4-cycle Systems of Line Graphs of Complete Multipartite Graphs

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A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
August 9, 2008

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Date of Graduation

Vita

Nidhi Sehgal was born on January 27th 1984, in a small town named Jallandhar, in North India. She grew up in the coastal city of Mumbai on the West coast of India. She completed her high school education at Conossa Convent, Mumbai. She then went on to pursue her Bachelor of Science at St. Xavier's College in Mumbai. She studied at five educational institutions due to the nature of her father's profession. This exposed her to the flavor of India and she learnt from such a diverse experience. Then she came to Auburn University, to pursue her doctoral studies in Mathematics, at the Mathematics department of the Auburn University.

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Thesis Abstract

4-cycle Systems of Line Graphs of Complete Multipartite Graphs

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Master of Science, August 9, 2008
(B.S., St. Xavier's College, Mumbai University, 2005)

29 Typed Pages

Directed by Chris A Rodger

Here we investigate the necessary and sufficient conditions for the existence of 4 - cycle systems of the line graphs of complete multipartite graphs.

ACKNOWLEDGMENTS

The author is greatly indebted to her Master's advisor Dr. Chris A. Rodger for his valuable guidance and inpiration. She would like to thank him for his patient endurance and support in times of dificulties without which it would be very difficult for her to complete this work. Also, she would like to thank her committee members Dr. Dean Hoffman and Dr. Charles C Lindner for their support. She is thankful to her grandfather, parents, sister-in-law and brother for their understanding and moral support. Finally she also expresses her gratitude towards her friends who have always been here for her.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

I dedicate this work to *maa* and *papa* for their unflagging support.

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Chapter 1

Introduction

1.1 Definitions

An m-cycle system of a graph G is a set of m-cycles, the edges in which partition the edge set of G. The *line graph* of a graph $G = (V, E)$ is the graph $L(G) = (E, E_1)$ where E_1 is the set of edges that join two vertices if and only if the corresponding edges in G are adjacent. The *complete multipartite graph* $K(a_1, a_2, \ldots, a_p)$ is the graph with vertex set partitioned into parts $\{V_1, V_2, \ldots, V_p\}$, with $|V_i| = a_i$, in which two vertices in E are adjacent if and only if they are in different parts. By solving this problem we investigate the existence of 4-cycle systems of $L(K(a_1, a_2, \ldots, a_p))$.

1.2 History

There is a long history of problems in this area. In a more general context, Dudeney [3] posed the following problem of seating n people at a dinner table on consecutive evenings so that no person was ever to have the same pair of neighbors more than once. Any solution to this problem is equivalent to finding a set of hamilton cycles of K_n with the property that each 2-path in K_n occurs in exactly one hamilton cycle. This problem was solved when n is even by Kobayashi, Kiyasu-Zen'iti and Nakamura[7], and some results exist when n is odd. And this set S of 4-cycles is known as the *Dudeney set*. This result was extended further by looking at the case when each pair of people is a neighbor twice. Which was equal to finding a set of hamilton cycles of K_n such that each two path in K_n occurs in exactly two hamilton cycles. And this was solved by Midori, Mutoh, Kiyasu-Zen'iti and Nakamura[9].

It is quite conceivable that the restaurant has many tables of a small size, say m, instead of just one big table. So it is natural to solve the related problem of finding a set of 2-factors in K_n, each cycle in each 2-factor having length m, such that each 2-path in K_n occurs in exactly one m-cycle. This problem was solved by Kobayashi and Nakamura when $k = 4$ [8]. Notice that in any solution to such a problem, taking the line graph of each 4-cycle produces a 4-cycle system of $L(K_n)$. In [6], Henrich and Nonay removed the requirement that the set of 4-cycles be resolvable (partitionable into 2-factors), finding necessary and sufficient conditions for the existence of a set of 4-cycles such that each 2-path is in exactly one 4-cycle; this provides a 4-cycle system of $L(K_n)$ with the property that every 4-cycle in $L(K_n)$ corresponds to a 4-cycle in K_n.

