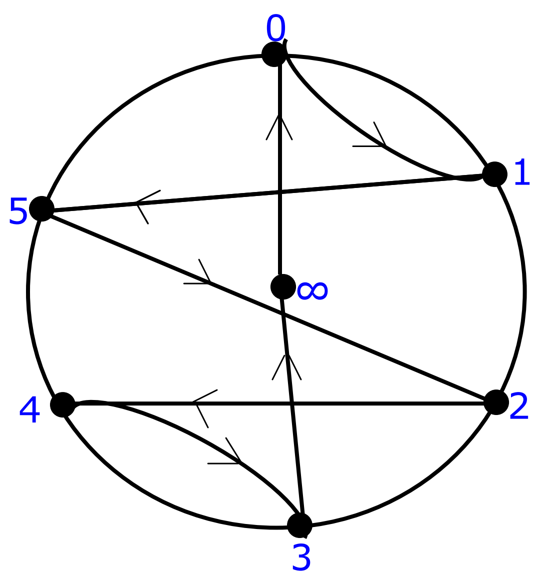

Figure 3.1: Example of C_0 in K_7

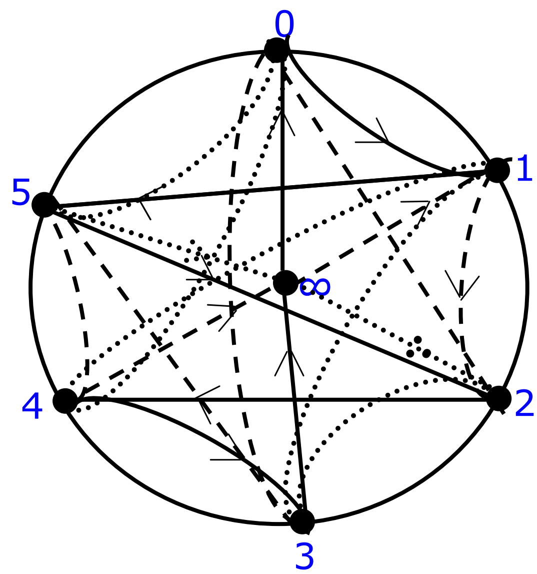

Figure 3.2: Example of Euler tour on K_7

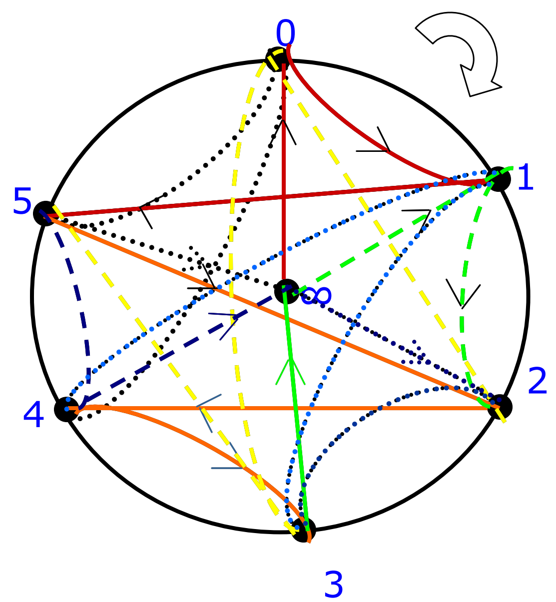

Figure 3.3: Example with $v = 7$ and $m = 3$

Case 1: Suppose $\{\infty, i\} \in E(S)$ for some i.

We will show that $\{\infty, i\} \to i+1$. By repeating this argument we can conclude that $W = V$.

Since $m \geq 3$ and $C_i = (\infty, i, i+1, \ldots, i+(v-1)/2+1, i+(v-1)/2, \infty)$, clearly $\{\infty, i\} \to i+1$ except possibly if

(a) $\{\infty, i\}$ is the last edge of some π_j and $i < (v-1)/2$, or

(b) $\{\infty, i\}$ is the first edge of some π_j and $i \geq (v-1)/2$.

We now consider each exceptional case in turn.

Case 1a: Suppose $\{\infty, i\}$ is the last edge of some π_j and $i < (v-1)/2$.

Since $m \geq 3$, in this case $\pi_j = (\ldots, i+(v-1)/2, i-1+(v-1)/2, \infty, i)$, so clearly

$$\{\infty, i\} \to i+(v-1)/2. \tag{3.4}$$

Then for some k, $\{\infty, i+(v-1)/2\}$ is in the path $\pi_{j+k} = (\ldots, i+(v-1)/2, \infty, i+1, \ldots)$ or in the path $\pi_{j+k} = (\ldots, i+(v-1)/2, \infty)$ in C.

In the first case Equation 3.4 implies that $V(\pi_{j+k}) \subseteq W$, so $i+1 \in W$ as required.

Otherwise $E(C_i) - \{\infty, i\} = \bigcup_{l=1}^{k} E(\pi_{j+l})$. So $|V| - 1 = |E(C_i) - \{\infty, i\}|$ is divisible by m. So this case only arises when $|V| \equiv 1 \pmod{m}$. So since $\pi_{j+k} = (\ldots, i+1+(v-1)/2, i+(v-1)/2, \infty)$, we have $\{\infty, i\} \to \{\infty, i+(v-1)/2\} \to i+1+(v-1)/2$. Then, since $|V| \equiv 1 \pmod{m}$, it follows that $E(C_{i+1}) - \{\infty, i+1+(v-1)/2\} = \bigcup_{l=k+1}^{2k} E(\pi_{j+l})$ and so $\pi_{j+(2k+1)} = (i+1+(v-1)/2, \infty, i+2, \ldots)$ implying that $\{\infty, i\} \to i+2$. But $\{i, i+2\}$ is in $\pi_{j+(k+1)} = (\infty, i+1, i+2, i, \ldots)$. Hence $\{\infty, i\} \to i+1$, so $i+1 \in W$ as required.

Case 1b: Suppose $\{\infty, i\}$ is the first edge of some π_j and $i \geq (v-1)/2$.

Since $m \geq 3$, in this case $\pi_j = (i, \infty, i-(v-1)/2+1, i-(v-1)/2+2, \dots)$, so clearly

$$\{\infty, i\} \to i-(v-1)/2+2. \tag{3.5}$$

Then for some k, $\{\infty, i-(v-1)/2+2\}$ is in the path $\pi_{j+k} = (\cdots, i+1, \infty, i-(v-1)/2+2, \dots)$ or in the path $\pi_{j+k} = (\infty, i-(v-1)/2+2, i-(v-1)/2+3, \dots)$ in C. So in either case Equation 3.5 implies that

$$V(\pi_{j+k}) \subseteq W \tag{3.6}$$

In the first case Equation 3.6 immediately implies that $i+1 \in W$ as required.

Otherwise $E(C_{i+1} = C_{i-(v-1)/2+1}) + \{\infty, i\} = \bigcup_{l=0}^{k-1} E(\pi_{j+l})$. So $|V|+1 = |E(C_{i+1}) + \{\infty, i\}|$ is divisible by m. So this case only arises when $|V|+1 \equiv 0 \pmod{m}$. Therefore $\pi_{j+(2k-1)} = (\dots, i+2, \infty, i-(v-1)/2+3)$. By Equation 3.6, $\{\infty, i-(v-1)/2+3\} \subseteq W$, which implies that $V(\pi_{j+2k-1}) \subseteq W$. Therefore $i+2 \in W$. Finally, notice that $\pi_{j+(k-1)} = (\dots, i, i+2, i+1, \infty)$. Hence $\{i, i+2\} \to \{i+1\}$ and so $i+1 \in W$ as required.

Case 2: Suppose $\{i, i+1\} \in E(S)$ for some $i \neq \infty$.

We will show that $\{i, i+1\} \to \{\infty, j\}$ for some j. Then the result follows by Case 1. Since the edge $\{i, i+1\}$ is either immediately precedes or follows $\{\infty, i\}$ in some C_x, clearly $\{i, i+1\} \to \{\infty, i\}$ except possibly if

(a) $\{i, i+1\}$ is the first edge of some π_j and $i < (v-1)/2$, or

(b) $\{i, i+1\}$ is the last edge of some π_j and $i \geq (v-1)/2$.

Observe that in both the exceptional cases $\{i, i+1\} \to i-1$, since $\pi_j = (i, i+1, i-1, \dots)$ or $\pi_j = (\dots, i-1, i+1, i)$ respectively. So for all x,

$$\text{either } \{x, x+1\} \to \{\infty, x\}, \text{ or } \{x, x+1\} \to \{x, x-1\}. \tag{3.7}$$

But, since $C_0 = (\infty, 0, 1, \dots)$ implies $\{0,1\} \to \{\infty, 0\}$, recursively applying the observation 3.7 implies that for all i $\{i, i+1\} \to \{\infty, j\}$ for some j (since at worst $j = 0$).

Case 3: Suppose $\{i, i+j\} \in E(S)$ for some $i \neq \infty, j > 1$.

Notice that if $\{i, i+j\}$ is in some path π_j then π_j contains at least one of the vertices $i-1, i+1, i+j-1$ or $i+j+1$. In any of these cases $\{i, i+j\} \to \{k, k+1\}$ for some $k \in \{i-1, i, i+j-1, i+j\}$. So the result follows by Case 2.

Case B: v is even.

We will solve this case by considering different subcases in turn depending on the length of the path.

Case 1: $m = v - 2$.

By Lemma 3.2 and from the fact that $m \geq 3$, we can conclude that P_m-decompositions of K_v contain no subsystems.

Case 2: $m = v - 3$.

