ITERATION METHODS FOR APPROXIMATION OF SOLUTIONS OF NONLINEAR EQUATIONS IN BANACH SPACES

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Chukwudi Chidume

Certificate of Approval:

Asheber Abebe
Associate Professor
Mathematics and Statistics

Geraldo Soares de Souza, Chair
Professor
Mathematics and Statistics

Narendra Govil
Professor
Mathematics and Statistics

Yonsheng Han
Professor
Mathematics and Statistics

George T. Flowers
Interim Dean, Graduate School

ITERATION METHODS FOR APPROXIMATION OF SOLUTIONS OF NONLINEAR EQUATIONS IN BANACH SPACES

Chukwudi Chidume

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Chukwudi Chidume

Signature of Author

Date of Graduation

Dissertation Abstract

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Chukwudi Chidume

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(B.S., University of Alabama in Huntsville, 2000)

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The objective in this manuscript is to study some iterative methods used to approximate solutions of nonlinear equations in Banach Spaces. In particular, we study a Halpern-type iterative scheme in relation to nonexpansive and asymptotically nonexpansive mappings and prove convergence theorems in both of these cases. We also study a hybrid steepest descent iterative scheme in relation to the variational inequality problem, and using this process, we prove convergence theorems for the approximation of the solution of the variational inequality problem in certain Banach spaces, in particular for Lp spaces.

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Style manual or journal used Journal of Approximation Theory (together with the style known as “aums”). Bibliograpy follows van Leunen’s *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

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Chapter 1

Introduction

1.1 Introduction

The contributions of this thesis fall within the general area of nonlinear operator theory, an area with vast amount of applicability in recent years, as such becoming the object of an increasing amount of study. We devote our attention to an important topic within the area: *iterative methods for approximating fixed points and solutions of variational inequality problems for nonexpansive and accretive-type nonlinear mappings.*

Let K be a nonempty subset of a real normed linear space E and let $T : K \to K$ be a map. A point $x \in K$ is said to be *a fixed point* of T if $Tx = x$. We shall denote the set of fixed points for an operator T by $F(T)$. Now, consider the differential equation $\frac{du}{dt} + Au(t) = 0$ which describes an evolution system where A is an *accretive map* from a Banach space to itself. At equilibrium state, $\frac{du}{dt} = 0$ and so a solution of $Au = 0$ describes the equilibrium or stable state of the system. This is very desirable in many applications in, for example, ecology, economics, physics, to name a few. Consequently, considerable research efforts have been devoted to methods of solving the equation $Au = 0$ when A is accretive. Since, in general A is a nonlinear operator, there is no closed form solution to this equation. The standard technique is to replace A by an operator $(I - T)$ where I is the identity map on E and T maps E to itself. Such a map T is called *a pseudo-contraction* (or is called *pseudo-contractive*). It is then clear that any zero of A is a fixed point of T. As a result of this, the study of fixed point theory for pseudo-contractive maps has attracted the interest of numerous scientists and has become a flourishing area of research, especially within the past 30 years or so, for numerous mathematicians. A very important subclass of the class of pseudo-contractive mappings is the *class of nonexpansive mappings*. In this dissertation, we shall devote attention particularly to this class of mappings. Interest in, and the importance of this class of mappings will become evident in the sequel.

One of the most important fixed point theorems in applications is the classical contraction mapping principle, or, in other words, the Banach-Cacciopoli [14] fixed point theorem which is the following:

Theorem 1.1.1 (Banach Contraction Mapping Principle) *Let (X, ρ) be a complete metric space and let $T : (X, \rho) \to (X, \rho)$ satisfy*

$$\rho(T(x), T(y)) \leq \kappa\rho(x, y) \tag{1.1}$$

for some nonnegative constant $k < 1$ and for each $x, y \in X$. Then, T has a unique fixed point in X. Moreover, starting with arbitrary $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n = T^n x_0, \qquad n \geq 1, \tag{1.2}$$

converges strongly to the unique fixed point.

The iterative technique of Theorem 1.1.1 is due to Picard [69]. A mapping T satisfying (1.1) is called a *strict contraction*. If $\kappa = 1$ in the relation (1.1), then T is called *nonexpansive*. If however, κ is an arbitrary fixed positive constant, then T is called a *Lipschitz* map or a κ-*Lipschitzian* map.

For the contractive condition (1.1), it was observed that if the condition $\kappa < 1$ on the operator T is weakened to $\kappa = 1$, the operator T may no longer have a fixed point and even when it does have a fixed point, the sequence $\{x_n\}$ defined by (1.2) may fail to converge to such a fixed point. This can be seen by considering an anti-clockwise rotation of the unit disc of $\mathbb{R}^2$ about the origin through an angle of say, $\frac{\pi}{4}$. This map is nonexpansive with the origin as the unique fixed point, but the Picard sequence fails to converge with any starting point $x_0 \neq (0, 0)$. Krasnosel'skii [58], however, showed that in this example, if the Picard iteration formula is replaced by the following formula,

$$x_0 \in K, \quad x_{n+1} = \frac{1}{2}(x_n + Tx_n), n \geq 0, \tag{1.3}$$

then the iterative sequence converges to the unique fixed point.

In general, if E is a normed linear space and T is a nonexpansive mapping, a generalization of (1.3) which has proved successful in the approximation of a fixed point of T (when it exists) was given by Schaefer [74] :

$$x_0 \in K, \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n \geq 0, \quad \lambda \in (0, 1). \tag{1.4}$$

However, the most general iterative formula for approximation of fixed points of nonexpansive mappings, which is called the *Mann iteration formula* (in the light of [63]), is the following:

$$x_0 \in K, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}$ is a real sequence in the interval $(0, 1)$ satisfying the following conditions: $(i) \quad \lim_{n\to\infty} \alpha_n = 0$ and $(ii) \sum_{n=1}^{\infty} \alpha_n = \infty$.

The recursion formula (1.4) is consequently called the *Krasnoselskii-Mann (KM)* formula for finding fixed points of nonexpansive mappings. This iterative process has become very important and applicable as noted below.

- "Many well-known algorithms in signal processing and image reconstruction are iterative in nature .... A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the (Krasnoselskii-Mann) iteration procedure, for particular choices of the (nonexpansive) operator...."

(Charles Byrne , [13]).

Apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important, as has been observed by Bruck [9], mainly for the following two reasons:

- Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.

- Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are *accretive* or *dissipative* and *minimally continuous*.

Nonexpansive maps have been studied, and are still being studied, extensively by numerous authors (see e.g., Bauschke [2], Belluce and Kirk [3], Browder [4], Bruck [7, 8], Chidume [19, 18], Chidume and Ali [21, 26], Chidume and Chidume [27], Chidume *et al.* [28, 34], Chidume and Shahzad [33], De Marr [43], Göhde [47]. Jung and Kim [49], Jung [50], Jung *et al.* [51], Khan and Fukharu-ud-din [53], Kirk [56], Lim [59], Matsuhita and Kuroiwa [64], O'Hara *et al.* [67], Oka [68], Reich [68], Senter and Dotson [77], Shahzad [78], Shahzad and Al-dubiban [79], Takahashi and Tamura [85], Takahashi and Kim [86], Tan and Xu [88], Xu and Yin [94], Zeng and Yao [100] and a host of other authors).

Let E be a real Banach space, K a closed convex subset of E and $T : K \to K$ a *non-expansive* mapping. For fixed $t \in (0,1]$ and arbitrary $u \in K$, define a map $T_t : K \to K$ by $T_t x := tu + (1-t)Tx,\ x \in K$. Then T_t is a strict contraction for every fixed constant $t \in (0,1]$. Denote the unique fixed point of T_t by $z_t \in K$, and assume $F(T) := \{x \in K : Tx = x\} \neq \emptyset$.

In 1967, Browder [5] proved that if $E = H$, a Hilbert space, then $\lim_{t\to 0} z_t$ exists and is a fixed point of T. In 1980, Reich [71] extended this result to *uniformly smooth Banach spaces*. In 1981, Kirk [57] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range. We have mentioned that every nonexpansive mapping is a pseudo-contractive mapping. Following this, in 2000, Morales and Jung [65] proved the same result for T a continuous pseudocontraction in a real reflexive Banach space with uniformly Gâteaux differentiable norm.

For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$

in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, n \geq 0. \tag{1.6}$$

The recursion formula (1.6) was first introduced in 1967 by Halpern [48] with $u = 0$, in the framework of Hilbert spaces. Under appropriate conditions on the domain of T, and some restrictions on the parameter $\{\alpha_n\}$ ($\alpha_n = n^{-a}$, $a \in (0,1)$), he proved strong convergence of $\{x_n\}$ to a fixed point of T. Iteration formulas of the form (1.6) are now said to be of the *Halpern-type*. Lions [62] considered a more general parameter $\{\alpha_n\}$ and improved the result of Halpern, still in Hilbert spaces. He proved strong convergence of $\{x_n\}$ to a fixed point of T where the real sequence $\{\alpha_n\}$ satisfies the conditions:

$$C1 : \lim \alpha_n = 0; \quad C2 : \sum \alpha_n = \infty; \quad C3 : \lim \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0.$$

In 1980, Reich [71] proved that the result of Halpern remains true when E is uniformly smooth. It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = \frac{1}{n+1}$. This was overcome in 1992 by Wittmann [93] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the conditions:

$$C1 : \lim \alpha_n = 0; \quad C2 : \sum \alpha_n = \infty; \quad C4 : \sum |\alpha_{n+1} - \alpha_n| < \infty.$$

In 1994, Reich [72] extended the result of Wittmann to Banach spaces which are uniformly smooth and have *weakly sequentially continuous duality maps* (e.g., l_p spaces, $1 < p < \infty$), where $\{\alpha_n\}$ satisfies $C1$ and $C2$ and is also required to be decreasing (and hence also satisfies

C4). These spaces exclude L_p spaces, $1 < p < \infty, p \neq 2$. Shioji and Takahashi [80] extended Wittmann's result to real Banach spaces with *uniformly Gâteaux differentiable norms* and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings (e.g., L_p spaces, $1 < p < \infty$). In 2002, Xu [95] (see also [96]) improved the result of Lions twofold. First, he weakened the condition C3 by removing the square in the denominator so that the canonical choice of $\alpha_n = \frac{1}{n+1}$ is possible. Secondly, he proved the strong convergence of the scheme (1.6) in the framework of real uniformly smooth Banach spaces. Xu also remarked ([95], Remark 3.2) that Halpern [48] observed that conditions (C1) and (C2) are necessary for the strong convergence of algorithm (1.6) for all nonexpansive mappings $T : K \to K$. It is not clear if they are sufficient. This brought about the following question which has been open for many years:

Question 1: Are the conditions $C1 : \lim \alpha_n = 0$ and $C2 : \sum \alpha_n = \infty$ sufficient for the strong convergence of algorithm (1.6) for all nonexpansive mappings $T : K \to K$?

In Chapter 2 of this dissertation, we modify the recursion formula (1.6) by introducing an auxiliary operator that has the same set of fixed points as T. With the help of this operator, we prove that conditions $C1$ and $C2$ are sufficient for the modified iteration algorithm to converge strongly to a fixed point of T, even in the more general setting where E is a real Banach space with uniformly Gâteaux differentiable norm. Consequently, our theorems in Chapter 2 (also see [27]), provide a partial answer to Question 1. The general question still remains open.

One important class of nonlinear mappings more general than the class of nonexpansive mappings which has been studied extensively by various authors is the class of *asymptotically nonexpansive mappings*. This class of mappings was introduced in 1972 by Goebel and Kirk [46].

Definition 1.1.2 *Let K be a nonempty subset of a normed linear space, a mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and $n = 1, 2, ...$.*

It was proved in [46] that if K is a nonempty closed, convex and bounded subset of a *uniformly convex real Banach space* and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point. It is clear that every nonexpansive mapping is asymptotically nonexpansive. The following example shows that the class of asymptotically nonexpansive mappings properly contains the class of nonexpansive mappings.

Example 1.1.3 (Goebel and Kirk, [46]) *Let B be a unit ball of the real Hilbert space l^2 and let $T : B \to B$ be defined by $T(\{x_1, x_2, ...\}) = \{0, x_1^2, a_2 x_2,\ a_3 x_3, ...\}$ where $\{a_n\}$ is a sequence of numbers such that $0 < a_n < 1$ and $\prod_{n=2}^{\infty} a_n = \frac{1}{2}$. Then T is Lipschitzian and $\|Tx - Ty\| \leq 2\|x - y\|$, for all $x, y \in B$ and moreover, $\|T^n x - T^n y\| \leq k_n \|x - y\|$, with $k_n := 2\prod_{i=2}^{n} a_i$. Observe that T is not nonexpansive and that $\lim_{n\to\infty} k_n = 1$, so that T is asymptotically nonexpansive map.*

In Chapter 3, we prove a strong convergence theorem for a Halpern-type iteration sequence for approximation of a fixed point of asymptotically nonexpansive mappings. Our main theorem in Chapter 3 (also see [35]) is proved in a real Banach space which has a uniformly Gâteaux differentiable norm. The main theorem extends some important known results from the class of nonexpansive mappings to the more general class of asymptotically nonexpansive mappings.

In the second half of this manuscript, we turn attention to the approximation of a solution of a variational inequality problem.

Definition 1.1.4 *Let E be a real normed linear space and E^* be its dual space. For some real number q $(1 < q < \infty)$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by*

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $\langle .,. \rangle$ denotes the duality pairing between elements of E and elements of E^*. If $q = 2$, then J_2 is called the normalized duality map on E.

Let K be a nonempty closed convex subset of E and $S : E \to E$ be a nonlinear operator. The variational inequality problem is formulated as follows: Find a point $x^* \in K$ such that

$$VI(S, K) \quad : \langle j_q(Sx^*), (y - x^*) \rangle \geq 0 \quad \forall y \in K. \tag{1.7}$$

If $E = H$, a real Hilbert space, the variational inequality problem reduces to the following: Find a point $x^* \in K$ such that

$$VI(S, K) \quad : \langle Sx^*, y - x^* \rangle \geq 0 \quad \forall y \in K. \tag{1.8}$$

Definition 1.1.5 *A mapping $G : E \to E$ is said to be accretive if $\forall x, y \in E$, there exists $j_q(x-y) \in J_q(x-y)$ such that*

$$\langle Gx - Gy, j_q(x-y)\rangle \geq 0. \tag{1.9}$$

Definition 1.1.6 *For some real number $\eta > 0$, G is called $\eta-$strongly accretive if $\forall x, y \in E$, there exists $j_q(x-y) \in J_q(x-y)$ such that*

$$\langle Gx - Gy, j_q(x-y)\rangle \geq \eta\|x-y\|^q. \tag{1.10}$$

In Hilbert spaces, accretive (strongly accretive) operators are called *monotone* (strongly monotone) where inequalities (1.9) and (1.10) hold with j_q replaced by the identity map of H.