In the same spirit Colby and Rodger[2] found necessary and sufficient conditions for the existence of a 4-cycle system of $L(K_n)$; when $n \equiv 1 \pmod 8$ no solutions can correspond to the existence of set of 4-cycles in K_n such that each 2-path in K_n is in exactly one 4-cycle.

The reader may be interested in a related problem posed by Dudeney [3].Twelve memebers of a club arranged to play bridge of eleven evenings, but no player was ever to have the same partner more than once or the same opponent more than twice. And the question was to find a scheme of seating them at three tables every evening. By dropping the requirement that each player partner each other player atmost once, the solution was equivalent to finding a set S of 4-cycles of K_{12} with the property that each 2-path in K_{12} occurs in exactly two 4-cycles. And taking the line graph of each 4-cycle in S produces a 4-cycle system of $2L(K_{12})$.

In this paper, we extend these results in the literature by investigating the existence of a 4-cycle system of $K(a_1, a_2, \ldots, a_p)$.

1.3 Notation

Throughout this paper, let $G = K(a_1, a_2, \ldots, a_p)$. For $1 \leq i \leq p$ let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,a_i}\}$. So the vertex set of $L(G)$ is $\{\{v_{i,x}, v_{j,y}\} \mid 1 \leq x \leq a_i, 1 \leq y \leq a_j, 1 \leq i < j \leq p\}$. Define $n = \sum_{1 \leq i \leq p} a_i$ to be the number of vertices in G. It will be useful to define $\widehat{a_i a_j} = n - a_i - a_j$.

Chapter 2

Necessary Conditions

In this chapter we investigate some neat necessary conditions which we conjecture are sufficient.

Lemma 2.1 *If there exists a 4-cycle system of $L(G)$ then*

1. $a_i \equiv a_j \pmod 2$ *for* $1 \le i < j \le p$, *and*
2. *If* a_i *is odd for* $1 \le i \le p$ *then,*
   (a) $p \equiv 1 \pmod 8$ *if* p *is odd, and*
   (b) $p \equiv n \pmod 8$ *if* p *is even.*

Proof. The degree of each vertex in $L(G)$ is clearly

$$d(\{v_{i,x}, v_{j,y}\}) = a_i + a_j - 2 + 2\widehat{a_i a_j}$$

since in G there are $a_j - 1 + \widehat{a_i a_j}$ edges incident with $v_{i,x}$ and $a_i - 1 + \widehat{a_i a_j}$ edges incident with $v_{j,y}$. Since we are assuming that a 4-cycle system of $L(G)$ exists, each vertex in $L(G)$ must have even degree. Therefore, we conclude that $a_i \equiv a_j \pmod 2$ and so condition (1) is necessary.

Now suppose that a_i is odd for $1 \le i \le p$. We consider the cases where p is odd and even in turn.

Case (1) If p is odd then clearly n is odd, being the sum of an odd number of odd numbers. Since there exists a 4-cycle system of $L(G)$, the number of edges in $L(G)$ must be divisible by 4. Each vertex in V_i in G is incident with $n - a_i$ edges. By considering, adjacent pairs of edges at each vertex in G in turn it follows that

$$|E(L(G))| = \sum_{i=1}^{p}(a_i(n-a_i)(n-a_i-1))/2.$$

So, 8 must divide

$$\begin{aligned} 2|E(L(G))| &= \sum_{i=1}^{p}(a_i n^2) - 2\sum_{i=1}^{p}(n a_i^2) + \sum_{i=1}^{p}(a_i^3) + \sum_{i=1}^{p}(a_i^2) - \sum_{i=1}^{p}(n a_i) \\ &= n^3 - (2n-1)\sum_{i=1}^{p}(a_i^2) + \sum_{i=1}^{p}(a_i^3) - n^2. \qquad (*) \end{aligned}$$

