An example on movable approximations of a minimal set in a continuous flow

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Dissertation Abstract

An example on movable approximations of a minimal set in a continuous flow

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In the present dissertation the study of flows on n-manifolds in particular in dimension three, e.g., $\mathbb{R}^3$, is motivated by the following question. Let A be a compact invariant set in a flow on X. Does every neighbourhood of A contain a movable invariant set M containing A? Here, a dynamical system (a flow) is the pair (X, π), where X, in general, is a manifold, $\pi : X \times \mathbb{R} \to X$ is continuous, $\pi(x, 0) = x$ and $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$, for each $x \in X$ and each $t_1, t_2 \in \mathbb{R}$. A nonempty set $A \subset X$ is *invariant* if $\pi(A, t) = A$ for each $t \in \mathbb{R}$. A compact invariant set $A \subset X$ is *stable* if for every neighbourhood U of A there exists a neighbourhood V of A with $V \subset U$, such that $\pi(V, t) \subset U$ for all $t \geq 0$. The topological notion of movability (also called the UV-property) is in the sense of K. Borsuk and is closely related to the notion of stability in dynamics. A continuum M in X is said to be *movable* if for every neighbourhood U of M there exists a neighbourhood $U_0 \subset U$ of M such that

for every neighbourhood W of M there is a continuous map $\varphi : U_0 \times I \to U$ satisfying the conditions $\varphi(x, 0) = x$ and $\varphi(x, 1) \in W$ for every point $x \in U_0$. It is known that a stable solenoid (an intersection of a nested sequence of solid tori positioned one inside another in some regular way) in a flow on a 3-manifold has approximating periodic orbits in each of its neighbourhoods. The solenoid with the approximating orbits form a movable set, although the solenoid is not movable. Not many such examples are known. The main part of the dissertation consists of constructing an example in $\mathbb{R}^3$ which uses Denjoy–like invariant approximating sets instead of periodic orbits. This gives a partial answer to the above question. The construction involves both, the adding machines and Denjoy maps, and the suspension of specially defined Cantor set homeomorphisms.

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Style manual or journal used Transactions of the American Mathematical Society (together with the style known as "auphd"). Bibliography follows van Leunen's *A Handbook for Scholars.*

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `auphd.sty`.

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Chapter 1

Introduction

The main subject of this dissertation is the study of continuous dynamical systems. The work is inspired by an open problem stated for invariant sets: Let A be a compact invariant set in a flow on an n–dimensional manifold. Does every neighbourhood of A contain a movable compact invariant set containing A?

It is known that the answer is positive for a stable set called a solenoid in dimension three. Such an example appeared in a paper by H. Bell and K. R. Meyer [1]. In their constructions the resulting stable solenoid has periodic orbits in every of its neighbourhoods. By a modification of this example they also proved that analogue result for a stable solenoid in higher dimension does not hold. Later M. Kulczycki showed in his dissertation [10], that it is possible to drop the stability assumption but only under some extra requirements on the flow. Another result by E. S. Thomas, Jr. in [18] guarantees that a minimal solenoid in dimension three is never an isolated invariant set, i.e., in every neighbourhood of the solenoid there are other invariant sets.

The author of this dissertation gives a partial answer to the above question by constructing an example in dimension three and by considering a set that is not stable and is not a solenoid. To the knowledge of the author, such a case has not been published yet.

In Chapter 2, first the definition of a dynamical system or what is also called a continuous flow is introduced. Definitions of a minimal set and almost periodicity are reviewed next. Later we recall the key notions of our study, in particular definitions of special minimal sets, solenoids, and Denjoy continua, and we summarize their basic properties. For the construction of these sets, we first need to discuss a map of a Cantor set that is known as the adding machine, and describe a process of blowing up orbits, that was first published in a paper by A. Denjoy in [7]. Then the notion of suspension is established. It is a continuous dynamical system obtained from a discrete dynamical system. All these objects and maps constitute a significant part of the example constructed in the last chapter of this dissertation. They have been a popular field of study of many authors.

Chapter 3 starts with introducing a dynamical system (a suspension) on a set Ω. The set Ω is minimal under the considered flow. The main original results provided in this chapter are the following.

The first two theorems describe the set Ω.

Theorem 1.1 *The set Ω is not a solenoid.*

Theorem 1.2 *The curve Ω is not movable.*

The next two theorems show that the set Ω is an invariant set in a flow in dimension three. Moreover, in every neighbourhood of Ω there is a compact invariant set that we call Denjoy–like. These Denjoy–like sets are proved to be movable. The set Ω cannot have approximating periodic orbits in each of its neighbourhoods as in the case of a stable solenoid in [1]. This is due to the fact that the flow defined on Ω is not almost periodic.

Theorem 1.3 *There exists an embedding of* Ω *in a mapping torus in* $\mathbb{R}^3$ *with the property that* Ω *is approximated by invariant Denjoy–like sets* $\mathcal{D}_n$, $n \in \mathbb{N}$.

Theorem 1.4 *Every Denjoy–like set* $\mathcal{D}_n$, $n \in \mathbb{N}$, *is movable.*

Finally we prove that Ω together with any sequence of its approximating movable sets is a movable set.

Theorem 1.5 *Let* $\mathcal{D}' = \bigcup_{n=k}^{\infty} \mathcal{D}_n$. *For any* $k \in \mathbb{N}$, *the union of* Ω *and* $\mathcal{D}'$ *is movable.*

To complete the description of the properties of Ω and the approximating sets $\mathcal{D}_n$, $n \in \mathbb{N}$, we show that none of those sets is stable. It is a corollary of a result by J. Buescu and I. Stewart [6].

Theorem 1.6 *The set* Ω *and the sets* $\mathcal{D}_n$, $n \in \mathbb{N}$, *are not stable.*

Although the sets Ω and $\mathcal{D}_n$, $n \in \mathbb{N}$, are not stable as the sets in the example by H. Bell and K. R. Meyer, and moreover Ω is not movable, the union of Ω and $\mathcal{D}'$ is movable. Therefore, this case still yields a kind of “stability”.

Chapter 2

Minimal sets

2.1 Preliminaries

In this section, we introduce the definition of a dynamical system that is sometimes also called a continuous flow, the definition of a minimal set, and we establish the notation.