Applications of variational inequalities span as diverse disciplines as differential equations, time-optimal control, optimization, mathematical programming, mechanics, finance and so on (see, for example, Kinderlehrer and Stampacchia [55], Noor [66] for more details).

It is known that if S is Lipschitz and *strongly accretive*, then problem $VI(S, K)$ has a unique solution. An important problem is how to find a solution of the problem $VI(S, K)$ whenever it exists. Considerable efforts have been devoted to this problem (see, e.g. Xu [98] , Yamada [99] and the references contained therein).

It is known that in a real Hilbert space, the problem $VI(S,K)$ is equivalent to the following fixed point equation

$$x^* = P_K(x^* - \delta S x^*), \tag{1.11}$$

where $\delta > 0$ is an arbitrary fixed constant and P_K is the *nearest point projection map* from H onto K, i.e., $P_K x = y$, where $\|x - y\| = \underset{u \in K}{inf}\|x - u\|$ for $x \in H$. Consequently, under appropriate conditions on S and δ, fixed point methods can be used to find or approximate a solution of problem $VI(S,K)$. For instance, if S is strongly monotone and Lipschitz then a mapping $G : H \to H$ defined by $Gx = P_K(x - \delta Sx)$, $x \in H$ with $\delta > 0$ sufficiently small is a strict contraction. Hence, the *Picard iteration,* $x_0 \in H$, $x_{n+1} = Gx_n, \quad n \geq 0$ of the classical Banach contraction mapping principle converges to the unique solution of the problem $VI(K,S)$.

In applications, however, the projection operator P_K in the fixed point formulation (1.11) may make the computation of the iterates difficult due to possible complexity of the convex set K. In order to reduce the possible difficulty with the use of P_K, Yamada [99] recently introduced a hybrid descent method for solving the problem $VI(K,S)$. Let $T : H \to H$ be a map and let $K := \{x \in H : Tx = x\} \neq \emptyset$. Let S be $\eta-$strongly monotone and $\kappa-$Lipschitz on H. Let $\delta \in (0, \frac{2\eta}{\kappa^2})$ be arbitrary but fixed real number and let a sequence $\{\lambda_n\}$ in $(0,1)$ satisfy the following conditions:

$$C1: \quad lim \lambda_n = 0; \qquad C2: \quad \sum \lambda_n = \infty; \quad \text{and} \quad C3: \quad lim \frac{\lambda_n - \lambda_{n+1}}{\lambda_n^2} = 0.$$

Starting with an arbitrary initial guess $x_0 \in H$, let a sequence $\{x_n\}$ be generated by the following algorithm

$$x_{n+1} = Tx_n - \lambda_{n+1}\delta S(Tx_n), \quad n \geq 0. \tag{1.12}$$

Then, Yamada [99] proved that $\{x_n\}$ converges strongly to the unique solution of $VI(K, S)$.

In the case that $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, where $\{T_i\}_{i=1}^{r}$ is a finite family of nonexpansive mappings, Yamada [99] studied the following algorithm,

$$x_{n+1} = T_{[n+1]}x_{n+1} - \lambda_{n+1}\delta S(T_{[n+1]}x_n), \quad n \geq 0, \tag{1.13}$$

where $T_{[k]} = T_{k \ mod \ r}$, for $k \geq 1$, with the *mod function* taking values in the set $\{1, 2, ..., r\}$, and where the sequence $\{\lambda_n\}$ satisfies the conditions $C1$, $C2$ and $C4: \sum |\lambda_n - \lambda_{n+N}| < \infty$. Under these conditions, he proved the strong convergence of $\{x_n\}$ to the unique solution of the $VI(K, S)$.

Recently, Xu and Kim [98] studied the convergence of the algorithms (1.12) and (1.13), still in the framework of Hilbert spaces, and proved strong convergence with condition C3 replaced by C5: $lim \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0$ and with condition $C4$ replaced by $C6: lim \frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$. These are improvements on the results of Yamada. In particular, the canonical choice $\lambda_n := \frac{1}{n+1}$ is applicable in the results of Xu and Kim but is not in the result of Yamada with condition C3.

In Chapter 4, we prove theorems that extend the results of Xu and Kim [98] (and consequently those of Wang [91], Xu and Kim [98], Yamada [99], Zheng and Yao [100]) from real Hilbert spaces to the more general real $q-$uniformly smooth Banach spaces, $q \geq 2$. In particular, our theorems are applicable in L_p spaces, $2 \leq p < \infty$. (see e.g., Chidume *et al.* [28]).

The condition $q > 2$, however, excludes the L_p spaces, $1 < p < 2$. In Section 4.4, we employ another tool to prove convergence theorems that extend the results of Xu and Kim to L_p spaces, $1 < p \leq 2$ (see Chidume and deSouza [36]). These theorems complement those of the first part of Chapter 4 to provide convergence theorems valid in all L_p spaces, $1 < p < \infty$.

In Chapter 5, we continue our interest in fixed points of nonexpansive mappings and solutions of variational inequality problems. In this chapter, we introduce a *new recursion formula* and prove strong convergence theorems for the unique solution of the variational inequality problem $VI(K,S)$ of Chapter 4, *requiring only conditions C1 and C2 on the parameter sequence* $\{\lambda_n\}$. Furthermore, in the case $T_i : E \to E, \quad i = 1, 2, ..., r$ is a family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, we prove a convergence theorem where condition $C6$ is replaced with $\lim_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. An example satisfying this condition is presented by Chidume and Ali in [22]. All our theorems in Chapter 5 (see also [29]) are proved in $q-$uniformly smooth Banach spaces, $q \geq 2$. In particular, they are applicable in L_p spaces, $2 \leq p < \infty$.

As in chapter 4, we also use a different tool to extend our theorems to include L_p spaces, $1 < p \leq 2$. Our theorems in Chapter 5 (see also [30]) still extend the results of Xu and Kim [98] (and consequently those of Wang [91], Xu and Kim [98], Yamada [99], Zheng and Yao [100]) from real Hilbert spaces to, in particular, the more general real L_p spaces, $1 < p < \infty$. Moreover, in this more general setting, the iteration parameter $\{\lambda_n\}$ is required to satisfy only conditions $C1$ and $C2$.

1.2 Preliminaries

Definition 1.2.1 *Let $S := \{x \in E : ||x|| = 1\}$ denote the unit sphere of the real Banach space E. The space E is said to have a Gâteaux differentiable norm if the limit*

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for each $x, y \in S$; and E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$, the limit is attained uniformly for $x \in S$.

Definition 1.2.2 *We shall denote a Banach limit by μ. Recall that μ is an element of $(l^\infty)^*$ such that $\|\mu\| = 1$, $\liminf_{n\to\infty} a_n \leq \mu_n a_n \leq \limsup_{n\to\infty} a_n$ and $\mu_n a_n = \mu_{n+1} a_n$ for all $\{a_n\}_{n\geq 0} \in l^\infty$ (see e.g. Chidume et al. [31], Chidume [17]).*

To motivate the definition of *modulus of smoothness* which will be used in the sequel, we begin with the following definition.

Definition 1.2.3 *A real Banach space is called smooth if for every x in X with $\|x\| = 1$, there exists a unique x^* in X^* such that $\|x^*\| = \langle x, x^* \rangle = 1$.*

Assume now that X is not smooth and take x in X and u^*, v^* in X^* such that $\|x\| = \|u^*\| = \|v^*\| = \langle x, u^*\rangle = \langle x, v^*\rangle = 1$ and $u^* \neq v^*$. Let y in X be such that $\|y\| = 1$, $\langle y, u^*\rangle > 0$ and $\langle y, v^*\rangle < 0$. Then for every $t > 0$ we have

$$1 + t\langle y, u^*\rangle = \langle x + ty, u^*\rangle \leq \|x + ty\|,$$

$$1 - t\langle y, v^*\rangle = \langle x - ty, v^*\rangle \leq \|x - ty\|$$

which imply

$$2 < 2 + t(\langle y, u^*\rangle - \langle y, v^*\rangle) \leq \|x + ty\| + \|x - ty\|$$

or equivalently

$$0 < t(\frac{\langle y, u^*\rangle - \langle y, v^*\rangle}{2}) \leq \frac{\|x + ty\| + \|x - ty\|}{2} - 1.$$

With this motivation we introduce the following definition.

Definition 1.2.4 *Let E be a normed space with* $\dim E \geq 2$. *The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| \ + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\}.$$

The space E is called uniformly smooth if and only if $\lim\limits_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

Definition 1.2.5 *For some positive constant q, E is called $q-$uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q, \quad t > 0$.*

L_p spaces, $1 < p < \infty$ are $p-$uniformly smooth (see e.g., Lindenstrauss and Tzafriri [61]). In fact, it is known that

$$L_p(or \quad l_p) \quad \text{spaces} \quad \text{are} \quad \begin{cases} 2- \quad \text{uniformly} \;\; \text{smooth}, \quad \text{if}, \quad 2 \leq p < \infty, \\ \\ p- \quad \text{uniformly} \;\; \text{smooth}, \quad \text{if}, \quad 1 < p \leq 2, \end{cases}$$

(see e.g., Lindenstrauss and Tzafriri [61]). It is well known that if E is smooth then the duality mapping is singled-valued, and if E is uniformly smooth then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

Definition 1.2.6 *Let E be a real Banach space and K be a nonempty, closed and convex subset of E. Let P be a mapping of E onto K. Then, P is said to be sunny if $P(Px+t(x-Px)) = Px$ for all $x \in E$ and $t \geq 0$. A mapping P of E into E is said to be a retraction if $P^2 = P$.*

Definition 1.2.7 *A subset K of E is said to be sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto K. A retraction P is said to be orthogonal if for each $x,\ x - P(x)$ is normal to K in the sense of James [54].*

It is well known (see Bruck [10]) that if E is uniformly smooth and there exists a nonexpansive retraction of E onto K, then there exists a nonexpansive projection of E onto K. If E is a real smooth Banach space, then P is an orthogonal retraction of E onto K if and only if $P(x) \in K$ and $\langle P(x)-x, j_q(P(x)-y)\rangle \leq 0$ for all $y \in K$. It is also known (see e.g., Shioji and Takahashi [81]) that if K is a convex subset of a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and $T : K \to K$ is nonexpansive with $F(T) \neq \emptyset$, then, $F(T)$ is a nonexpansive retract of K.

Let K be a nonempty closed convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := sup\{\|x-y\| : x, y \in K\}$. For each $x \in K$, let $r(x,K) := sup\{\|x-y\| : y \in K\}$ and let $r(K) := inf\{r(x,K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (see e.g. [12]) is defined by $N(E) := inf\Big\{\frac{d(K)}{r(K)} : K$ is a closed convex and bounded subset of E with $d(K) > 0\Big\}$. A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g., [60]).

Chapter 2

Iterative Approximation of Fixed Points of Nonexpansive Mappings

2.1 Introduction

Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$.

In this chapter, we prove that the conditions C1: $\lim \alpha_n = 0$ and C2: $\sum \alpha_n = \infty$ which are known to be necessary are, under appropriate conditions, also sufficient for the strong convergence of a Halpern-type iterative scheme to a fixed point of a nonexpansive mapping T. Our result gives a partial answer to Question 1 mentioned in the introduction.

We begin with the following well known theorem.

Theorem 2.1.1 *(Morales and Jung [65], Reich [71]) Let K be a nonempty closed convex subset of a Banach space E which has uniformly Gâteaux differentiable norm and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansve mappings. Then there exists a continuous path $t \to z_t, 0 < t < 1$ satisfying $z_t = tu + (1-t)Tz_t$, for arbitrary but fixed $u \in K$, which converges to a fixed point of T.*

Recently, Shioji and Takahashi [80] proved the following theorem.

Theorem 2.1.2 *(Shioji and Takahashi [80]) Let E be a real Banach space whose norm is uniformly Gâteaux differentiable and let K be a closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence* *which satisfies the following conditions:*

(i) $0 \leq \alpha_n \leq 1, \;\; \lim \alpha_n = 0;$

(ii) $\sum \alpha_n = \infty;$

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

Let $u \in K$ *and let* $\{x_n\}$ *be defined by* $x_0 \in K$*,*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \geq 0. \tag{2.1}$$

Assume that $\{z_t\}$ *converges strongly to* $z \in F(T)$ *as* $t \to 0$*, where for* $0 < t < 1$*,* z_t *is the unique element of* K *which satisfies* $z_t = tu + (1-t)Tz_t$*. Then,* $\{x_n\}$ *converges strongly to* z*.*

Xu [96] (see also [95]) proved the following theorem.

Theorem 2.1.3 *(Xu [96], Theorem 3.1) Let* E *be a uniformly smooth real Banach space,* K *a closed convex subset of* E*, and* $T : K \to K$ *a nonexpansive mapping with a fixed point. Let* $u, x_0 \in K$ *be given. Assume that* $\{\alpha_n\} \subset [0, 1]$ *satisfies the conditions:*

(1) $\lim \alpha_n = 0;$

(2) $\sum \alpha_n = \infty;$

(3) $\lim \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0.$

Then the sequence $\{x_n\}$ *generated by* $x_0 \in K$*,*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \geq 0,$$

converges strongly to a fixed point of T*.*

It is our purpose in this chapter to prove a significant improvement of Theorem 2.1.2 and Theorem 2.1.3 in the following sense. We prove the strong convergence of the algorithm (2.1) in the framework of real Banach spaces E with uniformly Gâteaux differentiable norms and without condition (iii) of Theorem 2.1.2. Our theorem then also extends Theorem 2.1.3 to the more general real Banach spaces with uniformly Gâteaux differentiable norms and at the same time dispenses with condition (3) of that theorem. Furthermore, our theorem gives a partial affirmative answer to Question 1 mentioned in Chapter 1.

2.2 Preliminaries

Lemma 2.2.1 *Let E be a real normed linear space. Then, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle \ \forall\ x, y \in E,\ \forall\ j(x+y) \in J(x+y).$$

In the sequel, we shall also make use of the following lemmas.