Notice that, since a_i is odd we can write

$$\begin{aligned} a_i &= 8z + l \qquad \text{for some} \quad l \in \{1,3,5,7\} \\ so,\ a_i^2 &= 64z^2 + 16zl + l^2, \\ so,\ a_i^2 &\equiv 1 \pmod 8 \end{aligned}$$

This also implies that $a_i^3 = a_i^2 a_i \equiv a_i \pmod 8$. Similarly, since n is also odd we can see that $n^2 \equiv 1 (\text{mod } 8)$ and $n^3 \equiv n \pmod 8$. Thus from (*), we can say that mod 8:

$$0 \equiv 2|E(L(G))| \equiv (n - (2n-1)p + n - 1) = ((2n-1)(1-p)).$$

So clearly $p \equiv 1 \pmod 8$. Hence condition $(2a)$ is necessary.

Case (2) Now supose that p is even and therefore, n is also even. Clearly $n^3 \equiv 0$ (mod 8).

Again we know that $2|E(L(G))|$ is divisible by 8 and so, from (*) we have mod 8:

$$
\begin{aligned}
0 &\equiv 2|E(L(G))| \\
&\equiv (0-(2n-1)p+n-n^2) \\
&\equiv (-(2n-1)p-n) \\
&\equiv (p-n)
\end{aligned}
$$

which implies that $p \equiv n \pmod 8$, thus proving that condition $(2b)$ is necessary.

$\blacksquare$

Chapter 3

all parts even

all parts odd: odd number of parts

In this chapter, we first show that the necessary conditions in Lemma 2.1 are sufficient when all vertices in G have even degree. Then we deal with the case when all parts are odd and there is an odd number of parts.

We begin with some useful decompositions. In [10] Sajna proved the necessary and sufficient conditions for the even length cycle decomposition of K_n when n is odd. Also, in [11] Sotteau proved a result regarding even length cycle decompositions of the complete graph $K_{x,y}$ These results were a more general solution to the following lemma, which is easy to obtain for 4-cycles.

Lemma 3.1 *There exists a* 4*-cycle system of:*

1. K_n *if and only if* $n \equiv 1$ *(mod* 8*), and*

2. *Of the complete bipartite graph* $K_{x,y}$ *if and only if* x *and* y *are even.*

In [1] Cavenagh and Billington investigated the necessary and sufficient conditions for the existence of an edge-disjoint decomposition of any complete multipartite graph into 4-cycles. The next result is again part of a more general result, and again follows quite readily from Lemma 3.1.

Theorem 3.1 *[1] There exists a* 4*-cycle system of* G *if and only if*

1. *All parts have even size, or*

2. *All parts have odd size and* $p \equiv 1 \pmod 8$*.*

The last result we need now is well known, but easy to prove here. Let $K - F$ be the graph formed from the graph K by removing the edges in F.

Lemma 3.2 *There exists a 4-cycle system of* $K_n - F$ *for any even* n *and any 1-factor* F*.*

Proof. Let $n = 2x$. Let the vertex set of K_n be $\{1, 2, \ldots, x\} \times \{1, 2\}$. The following 4-cycles form the required 4-cycle system:

$$\{((a, 1), (b, 1), (a, 2), (b, 2)) \mid 1 \leq a < b \leq x\}.$$

∎

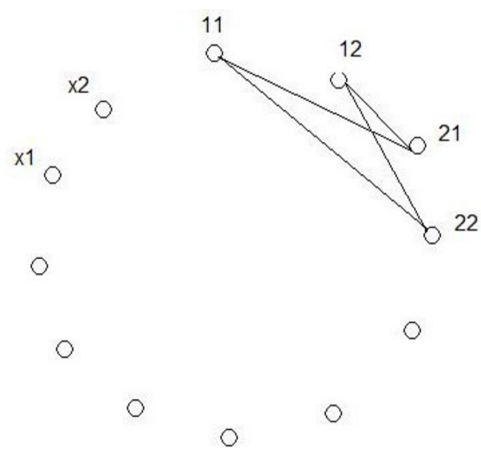

Figure 3.1: 4-cycle system

We are now ready to consider 4-cycle systems of $L(G)$.