Throughout the paper we usually consider metric spaces unless stated otherwise. The symbol $\mathbb{R}$ is the the real line, $\mathbb{Z}$ and $\mathbb{N}$ stand for all integer and all natural numbers, respectively. We denote by I the compact unit interval $[0, 1]$. Let $\overline{A}$ denotes the closure of a set A. By a neighbourhood of a set A we understand an open set containing A.

A *dynamical system* on X is the triplet $(X, \mathbb{R}, \pi)$ where π is a continuous map (also called a continuous flow) from the product space $X \times \mathbb{R}$ into the space X satisfying $\pi(x, 0) = x$ and $\pi(\pi(x, t_1), t_2) = \pi(x, t_1+t_2)$ for every $x \in X$ and $t_1, t_2 \in \mathbb{R}$.

The phase map π determines two other maps when one of the variables x or t is fixed. For a fixed $t \in \mathbb{R}$, the map $\pi^t : X \to X$ is defined by $\pi^t(x) = \pi(x, t)$ and is called a *motion* through x. For each $t \in \mathbb{R}$, π^t is a homeomorphism of X onto itself (see [2]). For a fixed $x \in X$, the map $\pi_x : \mathbb{R} \to X$ is given by $\pi_x(t) = \pi(x, t)$.

A *discrete dynamical system* on X is the triplet $(X, \mathbb{Z}, f)$ where f is a continuous map of X into itself. The dynamics is defined through iterations of f. The n–th iterate of f is the map $f^n = f \circ f^{n-1}$, $n \in \mathbb{N}$. The negative iterates are given by $f^{-n} = (f^n)^{-1}$, $n \in \mathbb{N}$. We use the notation $f^0 = f$.

The following definitions concern dynamical systems $(X, \mathbb{R}, \pi)$. The reader can easily reformulate all the notions for the discrete case. The *orbit* of a point $x \in X$ is the set $\{\pi^t(x) \mid t \in \mathbb{R}\}$ and the *positive half orbit* is the set $\{\pi^t(x) \mid t \geq 0\}$. A point $x \in X$ is said to be a *fixed point* (or a *critical point*) if $\pi(x,t) = x$ for all $t \in \mathbb{R}$. A point $x \in X$ is *periodic* if there is a $T \neq 0$ such that $\pi(x,t) = \pi(x, t+T)$ for all $t \in \mathbb{R}$. In this case the smallest such number $T \in \mathbb{R}$ will be called a *period* of x. A nonempty set $A \subset X$ is called *invariant* whenever $\pi(x,t) \in A$ for all $x \in A$ and $t \in \mathbb{R}$. A closed invariant set is *minimal* if it contains no proper closed invariant subset. It is easy to see that if A is compact, then A is minimal if and only if the positive half orbit of every point in A is dense in A. The simplest example of minimal sets are the orbits of fixed or periodic points.

Minimal sets can also arise in the following way. Suppose (X, d) is a metric space. A point $x \in X$ is said to be *almost periodic* (as defined in [15] on page 384) if, given $\varepsilon > 0$, there is a set $E \subseteq \mathbb{R}$ which is relatively dense such that $d(\pi^t(x), \pi^{t+\tau}(x)) < \varepsilon$ for all $\tau \in E$ and $t \in \mathbb{R}$. A set $E \subseteq \mathbb{R}$ is *relatively dense* means that for some number $L > 0$ every interval in $\mathbb{R}$ of length L contains a point of E. If x is almost periodic and the closure Γ of the orbit of x is compact and metrizable, then Γ is a minimal set (see [15], page 385). One–dimensional minimal sets of this type are described in the next section.

A compact invariant set $A \subset X$ is *stable* if for every neighbourhood U of A there exists a neighbourhood V of A with $V \subset U$, such that $\pi(V, t) \subset U$ for all $t \geq 0$.

2.2 Solenoidal and Denjoy minimal sets

The main construction of this paper involves solenoids and Denjoy continua. They are defined in this section. We also introduce some other well known objects and recount their basic properties. We use similar background as it can be found in [1], [6] and [17].

2.2.1 Adding machines and solenoids

First we recall the abstract definition, via symbolic dynamics, of the class of maps of the Cantor set called *adding machines*. Let $\mathfrak{k} = \{k_n\}_{n\geq 1}$ be a sequence of integers with $k_n > 1$ for all $n \in \mathbb{N}$. Let $\Sigma_{\mathfrak{k}} = \prod_{n=1}^{\infty}\{0, 1, 2, \ldots, k_n - 1\}$ be the space of all one–sided infinite sequences $\mathfrak{i} = \{i_n\}_{n\geq 1}$ such that $0 \leq i_n < k_n$ with the product topology. One can see that $\Sigma_{\mathfrak{k}}$ is metrizable and the metric

$$d(\mathfrak{i}, \mathfrak{j}) = \sum_{n=1}^{\infty} \frac{|i_n - j_n|}{k_n^n}$$

is compatible with this topology.

The *adding machine with base* $\mathfrak{k} = (k_1, k_2, \ldots)$ is the map

$$\alpha_{\mathfrak{k}} : \Sigma_{\mathfrak{k}} \to \Sigma_{\mathfrak{k}}$$

defined by $\alpha_{\mathfrak{k}}(\ldots, i_q, \ldots) = (\ldots, j_q, \ldots)$ in the following way

- if $i_q = k_q - 1$ for all q then $j_q = 0$ for all q, i.e. $\alpha_{\mathfrak{k}}(\ldots, i_q, \ldots) = (0, 0, \ldots)$;

  or

- if the first index q with $i_q < k_q - 1$ is r then $j_q = 0$ for $1 \leq q < r$, $j_r = i_r + 1$, and $i_q = j_q$ for $q > r$, i.e. $\alpha_{\mathfrak{k}}(\ldots, i_q, \ldots) = (0, 0, \ldots, i_r + 1, i_{r+1}, i_{r+2}, \ldots)$.

A familiar description of this operation is "add one and carry" because roughly speaking we add one to the first term of the sequence, and if the result is zero we add one to the next term, and so on. It is also well–known that $\alpha_{\mathfrak{k}}$ is a minimal homeomorphism of $\Sigma_{\mathfrak{k}}$ (cf., e.g., [6], page 277, [1], page 411–2, or [12], pages 242–3).