Lemma 2.2.2 *(Suzuki, [83]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \lim inf\beta_n \leq \lim sup\beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 0$ and $\lim sup(||y_{n+1}-y_n|| - ||x_{n+1}-x_n||) \leq 0$. Then, $lim||y_n - x_n|| = 0$.*

Lemma 2.2.3 *(Xu, [96]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where,

(i) $\{\alpha_n\} \subset [0,1],\ \sum \alpha_n = \infty;$

(ii) $\lim sup\ \sigma_n \leq 0;$

(iii) $\gamma_n \geq 0;$ $(n \geq 0),$ $\sum \gamma_n < \infty.$

Then, $a_n \to 0$ *as* $n \to \infty$*.*

2.3 Convergence Theorems

Theorem 2.3.1 *(C.E.Chidume and C.O.Chidume [27]) Let* K *be a nonempty closed convex subset of a real Banach space* E *which has a uniformly Gâteaux differentiable norm and* $T : K \to K$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$*. For a fixed* $\delta \in (0, 1)$*, define* $S : K \to K$ *by* $Sx := (1 - \delta)x + \delta Tx\ \forall\ x \in K$*. Assume that* $\{z_t\}$ *converges strongly to a fixed point* z *of* T *as* $t \to 0$*, where* z_t *is the unique element of* K *which satisfies* $z_t = tu + (1-t)Tz_t$ *for arbitrary* $u \in K$*. Let* $\{\alpha_n\}$ *be a real sequence in* $(0, 1)$ *which satisfies the conditions:* $C1 : \lim \alpha_n = 0;$ $C2 : \sum \alpha_n = \infty$*. For arbitrary* $x_0 \in K$*, let the sequence* $\{x_n\}$ *be defined iteratively by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n. \tag{2.2}$$

Then, $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Proof. Observe first that S is nonexpansive and has the same set of fixed points as T. Define

$$\beta_n := (1 - \delta)\alpha_n + \delta\ \forall\ n \geq 0;\ y_n := \frac{x_{n+1} - x_n + \beta_n x_n}{\beta_n},\ n \geq 0.$$

Observe also that $\beta_n \to \delta$ as $n \to \infty$, and that if $\{x_n\}$ is bounded, then $\{y_n\}$ is bounded. Let $x^* \in F(T) = F(S)$. One easily shows by induction that $||x_n - x^*|| \leq max\{||x_0 -$

$x^*||, ||u-x^*||\}$ for all integers $n \geq 0$, and so, $\{x_n\}, \{y_n\}$, $\{Tx_n\}$ and $\{Sx_n\}$ are all bounded. Also,

$$||x_{n+1} - Sx_n|| = \alpha_n ||u - Sx_n|| \to 0, \quad n \to \infty. \tag{2.3}$$

Observe that from the definitions of β_n and S, we obtain that

$$y_n = \frac{\alpha_n u + (1-\alpha_n)\delta T x_n}{\beta_n},$$

which implies

$$\begin{aligned} ||y_{n+1} - y_n|| &- ||x_{n+1} - x_n|| \\ &\leq \Big|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\Big|.||u|| + \frac{(1-\alpha_{n+1})}{\beta_{n+1}}\ \delta\ ||Tx_{n+1} - Tx_n|| \\ &+ \Big|\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\Big|\ \delta\ ||Tx_n|| - ||x_{n+1} - x_n||. \end{aligned}$$

Since $\{x_n\}$ and $\{Tx_n\}$ are bounded, we obtain (for some constants $M_1 > 0$, and $M_2 > 0$) that,

$$\begin{aligned} \lim sup(||y_{n+1} - y_n|| &- ||x_{n+1} - x_n||) \\ &\leq \lim sup\Big\{\Big|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\Big|.||u|| \\ &+ \Big|\frac{(1-\alpha_{n+1})}{\beta_{n+1}}\delta - 1\Big|\ M_1 \\ &+ \Big|\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\Big|\delta M_2\Big\} \leq 0. \end{aligned}$$

Hence, by Lemma 2.2.2, $||y_n - x_n|| \to 0$ as $n \to \infty$. Consequently,

$$\lim ||x_{n+1} - x_n|| = \lim \beta_n ||y_n - x_n|| = 0.$$

Combining this with (2.3) yields that

$$||x_n - Sx_n|| \to 0 \ as \ n \to \infty.$$

We now show that

$$\lim sup\langle u - z, j(x_n - z)\rangle \leq 0.$$

For each integer $n \geq 0$, let $t_n \in (0, 1)$ be such that

$$t_n \to 0, \text{ and } \frac{||x_n - Sx_n||}{t_n} \to 0, \ n \to \infty.$$

Let $z_{t_n} \in K$ be the unique fixed point of the contraction mapping S_{t_n} given by

$$S_{t_n}x = t_n u + (1 - t_n)Sx, \ x \in K.$$

Then,

$$z_{t_n} - x_n = t_n(u - x_n) + (1 - t_n)(Sz_{t_n} - x_n).$$

Using the inequality of Lemma 2.2.1, we compute as follows:

$$\begin{aligned}
||z_{t_n} - x_n||^2 &\leq (1-t_n)^2||Sz_{t_n} - x_n||^2 + 2t_n\langle u - x_n, j(z_{t_n} - x_n)\rangle \\
&\leq (1-t_n)^2(||Sz_{t_n} - Sx_n|| + ||Sx_n - x_n||)^2 + 2t_n(||z_{t_n} - x_n||^2 \\
&+ \langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle \\
&\leq (1+t_n^2)||z_{t_n} - x_n||^2 + ||Sx_n - x_n|| \times \\
&\quad (2||z_{t_n} - x_n|| + ||Sx_n - x_n||) \\
&+ 2t_n\langle u - z_{t_n}, j(z_{t_n} - x_n)\rangle,
\end{aligned}$$

and hence,

$$\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq \frac{t_n}{2}||z_{t_n} - x_n||^2 + \frac{||Sx_n - x_n||}{2t_n} \times (2||z_{t_n} - x_n|| + ||Sx_n - x_n||).$$

Since $\{x_n\}, \{z_{t_n}\}$ and $\{Sx_n\}$ are bounded and $\frac{||Sx_n - x_n||}{2t_n} \to 0, n \to \infty$, it follows from the last inequality that

$$\lim sup\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq 0.$$

Moreover, we have that

$$\begin{aligned}
\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle &= \langle u - z, j(x_n - z)\rangle + \langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \\
&+ \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle. \qquad (2.4)
\end{aligned}$$

But, by hypothesis, $z_{t_n} \to z \in F(S),\ n \to \infty$. Thus, using the boundedness of $\{x_n\}$ we obtain that

$$\langle z - z_{t_n}, j(x_n - z_{t_n})\rangle \to 0,\ n \to \infty. \tag{2.5}$$

Also,

$$\langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \to 0,\ n \to \infty,$$

since j is norm-to-weak* uniformly continuous on bounded subsets of E. Hence, we obtain from (2.4) and (2.5) that

$$\lim sup\langle u - z, j(x_n - z)\rangle \leq 0.$$

Furthermore, from the recurrence relation (2.2) we get that $x_{n+1} - z = \alpha_n(u - z) + (1 - \alpha_n)(Sx_n - z)$. It then follows that

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq (1 - \alpha_n)^2||Sx_n - z||^2 + 2\alpha_n\langle u - z, j(x_{n+1} - z)\rangle \\ &\leq (1 - \alpha_n)||x_n - z||^2 + \alpha_n\sigma_n, \end{aligned}$$

where $\sigma_n := 2\langle u - z, j(x_{n+1} - z)\rangle;\ \gamma_n \equiv 0\ \forall\ n \geq 0$. Thus, by Lemma 2.2.3, $\{x_n\}$ converges strongly to a fixed point of T. $\square$

Remark 2.3.2 *We note that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm and is such that every nonempty closed convex and bounded subset of E has the fixed point property for nonexpansive maps (see e.g., [1]).*

Remark 2.3.3 *Theorem 2.3.1 is a significant generalization of Theorem 2.1.2 and of Theorem 2.1.3 as has been explained in the introduction. Furthermore, our method of proof which is different from the method of Shioji and Takahashi [80] is of independent interest.*

Let $S_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} S^k x$. With this definition, Xu also proved the following theorem.

Theorem 2.3.4 *(Xu [96], Theorem 3.2) Assume that E is a real uniformly convex and uniformly smooth Banach space. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm:*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_n x_n, n \geq 0. \tag{2.6}$$

Assume that

(i) $\lim \alpha_n = 0$;

(i) $\sum \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to a fixed point of $S : K \to K$ nonexpansive.

Remark 2.3.5 *Theorem 2.3.1 is also a significant improvement of Theorem 2.3.4 in the sense that the recursion formula (2.2) is simpler and requires less computer time than the recursion formula (2.6). Moreover, the requirement that E is also uniformly convex imposed in Theorem 2.3.4 is dispensed with in Theorem 2.3.1. Furthermore, Theorem 2.3.1 is proved in the framework of the more general real Banach spaces with uniformly Gâteaux differentiable norms.*

Chapter 3

A Strong Convergence Theorem for Fixed Points of Asymptotically Nonexpansive Mappings in Banach Spaces

3.1 Introduction

In this chapter, we extend the result of Chapter 2 from the class of nonexpansive mappings to the class of *asymptotically nonexpansive* ones.

Recall that a mapping $T : K \to K$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}, k_n \geq 1$, such that $\lim_{n\to\infty} k_n = 1$ and $||T^n x - T^n y|| \leq k_n||x - y||$ holds for each $x, y \in K$ and for each integer $n \geq 1$.

This class of mappings has been studied extensively by various authors (see e.g.,Chidume and Ali [20],[22], [24], Chidume *et al.*[32, 33], Chang *et al.* [15], Falset *et al.* [45], Kaczor [52], Oka [68], Schu [75, 76], Wang [92], Qihou [70], Shioji and Takahashi [81], Sun [82],Tan and Xu [89, 87] and the references contained therein).

Suppose now K is a nonempty closed convex subset of real uniformly smooth Banach space E and $T : K \to K$ is an asymptotically nonexpansive mapping with sequence $k_n \geq 1$ for all $n \geq 1$. Fix $u \in K$ and define, for each integer $n \geq 1$, the contraction mapping $S_n : K \to K$ by

$$S_n(x) = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x,$$

where $\{t_n\} \subset [0, 1)$ is any sequence such that $t_n \to 1$. Then, by the Banach contraction mapping principle, there exists unique x_n such that

$$x_n = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x_n.$$

The question now arises as to whether or not this sequence converges to a fixed point of T. A partial answer was given in 1994 by Lim and Xu who proved the following theorem:

Theorem 3.1.1 *(Lim and Xu [60]) Suppose E is a real uniformly smooth Banach space and suppose $\{t_n\}$ is chosen such that $\lim_{n\to\infty}\left(\frac{k_n-1}{k_n-t_n}\right) = 0$. Suppose, in addition, the following condition holds:*

$$\lim ||x_n - Tx_n|| = 0.$$

Then, the sequence $\{x_n\}$ defined, for a fixed $u \in K$, by

$$x_n = \Big(1 - \frac{t_n}{k_n}\Big)u + \frac{t_n}{k_n}T^n x_n \tag{3.1}$$

converges strongly to a fixed point of T.

Remark 3.1.2 *Observe that equation (4.10) can be re-written as follows:*

$$x_n = \omega_n u + (1 - \omega_n)T^n x_n$$

where $\omega_n := 1 - \frac{t_n}{k_n}$ and $\omega_n \to 0$ as $n \to \infty$.

It is our purpose in this chapter to extend Theorem 2.3.1 of Chapter 2 to the more general class of *asymptotically nonexpansive* mappings.

3.2 Convergence Theorems

Theorem 3.2.1 *(Chidume and de Souza [35]) Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}, k_n \geq 1$ and $\lim k_n = 1$ such that $\sum(k_n^2 - 1) < \infty$ and $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For a fixed $\delta \in (0,1)$, define $S^n : K \to K$ by $S^n x := (1-\delta)x + \delta T^n x \ \forall \ x \in K$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of K which satisfies $z_t = tu + (1-t)T^n z_t$ for arbitrary $u \in K$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ which satisfies the following conditions: $C1 : \lim \alpha_n = 0$; $C2 : \sum \alpha_n = \infty$. For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)S^n x_n.$$

Assume $\{x_n\}$ is bounded and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Observe first that

$$\|S^n x - S^n y\| \leq (1-\delta)\|x-y\| + \delta\|T^n x - T^n y\| \leq (1 - \delta + k_n\delta)\|x-y\|.$$

Furthermore, $S^n x = x$ if and only if $T^n x = x$, and hence S is asymptotically nonexpansive and has the same set of fixed points as T. Define

$$\beta_n := (1-\delta)\alpha_n + \delta \ \forall \ n \geq 0; \ y_n := \frac{x_{n+1} - x_n + \beta_n x_n}{\beta_n}, \ n \geq 0. \tag{3.2}$$

Observe that $\beta_n \to \delta$ as $n \to \infty$, and that $\{x_n\}$, $\{y_n\}$, $\{Tx_n\}$ and $\{Sx_n\}$ are all bounded. Observe also that from the definitions of β_n and S^n, we obtain that $y_n = \frac{\alpha_n u+(1-\alpha_n)\delta T^n x_n}{\beta_n}$ so that,

$$
\begin{aligned}
||y_{n+1} - y_n|| \quad &- \quad ||x_{n+1} - x_n|| \\
&\leq \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| .||u|| + \frac{(1-\alpha_{n+1})}{\beta_{n+1}}\ \delta\ ||T^{n+1}x_{n+1} - T^n x_n|| \\
&+ \left|\left|\frac{(1-\alpha_{n+1})}{\beta_{n+1}}\ \delta T^{n+1}x_n - \frac{(1-\alpha_n)}{\beta_n}\ \delta T^{n+1}x_n\right.\right. \\
&+ \left.\left.\frac{(1-\alpha_n)}{\beta_n}\ \delta T^{n+1}x_n - \frac{(1-\alpha_n)}{\beta_n}\ \delta T^n x_n\right|\right| \\
&\leq \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right|\ ||u|| + \frac{(1-\alpha_{n+1})}{\beta_{n+1}}\ \delta k_{n+1}\ ||x_{n+1} - x_n|| \\
&+ \left|\left(\frac{(1-\alpha_{n+1})}{\beta_{n+1}} - \frac{(1-\alpha_n)}{\beta_n}\right)\ \delta\right|\ ||T^{n+1}x_n|| \\
&+ \left|(\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n})\ \delta K_n\right|\ ||x_n - Tx_n||.
\end{aligned}
$$

Hence we have for some constant $M_1 > 0$,

$$
\begin{aligned}
||y_{n+1} - y_n|| \quad &- \quad ||x_{n+1} - x_n|| \\
&\leq \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right|.||u|| + \left|\frac{(1-\alpha_{n+1})}{\beta_{n+1}}\ \delta k_{n+1} - 1\right|\ ||x_{n+1} - x_n|| \\
&+ \left(\frac{1-\alpha_{n+1}}{\beta_{n+1}} - \frac{1-\alpha_n}{\beta_n}\right)\ \delta M_1 + \left(\frac{1-\alpha_n}{\beta_n}\right)\delta k_n ||x_n - Tx_n||,
\end{aligned}
$$

and so,

$$
\lim sup(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \quad \leq \quad 0.
$$

Hence, by Lemma 2.2.2, $||y_n - x_n|| \to 0$ as $n \to \infty$. Consequently, $\lim ||x_{n+1} - x_n|| = \lim \beta_n ||y_n - x_n|| = 0$. Furthermore, $||x_{n+1} - S^n x_n|| = \alpha_n ||u - S^n x_n|| \to 0 \quad as \quad n \to \infty$. Hence, $||x_n - S^n x_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - S^n x_n|| \to 0 \quad as \quad n \to \infty$.

