Poroelasticity

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Poroelasticity

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Poroelasticity

Chadia Affane Aji

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Vita

Chadia Affane Aji, daughter of Abbes Affane Aji and Noufissa Hassan, was born October 2, 1968, in Fes, Morocco. She is married and has three children. In 1991, she got an associate degree in Chemical Engineering from Ecole Superieure de Technologie, E.S.T., in Fes, Morocco. She entered Texas A&M University in 1997 and graduated with the degree of Bachelor of Chemical engineering in 1999. Then she went back to her country where she taught Mathematics for two years at Ifrane School in Morocco. She came to Auburn in December 2001 when her husband joined the Computer Science faculty. She felt drawn to learn more mathematics of an applied kind thus she became a graduate student in fall 2002. She finished her Master of Applied Mathematics in December 2003. Since then she has been in the Ph. D. program in Mathematics.

Dissertation Abstract

Poroelasticity

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Poroelasticity is the study of elastic deformation of porous materials saturated with a fluid and the coupling between the fluid pressure and the solid deformation.
Considerable progress has been made in formulating analytical and numerical models of subsurface fluid flow, but only few models explain the interrelations between fluid-flow pressure changes and seismicity.

In this work, we describe the quasi-static poroelasticity system of partial differential equations consisting of the equilibrium equation for momentum conservation and the diffusion equation for Darcy flow. We prove existence and uniqueness of weak solutions to the equations of the quasi-static poroelasticity system and derive error estimates. We describe a coupled numerical algorithm that accounts for the interrelations between the fluid pressure changes and the deformation of the porous elastic material based on the finite element method using MATLAB.

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This work is dedicated to my lovely father Abbes Affane Aji (1927-1992). A special thanks to my lovely mother Noufissa Hassan, my brother Si Mohammed Affane, my sisters, and all my family for their ongoing encouragement and love.

Of all I am most thankful to my great kids Youssef, Rim, Yasmine, and I am very grateful to my husband Saad Biaz for the unconditioned love and care. I would not be where I am today without his invaluable support throughout my graduate studies.

Style manual or journal used Journal of Approximation Theory (together with the style known as “aums”). Bibliograpy follows van Leunen’s *A Handbook for Scholars.*

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

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Chapter 1

Introduction

A porous medium, such as rock, sediment, or artificial porous material, is a material with empty cavities called pores. These cavities may be filled with liquids or gases. Peat and clay are porous materials; about half of their volume consists of empty cavities. A kitchen sponge is an artificial porous medium. Due to its nature, a porous medium is usually elastic: when subjected to a force, a porous material may change its form but it will often return to its original shape when the force is removed. The notion and study of porous material in geology was first introduced in 1943 by Karl von Terzaghi (see [17]), the father of soil mechanics. When a saturated porous medium is deformed, the volume of the cavities changes, producing a change in the gas or liquid pressure. The relationship between the deformation and the pressure changes is of interest in many geologic and engineering applications.

Two mechanisms play a role in the interaction between fluid pressure changes and deformation of the porous elastic material: (1) dilation of the medium results in a decrease of pore pressure and, (2) compression of the material causes a rise of pore pressure, if the compression is faster than the fluid flow rate. For example, the water level in a well changes when a train passes nearby. In 1892, F. H. King (see [18]) noticed that the water level in a well near the train station at Whitewater, Wisconsin, went up when the train approached the station and it went down when the train left the station. The change of the water level depends on the weight of the train, that is, the water level increases more for a heavy loaded freight train than for a passenger train. Another example of the coupled pressure-deformation is a sponge whose pores are saturated with water. By compressing the sponge, its form changes. The decrease of the volume of the pores creates an overpressure. Therefore, the fluid is pressed out of the material and flows away because of the increase of the pore pressure. When releasing the sponge, i.e., reducing the pore pressure, the sponge returns to its original form. This is explained by the elastic behavior of the material. This coupled mechanism – namely the coupling of the stress in the solid with the pressure of the fluid – plays an essential role in poroelasticity.

Poroelasticity is the study of elastic deformation of porous materials saturated with a fluid and the coupling between the fluid pressure and the solid deformation. Anthony Biot was the first to develop a model for such a relationship. His seminal paper [1] in 1941 describes a linear theory of poroelasticity which relates the evolution of fluid pressure p (a scalar field) and the solid displacement u (a vector field). This system of equations is given as follows

$$\rho u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) + \alpha\nabla p = f, \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha\nabla\cdot u) - \nabla\cdot k\nabla p = h, \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

where Ω is an open bounded non-empty set in $\mathbb{R}^3$ and T is a positive time.

This system consists of the momentum balance equations for the displacement of the medium (1.1) and the mass balance equation for the pressure distribution (1.2). The coefficient ρ represents the local density of the porous medium. The constants, λ, called the Lame constant, and μ, the shear modulus, are a measure of the strength of the material, and are determined from the elasticity of the medium. The constant $\alpha > 0$ is the Biot-Willis constant and accounts for the mechanical coupling of the porous media and the fluid pressure. The coefficient $c_0 > 0$, called specific storage, is the amount of fluid which can be forced into or out the medium by a unit pressure increment under constant volume. The parameter k involves the permeability of the medium and the viscosity of the fluid in Darcy's law. The functions f and h are suitably given functions. Note that u_{tt} is the second order partial derivative of u with respect to time, Δ denotes the Laplace operator in $\mathbb{R}^3$, ∇ is the gradient, and $\nabla\cdot$ is the divergence. These mathematical terms are defined in Appendix A.

Biot's poroelasticity model is very general as it is independent of the application domain. This model was extended by Coussy (see [4]) to take into account the heat-convection phenomena. Other researchers have refined the model for specific engineering fields (see [11], [12], [13], and [20]) such as geomechanics or petroleum engineering. The Biot's system model is complex and, in general, does not have closed form solutions. In 1993, Gomberg and Ellis (see [7]) provided an algorithm dubbed 3D-DEF (a three-dimensional boundary element program) that approximates Biot's system for the displacement from which the strain ϵ and the stress σ can be calculated. In order to calculate pore pressure changes, Lee and Wolf proposed in 1998 an algorithm dubbed 3P-Flow (see [9]) that uses the above calculated strain ϵ and stress σ. The algorithm 3D-DEF approximates solutions of the quasi-static case of the elasticity equation (1.1) for the vector displacement u. Using these results, 3P-flow can approximate the pressure in the diffusion equation (1.2). Thus the two algorithms together do not treat the fully coupled system of the two partial differential equations. Furthermore, at the time there was no guarantee that a solution for the system exists.

In 2000, using abstract theory (non constructive), Showalter (see [14]) showed existence and uniqueness of strong solutions and weak solutions to Biot's system in the quasi-static case. A summary of his results is described in Section 3.1.

In this work, using a constructive approach (based on Babuska-Brezzi theory and Rothe's method of lines), we proved existence and uniqueness of weak solutions to the equations of quasi-static poroelasticity (1.1)-(1.2).

This approach (a constructive approach) suggests numerical approximation methods and allows derivation of error estimates.

Developing solvers for the coupled system (1.1)-(1.2) is an area of current research. To our knowledge, there is no rigorous 3-dimensional error analysis for this coupled system.

The main contribution of this work is the construction of two algorithms for approximating solutions of the quasi-static poroelasticity system of partial differential equations: a segregated algorithm where the solution is approximated by an iterative method and a coupled algorithm where the system is concurrently approximated for both the vector displacement u and the scalar pore pressure p.

This work is organized as follows. In the next chapter, we describe the mathematical model. That is, we describe the quasi-static poroelasticity system of partial differential equations. Existence and uniqueness of weak solutions are proved and error estimates are derived in Chapter 3. Chapter 4 describes numerical experiments. Finally, conclusion and proposed future work are given in Chapter 5.

Chapter 2

Poroelasticity model

A quasi-static poroelastic problem is described by the following basic variables: stress (σ); the normalized force, strain (ϵ); the symmetric part of the deformation gradient, the vector displacement (u), the scalar pore pressure (p), and the increment of fluid content (ξ).

In this section, we formulate the equations describing the coupling of elastic deformation and pore fluid pressure in a porous medium. We first consider poroelastic constitutive response – i.e., the dependence of strain and fluid content on stress and pore pressure – and the Darcy law for pore fluid transport. Then we formulate the governing field equations using considerations of stress equilibrium and mass conservation.

2.1 The elasticity equation

The equilibrium equation for momentum conservation will be formulated based on the force equilibrium equation and the linear constitutive equation, the dependence of strain and fluid content on stress. Therefore, we first need to define stress and strain to derive the elasticity equation.

2.1.1 Stress

Consider a volume, an infinitesimal cube with faces pointing in the coordinate directions. There are two types of external forces acting on the material body:

1. The force acting on volume elements of the body, called body force.

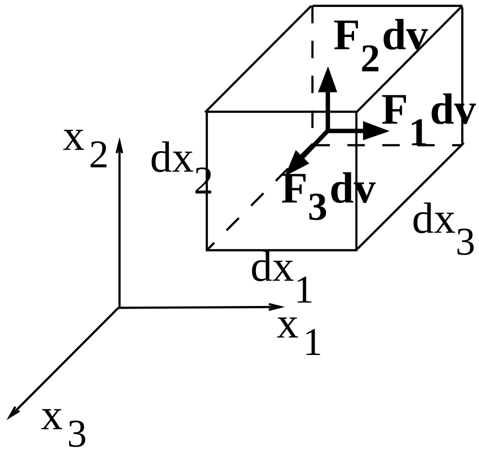

Figure 2.1: Body force $F_i dv$ ($dv = dx_1 dx_2 dx_3$, where dx_1, dx_2, dx_3 are lengths of the edges of the elements in x, y, and z-direction respectively) (see [6])

The force vector $F = [F_1, F_2, F_3]$ is called body force per unit volume. Examples of body forces that are due to the action at a distance, are gravitational forces and electromagnetic forces.

2. The forces acting on surface elements called stresses. Stresses can be defined with reference to an infinitesimal cube with faces pointing in the coordinate directions: σ_{ji} is the force in the x_j direction, per unit area, acting on a face of the cube whose normal points in the x_i direction (see [16]).

   Examples of stresses are aerodynamic pressure acting on a body and pressure due to mechanical contact of two bodies.

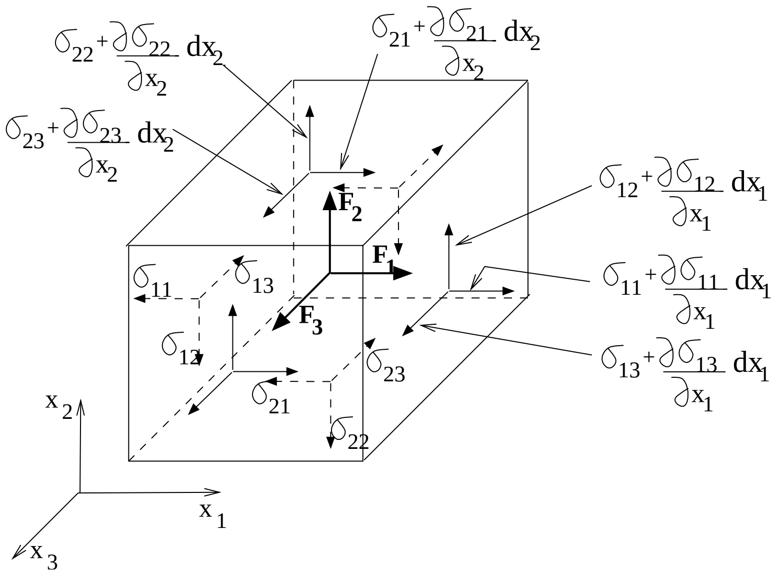

Figure 2.2: Stresses acting on surfaces and body force $F_i dx_1 dx_2 dx_3$

For example, as shown in Figure 2.2 (see [6]), the force $\sigma_{11} dx_2 dx_3$ acts on the left hand face of the cube, the force $(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1) dx_2 dx_3$ acts on the right hand face of the cube, and so forth.

Every stress component is a function of position, that is, σ_{11} is a function of (x_1, x_2, x_3). The value of the stress σ_{11} at a point slightly to the right of (x_1, x_2, x_3), namely (x_1+dx_1, x_2, x_3), is $\sigma_{11}(x_1+dx_1, x_2, x_3)$. Now, since σ_{11} is a continuously differentiable function of (x_1, x_2, x_3), then according to Taylor's theorem we have

$$\begin{aligned}\sigma_{11}(x_1 + dx_1, x_2, x_3) \quad &= \quad \sigma_{11}(x_1, x_2, x_3) \; + \; dx_1 \frac{\partial \sigma_{11}}{\partial x_1}(x_1, x_2, x_3) \\ &+ \; dx_1^2 \; \frac{1}{2} \; \frac{\partial^2 \sigma_{11}}{\partial x_1^2}(x_1 + \alpha dx_1, x_2, x_3), \qquad (2.1)\end{aligned}$$

where $0 \leq \alpha \leq 1$. The last term can be made arbitrarily small by choosing dx_1 sufficiently small.

Neglecting the last term in equation (2.1), we get

$$\sigma_{11}(x_1 + dx_1, x_2, x_3) \ = \sigma_{11}(x_1, x_2, x_3) \ + \ dx_1 \frac{\partial \sigma_{11}}{\partial x_1}(x_1, x_2, x_3).$$

Let us consider balance of forces in the x_1-direction:

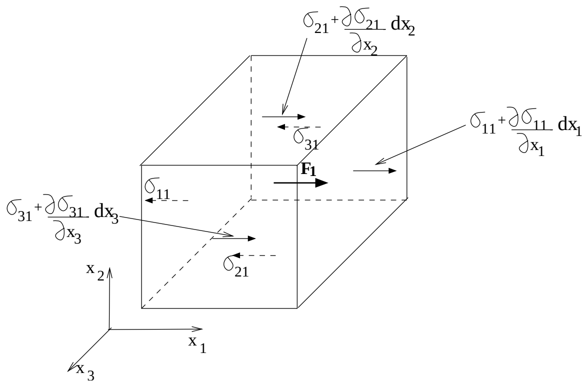

Figure 2.3: Stress components in x_1-direction

At equilibrium, the sum of forces on the body vanishes, then we have

$$(\sigma_{11} + \tfrac{\partial \sigma_{11}}{\partial x_1} dx_1) dx_2 dx_3 \ - \ \sigma_{11} dx_2 dx_3 \ + \ (\sigma_{21} + \tfrac{\partial \sigma_{21}}{\partial x_2} dx_2) dx_1 dx_3 \ - \ \sigma_{21} dx_3 dx_1$$
$$+ \ (\sigma_{31} + \tfrac{\partial \sigma_{31}}{\partial x_3} dx_3) dx_1 dx_2 - \ \sigma_{31} dx_1 dx_2 \ + \ F_1 dx_1 dx_2 dx_3 \ = 0.$$

Simplifying and dividing by $dx_1 dx_2 dx_3$, we obtain

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + F_1 \ = \ 0. \tag{2.2}$$

Repeating the same process (used in the x_1-direction), in x_2-direction and in x_3-direction, we get

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{32}}{\partial x_3} + F_2 \;=\; 0, \tag{2.3}$$

$$\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + F_3 \;=\; 0. \tag{2.4}$$

Equations (2.2)–(2.3) can be expressed in index notation as

$$\sum_{j=1}^{3} \frac{\partial\sigma_{ji}}{\partial x_j} \;+\; F_i \;=\; 0 \qquad i = 1, 2, 3. \tag{2.5}$$

Equation (2.5) expresses the force equilibrium where σ_{ji} is the total stress, per unit area (in the j-direction acting on the surface with normal in the i-direction) and F_i is the body force per unit volume.

The components σ_{11}, σ_{22}, and σ_{33} are the normal stresses, they are perpendicular to the face. The other three stresses are the shear stresses where the force is tangent to the face. Rotational equilibrium on all such infinitesimal elements of material requires that shear stresses be equal on adjoining faces, which is concisely expressed by requiring that $\sigma_{ji} = \sigma_{ij}$ for all i and j (see [18]) (the symmetry of the stress tensor).

2.1.2 Strain

Stresses cause solids to deform. The quantities describing deformations of the body are called strains (denoted by ϵ). Strains can be defined most simply in the case of extremely small deformations, in which case the coordinates directions x_1, x_2, and x_3 of material points are virtually the same before and after deformation. For normal strain in x_1-direction, let us consider two points A and B, a small distance dx_1 apart (see [6]),

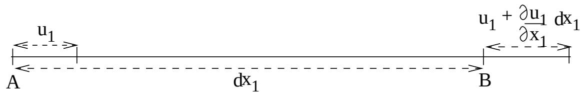

Figure 2.4: Normal strain in x_1-direction

and let u_1 be the x_1-displacement at A and $u_1 + \frac{\partial u_1}{\partial x_1} dx_1$ be the x_1-displacement at B. The unit displacement in the x_1-direction $\frac{\partial u_1}{\partial x_1}$ defines the normal strain denoted by ϵ_{11},

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}.$$

Similarly, the unit displacement in the x_2-direction and x_3-direction respectively are

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2},$$

and

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}.$$

The shear strains $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$ are the small changes of angle between the line segments in the x_1 and x_2-directions, x_1 and x_3-directions, and x_2 and x_3-directions respectively (see

[16]).

To illustrate this for the shear strain ϵ_{12}, consider the line segements AB and AC, initially making a right angle with dx_1 a small distance between A and B and dx_2 a small distance between A and C. After deformation the points are at A', B', and C' and the lines A'B' and A'C' no longer meet at a right angle at A'.

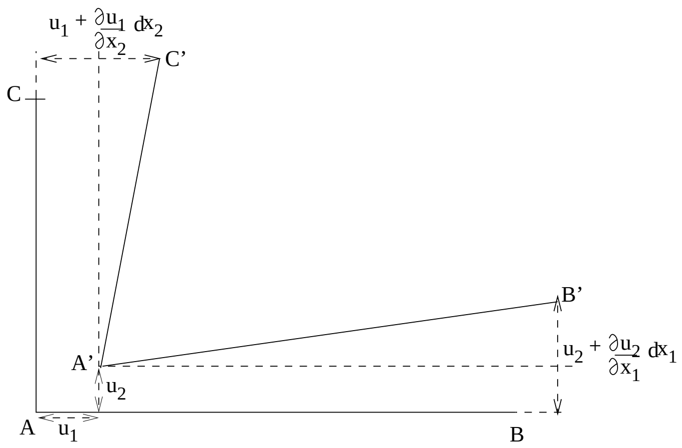

Figure 2.5: Shear strain in x_1, x_2-direction

The shear strain is defined as the average of the angles $\frac{\partial u_2}{\partial x_1}$ and $\frac{\partial u_1}{\partial x_2}$ that A'B' and A'C' make with the x_1 and x_2 directions respectively (see [6]),

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right).$$

Similarly,

$$\epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right),$$

and

$$\epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right).$$

The normal and shear strains can be compactly written using Einstein notation, in which repeated indices are summed, as follows

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i, j = 1, 2, 3. \tag{2.6}$$

Note that $\epsilon_{ij} = \epsilon_{ji}$ and that strains are dimensionless since they are the ratio of lengths.

2.1.3 Stress-strain relationship

The relationship between stress and strain can be derived using the following constitutive equation (see [18])

$$\text{trace}(\epsilon) = \frac{1}{3K}\text{trace}(\sigma) + \frac{1}{H}p, \tag{2.7}$$

where $\frac{1}{K}$ is the compressibility of the material (K is bulk modulus) measured under drained conditions. Drained conditions correspond to the deformation at fixed pressure p, with the fluid being allowed to flow in or out of the deforming element. The coefficient $\frac{1}{K}$ is obtained ($\frac{1}{K} = \frac{\delta\epsilon}{\delta\sigma}|_{p=0}$) by measuring the change in volumetric strain due to changes in applied stress while holding the pressure constant. The coefficient $\frac{1}{H}$ represents the poroelastic expansion coefficient. It describes how much the bulk volume changes due to a pore pressure change ($\frac{1}{H} = \frac{\delta\epsilon}{\delta p}|_{\sigma=0}$) while holding the applied stress constant (see [18]).

Equation (2.7), says that the fractional volume change is the result of change in applied stress and pore pressure.

In equation (2.7), $\frac{\text{trace}(\sigma)}{3}$ is the average of normal stresses and trace(ϵ) is the volumetric strain, i.e.,

$$\frac{\text{trace}(\sigma)}{3} = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3},$$

$$\text{trace}(\epsilon) = \epsilon_{11}+\epsilon_{22}+\epsilon_{33}.$$

Then

$$\epsilon_{11}+\epsilon_{22}+\epsilon_{33} = \frac{1}{K}\frac{(\sigma_{11}+\sigma_{22}+\sigma_{33})}{3}+\frac{p}{H}.$$

The coefficient K can be expressed in terms of Young's modulus E (see [18]) by

$$K = \frac{E}{3(1-2\nu)}.$$

Young's modulus is the measure of the stiffness of an elastic material and is defined as the ratio of the rate of change of stress with strain. The constant ν represents Poisson's ratio. When an elastic material is stretched or compressed in one direction, it deforms in perpendicular directions (becoming thicker or thinner), the measure of this deformation is given by the Poisson's ratio ν (see [16]).