Since $m < 2v/3$ and in this case $m = v - 3$, it follows that $v < 9$. By the necessary condition that m must divide $\binom{v}{2}$, the only situation that needs to be solved is when $v = 6$ and $m = 3$. If $v = 6$, let $Z(3)$ be the zigzag path defined by $(0, 2, 5, 3)$. So $(V, C) = (Z_6, \{\{Z(3) + i \mid i \in Z_3\} \cup (0, 1, 2, 3) \cup (3, 4, 5, 0)\})$ is a P_3-decomposition of K_6.

Suppose $S = (W, D)$ is any subsystem in this P_3-decomposition. Let $\pi_j \in D$ be any path of length 3. Suppose $\pi_j = Z(3) + i$ for some $i \in Z_3$. Each path $Z(3) + i$ contains the edge $\{k, k+3\}$ of half difference for some k. Since k and $k+3$ have different parity, it follows that $Z(3) + i$ contains both $k+1$ and $k+4$. So W must contain the vertices of the next half difference, which implies that $Z(3) + (i+1) \in D$. By repeating this argument we can conclude that $V(\{Z(3) + i | i \in Z_3\}) = W = V$.

Suppose $\pi_j = (0, 1, 2, 3)$ or $(3, 4, 5, 0)$. In either of these cases $\pi_j \to \{0, 3\} \to Z(3)+1 \in D$. Then the result follows by the above argument.

Case 3: $m \leq v - 4$ and $v - m \equiv 3 \pmod 4$.

Without loss of generality we can assume $m \leq v - 7$, because $v - m \not\equiv 3 \pmod 4$ when $m > v - 7$. Observe that in this case m is odd (since v is even in Case B and $v - m \equiv 3 \pmod 4$).

Let $Z(m)$ be the zigzag path $(v_0, v_1, \dots, v_m)$ defined by

$$v_i = \begin{cases} (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \leq i \leq \lfloor m/2 \rfloor, and \\ v_{m-i} + v/2 \text{ otherwise }, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z = \{Z(m) + i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{2, 3, \dots, \lceil m/2 \rceil\} \cup \{v/2\}$ (see figure 3.4).

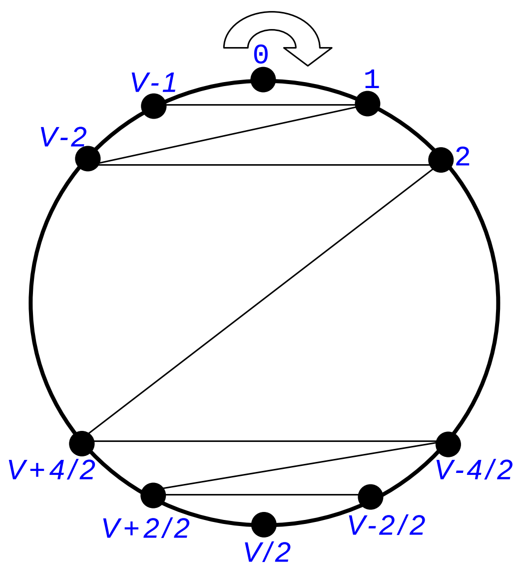

Figure 3.4: Example with $m = 7$ and general v

Let $T = (v_1, v_2, \dots, v_k)$ be the trail formed by the concatenation $I(m, 0)+C(v, \lceil m/2 \rceil + 1) + C(v, \lceil m/2 \rceil + 3) + \cdots + C(v, v/2 - 2)$. Apply Corollary 3.5 to T using $x = m$ and $i = \lceil m/2 \rceil + 1 = (m+3)/2$. Notice that in this case, if $m \geq 5$ then $x = m \geq (m+5)/2 = i+1$ and the condition of the Corollary 3.5 is met, and otherwise $m = 3$ in which case $x = m \leq i$, so clearly $m \leq \min\{2i, v+x\}$. Thus we can conclude that all trails in T/m are paths. Note that in Case 3 $v - m \equiv 3 \pmod 4$, so $\lceil m/2 \rceil + 1 \equiv v/2 - 2 (\mathrm{mod}\ 2)$. So T/m is a set of m-paths which partitions all the edges of differences in $\{\lceil m/2 \rceil + 1, \lceil m/2 \rceil + 2, \ldots, v/2 - 1\}$ and the $v - m$ edges of difference 1 from the vertex m forward to the vertex 0. So $(V, C) = (Z_v, \{Z \cup T/m \cup I(0, m)\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition of K_v. Let $\pi_j \in D$ be any path of length m. Now we consider various possibilities, arriving at the contradiction $W = V$ in each case.

Case 3a: Suppose that $\pi_j = Z(m) + i$ for some $i \in Z_{v/2}$.

Each path $Z(m) + i$ contains the edge $\{k, k + v/2\}$ of half difference for some k

Suppose $m \geq 5$. Then $Z(m) + i$ contains both $k + 1$ and $k + v/2 + 1$ (if $m \equiv 1 \pmod 4$) or both $k - 1$ and $k + v/2 - 1$ (if $m \equiv 3 \pmod 4$). So W must contain one pair of vertices in the next half difference, which implies that either $Z(m) + (i + 1)$ or $Z(m) + (i - 1) \in D$. By repeating this argument we can conclude that $V(Z) = W = V$.

Suppose $m = 3$. Then $Z(m) + i \rightarrow \{k - 2, k + v/2 - 2\} \rightarrow Z(m) + i - 2$. So recursively it follows that $X = \{k - 2i, k + v/2 - 2i \mid i \in Z_{v/2}\} \subseteq W$. Since $v - m \equiv 3 \pmod 4$ and $m = 3$, it follows that $v/2$ is odd so k and $k + v/2$ have different parity. So $X = V$; so $W = V$.

Case 3b: Suppose that $\pi_j \in I(0, m) \cup T/m \in D$.

We will show that $\pi_j \rightarrow Z(m) + i \in D$ for some i, then the result follows from Case 3a.

Suppose $m \geq 5$. Every $\pi_j \in I(0, m) \cup T/m$ contains a pair of vertices $\{k, k + 2\}$ for some k, and the edge $\{k, k + 2\} \in Z(m) + i$ for some i. So $Z(m) + i \in D$ as required.

Suppose $m = 3$. Then one of the following occurs.

(i) π_j contains the edge $\{k, k + 2\}$. So, as above, $Z(m) + i \in D$ for some i.

(i) $\pi_j = (k, k+l, k+1, k+l+1)$ is contained in a C trail. In this case the edge $\{k, k+1\}$ is in some π_n that must contain either $k-1$ or $k+2$. So S contains an edge of difference 2 (either $\{k, k+2\}$ or $\{k-1, k+1\}$). So S contains an edge in $Z(m)+i$ for some i.

(ii) $\pi_j = (k, 0, k+3, 1)$ straddles two C trails. In this case the edge $\{0, 1\} \in I(0, 3)$. So S contains the edge $\{0, 2\} \in Z(m) + 1$.

Hence the result follows by Case 3a.

Case 4: $m \leq v - 4$ and $v - m \equiv 1 \pmod 4$.

Without loss of generality we can assume $m \leq v - 5$, because $v - m \not\equiv 1 \pmod 4$ when $m > v - 5$. Observe that in this case m is odd (since v is even in Case B and $v - m \equiv 1 \pmod 4$).

Let $Z_1(m)$ be the zigzag path $(v_0, v_1, \ldots, v_m)$ defined by

$$v_i = \begin{cases} (-1)^{i+1}\lceil i/2 \rceil \text{ for } 0 \leq i \leq \lfloor m/2 \rfloor, and \\ v_{m-i} + v/2 \text{ otherwise }, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_1 = \{Z_1(m) + i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{1, 2, \ldots, \lfloor m/2 \rfloor\} \cup \{v/2\}$ (see figure 3.5).

Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $C(v, \lfloor m/2 \rfloor + 1) + C(v, \lfloor m/2 \rfloor + 3) + \cdots + C(v, v/2 - 2)$; note that in Case 4 $v - m \equiv 1 \pmod 4$, so $\lfloor m/2 \rfloor + 1 \equiv v/2 - 2 \pmod 2$. Using $i = \lfloor m/2 \rfloor + 1 = (m+1)/2$, clearly $m \leq 2i$, so Corollary 3.4 can be applied to T to conclude that all trails in T/m are paths. So T/m is a set of m-paths which partitions all the edges of differences in $\{\lfloor m/2 \rfloor + 1, \lfloor m/2 \rfloor + 2, \ldots, v/2 - 1\}$. So $(V, C) = (Z_v, \{Z_1 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Then either $\pi_j = Z_1(m) + i$ for some $i \in Z_{v/2}$ or $\pi_j \in T/m$. We will show that $W = V$ in both these cases.

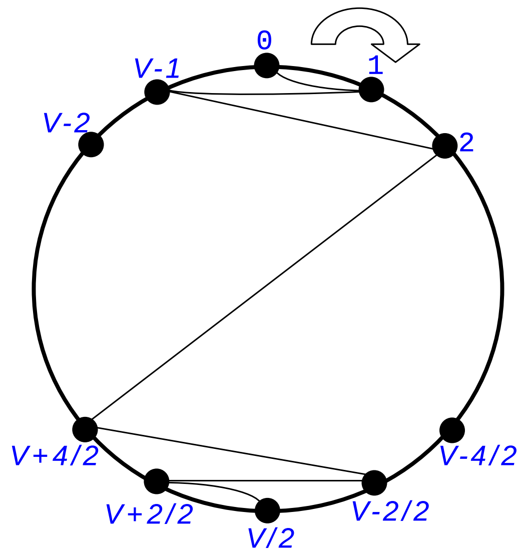

Figure 3.5: Example with $m = 7$ and general v

Suppose that $\pi_j = Z_1(m) + i$ for some $i \in Z_{v/2}$. Each path $Z_1(m) + i$ contains the edge $\{k, k + v/2\}$ of half difference for some k. Since $m \geq 3$, $Z_1(m) + i$ contains both $k + 1$ and $k + v/2 + 1$ (if $m \equiv 1 \pmod 4$) or both $k - 1$ and $k + v/2 - 1$ (if $m \equiv 3 \pmod 4$). So W must contain one pair of vertices in the next half difference, which implies that either $Z_1(m) + (i + 1)$ or $Z_1(m) + (i - 1) \in D$. By repeating this argument we can conclude that $V(Z_1) = W = V$.