Theorem 3.2 *There exists a 4-cycle system of* $L(G)$ *if*

1. a_i *is even for* $1 \leq i \leq p$*, or*

2. a_i *is odd for* $1 \leq i \leq p$ *and* $p \equiv 1 (mod\ 8)$*.*

Proof. The edges of $L(G)$ can be partitioned into sets that induce complete graphs, namely the complete graphs $K(v_{i,x})$ with vertex set $\{\{v_{i,x}, v_{j,y}\} \mid 1 \leq y \leq a_j$ and $1 \leq j \leq p, j \neq i\}$ for each vertex $v_{i,x}$ in $V(G)$. So the edges in $K(v_{i,x})$ correspond to all the 2-paths in G with middle vertex $v_{i,x}$.

By Theorem 3.1, there exists a 4-cycle system B of G. Consider the set of 4-cycles S in $L(G)$ formed by taking the line graph of each 4-cycle in B. For each vertex $v_{i,x}$ in $V(G)$, the edges in $K(v_{i,x})$ contained in 4-cycles in S form a 1-factor $F(v_{i,x})$ of $K(v_{i,x})$ (to see this, observe that the 4-cycles in P pair the edges incident with $v_{i,x}$ in $V(G)$, and each such pair produces an edge in $K(v_{i,x})$ which is vertex-disjoint from the other such pairs).

Also, each such complete graph has even order, so by Lemma 3.2 there exists a 4-cycle system $T(v_{i,x})$ of $K(v_{i,x}) - F(v_{i,x})$.

So the union of the $T(v_{i,x})$ over all the vertices $v_{i,x}$ in $V(G)$ together with the 4-cycles in S produce the required 4-cycle system.

∎

Chapter 4

All parts odd:even number of parts

4.1 Line Graphs of $K_n - K_u$

In this chapter we make progress in tackling the difficult last case, solving a problem that is of interest in its own right. Much progress solving existence problems for graph designs has been made by using decompositions of complete graphs with holes; that is, of $K_n - K_u$. Such decompositions are now of interest in their own right. This is a graph in the family we are considering in this paper, namely the graph G with $a_p = u$, and $a_i = 1$ for $1 \leq i \leq p-1$, and $p = n-u+1$. We begin with a result by Henrich and Nonay referred to in the introduction. Throughout this chapter we deal with the case where a_i is odd for $1 \leq i \leq p$ and p is even.

Theorem 4.1 *[6] Let p be even. There exists a* 4*-cycle system of* $L(K_p)$.

We shall also use some of the results by Fu, Fu and Rodger regarding 4-cycle systems of K_n - E(F) and $2K_n$ - E(F) for all 2 regular subgraphs F.

Theorem 4.2 *[4, 5] There exists a* 4*-cycle system of* $K_z - P$ *for any graph* P *of maximum degree at most* 3 *if and only if*

1. *z is odd,*
2. *the number of edges in $K_z - P$ is divisible by 4, and*
3. *if $z = 8x+1$ then P is not one of two exceptional graphs, both of which are 3-regular.*

We will use this result several times, including the following corollary. Let C_z denote a cycle of length z.

Corollary 4.1 *There exists a 4-cycle system of $K_z - P$ if:*

1. $z \equiv 1$ *(mod 8) and* $P = \emptyset$,
2. $z \equiv 3$ *(mod 8) and* $P = C_3$,
3. $z \equiv 5$ *(mod 8),* $z \neq 5$*, and* $P = C_6$*, and*
4. $z \equiv 7$ *(mod 8) and* $P = C_5$.

Our final preparatory result is needed just for the case when $p = 6$. Form the graph $G \vee H$ from $G \cup H$ by joining each vertex in G to each vertex in H.

Lemma 4.1 *There exists a set* $F = \{F(1), \ldots, F(4)\}$ *of four 1-factors in* K_{8x} *for which there exists a 4-cycle system of* $(K_{8x} - F) \vee K_1$.