From the above we get

$$\begin{aligned}\epsilon_{11}+\epsilon_{22}+\epsilon_{33} &= \frac{1}{E}(\sigma_{11}-\nu\sigma_{11}-\nu\sigma_{11})+\frac{1}{E}(\sigma_{22}-\nu\sigma_{22}-\nu\sigma_{22})\\ &\quad +\frac{1}{E}(\sigma_{33}-\nu\sigma_{33}-\nu\sigma_{33})+\frac{p}{H}.\end{aligned}$$

That is,

$$\begin{aligned}\epsilon_{11} &= \frac{1}{E}\sigma_{11}-\frac{\nu}{E}\sigma_{22}-\frac{\nu}{E}\sigma_{33}+\frac{p}{3H},\\ \epsilon_{22} &= -\frac{\nu}{E}\sigma_{11}+\frac{1}{E}\sigma_{22}-\frac{\nu}{E}\sigma_{33}+\frac{p}{3H},\end{aligned}$$

$$\epsilon_{33} = -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33} + \frac{p}{3H}.$$

This form is chosen (see [8]) to express the fact that one constant, $\frac{1}{E}$, connects strain and stress in the same direction. The other constant, $\frac{\nu}{E}$, relates strain and stress in two perpendicular directions.

We use the following expressions (see [18])

$$E = 2G(1+\nu) \qquad \text{and} \qquad \frac{1}{H} = \frac{\alpha}{K}.$$

The coefficient G is the shear modulus and is a quantity measuring the strength of the material defined as a ratio of shear stress to the shear strain. The positive constant α is the Biot-Willis coefficient, the ratio of volume of fluid that is added to storage and the change in bulk volume under the constraint that the pore pressure remains constant. Note that the constant fluid pressure condition means that the volume of fluid that goes into or out of storage is equal to the change in pore volume (see [16]).

Substituting the previous two expressions into the three normal strain equations, we obtain after simplification (using that $\frac{1}{1+\nu} = 1 - \frac{\nu}{1+\nu}$):

$$\epsilon_{11} = \frac{1}{2G}[\sigma_{11} - \frac{\nu}{1+\nu}\sigma_{kk}] + \frac{\alpha}{3K}p.$$

Similarly,

$$\epsilon_{22} = \frac{1}{2G}[\sigma_{22} - \frac{\nu}{1+\nu}\sigma_{kk}] + \frac{\alpha}{3K}p,$$

and

$$\epsilon_{33} = \frac{1}{2G}[\sigma_{33} - \frac{\nu}{1+\nu}\sigma_{kk}] + \frac{\alpha}{3K}p.$$

Because changes in pore pressure are assumed not to induce shear strain, the following shear strain and shear stress relationships are independent of pore pressure (see [18]).

$$\epsilon_{12} = \frac{1}{2G}\sigma_{12},$$

$$\epsilon_{23} = \frac{1}{2G}\sigma_{23},$$

and

$$\epsilon_{13} = \frac{1}{2G}\sigma_{13}.$$

Expressing the previous six strain-stress equations using Einstein summation convention, we get

$$\epsilon_{ij} = \frac{1}{2G}[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}] + \frac{\alpha}{3K}p\delta_{ij} \qquad i,j = 1,2,3 \tag{2.8}$$

where δ_{ij} is the Kronecker delta.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The above equation of strain in terms of stress and pore pressure may be inverted to solve for stress, that is,

$$\sigma_{ij} = 2G\epsilon_{ij} + \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij} - 2G\frac{\alpha}{3K}p\delta_{ij}. \tag{2.9}$$

We have

$$\epsilon_{11} = \frac{1}{2G}[\sigma_{11} - \frac{\nu}{1+\nu}\sigma_{kk}] + \frac{\alpha}{3K}p,$$

$$\epsilon_{22} = \frac{1}{2G}[\sigma_{22} - \frac{\nu}{1+\nu}\sigma_{kk}] + \frac{\alpha}{3K}p,$$

and

$$\epsilon_{33} = \frac{1}{2G}[\sigma_{33} - \frac{\nu}{1+\nu}\sigma_{kk}] + \frac{\alpha}{3K}p.$$

Adding these three equations yields

$$\epsilon_{kk} = \frac{1}{2G}\frac{(1-2\nu)}{(1+\nu)}\sigma_{kk} + \frac{\alpha}{K}p,$$

which implies that

$$\sigma_{kk} = 2G\frac{(1+\nu)}{(1-2\nu)}\epsilon_{kk} - 2G\frac{(1+\nu)}{(1-2\nu)}\frac{\alpha}{K}p.$$

Substituting σ_{kk} into (2.9) and simplifying, we get

$$\sigma_{ij} = 2G\epsilon_{ij} + 2G\frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij} - \alpha p\delta_{ij}. \quad (2.10)$$

Writing equation (2.10) explicitly for the normal stresses yields

$$\sigma_{11} = 2G\epsilon_{11} + 2G\frac{\nu}{1-2\nu}\epsilon_{kk} - \alpha p, \quad (2.11)$$

$$\sigma_{22} = 2G\epsilon_{22} + 2G\frac{\nu}{1-2\nu}\epsilon_{kk} - \alpha p, \quad (2.12)$$

$$\sigma_{33} = 2G\epsilon_{33} + 2G\frac{\nu}{1-2\nu}\epsilon_{kk} - \alpha p, \quad (2.13)$$

$$\sigma_{12} = 2G\epsilon_{12}, \quad (2.14)$$

$$\sigma_{13} = 2G\epsilon_{13}, \quad (2.15)$$

and

$$\sigma_{23} = 2G\epsilon_{23}. \quad (2.16)$$

We also have the force equilibrium equations

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + F_1 = 0, \tag{2.17}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + F_2 = 0, \tag{2.18}$$

and

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + F_3 = 0. \tag{2.19}$$

Substituting the six stress equations (2.11)–(2.16) into the force equilibrium equations (2.17)–(2.19), we obtain

$$2G\frac{\partial \epsilon_{11}}{\partial x_1} + 2G\frac{\nu}{(1-2\nu)}\frac{\partial \epsilon_{kk}}{\partial x_1} + 2G\frac{\partial \epsilon_{12}}{\partial x_2} + 2G\frac{\partial \epsilon_{13}}{\partial x_3} - \alpha\frac{\partial p}{\partial x_1} + F_1 = 0, \tag{2.20}$$

$$2G\frac{\partial \epsilon_{12}}{\partial x_1} + 2G\frac{\nu}{(1-2\nu)}\frac{\partial \epsilon_{kk}}{\partial x_2} + 2G\frac{\partial \epsilon_{22}}{\partial x_2} + 2G\frac{\partial \epsilon_{23}}{\partial x_3} - \alpha\frac{\partial p}{\partial x_2} + F_2 = 0, \tag{2.21}$$

and

$$2G\frac{\partial \epsilon_{13}}{\partial x_1} + 2G\frac{\nu}{(1-2\nu)}\frac{\partial \epsilon_{kk}}{\partial x_3} + 2G\frac{\partial \epsilon_{23}}{\partial x_2} + 2G\frac{\partial \epsilon_{33}}{\partial x_3} - \alpha\frac{\partial p}{\partial x_3} + F_3 = 0. \tag{2.22}$$

We write explicitly the strain in terms of the displacement

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \tag{2.23}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \tag{2.24}$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}, \tag{2.25}$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \tag{2.26}$$

$$\epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \tag{2.27}$$

and

$$\epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right), \tag{2.28}$$

where the displacements u_1, u_2, u_3 are the displacements in x_1, x_2, x_3 directions respectively. We first substitute equations (2.23)–(2.28) into equations (2.20)–(2.22) and simplify to get

$$G\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + \frac{G}{(1-2\nu)}\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) - \alpha\frac{\partial p}{\partial x_1} + F_1 = 0,$$

$$G\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + \frac{G}{(1-2\nu)}\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 x_3}{\partial x_2 \partial x_3}\right) - \alpha\frac{\partial p}{\partial x_2} + F_2 = 0,$$

and

$$G\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + \frac{G}{(1-2\nu)}\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_3^2}\right) - \alpha\frac{\partial p}{\partial x_3} + F_3 = 0.$$

These last three equations are expressed as

$$-G\triangle u_i - \frac{G}{(1-2\nu)}\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \alpha\frac{\partial p}{\partial x_i} = F_i \qquad i,k = 1,2,3.$$

This is just the conservation of momentum and can be written in vector form as

$$-G\triangle u - \frac{G}{(1-2\nu)}\nabla(\nabla\cdot u) + \alpha\nabla p = F. \tag{2.29}$$

An equivalent expression to the conservation of momentum equation (2.29) is

$$-G\nabla \cdot (\nabla u + (\nabla u)^T) - G\frac{2\nu}{(1-2\nu)}\nabla(\nabla \cdot u) + \alpha \nabla p \; = \; F. \tag{2.30}$$

Equation (2.30) is obtained by using

$$\triangle u \; = \; \nabla \cdot (\nabla u + \nabla u^T) - \nabla(\nabla \cdot u),$$

since $\nabla \cdot (\nabla u^T) = \nabla(\nabla \cdot u)$.

Then equation (2.29) becomes

$$-G\nabla \cdot (\nabla u + \nabla u^T) + G\nabla(\nabla \cdot u) - \frac{G}{(1-2\nu)}\nabla(\nabla \cdot u) + \alpha \nabla p \; = \; F.$$

By simplifying we get the equivalent equation for momentum conservation (2.30).

2.2 The pore pressure equation

The simplest mathematical description of the coupling of the pressure and deformation is the constitutive equation

$$\xi = \frac{\alpha}{3K}\text{trace}(\sigma) + \frac{1}{R}p, \tag{2.31}$$

(see [18]) where ξ is increment of fluid which is positive for fluid added to the control volume and negative for fluid withdrawn from the control volume. The coefficient $\frac{1}{K}$ is the compressibility of the material as defined previously. The coefficient $\frac{1}{R}$ $(\frac{\delta\xi}{\delta p}|_{\sigma=0})$ is the specific storage coefficient measured under conditions of constant applied stress.

The Skempton's coefficient $B = \frac{R}{H}$ is the ratio of the induced pore pressure to the change in applied stress for undrained condition, that is, no fluid is allowed to move into or out of the control volume (see [18]).

Using the Biot-Willis coefficient $\alpha = \frac{K}{H}$ and Skempton's coefficient $B = \frac{R}{H}$, we get $\frac{1}{R} = \frac{\alpha}{KB}$. Equation (2.31) can be expressed as

$$\xi = \frac{\alpha}{3K}\text{trace}(\sigma) + \frac{\alpha}{KB}p. \tag{2.32}$$

The average velocity, $v = \frac{q}{\phi}$, is interpreted as the relative velocity between the fluid and solid, that is,

$$v = \frac{1}{\phi}q = \nabla \cdot (U_f - U_s), \tag{2.33}$$

where U_f is the average displacement of the fluid, U_s is the average displacement of the solid, q is the fluid flux, and ϕ is the porosity (see [18]).

The increment of fluid is expressed by Biot and Willis (see [18]) (1957) in terms of U_f and U_s as

$$\xi = -\phi\nabla \cdot (U_f - U_s). \tag{2.34}$$

Taking derivative of equation (2.34) with respect to time and substituting equation (2.33) into it yields

$$\frac{\partial \xi}{\partial t} = -\nabla \cdot q.$$

Now, substituting q from Darcy's law: $q = -\frac{k}{\mu}\nabla p$ into this last equation (here k is the permeability of the rock and μ is the viscosity) gives

$$\frac{\partial \xi}{\partial t} = \nabla \cdot (\frac{k}{\mu}\nabla p).$$

Accounting for quantity of fluid from an external source Q, we have

$$\frac{\partial \xi}{\partial t} - \nabla \cdot (\frac{k}{\mu}\nabla p) = Q.$$

Finally, substituting equation (2.32) into the previous equation yields

$$\frac{\partial}{\partial t}\left[\frac{\alpha}{KB}p + \frac{\alpha}{3K}\sigma_{kk}\right] - \nabla \cdot (\frac{k}{\mu}\nabla p) = Q.$$

If in this last equation, displacement is chosen as the mechanical variable instead of mean normal stress then we get the general diffusion equation for Darcy flow

$$\frac{\partial}{\partial t}(Se\ p + \alpha\nabla \cdot u) - \nabla \cdot (\frac{k}{\mu}\nabla p) = Q. \tag{2.35}$$

Equation (2.35) was derived using the following steps:

- write the normal stress $\frac{\sigma_{kk}}{3}$ in terms of strains using equation (2.10),
- replace strain by displacement using equation (2.6),
- use $K = \frac{E}{3(1-2\nu)}$, $E = 2G(1+\nu)$, and the specific storage $Se = \frac{\alpha}{KB}$.

In summary, we derived the following system of partial differential equations:

$$\begin{aligned} -G\triangle u - \frac{G}{1-2\nu}\nabla(\nabla\cdot u) + \alpha\nabla p &= F, \qquad \text{in } \Omega\times(0,T), \\ \frac{\partial}{\partial t}(Se\ p + \alpha\nabla\cdot u) - \nabla\cdot(\frac{k}{\mu}\nabla p) &= Q, \qquad \text{in } \Omega\times(0,T). \end{aligned}$$

Equivalently,

$$\begin{aligned} -G\nabla\cdot(\nabla u + (\nabla u)^T) - G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u) + \alpha\nabla p &= F, \qquad \text{in } \Omega\times(0,T), \\ \frac{\partial}{\partial t}(Se\ p + \alpha\nabla\cdot u) - \nabla\cdot(\frac{k}{\mu}\nabla p) &= Q, \qquad \text{in } \Omega\times(0,T). \end{aligned}$$

The first equation represents the equilibrium equation for conservation of momentum and the second equation is the general diffusion equation.

2.3 Boundary and initial conditions

In this section, we discuss the boundary and initial conditions for the quasi-static poroelasticity system. Let Ω be a bounded open connected subset of $\mathbb{R}^3$ with Lipschitz boundary. Denote the boundary by $\Gamma = \partial\Omega$.

The boundary Γ is divided into two disjoint parts; the clamped boundary denoted by Γ_c with strictly positive measure and the traction boundary denoted by Γ_t. The boundary, Γ, can further be divided into drained boundary Γ_d and the flux boundary Γ_f.

Under certain geological conditions the boundary condition can belong to both Γ_t and Γ_f. Let us denote this boundary by Γ_{tf} ($\Gamma_{tf} = \Gamma_t \cap \Gamma_f$). The boundaries Γ_c and Γ_t correspond to the momentum equation, the boundaries Γ_d and Γ_f correspond to the fluid equation, and there is a coupling between the two equations on Γ_{tf}.

The initial boundary value problem (IBVP) becomes (see [15]):

$$
\begin{aligned}
-G\nabla\cdot(\nabla u + (\nabla u)^T) - G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u) + \alpha\nabla p &= F && \text{in } \Omega\times(0,T), && (2.36)\\
\frac{\partial}{\partial t}(Se\, p + \alpha\nabla\cdot u) - \nabla\cdot(\frac{k}{\mu}\nabla p) &= Q && \text{in } \Omega\times(0,T), && (2.37)\\
u &= u_c && \text{on } \Gamma_c\times(0,T), && (2.38)\\
[G(\nabla u + (\nabla u)^T) + G\frac{2\nu}{1-2\nu}\nabla\cdot uI]\hat{n} - \beta\alpha p\hat{n}\chi_{tf} &= \sigma_t && \text{on } \Gamma_t\times(0,T), && (2.39)\\
p &= p_d && \text{on } \Gamma_d\times(0,T), && (2.40)\\
-\frac{\partial}{\partial t}((1-\beta)\alpha u\cdot\hat{n})\chi_{tf} + \frac{k}{\mu}\nabla p\cdot\hat{n} &= h_1\chi_{tf} && \text{on } \Gamma_f\times(0,T), && (2.41)\\
Se\, p + \alpha\nabla\cdot u &= v_0 && \text{on } \Omega\times\{0\}, && (2.42)\\
(1-\beta)\alpha u\cdot\hat{n} &= v_1 && \text{on } \Gamma_{tf}\times\{0\}. && (2.43)
\end{aligned}
$$

Equations (2.36) and (2.37) are the partial differential equations for the quasi static poroelasticity system. Equation (2.36) represents the general force equilibrium equation and equation (2.37) is the general diffusion equation.

Boundary conditions (2.38) and (2.40) correspond to the clamped boundary Γ_c and the drained boundary Γ_d. The boundary conditions (2.39) and (2.41) consist of a balance forces on the traction boundary Γ_t and a balance of fluid mass on the flux boundary Γ_f. Motivated by the geological application and for simplicity, the boundary functions u_c, σ_t, and p_d are set equal to zero.

Here I is the identity tensor and $\hat{n}$ is the unit outward pointing normal vector on the boundary. The fraction $0 \leq \beta \leq 1$ defined on the boundary Γ_{tf}, the portion of the boundary which is neither clamped nor drained, denotes the surface fraction of the matrix pores which are sealed along Γ_{tf}. The remaining portion $(1-\beta)$ is exposed along the flux boundary Γ_f and contributes to the flux.

Here χ_{tf} denotes the characteristic function of Γ_{tf}, that is, $\chi_{tf} = 1$ on Γ_{tf} and 0 otherwise. The transverse flow on the flux boundary Γ_f is h_1. More specifically $h_1 = -(1-\beta)v(t) \cdot \hat{n}$, where $v(t)$ is the fluid velocity on the boundary Γ_f.

Finally, equations (2.42) and (2.43) represent the initial conditions where v_0 and v_1 are the given initial data.

In [14] Showalter showed that the system (2.36)-(2.43) has a unique strong solution under mild (smoothness) requirements on the data, these will be clarified later. He also proved existence and uniqueness of weak solutions for the system.

We will use a constructive approach to prove the existence and uniqueness of weak solutions for the system. Our approach immediately suggests a numerical algorithm which can be used to approximate solutions of the quasi-static poroelasticity system.

Chapter 3

Analysis

In the previous Chapter, we introduced the system of partial differential equations that consists of the equilibrium equation for momentum conservation and the diffusion equation for Darcy flows. We also discussed the boundary and initial conditions for the system. In this Chapter, we briefly recall existence and uniqueness of strong and weak solutions derived by Showalter in [14]. We give an alternative constructive proof of existence and uniqueness of weak solutions of the quasi-static poroelasticity system. We describe a discretization of the problem and derive error estimates.

3.1 Existence and uniqueness of solutions (Showalter)

We start this section by briefly recalling existence and uniqueness results for strong and weak solutions proved by Showalter (see [14]) for the system (3.1)–(3.8).

$$
\begin{aligned}
-G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= F && \text{in } \Omega\times(0,T), && (3.1)\\
\frac{\partial}{\partial t}(Se\, p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= Q && \text{in } \Omega\times(0,T), && (3.2)\\
u &= 0 && \text{on } \Gamma_c, && (3.3)\\
[G(\nabla u+(\nabla u)^T)+G\frac{2\nu}{1-2\nu}\nabla\cdot uI]\hat{n}-\beta\alpha p\hat{n}\chi_{tf} &= 0 && \text{on } \Gamma_t, && (3.4)\\
p &= 0 && \text{on } \Gamma_d, && (3.5)\\
-\frac{\partial}{\partial t}((1-\beta)\alpha u\cdot\hat{n})\chi_{tf}+\frac{k}{\mu}\nabla p\cdot\hat{n} &= h_1\chi_{tf} && \text{on } \Gamma_f, && (3.6)\\
Se\, p+\alpha\nabla\cdot u &= v_0 && \text{on } \Omega\times\{0\}, && (3.7)\\
(1-\beta)\alpha u\cdot\hat{n} &= v_1 && \text{on } \Gamma_{tf}\times\{0\}. && (3.8)
\end{aligned}
$$

Throughout the course of this work, we will use the following Hilbert spaces: $L^2(\Omega)$ which is the space of square integrable functions on Ω and $H^1(\Omega)$ which is the space of functions in $L^2(\Omega)$ whose first distribution derivatives are square integrable. The L^2 inner product and L^2 and H^1 norms are given by

$$(f,g)=\int_\Omega f(x)g(x)dx,$$

$$||f||_{L^2(\Omega)}=\left(\int_\Omega |f(x)|^2dx\right)^{\frac{1}{2}}=((f,f))^{\frac{1}{2}},$$

and

$$||f||_{H^1(\Omega)} = \left(||f||^2_{L^2(\Omega)} + ||\nabla f||^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

respectively.

The space $H^1_\Gamma(\Omega)$ is the closure of $\{v \in C^\infty(\Omega)^3 : v(x) = 0 \;\; for \;\; x \in \Gamma\}$ with respect to the $|| \cdot ||_1$-norm.

Notation: we denote the L^2 inner product by $(\cdot, \cdot)$, the L^2 norm by $|| \cdot ||$, and the H^1 norm by $|| \cdot ||_1$.