If $\pi_j \in T/m$, then since $m \geq 3$, $\pi_j = (k, k + l, k + 1, k + 1 + l, \dots)$ for some k and for some l, which implies that the edge $\{k, k + 1\}$ is in some $Z_1(m) + i \in D$. So the result follows by the previous argument.

Case 5: $m \leq v - 4$ and $v - m \equiv 2 \pmod 4$.

We will solve this case by considering two subcases in turn. Without loss of generality we can assume $m \leq v - 6$, because $v - m \not\equiv 2 \pmod 4$ when $m > v - 6$. Observe that in this case m is even (since v is even in Case B and $v - m \equiv 2 \pmod 4$).

Case 5a: $m \leq v/2$.

Let $Z_2(m)$ be the zigzag path $(v_0, v_1, \dots, v_m)$ define by

$$v_i = \begin{cases} (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \le i \le m/2-1, \\ v_{m-(i+1)} + v/2 \text{ for } m/2 \le i \le m-1, and \\ v_{m-1}+1 \text{ for } i = m, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_2 = \{Z_2(m) - i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{2, 3, \ldots, m/2\} \cup \{v/2\}$ and the $v/2$ edges of difference 1 in $I(0, v/2)$ (see figure 3.6).

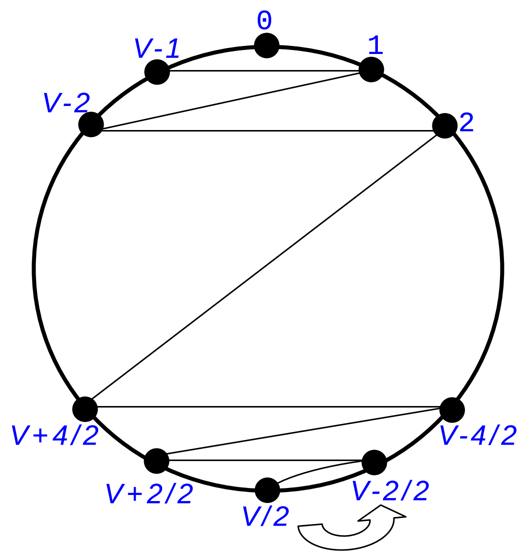

Figure 3.6: Example with $m = 8$ and general v

Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $I(v/2, 0) + C(v, m/2 + 1) + C(v, m/2 + 3) + \cdots + C(v, v/2 - 2)$. Apply Corollary 3.5 to T using $x = v/2$ and $i = m/2 + 1$; notice that in Case 5a $m \le v/2$, so $x = v/2 \ge m/2 + 2 = i + 1$ and $m \le \min\{2i, v + x - 2i - 2\}$ since $m \ge 4$. So clearly the condition of the Corollary 3.5 is met. Thus we can conclude that all trails in T/m are paths. Note that in Case 5 $v - m \equiv 2 \pmod 4$, so $m/2 + 1 \equiv v/2 - 2 \pmod 2$. So T/m is a set of m-paths which partitions all the edges of differences in $\{m/2 + 1, m/2 + 2, \ldots, v/2 - 1\}$ and the $v/2$ edges of difference 1 in $I(v/2, 0)$. So $(V, C) = (Z_v, \{Z_2 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Then either $\pi_j = Z_2(m) - i$ for some $i \in Z_{v/2}$ or $\pi_j \in T/m$. We will show that $W = V$ in both these cases.

Case 5a(i): Suppose that $\pi_j = Z_2(m) - i$ for some $i \in Z_{v/2}$.

Each path $Z_2(m) - i$ contains the edge $\{k, k + v/2\}$ of half difference for some k.

Suppose $m \geq 6$. Then $Z_2(m) - i$ contains both $k+1$ and $k+v/2+1$ (if $m \equiv 2 \pmod 4$) or both $k - 1$ and $k + v/2 - 1$ (if $m \equiv 0 \pmod 4$). So W must contain one pair of vertices in the next half difference, which implies that either $Z_2(m) - (i-1)$ or $Z_2(m) - (i+1) \in D$. By repeating this argument we can conclude that $V(Z_2) = W = V$.

Suppose $m = 4$. Then $Z_2(m) - i \rightarrow \{k-2, k+v/2-2\} \rightarrow Z_2(m) - i - 2$. So recursively it follows that $X = \{k - 2i, k + v/2 - 2i \mid i \in Z_{v/2}\} \subseteq W$. Since $v - m \equiv 2 \pmod 4$ and $m = 4$ it follows that $v/2$ is odd so k and $k + v/2$ have different parity. So $X = V$; so $W = V$.

Case 5a(ii): Suppose that $\pi_j \in T/m \in D$.

Since $m \geq 4$, every $\pi_j \in T/m$ contains a pair of vertices $\{k, k + 2\}$ for some k; since $\{k, k + 2\} \in E(Z_2(m) - i)$, it follows that $Z_2(m) - i \in D$ for some i. So $\pi_j \rightarrow Z(m) - i$ for some i, so the result follows by Case 5a(i).

Case 5b: $v/2 < m < 2v/3$.

First observe that in this case $m \geq 6$, since when $m = 4$ there is no even v which satisfies $v/2 < 4 < 2v/3$. Recall that $S(T, a, b)$ was defined to be a subtrail of T from a to b.

Let $D_1 = I(m, 0) + S(C(v, m/2 + 1), 0, m - v/2)$; it is easy to check D_1 is a path of length m. Denote by T_l, the final segment from $m - v/2$ to 0 remaining of $C(v, m/2 + 1)$; then note that $|E(T_l)| = 3v - 2m > m$. Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $I(0, m - v/2) + T_l + C(v, m/2 + 3) + \cdots + C(v, v/2 - 2)$. Note that in Case

5 $v - m \equiv 2 \pmod 4$, so $m/2 + 3 \equiv v/2 - 2 \pmod 2$, so T has all the edges of differences $m/2+1, \ldots, v/2-1$, and $v/2$ edges of difference 1 from the vertex m forward (through 0) to $m - v/2$. We now show that trails in T/m are paths by showing that if T contains consecutive vertices that form a cycle C then it has length more than m; so let C be a cycle formed by the consecutive vertices in T. Since $|E(T_l)| > m$, we need only consider 3 cases.

(i) Suppose C consists only of edges in $I(0, m - v/2) + T_l$. If C is in T_l then since T_l is a subgraph of $C(v, m/2+1)$ we can use Lemma 3.3 to conclude that the length of C is greater than m. If C contains edges from the path $I(0, m - v/2)$ then note that the first vertex to be repeated in T is either $(m - v/2) + m/2 + 2$ or 0. The number of edges between first two appearances of $3m/2 - v/2 + 2$ in T is $m + 4 > m$; and the number of edges between first two appearances of 0 in T is $(2v - m - 4) - (2m - v) + (m - v/2) = 5v/2 - 2m - 4 > m$ since $m < 2v/3$ and $v > 8$. So the length of C is greater than m.

(ii) If C is in $T_l + C(v, m/2+3) + \cdots + C(v, v/2-2)$ then C is in $C(v, m/2+1) + \cdots + C(v, v/2-2)$. So we can use Lemma 3.3 to conclude that the length of C is greater than m.

Therefore, by the above observations, it follows that, $D_1 \cup T/m$ is a set of m-paths which partitions all the edges of differences in $\{m/2+1, m/2+2, \ldots, v/2-1\}$ and the $v/2$ edges of difference 1 from the vertex m forward (through 0) to $m - v/2$.

Let $Z_3(m-1)$ be the zigzag path $(v_0, v_1, \ldots, v_{m-1})$ of length $m-1$ defined by

$$v_i = \begin{cases} (-1)^{i+1}\lceil (i+1)/2 \rceil \text{ for } 0 \leq i \leq m/2 - 1, \textit{and} \\ v_{m-(i+1)} + v/2 \text{ for } m/2 \leq i \leq m-1. \end{cases}$$

Observe that the paths in $D_1 \cup T/m$ include $v/2$ edges of difference 1, one in $L(x) = \{(x, x+1), (x + v/2, x + v/2 + 1)\}$ for each $x \in Z_{v/2}$. Thus the set L of the remaining $v/2$ edges of difference 1 also has exactly one edge in $L(x)$ for each $x \in Z_{v/2}$; so to each path in $\{Z_3(m-1)+i \mid i \in Z_{v/2}\}$ we can add one edge from L to form the set M of $v/2$ simple m-paths. Notice that the set of m-paths M partitions all the edges of differences in $\{2,3,\ldots,m/2\} \cup \{v/2\}$ and the remaining $v/2$ edges in $I(m-v/2, m)$ of difference 1. So $(V,C) = (Z_v, \{M \cup D_1 \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W,D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. Now we consider various possibilities for π_j, arriving at the contradiction $W = V$ in each case.