Claim: $\lim sup \langle u - z, j(x_n - z) \rangle \leq 0$.

For each integer $n \geq 0$, let $t_n \in (0, \frac{\alpha_n}{1-\alpha_n})$ be such that

$$\frac{k_n^2 - 1}{t_n} \to 0, \text{ and } \frac{||x_n - Sx_n||}{t_n} \to 0, \; n \to \infty. \tag{3.3}$$

Clearly $t_n \to 0 \quad as \quad n \to \infty$. Now observe that

$$z_{t_n} = t_n u + (1 - t_n) S^n z_{t_n}$$

so that

$$\begin{aligned}
||z_{t_n} - x_n||^2 &\leq (1 - t_n)^2 ||S^n z_{t_n} - x_n||^2 + 2t_n \langle u - z_{t_n}, j(z_{t_n} - x_n) \rangle \\
&\leq (1 - t_n)^2 \Big[||S^n z_{t_n} - S^n x_n|| + ||S^n x_n - x_n|| \Big]^2 \\
&+ 2t_n ||z_{t_n} - x_n||^2 + 2t_n \langle u - z_{t_n}, j(z_{t_n} - x_n) \rangle \\
&\leq (1 - t_n)^2 {k_n}^2 ||z_{t_n} - x_n||^2 + 2||S^n z_{t_n} - S^n x_n|| ||S^n x_n - x_n|| \\
&+ ||S^n x_n - x_n||^2 + 2t_n ||z_{t_n} - x_n||^2 + 2t_n \langle u - z_{t_n}, j(z_{t_n} - x_n) \rangle
\end{aligned}$$

and because

$$\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq \frac{1}{2}\Big[\frac{(1-t_n)^2k_n^2 + 2t_n - 1}{t_n}\Big]||z_{t_n} - x_n||^2 + \frac{||S^n x_n - x_n||M}{t_n},$$

for some constant $M > 0$, this yields that

$$\lim sup\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle \leq 0.$$

Moreover,

$$\begin{aligned}\langle u - z_{t_n}, j(x_n - z_{t_n})\rangle &= \langle u - z, j(x_n - z)\rangle + \langle u - z, j(x_n - z_{t_n}) - j(x_n - z)\rangle \\ &+ \langle z - z_{t_n}, j(x_n - z_{t_n})\rangle,\end{aligned}$$

and since j is norm-to-$weak^*$ uniformly continuous on bounded sets and $z_{t_n} \to z$, we obtain that

$$\lim sup\langle u - z, j(x_n - z)\rangle \leq 0,$$

establishing the claim. From $x_{n+1} = \alpha_n u + (1-\alpha_n)S^n x_n$ we have

$$||x_{n+1} - z||^2 \leq (1-\alpha_n)^2 k_n^2 ||x_n - z||^2 + 2\alpha_n\langle u - z, j(x_{n+1} - z\rangle.$$

Since $t_n \in (0, \frac{\alpha_n}{1-\alpha_n})$, there exists an integer $N_0 > 0$ such that,

$$||x_{n+1} - z||^2 \leq (1-\alpha_n)||x_n - z||^2 + \alpha_n\sigma_n,$$

for all $n \geq N_0$, where $\sigma_n := 2\langle u - z, j(x_{n+1} - z)\rangle \ \forall \ n \geq 0$. Thus, by Lemma 2.2.3, $\{x_n\}$ converges strongly to a fixed point of T. $\square$

Corollary 3.2.2 *Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For a fixed $\delta \in (0, 1)$, define $S : K \to K$ by $Sx := (1-\delta)x + \delta Tx, \ \forall \ x \in K$. Assume that $\{z_t\}$ converges strongly to a fixed point z of T as $t \to 0$, where z_t is the unique element of K which satisfies $z_t = tu + (1 - t)Tz_t$ for arbitrary $u \in K$. Let $\{\alpha_n\}$ be a real sequence in $(0, 1)$ which satisfies the conditions: $C1 : \lim \alpha_n = 0$; $C2 : \sum \alpha_n = \infty$. For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n. \tag{3.4}$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It is easy to see from equation (3.4) that $\{x_n\}$ is bounded and $\lim ||x_n - Tx_n|| = 0$. Hence, the result follows from Theorem 3.2.1.

Remark 3.2.3 *Theorem 3.2.1 extends Theorem 2.3.1 (and consequently, extends Theorem 2.1.2, Theorem 2.1.3 and Theorem 2.3.4, (see Remarks 3.1, 3.2 and 3.4 of [27]) to the more general class of asymptotically nonexpansive mappings.*

CHAPTER 4

CONVERGENCE OF A HYBRID STEEPEST DESCENT METHOD FOR VARIATIONAL INEQUALITIES IN BANACH SPACES

4.1 Introduction

In this chapter, we extend the results of Xu and Kim [98] from real Hilbert spaces to $q-$uniformly smooth real Banach spaces which are much more general than Hilbert spaces. In particular, our theorems will be applicable in L_p spaces, $1 < p < \infty$.

4.2 Preliminaries

We shall make use of the following lemmas.

Lemma 4.2.1 (Shoiji and Takahashi, [80]) *Let* $(a_0, a_1, ...) \in l^\infty$ *such that* $\mu_n(a_n) \leq 0$ *for all Banach limit* μ *and* $\limsup\limits_{n\to\infty} (a_{n+1} - a_n) \leq 0$. *Then,* $\limsup\limits_{n\to\infty} a_n \leq 0$.

Lemma 4.2.2 (Xu, [97]) *Let* E *be a q-uniformly smooth real Banach space for some* $q > 1$, *then there exists some positive constant* d_q *such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q \ \forall \ x, y \in E \text{ and } j_q(x) \in J_q(x).$$

Lemma 4.2.3 *(*(Lim and Xu,) *[60], Theorem 1) Suppose* E *is a Banach space with uniform normal structure,* K *is a nonempty bounded subset of* E, *and* $T : K \to K$ *is uniformly* $k-$*Lipschitzian mapping with* $k < N(E)^{\frac{1}{2}}$. *Suppose also there exists a nonempty bounded* *closed convex subset C of K with the property (P) :*

$$(P) \quad x \in C \quad \text{implies} \quad \omega_w(x) \subset C,$$

where $\omega_w(x)$ is the $\omega-$limi set of T at x, i.e., the set

$$\{y \in E : y = \text{weak} - \underset{j}{lim} T^{n_j} x \text{ for } \ j \to \infty\}.$$

Then, T has a fixed point in C.

Lemma 4.2.4 *Let X be a real reflexive Banach space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex $\boxed{\text{proper}}$ lower semi-continuous function. Suppose*

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

Then, $\exists\, \bar{x} \in X$ such that $f(\bar{x}) \leq f(x)$, $x \in X$, i.e.,

$$f(\bar{x}) = \inf_{x \in X} f(x).$$

4.3 Convergence Theorems

Lemma 4.3.1 *(Chidume et al. [28]) Let E be a $q-$uniformly smooth real Banach space with constant d_q, $q \geq 2$. Let $T : E \to E$ be a nonexpansive mapping and $G : E \to E$ be an $\eta-$ strongly accretive and $\kappa-$Lipschitzian map. For $\lambda \in \left(0, \frac{2}{q(q-1)}\right)$ and $\delta \in \left(0, \min\left\{\frac{q}{4\eta}, (\frac{q\eta}{d_q \kappa^q})^{\frac{1}{(q-1)}}\right\}\right)$, define a map $T^{\lambda} : E \to E$ by $T^{\lambda} x = Tx - \lambda \delta G(Tx), \quad x \in E.$*

Then, T^λ is a strict contraction. Furthermore,

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\alpha)\|x-y\| \ \forall\ x,y \in E, \tag{4.1}$$

where $\alpha := \frac{q}{2} - \sqrt{\frac{q^2}{4} - \delta(q\eta - \delta^{q-1}d_q\kappa^q)} \ \ \in (0,1)$.

Proof. For $x, y \in E$, using Lemma 4.2.2, we have:

$$\begin{aligned}
\|T^\lambda x - T^\lambda y\|^q &= \|Tx - Ty - \lambda\delta(G(Tx) - G(Ty))\|^q \\
&\leq \|Tx - Ty\|^q - q\lambda\delta\langle G(Tx) - G(Ty), j_q(Tx - Ty)\rangle \\
&+ d_q\lambda^q\delta^q\|G(Tx) - G(Ty)\|^q \\
&\leq \|Tx - Ty\|^q - q\lambda\delta\eta\|Tx - Ty\|^q + d_q\lambda^q\delta^q\kappa^q\|Tx - Ty\|^q \\
&\leq \Big[1 - \lambda\delta\big(q\eta - d_q\lambda^{q-1}\delta^{q-1}\kappa^q\big)\Big]\|x - y\|^q \\
&\leq \Big[1 - \lambda\delta\big(q\eta - d_q\delta^{q-1}\kappa^q\big)\Big]\|x - y\|^q.
\end{aligned}$$

Define

$$f(\lambda) := 1 - \lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) = (1-\lambda\tau)^q, \text{ for some } \tau \in (0,1), \text{ say.}$$

By Taylor development, there exists $\xi \in (0, \lambda)$ such that

$$1 - \lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) = 1 - q\tau\lambda + \frac{1}{2}q(q-1)(1-\xi\tau)^{q-2}\lambda^2\tau^2.$$

Using $\lambda \in \left(0, \frac{2}{q(q-1)}\right)$ which implies $\frac{1}{2}q(q-1)\lambda < 1$, we obtain that

$$1 - \lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) < 1 - q\tau\lambda + \frac{1}{2}q(q-1)\lambda^2\tau^2 < 1 - q\tau\lambda + \lambda\tau^2,$$

so that

$$\tau^2 - q\tau + \delta(q\eta - d_q\delta^{q-1}\kappa^q) > 0.$$

Solving this quadratic inequality in τ, we obtain, $\tau < \frac{q}{2} - \sqrt{\frac{q^2}{4} - \delta(q\eta - d_q\delta^{q-1}\kappa^q)}$.

Now, set

$$\alpha := \frac{q}{2} - \sqrt{\frac{q^2}{4} - \delta(q\eta - d_q\delta^{q-1}\kappa^q)}.$$

Observe that

$$\frac{q^2}{4} - \delta(q\eta - d_q\delta^{q-1}\kappa^q) = \Big(\frac{q^2}{4} - \delta q\eta\Big) + d_q\delta^{q-1}\kappa^q > 0,$$

since $\delta < \frac{q}{4\eta}$. Moreover, since $q \geq 2$ and $\lambda < \frac{2}{q(q-1)} < \frac{2}{q}$, we have

$$1 - \lambda\alpha = 1 - \frac{\lambda q}{2} + \sqrt{\frac{q^2\lambda^2}{4} - \lambda^2\delta(q\eta - d_q\delta^{q-1}\kappa^q)} \in (0,1).$$

The proof is complete.

We note that L_p spaces, $2 \leq p < \infty$, are $2-$uniformly smooth and the following inequality holds (see e.g., [97]): For each $x, y \in L_p$, $2 \leq p < \infty$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + (p-1)\|y\|^2.$$

It then follows that by setting $q = 2,\ d_q = (p-1)$ in Lemma 4.3.1, we obtain the following corollary.

Corollary 4.3.2 *Let $E = L_p$, $2 \leq p < \infty$. Let $T : E \to E$, be a nonexpansive map and $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian map. For $\lambda \in \Big(0, 1\Big)$ and $\delta \in \Big(0, \min\Big\{\frac{1}{2\eta}, \frac{2\eta}{(p-1)\kappa^2}\Big\}\Big)$, define a map $T^\lambda : E \to E$ by $T^\lambda x = Tx - \lambda\delta G(Tx), \quad x \in E$.*

Then, T^λ is a strict contraction. In particular,

$$\|T^\lambda x - T^\lambda y\| \le (1-\lambda\alpha)\|x-y\|, \quad x, y \in E, \tag{4.2}$$

where $\alpha := 1 - \sqrt{1-\delta(2\eta-(p-1)\delta\kappa^2)} \quad \in (0,1)$.

By setting $p = 2$ in Corollary 4.3.2, we obtain the following corollary.

Corollary 4.3.3 *Let H be a real Hilbert space, $T : H \to H$ be a nonexpansive map and $G : H \to H$ be an $\eta-$strongly monotone and $\kappa-$Lipschitzian map. For $\lambda \in \Big(0,1\Big)$ and $\delta \in \Big(0, \min\Big\{\frac{1}{2\eta}, \frac{2\eta}{\kappa^2}\Big\}\Big)$, define a map $T^\lambda : H \to H$ by $T^\lambda x = Tx - \lambda\delta G(Tx), \quad x \in H$. Then, T^λ is a strict contraction. In particular,*

$$\|T^\lambda x - T^\lambda y\| \le (1-\lambda\alpha)\|x-y\|, \quad x, y \in H, \tag{4.3}$$

where $\alpha := 1 - \sqrt{1-\delta(2\eta-\delta\kappa^2)} \quad \in (0,1)$.

Remark 4.3.4 *Corollary 4.3.3 is a result of Yamada [99] and is the main tool used in Wang [91], Xu and Kim [98], Yamada [99], Zheng and Yao [100]. Lemma 4.3.1 and Corollary 4.3.2 which extend this result to q-uniformly smooth spaces, $q \ge 2$, and L_p spaces, $2 \le p < \infty$, respectively, are new.*

We prove the following theorem for family of nonexpansive maps. In the theorem, d_q is the constant which appears in Lemma 4.2.2.

Theorem 4.3.5 *(Chidume et al. [28]) Let E be a $q-$uniformly smooth real Banach space with constant d_q, $q \ge 2$. Let $T_i : E \to E, \quad i = 1, 2, ..., r$ be a finite family of nonexpansive* *mappings with* $K := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. *Let* $G : E \to E$ *be an* $\eta-$*strongly accretive map which is also* $\kappa-$*Lipschitzian. Let* $\{\lambda_n\}$ *be a real sequence in* $[0,1]$ *satisfying*

$$C1 : lim\lambda_n = 0;\ C2 : \sum \lambda_n = \infty;\ C6 : lim\frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0.$$

For $\delta \in \left(0, \min\left\{\frac{q}{4\eta}, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{(q-1)}}\right\}\right)$, *define a sequence* $\{x_n\}$ *iteratively in* E *by* $x_0 \in E$,

$$x_{n+1} = T_{[n+1]}^{\lambda_{n+1}} x_n = T_{[n+1]}x_n - \delta\lambda_{n+1}G(T_{[n+1]}x_n), \quad n \geq 0, \tag{4.4}$$

where $T_{[n]} = T_{n\ mod\ r}$. *Assume also that*

$$K = F(T_rT_{r-1}...T_1) = F(T_1T_r...T_2) = ... = F(T_{r-1}T_{r-2}...T_r).$$

Then, $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality* $VI(G,K)$.