In addition to the above spaces, we will need the subspaces

$$V \;=\; \{v \in (H^1(\Omega))^3 : \quad v \;=\; 0 \qquad \text{on } \Gamma_c\}$$

and

$$M \;=\; \{q \in H^1(\Omega) : \quad q \;=\; 0 \qquad \text{on } \Gamma_d\}.$$

For the strong solution and in the special case where $Se \neq 0$, Showalter's theorem (see theorem 3.1 [14]) becomes

Theorem 3.1 *Let $T > 0$, $v_0 \in L^2(\Omega)$, $v_1 \in L^2(\Gamma_{tf})$, and the Holder continuous functions $F, Q \in C^\alpha([0,T], L^2(\Omega))$, $h_1(.) \in C^\alpha([0,T], L^2(\Gamma_{tf}))$ be given, then there exists a pair of functions $p : (0,T] \to M$ and $u : (0,T] \to V$ for which $(Sep + \nabla \cdot u) \in C^0([0,T], L^2(\Omega)) \cap C^1([0,T], L^2(\Omega))$ and $u \in C^0([0,T], V) \cap C^1([0,T], V)$. The system (3.1)–(3.8) is satisfied and the function u is unique. Furthermore, if the measure of Γ_c is strictly positive then p is unique.*

In [14] Showalter also proved existence and uniqueness of weak solutions to the system under weaker assumptions on the data compared to the strong solution. His results is given in the following theorem (see theorem 4.1 [14]).

Theorem 3.2 *Let $T > 0$, $v_0 \in M'$, and $Q \in C^{\alpha}([0,T], M')$ be given. Then there exists a unique pair of functions $p : (0,T] \to M$ and $u : (0,T] \to V$ for which the system (3.1)–(3.8) is satisfied in a weak sense. The function u is unique. Furthermore, if the measure of Γ_c is strictly positive then p is unique.*

3.2 Existence and uniqueness of weak solutions

We will prove existence and uniqueness of weak solutions of the quasi-static poroelasticity system (3.1)–(3.8). We will use Rothe's method of lines, and at each time step we will use Babuska-Brezzi theory to show that the elliptic system has a unique solution. We first derive a weak formulation of the following system of partial differential equations:

$$\begin{aligned} -G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= F, &\text{in } \Omega\times(0,T),\\ \frac{\partial}{\partial t}(Se\ p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= Q, &\text{in } \Omega\times(0,T). \end{aligned}$$

Let the Hilbert spaces V and M such that:

$$\begin{aligned} V &= \{v\in(H^1(\Omega))^3:\ v\ =\ 0\ \text{ on } \Gamma_c\},\\ M &= \{q\in H^1(\Omega):\ q\ =\ 0\ \text{ on } \Gamma_d\}. \end{aligned}$$

Multiply the previous two partial differential equations by the test functions $v\in V$ and $q\in M$ respectively and integrate over Ω, to get

$$\begin{aligned} \int_\Omega -G\nabla\cdot(\nabla u\ +\ (\nabla u)^T)v\ -\ \int_\Omega G\frac{2\nu}{1-2\nu}\nabla(\nabla\cdot u)v\ +\ \int_\Omega \alpha\nabla p\ v\ &=\ \int_\Omega F\ v \quad \forall\, v\in V,\\ \int_\Omega Sep_tq\ +\ \int_\Omega \alpha\nabla\cdot u_tq\ -\ \int_\Omega \frac{k}{\mu}\Delta p\ q\ &=\ \int_\Omega Q\ q \quad \forall\, q\in M. \end{aligned}$$

Applying Green's formula:

$$-\int_\Omega \nabla\cdot(\nabla u\ +\ (\nabla u)^T)v\ =\ \int_\Omega(\nabla u\ +\ (\nabla u)^T):\nabla v\ -\ \int_{\Gamma_t}(\nabla u\ +\ (\nabla u)^T)\cdot\hat{n}v, \quad \text{for } u,v\in V,$$

and

$$-\int_\Omega \nabla(\nabla\cdot u)v \;=\; \int_\Omega (\nabla\cdot u)(\nabla\cdot v) \;-\; \int_{\Gamma_t}(\nabla\cdot u)\hat{n}\cdot v, \quad \text{for } u,v\in V.$$

Apply Green's formula

$$-\int_\Omega \Delta p\cdot q \;=\; \int_\Omega \nabla p\cdot\nabla q - \int_{\Gamma_f}\nabla p\cdot\hat{n}q, \quad \text{for } p,q\in M.$$

Therefore, the system becomes:

$$\int_\Omega [G(\nabla u + (\nabla u)^T):\nabla v + G\frac{2\nu}{1-2\nu}(\nabla\cdot u)(\nabla\cdot v)] \;+\; \int_\Omega \alpha\nabla p\; v \;=$$

$$\int_\Omega F\; v \;+\; \int_{\Gamma_t} G(\nabla u \;+\; (\nabla u)^T)\cdot\hat{n}v \;+\; \int_{\Gamma_t} G\frac{2\nu}{1-2\nu}(\nabla\cdot u)\hat{n}\cdot v,$$

$$\int_\Omega Se\; p_t\; q \;+\; \int_\Omega \alpha(\nabla\cdot u)_t q \;+\; \int_\Omega \frac{k}{\mu}\nabla p\cdot\nabla q \;=\; \int_\Omega Q\; q + \int_{\Gamma_f}\frac{k}{\mu}\nabla p\cdot\hat{n}q.$$

Discretizing in time using θ-scheme, we get

$$\int_\Omega [G(\nabla u^{n+1} + (\nabla u^{n+1})^T):\nabla v + G\frac{2\nu}{1-2\nu}(\nabla\cdot u^{n+1})(\nabla\cdot v)] \;+\; \int_\Omega \alpha\nabla p^{n+1}\; v \;=$$

$$\int_\Omega F^{n+1}v \;+\; \int_{\Gamma_t} G(\nabla u^{n+1} \;+\; (\nabla u^{n+1})^T)\cdot\hat{n}v \;+\; \int_{\Gamma_t} G\frac{2\nu}{1-2\nu}(\nabla\cdot u^{n+1})\hat{n}\cdot v,$$

$$\int_\Omega Se\ \frac{p^{n+1}-p^n}{\tau}\ q\ +\ \int_\Omega \alpha\frac{\nabla\cdot u^{n+1}-\nabla\cdot u^n}{\tau}\cdot q\ +\ \int_\Omega \frac{k}{\mu}(\theta\nabla p^{n+1}+(1-\theta)\nabla p^n)\cdot\nabla q\ =$$

$$\int_\Omega(\theta Q^{n+1}+(1-\theta)Q^n)\ q+\int_{\Gamma_f}\frac{k}{\mu}(\theta\nabla p^{n+1}+(1-\theta)\nabla p^{n+1})\cdot\hat{n}q.$$

Here $0\leq\theta\leq 1$, the superscript n denotes the discrete time level at which the functions are evaluated, and τ is the time step. That is, $\tau=\frac{T}{N}$ where N is the number of time steps. Hence $u^n=u(t_n)$ where $t_n=n*\tau$.

Using the divergence theorem: $\int_\Omega\nabla\cdot uq\ =\ \int_{\Gamma_f}u\cdot\hat{n}q\ -\ \int_\Omega u\cdot\nabla q$, and rearranging the second equation of the previous system, we get

$$\int_\Omega[G(\nabla u^{n+1}+(\nabla u^{n+1})^T):\nabla v+G\frac{2\nu}{1-2\nu}(\nabla\cdot u^{n+1})(\nabla\cdot v)]+\int_\Omega\alpha\nabla p^{n+1}\ v\ =$$

$$\int_\Omega F^{n+1}v\ +\ \int_{\Gamma_t}G(\nabla u^{n+1}\ +\ (\nabla u^{n+1})^T)\cdot\hat{n}v\ +\ \int_{\Gamma_t}G\frac{2\nu}{1-2\nu}(\nabla\cdot u^{n+1})\hat{n}\cdot v,$$

$$-\int_\Omega\alpha u^{n+1}\cdot\nabla q\ +\ \int_\Omega[Se\ p^{n+1}q\ +\ \frac{k\tau}{\mu}\theta\nabla p^{n+1}\cdot\nabla q]\ =$$

$$\int_\Omega[\tau(\theta Q^{n+1}+(1-\theta)Q^n)\ +\ \alpha\nabla\cdot u^n\ +\ Se\ p^n]q\ -\ \int_\Omega\frac{k\tau}{\mu}(1-\theta)\nabla p^n\nabla q$$

$$-\int_{\Gamma_f}\alpha u^{n+1}\cdot\hat{n}q+\int_{\Gamma_f}\frac{k}{\mu}(\theta\nabla p^{n+1}+(1-\theta)\nabla p^{n+1})\cdot\hat{n}q.$$

We introduce the bilinear forms a, b, and c as follows:

$$
\begin{aligned}
a(u,v) &:= \int_\Omega [G(\nabla u \;+\; (\nabla u)^T) : \nabla v \;+\; G\frac{2\nu}{1-2\nu}(\nabla\cdot u)(\nabla\cdot v)],\\
b(v,p) &:= \int_\Omega \alpha\nabla p\; v,\\
c(p,q) &:= \int_\Omega [Se\; pq \;+\; \frac{k\tau}{\mu}\theta\nabla p\cdot\nabla q],\\
&\quad \textit{and}\\
l_1(F,g,v) &:= \int_\Omega Fv \;+\; \int_{\Gamma_t} G(\nabla g \;+\; (\nabla g)^T)\cdot\hat{n}v \;+\; \int_{\Gamma_t} 2G\frac{\nu}{1-2\nu}(\nabla\cdot g)\hat{n}\cdot v,\\
l_2(Q_1,r_1,s_1,q) &:= \int_\Omega [\tau(1-\theta)Q_1 \;+\; \alpha\nabla\cdot r_1 \;+\; Se\; s_1]q \;-\; \int_\Omega \frac{k\tau}{\mu}(1-\theta)\Delta\nabla s_1\nabla q,\\
l_3(Q_2,r_2,s_2,q) &:= \int_\Omega \tau\theta Q_2 q \;-\; \int_{\Gamma_f} \alpha r_2\cdot\hat{n}q \;+\; \int_{\Gamma_f}\frac{k}{\mu}(\theta\nabla s_2 + (1-\theta)\nabla s_2)\cdot\hat{n}q.
\end{aligned}
$$

The weak formulation of this problem is: find $(u^{n+1}, p^{n+1}) \in V \times M$ such that:

$$
\begin{aligned}
a(u^{n+1},v) \;+\; b(v,p^{n+1}) &= l_1(F^{n+1},u^{n+1},v) && \forall\; v\in V,\\
-b(u^{n+1},q) \;+\; c(p^{n+1},q) &= l_2(Q^n,u^n,p^n,q)\\
&\quad +\; l_3(Q^{n+1},u^{n+1},p^{n+1},q) && \forall\; q\in M.
\end{aligned}
$$

That is,

$$
\begin{aligned}
a(u^{n+1},v) \;+\; b(v,p^{n+1}) &= l_1(F^{n+1},u^{n+1},v) && \forall\; v\in V, && (3.9)\\
b(u^{n+1},q) \;-\; c(p^{n+1},q) &= -l_2(Q^n,u^n,p^n,q)\\
&\quad -\; l_3(Q^{n+1},u^{n+1},p^{n+1},q) && \forall\; q\in M. && (3.10)
\end{aligned}
$$

Using Babuska-Brezzi theory (see [3]), we will show that the system (3.9)–(3.10) has a unique solution.

Definition 1 *: The bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and the linear form $b(\cdot,\cdot) : V \times M \to \mathbb{R}$ are continuous provided that positive constants β and γ exist such that:*

$$|a(u,v)| \leq \beta ||u||_V ||v||_V \qquad \forall\ u, v \in V.$$

and

$$|b(u,v)| \leq \gamma ||u||_V ||v||_M \qquad \forall u \in V,\ \forall v \in M.$$

Definition 2 *: The bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is coercive (or V-elliptic) provided that a positive constant α exists such that:*

$$a(v,v) \geq \alpha ||v||_V^2 \qquad \forall\ \ v \in V.$$

Theorem 3.3 *: If the bilinear forms $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is continuous and coercive, $c(\cdot,\cdot) : M \times M \to \mathbb{R}$ is continuous and coercive, and $b(\cdot,\cdot) : V \times M \to \mathbb{R}$ is continuous then for every $f \in V'$ and $g \in M'$*

$$\begin{aligned} a(u,v) \ + \ b(v,p) \ &= \ (f,v) \\ b(u,q) \ - \ c(p,q) \ &= \ (g,q) \end{aligned}$$

has a unique solution (u,p).

We now show that the bilinear forms $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are continuous and coercive and the bilinear form $b(\cdot,\cdot)$ is continuous on the respective spaces, hence there exist unique solutions u and p of the semi-discrete problem.

Recall that:

$$\begin{aligned}\nabla u : \nabla v &= \sum_{i,j} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial v_i}{\partial x_j}, \\ \nabla \cdot u &= \sum_i \frac{\partial u_i}{\partial x_i},\end{aligned}$$

and

$$\begin{aligned} V &= \{v \in (H^1(\Omega))^3 : \; v \; = \; 0 \;\; \text{on } \Gamma_c\}, \\ M &= \{q \in H^1(\Omega) : \; q \; = \; 0 \;\; \text{on} \;\; \Gamma_d\}.\end{aligned}$$

Continuity and coercivity of $a(\cdot,\cdot)$

$$|a(u,v)| \; = \; \left| \int_\Omega G\nabla u : \nabla v \; + \; G\nabla u^T : \nabla v \; + \; G\frac{2\nu}{1-2\nu}(\nabla \cdot u)(\nabla \cdot v) \right|$$

Using the triangle inequality

$$|a(u,v)| \leq \left| \int_\Omega G \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_i}{\partial x_j}\right) \right| + \left| \int_\Omega G \sum_{i,j} \left(\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}\right) \right| + \left| \int_\Omega G \frac{2\nu}{1-2\nu} \sum_i \frac{\partial u_i}{\partial x_i} \sum_i \frac{\partial v_i}{\partial x_i} \right|$$

The inner product

$$\begin{aligned}(\nabla u, \nabla v) &= \int_\Omega \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_i}{\partial x_j}\right) \\ &\leq \int_\Omega |\nabla u||\nabla v| \\ &\leq ||\nabla u|| \; ||\nabla v|| \qquad \text{(By Hölder's inequality (Appendix A))} \\ &\leq ||u||_1 ||v||_1 \qquad \text{(Since } ||u||_1 = ||\nabla u|| + ||u|| \text{ so } ||\nabla u|| \leq ||u||_1).\end{aligned}$$

Thus

$$| a(u,v) | \leq \; G||\nabla u|| \; ||\nabla v|| \; + \; G||\nabla u|| \; ||\nabla v|| \; + \; G\frac{2\nu}{1-2\nu}||\nabla \cdot u|| \; ||\nabla \cdot v||.$$

Since $||\nabla \cdot u|| \leq \sqrt{3}||\nabla u||$ (is shown below),

$$| a(u,v) | \leq \; 2G||\nabla u|| \; ||\nabla v|| \; + \; \frac{6G\nu}{1-2\nu}||\nabla u|| \; ||\nabla v||,$$

hence,

$$| a(u,v) | \leq \max\left(2G, \frac{6G\nu}{1-2\nu}\right) \quad || \, u \, ||_1 || \, v \, ||_1 \qquad \forall \; u, v \in (H^1(\Omega))^3. \tag{3.11}$$

Hence $a(u,v)$ is continuous.

The inequality $||\nabla \cdot u|| \leq \sqrt{3}||\nabla u||$

We have

$$\nabla \cdot u \;=\; \sum_i \frac{\partial u_i}{\partial x_i},$$

thus,

$$(\nabla \cdot u)(\nabla \cdot v) \;=\; \left(\frac{\partial u_1}{\partial x_1} \;+\; \frac{\partial u_2}{\partial x_2} \;+\; \frac{\partial u_3}{\partial x_3}\right)\left(\frac{\partial v_1}{\partial x_1} \;+\; \frac{\partial v_2}{\partial x_2} \;+\; \frac{\partial v_3}{\partial x_3}\right),$$

$$\begin{aligned}(\nabla \cdot u)(\nabla \cdot v) \;=\; & \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_1}{\partial x_1}\frac{\partial v_3}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial v_3}{\partial x_3} \\ & + \; \frac{\partial u_3}{\partial x_3}\frac{\partial v_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\frac{\partial v_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial v_3}{\partial x_3}.\end{aligned}$$

Using the fact that $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$

$$(\nabla\cdot u)(\nabla\cdot v) \;\leq\; \frac{3}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \frac{3}{2}\left(\frac{\partial u_2}{\partial x_2}\right)^2 + \frac{3}{2}\left(\frac{\partial u_3}{\partial x_3}\right)^2 + \frac{3}{2}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \frac{3}{2}\left(\frac{\partial v_2}{\partial x_2}\right)^2 + \frac{3}{2}\left(\frac{\partial v_3}{\partial x_3}\right)^2.$$

Therefore,

$$(\nabla \cdot u)(\nabla \cdot v) \;\leq\; \frac{3}{2}\sum_i \left(\frac{\partial u_i}{\partial x_i}\right)^2 \;+\; \frac{3}{2}\sum_i \left(\frac{\partial v_i}{\partial x_i}\right)^2.$$

That is,

$$(\nabla \cdot u)(\nabla \cdot u) \;\leq\; 3\sum_i \left(\frac{\partial u_i}{\partial x_i}\right)^2,$$

thus,

$$||\nabla \cdot u||^2 \leq 3\int_\Omega \sum_i \left(\frac{\partial u_i}{\partial x_i}\right)^2 \leq 3||\nabla u||^2.$$

Hence

$$||\nabla \cdot u|| \;\leq \sqrt{3}||\nabla u||. \tag{3.12}$$

Coercivity of $a(\cdot,\cdot)$,

$$a(u,v) \;=\; \int_\Omega G(\nabla u \;+\; \nabla u^T) : \nabla v \;+\; G\frac{2\nu}{1-2\nu}(\nabla \cdot u)(\nabla \cdot v),$$

and

$$\begin{aligned}(\nabla u \;+\; \nabla u^T) : \nabla v \;&=\; \nabla u : \nabla v \;+\; \nabla u^T : \nabla v \\ &=\; \frac{1}{2}[\nabla u : \nabla v \;+\; \nabla u^T : \nabla v^T \;+\; \nabla u^T : \nabla v \;+\; \nabla u : \nabla v^T] \\ &=\; \frac{1}{2}(\nabla u \;+\; \nabla u^T) : (\nabla v \;+\; \nabla v^T). \end{aligned} \tag{3.13}$$

Note that the strain

$$\epsilon(u) \;=\; \frac{1}{2}(\nabla u \;+\; \nabla u^T),$$

and

$$a(u,v) \;=\; \int_\Omega 2G(\frac{1}{2}(\nabla u \;+\; \nabla u^T))(\frac{1}{2}(\nabla v \;+\; \nabla v^T)) + G\frac{2\nu}{1-2\nu}(\nabla\cdot u)(\nabla\cdot v),$$

where again we use Einsteins' summation convention.

Therefore,

$$a(v,v) \;\geq\; \int_\Omega 2G(\epsilon(v))^2.$$

Korn's inequality (see [2]): Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with piecewise smooth boundary. In addition, suppose $\Gamma_0 \subset \partial\Omega$ has positive two-dimentional measure. Then there exists a positive constant $c = c(\Omega, \Gamma_0)$ such that:

$$\int_\Omega \epsilon(v) : \epsilon(v) \geq c||v||_1^2 \quad \text{for all} \;\; v \in H^1_\Gamma(\Omega).$$

Since we assumed that the measure of the clamped boundary Γ_c is positive, then from Korn's inequality $a(\cdot,\cdot)$ is coercive.

$$a(v,v) \;\geq\; 2G\, ||\, v\, ||_1^2 \qquad \forall\, v \in (H^1(\Omega))^3, \tag{3.14}$$

where $G > 0$ is the shear modulus.

Furthermore, the bilinear form $a(.,.)$ is symmetric. Recall that

$$a(u,v) \;=\; \int_\Omega G(\nabla u \;+\; \nabla u^T) : \nabla v \;+\; G\frac{2\nu}{1-2\nu}(\nabla\cdot u)(\nabla\cdot v),$$

and from (3.13)

$$(\nabla u \ + \ \nabla u^T) : \nabla v \ = \ \frac{1}{2}(\nabla u \ + \ \nabla u^T) : (\nabla v \ + \ \nabla v^T).$$

Therefore,

$$\begin{aligned}
a(u,v) &= \int_\Omega \frac{1}{2}G(\nabla u \ + \ \nabla u^T) : (\nabla v \ + \ \nabla v^T) \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot u)(\nabla \cdot v) \\
&= \int_\Omega \frac{1}{2}G(\nabla u : \nabla v + \nabla u : \nabla v^T + \nabla u^T : \nabla v + \nabla u^T : \nabla v^T) \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot v)(\nabla \cdot u) \\
&= \int_\Omega \frac{1}{2}G(\nabla v : \nabla u + \nabla v^T : \nabla u + \nabla v : \nabla u^T + \nabla v^T : \nabla u^T) \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot v)(\nabla \cdot u) \\
&= \int_\Omega \frac{1}{2}G(\nabla v \ + \ \nabla v^T) : (\nabla u \ + \ \nabla u^T) \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot v)(\nabla \cdot u) \\
&= \int_\Omega G(\nabla v \ + \ \nabla v^T) : \nabla u \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot v)(\nabla \cdot u) \\
&= a(v,u). \qquad (3.15)
\end{aligned}$$

Continuity of $b(\cdot,\cdot)$

Recall that

$$b(v,p) \ := \ \int_\Omega \alpha(\nabla p)v,$$

hence,

$$\begin{aligned}
\mid b(v,p) \mid &\leq \alpha ||\nabla p|| \, ||v|| \\
&\leq \alpha \, || \, p \, ||_1 || \, v \, ||_1 \qquad \forall \, p \in H^1(\Omega) \ \text{ and } \ v \in (H^1(\Omega))^3, \qquad (3.16)
\end{aligned}$$

where the constant $\alpha > 0$ is the Biot-Willis coefficient.