Suppose that $\pi_j \in M$. Each path $\pi_j \in M$ contains the edge $\{k, k+v/2\}$ of half difference for some k. Since $m \geq 6$, π_j also contains both $k+1$ and $k+v/2+1$ (if $m \equiv 2 \pmod 4$) or both $k-1$ and $k+v/2-1$ (if $m \equiv 0 \pmod 4$). So W must contain one pair of vertices in the next half difference. By repeating the argument we can conclude that $V(M) = W = V$.

Suppose that $\pi_j \in D_1 \cup T/m$. Since $m \geq 6$, every $\pi_j \in D_1 \cup T/m$ contains a pair of vertices $\{k, k+2\}$ for some k, which implies that D contains the path $\pi_i \in M$ which contains the edge $\{k, k+2\}$. Then the result follows by the previous argument.

Case 6: $m \leq v-4$ and $v-m \equiv 0 \pmod 4$.

We will solve this case by considering three subcases in turn. Observe that in this case m is even (since v is even in Case B and $v-m \equiv 0 \pmod 4$).

Case 6a: $m < v/2$.

Let $Z_4(m)$ be the tailed zigzag path $(v_0, v_1, \ldots, v_m)$ defined by

$$v_i = \begin{cases} (-1)^i \lceil (i+2)/2 \rceil \text{ for } 0 \leq i \leq m/2-1, \\ v_{m-(i+1)} + v/2 \text{ for } m/2 \leq i \leq m-1, and \\ v_{m-1} - 1 = v/2 \text{ for } i = m, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_4 = \{Z_4(m) - i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{3, 4, \ldots, m/2+1\} \cup \{v/2\}$ and the $v/2$ edges of difference 1 in $I(1, v/2+1)$ (see figure 3.7).

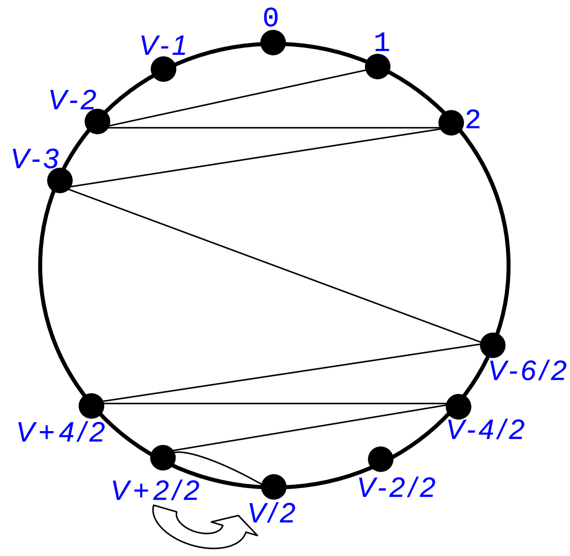

Figure 3.7: Example with $m = 8$ and general v

Let $C_2(x) = (x, x+2, x+4, \ldots, x)$ be the trail (it is a cycle) of length $v/2$. Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $C_2(v/2+1) + \{v/2+1, v/2+2\} + C_2(v/2+2) + I(v/2+2, 1) + \{C(v, m/2+2)+1\} + \cdots + \{C(v, v/2-2)+1\}$; note that in Case 6 $v - m \equiv 0 \pmod 4$, so $m/2+2 \equiv v/2-2 \pmod 2$. So T has all the edges of difference 2, of differences $m/2+2, \ldots, v/2-1$, and the edges in $I(1, v/2+1)$. If T contains consecutive vertices that form a cycle C then we now show that the length of C is more than m by considering the following 3 cases.

(i) Suppose C consists only of edges in $C_2(v/2+1) + \{v/2+1, v/2+2\} + C_2(v/2+2) + I(v/2+2, 1)$. Observe that the least number of edges between two appearances of any vertex in $C_2(v/2+1) + \{v/2+1, v/2+2\} + C_2(v/2+2) + I(v/2+2, 1)$ is clearly at least $v/2$. Since $m < v/2$, it follows that the length of C is greater than m.

(ii) Suppose C is in $I(v/2+2,1)+\{C(v,m/2+2)+1\}$. Since $C(v,m/2+2)+1$ is isomorphic to $C(v,m/2+2)$, we can apply Lemma 3.3 to conclude that any cycle consisting only of edges in $C(v,m/2+2)+1$ has length $m+6$. If C contains edges from the path $I(v/2+2,1)$ then since the second vertex in $C(v,m/2+2)+1$ is less than $v/2+1$ (all differences in T are at most $v/2$) which is not in $I(v/2+2,1)$, it follows that the length of C is greater than $v/2>m$.

(iii) Suppose C is in $\{C(v,m/2+2)+1\}+\cdots+C\{(v,v/2-2)+1\}$. Then observe that $C(v,m/2+j)+1$ is isomorphic to $C(v,m/2+j)$ for all j, so we can use Lemma 3.3 to conclude that the length of C is greater than m.

Therefore, by the above observations, T/m partitions into paths of length m all the edges of differences in $\{m/2+2,\ldots,v/2-1\}\cup\{2\}$ and the $v/2$ edges of difference 1 in $I(v/2+1,1)$. So $(V,C)=(Z_v,\{Z_4\cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S=(W,D)$ is any subsystem in this P_m-decomposition. Let $\pi_j\in D$ be any path of length m. Then either $\pi_j=Z_4(m)-i$ for some $i\in Z_{v/2}$ or $\pi_j\in T/m$. We will show that $W=V$ in both these cases.

Case 6a(i): Suppose that $\pi_j=Z_4(m)-i$ for some $i\in Z_{v/2}$.

Each path $Z_4(m)-i$ contains the edge $\{k,k+v/2\}$ of half difference for some k.

Suppose $m\geq 6$. Then $Z_4(m)-i$ contains both $k+1$ and $k+v/2+1$ (if $m\equiv 0 \pmod 4$) or both $k-1$ and $k+v/2-1$ (if $m\equiv 2 \pmod 4$). So W must contain one pair of vertices in the next half difference which implies that either $Z_4(m)-(i+1)$ or $Z_4(m)-(i-1)\in D$. By repeating this argument we can conclude that $V(Z_4)=W=V$.

Suppose $m=4$ (so, being Case 6, $v\equiv m\equiv 0 \pmod 4$). Then $Z_4(m)-i\to\{k+3,k+v/2+3\}\to Z_4(m)-i+3$. So recursively it follows that $X=\{k+3i,k+v/2+3i \mid i\in Z_{v/2}\}\subseteq W$. So $X=V=W$ unless $v\equiv 0 \pmod 3$.

Hence the only case that remains to be solved is when $m = 4$ and $v \equiv 0 \pmod 3$ (so actually $v \equiv 0 \pmod{12}$, since in this case $v \equiv 0 \pmod 4$ as well). So finally suppose that $v \equiv 0 \pmod{12}$ and $m = 4$. Notice that in this exceptional case,

$$\text{for any } x \in V, \{x, x + v/2\} \to \{x + 3, x + 3 + v/2\}. \tag{3.8}$$

So if $\{x, x + v/2\} \subseteq W$ then we can recursively apply Equation 3.8 to $\{x, x + v/2\}$ to see that $\{y \in V \mid y \equiv x \pmod 3\} \subseteq W$. In particular, since $\{k, k + v/2\} \subseteq W$, it follows that $A = \{a \in V \mid a \equiv k \pmod 3\} \subseteq W$. But since each path containing an edge of half difference joining vertices in A also contains an edge of difference 1, we in fact know that $A' = \{b \in V \mid b \equiv k - 1 \pmod 3, 1 \leq b \leq v/2\} \subseteq W$. Then observe that if $b \geq 4$ and $b \in A'$ then $\{b, b - 3\} \to \{b - 3, b - 3 + v/2\}$. So by applying Equation 3.8 recursively to $\{b-3, b-3+v/2\}$, where $b \in A'$ and $b \geq 4$ we will get that $B = \{b \in V \mid b \equiv k-1 \pmod 3\} \subseteq W$. So $\{k, k+v/2\} \to \{a \mid a \equiv k \textit{ or } k-1 \pmod 3\}$. In particular $\{k-1, k+v/2-1\} \subseteq W$, so similarly $\{k - 1, k + v/2 - 1\} \to \{a \mid a \equiv k - 1 \textit{ or } k - 2 \pmod 3\}$. Hence $W = V$ in this case.

Case 6a(ii): Suppose that $\pi_j \in T/m$.

Now we consider various possibilities for π_j. If π_j contains two vertices that are joined by an edge that occurs in $Z_4(m) - i$ for some i, then the result follows from Case 6a(i). Notice that if $m \geq 4$ then each edge of difference 3 occurs in $Z_4(m) - i$ for some i, and if $m \geq 6$ then each edge of difference 4 occurs in $Z_4(m) - i$ for some i.

Suppose $m \geq 6$. Every $\pi_j \in T/m$ contains the vertices in $\{x, x + 3\}$ or $\{x, x + 4\}$ for some x. So every subsystem S containing π_j contains an edge of difference 3 or 4; so S contains an edge in $Z_4(m) - i$ for some i.

Suppose $m = 4$. Then one of the following occurs.

(i) $\pi_j = (\dots, k, k+2, k+4, \dots)$. In this case the edge $\{k, k+4\}$ is in a path that must contain either $k-1$ or $k+5$. So S contains an edge of difference 3 (either $\{k-1, k+2\}$ or $\{k+2, k+5\}$), so S contains $Z_4(m) - i$ for some i.