Let us now construct a Cantor set by the following common algorithm. It is especially known for the ternary (or so called middle–third) Cantor set which can be seen as all members in the compact unit interval $I = [0, 1]$ with ternary expansion using only digits 0 and 2.

Take the interval I and let $\mathfrak{k} = (k_1, k_2, \ldots)$ be as previously. In the first step remove from I a collection of $k_1 - 1$ nonempty, open intervals with pairwise disjoint closure and not containing 0 or 1 as an endpoint. Moreover, the intervals that are removed and that remain must all have the same length. Inductively, at the n–th step remove from each of the remaining intervals $k_n - 1$ intervals in the same way and denote the remaining collection of closed intervals by I_n. At each step we obtain a compact set that is a subset of the compact set resulting from the previous step. As a limit of this process we take the intersection of this nested sequence of compact sets and denote it by C, i.e. $C = \bigcap_{n=1}^{\infty} I_n$. It is well known that C is a non–empty, perfect, totally disconnected compact metric space called the *Cantor set*.

We can easily see that the space $\Sigma_{\mathfrak{k}}$ is homeomorphic to such a Cantor set. Indeed, any point $c \in C$ is "coded" as follows to obtain a point $\mathfrak{i} \in \Sigma_{\mathfrak{k}}$. If c lies in the (i_1+1)–th interval from the left of the collection of intervals I_1 (let's denote this interval by $I_1^{i_1}$) then the first coordinate of $\mathfrak{i}$ is i_1. Inductively, in the n–th step, if c lies in the (i_n+1)–th interval from the left of the collection of intervals $I_{n-1}^{i_{n-1}}$ (let's denote this interval by $I_n^{i_n}$) then the n–th coordinate of $\mathfrak{i}$ is i_n.

Adding machines occur in a natural way in the study of solenoids. To see it, we need to introduce some auxiliary definitions.

An *inverse sequence* $\{X_i, f_i^j\}$ of topological groups is a sequence of topological groups $\{X_i\}_{i\in\mathbb{N}}$ together with a collection of continuous homomorphisms $\{f_i^j : X_j \to X_i\}_{i\leq j}$ satisfying

- $f_i^i : X_i \to X_i$ is the identity for all $i \in \mathbb{N}$; and
- $f_i^k = f_i^j \circ f_j^k$ for all $i \leq j \leq k$, $i, j, k \in \mathbb{N}$.

Notice, that it is sufficient to define f_i^{i+1} (called bonding maps) for each $i \in \mathbb{N}$ to determine all f_i^j by the second part above.

The *inverse limit* of an inverse sequence $\{X_i, f_i^j\}$ is the topological group

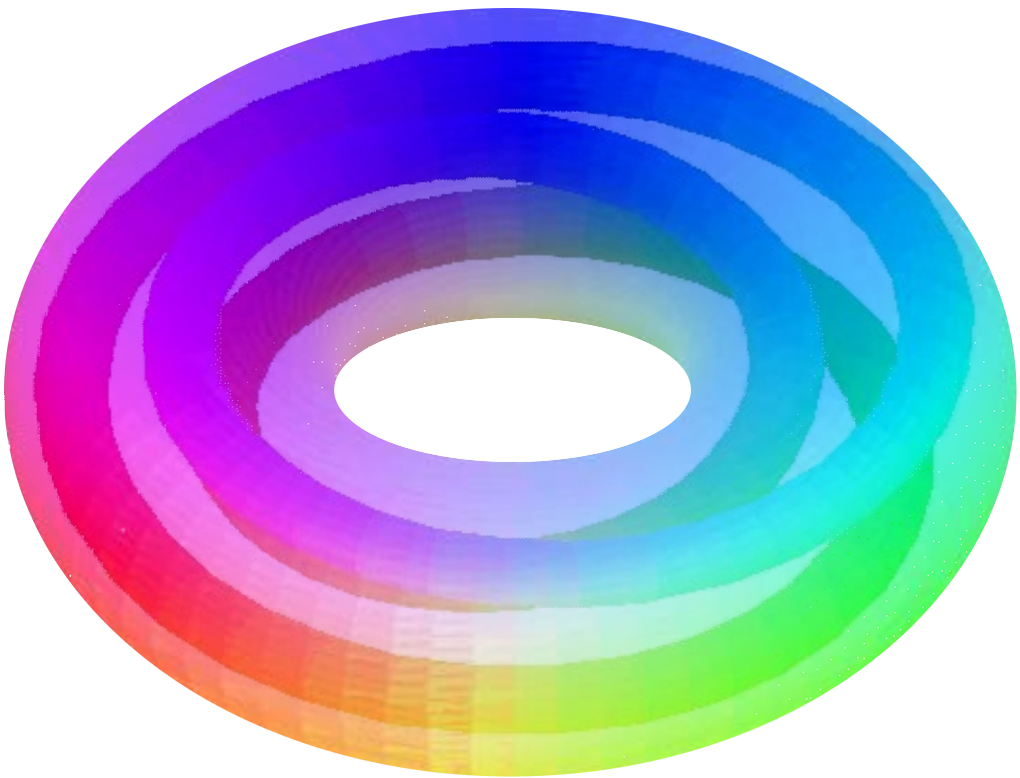
$$X = \varprojlim\{X_i, f_i^j\} = \{(x_1, x_2, \ldots) \in \prod_{i\in\mathbb{N}} X_i \mid x_i = f_i^{i+1}(x_{i+1}) \text{ for all } i \in \mathbb{N}\}$$

with the topology inherited from the product $\prod_{i\in\mathbb{N}} X_i$ with the product topology.

A solenoid can be defined in several ways. The presented definitions disclose homeomorphic objects, we omit the technical proof.

For example, by a *solenoid* we mean a space that is homeomorphic to the inverse limit of a sequence of bonding maps $f_i^{i+1} : S^1 \to S^1$ given by $f_i^{i+1}(z) = z^{n_i}$, where S^1 is the unit circle in the complex plane and $n_i \in \{2, 3, \ldots\}$.

Geometrically, a solenoid is the intersection of a nested sequence of solid tori in $\mathbb{R}^3$ such that each torus is positioned in a specific way inside the previous one as on the picture below.



Before we discuss another way to construct a solenoid, we need the definition of a suspension on a mapping torus.