Continuity and coercivity of $c(\cdot,\cdot)$

We have

$$c(p,q) \ := \ \int_\Omega Se\ p\ q + \frac{k\tau}{\mu}\theta(\nabla p)(\nabla q),$$

thus,

$$\begin{aligned} \mid c(p,q) \mid \ &\leq \ Se||p||\ ||q|| + \frac{k\tau}{\mu}\theta||\nabla p||\ ||\nabla q|| \\ &\leq \ \max(Se, \frac{k\tau}{\mu}\theta)\ ||\ p\ ||_1||\ q\ ||_1 \qquad \forall\ p, q \in H^1(\Omega). \end{aligned} \tag{3.17}$$

Thus c is continuous.

Now,

$$\begin{aligned} c(q,q) \ &= \ \int_\Omega Se\ q^2 \ + \ \frac{k\tau}{\mu}\theta(\nabla q)^2 \\ &\geq \ Se||q||^2 \ + \ \frac{k\tau}{\mu}\theta||\nabla q||^2 \\ &\geq \ \min(Se, \frac{k\tau}{\mu})\ ||\ q\ ||_1^2 \qquad \forall\ q \in H^1(\Omega). \end{aligned} \tag{3.18}$$

Thus c is coercive.

corollary 1 *: The bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is continuous. If the measure of the clamped boundary Γ_c is positive, then $a(\cdot,\cdot)$ is coercive. The bilinear forms $c(\cdot,\cdot) : M \times M \to \mathbb{R}$ is continuous and coercive, and $b(\cdot,\cdot) : V \times M \to \mathbb{R}$ is continuous. Hence for every $F \in V'$ and $Q \in M'$ the semi-discrete system (3.9)–(3.10) has a unique weak solution (u,p).*

3.3 Rothe's method of lines

Using the previous result (Corollary 1), we can now use Rothe's method to prove existence and uniqueness of weak solutions of the equations of poroelasticity:

$$
\begin{aligned}
-G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= F && \text{in } \Omega\times(0,T), && (3.19)\\
\frac{\partial}{\partial t}(Se\,p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= Q && \text{in } \Omega\times(0,T), && (3.20)\\
u &= 0 && \text{on } \Gamma_c, && (3.21)\\
[G(\nabla u+(\nabla u)^T)+G\frac{2\nu}{1-2\nu}\nabla\cdot uI]\hat{n}-\beta\alpha p\hat{n}\chi_{tf} &= 0 && \text{on } \Gamma_t, && (3.22)\\
p &= 0 && \text{on } \Gamma_d, && (3.23)\\
-\frac{\partial}{\partial t}((1-\beta)\alpha u\cdot\hat{n})\chi_{tf}+\frac{k}{\mu}\nabla p\cdot\hat{n} &= h_1\chi_{tf} && \text{on } \Gamma_f, && (3.24)\\
Se\,p+\alpha\nabla\cdot u &= v_0 && \text{on } \Omega\times\{0\}, && (3.25)\\
(1-\beta)\alpha u\cdot\hat{n} &= v_1 && \text{on } \Gamma_{tf}\times\{0\}. && (3.26)
\end{aligned}
$$

As defined above, Ω is a bounded open connected subset of $\mathbb{R}^3$ with Lipschitz boundary and T is a positive time. Given functions F, $Q\in C^{0,1}(0,T;L^2(\Omega))$, $h_1\in C^{0,1}(0,T;L^2(\Gamma_f))$, $v_0\in L^2(0,T;H^1(\Omega))$ and $v_1\in L^2(0,T;L^2(\Gamma_{tf}))$.

Since the problem (3.19)–(3.26) is linear its solution can be written as the sum of the solutions of the following three problems.

Probelem I:

$$
\begin{aligned}
-G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= F \qquad \text{in } \ \Omega\times(0,T), && (3.27)\\
\frac{\partial}{\partial t}(Se\ p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= Q \qquad \text{in } \ \Omega\times(0,T), && (3.28)\\
u &= 0 \qquad \text{on } \ \Gamma_c, && (3.29)\\
[G(\nabla u+(\nabla u)^T)+G\frac{2\nu}{1-2\nu}\nabla\cdot uI]\hat{n}-\beta\alpha p\hat{n}\chi_{tf} &= 0 \qquad \text{on } \ \Gamma_t, && (3.30)\\
p &= 0 \qquad \text{on } \ \Gamma_d, && (3.31)\\
-\frac{\partial}{\partial t}((1-\beta)\alpha u\cdot\hat{n})\chi_{tf}+\frac{k}{\mu}\nabla p\cdot\hat{n} &= 0 \qquad \text{on } \ \Gamma_f, && (3.32)\\
Se\ p+\alpha\nabla\cdot u &= 0 \qquad \text{on } \ \Omega\times\{0\}, && (3.33)\\
(1-\beta)\alpha u\cdot\hat{n} &= 0 \qquad \text{on } \ \Gamma_{tf}\times\{0\}, && (3.34)
\end{aligned}
$$

problem II:

$$
\begin{aligned}
-G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= 0 \qquad \text{in } \ \Omega\times(0,T), && (3.35)\\
\frac{\partial}{\partial t}(Se\ p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= 0 \qquad \text{in } \ \Omega\times(0,T), && (3.36)\\
u &= 0 \qquad \text{on } \ \Gamma_c, && (3.37)\\
[G(\nabla u+(\nabla u)^T)+G\frac{2\nu}{1-2\nu}\nabla\cdot uI]\hat{n}-\beta\alpha p\hat{n}\chi_{tf} &= 0 \qquad \text{on } \ \Gamma_t, && (3.38)\\
p &= 0 \qquad \text{on } \ \Gamma_d, && (3.39)\\
-\frac{\partial}{\partial t}((1-\beta)\alpha u\cdot\hat{n})\chi_{tf}+\frac{k}{\mu}\nabla p\cdot\hat{n} &= 0 \qquad \text{on } \ \Gamma_f, && (3.40)\\
Se\ p+\alpha\nabla\cdot u &= v_0 \qquad \text{on } \ \Omega\times\{0\}, && (3.41)\\
(1-\beta)\alpha u\cdot\hat{n} &= v_1 \qquad \text{on } \ \Gamma_{tf}\times\{0\}, && (3.42)
\end{aligned}
$$

and problem III:

$$
\begin{aligned}
-G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= 0 && \text{in } \Omega\times(0,T), && (3.43)\\
\frac{\partial}{\partial t}(Se\ p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= 0 && \text{in } \Omega\times(0,T), && (3.44)\\
u &= 0 && \text{on } \Gamma_c, && (3.45)\\
[G(\nabla u+(\nabla u)^T)+G\frac{2\nu}{1-2\nu}\nabla\cdot uI]\hat{n}-\beta\alpha p\hat{n}\chi_{tf} &= 0 && \text{on } \Gamma_t, && (3.46)\\
p &= 0 && \text{on } \Gamma_d, && (3.47)\\
-\frac{\partial}{\partial t}((1-\beta)\alpha u\cdot\hat{n})\chi_{tf}+\frac{k}{\mu}\nabla p\cdot\hat{n} &= h_1\chi_{tf} && \text{on } \Gamma_f, && (3.48)\\
Se\ p+\alpha\nabla\cdot u &= 0 && \text{on } \Omega\times\{0\}, && (3.49)\\
(1-\beta)\alpha u\cdot\hat{n} &= 0 && \text{on } \Gamma_{tf}\times\{0\}. && (3.50)
\end{aligned}
$$

3.3.1 Existence and uniqueness of weak solutions for homogeneous initial and boundary conditions

We first consider problem (3.27)–(3.34) which has homogeneous boundary and initial conditions.

$$
\begin{aligned}
-G\nabla\cdot(\nabla u+(\nabla u)^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u)+\alpha\nabla p &= F, && \text{in } \Omega\times(0,T), && (3.51)\\
\frac{\partial}{\partial t}(Se\ p+\alpha\nabla\cdot u)-\nabla\cdot(\frac{k}{\mu}\nabla p) &= Q, && \text{in } \Omega\times(0,T). && (3.52)
\end{aligned}
$$

Construct a mesh d_1 on the interval $I=[0,T]$; divide I into m subintervals $I_j:=[t_{j-1},t_j]$ each of length $h=\frac{T}{m}$ and $t_j=jh$, $j=1,...,m$.

Using finite difference backward time discretization, we get

$$
\begin{aligned}
-G\nabla\cdot(\nabla u_j + (\nabla u_j)^T) - G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot u_j) + \alpha\nabla p_j &= F_j, \qquad \text{in } \ \Omega\times(0,T),\\
\frac{Se}{h}(p_j - p_{j-1}) + \frac{\alpha}{h}(\nabla\cdot u_j - \nabla\cdot u_{j-1}) - \nabla\cdot\frac{k}{\mu}\nabla p_j &= Q_j, \qquad \text{in } \ \Omega\times(0,T).
\end{aligned}
$$

Let $w_j \in L^2(0,t;V)$ and $z_j \in L^2(0,t;M)$ be the approximates solutions of the system, i.e., $w_j = u_j$ and $z_j = p_j$, for $j = 1,...,m$. Here $w_j = w(t_j)$ and $z_j = z(t_j)$, so the system (in terms of w_j and z_j) is

$$
\begin{aligned}
-G\nabla\cdot(\nabla w_j + (\nabla w_j)^T) - G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot w_j) + \alpha\nabla z_j &= F_j, \qquad \text{in } \ \Omega\times(0,T),\\
\frac{Se}{h}(z_j - z_{j-1}) + \frac{\alpha}{h}(\nabla\cdot w_j - \nabla\cdot w_{j-1}) - \nabla\cdot\frac{k}{\mu}\nabla z_j &= Q_j, \qquad \text{in } \ \Omega\times(0,T).
\end{aligned}
$$

The weak formulation of this problem is: find $w_j \in L^2(0,t;V)$ and $z_j \in L^2(0,t;M)$ such that:

$$
\int_\Omega [G(\nabla w_j \ + \ (\nabla w_j)^T) : \nabla v \ + \ G\frac{2\nu}{1-2\nu}(\nabla\cdot w_j)(\nabla\cdot v)] \ + \ \int_\Omega \alpha\nabla z_j\cdot v \ =
$$

$$
\int_\Omega F\ v \ + \ \int_{\Gamma_t} [G(\nabla w_j \ + \ (\nabla w_j)^T) + G\frac{2\nu}{1-2\nu}\nabla\cdot w_j I]\hat{n}\cdot v, \qquad \forall v\in V,
$$

$$
\int_\Omega \frac{Se}{h}(z_j - z_{j-1})q + \int_\Omega \frac{\alpha}{h}(\nabla\cdot w_j - \nabla\cdot w_{j-1})q + \int_\Omega \frac{k}{\mu}\nabla z_j\cdot\nabla q \ =
$$

$$
\int_\Omega \ Q\ q + \int_{\Gamma_f} \frac{k}{\mu}\nabla z_j\cdot\hat{n}q, \qquad \forall q\in M.
$$

We have (using the divergence theorem $\int_\Omega z_j \ \nabla \cdot v \ = \ \int_{\Gamma_t} z_j \cdot \hat{n} v \ - \ \int_\Omega \nabla z_j \cdot v$)

$$
\begin{aligned}
\int_\Omega [G(\nabla w_j \ + \ (\nabla w_j)^T) : \nabla v \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot w_j)(\nabla \cdot v)] \ - \ \int_\Omega \alpha z_j \nabla \cdot v \ &= \\
\int_\Omega F_j v \ + \ \int_{\Gamma_t} \beta \alpha z_j \hat{n} \chi_{tf} \cdot v \ - \ \int_{\Gamma_t} \alpha z_j \hat{n} \cdot v, & \qquad (3.53) \\
\int_\Omega \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}) q + \int_\Omega Se(z_j - z_{j-1}) q + h \int_\Omega \frac{k}{\mu} \nabla z_j \cdot \nabla q \ &= \\
h \int_\Omega \ Q_j q \ + \ h \int_{\Gamma_f} h_1 \chi_{tf} q \ + \ \int_{\Gamma_f} (1-\beta)\alpha(w_j - w_{j-1}) \chi_{tf} \cdot q. & \qquad (3.54)
\end{aligned}
$$

Let

$$
\begin{aligned}
a(w_j, v) \ &= \ \int_\Omega [G(\nabla w_j \ + \ (\nabla w_j)^T) : \nabla v \ + \ G\frac{2\nu}{1-2\nu}(\nabla \cdot w_j)(\nabla \cdot v)], \\
b(v, z_j) \ &= \ -\int_\Omega \alpha z_j \nabla \cdot v, \\
c(z_j, q) \ &= \ \int_\Omega [Se \ z_j q + \frac{k}{\mu} h \nabla z_j \cdot \nabla q].
\end{aligned}
$$

Hence the system (3.51)–(3.52) is in the form

$$
\begin{aligned}
a(w_j, v) \ + \ b(v, z_j) \ &= \ \int_\Omega F_j v \ - \ \int_{\Gamma_{tf}} (1-\beta)\alpha z_j \hat{n} v, \qquad (3.55) \\
b(w_j, q) \ - \ c(z_j, q) \ &= \ h \int_\Omega Q_j q \ + \ h \int_{\Gamma_{tf}} h_1 q \ + \ \int_{\Gamma_{tf}} (1-\beta)\alpha(w_j - w_{j-1}) q \\
& \quad + \ \int_\Omega (\alpha \nabla \cdot w_{j-1} + Se \ z_{j-1}) q, \qquad (3.56)
\end{aligned}
$$

It was shown in section 3.2 that the bilinear forms $a(.,.)$ and $c(.,.)$ are continuous and coercive and the linear form $b(.,.)$ is continuous. Therefore, from Corollary 1 (3.55)-(3.56) has a unique solution $(z_j, \ w_j) \in V \times M$.

The functions $z_j \in M$ and $w_j \in V$, $j = 1, ..., m$, are the approximates to the functions p and u.

Define the Rothe functions for the mesh d_1 (recall that the mesh d_1 corresponds to the division of the interval I into m subintervals I_j) by

$$\begin{aligned} p_1(x,t) &= z_{j-1} + \frac{z_j - z_{j-1}}{h}(t - t_{j-1}), \\ u_1(x,t) &= w_{j-1} + \frac{w_j - w_{j-1}}{h}(t - t_{j-1}), \end{aligned}$$

for all $t \in I_j = [t_{j-1}, t_j]$ and $j = 1, \dots, m$.

Instead of the mesh d_1 (division of the interval I into m subintervals I_j of lengths $h = \frac{T}{m}$), consider the mesh d_n, $n = 2, 3, \dots,$ which consists of $m2^{n-1}$ subintervals $I_j^n := [t_{j-1}^n, t_j^n]$, $j = 1, \dots, m2^{n-1}$, each of length $h_n = \frac{T}{m2^{n-1}}$. (Note that the superscript n corresponds to the mesh d_n).

The Rothe functions p_n and u_n which correspond to the mesh d_n are defined as follows

$$\begin{aligned} p_n(x,t) &= z_{j-1}^n + \frac{z_j^n - z_{j-1}^n}{h_n}(t - t_{j-1}^n), \\ u_n(x,t) &= w_{j-1}^n + \frac{w_j^n - w_{j-1}^n}{h_n}(t - t_{j-1}^n), \end{aligned}$$

for all $t \in I_j^n$, $j = 1, \dots, m2^{n-1}$.

We constructed the sequences $\{p_n(x,t)\}$ and $\{u_n(x,t)\}$, we will show that these sequences converge to the solution $p(x,t)$ and $u(x,t)$ of problem I.

Consider the system of equations (3.53)-(3.54)

$$\begin{aligned}
\int_\Omega [G(\nabla w_j \;+\; (\nabla w_j)^T) : \nabla v \;+\; G\frac{2\nu}{1-2\nu}(\nabla\cdot w_j)(\nabla\cdot v)] \;-\; \int_\Omega \alpha z_j \nabla\cdot v \;&=\\
\int_\Omega F_j v \;+\; \int_{\Gamma_t} \beta\alpha z_j \hat{n}\chi_{tf} v \;-\; \int_{\Gamma_t} \alpha z_j \hat{n} v&\\
\int_\Omega Se(z_j - z_{j-1})q + \int_\Omega \alpha(\nabla\cdot w_j - \nabla\cdot w_{j-1})q + h\int_\Omega \frac{k}{\mu}\nabla z_j\cdot\nabla q \;&=\\
h\int_\Omega Q_j q \;+\; h\int_{\Gamma_f} h_1\chi_{tf} q \;+\; \int_{\Gamma_f}(1-\beta)\alpha(w_j - w_{j-1})\chi_{tf} q&
\end{aligned}$$

We first consider the case that $F = 0$ and then return to the case of an arbitrary F. Recall the bilinear form

$$a(u,v) \;=\; G\left[\left(\nabla u + \nabla u^T : \nabla v \nabla v^T\right) + \frac{2\nu}{1-2\nu}\left(\nabla\cdot u, \nabla\cdot v\right)\right],$$

it induces a norm

$$[[u]] \;=\; \left[G|u|_1^2 + G|u^T|_1^2 + G\frac{2\nu}{1-2\nu}||\nabla\cdot u||_0^2\right]^{\frac{1}{2}}.$$

Thus

$$a(u,v) \;\le\; [[u]]\;[[v]] \qquad \text{and} \qquad a(u,u) \;=\; [[u]]^2.$$

Recall (3.11) and (3.14)

$$a(u,v) \;\le\; \max\left(2G, \frac{6G\nu}{1-2\nu}\right)||u||_1||v||_1 \qquad \text{the continuity of } a,$$

$$a(u,u) \;=\; [[u]]^2 \;\ge 2G||u||_1^2 \qquad \text{the coercivity of } a.$$

Let us denote the boundary integrals by $< \cdot, \cdot >$, that is,

$$< f, g >= \int_{\Gamma_{tf}} f\hat{n} \cdot g.$$

Then the system (3.51)-(3.52) (with $F = 0$) can be written at time j, ($j = 1, \ldots, m$) as

$$a(w_j, v) \;-\; \alpha(z_j, \nabla \cdot v) \;=\; -(1-\beta)\alpha < z_j, v > \tag{3.57}$$

$$Se(z_j - z_{j-1}, q) \;+\; \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q) \;+\; h\frac{k}{\mu}(\nabla z_j, \nabla q)$$

$$=\; h(Q_j, q) + \; (1-\beta)\alpha < w_j - w_{j-1}, q > \tag{3.58}$$

Recall that in problem I, we have homogeneous boundary conditions, i.e., $h_1(t) = 0$ and homogeneous initial conditions, i.e., $v_0 = 0$ and $v_1 = 0$.

Equation (3.57) can be written at time $(j-1)$ as

$$a(w_{j-1}, v) \;-\; \alpha(z_{j-1}, \nabla \cdot v) \;=\; -(1-\beta)\alpha < z_{j-1}, v > \tag{3.59}$$

Subtracting equation (3.59) from equation (3.57) and using equation (3.58), we get

$$a(w_j - w_{j-1}, v) \;-\; \alpha(z_j - z_{j-1}, \nabla \cdot v) \;=\; -(1-\beta)\alpha < z_j - z_{j-1}, v >, \tag{3.60}$$

$$Se(z_j - z_{j-1}, q) \;+\; \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q) \;+\; h\frac{k}{\mu}(\nabla z_j, \nabla q)$$

$$=\; h(Q_j, q) + \; (1-\beta)\alpha < w_j - w_{j-1}, q >, \tag{3.61}$$

for $j = 1, \ldots, m$. Assuming that the the system (3.51)-(3.52) holds at time zero.