Proof. Let the vertex set be $(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_x) \cup \{v\}$. The required cycle system can be formed by taking:

1. $F(k) = \{\{(j, 0, i), (j + k, 1, i)\} \mid j \in \mathbb{Z}_4, i \in \mathbb{Z}_x\}$ for each $k \in \{1, 2, 3\}$,
2. $F(4) = \{\{(j, k, i), (j + 2, k, i)\} \mid j, k \in \mathbb{Z}_2, i \in \mathbb{Z}_x\}$,
3. $B(1) = \{((0, 0, i), (1, 0, i), (2, 0, i), (3, 0, i)), (v, (j, 0, i), (j, 1, i), (j + 1, 1, i)) \mid j \in \mathbb{Z}_4, i \in \mathbb{Z}_x\}$, and
4. A 4-cycle system $B(y, z)$ of $K_{8,8}$ with bipartition $\{\{\mathbb{Z}_4 \times \mathbb{Z}_2 \times \{y\}\}, \{\mathbb{Z}_4 \times \mathbb{Z}_2 \times \{z\}\}$ for $0 \leq y < z < x$ (see Lemma 3.1).

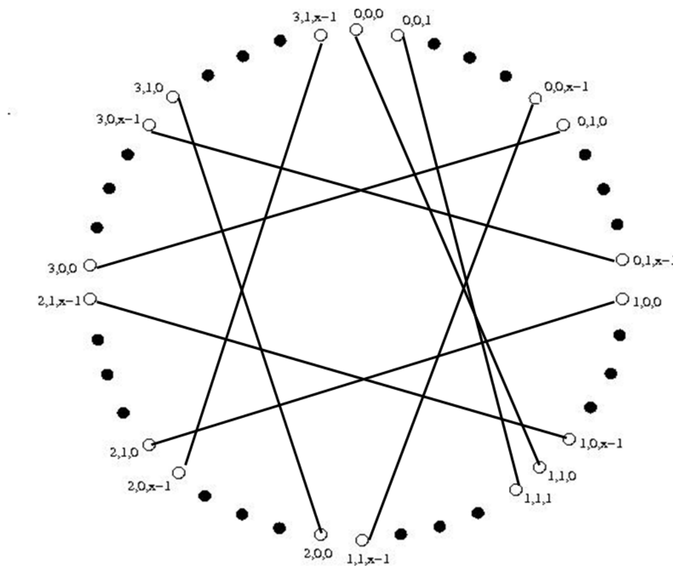

Figure 4.1: F(1): 1 factor of K_{8x}

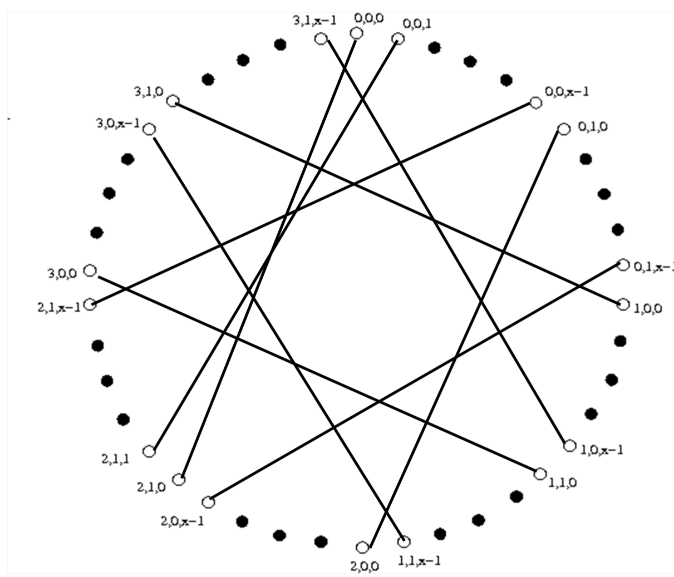

Figure 4.2: F(2): 1 factor of K_{8x}

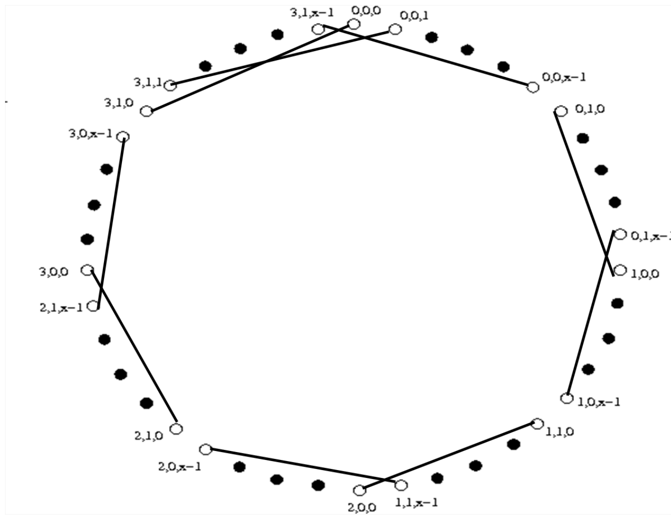

Figure 4.3: F(3): 1-factor of K_{8x}

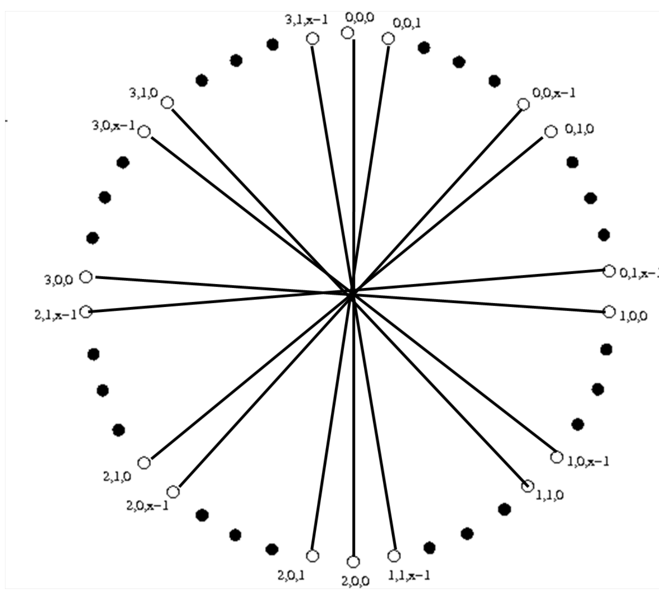