(ii) π_j contains 3 consecutive edges in $I(v/2+2, 1)$. In this case π_j contains a pair of vertices distance 3 apart, so S contains an edge of difference 3. So S contains $Z_4(m) - i$ for some i.

(iii) π_j contains edges in $I(v/2+2, 1) + C(v, m/2+2) + 1$. In view of the Case(ii) we can assume that π_j contains at most 2 edges from $I(v/2+2, 1)$. So S contains an edge of difference 2 or 3. If S contains an edge of difference 2 then S contains a path that was just considered in Case(i). If S contains an edge of difference 3, then S contains $Z_4(m) - i$ for some i.

(iv) $\pi_j = (k, k+l, k+1, k+l+1, k+2)$. In this case the edge $\{k, k+2\}$ is in a path that was just considered in Case(i). So S contains $Z_4(m) - i$ for some i.

Case 6b: $v/2 \le m < 3v/4$, and $m \le v - 8$.

First define the sub zigzag path $(v'_0, v'_1, \dots, v'_{m/2-1})$ of length $m/2 - 1$ by

$$v'_i = (-1)^{i+1} \lceil (i+1)/2 \rceil \text{ for } 0 \le i \le m/2 - 1.$$

Then let $Z_5(m)$ be the zigzag path $(v_0, v_1, \dots, v_m)$ defined by

$$v_i = \begin{cases} v'_{(m/2-1)-i} \text{ for } 0 \le i \le m/2 - 1, \\ v_{m/2-1} + 1 \text{ for } i = m/2, \mathit{and} \\ v_{m-i} + v/2 \text{ for } m/2 + 1 \le i \le m, \end{cases}$$

where each sum is reduced modulo v. Notice that the set of m-paths $Z_5 = \{Z_5(m) - i \mid i \in Z_{v/2}\}$ partitions all the edges of differences in $\{2, 3, \dots, m/2\}$, $v/2$ edges of difference $v/2 - 1$, and the $v/2$ edges of difference 1 in $I(v/2, 0)$ (see figure 3.8).

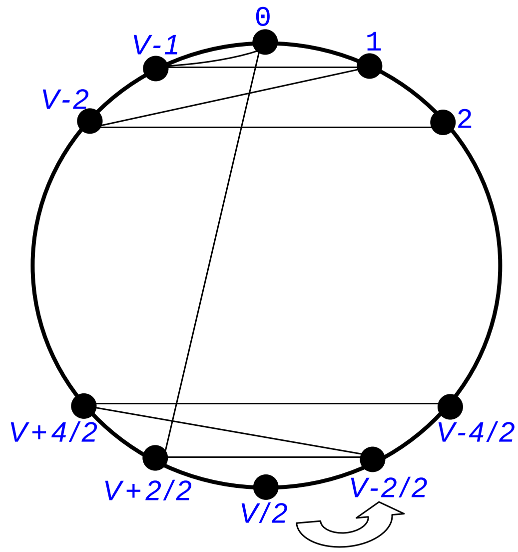

Figure 3.8: Example with $m = 8$ and general v

Let A be the trail defined by $(0, v/2, 1, v/2+1, 2, \ldots, v-1, v/2)$ of length v which covers the edges of difference $v/2$ and remaining edges of difference $v/2-1$. Notice that the only vertex appearing more than once in A is $v/2$ which appears twice.

If $m > v/2$ then let $C = S(A, 0, v-m) + I(v-m, v/2)$. C is a trail of length $3v/2 - m > m$. Let T_l be the final segment of A after the subtrail $S(A, 0, v-m)$ has been removed. Observe that the length of T_l is $2m - v$. Clearly $B = I(0, v-m) + T_l$ is an m-path.

Note that if $m = v/2$ then let $B = I(0, v/2)$ and $C = A$.

In either case, let F be the subtrail of C containing the last m edges and let E be the subtrail of C formed by removing F. F is an m-path because the only vertex repeated in C is $v/2$ which appears as both the second and the last vertex. Note that

$$E = \begin{cases} S(C, 0, (3v/2 - 2m)/2) \text{ if } 3v/2 - 2m \text{ is even, } \textit{and} \\ S(C, 0, v/2 + \lfloor (3v/2 - 2m)/2 \rfloor) \text{ otherwise.} \end{cases}$$

Let $T = (v_1, v_2, \ldots, v_k)$ be the trail formed by the concatenation $C(v, m/2+1) + C(v, m/2+3) + \cdots + C(v, v/2-3) + E$. Again we show that the trails in T/m are paths by showing that each cycle C in T has length more than m; so let C be a cycle in T.

(i) Suppose C consists only of edges in $C(v, m/2+1)+C(v, m/2+3)+\cdots+C(v, v/2-3)$ then by Lemma 3.3 we can conclude that the length of C is greater than m.

(ii) C contains edges from both $C(v, v/2-3)$ and E. First observe that, since $m \geq \max\{4, v/2\}$ and since $m \leq v-8$, it follows that $v \geq 16$. Hence $3v/4-m < v/2-3 < v/2+\lfloor(3v/2-2m)/2\rfloor$. The first vertex to be repeated in $C(v, v/2-3)+E$ is $v/2-3$ and the number of edges between it's appearances is $v-5(>m)$, which implies that the length of C is greater than m.

Therefore, by the above observations, we can conclude that all trails in T/m are paths. Note that in Case 5 $v-m \equiv 2 \pmod 4$, so $m/2+3 \equiv v/2-3 \pmod 2$. So $T/m \cup B \cup F$ is a set of m-paths which partitions all the edges of differences in $\{m/2+1, m/2+2, \ldots, v/2-2\} \cup \{v/2\}$, the remaining $v/2$ edges of difference $v/2-1$, and the $v/2$ edges of difference 1 in $I(0, v/2)$. So $(V, C) = (Z_v, \{Z_5 \cup B \cup F \cup T/m\})$ is a P_m-decomposition of K_v.

Suppose $S = (W, D)$ is any subsystem in this P_m-decomposition. Let $\pi_j \in D$ be any path of length m. We will now consider various possibilities for π_j and show that $W = V$ in each possibility. First observe that since $v \geq 16$, $m \geq v/2$ implies that $m \geq 8$.

Suppose that $\pi_j = Z_5(m) - i$ for some $i \in Z_{v/2}$.

Each $Z_5(m) - i$ contains the edge $\{k, k+(v/2-1)\}$ for some k. Therefore π_j contains both $k-1$ and $k-1+(v/2-1)$. So W must contain the edge $\{k-1, k-1+(v/2-1)\}$, which implies that $Z_5(m)-(i+1) \in D$. By repeating this argument we can conclude that $V(Z_5) = W = V$.

Suppose that $\pi_j \in B \cup F \cup T/m$.

Every $\pi_j \in B \cup F \cup T/m$ contains a pair of vertices $\{k, k+2\}$ for some k, which implies that the edge $\{k, k+2\}$ is in some $Z_5(m) - i$ for some i. Hence the result follows by the previous argument.

Case 6c: $v/2 \leq m \leq 2v/3, m > v-8$.

Without loss of generality we can assume that $m = v-4$, because $v-m \equiv 0 \pmod 4$ in Case 6. By Lemma 3.2 $m \leq 2v/3$, so in this case $v/2 \leq v-4 \leq 2v/3$, implying $8 \leq v \leq 12$. Since v is even and m has to divide $\binom{v}{2}$, this implies that the only exceptional case that needs to be solved is when $v = 8$ and $m = 4$.

So finally suppose that $v = 8$ and $m = 4$. Then let $(V, C) = (Z_7 \cup \{\infty\}, \{Z_6 + i \mid i \in Z_7\})$ is a P_4- decomposition of K_8 where $Z_6 = (\infty, 0, 6, 1, 5)$. This decomposition contains no subsystems because whenever $\{\infty, x\}$ is in any subsystem, $\{\infty, x\} \rightarrow x+1$ for some $x \in Z_7$, which implies that whenever $Z_6 + i$ is in any subsystem $Z_6 + (i+1)$ is also in the same subsystem. By repeating this argument we can conclude that $V(\{Z_6 + i \mid i \in Z_7\}) = V$.

□

Chapter 4

Decomposition of a K_v and $K_v - I$ into diagonally switchable 4-cycle systems

In this chapter we solve the problem of decomposing a complete graph K_v and $K_v - F$, where F is any one factor, into 4-cycles having the property of being diagonally switchable.

4.1 Introduction

A C_4-decomposition of G is also known as a 4-cycle system of order G. A C_4-decomposition of K_v is said to be a 4-cycle system of order v and is denoted by $4CS(v)$. It is already known that the spectrum of $4CS(v)$ is precisely the set of all $v \equiv 1 \pmod 8$ [13]. In this chapter we consider a class of 4-cycle systems with diagonally switchable property.

In order to define diagonally switchable property, first let (a, b, c, d) denote the 4-cycle induced by the edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$. The 4-cycle (a, b, c, d) is said to have diagonals $\{a, c\}$ and $\{b, d\}$. Using the four points a, b, c, d two more new 4-cycles (a, c, b, d) and (a, b, d, c) can be constructed by replacing, respectively, each pair of non-adjacent edges of the original 4-cycle (a, b, c, d) by its diagonals. We will call such transformations diagonal switches (see figure 4.1).

A 4-cycle system (V, F) of G is said to be diagonally switchable if each element of I can be replaced by one of its diagonal switches to get a new set of 4-cycles $\bar{F}$ such that $(V, \bar{F})$ is an another 4-cycle system of G (we use $\bar{F}$ throughout the rest of the chapter to denote the set of 4-cycles formed from I after performing diagonal switches, which produce another 4-cycle system).