Setting $j = 1$ in (3.60) with $v = w_1 - w_0$ (this is possible since $v \in V$), we obtain

$$a(w_1 - w_0, w_1 - w_0) \; - \; \alpha(z_1 - z_0, \nabla \cdot w_1 - \nabla \cdot w_0) \; = \; -(1-\beta)\alpha < z_1 - z_0, w_1 - w_0 >, \quad (3.62)$$

Substituting $q = z_1 - z_0$ and $q = z_0$ (again we can do this since $q \in M$) into (3.61) respectively, we get

$$\begin{aligned} Se(z_1 - z_0, z_1 - z_0) \; + \; \alpha(\nabla \cdot w_1 - \nabla \cdot w_0, z_1 - z_0) \; + \; h\frac{k}{\mu}(\nabla z_1, \nabla z_1 - \nabla z_0) \\ = h(Q_1, z_1 - z_0) \; + \; (1-\beta)\alpha < w_1 - w_0, z_1 - z_0 >, \qquad (3.63) \end{aligned}$$

and

$$\begin{aligned} Se(z_1 - z_0, z_0) \; + \; \alpha(\nabla \cdot w_1 - \nabla \cdot w_0, z_0) \; + \; h\frac{k}{\mu}(\nabla z_1, \nabla z_0) \\ = h(Q_1, z_0) \; + \; (1-\beta)\alpha < w_1 - w_0, z_0 > . \qquad (3.64) \end{aligned}$$

Using equation (3.59) with $v = w_1 - w_0$, we have

$$a(w_0, w_1 - w_0) \; - \; \alpha(z_0, \nabla \cdot w_1 - \nabla \cdot w_0) \; = \; -(1-\beta)\alpha < w_1 - w_0, z_0 > . \qquad (3.65)$$

Adding (3.62)–(3.65) and simplifying, we get

$$[[w_1 - w_0]]^2 \; + \; Se||z_1 - z_0||^2 \; + \; a(w_0, w_1 - w_0) \; + \; Se(z_1 - z_0, z_0) \; + \; h\frac{k}{\mu}||\nabla z_1||^2 \; = \; h(Q_1, z_1).$$

Since $h\frac{k}{\mu}||\nabla z_1||^2 \geq 0$,

$[[w_1 - w_0]]^2 \ + \ Se||z_1 - z_0||^2 \ \leq \ [[w_0]]\ [[w_1 - w_0]] \ + \ Se||z_1 - z_0||\ ||z_0|| \ + \ h||Q_1||\ ||z_1||,$

hence,

$$[[w_1 - w_0]]^2 + Se||z_1 - z_0||^2 \ \leq \ \Big([[w_1 - w_0]]^2 + Se||z_1 - z_0||^2\Big)^{\frac{1}{2}}\Big([[w_0]]^2 + Se||z_0||^2\Big)^{\frac{1}{2}} + h||Q_1||\ ||z_1||. \tag{3.66}$$

Taking (3.57)–(3.58) with $j = 1$ and $v = w_1$, $q = z_1$ we get

$a(w_1, w_1) \ - \ \alpha(z_1, \nabla\cdot w_1) \ = \ -(1-\beta)\alpha < z_1, w_1 >,$

$Se(z_1 - z_0, z_1) \ + \ \alpha(\nabla\cdot w_1 - \nabla\cdot w_0, z_1) \ + \ h\frac{k}{\mu}(\nabla z_1, \nabla z_1)$

$\qquad = \ h(Q_1, z_1) \ + \ (1-\beta)\alpha < w_1 - w_0, z_1 > .$

Adding these two equations, we have

$a(w_1, w_1) \ - \ \alpha(\nabla\cdot w_0, z_1) \ - \ Se(z_0, z_1) \ + \ Se(z_1, z_1) \ + \ h\frac{k}{\mu}(\nabla z_1, \nabla z_1)$

$\qquad = \ h(Q_1, z_1) - (1-\beta)\alpha < w_0, z_1 >,$

that is,

$a(w_1, w_1) \ - \ (Se\ z_0 + \alpha\nabla\cdot w_0, z_1) \ + \ Se(z_1, z_1) \ + \ h\frac{k}{\mu}(\nabla z_1, \nabla z_1)$

$\qquad = \ h(Q_1, z_1) - < (1-\beta)\alpha w_0, z_1 > .$

Using the initial conditions $Se\ z_0 + \alpha\nabla\cdot w_0 = 0$ and $(1-\beta)\alpha w_0\cdot\hat{n} = 0$, we get

$$[[w_1]]^2 \ + \ Se||z_1||^2 \ \leq \ h||Q_1||\ ||z_1||,$$

which implies

$$||z_1|| \ \leq \ h\frac{||Q_1||}{Se}. \tag{3.67}$$

The initial conditions $Se\ z_0 + \alpha\nabla \cdot w_0 = v_0 = 0$ obviously implies that

$$- \alpha(\nabla \cdot w_0, z_0) \ = \ Se(z_0, z_0). \tag{3.68}$$

From (3.59) with $j = 1$ and $v = w_0$, we have

$$a(w_0, w_0) \ - \ \alpha(z_0, \nabla \cdot w_0) \ = \ -(1-\beta)\alpha < z_0, w_0 > .$$

Using (3.68) and homogeneous initial condition ($(1-\beta)\alpha w_0 \cdot \hat{n} = 0$) implies that

$$a(w_0, w_0) \ + \ Se(z_0, z_0) \ = \ 0.$$

Therefore,

$$[[w_0]]^2 \ + \ Se||z_0||^2 \ = \ 0, \tag{3.69}$$

since each term on the left hand side of equation (3.69) is positive

$$w_0 \ = \ z_0 \ = \ 0. \tag{3.70}$$

Hence (3.66) becomes

$$[[w_1 - w_0]]^2 \ + \ Se||z_1 - z_0||^2 \ \leq \ h||Q_1|| \ ||z_1||,$$

and using (3.67)

$$[[w_1 - w_0]]^2 \ + \ Se||z_1 - z_0||^2 \ \leq \ h^2 \frac{||Q_1||^2}{Se}. \tag{3.71}$$

At time j, $(j = 2, \ldots, m)$,

$$a(w_j, v) \; - \; \alpha(z_j, \nabla \cdot v) = -(1-\beta)\alpha < z_j, v >, \tag{3.72}$$

$$Se(z_j - z_{j-1}, q) + \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q) + h\frac{k}{\mu}(\nabla z_j, \nabla q)$$
$$= h(Q_j, q) \; + \; (1-\beta)\alpha < w_j - w_{j-1}, q >, \tag{3.73}$$

and at time $(j-1)$, $(j = 2, \ldots, m)$,

$$a(w_{j-1}, v) \; - \; \alpha(z_{j-1}, \nabla \cdot v) = -(1-\beta)\alpha < z_{j-1}, v >, \tag{3.74}$$

$$Se(z_{j-1} - z_{j-2}, q) \; + \; \alpha(\nabla \cdot w_{j-1} - \nabla \cdot w_{j-2}, q) \; + \; h\frac{k}{\mu}(\nabla z_{j-1}, \nabla q)$$
$$= h(Q_{j-1}, q) \; + \; (1-\beta)\alpha < w_{j-1} - w_{j-2}, q > . \tag{3.75}$$

Subtract (3.74) from (3.72) and (3.75) from (3.73) to get

$$a(w_j - w_{j-1}, v) \; - \; \alpha(z_j - z_{j-1}, \nabla \cdot v) = -(1-\beta)\alpha < z_j - z_{j-1}, v >, \tag{3.76}$$

$$Se(z_j - z_{j-1}, q) - Se(z_{j-1} - z_{j-2}, q) + \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q)$$
$$-\alpha(\nabla \cdot w_{j-1} - \nabla \cdot w_{j-2}, q) + h\frac{k}{\mu}(\nabla z_j - \nabla z_{j-1}, \nabla q)$$
$$= h(Q_j, q) - h(Q_{j-1}, q) + (1-\beta)\alpha < w_j - w_{j-1}, q >$$
$$-(1-\beta)\alpha < w_{j-1} - w_{j-2}, q > . \tag{3.77}$$

Adding equations (3.76) and (3.77) with $v = w_j - w_{j-1}$, $q = z_j - z_{j-1}$, subtracting the following equation ((3.76) with $v = w_{j-1} - w_{j-2}$)

$a(w_j - w_{j-1}, w_{j-1} - w_{j-2}) \; - \; \alpha(z_j - z_{j-1}, \nabla \cdot w_{j-1} - \nabla \cdot w_{j-2}) =$

$- \, (1-\beta)\alpha < z_j - z_{j-1}, w_{j-1} - w_{j-2} >,$

and simplifying, we obtain

$$\begin{aligned}[[w_j - w_{j-1}]]^2 \;&+\; Se||z_j - z_{j-1}||^2 \\ &\le\; [[w_j - w_{j-1}]]\,[[w_{j-1} - w_{j-2}]] \;+\; Se||z_j - z_{j-1}||\,||z_{j-1} - z_{j-2}|| \\ &\qquad + h\left(||Q_j|| - ||Q_{j-1}||\right)||z_j - z_{j-1}|| \\ &\le \Big[[[w_j - w_{j-1}]]^2 \;+\; Se||z_j - z_{j-1}||^2\Big]^{\frac{1}{2}} \\ &\quad \Big[[[w_{j-1} - w_{j-2}]]^2 \;+\; Se\Big(||z_{j-1} - z_{j-2}|| \;+\; h\tfrac{||Q_j|| \;-\; ||Q_{j-1}||}{Se}\Big)^2\Big]^{\frac{1}{2}}.\end{aligned}$$

Squaring both sides and simplifying, we get

$$\begin{aligned}[[w_j - w_{j-1}]]^2 \;&+\; Se||z_j - z_{j-1}||^2 \\ &\le\; [[w_{j-1} - w_{j-2}]]^2 \;+\; Se\left(||z_{j-1} - z_{j-2}|| \;+\; h\tfrac{||Q_j|| \;-\; ||Q_{j-1}||}{Se}\right)^2\end{aligned}$$

$$\begin{aligned}[[w_j - w_{j-1}]]^2 + Se||z_j - z_{j-1}||^2 \le\; &[[w_{j-1} - w_{j-2}]]^2 + Se||z_{j-1} - z_{j-2}||^2 \\ &+h^2\frac{\left(||Q_j|| - ||Q_{j-1}||\right)^2}{Se} \\ &+2h(||Q_j|| - ||Q_{j-1}||)||z_{j-1} - z_{j-2}||. \qquad (3.78)\end{aligned}$$

Recalling inequality (3.71)

$$[[w_1 - w_0]]^2 \;+\; Se||z_1 - z_0||^2 \;\le\; h^2\frac{||Q_1||^2}{Se},$$

and setting $j = 2$ in (3.78)

$$\begin{aligned}[[w_2 - w_1]]^2 \;&+\; Se||z_2 - z_1||^2 \\ &\le\; [[w_1 - w_0]]^2 + Se||z_1 - z_0||^2 + h^2\tfrac{\left(||Q_2||-||Q_1||\right)^2}{Se} \\ &\quad + 2h\big(||Q_2|| - ||Q_1||\big)||z_1 - z_0|| \\ &\le\; h^2\tfrac{||Q_1||^2}{Se} + h^2\tfrac{\left(||Q_2||-||Q_1||\right)^2}{Se} + 2h\big(||Q_2|| - ||Q_1||\big)||z_1 - z_0||,\end{aligned}$$

also from inequality (3.71): $||z_1 - z_0|| \;\leq\; h\frac{||Q_1||}{Se}$, hence,

$$\begin{aligned}
[[w_2 - w_1]]^2 \;&+\; Se||z_2 - z_1||^2 \\
&\leq\; h^2\tfrac{||Q_1||^2}{Se} \;+\; h^2\tfrac{\left(||Q_2|| - ||Q_1||\right)^2}{Se} \;+\; 2h^2\left(||Q_2|| - ||Q_1||\right)\tfrac{||Q_1||}{Se} \\
&=\; \tfrac{h^2}{Se}\Big(||Q_1|| \;+\; \left(||Q_2|| - ||Q_1||\right)\Big)^2 \\
&=\; \tfrac{h^2}{Se}||Q_2||^2.
\end{aligned}$$

Repeating the same process, we get for $j = 2, \ldots, m$

$$[[w_j - w_{j-1}]]^2 \;+\; Se||z_j - z_{j-1}||^2 \;\leq\; \frac{h^2}{Se}||Q_j||^2. \tag{3.79}$$

Let us now define the norm:

$$|||(w_j, z_j) - (w_{j-1}, z_{j-1})||| \;=\; \Big([[w_j - w_{j-1}]]^2 \;+\; Se||z_j - z_{j-1}||^2\Big)^{\frac{1}{2}}$$

So,

$$\begin{aligned}
|||(w_j, z_j) - (w_0, z_0)||| \;&=\; |||(w_j, z_j) - (w_{j-1}, z_{j-1}) + \;\cdots\; + (w_1, z_1) - (w_0, z_0)||| \\
&\leq\; |||(w_j, z_j) - (w_{j-1}, z_{j-1})||| + \;\cdots\; + |||(w_1, z_1) - (w_0, z_0)|||,
\end{aligned}$$

by the triangle inequality. Then using (3.71) and (3.79), we obtain

$$|||(w_j, z_j) - (w_0, z_0)||| \;\leq\; \frac{h}{\sqrt{Se}}\Big(||Q_1|| \;+\; ||Q_2|| \;+\; \cdots \;+\; ||Q_j||\Big). \tag{3.80}$$

Since $Q(t) \in C^{0,1}(0, T; L^2(\Omega))$, then for all t in I, there exists a constant d such that $||\frac{Q(t+h)-Q(t)}{h}|| \leq d$, for all $t,\; t + h \in I$, see (see [10]). Then $||Q(t)||$ is a continuous function on I and so $||Q(t)||$ attains a maximum on I, say $||Q||$, i.e.,

$$\max_{t \,\in\, I} ||Q(t)|| \;=\; ||Q||.$$

From (3.80),

$$|||(w_j, z_j) - (w_0, z_0)||| \;\leq\; jh\frac{||Q||}{\sqrt{Se}},$$

and

$$|||(w_j, z_j) - (w_0, z_0)|||^2 \;\leq\; j^2h^2\frac{||Q||^2}{Se}.$$

Therefore

$$[[w_j - w_0]]^2 \;+\; Se||z_j - z_0||^2 \;\leq\; j^2h^2\frac{||Q||^2}{Se}.$$

Using the fact that $z_0 = w_0 = 0$ (from (3.70))

$$||z_j|| \;\leq\; jh\frac{||Q||}{Se} \qquad \text{and} \qquad ||w_j||_1 \;\leq\; jh\frac{||Q||}{\sqrt{2GSe}}.$$

Since $h = \frac{T}{m}$,

$$||z_j|| \;\leq\; T\frac{||Q||}{Se} \qquad\qquad ||w_j||_1 \;\leq\; T\frac{||Q||}{\sqrt{2GSe}}. \tag{3.81}$$

The estimates in (3.81) are obviously independent of h, thus remain valid for an arbitrary mesh d_n. Thus for every positive integer n and $j = 1, \ldots, m2^{n-1}$, we have

$$||z_j^n|| \;\leq\; T\frac{||Q||}{Se}, \qquad\qquad ||w_j^n||_1 \;\leq\; T\frac{||Q||}{\sqrt{2GSe}}. \tag{3.82}$$

Let $Z_j = \frac{z_j - z_{j-1}}{h}$ and $W_j = \frac{w_j - w_{j-1}}{h}$, $j = 1, \cdots, m$. From (3.79),

$$[[w_j - w_{j-1}]]^2 \ + \ Se||z_j - z_{j-1}||^2 \ \leq \ h^2 \frac{||Q_j||^2}{Se},$$

we get,

$$[[W_j]]^2 \ + \ Se||Z_j||^2 \ \leq \ \frac{||Q_j||^2}{Se}.$$

Using (3.14), $[[u]]^2 > 2G||u||_1^2$, we have

$$2G||W_j||_1^2 \ + \ Se||Z_j||^2 \ \leq \ \frac{||Q_j||^2}{Se}.$$

Therefore,

$$||Z_j|| \leq \frac{||Q_j||}{Se} \leq \frac{||Q||}{Se} \quad \text{and} \quad ||W_j||_1 \leq \frac{||Q_j||}{\sqrt{2GSe}} \leq \frac{||Q||}{\sqrt{2GSe}}. \tag{3.83}$$

Again, the estimates in (3.83) are independent of h, and so remain valid for an arbitrary mesh d_n. Thus

$$||Z_j^n|| \leq \frac{||Q||}{Se} \quad \text{and} \quad ||W_j^n||_1 \leq \frac{||Q||}{\sqrt{2GSe}}. \tag{3.84}$$

From (3.82) and (3.84), we see that the norms (in $L^2(\Omega)$ and $H^1(\Omega)$) of the functions z_j^n, Z_j^n and w_j^n, W_j^n are uniformly bounded with respect to j and n, thus independently of the mesh d_n. Hence

$$||w_j^n||_1 \leq c_1, \qquad \forall j = 0, 1, \cdots, m2^{n-1}, \ n = 1, 2, \cdots, \tag{3.85}$$

$$||z_j^n|| \leq c_2, \qquad \forall j = 0, 1, \cdots, m2^{n-1}, \ n = 1, 2, \cdots. \tag{3.86}$$

We now examine the Rothe sequences $\{u_n(x,t)\}$ and $\{p_n(x,t)\}$ in the spaces $L^2(I,V)$ and $L^2\big(I, L^2(\Omega)\big)$ of abstract functions which are square integrable in the Bochner sense. See Appendix B for the definitions of abstract function, Bochner integral and square integrability in the Bochner sense.

The Rothe functions are

$$\begin{aligned} u_n(x,t) &= w_{j-1}^n \ + \ (t-t_{j-1}^n)\frac{w_j^n - w_{j-1}^n}{h_n}, \quad \text{in } \ I_j^n = [t_{j-1}^n, t_j^n], \\ p_n(x,t) &= z_{j-1}^n \ + \ (t-t_{j-1}^n)\frac{z_j^n - z_{j-1}^n}{h_n}, \quad \text{in } \ I_j^n = [t_{j-1}^n, t_j^n]. \end{aligned}$$

Since $0 \leq \frac{t-t_{j-1}^n}{h_n} \leq 1$ in I_j^n, for arbitrary $t \in I$,

$$\begin{aligned} ||u_n(t)||_V &= \left|\left|\left(1 - \frac{t-t_{j-1}^n}{h_n}\right) w_{j-1}^n + \frac{t-t_{j-1}^n}{h_n} w_j^n\right|\right|_V \\ &\leq \left|\left|\left(1 - \frac{t-t_{j-1}^n}{h_n}\right) w_{j-1}^n\right|\right|_V + \left|\left|\frac{t-t_{j-1}^n}{h_n} w_j^n\right|\right|_V \\ &\leq c_1 \left(1 - \frac{t-t_{j-1}^n}{h_n}\right) \ + \ c_1 \frac{t-t_{j-1}^n}{h_n} \\ ||u_n(t)||_V &\leq c_1. \end{aligned}$$

Similarly, we get that $||p_n(t)|| \leq c_2$.

Hence from (B.1),

$$||u_n(t)||_{L^2(I,V)}^2 = \int_0^T ||u_n(t)||_V^2 dt \leq c_1^2 T,$$

and

$$||p_n(t)||_{L^2(I,L^2(\Omega))}^2 = \int_0^T ||p_n(t)||^2 dt \leq c_2^2 T.$$

Therefore the Rothe sequences $\{u_n\}$ and $\{p_n\}$ are bounded in the spaces $L^2(I,V)$ and $L^2\big(I,L^2(\Omega)\big)$ respectively. Since these spaces are Hilbert spaces (see [10]), there exist subsequences $\{u_{n_k}\}$ and $\{p_{n_k}\}$ which converge weakly to abstract functions u in $L^2(I,V)$ and p in $L^2\big(I,L^2(\Omega)\big)$ respectively.

Let $W_j^n(t) = \frac{w_j^n(t)-w_{j-1}^n(t)}{h_n}$ and $Z_j^n(t) = \frac{z_j^n(t)-z_{j-1}^n(t)}{h_n}$, we have

$$u_n(t) \;=\; w_{j-1}^n \;+\; (t-t_{j-1}^n)W_j^n, \qquad \text{in } I_j^n, \tag{3.87}$$

$$p_n(t) \;=\; z_{j-1}^n \;+\; (t-t_{j-1}^n)Z_j^n, \qquad \text{in } I_j^n. \tag{3.88}$$

Define the abstract functions $U_n(t):\ I \to H^1(\Omega)$ and $P_n(t):\ I \to L^2(\Omega)$ by

$$\begin{aligned} U_n(0) &= W_1^n \\ U_n(t) &= W_j^n \qquad \text{for } t \in \tilde{I}_j^n := (t_{j-1}^n, t_j^n],\quad j = 1,\cdots,m2^{n-1}, \end{aligned} \tag{3.89}$$

and

$$\begin{aligned} P_n(0) &= Z_1^n \\ P_n(t) &= Z_j^n \qquad \text{for } t \in \tilde{I}_j^n =: (t_{j-1}^n, t_j^n],\quad j = 1,\cdots,m2^{n-1}. \end{aligned} \tag{3.90}$$

Since $||W_j^n||_1 \le \frac{||Q||}{\sqrt{2GSe}}$ and $||Z_j^n|| \le \frac{||Q||}{Se}$, the sequences $\{U_n\}$ and $\{P_n\}$ are bounded in the spaces $L^2\big(I,H^1(\Omega)\big)$ and $L^2\big(I,L^2(\Omega)\big)$ respectively. Since these spaces are Hilbert spaces, there exist subsequences $\{U_{n_k}\}$ and $\{P_{n_k}\}$ which converge weakly to U in $L^2\big(I,H^1(\Omega)\big)$ and to P in $L^2\big(I,L^2(\Omega)\big)$ respectively (see [10]).