Let A be a set and $h : A \to A$ a homeomorphism. The *mapping torus* T_A of the homeomorphism h is the set obtained by the following identification. Consider the set $A \times I$. For each $x \in A$ we identify the point $(x, 1)$ with the point $(h(x), 0)$. We define a dynamical system on T_A by $\pi_{T_A}((x,0),t) = (x,t)$ for each $x \in A$ and each $t \in [0,1]$ and extend π_{T_A} in a unique way to a dynamical system on the whole of T_A by the equivalence relation $\sim$

$$(x,t) \sim (y,s) \text{ if and only if}$$

$$(x = y \text{ and } t = s) \text{ or}$$

$$(t = 1, s = 0 \text{ and } h(x) = y) \text{ or}$$

$$(t = 0, s = 1 \text{ and } h(y) = x).$$

A dynamical system defined as above for any homeomorphism h of an arbitrary set is called a *suspension* of h on the mapping torus T_A (see also [17], Appendix).

Now we are ready to construct a solenoid Σ. Consider the space $S \times [0,1]$, where S is a Cantor set, and a homeomorphism $h_{\alpha_{\mathfrak{k}}} : S \to S$ that is the adding machine as defined above. Denote by Σ the mapping torus of the homeomorphism involved and by π_Σ the dynamical system on Σ that is given by the suspension of h on the mapping torus Σ.

Because the orbit of every point in any adding machine is dense, the whole Σ is minimal under π_Σ.

To define a dynamical system on $\mathbb{R}^3$ with a subspace homeomorphic to a solenoid as a minimal set see Section 2 in [1].

2.2.2 Irrational rotation, blowing up orbits, and Denjoy continuum

This section is devoted to a construction of another useful minimal set. We start with a rotation through the angle $2\pi\theta$ of the unit circle $r_\theta : S^1 \to S^1$, where θ is an irrational number. We will change this map and obtain a new homeomorphism h_{r_θ} with a minimal set which is neither a single closed orbit, nor the whole space. Let us consider the circle S^1 to be obtained from the interval $[0,1]$ by identifying its endpoints. We choose a point $x_0 \in S^1$, and at each point $x_n = r_\theta^n(x_0)$ of its orbit we insert a small closed interval I_n into the circle. To fit again into a new circle of circumference $1+a$ denoted by S^1_a, the intervals I_n have to satisfy the condition $a = \sum_{n\in\mathbb{Z}} \text{length}(I_n) < \infty$. There is a continuous onto map $g : S^1_a \to S^1$ which collapses each interval $I_n \subset S^1_a$ to the corresponding point $x_n \in S^1$ and is one–to–one otherwise. We can now define the new map $h_{r_\theta} : S^1_a \to S^1_a$, which is *topologically semi–conjugate* to r_θ under a *topological semi–conjugacy* g, i.e.

$$g \circ h_{r_\theta} = r_\theta \circ g \tag{2.1}$$

and g is continuous and onto by definition. This semi–conjugacy determines h_{r_θ} at all points at which g is one–to–one. We can define g at the remaining points such that h_{r_θ} is a homeomorphism. Moreover, it is possible to obtain a C^1 diffeomorphism h_{r_θ}, for details see [17]. It is an easy exercise to show that the orbits of r_θ are mapped onto orbits of h_{r_θ} by means of a topological semi–conjugacy g, thus "the dynamics is preserved".

The irrationality of θ implies that r_θ and, by 2.1, also h_{r_θ} have no periodic points. Hence, the compact invariant set $S^1_a \setminus \text{Int}\bigcup_{n\in\mathbb{Z}} I_n$ contains a minimal set (under h_{r_θ})

D which is clearly a Cantor set and is neither a single closed orbit, nor the whole space S^1_a.

Remark 2.1 Note that to be a *topological conjugacy* the map g has to be a homeomorphism.

Take again the suspension π_Δ of h_{r_θ} (restricted to D) on the mapping torus Δ obtained from D. The whole Δ is minimal under π_Δ. The set Δ is referred to as a *Denjoy continuum*. The process of inserting intervals is called "blowing up orbits". The construction of π_Δ was first described by A. Denjoy in [7], page 352–5. For details of this construction see [17], Appendix or [14].

Chapter 3

An example of a suspension on a mapping torus Ω

We construct the following example of a suspension. Suppose $h_{\alpha_{\mathfrak{k}}} : S \to S$, $h_{r_\theta} : D \to D$, π_Σ and π_Δ are as in the previous Section 2.2.

Take the product $h_{\alpha_{\mathfrak{k}}} \times h_{r_\theta}$ and denote it by $F : S \times D \to S \times D$. Let Ω be the mapping torus of F and consider the suspension π_Ω of F on Ω.

In this chapter we will show that Ω is not a solenoid (and that F is not an adding machine) and that Ω is not a movable set. Then we will embed Ω in $\mathbb{R}^3$ and we will discuss the properties of this embedding. We will also state that Ω and its approximating sets are not stable.

3.1 The set Ω is not a solenoid

Using the fact that π_Δ is not almost periodic for any point we will show that Ω is not a solenoid.

Lemma 3.1 *Every point is almost periodic for π_Σ.*

Proof. The proof can be found in [15]. It also follows from [6], page 277. □

The proof of the next lemma uses Theorem 1 by E. S. Thomas, Jr. [18].

Theorem 3.2 (Thomas) *If Γ is a compact 1–dimensional metric space which is minimal under some flow and if some point of Γ is almost periodic, then Γ is a solenoid or a circle.*

Lemma 3.3 *There are no almost periodic points for π_Δ.*

Proof. Suppose there exits an almost periodic of π_Δ. Then by Theorem 3.2 the set Δ is a solenoid or a circle. But it is clearly not a circle and, by [6] Remark 7.9, $h_{r_\theta} : D \to D$ is not an adding machine (and not topologically conjugate to one). Contradiction. □

Proposition 3.4 *There are no almost periodic points for π_Ω. Hence, F is not (topologically conjugate to) an adding machine and Ω is not a solenoid.*

Proof. Let $((x_1, y_1), t_1), ((x_2, y_2), t_2) \in \Omega$. We denote and define a metric on Ω by

$$d_\Omega(((x_1, y_1), t_1), ((x_2, y_2), t_2)) = d_\Sigma((x_1, t_1), (x_2, t_2)) + d_\Delta((y_1, t_1), (y_2, t_2)), \quad (3.1)$$

where d_Σ and d_Δ is a metric on Σ, and on Δ, respectively.