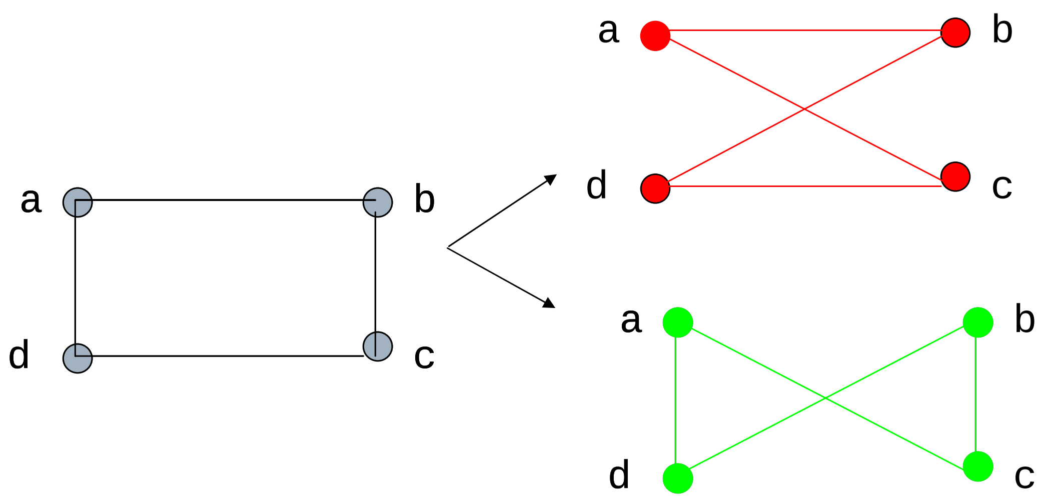

Figure 4.1: Diagonal Switches

A 4-cycle system (V, F) of K_v in which F is diagonally switchable is denoted by $DS4CS(v)$. A pair of 4-cycles (a, b, c, d) and (a', b', c', d') is said to have a double-diamond configuration D if they have a common diagonal. In order $DS4CS(v)$ to exist, no two 4-cycles in the original $4CS(v)$ can share a diagonal as all diagonals of the original 4-cycle system become edges of the transformed system. So diagonally switchable 4-cycle decompositions must be double-diamond avoiding decompositions. Configurations in 4-cycle systems were studied by Bryant, Grannell, Griggs and Mačaj; among other results they proved the following theorem [5].

Theorem 4.1. *There exists a double-diamond-avoiding $4CS(v)$ for all $v \equiv 1 \pmod 8$.*

The existence spectrum of $DS4CS(v)$s was determined by Adams, Bryant, Grannell and Griggs [1]. In this chapter we give an alternative proof of their result. This construction not only solves the case for K_v in a more efficient way, but is also powerful enough to easily prove a new result, considering the case for $K_v - F$, where F is any 1-factor of K_v. The constructions used here are recursive in nature, requiring fewer special cases than the proof in [1]. The basic building blocks in our constructions are holey self-orthogonal latin squares or holey SOLS. The method is then applied to the releated problem of finding 4-cycle systems of $K_v - F$ with the diagonally switchable property using self-orthogonal latin squares.

A self-orthogonal latin square of order v, or SOLS(v) is a latin square of order v which is orthogonal to its transpose. It is well known [4] that an SOLS(v) exists for all values of v, $v \neq 2$, 3, or 6.

Let V be a set and $H = \{H_1, H_2, \ldots, H_k\}$ be a set of nonempty subsets which partitions the set V. A holey SOLS or HSOLS of type $h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k}$ is an ordered pair $L = (V, \circ)$ of order $|V| = v = \sum_{1 \leq i \leq k} n_i h_i$ in which:

(1) every cell of L is either empty or contains a symbol of V;

(2) every symbol of V occurs at most once in any row or column of L;

(3) the subarrays $H_i \times H_i$ are empty for $1 \leq i \leq k$ (these subarrays are referred to as holes);

(4) the symbol $x \in V$ occurs in a row or column y if and only if $(x, y) \in (V \times V) \setminus \cup_{i=1}^{k} (H_i \times H_i)$;

(5) the superposition of L with its transpose yields every ordered pair in $(V \times V) \setminus \cup_{i=1}^{k} (H_i \times H_i)$.

We briefly denote a holey SOLS of type $h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k}$ by HSOLS($h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k}$). Finding necessary and sufficient conditions for the existence of HSOLS($h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k}$) is still an open problem. For the purposes of our proof the following results are sufficient.

Theorem 4.2.

(1) For $h \geq 2$, there exists an HSOLS(h^n) if and only if $n \geq 4$ [17] .

(2) Suppose that n, u are positive integers and $u \neq 12$. Then there exists an HSOLS($12^n u^1$) if and only if $n \geq 4$ and $n \geq 1 + u/6$ (Theorem7.1 in [20]).

4.2 Preliminary Results

We begin with a result from [1], where each system referred to in the following result is constructed explicitly (they also constructed systems of order 177 and 209 but these special cases are not needed in the constructions presented here).

Lemma 4.3. [*1*] *For all* $v \equiv 1$ *(mod* 8*) with* $25 \leq v \leq 137$*,* $v \neq \{97, 121, 129\}$*, there exists a* $DS4CS(v)$*.*

The following result was known to the authors of [1] but was accidentally omitted in [1].

Lemma 4.4. *There does not exist a diagonally switchable* 4*-cycle system of order* 9*.*

Proof. First note that, there are only 8 non-isomorphic $4CS(9)$s [7], of which seven have double-diamond configurations. In view of the discussion in the introduction, there is, therefore only one candidate for being diagonally switchable, namely $(V, F) = (Z_9, \{(0, 1, 5, 2)+i \mid i \in Z_9\})$, where each sum is reduced modulo 9.

Observe that F contains the 4-cycles (0, 1, 5, 2), (5, 6, 1, 7) and (8, 0, 4, 1). No matter how (8, 0, 4, 1) is switched, the resulting 4-cycle contains the edge {0, 1}. So (0, 1, 5, 2) must be switched to (0, 5, 1, 2) (not switched to (0, 5, 2, 1)). But this 4-cycle contains the edge {5, 1} and hence when the 4-cycle (5, 6, 1, 7) is switched, the edge {5, 1} is covered twice. Hence there does not exist a $DS4CS(9)$. □

4.3 Constructions

We consider three cases in turn, $v \equiv 1 \pmod{24}$ and $v \geq 97$, $v \equiv 9 \pmod{24}$ and $v \geq 129$, and $v \equiv 17 \pmod{24}$ and $v \geq 161$.

4.3.1 Case A: $v = 24s + 1, s \geq 4$.

We begin with a construction of a $4CS(v)$. To deal with this case we need HSOLS(12^s) $s \geq 4$, which is known to exist (see Theorem 4.2).

The $24s + 1$ Construction. Let $s \geq 4$. Let $V = \{\{\infty\} \cup (Z_{12s} \times \{1, 2\})\}$ and let G be a copy of K_{24s+1} defined on the vertex set V. Let $(Z_{12s}, \circ)$ be a HSOLS(12^s) having the hole set $H = \{H_i \mid i \in Z_s\}$ where $H_i = \{12i, 12i + 1, \ldots, 12i + 11\}$. Define a collection F of copies of 4-cycles as follows.

(1) **Type 1:** For each $i \in Z_s$, let

$S_i = (\{\infty\} \cup (H_i \times \{1, 2\}), F_i)$ be a $DS4CS(25)$ (see Lemma 4.3).

Let $F_i \subseteq F$.

(2) **Type 2:** For each $\{a, b\} \subseteq Z_{12s}$, $\{a, b\} \not\subseteq H_i$ and for each $i \in Z_s$, let

$((a, 1), (b, 1), (a \circ b, 2), (b \circ a, 2)) \in F$.

Proposition 4.3.1. *The $24s+1$ Construction produces a diagonally switchable $4CS(24s+1)$.*

Proof. The total number of 4-cycles in a 4-cycle system of order $v = 24s+1$ is $\binom{v}{2}/4 = 3s(24s + 1)$. We begin by counting the number of 4-cycles in F.

The number of Type 1 4-cycles is clearly $\sum_{i \in Z_s} 75|F_i| = 75s$. For each $\{a, b\} \subseteq Z_{12s}$, $\{a, b\} \not\subseteq H_i$ there are $\binom{12s}{2} - s\binom{12}{2} = 72s(s - 1)$ choices for a and b. Therefore $|F| = 75s + 72s(s - 1) = 3s(24s+1)$ as required.

To see (V, F) is a 4-cycle system, it remains to show that each edge e in $E(K_{24s+1})$ occurs in some 4-cycle in F. If $e = \{\infty, (x, j)\}$ or $\{(x, 1), (y, 1)\}$, or $\{(x, 1), (y, 2)\}$ where $\{x, y\} \subseteq H_i$ for $i \in Z_s$ and $1 \leq j \leq 2$ then e occurs in a Type 1 cycle. Now suppose $\{x, y\} \not\subseteq H_i$ for $i \in Z_s$. Clearly $\{(x, 1), (y, 1)\}$ occurs in a Type-2 cycle. If $e = \{(x, 1), (y, 2)\}$

then e occurs in the Type 2 cycle $((x,1),(b,1),(x\circ b,2),(b\circ x,2))$ where b is chosen to satisfy $b\circ x=y$. If $e=\{(x,2),(y,2)\}$ then e occurs in the Type 2 cycle $((a,1),(b,1),(a\circ b,2),(b\circ a,2))$ where a and b are chosen by the self-orthogonal property (5) to satisfy $a\circ b=x$ and $b\circ a=y$.