Proof. Let $x^* \in K$, then the sequence $\{x_n\}$ satisfies

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\|\right\},\ n \geq 0.$$

It is obvious that this is true for $n = 0$. Assume it is true for $n = k$ for some $k \in \mathbb{N}$. From the recursion formula (4.4) and condition C1, we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|T^{\lambda_{k+1}}_{[k+1]} x_k - x^*\| \\
&\leq \|T^{\lambda_{k+1}}_{[k+1]} x_k - T^{\lambda_{k+1}}_{[k+1]} x^*\| + \|T^{\lambda_{k+1}}_{[k+1]} x^* - x^*\| \\
&\leq (1 - \lambda_{k+1}\alpha)\|x_k - x^*\| + \lambda_{k+1}\delta\|G(x^*)\| \\
&\leq \max\left\{\|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\|\right\},
\end{aligned}$$

and the claim follows by induction. Thus the sequence $\{x_n\}$ is bounded and so are $\{T_{[n+1]}x_n\}$ and $\{G(T_{[n+1]}x_n)\}$. Using the recursion formula (4.4) we get,

$$\|x_{n+1} - T_{[n+1]}x_n\| = \lambda_{n+1}\delta\|G(T_{[n+1]}x_n)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Also,

$$\begin{aligned}
\|x_{n+r} - x_n\| &= \|T^{\lambda_{n+r}}_{[n+r]} x_{n+r-1} - T^{\lambda_n}_{[n]} x_{n-1}\| \\
&\leq \|T^{\lambda_{n+r}}_{[n+r]} x_{n+r-1} - T^{\lambda_{n+r}}_{[n+r]} x_{n-1}\| + \|T^{\lambda_{n+r}}_{[n+r]} x_{n-1} - T^{\lambda_n}_{[n]} x_{n-1}\| \\
&\leq (1 - \lambda_{n+r}\alpha)\|x_{n+r-1} - x_{n-1}\| \\
&+ \alpha\lambda_{n+r}\Big(\frac{|\lambda_{n+r} - \lambda_n|}{\alpha\lambda_{n+r}}\delta\|G(T_{[n]}x_{n-1})\|\Big).
\end{aligned}$$

By Lemma 2.2.3 and condition $C6$, we have

$$\|x_{n+r} - x_n\| \to 0 \text{ as } n \to \infty. \tag{4.5}$$

In particular,

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{4.6}$$

Replacing n by $n + r - 1$ in (4.4) we have,

$$||x_{n+r} - T_{n+r}x_{n+r-1}|| \quad = \quad \delta\lambda_{n+r}||G(T_{[n+r]}x_{n+r-1})|| \to 0, \;\; n \to \infty.$$

Using the fact that T_i is nonexpansive for each i, we obtain the following finite table:

$$x_{n+r} - T_{n+r}x_{n+r-1} \to 0 \quad as \quad n \to \infty;$$

$$T_{n+r}x_{n+r-1} - T_{n+r}T_{n+r-1}x_{n+r-2} \to 0 \quad as \quad n \to \infty;$$

$$\vdots$$

$$T_{n+r}T_{n+r-1}...T_{n+2}x_{n+1} - T_{n+r}T_{n+r-1}...T_{n+2}T_{n+1}x_n \to 0 \quad as \quad n \to \infty;$$

and adding up the table yields

$$x_{n+r} - T_{n+r}T_{n+r-1}...T_{n+1}x_n \to 0 \quad as \quad n \to \infty.$$

Using this and (4.5) we get that

$$\lim_{n\to\infty} ||x_n - T_{n+r}T_{n+r-1}...T_{n+1}x_n|| = 0. \tag{4.7}$$

Define a map $\varphi : E \to \mathbb{R}$ by $\varphi(y) = \mu_n\|x_{n+1} - y\|^2$, where μ_n denotes a Banach limit. Then, φ is continuous, convex and $\varphi(y) \to +\infty$ as $\|y\| \to +\infty$. Thus, since E is a reflexive Banach space, there exists $y^* \in E$ such that $\varphi(y^*) = \min_{u \in E} \varphi(u)$. So, the set $K^* := \{x \in E : \varphi(x) = \min_{u \in E} \varphi(u)\} \neq \emptyset$. We now show T_i has a fixed point in K^* for each $i = 1, 2, ..., r$. We shall assume, from equation (4.7), that $\forall i$,

$$\lim_{n \to \infty} ||x_n - T_i x_n|| = 0. \tag{4.8}$$

We shall make use of Lemma 4.2.3. If x is in K^* and $y := \omega - \lim_j T_i^{m_j} x$, belongs to the weak $\omega - limit$ set $\omega_w(x)$ of T_i at x, then, from the w-l.s.c. of φ and equation (4.8), we have, (since equation (4.8) implies $||x_n - T_i^m x_n|| \to 0$ as $n \to \infty$, this is easily proved by induction),

$$\begin{aligned}
\varphi(y) &\leq \liminf_j \varphi\Big(T_i^{m_j} x\Big) \leq \limsup_m \varphi\Big(T_i^m x\Big) \\
&= \limsup_m \Big(\mu_n ||x_n - T_i^m x||^2\Big) \\
&= \limsup_m \Big(\mu_n ||x_n - T_i^m x_n + T_i^m x_n - T_i^m x||^2\Big) \\
&\leq \limsup_m \Big(\mu_n ||T_i^m x_n - T_i^m x||^2\Big) \leq \limsup_m \Big(\mu_n ||x_n - x||^2\Big) = \varphi(x) \\
&= \inf_{u \in E} \varphi(u).
\end{aligned}$$

So, $y \in K^*$. By Lemma 4.2.3, T_i has a fixed point in K^* $\forall$ i and so $K^* \cap K \neq \emptyset$.

Let $x^* \in K^* \cap K$ and $t \in (0, 1)$. It then follows that $\varphi(x^*) \leq \varphi(x^* - tG(x^*))$. Using the inequality of Lemma 2.2.1, we have that

$$\|x_n - x^* + tG(x^*)\|^2 \leq \|x_n - x^*\|^2 + 2t\langle G(x^*), j(x_n - x^* + tG(x^*))\rangle.$$

Thus, taking Banach limits over $n \geq 1$ gives

$$\begin{aligned} \mu_n\|x_n - x^* + tG(x^*)\|^2 &\leq \mu_n\|x_n - x^*\|^2 \\ &+ 2t\mu_n\langle G(x^*), j(x_n - x^* + tG(x^*))\rangle. \end{aligned}$$

This implies,

$$\mu_n\langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle \leq \varphi(x^*) - \varphi(x^* - tG(x^*)) \leq 0.$$

This therefore implies that

$$\mu_n\langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle \leq 0 \;\forall\; n \geq 1.$$

Since the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of E, we obtain, as $t \to 0$, that

$$\langle -G(x^*), j(x_n - x^*)\rangle - \langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle \to 0.$$

Hence, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0, \delta)$ and for all $n \geq 1$,

$$\langle -G(x^*), j(x_n - x^*)\rangle < \langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle + \varepsilon.$$

Consequently,

$$\mu_n\langle -G(x^*), j(x_n - x^*)\rangle \leq \mu_n\langle -G(x^*), j(x_n - x^* + tG(x^*))\rangle + \varepsilon \leq \varepsilon.$$

Since ε is arbitrary, we have

$$\mu_n\langle -G(x^*), j(x_n - x^*)\rangle \leq 0.$$

Moreover, from the norm-to-norm uniform continuity of j on bounded sets, we obtain, that

$$\lim_{n\to\infty}\Big(\langle -G(x^*), j(x_{n+1} - x^*)\rangle - \langle -G(x^*), j(x_n - x^*)\rangle\Big) = 0.$$

Thus, the sequence $\{\langle -G(x^*), j(x_n - x^*)\rangle\}$ satisfies the conditions of Lemma 4.2.1. Hence, we obtain that

$$\limsup_{n\to\infty}\langle -G(x^*), j(x_n - x^*)\rangle \leq 0.$$

Define

$$\varepsilon_n := \max\ \{\langle -G(x^*), j(x_{n+1} - x^*)\rangle, 0\}.$$

Then, $\lim \varepsilon_n = 0$, and $\langle -G(x^*), j(x_{n+1} - x^*)\rangle \leq \varepsilon_n$. From the recursion formula (4.4), and Lemma 2.2.1, we have,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|T^{\lambda_{n+1}}_{[n+1]}x_n - T^{\lambda_{n+1}}_{[n+1]}x^* + T^{\lambda_{n+1}}_{[n+1]}x^* - x^*\|^2 \\
&\leq \|T^{\lambda_{n+1}}_{[n+1]}x_n - T^{\lambda_{n+1}}_{[n+1]}x^*\|^2 + 2\lambda_{n+1}\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle \\
&\leq (1 - \lambda_{n+1}\alpha)\|x_n - x^*\|^2 + 2\lambda_{n+1}\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle
\end{aligned}$$

and by Lemma 2.2.3, we have that $x_n \to x^*$ as $n \to \infty$. This completes the proof.$\Box$

The following corollaries follow from Theorem 4.3.5.

Corollary 4.3.6 *Let $E = L_p$, $2 \leq p < \infty$. Let $T_i : E \to E, \quad i = 1, 2, ..., r$ be a finite family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive map which is also $\kappa-$Lipschitzian. Let $\{\lambda_n\}$ be a real sequence in $[0, 1]$ that satisfies conditions $C1$, $C2$ and $C6$ as in theorem 4.3.5. For $\delta \in \left(0, \min\left\{\frac{1}{2\eta}, \frac{2\eta}{(p-1)\kappa^2}\right\}\right)$, define a sequence $\{x_n\}$ iteratively in E by (4.4). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI(G, K)$.*

Corollary 4.3.7 *Let H be a real Hilbert space. Let $T_i : H \to H, \quad i = 1, 2, ..., r$ be a finite family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : H \to H$ be an $\eta-$strongly monotone map which is also $\kappa-$Lipschitzian. Let $\{\lambda_n\}$ be a real sequence in $[0, 1]$ that satisfies conditions $C1$, $C2$ and $C6$ as in theorem 4.3.6. For $\delta \in \left(0, \min\left\{\frac{1}{2\eta}, \frac{2\eta}{\kappa^2}\right\}\right)$, define a sequence $\{x_n\}$ iteratively in H by (4.4). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI(G, K)$.*

Theorem 4.3.8 *(Chidume et al. [28]) Let E be a real $q-$uniformly smooth Banach space with constant d_q, $q \geq 2$. Let $T : E \to E$ be a nonexpansive map. Assume that $K := F(T) = \{x \in E : Tx = x\} \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian map. Let $\{\lambda_n\}$ be a real sequence in $[0, 1]$ satisfying the following conditions:*

$$C1 : lim \lambda_n = 0; \quad C2 : \sum \lambda_n = \infty; \quad C5 : lim \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_{n+1}} = 0.$$

For $\delta \in \left(0, \min\left\{\frac{q}{4\eta}, (\frac{q\eta}{d_q \kappa^q})^{\frac{1}{(q-1)}}\right\}\right)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$,

$$x_{n+1} = T^{\lambda_{n+1}} x_n = Tx_n - \delta \lambda_{n+1} G(Tx_n), \quad n \geq 0. \tag{4.9}$$

Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI(G, K)$.

Proof. Take $T_1 = T_2 = ... = T_r = T$ in Theorem 4.3.5 and the result follows.

The following corollaries follow from Theorem 4.3.8.

Corollary 4.3.9 *Let $E = L_p$, $2 \leq p < \infty$. Let $T : E \to E$, be a nonexpansive map. Assume that $K := F(T) = \{x \in E : Tx = x\} \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian map. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions $C1$, $C2$ and $C5$ as in theorem 4.3.8. For $\delta \in \left(0, \min\left\{\frac{1}{2\eta}, \frac{2\eta}{(p-1)\kappa^2}\right\}\right)$, define a sequence $\{x_n\}$ iteratively in E by (4.9). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI(G, K)$.*

Corollary 4.3.10 *Let H be a real Hilbert space. Let $T : H \to H$, be a nonexpansive map. Assume that $K := F(T) = \{x \in E : Tx = x\} \neq \emptyset$. Let $G : H \to H$ be an $\eta-$strongly monotone and $\kappa-$Lipschitzian map. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions $C1$, $C2$ and $C5$ as in theorem 4.3.8. For $\delta \in \left(0, \min\left\{\frac{1}{2\eta}, \frac{2\eta}{\kappa^2}\right\}\right)$, define a sequence $\{x_n\}$ iteratively in H by (4.9). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI(G, K)$.*

4.4 The case of L_p spaces, $1 < p \leq 2$.

We begin with the following definition.

Definition 4.4.1 *A Banach space E is called a lower weak parallelogram space with constant $b \geq 0$ or, briefly, E is $LWP(b)$, in the terminology of Bynum [12] if*

$$||x + y||^2 + b||x - y||^2 \leq 2(||x||^2 + ||y||^2) \tag{4.10}$$

holds for all $x, y \in E$.

It is proved in [12] that l_p space, $1 < p \leq 2$, is a lower weak parallelogram space with $(p-1)$ as the largest number b for which (4.10) holds. Furthermore, if $L_p,\ (1 < p \leq 2)$ has at least two disjoint sets of positive finite measure, then it is a lower weak parallelogram space with $(p-1)$ as the largest number b for which (4.10) holds. We shall assume, without loss of generality, that $L_p,\ (1 < p \leq 2)$ has at least two disjoint sets of positive finite measure. In the sequel, we shall state all our theorems and lemmas only for L_p spaces, $1 < p \leq 2$, with the understanding that they also hold for l_p spaces, $1 < p \leq 2$.

In terms of the normalized duality mapping, Bynum [12] proved that a real Banach space is a lower weak parallelogram space if and only if for each $x, y \in E$ and $f \in J(x)$, the following inequality holds:

$$||x+y||^2 \geq ||x||^2 + b||y||^2 + 2\langle y, f\rangle. \tag{4.11}$$

In particular, for $E = L_p,\ 1 < p \leq 2$, the following inequality holds:

$$||x+y||^2 \geq ||x||^2 + (p-1)||y||^2 + 2\langle y, j(x)\rangle \ \forall\ x, y \in E. \tag{4.12}$$

We now prove the following lemmas which will be central in the sequel.