Figure 4.4: F(4): 1-factor of K_{8x}

$\blacksquare$

We are now ready to find 4-cycle systems of $L(K_n - K_u)$, which we state in the following form. By Lemma 2.1, $u \equiv 1 \pmod 8$ is a necessary condition when p is even. The case where p is odd is handled in the previous chapter.

4.2 4-cycle system of L(K(1,1,...,1,8x + 1)), p $\neq$ 6

Theorem 4.3 *There exists a 4-cycle system of $L(G)$ if p is even, $a_i = 1$ for $1 \leq i \leq p-1$ and $a_p = 8x+1$.*

Proof. Let $V_j = \{t(j)\}$ for $1 \leq j \leq p-1$ and let $V_p = \{t(0), s(i) \mid 1 \leq i \leq 8x\}$. For each vertex $w \in V(G)$, let $K(w)$ be the complete subgraph of $L(G)$ induced by the vertex set $\{\{w, w'\} \mid w' \in V(G) \setminus \{w\}\}$. So $K(w)$ contains $p-1$ vertices if $w \in V_p$, and $K(w)$ contains $p + 8x - 1$ vertices otherwise. For $1 \leq j \leq p-1$ it will also be useful to define $K'(t(j))$ to be the subgraph of $K(t(j))$ induced by the vertices in $\{\{t(j), s(i)\} \mid 1 \leq i \leq 8x\}$.