An easy check verifies that d_Ω is a well defined metric on Ω. Indeed, let $d_\Omega(((x_1, y_1), t_1), ((x_2, y_2), t_2)) = 0$. Then by (3.1) and the fact that both d_Σ and d_Δ are metrics, we have $d_\Sigma = d_\Delta = 0$. It means that $(x_1, t_1) = (x_2, t_2)$ and $(y_1, t_1) = (y_2, t_2)$. Consequently, $x_1 = x_2$, $y_1 = y_2$ and $t_1 = t_2$, i.e. $((x_1, y_1), t_1) = ((x_2, y_2), t_2)$. The converse is trivial. This completes the proof of positivity of the metric d_Ω. Symmetry and triangular inequality are immediate using (3.1) and symmetry and triangular inequality of d_Σ and d_Δ.

A more natural way to define a metric on Ω would be to establish a general metric for any suspension. Roughly, such a metric would reflect naturally the length of the orbit of a point in the direction of the flow. But since we want to avoid technicalities, the presented metric is more convenient for our purpose.

We need to introduce projections p_1 and p_2 of Ω on Σ and on Δ, respectively. These projections $p_1 : \Omega \to \Sigma$ and $p_2 : \Omega \to \Delta$ are defined by $p_1(\omega) = \sigma$ and $p_2(\omega) = \delta$ where $\omega = ((x, y), t)$, $\sigma = (x, t)$ and $\delta = (y, t)$. The maps p_1 and p_2 are well defined continuous, surjective maps preserving the suspension. Indeed, let $((x_1, y_1), t_1), ((x_2, y_2), t_2) \in \Omega$ and $\pi_\Omega(((x_1, y_1), t_1), t) = ((x_2, y_2), t_2)$ for some $t \in \mathbb{R}$. By definition of suspension, it means that $F^{(t_1+t) \text{ div } 1}(x_1, y_1) = (x_2, y_2)$ and $t_2 = (t_1 + t) \bmod 1$, where $t_1+t = (t_1+t) \text{ div } 1+(\mathrm{t_1+t}) \bmod 1$. Recall that $F = h_{\alpha_\mathfrak{t}} \times h_{r_\theta}$. To prove that the projections are well defined we must prove that $\pi_\Sigma(p_1((x_1, y_1), t_1), t) = p_1((x_2, y_2), t_2)$, and similarly for p_2. We have $\pi_\Sigma(p_1((x_1, y_1), t_1), t) = \pi_\Sigma((x_1, t_1), t) = (x_2, t_2) = p_1((x_2, y_2), t_2)$, where again, by the defition of suspension, $h_{\alpha_\mathfrak{t}}^{(t_1+t) \text{ div } 1}(x_1) = (x_2)$ and $t_2 = (t_1 + t) \bmod 1$. The proof for p_2 is analogous. Surjectivity and continuity are obvious.

Suppose $\omega \in \Omega$ is almost periodic with respect to π_Ω. Let $\varepsilon > 0$. Then by definition, there is a relatively dense set $E \subset \mathbb{R}$ such that $d_\Omega(\pi_\Omega^t(\omega), \pi_\Omega^{t+\tau}(\omega)) < \varepsilon$ for every $\tau \in E$ and every $t \in \mathbb{R}$. Since π_Ω, π_Σ and π_Δ are suspensions and by (3.1) we have

$$d_\Omega(\pi_\Omega^t(\omega), \pi_\Omega^{t+\tau}(\omega)) = d_\Sigma(\pi_\Sigma^t(\sigma), \pi_\Sigma^{t+\tau}(\sigma)) + d_\Delta(\pi_\Delta^t(\delta), \pi_\Delta^{t+\tau}(\delta)) < \varepsilon.$$

Hence, $d_\Delta(\pi^t_\Delta(\delta), \pi^{t+\tau}_\Delta(\delta)) < \varepsilon$. But it is not possible by Lemma 3.3. □

The fact that the flow π_Ω on Ω is not almost periodic implies that Ω cannot have approximating orbits in each of its neighbourhoods as in the case of a stable solenoid in [1].

Remark 3.5 Let f be a homeomorphism defined on a Cantor set that is minimal under f. We have proved that if the product of an adding machine with the function f is (topologically conjugate to) an adding machine, then f must also be (topologically conjugate to) an adding machine. The reader can also convince himself, that a product of two adding machines is (topologically conjugate to) an adding machine. But we will not need it in this dissertation.

3.2 The set Ω is not movable

As a corollary of results by K. Borsuk, J. Krasinkiewicz and A. Trybulec, we will state that Ω is not movable.

The notion of movability and n–movability was introduced by K. Borsuk (see [4] and [5]) and is closely related to stability in dynamical systems.

Definition 3.6 A set which is both compact and connected is called a *continuum.*

Definition 3.7 A continuous map $r : X \to A$ is said to be a *retraction* of X to A if $A \subset X$ and $r(A) = A$. In this case, A is said to be a *retract* of X. A space Y is said to be an *absolute retract* (abbreviated AR), provided that for each homeomorphism h mapping Y onto a closed subset $h(Y)$ of a space X the set $h(Y)$ is a retract of

X. A space Y is called an *absolute neighbourhood retract* (abbreviated ANR), if for every homeomorphism h mapping Y onto a closed subset of a space X there is a neighbourhood U of the set $h(Y)$ in the space X such that $h(Y)$ is a retract of U.

Definition 3.8 Let X be an ANR. A continuum $M \subset X$ is said to be *movable in X* if for every neighbourhood U of M there exists a neighbourhood $U_0 \subset U$ of M such that for every neighbourhood W of M there is a continuous map $\varphi : U_0 \times I \to U$ satisfying the condition $\varphi(x, 0) = x$ and $\varphi(x, 1) \in W$ for every point $x \in U_0$.

In several places we will need a result by Borsuk (see [4], page 142) about independence of movability on the embedding.