Lemma 4.4.2 *Let $E = L_p, 1 < p \leq 2$. Then, for all $x, y \in E$, the following inequality holds:*

$$(p-1)||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + ||y||^2. \tag{4.13}$$

Proof. Observe first that E is smooth so that the normalized duality map on E is single-valued. Now, replacing x by $(-x)$ and y by $(x+y)$ in inequality (4.12), we obtain

$$||y||^2 \geq ||x||^2 + 2\langle x+y, j(-x)\rangle + (p-1)||x+y||^2,$$

which implies

$$\begin{aligned}(p-1)||x+y||^2 &\leq ||y||^2 - ||x||^2 + 2\langle x+y, j(x)\rangle \\ &= ||x||^2 + 2\langle y, j(x)\rangle + ||y||^2,\end{aligned}$$

establishing inequality (4.13) and completing proof of the lemma.□

Lemma 4.4.3 *(Chidume and de Souza [36]) Let* $E = L_p,\ 1 < p \leq 2,\ T : E \to E$ *a non-expansive mapping and* $G : E \to E$ *an* $\eta-$*strongly accretive and* κ- *Lipschitzian mapping. For,*

$$\lambda \in \Big(0, \frac{1}{p-1}\Big),\ \text{and}\ \delta \in \Big(0\ ,\ min\Big\{\frac{2\eta(p-1)}{\kappa^2}\ ,\ \frac{(p-1)^2}{\eta}\Big\}\Big),$$

define a map $T^\lambda : E \to E$ *by:* $T^\lambda x := Tx - \lambda\delta G(Tx),\ x \in E$. *Then,* T^λ *is a strict contraction. Furthermore,*

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\alpha)\|x-y\| \quad \forall \quad x, y \in E, \tag{4.14}$$

where

$$\alpha := (p-1) - \sqrt{(p-1)^2 - \delta[2\eta - \delta(p-1)^{-1}\kappa^2]} \quad \in (0,1).$$

Proof. For $x, y \in E$, using Lemma 4.4.2, we have,

$$\begin{aligned}
&\|T^{\lambda}x - T^{\lambda}y\|^2 = \|Tx - Ty - \lambda\delta(G(Tx) - G(Ty))\|^2 \\
\leq \;& \frac{1}{(p-1)}\Big[\|Tx - Ty\|^2 - 2\lambda\delta\langle G(Tx) - G(Ty), j(Tx - Ty)\rangle \\
+ \;& \lambda^2\delta^2\|G(Tx) - G(Ty)\|^2\Big] \\
\leq \;& \frac{1}{(p-1)}\Big[\|Tx - Ty\|^2 - 2\lambda\delta\eta\|Tx - Ty\|^2 + \lambda^2\delta^2\kappa^2\|Tx - Ty\|^2\Big] \\
\leq \;& \frac{1}{(p-1)}\Big[1 - \lambda\delta[2\eta - \delta(p-1)^{-1}\kappa^2]\Big]\|x - y\|^2, \quad \textit{since } \lambda < \frac{1}{(p-1)}.
\end{aligned}$$

Define

$$f(\lambda) := \frac{1}{(p-1)}\Big[1 - \lambda\delta[2\eta - \delta(p-1)^{-1}\kappa^2]\Big].$$

If $f(\lambda) = (1 - \lambda\tau)^2$ for some $\tau \in (0, 1)$ then, $\tau^2 - 2(p-1)\tau + \sigma \geq 0$, where $\sigma := \delta[2\eta - \delta(p-1)^{-1}\kappa^2]$. Thus we obtain that

$$\tau \leq (p-1) - \sqrt{(p-1)^2 - \delta[2\eta - \delta(p-1)^{-1}\kappa^2]} \quad \in (0, 1).$$

Now set

$$\alpha := (p-1) - \sqrt{(p-1)^2 - \delta[2\eta - \delta(p-1)^{-1}\kappa^2]} \quad \in (0, 1),$$

and the proof is complete. □

Remark 4.4.4 *In Hilbert space, by putting $p = 2$ and observing that η can always be assumed to be arbitrarily small, without any loss of generality, we get, min*$\Big\{\frac{2\eta(p-1)}{\kappa^2}, \frac{(p-1)^2}{\eta}\Big\} = \frac{2\eta}{\kappa^2}$.

Thus, we have the following corollary.

Corollary 4.4.5 *Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping, $G : H \to H$ an $\eta-$strongly monotone and $\kappa-$Lipschitzian map. For $\lambda \in (0,1)$ and $\delta \in (0, \frac{2\eta}{\kappa^2})$, define a map $T^\lambda : H \to H$ by: $T^\lambda x = Tx - \lambda\delta G(Tx), \quad x \in H$. Then, T^λ is a strict contraction. In particular,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\alpha)\|x - y\| \;\; \forall \;\; x, y \in H, \tag{4.15}$$

where $\alpha := 1 - \sqrt{1 - \delta(2\eta - \delta\kappa^2)} \quad \in (0,1)$.

Proof. Set $p = 2$ in Lemma 4.4.3 and the result follows.

Remark 4.4.6 *Corollary 4.4.5 is a result of Yamada [99] and is the main tool used in Wang [91], Xu and Kim [98], Yamada [99] and Zheng and Yao [100]. Consequently, Lemma 4.4.3 is an important extension of these results to L_p spaces, $1 < p \leq 2$.*

We now prove the following theorems. In the theorem, $F(T_i) := \{x \in E : T_i x = x\}$.

Theorem 4.4.7 *(Chidume and de Souza [36]) Let $E = L_p$, $1 < p \leq 2$, $\;T : E \to E$ a nonexpansive mapping. Assume $K := \{x \in E : Tx = x\} \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ satisfying the following conditions:*

$$C1 : lim\lambda_n = 0; \quad C2 : \sum \lambda_n = \infty; C3 : lim\frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0.$$

For $\delta \in \Big(\ 0,\ min\Big\{\frac{2\eta(p-1)}{\kappa^2}\ ,\ \frac{(p-1)^2}{\eta}\Big\}\Big)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$,

$$x_{n+1} = T^{\lambda_{n+1}} x_n = Tx_n - \delta\lambda_{n+1} G(Tx_n), \quad n \geq 0. \tag{4.16}$$

Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G,K)$.

Proof. This follows using Lemma 4.4.3.

The following corollary follows from Theorem 4.4.7.

Corollary 4.4.8 *Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping. Assume $K := \{x \in E : Tx = x\} \neq \emptyset$. Let $G : H \to H$ be an $\eta-$strongly monotone and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions $C1$, $C2$ and $C3$ as in Theorem 4.4.7. For $\delta \in (0, \frac{2\eta}{\kappa^2})$, define a sequence $\{x_n\}$ iteratively in H by (4.16). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G,K)$.*

Following the method of section 4.3 and using Lemma 4.4.3, the following theorem and corollary are easily proved.

Theorem 4.4.9 *Let $E = L_p$, $1 < p \leq 2$, $T_i : E \to E, i = 1, 2, ..., r$ a finite family of nonexpansive mappings with $K := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ satisfying the conditions:*

$$C1 : lim \lambda_n = 0; \quad C2 : \sum \lambda_n = \infty; \quad C6 : lim \frac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0.$$

For $\delta \in \left(0,\ min\left\{\frac{2\eta(p-1)}{\kappa^2}\ ,\ \frac{(p-1)^2}{\eta}\right\}\right)$, define a sequence $\{x_n\}$ iteratively in E by: $x_0 \in E$,

$$x_{n+1} = T_{[n+1]}^{\lambda_{n+1}} x_n = T_{[n+1]} x_n - \delta \lambda_n G(T_{[n+1]} x_n), \quad n \geq 0, \tag{4.17}$$

where $T_{[n]} = T_{n \ mod \ r}$. *Assume also that*

$$K = F(T_rT_{r-1}...T_1) = F(T_1T_r...T_2) = ... = F(T_{r-1}T_{r-2}...T_r)$$

and $\lim\limits_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. *Then,* $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality problem* $VI(G, K)$.

Corollary 4.4.10 *Let* H *be a real Hilbert space,* $T_i : H \to H, \quad i = 1, 2, ..., r$ *a finite family of nonexpansive mappings with* $K := \overset{r}{\underset{i=1}{\cap}} F(T_i) \neq \emptyset$. *Let* $G : H \to H$ *be an* $\eta-$*strongly monotone and* $\kappa-$*Lipschitzian mapping. Let* $\{\lambda_n\}$ *be a real sequence in* $[0, 1]$ *that satisfies conditions* $C1$, $C2$ *and* $C6$ *as in Theorem 4.4.9 and let* $\lim\limits_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. *For* $\delta \in (0, \frac{2\eta}{\kappa^2})$, *define a sequence* $\{x_n\}$ *iteratively in* H *by* (5.9). *Then,* $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality problem* $VI(G, K)$.

Remark 4.4.11 *Our theorems in this chapter which are extensions of the results of Yamada [99], Wang [91], Xu and Kim [98], Zeng and Yao [100] from real Hilbert spaces to* L_p *spaces,* $1 < p \leq 2$ *complement the theorems earlier in the chapter (see also Chidume et al. [28]) to provide convergence theorems, for the problems considered here, in all* L_p *spaces,* $1 < p < \infty$.

CHAPTER 5

APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE MAPPINGS AND SOLUTIONS OF VARIATIONAL INEQUALITIES

5.1 Introduction

In Chapter 4, we extended the results of Xu and Kim [98] to $q-$uniformly smooth Banach spaces, $q \geq 2$. In particular, we proved theorems which are applicable in L_p spaces, $1 < p < \infty$ under conditions $C1, C2$ and $C5$ or $C6$ (as in the result of Xu and Kim).

In this chapter, we introduce *new recursion formulas* and prove strong convergence theorems for the unique solution of the variational inequality problem $VI(K, S)$, *requiring only conditions C1 and C2 on the parameter sequence* $\{\lambda_n\}$. Furthermore in the case $T_i : E \to E \;\; i = 1, 2, ..., r$ is a family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, we prove a convergence theorem where condition $C6$ is replaced with $\lim_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. An example satisfying this condition is given in [21]. All our theorems are proved in $q-$uniformly smooth Banach spaces, $q \geq 2$. In particular, our theorems are applicable in L_p spaces, $1 < p < \infty$.

5.2 Convergence Theorems

We first prove the following lemma which will be central in the sequel.

Lemma 5.2.1 *(Chidume et al. [29]) Let E be a $q-$uniformly smooth real Banach space with constant d_q, $\;\; q \geq 2, \;\; T : E \to E$ a nonexpansive mapping and $G : E \to E$ an*

$\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. For

$$\delta \in \Big(0, \min\Big\{\frac{q}{4\sigma\eta}, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{(q-1)}}\Big\}\Big), \quad \sigma, \lambda \in (0,1),$$

define a mapping $T^\lambda : E \to E$ by:

$$T^\lambda x := (1-\sigma)x + \sigma[Tx - \lambda\delta G(Tx)], \quad x \in E.$$

Then, T^λ is a strict contraction. Furthermore,

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\alpha)\|x-y\| \quad x, y \in E, \tag{5.1}$$

where

$$\alpha := \frac{q}{2} - \sqrt{\frac{q^2}{4} - \sigma\delta(q\eta - \delta^{q-1}d_q\kappa^q)} \quad \in (0,1).$$

Proof. For $x, y \in E$, using the convexity of $||.||^q$ and Lemma 4.2.2, we have,

$$\begin{aligned}
\|T^\lambda x - T^\lambda y\|^q &= \|(1-\sigma)(x-y) + \sigma[Tx - Ty - \lambda\delta(G(Tx) - G(Ty))]\|^q \\
&\leq (1-\sigma)\|x-y\|^q + \sigma\Big[\|Tx-Ty\|^q - q\lambda\delta\langle G(Tx) - G(Ty), j_q(Tx-Ty)\rangle \\
&+ d_q\lambda^q\delta^q\|G(Tx) - G(Ty)\|^q\Big] \\
&\leq (1-\sigma)\|x-y\|^q + \sigma\Big[\|Tx-Ty\|^q - q\lambda\delta\eta\|Tx-Ty\|^q \\
&+ d_q\lambda^q\delta^q\kappa^q\|Tx-Ty\|^q\Big] \\
&\leq \Big[1 - \sigma\lambda\delta\big(q\eta - d_q\lambda^{q-1}\delta^{q-1}\kappa^q\big)\Big]\|x-y\|^q \\
&\leq \Big[1 - \sigma\lambda\delta\big(q\eta - d_q\delta^{q-1}\kappa^q\big)\Big]\|x-y\|^q.
\end{aligned}$$

Define

$$f(\lambda) := 1 - \sigma\lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) = (1-\lambda\tau)^q, \quad \text{for some} \quad \tau \in (0,1), \quad \text{say}.$$

Then, there exists $\xi \in (0, \lambda)$ such that

$$1 - \sigma\lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) = 1 - q\tau\lambda + \frac{1}{2}q(q-1)(1-\xi\tau)^{q-2}\lambda^2\tau^2,$$

and since $q \geq 2$, this implies

$$1 - \sigma\lambda\delta(q\eta - d_q\delta^{q-1}\kappa^q) \leq 1 - q\tau\lambda + \frac{1}{2}q(q-1)\lambda^2\tau^2,$$

which yields,

$$\tau^2 - q\tau + \sigma\delta\Big(q\eta - d_q\delta^{q-1}\kappa^q\Big) > 0,$$

since $\lambda \in \Big(0, \frac{2}{q(q-1)}\Big)$. Thus we have,

$$\tau \leq \frac{q}{2} - \sqrt{\frac{q^2}{4} - \sigma\delta(q\eta - \delta^{q-1}d_q\kappa^q)} \quad \in (0,1).$$

Set

$$\alpha := \frac{q}{2} - \sqrt{\frac{q^2}{4} - \sigma\delta(q\eta - \delta^{q-1}d_q\kappa^q)} \quad \in (0,1).$$

and the proof is complete. □

We note that in L_p spaces, $2 \leq p < \infty$, the following inequality holds (see e.g., [17]):

For each $x, y \in L_p$, $2 \leq p < \infty$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + (p-1)\|y\|^2.$$

It then follows that by setting $q = 2,\ d_q = p - 1$ in Lemma 5.2.1, the following corollary is easily proved.

Corollary 5.2.2 *Let $E = L_p,\quad 2 \leq p < \infty,\quad T : E \to E$ a nonexpansive mapping and $G : E \to E$ an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. For $\lambda,\ \sigma \in (0,1)$, and $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{(p-1)\kappa^2}\right\}\right)$, define a mapping $T^\lambda : E \to E$ by:*

$$T^\lambda x := (1-\sigma)x + \sigma[Tx - \lambda\delta G(Tx)] \ \forall\ x \in E.$$

Then, T^λ is a strict contraction. In particular,

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\alpha)\|x-y\| \quad x, y \in H, \tag{5.2}$$

where $\alpha := 1 - \sqrt{1 - \sigma\delta(2\eta - (p-1)\delta\kappa^2)} \quad \in (0,1)$.