If $p = 2$ then $L(G)$ is isomorphic to K_{8x+1}, and if $x = 0$ then $L(G)$ is isomorphic to K_1, so the result follows from Lemma 2.1.

Now assume that $x \geq 1$ and $p \geq 4$. We will handle the case $p = 6$ last, so for now assume that $p \neq 6$. Let $F = \{F(i) \mid 1 \leq i \leq 3\}$ be a set of 3 edge disjoint 1-factors in K_{8x} defined on the vertex set $\{1, 2, \ldots, 8x\}$. The construction contains 5 types of 4-cycles.

Type 1. Let $B(1)$ be a 4-cycle system of $L(K_p)$ in which K_p is defined on the vertex set $\{t(j) \mid 0 \leq j \leq p-1\}$. This exists by Theorem 4.1.

Type 2. For $1 \leq i \leq 8x$ let $B(2,i)$ be a 4-cycle system of $K(s(i)) - C(s(i))$, where $C(s(i))$ is the cycle $(\{s(i), t(1)\}, \{s(i), t(2)\}, \ldots, \{s(i), t(\alpha)\})$, and where $\alpha = 0, 3, 6$ or 5 when $p-1 \equiv 1, 3, 5$ or $7 \pmod 8$ respectively. Such a 4-cycle system exists by Corollary 4.1. Notice that since $p \neq 6$, $p-1 \geq \alpha$, so $K(s(i)) - C(s(i))$ is well defined.

Type 3. If $\alpha > 0$ then define the following 4-cycle systems. For $1 \leq i \leq 8x$, alternately color the edges of $C(s(i))$ with 1 and 2, except if α is odd then the last edge $\{\{s(i), t(1)\}, \{s(i), t(\alpha)\}\}$ is colored 3; so the same proper edge-coloring is used on each of the $8x$ cycles. For each edge $\{\{s(i), t(j)\}, \{s(i), t(j+1)\}\}$ (reducing $j+1$ mod α) in $C(s(i))$ colored k, form the 4-cycle $(\{s(i), t(j)\}, \{s(i), t(j+1)\}, \{s(i_1), t(j+1)\}, \{s(i_1), t(j)\})$, where $\{i, i_1\}$ is the edge incident with vertex i in $F(k)$. (This same 4-cycle is defined again when i_1 is used instead of i, but we only use it once, of course, in the following union.) Let $B(3)$ be the union of all such 4-cycles. Note that the edges in the 4-cycles in $B(3)$ contain:

1. All the edges in $C(s(i))$ for $1 \leq i \leq 8x$, and

2. The edges in a 2-factor $R(t(j))$ of $K'(t(j))$ for $1 \leq j \leq \alpha$.

The second property holds since, when $j \leq \alpha$, for each vertex $\{t(j), s(i)\}$ in $K'(t(j))$, the two 4-cycles containing the edges $\{\{s(i), t(j-1)\}, \{s(i), t(j)\}\}$, and $\{\{s(i), t(j)\}, \{s(i), t(j+1)\}\}$ (reducing sums mod α) colored say a and b also contain the 2 edges $\{\{t(j), s(i)\}, \{t(j), s(i_a)\}\}$, and $\{\{t(j), s(i)\}, \{t(j), s(i_b)\}\}$ where $\{i, i_a\}$ and $\{i, i_b\}$ are edges in $F(a)$ and $F(b)$ respectively. So by this explanation, in fact $R(t(j))$ is isomorphic to $F(a) \cup F(b)$. If $j > \alpha$ or if $p-1 \equiv 1 \pmod 8$ (this is the case where $\alpha = 0$) then define $R(t(j)) = \emptyset$.