To see that (V,F) is diagonally switchable, observe that by replacing F_i with $\bar{F}_i$ for each $i\in Z_s$, and replacing each 4-cycle $((a,1),(b,1),(a\circ b,2),(b\circ a,2))$ by $((a,1),(b,1),(b\circ a,2),(a\circ b,2))$ for each $\{a,b\}\subseteq Z_{12s}$, $\{a,b\}\not\subseteq H_i$ for $i\in Z_s$, we get a new set of 4-cycles $\bar{F}$ which can be seen to form another 4-cycle system of K_{24s+1} using essentially the same proof that showed (V,F) is a 4-cycle system. □

4.3.2 Case B: $v=24s+9, s\geq 5$.

We begin with a construction of a $4CS(v)$. To deal with this case we need HSOLS($12^{s-1}16^1$) $s-1\geq 4$, which is known to exist (see Theorem 4.2).

The $24s+9$ **Construction**. Let $s\geq 5$. Let $V=\{\{\infty\}\cup(Z_{12s+4}\times\{1,2\})\}$ and let G be a copy of K_{24s+9} defined on the vertex set V. Let $(Z_{12s+4},\circ)$ be a HSOLS($12^{s-1}16^1$) $s-1\geq 4$ having the hole set $H=\{H_i\mid i\in Z_s\}$ where $H_{s-1}=\{12s-12,12s-11,\ldots,12s+3\}$ and $H_i=\{12i,12i+1,\ldots,12i+11\}$ for each $i\in Z_{s-1}$. Define a collection F of copies of 4-cycles as follows.

(1) **Type 1:** Let $S_{s-1}=(\{\infty\}\cup(H_{s-1}\times\{1,2\}),F_{s-1})$ be a $DS4CS(33)$, and for each $i\in Z_{s-1}$, let $S_i=(\{\infty\}\cup(H_i\times\{1,2\}),F_i)$ be a $DS4CS(25)$(see Lemma 4.3). Let $F_i\subseteq F$ for each $i\in Z_s$.

(2) **Type 2:** For each $\{a,b\}\subseteq Z_{12s+4}$, $\{a,b\}\not\subseteq H_i$ and for each $i\in Z_s$, let $((a,1),(b,1),(a\circ b,2),(b\circ a,2))\in F$.

Proposition 4.3.2. *The* $24s+9$ *Construction produces a diagonally switchable* $4CS(24s+9)$.

Proof. The total number of 4-cycles in a 4-cycle system of order $v = 24s+9$ is $\binom{v}{2}/4 = (3s+1)(24s+9)$. We begin by counting the number of 4-cycles in F.

The number of Type 1 4-cycles is clearly $\sum_{i \in Z_{s-1}} 75|F_i| + |F_{s-1}| = 75(s-1)+132 = 75s+57$. For each $\{a,b\} \subseteq Z_{12s+4}$, $\{a,b\} \not\subseteq H_i$ there are $\binom{12s+4}{2} - (s-1)\binom{12}{2} - \binom{16}{2} = 72s^2 - 24s - 48$ choices for a and b. Therefore $|F| = 75s+57+ 72s^2$ - $24s$ - 48 = $(3s+1)$ $(24s+9)$ as required.

To see (V,F) is a 4-cycle system, it remains to show that each edge e in $E(K_{24s+9})$ occurs in some 4-cycle in F. If $e = \{\infty,(x,j)\}$ or $\{(x,1),(y,1)\}$, or $\{(x,1),(y,2)\}$ where $\{x,y\} \subseteq H_i$ for $i \in Z_s$ and $1 \leq j \leq 2$ then e occurs in a Type 1 cycle. Now suppose $\{x,y\} \not\subseteq H_i$ for $i \in Z_s$. Clearly $\{(x,1),(y,1)\}$ occurs in a Type-2 cycle. If $e = \{(x,1),(y,2)\}$ then e occurs in the Type 2 cycle $((x,1),(b,1),(x \circ b,2),(b \circ x,2))$ where b is chosen to satisfy $b \circ x = y$. If $e = \{(x,2),(y,2)\}$ then e occurs in the Type 2 cycle $((a,1),(b,1),(a \circ b,2),(b \circ a,2))$ where a and b are chosen by the self-orthogonal property (5) to satisfy $a \circ b = x$ and $b \circ a = y$.

To see that (V,F) is diagonally switchable, observe that by replacing F_i with $\bar{F_i}$ for each $i \in Z_s$, and replacing each 4-cycle $((a,1),(b,1),(a \circ b,2),(b \circ a,2))$ by $((a,1),(b,1),(b \circ a,2),(a \circ b,2))$ for each $\{a,b\} \subseteq Z_{12s+4}$, $\{a,b\} \not\subseteq H_i$ for $i \in Z_s$, we get a new set of 4-cycles $\bar{F}$ which can be seen to form another 4-cycle system of K_{24s+9} using essentially the same proof that showed (V,F) is a 4-cycle system. □

4.3.3 Case C: $v = 24s+17, s \geq 6$.

We begin with a construction of a $4CS(v)$. To deal with this case we need HSOLS($12^{s-1}20^1$) $s-1 \geq 5$, which is known to exist [20].

The $24s + 17$ **Construction**. Let $s \geq 6$. Let $V = \{\{\infty\} \cup (Z_{12s+8} \times \{1, 2\})\}$ and let G be a copy of K_{24s+17} defined on the vertex set V. Let $(Z_{12s+8}, \circ)$ be a HSOLS$(12^{s-1}20^1)$ $s-1 \geq 5$ having the hole set $H = \{H_i \mid i \in Z_s\}$ where $H_{s-1} = \{12s-12, 12s-11, \ldots, 12s+7\}$ and $H_i = \{12i, 12i + 1, \ldots, 12i + 11\}$ for each $i \in Z_{s-1}$. Define a collection F of copies of 4-cycles as follows.

(1) **Type 1:** Let $S_{s-1} = (\{\infty\} \cup (H_{s-1} \times \{1, 2\}), F_{s-1})$ be a $DS4CS(41)$, and for each $i \in Z_{s-1}$, let $S_i = (\{\infty\} \cup (H_i \times \{1, 2\}), F_i)$ be a $DS4CS(25)$(see Lemma 4.3). Let $F_i \subseteq F$ for each $i \in Z_s$.

(2) **Type 2:** For each $\{a, b\} \subseteq Z_{12s+8}$, $\{a, b\} \not\subseteq H_i$ and for each $i \in Z_s$, let

$((a, 1), (b, 1), (a \circ b, 2), (b \circ a, 2)) \in F$.

Proposition 4.3.3. *The* $24s+17$ *Construction produces a diagonally switchable* $4CS(24s+17)$.

Proof. The total number of 4-cycles in a 4-cycle system of order $v = 24s + 17$ is $\binom{v}{2}/4 = (3s + 2)(24s + 17)$. We begin by counting the number of 4-cycles in F.

The number of Type 1 4-cycles is clearly $\sum_{i \in Z_{s-1}} 75F_i + |F_{s-1}| = 75(s-1)+205 = 75s+130$. For each $\{a, b\} \subseteq Z_{12s+8}$, $\{a, b\} \not\subseteq H_i$ there are $\binom{12s+8}{2} - (s-1)\binom{12}{2} - \binom{20}{2} = 72s^2+24s-96$ choices for a and b. Therefore $|F| = 75s+130+ 72s^2+ 24s\text{-}96 = 72s^2+99s+34 = (3s+2)\ (24s+17)$ as required.

To see (V, F) is a 4-cycle system, it remains to show that each edge e in $E(K_{24s+17})$ occurs in some 4-cycle in F. If $e = \{\infty, (x, j)\}$ or $\{(x, 1), (y, 1)\}$, or $\{(x, 1), (y, 2)\}$ where $\{x, y\} \subseteq H_i$ for $i \in Z_s$ and $1 \leq j \leq 2$ then e occurs in a Type 1 cycle. Now suppose $\{x, y\} \not\subseteq H_i$ for $i \in Z_s$. Clearly $\{(x, 1), (y, 1)\}$ occurs in a Type-2 cycle. If $e = \{(x, 1), (y, 2)\}$ then e occurs in the Type 2 cycle $((x, 1), (b, 1), (x \circ b, 2), (b \circ x, 2))$ where b is chosen to satisfy $b \circ x = y$. If $e = \{(x, 2), (y, 2)\}$ then e occurs in the Type 2 cycle $((a, 1), (b, 1), (a \circ b, 2), (b \circ$

$a, 2))$ where a and b are chosen by the self-orthogonal property (5) to satisfy $a \circ b = x$ and $b \circ a = y$.

To see that (V, F) is diagonally switchable, observe that by replacing F_i with $\bar{F}_i$ for each $i \in Z_s$, replacing, and replacing each 4-cycle $((a, 1), (b, 1), (a \circ b, 2), (b \circ a, 2))$ by $((a, 1), (b, 1), (b \circ a, 2), (a \circ b, 2))$ for each $\{a, b\} \subseteq Z_{12s+8}$, $\{a, b\} \not\subseteq H_i$ for $i \in Z_s$, we get a new set of 4-cycles $\bar{F}$ which can be seen to form another 4-cycle system of K_{24s+17} using essentially the same proof that showed (V, F) is a 4-cycle system. □

4.3.4 The Main Result

Now we will state and prove the main theorem

Theorem 4.5. *There exists a diagonally switchable 4-cycle system of order v ($DS4CS(v)$) if and only if $v \equiv 1 \pmod 8$, $v \geq 17$, with the possible exception of $v = 17$.*

Proof. In view of Lemmas 4.3 and 4.4, we can assume $v \geq 145$ or $v \in \{97, 121, 129\}$.