Hence the integrals

$$\int_0^t U(\tau)d\tau = r(t) \qquad \text{and} \qquad \int_0^t P(\tau)d\tau = s(t) \tag{3.91}$$

exist. From (3.87), (3.88) and (3.89), (3.90), we get that

$$\int_0^t U_{n_k}(\tau)d\tau = u_{n_k}(t) \qquad \text{and} \qquad \int_0^t P_{n_k}(\tau)d\tau = p_{n_k}(t). \tag{3.92}$$

We now show that

$$r = u \quad \text{in} \quad L^2(I, H^1(\Omega)) \qquad \text{and} \qquad s = p \quad \text{in} \quad L^2(I, L^2(\Omega)). \tag{3.93}$$

It is sufficient to show that $u_{n_k} \rightharpoonup r$ in $L^2\big(I, H^1(\Omega)\big)$. We show that

$$\lim_{n_k\to\infty} \int_0^T \big(u_{n_k}(t), v(t)\big)dt \; - \; \int_0^T \big(r(t), v(t)\big)dt \; = \; 0 \quad \forall v \in L^2\big(I, H^1(\Omega)\big).$$

Let $v(t)$ be the constant v in $H^1(\Omega)$ for all t in I, then

$$\begin{aligned}
\lim_{n_k\to\infty} \int_0^T \big(u_{n_k}(t) - r(t), v\big) &= \lim_{n_k\to\infty} \Big(\int_0^T \big(U_{n_k}(\tau) - U(\tau)\big)d\tau, v\Big) \qquad \text{by (3.92) and (3.91)} \\
&= \lim_{n_k\to\infty} \int_0^T \Big(U_{n_k}(\tau) - U(\tau), v\Big)d\tau \\
&= 0 \qquad\qquad (\text{since } U_{n_k} \rightharpoonup U)
\end{aligned}$$

We can now apply the Lebesgue theorem that

$$0 = \lim_{n_k\to\infty} \int_0^T \Big(u_{n_k}(t) - r(t), v\Big)dt = \int_0^T \Big[\lim_{n_k\to\infty} \big(u_{n_k}(t) - r(t), v\big)\Big]dt.$$

implies

$$u_{n_k} \rightharpoonup r.$$

The same proof also applies for piecewise constant functions $v(t)$, $t \in I$ and so for every function $v \in L^2(I, H^1(\Omega))$ (since the piecewise constant functions are dense in $L^2(I, H^1(\Omega))$). Hence $r = u$.

In the same manner, it can be shown that $s = p$.

From (3.91) and (3.93), we get that $\int_0^t U(\tau)d\tau = u$ and $\int_0^t P(\tau)d\tau = p$, hence

$$u \in AC(I, H^1(\Omega)), \qquad p \in AC(I, L^2(\Omega))$$

and

$$u_t(t) = U(t), \qquad p_t(t) = P(t),$$

in $H^1(\Omega)$ and $L^2(\Omega)$, respectively, for almost all $t \in I$.

Since $u(t) = \int_0^t U(\tau)d\tau$ and $p(t) = \int_0^t P(\tau)d\tau$ then $u(0) = 0$ and $p(0) = 0$ in $C(I, H^1(\Omega))$ and $C(I, L^2(\Omega))$. Thus the initial conditions of the problem are satisfied. Since $u \in L^2(I, V)$ and $p \in L^2(I, L^2(\Omega))$, then for almost all $t \in I$, $u \in V$ and $p \in M$, which imply that the boundary conditions are satisfied in the sense of traces.

We now have to show that the functions u and p satisfy the system of partial differential equations.

Define sequences $\{\tilde{u}_{n_k}(t)\}$ and $\{\tilde{p}_{n_k}(t)\}$ by

$$\begin{aligned} \tilde{u}_{n_k}(0) &= w_1^n \\ \tilde{u}_{n_k}(t) &= w_j^n \qquad \text{for } t \in \tilde{I}_j^n = (t_{j-1}^n, t_j^n], \quad j = 1, \cdots, m2^{n-1}, \end{aligned}$$

and

$$
\begin{aligned}
\tilde{p}_{n_k}(0) &= z_1^n \\
\tilde{p}_{n_k}(t) &= z_j^n \qquad\qquad \text{for } t \in \tilde{I}_j^n = (t_{j-1}^n, t_j^n], \quad j = 1, \cdots, m2^{n-1},
\end{aligned}
$$

We show that if $u_{n_k} \rightharpoonup u$ in $L^2(I, V)$ and $p_{n_k} \rightharpoonup p$ in $L^2(I, L^2(\Omega))$, then $\tilde{u}_{n_k} \rightharpoonup u$ in $L^2(I, V)$ and $\tilde{p}_{n_k} \rightharpoonup p$ in $L^2(I, L^2(\Omega))$.

We show that

$$
\lim_{n_k \to \infty} \int_0^T \big(u(t) - \tilde{u}_{n_k}(t), v(t)\big)_V dt \;=\; 0 \quad \forall v \in L^2(I, V)
$$

and

$$
\lim_{n_k \to \infty} \int_0^T \big(p(t) - \tilde{p}_{n_k}(t), q(t)\big)_M dt \;=\; 0 \quad \forall q \in L^2(I, M)
$$

$$
\begin{aligned}
\lim_{n_k \to \infty} \int_0^T \big(u(t) - \tilde{u}_{n_k}(t), v(t)\big)_V dt &= \lim_{n_k \to \infty} \int_0^T \Big(u(t) - u_{n_k}(t) + u_{n_k}(t) - \tilde{u}_{n_k}(t), v(t)\Big)_V dt \\
&= \lim_{n_k \to \infty} \int_0^T \big(u(t) - u_{n_k}(t), v(t)\big)_V dt \\
&\quad + \lim_{n_k \to \infty} \int_0^T \big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v(t)\big)_V dt.
\end{aligned}
$$

The limit of the first term on the right hand side is zero since $u_{n_k} \rightharpoonup u$, then we have to show that the limit of the second term is equal to zero.

Let K be a set of abstract functions $v \in L^2(I, V)$ such that $v = g$ where $g \in V$ is a certain function on an interval $[\alpha, \beta] \subset I$ and $v = 0$ on $I \setminus [\alpha, \beta]$.

Assume that for sufficiently large n

$$\alpha = \tilde{\alpha} h_n, \qquad \beta = \tilde{\beta} h_n, \qquad \text{where } \ 0 \le \tilde{\alpha} \le \tilde{\beta} \le m2^{n-1}.$$

Let X be the set of all linear combinations of the functions from K. The set X is dense in $L^2(I, V)$.

To show that

$$\lim_{n_k \to \infty} \int_0^T \big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v(t)\big)_V dt = 0 \quad \forall v \in L^2(I, V),$$

it is sufficient to show that

$$\lim_{n_k \to \infty} \int_0^T \big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v(t)\big)_V dt = 0 \quad \forall v \in K,$$

since each of the functions from X is a linear combination of functions from K.

Fix a function $v(t)$ from K and assume that n_k is sufficiently large, so

$$\alpha = \tilde{\alpha} h_{n_k}, \qquad \beta = \tilde{\beta} h_{n_k}, \qquad \text{where } \ 0 \le \tilde{\alpha} \le \tilde{\beta} \le m2^{n_k-1}.$$

Then $\forall v \in K$

$$\begin{aligned} \int_0^T \big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v\big)_V dt &= \int_\alpha^\beta \big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v\big)_V dt \\ &= \int_{\tilde{\alpha} h_{n_k}}^{\tilde{\beta} h_{n_k}} \big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v\big)_V dt. \end{aligned}$$

Recall that

$$u_{n_k}(t) \; = \; w_{j-1}^{n_k} + (w_j^{n_k} - w_{j-1}^{n_k})\frac{t - t_{j-1}^{n_k}}{h_{n_k}}, \quad t \in \tilde{I}_j^{n_k} = (t_{j-1}^{n_k}, t_j^{n_k}],$$

and

$$\tilde{u}_{n_k}(t) \; = \; w_j^{n_k}, \quad t \in \tilde{I}_j^{n_k} = (t_{j-1}^{n_k}, t_j^{n_k}].$$

This implies that

$$\begin{aligned} u_{n_k}(t) - \tilde{u}_{n_k}(t) &= (w_j^{n_k} - w_{j-1}^{n_k})\left[\frac{t - t_{j-1}^{n_k}}{h_{n_k}} - 1\right] \\ &= (w_j^{n_k} - w_{j-1}^{n_k})\frac{t - t_j^{n_k}}{h_{n_k}} \qquad (\text{since} \;\; h_{n_k} = t_j^{n_k} - t_{j-1}^{n_k}). \end{aligned}$$

Then we have

$$\begin{aligned} \int_{t_{j-1}^{n_k}}^{t_j^{n_k}} \left((w_j^{n_k} - w_{j-1}^{n_k})\frac{t - t_j^{n_k}}{h_{n_k}}, v\right)_V dt &= \int_{t_{j-1}^{n_k}}^{t_j^{n_k}} \left(w_j^{n_k} - w_{j-1}^{n_k}, v\right)_V \frac{t - t_j^{n_k}}{h_{n_k}} dt \\ &= \left(w_j^{n_k} - w_{j-1}^{n_k}, v\right)_V \frac{1}{h_{n_k}} \left[\frac{(t - t_j^{n_k})^2}{2}\right]_{t_{j-1}^{n_k}}^{t_j^{n_k}}. \end{aligned}$$

Since $h_{n_k} = t_j^{n_k} - t_{j-1}^{n_k}$,

$$\int_{t_{j-1}^{n_k}}^{t_j^{n_k}} \left((w_j^{n_k} - w_{j-1}^{n_k})\frac{t - t_j^{n_k}}{h_{n_k}}, v\right)_V dt \; = \; \left(w_{j-1}^{n_k} - w_j^{n_k}, v\right)_V \frac{h_{n_k}}{2}. \tag{3.94}$$

From which it follows that

$$\int_{\alpha}^{\beta} \Big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v\Big)_V dt$$
$$= \frac{h_{n_k}}{2}\Big((w_{\tilde{\alpha}}^{n_k} - w_{\tilde{\alpha}+1}^{n_k}) + (w_{\tilde{\alpha}+1}^{n_k} - w_{\tilde{\alpha}+2}^{n_k}) + \cdots + (w_{\tilde{\beta}-1}^{n_k} - w_{\tilde{\beta}}^{n_k}), v\Big)_V$$
$$= \frac{h_{n_k}}{2}\Big(w_{\tilde{\alpha}}^{n_k} - w_{\tilde{\beta}}^{n_k}, v\Big)_V.$$

Recall (3.85), $||w_j^n||_1 \leq c_1$, then

$$\Big|(w_{\tilde{\alpha}}^{n_k} - w_{\tilde{\beta}}^{n_k}, v)\Big| \leq ||v||_V ||w_{\tilde{\alpha}}^{n_k} - w_{\tilde{\beta}}^{n_k}|| \leq ||v||_V \Big(||w_{\tilde{\alpha}}^{n_k}|| + ||w_{\tilde{\beta}}^{n_k}||\Big) \leq 2c_1 ||v||_V$$

Now as $n_k \to \infty$, $h_{n_k} \to 0$, therefore since v is fixed

$$\lim_{n_k \to \infty} \int_0^T \Big(u_{n_k}(t) - \tilde{u}_{n_k}(t), v\Big)_V dt \;=\; 0.$$

Similarly, we can show that this limit is zero when $v(t)$ is a piecewise constant function of $t \in I$. Since the piecewise constant functions are dense in X, this proof is also valid for every function $v \in L^2(I, V)$. Therefore, $\tilde{u}_{n_k} \rightharpoonup u$ in $L^2(I, V)$.

Similarly, using the same approach $\tilde{p}_{n_k} \rightharpoonup p$ in $L^2(I, L^2(\Omega))$.

We now consider the question in which sense the functions $u(t)$ and $p(t)$ satisfy the given system of partial differential equations. We have by (3.57) and (3.58) the system for $j = 1, \cdots, m2^{n_k - 1}$

$$a(w_j^{n_k}, v) \;-\; \alpha(z_j^{n_k}, \nabla \cdot v) \;=\; -(1-\beta)\alpha < z_j^{n_k}, v > \quad \forall v \in V$$

$$Se\frac{1}{h_{n_k}}(z_j^{n_k} - z_{j-1}^{n_k}, q) \ + \ \alpha\frac{1}{h_{n_k}}(\nabla\cdot w_j^{n_k} - \nabla\cdot w_{j-1}^{n_k}, q) \ + \ \frac{k}{\mu}(\nabla z_j^{n_k}, \nabla q)$$

$$= \ (Q_j, q) \ + \ (1-\beta)\alpha\frac{1}{h_{n_k}} < w_j^{n_k} - w_{j-1}^{n_k}, q > \quad \forall q \in M$$

Define the abstract function $Q(t)$ to be the constant Q for all $t \in [0, T]$ and let $v(t)$ and $q(t)$ be arbitrary functions in $L^2(I, V)$ and $L^2(I, M)$ respectively. With $W_j^{n_k} = \frac{w_j^{n_k} - w_{j-1}^{n_k}}{h_{n_k}}$ and $P_j^{n_k} = \frac{p_j^{n_k} - p_{j-1}^{n_k}}{h_{n_k}}$, we get

$$a(w_j^{n_k}, v) \ - \ \alpha(z_j^{n_k}, \nabla\cdot v) \ = \ -(1-\beta)\alpha < z_j^{n_k}, v >, \tag{3.95}$$

$$Se(Z_j^{n_k}, q) \ + \ \alpha(\nabla\cdot W_j^{n_k}, q) \ + \ \frac{k}{\mu}(\nabla z_j^{n_k}, \nabla q)$$

$$= \ (Q, q) \ + \ (1-\beta)\alpha < W_j^{n_k}, q > . \tag{3.96}$$

Recall that

$$\tilde{u}_{n_k}(t) = w_j^{n_k}, \qquad U_{n_k}(t) = W_j^{n_k}, \qquad \text{for } t \in \tilde{I}_j^n = (t_{j-1}^{n_k}, t_j^{n_k}], \quad j = 2, \cdots, m2^{n_k - 1},$$

$$\tilde{p}_{n_k}(t) = z_j^{n_k}, \qquad P_{n_k}(t) = Z_j^{n_k}, \qquad \text{for } t \in \tilde{I}_j^n = (t_{j-1}^{n_k}, t_j^{n_k}], \quad j = 2, \cdots, m2^{n_k - 1}.$$

Integrate from 0 to T to obtain

$$\int_0^T a\Big(\tilde{u}_{n_k}(t), v(t)\Big)dt \ - \ \int_0^T \alpha\Big(\tilde{p}_{n_k}(t), \nabla\cdot v(t)\Big)dt \ = \ -\int_0^T (1-\beta)\alpha < \tilde{p}_{n_k}(t), v(t) > dt,$$

$$\int_0^T Se\Big(P_{n_k}(t), q(t)\Big)dt \ + \ \int_0^T \alpha\Big(\nabla\cdot U_{n_k}(t), q(t)\Big)dt \ + \ \int_0^T \frac{k}{\mu}\Big(\nabla\tilde{p}_{n_k}(t), \nabla q(t)\Big)dt$$

$$= \ \int_0^T \big(Q, q(t)\big)dt \ + \ \int_0^T (1-\beta)\alpha < U_{n_k}(t), q(t) > dt.$$

Each of these integrals exists since $v \in L^2(I,V)$, $q \in L^2(I,L^2(\Omega))$ (consequently, $v \in L^2(I,H^1(\Omega))$ and $q \in L^2(I,H^1(\Omega)))$, $\tilde{p}_{n_k} \in L^2(I,L^2(\Omega))$, $U_{n_k} \in L^2(I,H^1(\Omega))$, $P_{n_k} \in L^2(I,H^1(\Omega))$, and $Q \in L^2(I,L^2(\Omega))$.

The integral $\int_0^T a(\tilde{u}_{n_k}, v)dt$ defines a bounded linear functional on $L^2(I,V)$ since $a(.,.)$ is a bounded bilinear form on V. For a fixed $v \in L^2(I,V)$,

$$\begin{aligned}\left(\int_0^T a(\tilde{u}_{n_k}(t), v(t))dt\right)^2 &\leq C^2 \left(\int_0^T ||\tilde{u}_{n_k}(t)||_V ||v(t)||_V dt\right)^2 \\ &\leq C^2 \int_0^T ||\tilde{u}_{n_k}(t)||_V^2 dt \int_0^T ||v(t)||_V^2 dt \\ &\leq C^2 ||\tilde{u}_{n_k}||^2_{L^2(I,V)} ||v||^2_{L^2(I,V)}.\end{aligned}$$

Thus for a fixed v, the integral $\int_0^T a(\tilde{u}_{n_k}(t), v(t))dt \leq C||\tilde{u}_{n_k}||_{L^2(I,V)}$.

For $n_k \to \infty$, $\tilde{u}_{n_k} \rightharpoonup u$ in $L^2(I,V)$, thus

$$\int_0^T a\Big(\tilde{u}_{n_k}(t), v(t)\Big)dt \to \int_0^T \Big(u(t), v(t)\Big)dt, \qquad \text{for } n_k \to \infty.$$

Furthermore, for $n_k \to \infty$ we have that

$$\tilde{p}_{n_k} \rightharpoonup p \qquad \text{in } L^2(I, L^2(\Omega)),$$

hence,

$$\int_0^T \alpha\Big(\tilde{p}_{n_k}(t), \nabla \cdot v(t)\Big)dt \to \int_0^T \alpha\Big(p(t), \nabla \cdot v(t)\Big)dt,$$

and

$$\int_0^T (1-\beta) < \tilde{p}_{n_k}(t), v(t) > dt \to \int_0^T (1-\beta) < p(t), v(t) > dt,$$

and

$$\int_0^T \frac{k}{\mu}\Big(\nabla\tilde{p}_{n_k}(t), \nabla q(t)\Big)dt \to \int_0^T \frac{k}{\mu}\Big(\nabla p(t), \nabla q(t)\Big)dt,$$

as $n_k \to \infty$.

For $n_k \to \infty$, $P_{n_k} \rightharpoonup p_t$ in $L^2(I, L^2(\Omega))$, thus

$$\int_0^T \Big(P_{n_k}(t), \nabla\cdot v(t)\Big)dt \to \int_0^T \Big(p_t(t), \nabla\cdot v(t)\Big)dt, \qquad \text{as } n_k \to \infty.$$

Furthermore, as $n_k \to \infty$, we have $U_{n_k} \rightharpoonup u_t$ in $L^2(I, H^1(\Omega))$, hence

$$\int_0^T \alpha\Big(\nabla\cdot U_{n_k}(t), q(t)\Big)dt \to \int_0^T \alpha\Big(\nabla\cdot u_t(t), q(t)\Big)dt$$

and

$$\int_0^T (1-\beta)\alpha < U_{n_k}(t), q(t) > dt \to \int_0^T (1-\beta)\alpha < u_t(t), q(t) > dt,$$

for $n_k \to \infty$.

Since $v(t)$ and $q(t)$ were arbitrary functions from $L^2(I, V)$ and $L^2(I, M)$, we have that

$$\int_0^T a\Big(u(t), v(t)\Big)dt - \int_0^T \alpha\Big(p(t), \nabla\cdot v(t)\Big)dt$$
$$= -\int_0^T (1-\beta) < p(t), v(t) > dt \qquad \forall v \in L^2(I, V),$$

and

$$\int_0^T \alpha\Big(\nabla\cdot u_t(t), q(t)\Big)dt + \int_0^T Se\Big(p_t(t), q(t)\Big)dt + \int_0^T \frac{k}{\mu}\Big(\nabla p(t), \nabla q(t)\Big)dt$$
$$= \int_0^T \Big(Q, q(t)\Big)dt + \int_0^T (1-\beta)\alpha < u_t(t), q(t) > dt \qquad \forall q \in L^2(I, M).$$

Therefore, $u(t)$ and $p(t)$ satisfy the given system of partial differential equations weakly and have the following properties

$$\begin{aligned}
u &\in L^2(I, V), & p &\in L^2(I, L^2(\Omega)), \\
u &\in AC(I, H^1(\Omega)), & p &\in AC(I, H^1(\Omega)), \\
u_t &\in AC(I, H^1(\Omega)), & p_t &\in AC(I, H^1(\Omega)), \\
u(0) &= 0 \text{ in } C(I, H^1(\Omega)), & p(0) &= 0 \text{ in } C(I, H^1(\Omega)),
\end{aligned}$$

$$\int_0^T a(u,v)dt - \int_0^T \alpha(p, \nabla \cdot v)dt = -\int_0^T (1-\beta) < p, v > dt \qquad \forall v \in L^2(I, V),$$

$$\int_0^T Se(p_t, q)dt + \int_0^T \alpha(\nabla \cdot u_t, q)dt + \int_0^T \frac{k}{\mu}(\nabla p, \nabla q)dt = \int_0^T (Q, q)dt + \int_0^T (1-\beta)\alpha < u_t, q > dt \qquad \forall q \in L^2(I, M).$$

In conclusion, we just proved existence of weak solutions $u(t)$ and $p(t)$ for problem (3.27)–(3.34).

Uniqueness of weak solutions

Let $(\tilde{u}, \tilde{p})$ and $(\hat{u}, \hat{p})$ be two solutions of problem (3.27)–(3.34). Then $u = \tilde{u} - \hat{u}$ and $p = \tilde{p} - \hat{p}$ are also solutions of this problem.