Theorem 3.9 (Borsuk) *Movability is a topological property. Thus, a continuum is movable if it is homeomorphic to a continuum movable in the previous sense.*

Definition 3.10 By a *curve* we understand any 1–dimensional continuum.

The following theorem combines Theorem 4.1 in [9] (see also [11]) with a theorem in [19].

Theorem 3.11 (Krasinkiewicz, Trybulec) *If f is a continuous map from a movable curve X onto a curve Y, then Y is movable.*

The proof of the next theorem appears in [4].

Theorem 3.12 (Borsuk) *If Γ is a solenoid then Γ is not movable.*

Corollary 3.13 *The curve Ω is not movable.*

Proof. Let Σ is a solenoid given by a mapping torus obtained from a Cantor set S as presented in Section 2.2.1. Suppose that the Cantor set S here is the same one that is used in construction of Ω. Notice that both, Σ and Ω, are curves. Let $p_1 : \Omega \to \Sigma$ be a function defined by $p_1((x, y), t) = (x, t)$ (see the proof of Proposition 3.4). It is a continuous well–defined map of Ω onto Σ, therefore Ω is not movable by Theorems 3.12 and 3.11. □

3.3 Embedding of Ω in $\mathbb{R}^3$

In this section, we will first show that Ω can be embedded in a flow in $\mathbb{R}^3$ in such a way that it is approximated by Denjoy–like sets that are movable. We construct them as a mapping torus of the product of the Denjoy map h_{r_θ} on D and a map that constitutes just of one periodic orbit O of a point in a discrete dynamical system. These Denjoy–like sets (orbits) are "stretched along" the orbits of the points from Ω, i.e. for every point in Ω we can find a point of the same Denjoy–like set that is as close to the selected point in Ω as we like if the Denjoy–like set is chosen sufficiently long (in the sense that the periodic orbit O is sufficiently long) and sufficiently close to Ω in the sense of Hausdorff metric.

Then we will prove that although Ω is not movable, its union with the approximating Denjoy–like sets is movable. We will complete the description by a corollary giving that none of the sets Ω and its approximating Denjoy–like sets are stable.

For the formulation of the theorems of this section we need some auxiliary definitions.

Definition 3.14 Let O be a periodic orbit in a discrete dynamical system. Consider the product $D \times O$ with the product of the corresponding maps. We say that $\mathcal{D}$ is a *Denjoy–like* set if it is the mapping torus of this product.

Definition 3.15 Let M be a complete metric space with a metric d, and $\mathcal{C}_M$ be the collection of all compact subsets of M. The *Hausdorff metric* d_H on $\mathcal{C}_M$ is defined as follows. For $A, B \in \mathcal{C}_M$,

$$d_H = \sup\{d(a, B), d(b, A) : a \in A, b \in B\},$$

where

$$d(b, A) = \inf\{d(b, a) : a \in A\}$$

and similarly for $d(a, B)$.

Definition 3.16 We say that Ω is *approximated* by Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$, if every for every $\varepsilon > 0$ there is a Denjoy–like set $\mathcal{D}_j$, for some $j \in \mathbb{N}$, such that $d_H(\Omega, \mathcal{D}_j) < \varepsilon$.

The sets S and D can be embedded in $\mathbb{R}$ and therefore Ω and the sets $\mathcal{D}_n$, $n \in \mathbb{N}$, can be embedded in $\mathbb{R}^3$. Let the metric d needed in the previous definition be the Euclidean metric of $\mathbb{R}^3$.

Before we state the main Theorem 3.18 of this section, we need the following theorem that is proved, e.g., in [13] in Chapter 12 and in more general settings also in Chapter 13.

Theorem 3.17 *Let C_1 and C_2 be Cantor sets in $\mathbb{R}^2$ and $h : C_1 \to C_2$ a homeomorphism. Then there exists an orientation preserving homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H_{|C_1} = h$.*

Theorem 3.18 *There exists an embedding of Ω in a mapping torus in $\mathbb{R}^3$ with the property that Ω is approximated by invariant Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$.*

Proof. We can consider the Cantor set S being embedded in $\mathbb{R}$, such coding is described in Section 2.2.1. We approximate S by periodic orbits O_n, $n \in \mathbb{N}$, in $\mathbb{R}$, in the following way. Let $O_n = \{o_1^n, o_2^n, \ldots, o_{m_n}^n\}$, where the last lower index $m_n = (k_i - 1) \cdot k_{i-1} \cdot k_{i-2} \cdot \ldots \cdot k_2 \cdot k_1$ with the notation from the algorithm in Section 2.2.1. The set O_n is a subset of the union of the intervals that are removed at i–th step (there are exactly m intervals removed at this step), every point from O_n lying in a different of these intervals. Hence, the sets O_n, $n \in \mathbb{N}$ are pairwise disjoint.

It means that there is a homeomorphism $h'_{\alpha_\mathfrak{k}} : S \cup \bigcup_{n=1}^{\infty} O_n \to S \cup \bigcup_{n=1}^{\infty} O_n$ such that $h'_{\alpha_\mathfrak{k}|S} = h_{\alpha_\mathfrak{k}}$, and $h'_{\alpha_\mathfrak{k}|O_n} = O_n$, for each n. Then $(S \cup \bigcup_{n=1}^{\infty} O_n) \times D$ is a Cantor set that can be embedded in $\mathbb{R}^2$. By Lemma 3.17, $h'_{\alpha_\mathfrak{k}} \times h_{r_\theta}$ has an extension $F' : \mathbb{R}^2 \to \mathbb{R}^2$ which is also a homeomorphism. We notice that F' is also an extension of F. Therefore, Ω is a subset of the mapping torus Ω' of F', and the suspension $\pi_{\Omega'}$ of F' on Ω' is an extension of π_Ω. The verification of the fact that Ω is approximated by pairwise disjoint invariant Denjoy–like sets $\mathcal{D}_n$ is immediate from the construction. Finally, we remark that it is possible to extend the flow $\pi_{\Omega'}$ extended onto the whole $\mathbb{R}^3$ so that the properties of the embedding mentioned in this theorem are preserved. $\square$

The next result by J. Krasinkiewicz [9] and also R. D. McMillan [11] generalize a theorem of K. Borsuk [4] on movability of plane continua. By a surface we understand a compact two dimensional manifold.