Let $v = 24s + h$ where $h \in \{1, 9, 17\}$. If $h = 1$ then use $s \geq 4$, which implies that $v \equiv 1 \pmod{24}$ and $v \geq 97$ which is covered by Case A. If $h = 9$ then use $s \geq 5$, which implies that $v \equiv 9 \pmod{24}$ and $v \geq 129$ which is covered by Case B. If $h = 17$ then use $s \geq 6$, which implies that $v \equiv 17 \pmod{24}$ and $v \geq 161$, which is covered by Case C.

□

4.4 Decompositions of $K_v - I$

Now we will use the same proof technique to decompose $K_v - I$ into diagonally switchable 4-cycles, where I is any 1-factor of K_v. The basic building blocks in our constructions are self-orthogonal latin squares or SOLS.

Theorem 4.6. *There exists a 4-cycle system of $K_v - I$ having the diagonally switchable property if and only if v is even and $v \notin \{4, 6\}$.*

Proof. To prove the necessary condition first note that in order to have a 1-factor v has to be even. Secondly observe that any 4-cycle in $K_4 - I$ is going to cover both the edges of the 1-factor after the diagonal switch. Hence $K_4 - I$ cannot have a diagonally switchable 4-cycle system.

Now consider $K_6 - I$ where I is any 1-factor of K_6. Let $I = \{\{a,b\},\{c,d\},\{e,f\}\}$ be any 1-factor of K_6. Let (V,F) be any diagonally switchable 4-cycle system of $K_6 - I$. Clearly a and b cannot be in the same 4-cycle in F, since $\{a,b\}$ cannot be an edge in any 4-cycle nor in any of their diagonal switches. Similarly c,d and e,f cannot be in the same 4-cycle. Now consider the 4-cycle containing the edge $\{e,d\}$, by the above observation that 4-cycle cannot have c and f as it's vertices. Therefore it should contain the edge $\{a,b\}$, a contradiction. Hence $K_6 - I$ has no diagonally switchable 4-cycle system.

In order to prove the sufficiency we now consider two cases.

Case A: $v = 12$.

In the following construction, in any ordered pair reduce arithmetic operations modulo 5 in the second component. Let $V = \{\{\infty_1,\infty_2\} \cup (Z_5 \times \{1,2\})\}$ and let G be a copy of $K_{12} - I$ defined on the vertex set V where $I = \{\{(i,1),(i,2)\} \mid i \in Z_5\}$ and define a collection F of copies of 4-cycles as follows (see figure 4.2).

(1) **Type 1:** for each $i \in Z_5$, let

$C_1(i) = ((i,1),(i+1,1),(i+4,1),(i+2,2)) \in F.$

(2) **Type 2:** for each $i \in Z_5$, let

$C_2(i) = (\infty_1,(i+1,2),(i+2,2),(i+3,1)) \in F.$

(3) **Type 3:** for each $i \in Z_5$, let

$C_3(i) = (\infty_2,(i+1,2),(i+3,2),(i+2,1)) \in F.$

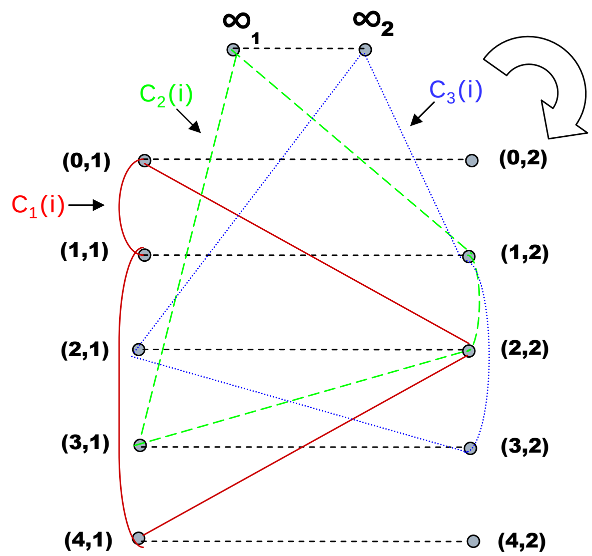

Figure 4.2: 4-cycle system of $K_{12} - I$

Now we prove that this construction produces a 4-cycle system of $K_{12} - I$. We begin by counting the number of 4-cycles in F. There are five 4-cycles of each type. The total number of 4-cycles in a 4-cycle system of $K_{12} - I$ is $(\binom{12}{2} - 6)/4 = 15$. Hence $|F| = 3.5 = 15$ as required.

To see (V, F) is a 4-cycle system, it remains to show that each edge e in $E(K_{12} - I)$ occurs in some 4-cycle in F. If $e = \{\infty_i, (x, j)\}$ with $x \in Z_5$ and $1 \leq i, j \leq 2$ then e occurs in some $C_{i+1}(k)$. If $e = \{(x, 1), (y, 1)\}$ with $x, y \in Z_5$ then e occurs in some $C_1(k)$. If $e = \{(x, 1), (x + 2, 2)\}$ with $x \in Z_5$ then e occurs in $C_1(x)$. If $e = \{(x, 1), (x + 1, 2)\}$ with $x \in Z_5$ then e occurs in $C_3(x - 2)$. If $e = \{(x, 2), (x + 1, 1)\}$ with $x \in Z_5$ then e occurs in $C_2(x - 2)$. If $e = \{(x, 2), (x + 2, 1)\}$ with $x \in Z_5$ then e occurs in $C_1(x - 2)$. If $e = \{(x, 2), (x + 1, 2)\}$ with $x \in Z_5$ then e occurs in $C_2(x - 1)$. If $e = \{(x, 2), (x + 2, 2)\}$ with $x \in Z_5$ then e occurs in $C_3(x - 1)$.

To prove (V, F) is diagonally switchable, observe that no two 4-cycles in F share a diagonal. Now by replacing each $C_1(i) = ((i, 1), (i + 1, 1), (i + 4, 1), (i + 2, 2)) \in C$ with $C_1'(i) = ((i, 1), (i + 4, 1), (i + 1, 1), (i + 2, 2))$, and replacing each $C_2(i) = (\infty_1, (i + 1, 2), (i +$

$2,2),(i+3,1)) \in C$ with $C_2'(i) = (\infty_1,(i+2,2),(i+1,2),(i+3,1))$, and each $C_3(i) = (\infty_2,(i+1,2),(i+3,2),(i+2,1)) \in C$ with $C_3'(i) = (\infty_2,(i+3,2),(i+1,2),(i+2,1))$, we get a new set of 4-cycles $\bar{F}$ (see figure 4.3).

It is easy to check that $(V, \bar{F})$ forms another 4-cycle system.

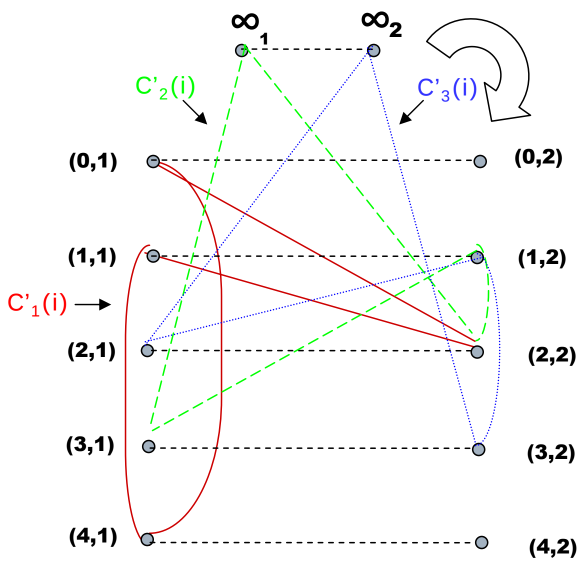

Figure 4.3: $DS4CS(K_{12} - I)$

Case B: $v \neq$ **4, 6, 12**

Let $V = Z_{v/2} \times \{1,2\}$ and let G be a copy of $K_v - I$ defined on the vertex set V where $I = \{\{(i,1),(i,2)\} \mid i \in Z_{v/2}\}$. Let $(Z_{v/2}, \circ)$ be a SOLS($v/2$) (this is known to exist since v is even and $v \neq 4, 6, 12$). Define a collection F of copies of 4-cycles as follows. For each $\{a,b\} \subseteq Z_{v/2}$, let $((a,1),(b,1),(a \circ b,2),(b \circ a,2)) \in F$.

Now we prove that this construction produces a 4-cycle system of $K_v - I$. The total number of 4-cycles in a 4-cycle system of $K_v - I$ is $(\binom{v}{2} - (v/2))/4 = v(v-2)/8$. We begin by counting the number of 4-cycles in F.

For each $\{a, b\} \subseteq Z_{v/2}$, there are $\binom{v/2}{2}$ choices for a and b. Therefore $|F| = v(v-2)/8$ as required. Using the same argument as in the proof of Proposition 4.3.3, it is easy to see that each edge e of $K_v - I$ is in a 4-cycle in F.

To see (V, F) is diagonally switchable, observe that by replacing each 4-cycle $((a,1),(b,1),(a \circ b,2),(b \circ a,2))$ by $((a,1),(b,1),(b \circ a,2),(a \circ b,2))$ for each $\{a,b\} \subseteq Z_{v/2}$, we get a new set of 4-cycles $\bar{F}$ which can be easily seen to form another 4-cycle system of $K_{v/2} - I$.

□

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