We have

$$a(\tilde{u}, v) \;-\; \alpha\big(\tilde{p}, \nabla \cdot v\big) \;=\; (F, v) - (1-\beta)\alpha < \tilde{p}, v > \tag{3.97}$$

$$Se(\tilde{p}_t, q) \;+\; \alpha\big((\nabla \cdot \tilde{u})_t, q\big) \;+\; \frac{k}{\mu}(\nabla \tilde{p}, \nabla q) \;=\; (Q, q)$$
$$+ \; < h_1, q > + (1-\beta)\alpha < \tilde{u}_t, q > \tag{3.98}$$

and

$$a(\hat{u}, v) \;-\; \alpha\big(\hat{p}, \nabla \cdot v\big) \;=\; (F, v) - (1-\beta)\alpha < \hat{p}, v > \tag{3.99}$$

$$Se(\hat{p}_t, q) \;+\; \alpha\big((\nabla \cdot \hat{u})_t, q\big) \;+\; \frac{k}{\mu}(\nabla \hat{p}, \nabla q) \;=\; (Q, q)$$
$$+ \; < h_1, q > + (1-\beta)\alpha < \hat{u}_t, q > \tag{3.100}$$

Setting $u = \tilde{u} - \hat{u}$ and $p = \tilde{p} - \hat{p}$, we subtract (3.99) from (3.97) and (3.100) from (3.98) then integrate over I to get

$$\int_0^T a(u, v)dt \;-\; \int_0^T \alpha\big(p, \nabla \cdot v\big)dt \;=\; -\int_0^T (1-\beta)\alpha < p, v > dt \tag{3.101}$$

$$\int_0^T Se(p_t, q)dt \;+\; \int_0^T \alpha\big((\nabla \cdot u)_t, q\big)dt \;+\; \int_0^T \frac{k}{\mu}(\nabla p, \nabla q)dt \;=$$
$$\int_0^T (1-\beta)\alpha < u_t, q > dt. \tag{3.102}$$

Choose an arbitrary $a \in I$ and let

$$v(t) = \begin{cases} u_t(t) & \text{if } 0 \leq t \leq a, \\ 0 & \text{if } a < t \leq T, \end{cases}$$

and

$$q(t) = \begin{cases} p(t) & \text{if } 0 \leq t \leq a, \\ 0 & \text{if } a < t \leq T. \end{cases}$$

Hence

$$\int_0^a a(u, u_t)dt \ - \ \int_0^a \alpha\big(p, (\nabla \cdot u)_t\big)dt \ = \ -\int_0^a (1-\beta)\alpha < p, u_t > dt \tag{3.103}$$

$$\int_0^a Se(p_t, p)dt \ + \ \int_0^a \alpha\big((\nabla \cdot u)_t, p\big)dt \ + \ \int_0^a \frac{k}{\mu}(\nabla p, \nabla p)dt \ = \ \int_0^a (1-\beta)\alpha < u_t, p > dt. \tag{3.104}$$

Adding (3.103) and (3.104), we get

$$\int_0^a Se(p_t, p)dt \ + \ \int_0^a a(u, u_t)dt \ + \ \int_0^a \frac{k}{\mu}(\nabla p, \nabla p)dt \ = \ 0 \tag{3.105}$$

Obviously,

$$\int_0^a \frac{k}{\mu}(\nabla p, \nabla p)dt \ = \ \int_0^a \frac{k}{\mu}||\nabla p||^2 \ \geq \ 0, \tag{3.106}$$

$$\int_0^a Se(p_t, p)dt \ = \ \frac{1}{2}Se||p(a)||^2 - \frac{1}{2}Se||p(0)||^2 \ = \ \frac{1}{2}Se||p(a)||^2 \ \geq \ 0, \tag{3.107}$$

and

$$\int_0^a a(u, u_t)dt \ = \ \frac{1}{2}[[u(a)]]^2 - \frac{1}{2}[[u(0)]]^2 \ = \ \frac{1}{2}[[u(a)]]^2 \ \geq \ 0. \tag{3.108}$$

Hence we conclude that

$$||u(a)|| \;=\; 0 \qquad \text{and} \qquad ||p(a)|| \;=\; 0.$$

And since a was arbitrary then

$$||u(t)|| \;=\; 0 \qquad \text{in } I,$$

$$||p(t)|| \;=\; 0 \qquad \text{in } I.$$

Therefore, $\tilde{u}(t) = \hat{u}(t)$ and $\tilde{p}(t) = \hat{p}(t)$ and we conclude that the system has a unique solution $(u(t), p(t))$.

3.3.2 Energy norm estimate for homogeneous initial and boundary conditions

Given the quasi-static poroelasticity system of partial differential equations with homogeneous initial and boundary conditions then the weak formulation yields

$$
\begin{aligned}
a(u,v) \; - \; \alpha(p, \nabla\cdot v) &= -(1-\beta)\alpha < p, v >, \qquad \forall v \in V,\\
Se(p_t, q) \; + \; \alpha((\nabla\cdot u)_t, q) \; + \; \frac{k}{\mu}(\nabla p, \nabla q) &= (Q(t), q)\\
&\quad + (1-\beta)\alpha < u_t, q >, \qquad \forall q \in M,
\end{aligned}
$$

for almost every $t \in I$.

Let $v = u_t$ and $q = p$ and add the two equations to obtain

$$a(u,u_t) \; + \; Se(p_t,p) \; + \; \tfrac{k}{\mu}(\nabla p, \nabla p) \; = \; (Q,p),$$

and since $\frac{k}{\mu}||\nabla p||^2 \geq 0$

$$\frac{1}{2}\frac{d}{dt}[[u]]^2 + \frac{Se}{2}\frac{d}{dt}||p||^2 \leq ||Q|| \; ||p||, \tag{3.109}$$

That is,

$$
\begin{aligned}
\frac{d}{dt}\left([[u]]^2 + ||p||^2\right) &\leq \frac{2}{\min(1,Se)}||Q|| \; ||p||,\\
&\leq \frac{||Q||^2}{(\min(1,Se))^2} + ||p||^2, \quad (\text{using } 2ab \leq a^2 + b^2)\\
&\leq \frac{||Q||^2}{(\min(1,Se))^2} + ||p||^2 + ||u||_1^2. \qquad (3.110)
\end{aligned}
$$

Lemma 1 *(Gronwall's Inequality (see [5])): Let η be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds\right],$$

for all $0 \leq t \leq T$.

Therefore applying Gronwall's inequality to (3.109), we obtain

$$[[u]]^2 + ||p(t)||^2 \leq e^{\int_0^t ds} \left[[[u]]^2 + ||p(0)||^2 + \int_0^t \frac{||Q||^2}{(\min(1, Se))^2} ds\right]. \tag{3.111}$$

Using the fact that $u(0) = p(0) = 0$ (from (3.70)), we get

$$[[u]]^2 + ||p(t)||^2 \leq e^{\int_0^t ds} \left[\int_0^t \frac{||Q||^2}{(\min(1, Se))^2} ds\right].$$

Then

$$||p(t)||^2 \ \leq \ e^{\int_0^t ds} \left[\int_0^t \frac{||Q||^2}{(\min(1, Se))^2} ds\right],$$

and

$$2G||u(t)||_1^2 \leq \ e^{\int_0^t ds} \left[\int_0^t \frac{||Q||^2}{(\min(1, Se))^2} ds\right].$$

Therefore,

$$\max_{0 \leq t \leq T} ||p(t)||^2 \ \leq \ \frac{e^T}{(\min(1, Se))^2} ||Q||^2_{L^2(0,T;L^2(\Omega))},$$

and

$$\max_{0\leq t\leq T} ||u(t)||_1^2 \;\leq\; \frac{e^T}{2G(\min(1,Se))^2}||Q||^2_{L^2(0,T;L^2(\Omega))},$$

Hence

$$||p(t)|| \;\leq\; \frac{\sqrt{e^T}}{\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))}, \tag{3.112}$$

and

$$||u(t)||_1 \;\leq\; \frac{\sqrt{e^T}}{\sqrt{2G}\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))}. \tag{3.113}$$

The right hand side of the elasticity equation

Let $\bar{u}$ be the solution of the stationary elasticity problem

$$-G\nabla\cdot(\nabla\bar{u}+(\nabla\bar{u})^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot\bar{u}) \;=\; F.$$

The weak formulation of this problem is: find $\bar{u}\in V$ such that

$$a(\bar{u},v) \;=\; (F,v)+\int_{\Gamma_t}\left[G(\nabla\bar{u}+(\nabla\bar{u})^T)+G\frac{2\nu}{1-2\nu}(\nabla\cdot\bar{u})\right]\cdot\hat{n}v \qquad \forall v\in V, \tag{3.114}$$

where $a(\bar{u},v) \;=\; \int_\Omega[G(\nabla\bar{u} \;+\; (\nabla\bar{u})^T):\nabla v \;+\; G\frac{2\nu}{1-2\nu}(\nabla\cdot\bar{u})(\nabla\cdot v)]$.

And let $\tilde{u}$ be the solution of the quasi-static poroelasticity system with $F=0$:

$$\begin{aligned}-G\nabla\cdot(\nabla\tilde{u}+(\nabla\tilde{u})^T)-G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot\tilde{u})+\alpha\nabla p &= 0,\\ \frac{\partial}{\partial t}(Se\; p+\alpha\nabla\cdot\tilde{u})-\frac{k}{\mu}\Delta p &= Q.\end{aligned}$$

The weak formulation of this problem is: find $\tilde{u}\in V$ and $p\in M$ such that

$$a(\tilde{u},v) \;+\; \alpha(\nabla p,v) \;=\; \int_{\Gamma_t}\left[G(\nabla\tilde{u}+(\nabla\tilde{u})^T)+G\frac{2\nu}{1-2\nu}(\nabla\cdot\tilde{u})\right]\cdot\hat{n}v \quad \forall v\in V, \tag{3.115}$$

$$Se(p_t,q)+\alpha\Big((\nabla\cdot\tilde{u})_t,q\Big)+\frac{k}{\mu}(\nabla p,\nabla q) \;=\; (Q,q)+\int_{\Gamma_f}\frac{k}{\mu}\nabla p\cdot\hat{n}q \quad \forall q\in M, \tag{3.116}$$

The above equation holds for almost every $t \in I$. We will show that $u = \bar{u} + \tilde{u}$ satisfies the poroelasticity system with $F \neq 0$

$$
\begin{aligned}
-G\nabla\cdot(\nabla\tilde{u} + (\nabla\tilde{u})^T) - G\frac{2\nu}{(1-2\nu)}\nabla(\nabla\cdot\tilde{u}) + \alpha\nabla p &= F, \\
\frac{\partial}{\partial t}(Se\ p + \alpha\nabla\cdot\tilde{u}) - \frac{k}{\mu}\Delta p &= Q.
\end{aligned}
$$

Adding equations (3.114), (3.115), and $(\alpha(\nabla\cdot\bar{u})_t, q) = 0$ (which is zero since $\bar{u}$ is the solution to a stationary problem) to (3.116), we get

$$
\begin{aligned}
a(\bar{u}, v) \ + \ a(\tilde{u}, v) \ + \ \alpha(\nabla p, v) \ &= \ (F, v) + \int_{\Gamma_t}\left[G(\nabla\bar{u} + (\nabla\bar{u})^T) + G\frac{2\nu}{1-2\nu}(\nabla\cdot\bar{u})\right]\cdot\hat{n}v \\
&\quad + \int_{\Gamma_t}\left[G(\nabla\tilde{u} + (\nabla\tilde{u})^T) + G\frac{2\nu}{1-2\nu}(\nabla\cdot\tilde{u})\right]\cdot\hat{n}v, \\
Se(p_t, q) + \alpha\Big((\nabla\cdot\bar{u})_t, q\Big) + \alpha\Big((\nabla\cdot\tilde{u})_t, q\Big) + \frac{k}{\mu}(\nabla p, \nabla q) \ &= \ (Q, q) + \int_{\Gamma_f}\frac{k}{\mu}\nabla p\cdot\hat{n}q.
\end{aligned}
$$

That is,

$$
\begin{aligned}
a(\bar{u} + \tilde{u}, v) + \alpha(\nabla p, v) \ &= \ \textstyle\int_{\Gamma_t}\big[G\nabla(\bar{u} + \tilde{u}) + (\nabla(\bar{u} + \tilde{u}))^T \\
&\qquad + G\tfrac{2\nu}{1-2\nu}\nabla\cdot(\bar{u} + \tilde{u})\big]\cdot\hat{n}v + (F, v) \\
Se(p_t, q) + \alpha\Big((\nabla\cdot(\bar{u} + \tilde{u}))_t, q\Big) + \tfrac{k}{\mu}(\nabla p, \nabla q) \ &= \ (Q, q) + \textstyle\int_{\Gamma_f}\tfrac{k}{\mu}\nabla p\cdot\hat{n}q.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a(u, v) \ + \ \alpha(\nabla p, v) \ &= \ (F, v) + \int_{\Gamma_t}\left[G(\nabla u + (\nabla u)^T) + G\frac{2\nu}{1-2\nu}(\nabla\cdot u)\right]\cdot\hat{n}v \qquad \forall v \in V \\
Se(p_t, q) + \alpha\Big((\nabla\cdot u)_t, q\Big) + \frac{k}{\mu}(\nabla p, \nabla q) \ &= \ (Q, q) + \int_{\Gamma_f}\frac{k}{\mu}\nabla p\cdot\hat{n}q, \qquad \forall q \in M.
\end{aligned}
$$

Again the above equation holds for almost every $t \in I$. Hence we got the weak formulation of the poroelasticity system with $F \neq 0$.

3.3.3 Existence and uniqueness of weak solutions for nonhomogeneous initial conditions

Consider problem II (3.35)–(3.42) (with nonhomogeneous initial conditions). From equations (3.57) and (3.58), we have for $j = 1, \cdots, m$

$$a(w_j, v) \ - \ \alpha(z_j, \nabla \cdot v) \ = \ -(1-\beta)\alpha < z_j, v > \quad \forall v \in V, \tag{3.117}$$

$$Se(z_j - z_{j-1}, q) \ + \ \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q) \ + \ h\frac{k}{\mu}(\nabla z_j, \nabla q) \ =$$

$$(1-\beta)\alpha < w_j - w_{j-1}, q > \quad \forall q \in M, \tag{3.118}$$

with initial conditions

$$\begin{aligned} Se\ z_0 + \alpha \nabla \cdot w_0 &= v_0, & \text{on } \Omega, \\ (1-\beta)\alpha w_0 \cdot \hat{n} &= v_1, & \text{on } \Gamma_{tf}. \end{aligned}$$

Let

$$\tilde{z}_j \ = \ z_j - \frac{v_0}{se}$$

and

$$\tilde{w}_j \ = \ w_j,$$

then the system becomes

$$a\left(\tilde{w}_j, v\right) \ - \ \alpha\left(\tilde{z}_j + \frac{v_0}{se}, \nabla \cdot v\right) \ = \ -(1-\beta)\alpha < \tilde{z}_j + \frac{v_0}{se}, v >,$$

$$Se(\tilde{z}_j - \tilde{z}_{j-1}, q) + \alpha(\nabla \cdot \tilde{w}_j - \nabla \cdot \tilde{w}_{j-1}, q) + h\frac{k}{\mu}\left(\nabla \tilde{z}_j + \nabla \frac{v_0}{Se}, \nabla q\right) =$$

$$(1-\beta)\alpha < \tilde{w}_j - \tilde{w}_{j-1}, q > .$$

That is,

$$a(\tilde{w}_j, v) \; - \; \alpha(\tilde{z}_j, \nabla \cdot v) \; = \; \alpha\left(\frac{v_0}{Se}, \nabla \cdot v\right)$$

$$-(1-\beta)\alpha < \frac{v_0}{Se}, v > -(1-\beta)\alpha < \tilde{z}_j, v >, \tag{3.119}$$

and

$$Se(\tilde{z}_j - \tilde{z}_{j-1}, q) \; + \; \alpha(\nabla \cdot \tilde{w}_j - \nabla \cdot \tilde{w}_{j-1}, q) \; + \; h\frac{k}{\mu}(\nabla \tilde{z}_j, \nabla q) \; =$$

$$-h\frac{k}{\mu}\left(\nabla \frac{v_0}{Se}, \nabla q\right) + (1-\beta)\alpha < \tilde{w}_j - \tilde{w}_{j-1}, q > . \tag{3.120}$$

With initial conditions

$$Se \; \left(\tilde{z}_0 + \frac{v_0}{Se}\right) + \alpha \nabla \cdot \tilde{w}_0 \; = \; v_0,$$

$$(1-\beta)\alpha \tilde{w}_0 \cdot \hat{n} \; = \; v_1,$$

which implies that

$$Se \; \tilde{z}_0 + \alpha \nabla \cdot \tilde{w}_0 \; = \; 0, \tag{3.121}$$

$$(1-\beta)\alpha \tilde{w}_0 \cdot \hat{n} \; = \; v_1. \tag{3.122}$$

Therefore, we transformed the given problem (3.35)–(3.42) to an equivalent one with homogeneous initial condition (3.121). Furthermore, each of these integrals $\left(\frac{v_0}{Se}, \nabla \cdot v\right)$, $< \frac{v_0}{Se}, v >$, and $\left(\nabla \frac{v_0}{Se}, \nabla q\right)$ exists since $v \in V$, $q \in M$ (consequently, $v \in L^2(I, H^1(\Omega))$, $q \in L^2(I, H^1(\Omega)))$, and $v_0 \in L^2(I, H^1(\Omega))$, ($\alpha, \beta, Se$ are positive constants).

At time $j-1$, $j = 2, \cdots, m$, (3.119) and (3.120) become

$$a(\tilde{w}_{j-1}, v) - \alpha(\tilde{z}_{j-1}, \nabla\cdot v) \; = $$

$$\alpha\left(\frac{v_0}{Se}, \nabla \cdot v\right) - (1-\beta)\alpha < \frac{v_0}{Se}, v > -(1-\beta)\alpha < \tilde{z}_{j-1}, v >, \tag{3.123}$$

$$Se(\tilde{z}_{j-1} - \tilde{z}_{j-2}, q) + \alpha(\nabla\cdot\tilde{w}_{j-1} - \nabla\cdot\tilde{w}_{j-2}, q) + h\frac{k}{\mu}(\nabla\tilde{z}_{j-1}, \nabla q) \; =$$

$$-h\frac{k}{\mu}\left(\nabla\frac{v_0}{Se}, \nabla q\right) + (1-\beta)\alpha < \tilde{w}_{j-1} - \tilde{w}_{j-2}, q > . \tag{3.124}$$

Subtracting (3.123) from (3.119) and (3.124) from (3.120), we obtain

$$a(\tilde{w}_j - \tilde{w}_{j-1}, v) \; - \; \alpha(\tilde{z}_j - \tilde{z}_{j-1}, \nabla \cdot v) \; = \; -(1-\beta)\alpha < \tilde{z}_j - \tilde{z}_{j-1}, v >, \tag{3.125}$$

$$Se\Big(\tilde{z}_j - \tilde{z}_{j-1} - (\tilde{z}_{j-1} - \tilde{z}_{j-2}), q\Big) + \alpha\Big(\nabla\cdot\tilde{w}_j - \nabla\cdot\tilde{w}_{j-1} - (\nabla\cdot\tilde{w}_{j-1} - \nabla\cdot\tilde{w}_{j-2}), q\Big)$$

$$+ \; h\frac{k}{\mu}(\nabla\tilde{z}_j - \nabla\tilde{z}_{j-1}, \nabla q) \; = \; (1-\beta)\alpha < \tilde{w}_j - \tilde{w}_{j-1} - (\tilde{w}_{j-1} - \tilde{w}_{j-2}), q > . \tag{3.126}$$

Adding (3.125) and (3.126) with $v = \tilde{w}_j - \tilde{w}_{j-1}$ and $q = \tilde{z}_j - \tilde{z}_{j-1}$, we get

$$a(\tilde{w}_j-\tilde{w}_{j-1},\tilde{w}_j-\tilde{w}_{j-1})-\alpha(\nabla\cdot\tilde{w}_{j-1}-\nabla\cdot\tilde{w}_{j-2},\tilde{z}_j-\tilde{z}_{j-1})+Se(\tilde{z}_j-\tilde{z}_{j-1},\tilde{z}_j-\tilde{z}_{j-1})$$

$$-Se(\tilde{z}_{j-1}-\tilde{z}_{j-2},\tilde{z}_j-\tilde{z}_{j-1})+h\frac{k}{\mu}(\nabla\tilde{z}_j-\nabla\tilde{z}_{j-1},\nabla\tilde{z}_j-\nabla\tilde{z}_{j-1}) =$$

$$-(1-\beta)\alpha < \tilde{w}_{j-1}-\tilde{w}_{j-2},\tilde{z}_j-\tilde{z}_{j-1} > . \tag{3.127}$$

Equation (3.125) with $v=\tilde{w}_{j-1}-\tilde{w}_{j-2}$, $j=2,\cdots,m$, is

$$a(\tilde{w}_j-\tilde{w}_{j-1},\tilde{w}_{j-1}-\tilde{w}_{j-2})-\alpha(\tilde{z}_j-\tilde{z}_{j-1},\nabla\cdot\tilde{w}_{j-1}-\nabla\cdot\tilde{w}_{j-2}) =$$

$$-(1-\beta)\alpha < \tilde{z}_j-\tilde{z}_{j-1},\tilde{w}_{j-1}-\tilde{w}_{j-2} > . \tag{3.128}$$