Theorem 3.19 (Krasinkiewicz, McMillan) *Every continuum that can be embedded in a surface is movable.*

In the following, the Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$, are the sets constructed in the proof of Theorem 3.18.

Theorem 3.20 *Every Denjoy–like set $\mathcal{D}_n$, $n \in \mathbb{N}$, is movable.*

Proof. We construct an embedding of $\mathcal{D}_n$ in a surface. Let $h_{r_\theta} : S^1_a \to S^1_a$ be as in Section 2.2.2. Consider n copies of S^1_a, i.e. the product $S^1_a \times O_n$, where O_n is a periodic orbit as in the proof of Theorem 3.18. We define a map $g : S^1_a \times O_n \to S^1_a \times O_n$ to be the product of the corresponding maps on S^1_a and O_n, respectively. The mapping torus of the homeomorphism g is a surface homeomorphic to a surface of a torus which is wrapped n–times. It is easy to see that this surface is homeomorphic to $\mathcal{D}_n$. By Theorem 3.19, $\mathcal{D}_n$ is movable. □

The following definitions and Theorem 3.22 are necessary for the proof of Theorem 3.24.

Definition 3.21 Let X and Y be topological spaces and let f_0 and f_1 be continuous maps of X to Y. If there is a continuous map $h : X \times I \to Y$ such that $h(x, i) = f_i(x)$ for $i = 0, 1$, then we say that the maps f_0 and f_1 are *homotopic.* The map h is called a *homotopy* between f_0 and f_1.

Theorem 3.22 (Borsuk's homotopy extension theorem) *Let M be a closed subspace of a metrizable space X and f_0 and f_1 two homotopic maps of M to an ANR. Then if f_0 is continuously extendable over X, then f_1 is also continuously extendable over X. Moreover, for every extension of f_0 one can find an extension of f_1 homotopic to it.*

The proof of Borsuk's homotopy extension theorem can be found, e.g., in [3].

Definition 3.23 The map $p_n : \Omega \to \mathcal{D}_n$, $n \in \mathbb{N}$, defined bellow is called the *n–th projection of Ω on $\mathcal{D}_n$.* For any $\omega = ((x, y), t) \in \Omega \subset \mathbb{R}^3$ with $x \in S$, $y \in D$ and $t \in [0, 1]$, the n–th projection is defined by $p_n(\omega) = ((o_l^n, y), t)$. See the proof of Theorem 3.18 for the construction of periodic points o_l^n. The index $l \in \{1, 2, \ldots, m_n\}$ is such that the point $x = (i_1, i_2, \ldots, i_n, \ldots) \in S$ is mapped by p_n to the closest point $o_l^n \in I_{n-1}^{i_{n-1}}$ on the right of x, or if there is no such point on the right then to the left. The intervals $I_{n-1}^{i_{n-1}}$ are described in Section 2.2.1. By construction, p_n is continuous. Similarly are defined continuous *projection of $\mathcal{D}_q$ on $\mathcal{D}_n$, $q > n$.* Let $p_n^q : \mathcal{D}_q \to \mathcal{D}_n$ be such that $p_n^q((o_k^q, y), t) = ((o_l^n, y), t)$. For any index $k \in \{1, 2, \ldots, m_q\}$ the index $l \in \{1, 2, \ldots, m_n\}$ is such that the point o_k^q is mapped by p_n^q to the closest point o_l^n on the right of o_k^q, or if there is no such point on the right then to the left. Therefore, for any $q > n$,

$$p_n^q \circ p_q = p_n.$$

Theorem 3.24 *Let $\mathcal{D}' = \bigcup_{n=1}^{\infty} \mathcal{D}_n$. The union of Ω and $\mathcal{D}'$ is movable.*

Proof. By definition of movability, we have to prove the following statement. For every neighbourhood U of $\Omega \cup \mathcal{D}'$ there is a neighbourhood $U_0 \subset U$ of $\Omega \cup \mathcal{D}'$ such that for each neighbourhood W of $\Omega \cup \mathcal{D}'$, there is a continuous map φ satisfying the conditions

$$\varphi : U_0 \times I \to U, \varphi(x, 0) = x \text{ and } \varphi(x, 1) \in W \text{ for every point } x \in U_0. \tag{3.2}$$

We say in this case that U_0 can be deformed to W within U.

Actually, we will prove a stronger statement: For every neighbourhood U of $\Omega \cup \mathcal{D}'$ there is a number $N \in \mathbb{N}$ and a neighbourhood $U_0 \subset U$ of $\Omega \cup \mathcal{D}'$ such that for every neighbourhood W of $\bigcup_{j=1}^{N} \mathcal{D}_j$, there is a continuous map φ satisfying the conditions (3.2).

For a given neighbourhood U of $\Omega \cup \mathcal{D}'$ we will construct the neighbourhood U_0 of $\Omega \cup \mathcal{D}'$ as a finite union of pairwise disjoint neighbourhoods $U_1, U_2, \ldots, U_N$, where U_j is a neighbourhood of $\mathcal{D}_j$, $j < N$, and U_N is a neighbourhood of the set $\Omega \cup \bigcup_{j=N}^{\infty} \mathcal{D}_j$. Then we deform each set U_j, $1 \leq j \leq N$, into W within U.

Let U be a neighbourhood of $\Omega \cup \mathcal{D}'$. Then there is an $\varepsilon > 0$ such that every open ball with radius at most ε centered at a point from $\Omega \cup \mathcal{D}'$ is contained in U.

By Theorem 3.18, Ω is approximated, in the sense of Hausdorff metric, by pairwise disjoint Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$. Therefore, there exists a number $N' \in \mathbb{N}$ such that $d_H(\Omega, \mathcal{D}_{N'}) < \varepsilon$ and $d(p_{N'}(\omega), \omega) < \varepsilon$, for each $\omega \in \Omega$. By definition, the projection $p_{N'} : \Omega \to U$ satisfies $p_{N'}(\Omega) = \mathcal{D}_{N'}$. Note that U is an open set in $\mathbb{R}^3$, and therefore an ANR (see [8]). Hence, the identity on Ω is homotopic within U to $p_{N'}$. The corresponding homotopy $h : \Omega \times I \to U$ is given by $h(\omega, t) = (1-t)\omega + tp_{N'}(\omega)$, where $h(\omega, t) \in U$ for each $\omega \in \Omega$ and $t \in I$. By Borsuk's homotopy extension theorem 3.22 there is an extension $P_{N'} : U \to U$ of $p_{N'}$ homotopic to the identity on U. Hence, we have an extension $H : U \times I \to U$ of h.