We also have the following corollary.

Corollary 5.2.3 *Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping, $G : H \to H$ an $\eta-$strongly monotone and $\kappa-$Lipschitzian mapping. For $\lambda, \sigma \in (0,1)$ and $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{\kappa^2}\right\}\right)$, define a mapping $T^\lambda : H \to H$ by:*

$$T^\lambda x = (1-\sigma)x + \sigma[Tx - \lambda\delta G(Tx)] \ \forall\ x \in H.$$

Then, T^λ is a strict contraction. In particular,

$$\|T^\lambda x - T^\lambda y\| \le (1-\lambda\alpha)\|x-y\| \;\; \forall \;\; x, y \in H, \tag{5.3}$$

where $\alpha := 1 - \sqrt{1-\sigma\delta(2\eta - \delta\kappa^2)} \;\; \in (0,1)$.

Proof. Set $p = 2$ in Corollary 5.2.2 and the result follows.

We now prove the following convergence theorems.

Theorem 5.2.4 *(Chidume et al. [29]) Let E be a $q-$uniformly smooth real Banach space with constant d_q, $q \ge 2$ and $T : E \to E$ a nonexpansive mapping. Assume $K := \{x \in E : Tx = x\} \ne \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ satisfying the conditions:*

$$C1 : lim\lambda_n = 0; \quad C2 : \sum \lambda_n = \infty.$$

For $\delta \in \Big(0, \min\Big\{\frac{q}{4\sigma\eta}, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{(q-1)}}\Big\}\Big)$, $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$,

$$x_{n+1} = T^{\lambda_{n+1}} x_n = (1-\sigma)x_n + \sigma[Tx_n - \delta\lambda_{n+1}G(Tx_n)], \quad n \ge 0. \tag{5.4}$$

Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality $VI(G,K)$.

Proof. Let $x^* \in K := F(T)$, then the sequence $\{x_n\}$ satisfies

$$\|x_n - x^*\| \le max\Big\{\|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\|\Big\}, \quad n \ge 0.$$

It is obvious that this is true for $n = 0$. Assume it is true for $n = k$ for some $k \in \mathbb{N}$. From the recursion formula (5.4), we have

$$
\begin{aligned}
\|x_{k+1} - x^*\| &= \|T^{\lambda_{k+1}} x_k - x^*\| \\
&\leq \|T^{\lambda_{k+1}} x_k - T^{\lambda_{k+1}} x^*\| + \|T^{\lambda_{k+1}} x^* - x^*\| \\
&\leq (1 - \lambda_{k+1}\alpha)\|x_k - x^*\| + \lambda_{k+1}\delta\|G(x^*)\| \\
&\leq max\Big\{\|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\|\Big\},
\end{aligned}
$$

and the claim follows by induction. Thus the sequence $\{x_n\}$ is bounded and so are the sequences $\{Tx_n\}$ and $\{G(Tx_n)\}$.

Define two sequences $\{\beta_n\}$ and $\{y_n\}$ by $\beta_n := (1-\sigma)\lambda_{n+1} + \sigma$ and $y_n := \frac{x_{n+1} - x_n + \beta_n x_n}{\beta_n}$. Then,

$$
y_n = \frac{(1-\sigma)\lambda_{n+1} x_n + \sigma[Tx_n - \lambda_{n+1}\delta G(Tx_n)]}{\beta_n}.
$$

Observe that $\{y_n\}$ is bounded and that

$$
\begin{aligned}
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| &\leq \Big|\frac{\sigma}{\beta_{n+1}} - 1\Big| \ \|x_{n+1} - x_n\| \\
&+ \ \Big|\frac{\sigma}{\beta_{n+1}} - \frac{\sigma}{\beta_n}\Big|\|Tx_n\| + \frac{\lambda_{n+2}(1-\sigma)}{\beta_{n+1}}\|x_{n+1} - x_n\| \\
&+ \ (1-\sigma)\Big|\frac{\lambda_{n+2}}{\beta_{n+1}} - \frac{\lambda_{n+1}}{\beta_n}\Big|\|x_n\| + \frac{\lambda_{n+1}\sigma\delta}{\beta_n}\|G(Tx_n) - G(Tx_{n+1})\| \\
&+ \ \sigma\delta\Big|\frac{\lambda_{n+1}}{\beta_n} - \frac{\lambda_{n+2}}{\beta_{n+1}}\Big|\|G(Tx_{n+1})\|.
\end{aligned}
$$

This implies, $\underset{n\to\infty}{limsup}\;(||y_{n+1}-y_n||-||x_{n+1}-x_n||)\leq 0$, and therefore by Lemma 2.2.2,

$$\lim_{n\to\infty}||y_n-x_n||=0.$$

Hence,

$$||x_{n+1}-x_n||=\beta_n||y_n-x_n||\to 0 \quad \text{as} \quad n\to\infty. \tag{5.5}$$

From the recursion formula (5.4), we have that

$$\sigma\|x_{n+1}-Tx_n\|\leq(1-\sigma)\|x_{n+1}-x_n\|+\lambda_{n+1}\sigma\delta\|G(Tx_n)\|\to 0 \quad \text{as} \quad n\to\infty.$$

which implies,

$$\|x_{n+1}-Tx_n\|\to 0 \quad \text{as} \quad n\to\infty. \tag{5.6}$$

From (5.5) and (5.6) we have

$$\|x_n-Tx_n\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|\to 0 \;\text{ as }\; n\to\infty. \tag{5.7}$$

We now prove that

$$\underset{n\to\infty}{limsup}\langle -G(x^*),j(x_{n+1}-x^*)\rangle\leq 0.$$

Define a map $\phi:E\to\mathbb{R}$ by

$$\phi(x)=\mu_n\|x_n-x\|^2 \quad \forall\, x\in E,$$

where μ_n is a Banach limit for each n. Then, $\phi(x) \to \infty$ as $\|x\| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $y^* \in E$ such that $\phi(y^*) = \min_{u \in E} \phi(u)$. Hence, the set

$$K^* := \Big\{x \in E : \phi(x) = \min_{u \in E} \phi(u)\Big\} \neq \emptyset.$$

We now show T has a fixed point in K^*. We know

$$\lim_{n \to \infty} ||x_n - T_i x_n|| = 0. \tag{5.8}$$

We shall make use of Lemma 4.2.3. If x is in K^* and $y := \omega - \lim_j T^{m_j} x$, belongs to the weak $\omega - limit$ set $\omega_w(x)$ of T at x, then, from the w-l.s.c. (since φ is l.s.c. and convex) of φ and equation (5.8), we have,

$$\begin{aligned}
\varphi(y) &\leq \liminf_j \varphi\Big(T^{m_j} x\Big) \leq \limsup_m \varphi\Big(T^m x\Big) \\
&= \limsup_m \Big(\mu_n ||x_n - T^m x||^2\Big) \\
&= \limsup_m \Big(\mu_n ||x_n - T^m x_n + T^m x_n - T^m x||^2\Big) \\
&\leq \limsup_m \Big(\mu_n ||T^m x_n - T^m x||^2\Big) \leq \limsup_m \Big(\mu_n ||x_n - x||^2\Big) = \varphi(x) \\
&= \inf_{u \in E} \varphi(u).
\end{aligned}$$

So, $y \in K^*$. By Lemma 4.2.3, T has a fixed point in K^* and so $K^* \cap K \neq \emptyset$.

By Lemma 4.2.3, $K^* \cap K \neq \emptyset$. Let $x^* \in K^* \cap K$ and let $t \in (0, 1)$. Then, it follows that $\phi(x^*) \leq \phi(x^* - tG(x^*))$ and using Lemma 2.2.1, we obtain that

$$\|x_n - x^* + tG(x^*)\|^2 \leq \|x_n - x^*\|^2 + 2t\langle G(x^*), j(x_n - x^* + tG(x^*))\rangle$$

which implies,

$$\mu_n \big\langle -G(x^*), j(x_n - x^* + tG(x^*)) \big\rangle \leq 0.$$

The rest now follows exactly as in the proof of Theorem 4.3.5 to yield that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

The following corollaries follow from Theorem 5.2.4.

Corollary 5.2.5 *Let $E = L_p$, $2 \leq p < \infty$, $T : E \to E$ a nonexpansive mapping. Assume $K := \{x \in E : Tx = x\} \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions $C1$ and $C2$ as in theorem 5.2.4. For $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{(p-1)\kappa^2}\right\}\right)$, $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in E by (5.4). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G,K)$.*

Corollary 5.2.6 *Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping. Assume $K := \{x \in H : Tx = x\} \neq \emptyset$. Let $G : H \to H$ be an $\eta-$strongly monotone $\kappa-$Lipschitzian mapping. Further, let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions $C1$ and $C2$ as in Theorem 5.2.4. For $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{\kappa^2}\right\}\right)$, $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in H by (5.4). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G,K)$.*

Finally, we prove the following theorem for a finite family of nonexpansive mappings.

Theorem 5.2.7 *(Chidume et al. [29]) Let E be a $q-$uniformly smooth real Banach space with constant d_q, $q \geq 2$, $T_i : E \to E$, $i = 1, 2, ..., r$ a finite family of nonexpansive mappings with $K := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping, and $\{\lambda_n\}$ a real sequence in $[0,1]$ satisfying the conditions:*

$$C1 : lim \lambda_n = 0; \quad C2 : \sum \lambda_n = \infty.$$

For a fixed real number $\delta \in \left(0, \min\left\{\frac{q}{4\sigma\eta}, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{(q-1)}}\right\}\right)$, $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$,

$$x_{n+1} = T_{[n+1]}^{\lambda_{n+1}} x_n = (1-\sigma)x_n + \sigma[T_{[n+1]}x_n - \delta\lambda_n G(T_{[n+1]}x_n)], \quad n \geq 0, \tag{5.9}$$

where $T_{[n]} = T_{n \ mod \ r}$. Assume also that

$$K = F(T_r T_{r-1} ... T_1) = F(T_1 T_r ... T_2) = ... = F(T_{r-1} T_{r-2} ... T_r)$$

and $\lim\limits_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G, K)$.

Proof. Let $x^* \in K$, then the sequence $\{x_n\}$ satisfies

$$\|x_n - x^*\| \leq max\Big\{\|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\|\Big\}, \quad n \geq 0.$$

It is obvious that this is true for $n = 0$. Assume it is true for $n = k$ for some $k \in \mathbb{N}$. From the recursion formula (5.9), we have

$$
\begin{aligned}
\|x_{k+1} - x^*\| &= \|T^{\lambda_{k+1}}_{[k+1]} x_k - x^*\| \\
&\leq \|T^{\lambda_{k+1}}_{[k+1]} x_k - T^{\lambda_{k+1}}_{[k+1]} x^*\| + \|T^{\lambda_{k+1}}_{[k+1]} x^* - x^*\| \\
&\leq (1 - \lambda_{k+1}\alpha)\|x_k - x^*\| + \lambda_{k+1}\delta\|G(x^*)\| \\
&\leq max\Big\{\|x_0 - x^*\|, \frac{\delta}{\alpha}\|G(x^*)\|\Big\},
\end{aligned}
$$

and the claim follows by induction. Thus the sequence $\{x_n\}$ is bounded and so are $\{T_{[n]}x_n\}$ and $\{G(T_{[n]}x_n)\}$.

Define two sequences $\{\beta_n\}$ and $\{y_n\}$ by $\beta_n := (1 - \sigma)\lambda_{n+1} + \sigma$ and $y_n := \frac{x_{n+1} - x_n + \beta_n x_n}{\beta_n}$. Then,

$$
y_n = \frac{(1 - \sigma)\lambda_{n+1} x_n + \sigma[T_{[n+1]} x_n - \lambda_{n+1}\delta G(T_{[n+1]} x_n)]}{\beta_n}.
$$

Observe that $\{y_n\}$ is bounded and that

$$
\begin{aligned}
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| &\leq \Big|\frac{\sigma}{\beta_{n+1}} - 1\Big| \; \|x_{n+1} - x_n\| \\
+ \quad & \frac{\sigma}{\beta_{n+1}}\|T_{[n+2]} x_n - T_{[n+1]} x_n\| + \Big|\frac{\sigma}{\beta_{n+1}} - \frac{\sigma}{\beta_n}\Big|\|T_{[n+1]} x_n\| \\
+ \quad & \frac{\lambda_{n+2}(1 - \sigma)}{\beta_{n+1}}\|x_{n+1} - x_n\| + (1 - \sigma)\Big|\frac{\lambda_{n+2}}{\beta_{n+1}} - \frac{\lambda_{n+1}}{\beta_n}\Big|\|x_n\| \\
+ \quad & \frac{\lambda_{n+1}\sigma\delta}{\beta_n}\|G(T_{[n+1]} x_n) - G(T_{[n+2]} x_{n+1})\| \\
+ \quad & \sigma\delta\Big|\frac{\lambda_{n+1}}{\beta_n} - \frac{\lambda_{n+2}}{\beta_{n+1}}\Big|\|G(T_{[n+2]} x_{n+1})\|.
\end{aligned}
$$

This implies,

$$\limsup_{n\to\infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0,$$

and by Lemma 2.2.2,

$\lim_{n\to\infty} ||y_n - x_n|| = 0$. Hence,

$$||x_{n+1} - x_n|| = \beta_n ||y_n - x_n|| \to 0 \tag{5.10}$$

as $n \to \infty$. From the recursion formula (5.9), we have that

$$\sigma\|x_{n+1} - T_{[n+1]}x_n\| \leq (1-\sigma)\|x_{n+1} - x_n\| + \lambda_{n+1}\sigma\delta\|G(T_{[n+1]}x_n)\| \to 0,$$

as $n \to \infty$, which implies,

$$\|x_{n+1} - T_{[n+1]}x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Note that from (5.10) and (5.11) we have

$$\|x_n - T_{[n+1]}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_{[n+1]}x_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{5.12}$$

Also,

$$\|x_{n+r} - x_n\| \leq \|x_{n+r} - x_{n+r-1}\| + \|x_{n+r-1} - x_{n+r-2}\| + \cdots + \|x_{n+1} - x_n\|$$

and so,

$$\|x_{n+r} - x_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{5.13}$$

Using the fact that T_i is nonexpansive for each i, we obtain the following finite table:

$$x_{n+r} - T_{n+r}x_{n+r-1} \to 0 \quad as \quad n \to \infty;$$

$$T_{n+r}x_{n+r-1} - T_{n+r}T_{n+r-1}x_{n+r-2} \to 0 \quad as \quad n \to \infty;$$

$$\vdots$$

$$T_{n+r}T_{n+r-1}\cdots T_{n+2}x_{n+1} - T_{n+r}T_{n+r-1}\cdots T_{n+2}T_{n+1}x_n \to 0 \quad as \quad n \to \infty;$$

and adding up the table yields

$$x_{n+r} - T_{n+r}T_{n+r-1}\cdots T_{n+1}x_n \to 0 \quad as \quad n \to \infty.$$

Using this and (5.13) we get that $\underset{n\to\infty}{lim}||x_n - T_{n+r}T_{n+r-1}\cdots T_{n+1}x_n|| = 0.$

Carrying out similar arguments as in the proof of Theorem 5.2.4, we easily get that

$$\underset{n\to\infty}{limsup}\langle -G(x^*), j(x_{n+1} - x^*)\rangle \leq 0.$$

From the recursion formula (5.9), and Lemma 2.2.1 we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|T^{\lambda_{n+1}}_{[n+1]}x_n - T^{\lambda_{n+1}}_{[n]}x^* + T^{\lambda_{n+1}}_{[n+1]}x^* - x^*\|^2 \\ &\leq \|T^{\lambda_{n+1}}_{[n+1]}x_n - T^{\lambda_{n+1}}_{[n+1]}x^*\|^2 + 2\lambda_{n+1}\sigma\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle \\ &\leq (1 - \lambda_{n+1}\alpha)\|x_n - x^*\|^2 + 2\lambda_{n+1}\sigma\delta\langle -G(x^*), j(x_{n+1} - x^*)\rangle \end{aligned}$$

which by using Lemma 2.2.3, gives that $x_n \to x^*$ as $n \to \infty$, completing the proof. $\square$

The following corollaries follow from Theorem 5.2.7.