Type 4. For $1 \leq j \leq p-1$, let $B(4, j)$ contain the 4-cycles in a 4-cycle system of $K_{8x+1} - R(t(j))$ defined on the vertex set $V(K'(t(j))) \cup \{\{t(j), t(0)\}\}$. This exists by Theorem 4.2 since:

1. Each vertex has degree $8x$ or $8x-2$ which is even,

2. The number of edges is $(8x+1)4x$ if $R(t(j)) = \emptyset$ and is $(8x+1)4x - 8x = (8x-1)4x$ otherwise, so is divisible by 4, and

3. $R(t(j))$ for $1 \leq j \leq \alpha$ is not one of the exceptional graphs since it is 2-regular.

Type 5. For $1 \leq j \leq p-1$ let $B(5,j)$ be a 4-cycle system of the complete bipartite graph $K_{p-2,8x}$ with bipartition of the vertex set $\{\{\{t(j), t(z)\} \mid 1 \leq z \leq p-1, z \neq j\}, V(K'(t(j)))\}$. This exists by condition (2) of Lemma 3.1.

Then

$$B(1) \cup (\cup_{1\leq i\leq 8x} B(2,i)) \cup B(3) \cup (\cup_{1\leq j\leq p-1} B(4,j)) \cup (\cup_{1\leq j\leq p-1} B(5,j))$$

provides the required 4-cycle system.

∎

4.3 4-cycle system of L(K(1,1,...,1,8x + 1)), p = 6

Finally, suppose that $p = 6$. The difficulty here is that the Type 2 4-cycles must be different because it is impossible to fit a 6-cycle in a graph with only 5 vertices. This can be overcome by the use of 4 1-factors in $K'(t(1))$, for example. Nevertheless, the construction is very similar, so a brief description follows, again defining the five types of 4-cycles in turn. If the same set of cycles is used, we simply state that. In this case, let $\{F(1), \ldots, F(4)\}$ be a copy of the 4 1-factors defined in Lemma 4.1 on the vertex set $\{1, 2, \ldots, 8x\}$.

Type 1. Same as before.

Type 2. Let $B(2) = \{(\{s(i), t(2)\}, \{s(i), t(4)\}, \{s(i), t(3)\}, \{s(i), t(5)\}) \mid 1 \le i \le 8x\}$. Let $C(s(i))$ be the set of 6 edges occurring in no 4-cycle in $B(2)$ (these edges induce two copies of K_3 with one vertex in common).

Type 3. For $1 \le i \le x$, properly color the edges of $C(s(i))$ with the 4 colors in $\{1, 2, 3, 4\}$. For each edge $\{\{s(i), v_1\}, \{s(i), v_2\}\}$ in $C(s(i))$ colored k, form the 4-cycle $(\{s(i), t(j)\}, \{s(i), t(j+1)\}, \{s(i_1), t(j+1)\}, \{s(i_1), t(j)\})$, where $\{i, i_1\}$ is the edge incident with vertex i in $F(k)$. These 4-cycles use the edges forming:

1. a 4-factor $R(t(1))$ isomorphic to $\cup_{1\le k\le 4} F(k)$ in $K'(t(1))$, and
2. for $2 \le j \le 5$ a 2-factor $R(t(j))$ in $K'(f(j))$, each being isomorphic to the union of two of these four 1-factors.

Type 4. For $1 \le j \le p-1$, let $B(4, j)$ contain the 4-cycles in a 4-cycle system of $K_{8x+1} - R(t(j))$ defined on the vertex set $V(K'(t(j))) \cup \{\{t(j), t(0)\}\}$. This exists by Lemma 4.1 if $j = 1$ and by Theorem 4.2 otherwise.

Type 5. Same as before.

∎

Chapter 5

Conclusion

By pooling all the results in this thesis we see that there exists a 4-cycle system of the *line graph* of G for the following cases:

1. All parts are even.
2. All parts are odd and
   (a) $p \equiv 1 \pmod 8$, when p is odd
   (b) $a_i = 1$ for $1 \leq i \leq p-1$ and $a_p = 8x + 1$, when p is even,

Finally, we conjecture the following:

Conjecture 5.1 *There exists a 4-cycle system of the line graph of G for the following case:*

1. All parts are odd and p is even.

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