Theorem 3.20 provides movability of all Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$. Therefore, by definition of movability, for every neighbourhood U of $\mathcal{D}_n$ there is a neighbourhood $V_n \subset U$ of $\mathcal{D}_n$ such that for each neighbourhood W of $\mathcal{D}_n$, there is a map φ_n satisfying the conditions (3.2) with U_0 replaced by V_n and φ replaced by φ_n. Because the sets Ω and $\mathcal{D}_i$, $i \in \mathbb{N}$, are pairwise disjoint, we can assume that $\overline{V_n} \neq \overline{V_m}$, for $n \neq m$, and that $\overline{V_n} \cap \Omega = \emptyset$, for every $n \in \mathbb{N}$.

Now we will construct a neighbourhood U_N, for some $N \in \mathbb{N}$, of $\Omega \cup \bigcup_{j=N}^{\infty} \mathcal{D}_j$ that is disjoint with every V_j, $j < N$.

For the rest of the proof, we use the following notation. If $f : X \times I \to X$ is a map, we denote by $\dot{f} : X \to X$ the map given by $\dot{f}(x) = f(x, 1)$, for each $x \in X$.

Let $U' = U \cap \dot{H}^{-1}(V_{N'})$, and further let $U_N = U' \setminus \overline{\bigcup_{j<N} V_j}$. Then U_N is a neighbourhood of Ω. The index N is given as follows. By Theorem 3.18, U_N contains all $\mathcal{D}_j$, for $j \geq N$. Let $U_j = V_j$, for $j < N$. Clearly, the open sets $U_1, U_2, \ldots, U_N$ are pairwise disjoint. We put $U_0 = \bigcup_{j=1}^{N} U_j$.

Let W be any neighbourhood of $\Omega \cup \mathcal{D}'$. Finally, we define the map φ satisfying (3.2). Let $\varphi_{|U_j \times I} = \varphi_j$, for each $j < N$. It remains to define the map $\varphi_{|U_N \times I}$. Indeed, let $\dot{\varphi}_{|U_N} = \dot{\varphi}_{N'} \circ \dot{H}_{|U_N}$.

Since the sets $U_1, U_2, \ldots, U_N$ are pairwise disjoint, φ is a well–defined continuous map. □

Remark 3.25 The author of this dissertation is convinced that the following stronger statement is not true. For every neighbourhood U of $\Omega \cup \mathcal{D}'$ there is a neighbourhood $U_0 \subset U$ of $\Omega \cup \mathcal{D}'$ such that there is an $N \in \mathbb{N}$ and a map $\varphi : U_0 \times I \to U$ satisfying the condition $\varphi(x, 0) = x$ and $\varphi(x, 1) \in \bigcup_{n=1}^{N} \mathcal{D}_n$ for every point $x \in U_0$. Analogical statement is true for the union of a solenoid and its approximating orbits. But our proof shows that we can at least deform U_0 arbitrarily close, in the Hausdorff metric, to $\bigcup_{n=1}^{N} \mathcal{D}_n$, for some $N \in \mathbb{N}$.

By Definition 3.23, it is easy to proof the following statement about the structure of the set Ω .

Observation 3.26 *The set Ω is the inverse limit of the inverse sequence $\{\mathcal{D}_n, p_n^q\}$, i.e.,*

$$\Omega = \lim_{\leftarrow}\{\mathcal{D}_n, p_n^q\}.$$

Using this observation, the proof of Theorem 3.24 can be generalized in the sense of the next corollary. In this form it is a generalization of the "star" construction by R. Overton and J. Segal in [16]. Unlike their theorem, we are not requiring the sets X_n, $n \in N$, to be absolute neighbourhood retracts.

Corollary 3.27 *Let $X = \lim_{\leftarrow}\{X_n, f_n^q\}$, where X_n movable for each $n \in N$. Let $X' = \bigcup_{n=1}^{\infty} X_n$. Then the union of X and X' is movable.*

We have already discussed in the Introduction that it is known that the answer to our original questions is positive for a stable solenoid in dimension three. Such an example appears in a paper by H. Bell and K. R. Meyer [1]. As a corollary of a theorem by J. Buescu and I. Stewart in [6], page 278, we obtain that Ω and its approximating Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$, are not stable.

To understand the next theorem for discrete dynamical systems, we define transitivity and some necessary formalisms.

Definition 3.28 Let A be a compact set in a discrete dynamical system on a space X. We say that A is *transitive,* if there exists a point in A with dense positive half orbit.

Let X be a locally compact metric space, and let $f : X \to X$ is a continuous map. Suppose that X has a compact subset A that is transitive under f. Let $\sim$ be the equivalence relation on A determined by its connected components, i.e. $x \sim y$ if and only if x and y lie in the same component of A. Let $K = A/\sim$ with the identification topology. Then $i \circ f = \tilde{f} \circ i$, where i is the identification map and $\tilde{f}$ is the map induced by f.

Theorem 3.29 (Buescu, Stewart) *Suppose that X is a locally connected, locally compact metric space, $f : X \to X$ is a continuous map, and A is a compact transitive set. Assume A is stable and has infinitely many components. Then the map $\tilde{f} : K \to K$ is topologically conjugate to an adding machine.*

Corollary 3.30 *None of the sets Ω and its approximating Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$, is stable.*

Proof. Let $F : S \times D \to S \times D$ be the map defined at the beginning of this chapter and let the maps F' and $\pi_{\Omega'}$ are as in the proof of Theorem 3.18. Since the map F is not topologically conjugate to an adding machine (see Proposition 3.4), the set $S \times D$ is not stable with respect to the map F'. Thus, applying the definition of stability for flows, Ω is not stable with respect to $\pi_{\Omega'}$. The proof is similar for each Denjoy–like sets $\mathcal{D}_n$, $n \in \mathbb{N}$. We use the fact, that $F'_{|\mathcal{D}_n}$, for each $n \in \mathbb{N}$, is not topologically conjugate to an adding machine. The proof of this statement is analogue to the proof of Proposition 3.4. □

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