Corollary 5.2.8 *Let $E = L_p$, $2 \leq p < \infty$, $T_i : E \to E$, $i = 1, 2, ..., r$ a finite family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0, 1]$ that satisfies conditions $C1$ and $C2$ as in Theorem 5.2.7 and let $\lim_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. For $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{(p-1)\kappa^2}\right\}\right)$, $\sigma \in (0, 1)$, define a sequence $\{x_n\}$ iteratively in E by (5.9). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G, K)$.*

Corollary 5.2.9 *Let H be a real Hilbert space, $T_i : H \to H$, $i = 1, 2, ..., r$ a finite family of nonexpansive mappings with $K = \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : H \to H$ be an $\eta-$strongly monotone and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0, 1]$ that satisfies conditions $C1$ and $C2$ as in theorem 5.2.7 and let $\lim_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. For $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{\kappa^2}\right\}\right)$, $\sigma \in (0, 1)$, define a sequence $\{x_n\}$ iteratively in H by (5.9). Then, $\{x_n\}$ converges strongly to the variational inequality problem $VI(G, K)$.*

5.3 The case of L_p spaces, $1 < p \leq 2$.

We first prove the following lemmas.

We begin with the following definition. A Banach space E is called a *lower weak parallelogram space with constant* $b \geq 0$ or, briefly, E is $LWP(b)$, in the terminology of Bynum

[11] if

$$||x+y||^2 + b||x-y||^2 \leq 2(||x||^2 + ||y||^2) \tag{5.14}$$

holds for all $x, y \in E$. It is proved in [11] that l_p space, $1 < p \leq 2$, is a lower weak parallelogram space with $(p-1)$ as the largest number b for which (5.14) holds. Furthermore, if L_p, $(1 < p \leq 2)$, has at least two disjoint sets of positive finite measure, then it is a lower weak parallelogram space with $(p-1)$ as the largest number b for which (5.14) holds. We shall assume, without loss of generality, that L_p, $(1 < p \leq 2)$, has at least two disjoint sets of positive finite measure. In the sequel, we shall state all our theorems and lemmas only for L_p spaces, $1 < p \leq 2$, with the understanding that they also hold for l_p spaces, $1 < p \leq 2$.

In terms of the normalized duality mapping, Bynum [11] proved that a real Banach space is a lower weak parallelogram space if and only if for each $x, y \in E$ and $f \in J(x)$, the following inequality holds:

$$||x+y||^2 \geq ||x||^2 + b||y||^2 + 2\langle y, f\rangle. \tag{5.15}$$

In particular, for $E = L_p$, $1 < p \leq 2$, the following inequality holds:

$$||x+y||^2 \geq ||x||^2 + (p-1)||y||^2 + 2\langle y, j(x)\rangle \ \forall\ x, y \in E. \tag{5.16}$$

We now obtain the following lemmas which will be central in the sequel.

Lemma 5.3.1 *Let $E = L_p, 1 < p \leq 2$. Then, for all $x, y \in E$, the following inequality holds:*

$$(p-1)||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + ||y||^2. \tag{5.17}$$

Proof. Observe first that E is smooth so that the normalized duality map on E is single-valued. Now, replacing x by $(-x)$ and y by $(x+y)$ in inequality (5.16), we obtain $||y||^2 \geq ||x||^2 + 2\langle x+y, j(-x)\rangle + (p-1)||x+y||^2$, so that

$$\begin{aligned}(p-1)||x+y||^2 &\leq ||y||^2 - ||x||^2 + 2\langle x+y, j(x)\rangle \\ &= ||x||^2 + 2\langle y, j(x)\rangle + ||y||^2,\end{aligned}$$

establishing the lemma. $\square$

Lemma 5.3.2 *(Chidume et al. [30]) Let $E = L_p,\ 1 < p \leq 2,\ \ T : E \to E$ be a nonexpansive mapping and $G : E \to E$ be an $\eta-$strongly accretive and κ- Lipschitzian mapping. For,*

$$\lambda \in (0,1), \quad \sigma \in (0,1),\ \delta \in \Big(0\ ,\ min\Big\{\frac{2\eta}{\kappa^2}\ ,\ \frac{(p-1)^2}{2\eta\sigma}\Big\}\Big),$$

define a map $T^\lambda : E \to E$ by

$$T^\lambda x := (1-\sigma)x + \sigma\Big[Tx - \lambda\delta G(Tx)\Big],\ x \in E.$$

Then, T^λ is a strict contraction. Furthermore,

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\alpha)\|x-y\| \ \ \forall\ \ x, y \in E, \tag{5.18}$$

where

$$\alpha := (p-1) - \sqrt{(p-1)^2 - \sigma\delta(2\eta - \delta\kappa^2)} \quad \in (0,1).$$

Proof. For $x, y \in E$, using the convexity of $||.||^2$, and Lemma 5.3.1, we have,

$$\begin{aligned}
\|T^\lambda x - T^\lambda y\|^2 &= \|(1-\sigma)(x-y) + \sigma[Tx - Ty - \lambda\delta(G(Tx) - G(Ty))]\|^2 \\
&\leq (1-\sigma)\|x-y\|^2 + \frac{\sigma}{(p-1)}\Big[\|Tx - Ty\|^2 \\
&- 2\lambda\delta\langle G(Tx) - G(Ty), j(Tx - Ty)\rangle + \lambda^2\delta^2\|G(Tx) - G(Ty)\|^2\Big] \\
&\leq (1-\sigma)\|x-y\|^2 + \frac{\sigma}{(p-1)}\Big[\|Tx - Ty\|^2 - 2\lambda\delta\eta\|Tx - Ty\|^2 \\
&+ \lambda^2\delta^2\kappa^2\|Tx - Ty\|^2\Big] \\
&\leq \Big[1 + \sigma\Big(\frac{1}{p-1} - 1\Big) - \frac{2\sigma\lambda\delta\eta}{(p-1)} + \frac{\sigma\lambda\delta^2\kappa^2}{(p-1)}\Big]\|x-y\|^2, \quad (\lambda < 1).
\end{aligned}$$

Define

$$f(\lambda) := 1 + \sigma\Big(\frac{1}{p-1} - 1\Big) - \frac{2\sigma\lambda\delta\eta}{(p-1)} + \frac{\sigma\lambda\delta^2\kappa^2}{(p-1)} = (1-\lambda\tau)^2,$$

for some $\tau \in (0,1),$ say. Since $\Big(\frac{1}{p-1} - 1\Big) > 0$, and $\lambda(p-1) \leq 1$, this implies,

$$-\frac{2\sigma\delta\eta}{(p-1)} + \frac{\sigma\delta^2\kappa^2}{(p-1)} \leq -2\tau + \tau^2,$$

which yields

$$\tau^2 - 2(p-1)\tau + 2\sigma\delta\eta - \sigma\delta^2\kappa^2 \geq 0,$$

implying that

$$\tau \leq (p-1) - \sqrt{(p-1)^2 - \sigma\delta(2\eta - \delta\kappa^2)} \quad \in (0,1).$$

Now set

$$\alpha := (p-1) - \sqrt{(p-1)^2 - \sigma\delta(2\eta - \delta\kappa^2)} \quad \in (0,1),$$

and the proof is complete. $\Box$

Remark 5.3.3 *In a Hilbert space, by putting $p = 2$ and observing that η can always be assumed to be arbitrarily small, without any loss of generality, we get,* $\min\left\{\frac{2\eta(p-1)}{\kappa^2}, \frac{(p-1)^2}{\eta}\right\} = \frac{2\eta}{\kappa^2}$.

By Remark 5.3.3, we have the following corollary.

Corollary 5.3.4 *Let H be a real Hilbert space, $T : H \to H$ be a nonexpansive mapping, $G : H \to H$ be an $\eta-$strongly $\kappa-$Lipschitzian mapping. For $\lambda \in (0,1)$ and $\delta \in \left(0, \min\left\{\frac{1}{2\sigma\eta}, \frac{2\eta}{\kappa^2}\right\}\right)$, $\sigma \in (0,1)$, define a mapping $T^\lambda : H \to H$ by: $T^\lambda x = (1-\sigma)x + \sigma[Tx - \lambda\delta G(Tx)]$, $x \in H$. Then, T^λ is a strict contraction. In particular,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\alpha)\|x - y\| \;\; \forall \;\; x, y \in H, \tag{5.19}$$

where $\alpha := 1 - \sqrt{1 - \sigma\delta(2\eta - \delta\kappa^2)} \quad \in (0,1)$.

Proof. Set $p = 2$ in Lemma 5.3.2 and the result follows.

We now prove the following theorem.

Theorem 5.3.5 *(Chidume et al. [30]) Let $E = L_p$, $1 < p \leq 2$, $T : E \to E$ be a nonexpansive mapping. Assume $K := \{x \in E : Tx = x\} \neq \emptyset$. Let $G : E \to E$ be an $\eta-$strongly accretive and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$*

satisfying the conditions:

$$C1 : lim\lambda_n = 0; \quad C2 : \sum \lambda_n = \infty.$$

For $\sigma \in (0,1)$*, and* $\delta \in \Big(\ 0,\ min\Big\{\frac{2\eta}{\kappa^2}\ ,\ \frac{(p-1)^2}{2\eta\sigma}\Big\}\Big)$*, define a sequence* $\{x_n\}$ *iteratively in* E *by* $x_0 \in E$,

$$x_{n+1} = T^{\lambda_{n+1}}x_n = (1-\sigma)x_n + \sigma[Tx_n - \delta\lambda_{n+1}G(Tx_n)], \quad n \geq 0. \tag{5.20}$$

Then, $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality problem* $VI(G,K)$.

Proof. This follows exactly as in the proof of theorem 5.2.4, using Lemma 5.3.2.

The following corollary follows from Theorem 5.3.5.

Corollary 5.3.6 *Let* H *be a real Hilbert space,* $T : H \to H$ *be a nonexpansive mapping. Assume* $K := \{x \in E : Tx = x\} \neq \emptyset$*. Let* $G : H \to H$ *be an* $\eta-$*strongly monotone* $\kappa-$*Lipschitzian mapping. Let* $\{\lambda_n\}$ *be a real sequence in* $[0,1]$ *that satisfies conditions* $C1$ *and* $C2$ *as in theorem 5.2.4. For* $\delta \in \Big(0, \min\Big\{\frac{1}{2\sigma\eta}, \frac{2\eta}{\kappa^2}\Big\}\Big)$*,* $\sigma \in (0,1)$*, define a sequence* $\{x_n\}$ *iteratively in* H *by* (5.4)*. Then,* $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality* $VI(G,K)$.

Following the method of Section 5.2, the following theorem and corollary are easily proved.

Theorem 5.3.7 *Let* $E = L_p,\ 1 < p \leq 2$*, and* $T_i : E \to E, i = 1,2,...,r$ *be a finite family of nonexpansive mappings with* $K := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$*. Let* $G : E \to E$ *be an* $\eta-$*strongly accretive and* $\kappa-$*Lipschitzian mapping. Let* $\{\lambda_n\}$ *be a real sequence in* $[0,1]$ *satisfying the conditions:*

$C1: lim\lambda_n = 0$; $\quad$ *$C2: \sum \lambda_n = \infty$. For $\sigma \in (0,1)$, and $\delta \in \Big(0,\ min\Big\{ \frac{2\eta}{\kappa^2}\ ,\ \frac{(p-1)^2}{2\eta\sigma} \Big\}\Big)$, define a sequence $\{x_n\}$ iteratively in E by $x_0 \in E$,*

$$x_{n+1} = T^{\lambda_{n+1}}_{[n+1]} x_n = (1-\sigma)x_n + \sigma[T_{[n+1]}x_n - \delta\lambda_n G(T_{[n+1]}x_n)], \quad n \geq 0, \tag{5.21}$$

where $T_{[n]} = T_{n \ mod \ r}$. Assume also that

$$K = F(T_r T_{r-1} ... T_1) = F(T_1 T_r ... T_2) = ... = F(T_{r-1} T_{r-2} ... T_r)$$

and $\lim_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G,K)$.

Corollary 5.3.8 *Let H be a real Hilbert space, $T_i : H \to H, \quad i = 1,2,...,r$ be a finite family of nonexpansive mappings with $K := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $G : H \to H$ be an $\eta-$strongly monotone and $\kappa-$Lipschitzian mapping. Let $\{\lambda_n\}$ be a real sequence in $[0,1]$ that satisfies conditions $C1$ and $C2$ as in theorem 5.3.7 and let $\lim_{n\to\infty} \|T_{[n+2]}x_n - T_{[n+1]}x_n\| = 0$. For $\delta \in \Big(0, \min\Big\{\frac{1}{2\sigma\eta}, \frac{2\eta}{\kappa^2}\Big\}\Big)$, $\sigma \in (0,1)$, define a sequence $\{x_n\}$ iteratively in H by (5.21). Then, $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality problem $VI(G,K)$.*

Remark 5.3.9 *Our theorems in this chapter are extensions of the results of Yamada [99], Wang [91], Xu and Kim [98], Zeng and Yao [100] from real Hilbert spaces to L_p spaces, $1 < p < \infty$. Moreover, in this our general setting, the iteration parameter is required to satisfy only conditions C1 and C2.*

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