Subtracting (3.128) from (3.127), we get

$$a(\tilde{w}_j-\tilde{w}_{j-1},\tilde{w}_j-\tilde{w}_{j-1})+Se(\tilde{z}_j-\tilde{z}_{j-1},\tilde{z}_j-\tilde{z}_{j-1})+h\frac{k}{\mu}(\nabla\tilde{z}_j-\nabla\tilde{z}_{j-1},\nabla\tilde{z}_j-\nabla\tilde{z}_{j-1}) =$$

$$a(\tilde{w}_j-\tilde{w}_{j-1},\tilde{w}_{j-1}-\tilde{w}_{j-2})+Se(\tilde{z}_j-\tilde{z}_{j-1},\tilde{z}_{j-1}-\tilde{z}_{j-2}),$$

which implies that,

$$2G||\tilde{w}_j-\tilde{w}_{j-1}||_1^2+Se||\tilde{z}_j-\tilde{z}_{j-1}||^2$$

$$\leq C_1||\tilde{w}_j-\tilde{w}_{j-1}||_1||\tilde{w}_{j-1}-\tilde{w}_{j-2}||_1+Se||\tilde{z}_j-\tilde{z}_{j-1}||\ ||\tilde{z}_{j-1}-\tilde{z}_{j-2}||,$$

where $C_1 = \left(2G, \frac{6G\nu}{1-2\nu}\right)$ is the continuity constant of the bilinear form $a(\cdot,\cdot)$ (3.11). Then,

$$||\tilde{w}_j - \tilde{w}_{j-1}||_1^2 + ||\tilde{z}_j - \tilde{z}_{j-1}||^2$$

$$\leq \frac{\max(C_1, Se)}{\min(2G, Se)} \Big(||\tilde{w}_j - \tilde{w}_{j-1}||_1^2 + ||\tilde{z}_j - \tilde{z}_{j-1}||^2\Big)^{\frac{1}{2}} \Big(||\tilde{w}_{j-1} - \tilde{w}_{j-2}||_1^2 + \ ||\tilde{z}_{j-1} - \tilde{z}_{j-2}||^2\Big)^{\frac{1}{2}}.$$

Squaring both sides and simplifying, we obtain

$$||\tilde{w}_j - \tilde{w}_{j-1}||_1^2 + ||\tilde{z}_j - \tilde{z}_{j-1}||^2$$

$$\leq \left(\frac{\max(C_1, Se)}{\min(2G, Se)}\right)^2 \Big[||\tilde{w}_{j-1} - \tilde{w}_{j-2}||_1^2 + \ ||\tilde{z}_{j-1} - \tilde{z}_{j-2}||^2\Big]. \tag{3.129}$$

Setting $j = 1$ in (3.125) and letting $v = \tilde{w}_1 - \tilde{w}_0$, we get

$$a(\tilde{w}_1 - \tilde{w}_0, \tilde{w}_1 - \tilde{w}_0) - \alpha(\tilde{z}_1 - \tilde{z}_0, \nabla\cdot\tilde{w}_1 - \nabla\cdot\tilde{w}_0) = -(1-\beta)\alpha < \tilde{z}_1 - \tilde{z}_0 . \tilde{w}_1 - \tilde{w}_0 >, \tag{3.130}$$

Setting $j = 1$ in (3.120) and letting $q = \tilde{z}_1 - \tilde{z}_0$, we have

$$Se(\tilde{z}_1 - \tilde{z}_0, \tilde{z}_1 - \tilde{z}_0) + \alpha(\nabla\cdot\tilde{w}_1 - \nabla\cdot\tilde{w}_0, \tilde{z}_1 - \tilde{z}_0) + h\frac{k}{\mu}(\nabla\tilde{z}_1, \nabla\tilde{z}_1 - \nabla\tilde{z}_0) =$$

$$-h\frac{k}{\mu}\left(\nabla\frac{v_0}{Se}, \nabla\tilde{z}_1 - \nabla\tilde{z}_0\right) + (1-\beta)\alpha < \tilde{w}_1 - \tilde{w}_0, \tilde{z}_1 - \tilde{z}_0 > . \tag{3.131}$$

Using $(\nabla\tilde{z}_1, \nabla\tilde{z}_1 - \nabla\tilde{z}_0) = (\nabla\tilde{z}_1 - \nabla\tilde{z}_0 + \nabla\tilde{z}_0, \nabla\tilde{z}_1 - \nabla\tilde{z}_0) = (\nabla\tilde{z}_1 - \nabla\tilde{z}_0, \nabla\tilde{z}_1 - \nabla\tilde{z}_0) + (\nabla\tilde{z}_0, \nabla\tilde{z}_1 - \nabla\tilde{z}_0)$, then adding (3.130) and (3.131), we obtain

$$2G||\tilde{w}_1 - \tilde{w}_0||_1^2 + Se||\tilde{z}_1 - \tilde{z}_0||^2 + h\frac{k}{\mu}||\nabla\tilde{z}_1 - \nabla\tilde{z}_0||^2$$

$$\leq h\frac{k}{\mu}\frac{||\nabla v_0||}{Se}||\nabla\tilde{z}_1 - \nabla\tilde{z}_0|| + h\frac{k}{\mu}||\nabla\tilde{z}_0||\ ||\nabla\tilde{z}_1 - \nabla\tilde{z}_0||$$

$$\leq h\frac{k}{\mu}\left[\frac{||\nabla v_0||}{Se} + ||\nabla\tilde{z}_0||\right]||\nabla\tilde{z}_1 - \nabla\tilde{z}_0||.$$

Since z_j solves the homogeneous initial conditions poroelasticity problem, then from problem I ($z_0 = w_0 = 0$) we have

$$\tilde{z}_0 = z_o - \frac{v_0}{Se} = -\frac{v_0}{Se}, \qquad \text{and} \qquad \tilde{w}_0 = w_0 = 0. \tag{3.132}$$

Therefore,

$$2G||\tilde{w}_1 - \tilde{w}_0||_1^2 + Se||\tilde{z}_1 - \tilde{z}_0||^2 + h\frac{k}{\mu}||\nabla\tilde{z}_1 - \nabla\tilde{z}_0||^2$$

$$\leq \frac{1}{2\epsilon}\left[h\frac{k}{\mu}\left(\frac{2||\nabla v_0||}{Se}\right)\right]^2 + \frac{\epsilon}{2}||\nabla\tilde{z}_1 - \nabla\tilde{z}_0||^2.$$

Choose ϵ small enough such that $h\frac{k}{\mu} - \frac{\epsilon}{2} > 0$, then $(h\frac{k}{\mu} - \frac{\epsilon}{2})||\nabla\tilde{z}_1 - \nabla\tilde{z}_0||^2 \geq 0$. Hence

$$||\tilde{w}_1 - \tilde{w}_0||_1^2 + ||\tilde{z}_1 - \tilde{z}_0||^2 \leq h^2\frac{2}{\epsilon\ \min(2G, Se)}\left[\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right]^2. \tag{3.133}$$

Using (3.129) and (3.133), we obtain ($j = 1, \cdots, m$)

$$||\tilde{w}_j - \tilde{w}_{j-1}||_1^2 + ||\tilde{z}_j - \tilde{z}_{j-1}||^2 \leq h^2\left(\frac{\max(C_1, Se)}{\min(2G, Se)}\right)^2\frac{2}{\epsilon\ \min(2G, Se)}\left[\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right]^2. \tag{3.134}$$

Let

$$C = \sqrt{\frac{2}{\epsilon\ \min(2G, Se)} \frac{\max(C_1, Se)}{\min(2G, Se)} \frac{k}{\mu} \frac{||\nabla v_0||}{Se}},$$

therefore

$$||\tilde{w}_j - \tilde{w}_{j-1}||_1^2 + ||\tilde{z}_j - \tilde{z}_{j-1}||^2 \leq h^2 C^2. \tag{3.135}$$

Recall the norm:

$$|||(\tilde{w}_j, \tilde{z}_j) - (\tilde{w}_{j-1}, \tilde{z}_{j-1})||| \ = \ \Big(||\tilde{w}_j - \tilde{w}_{j-1}||_1^2 \ + \ ||\tilde{z}_j - \tilde{z}_{j-1}||^2 \Big)^{\frac{1}{2}},$$

then,

$$\begin{aligned} |||(\tilde{w}_j, \tilde{z}_j) - (\tilde{w}_0, \tilde{z}_0)||| \quad &= \quad |||(\tilde{w}_j, \tilde{z}_j) - (\tilde{w}_{j-1}, \tilde{z}_{j-1}) + \ \cdots \ + (\tilde{w}_1, \tilde{z}_1) - (\tilde{w}_0, \tilde{z}_0)||| \\ &\leq \quad |||(\tilde{w}_j, \tilde{z}_j) - (\tilde{w}_{j-1}, \tilde{z}_{j-1})||| + \ \cdots \ + |||(\tilde{w}_1, \tilde{z}_1) - (\tilde{w}_0, \tilde{z}_0)|||, \\ &\leq \quad \text{jhC} \end{aligned}$$

From which it follows that

$$||\tilde{w}_j - \tilde{w}_0||_1^2 + ||\tilde{z}_j - \tilde{z}_0||^2 \ \leq \ j^2 h^2 C^2 \tag{3.136}$$

Since $h = \frac{T}{m}$, then

$$||\tilde{w}_j - \tilde{w}_0||_1^2 \leq T^2 C^2$$

and

$$||\tilde{z}_j - \tilde{z}_0||^2 \leq T^2 C^2,$$

that is,

$$||\tilde{w}_j||_1 \leq TC + ||\tilde{w}_0||_1$$

and

$$||\tilde{z}_j|| \leq TC + ||\tilde{z}_0||.$$

Hence using the fact that $\tilde{z}_0 = -\frac{v_0}{Se}$ and $\tilde{w}_0 = 0$ (3.132), we get

$$||\tilde{w}_j||_1 \leq TC \qquad \text{and} \qquad ||\tilde{z}_j|| \leq TC + \frac{||v_0||}{Se}. \tag{3.137}$$

Using (3.135) with $\tilde{W}_j = \frac{\tilde{w}_j - \tilde{w}_{j-1}}{h}$ and $\tilde{Z}_j = \frac{\tilde{z}_j - \tilde{z}_{j-1}}{h}$, we obtain

$$||\tilde{W}_j||_1^2 + ||\tilde{Z}_j||^2 \leq C^2.$$

Hence

$$||\tilde{W}_j||_1 \leq C \qquad \text{and} \qquad ||\tilde{Z}_j|| \leq C. \tag{3.138}$$

The estimates obtained in (3.137) and (3.138) are independent of h, thus remain valid for an arbitrary mesh d_n. That is, for every positive integer n and $j = 1, \cdots, m2^{n-1}$

$$||\tilde{w}_j^n||_1 \ \leq \ TC \qquad\qquad ||\tilde{z}_j^n|| \ \leq \ TC + \frac{||v_0||}{Se}, \tag{3.139}$$

$$||\tilde{W}_j^n||_1 \ \leq \ C, \qquad \text{and} \qquad ||\tilde{Z}_j^n|| \ \leq \ C. \tag{3.140}$$

Recall that $\tilde{w}_j = w_j$, $\tilde{z}_j = z_j - \frac{v_0}{Se}$, and the constant

$$C = \sqrt{\frac{2}{\epsilon \ \min(2G, Se)}\frac{\max(C_1, Se)}{\min(2G, Se)}\frac{k}{\mu}\frac{||\nabla v_0||}{Se}},$$

thus the estimates (3.139) and (3.140) become

$$||w_j^n||_1 \ \le \ \frac{2Tk}{\sqrt{\epsilon}\mu} \ \frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}} \frac{||\nabla v_0||}{Se}, \tag{3.141}$$

$$||z_j^n|| \ \le \ \frac{2Tk}{\sqrt{\epsilon}\mu} \ \frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}} \frac{||\nabla v_0||}{Se} + 2\frac{||v_0||}{Se}, \tag{3.142}$$

$$||W_j^n||_1 \ \le \ \frac{2k}{\sqrt{\epsilon}\mu} \ \frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}} \frac{||\nabla v_0||}{Se}, \tag{3.143}$$

and

$$||Z_j^n|| \ \le \ \frac{2k}{\sqrt{\epsilon}\mu} \ \frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}} \frac{||\nabla v_0||}{Se}. \tag{3.144}$$

These basic estimates can then be used to show existence and uniqueness of weak solutions as done for problem I (3.27)–(3.34).

3.3.4 Energy norm estimate for nonhomogeneous initial conditions

For the given quasi-static poroelasticity system with nonhomogeneous initial conditions, we transformed the problem to an equivalent one with nonhomogeneous right hand side and homogeneous initial condition (3.121). Then the weak formulation of the transformed problem is: find $u \in V$ and $p \in M$ such that

$$a(u,v) \; - \; \alpha\left(p+\frac{v_0}{Se}, \nabla\cdot v\right) \;=\; -(1-\beta)\alpha < p+\frac{v_0}{Se}, v >, \qquad \forall\, v \in V, \tag{3.145}$$

$$Se(p_t,q) \;+\; \alpha(\nabla\cdot u_t, q) \;+\; \frac{k}{\mu}(\nabla p + \nabla\frac{v_0}{Se}, \nabla q) \;=\; (1-\beta)\alpha < u_t, q >, \qquad \forall\, q \in M, \tag{3.146}$$

for almost every $t \in I$.

Letting $v = u_t$ and $q = p + \frac{v_0}{Se}$ and adding (3.145) and (3.146), we obtain

$$a(u,u_t) \;+\; Se\left(p_t + \frac{v_{0_t}}{Se}, p+\frac{v_0}{Se}\right) \;+\; \frac{k}{\mu}\left(\nabla p + \nabla\frac{v_0}{Se}, \nabla p + \nabla\frac{v_0}{Se}\right) \;=\; Se\left(\frac{v_{0_t}}{Se}, p+\frac{v_0}{Se}\right),$$

therefore

$$\frac{1}{2}\frac{d}{dt}[[u]]^2 + \frac{Se}{2}\frac{d}{dt}\left|\left|p+\frac{v_0}{Se}\right|\right|^2 + \frac{k}{\mu}\left|\left|\nabla p + \nabla\frac{v_0}{Se}\right|\right|^2 \le ||v_{0_t}|| \; \left|\left|p+\frac{v_0}{Se}\right|\right|.$$

That is,

$$\frac{d}{dt}[[u]]^2 + \frac{d}{dt}\left|\left|p+\frac{v_0}{Se}\right|\right|^2 \le 2\frac{||v_{0_t}||}{\min(1,Se)} \; \left|\left|p+\frac{v_0}{Se}\right|\right|.$$

Then

$$\frac{d}{dt}\left([[u]]^2 + \left|\left|p+\frac{v_0}{Se}\right|\right|^2\right) \le \left(\frac{||v_{0_t}||}{\min(1,Se)}\right)^2 + \left|\left|p+\frac{v_0}{Se}\right|\right|^2 + [[u]]^2.$$

Integrating from 0 to t, where $t \in [0, T]$, using Gronwall's inequality, and using the initial conditions for the transformed problem $u(0) = 0$ and $p(0) = -\frac{v_0}{Se}$, we obtain

$$[[u]]^2 + \left|\left| p + \frac{v_0}{Se} \right|\right|^2 \leq \frac{e^T}{(\min(1, Se))^2} ||v_{0_t}||^2_{L^2(0,T;H^1(\Omega))}.$$

Hence

$$||p(t)|| \ \leq \ \frac{\sqrt{e^T}}{\min(1, Se)} ||v_{0_t}||_{L^2(0,T;H^1(\Omega))} + \frac{||v_0||_{L^2(0,T;H^1(\Omega))}}{Se}, \tag{3.147}$$

and

$$||u(t)||_1 \ \leq \ \frac{\sqrt{e^T}}{\sqrt{2G}\min(1, Se)} ||v_{0_t}||_{L^2(0,T;H^1(\Omega))}. \tag{3.148}$$

3.3.5 Existence and uniqueness of weak solutions for nonhomogeneous boundary conditions

For problem III (3.43)–(3.50) (with nonhomogeneous boundary conditions, i.e., $h_1 \neq 0$), we assume that $h_1(t) \in C^{0,1}(0,T; L^2(\Gamma_f))$ and use the Rothe method to approximate the solution as done for problem I (3.27)–(3.34). Then for $j = 2, \cdots, m$ we have

$$a(w_j, v) \; - \; \alpha(z_j, \nabla \cdot v) \; = \; -(1-\beta)\alpha < z_j, v > \qquad \forall v \in V, \tag{3.149}$$

and

$$Se(z_j - z_{j-1}, q) \; + \; \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q) \; + \; h\frac{k}{\mu}(\nabla z_j, \nabla q) \; =$$

$$h < h_{1_j}, q > + (1-\beta)\alpha < w_j - w_{j-1}, q > \quad \forall q \in M. \tag{3.150}$$

At time $(j-1)$, $j = 2, \cdots, m$

$$a(w_{j-1}, v) \; - \; \alpha(z_{j-1}, \nabla \cdot v) \; = \; -(1-\beta)\alpha < z_{j-1}, v > \tag{3.151}$$

Subtracting (3.151) from (3.149), we get

$$a(w_j - w_{j-1}, v) \; - \; \alpha(z_j - z_{j-1}, \nabla \cdot v) \; = \; -(1-\beta)\alpha < z_j - z_{j-1}, v >, \tag{3.152}$$

and

$$Se(z_j - z_{j-1}, q) \; + \; \alpha(\nabla \cdot w_j - \nabla \cdot w_{j-1}, q) \; + \; h\frac{k}{\mu}(\nabla z_j, \nabla q) \; =$$

$$h < h_{1_j}, q > \; + \; (1-\beta)\alpha < w_j - w_{j-1}, q > . \tag{3.153}$$

Setting $j = 1$, assuming that (3.149) and (3.150) hold at $j = 0$, and using the previous equations (3.151)–(3.153), we obtain

$$a(w_1-w_0, w_1-w_0) \; - \; \alpha(z_1-z_0, \nabla\cdot w_1-\nabla\cdot w_0) \; = \; -(1-\beta)\alpha < z_1-z_0, w_1-w_0 >, \quad (3.154)$$

$$Se(z_1 - z_0, z_1 - z_0) \; + \; \alpha(\nabla\cdot w_1 - \nabla\cdot w_0, z_1 - z_0) \; + \; h\frac{k}{\mu}(\nabla z_1, \nabla z_1 - \nabla z_0) \; =$$

$$h < h_{1_1}, z_1 - z_0 > \; + \; (1-\beta)\alpha < z_1 - z_0, w_1 - w_0 >, \quad (3.155)$$

$$Se(z_1 - z_0, z_0) \; + \; \alpha(\nabla\cdot w_1 - \nabla\cdot w_0, z_0) \; + \; h\frac{k}{\mu}(\nabla z_1, \nabla z_0) \; =$$

$$h < h_{1_1}, z_0 > \; + \; (1-\beta)\alpha < z_0, w_1 - w_0 >, \quad (3.156)$$

$$a(w_0, w_1 - w_0) \; - \; \alpha(z_0, \nabla\cdot w_1 - \nabla\cdot w_0) \; = \; -(1-\beta)\alpha < z_0, w_1 - w_0 > . \quad (3.157)$$

Adding (3.154)–(3.157) and simplifying, we get

$$Se||z_1-z_0||^2 + [[w_1-w_0]]^2 + h\frac{k}{\mu}||\nabla z_1||^2 \le \; [[w_1-w_0]][[w_0]] + Se||z_1-z_0|| \, ||z_0|| + h < h_{1_1}, z_1 >$$

Using the homogeneous initial conditions ($w_0 = z_0 = 0$ (3.70)), we obtain

$$Se||z_1 - z_0||^2 \; + \; [[w_1 - w_0]]^2 \; + \; h\frac{k}{\mu}||\nabla z_1||^2 \le h||h_{1_1}||_{L^2(\Gamma_f)}||z_1||_{L^2(\Gamma_f)}. \quad (3.158)$$

Given $q \in M$ and applying theorem B.1, we have

$$||q||_{L^2(\Gamma_f)} \le c||q||_M, \qquad \text{for some } c > 0, \text{and } \forall q \in M. \quad (3.159)$$

We have by Poincare's inequality (Appendix A)

$$(\nabla q, \nabla q) = ||\nabla q||^2 \geq \delta ||q||_M^2, \qquad \text{for some } \delta > 0,\ \forall q \in M. \tag{3.160}$$

Therefore, using (3.159) and (3.160) with $q = z_1$, inequality (3.158) becomes

$$\begin{aligned} Se||z_1 - z_0||^2 + [[w_1 - w_0]]^2 + h\frac{k}{\mu}\delta||z_1||_M^2 &\leq hc||h_{1_1}||_{L^2(\Gamma_f)}||z_1||_M \\ &\leq hc\frac{\epsilon}{2}||h_{1_1}||_{L^2(\Gamma_f)}^2 + \frac{hc}{2\epsilon}||z_1||_M^2 \\ &\leq h\eta||h_{1_1}||_{L^2(\Gamma_f)}^2 + h\eta_1||z_1||_M^2. \end{aligned} \tag{3.161}$$

Where η and η_1 are positive constants that can be found with $\eta_1 < \frac{k}{\mu}\delta$, then

$$h\frac{k}{\mu}\delta||z_1||_M^2 \ - \ h\eta_1||z_1||_M^2 \ \geq \ 0, \tag{3.162}$$

and (3.161) yields

$$Se||z_1 - z_0||^2 \ + \ [[w_1 - w_0]]^2 \ \leq \ h\eta||h_{1_1}||_{L^2(\Gamma_f)}^2. \tag{3.163}$$

And ($w_0 = z_0 = 0$ (3.70))

$$Se||z_1||^2 \ + \ [[w_1]]^2 \ \leq \ h\eta||h_{1_1}||_{L^2(\Gamma_f)}^2. \tag{3.164}$$

Setting now $j = 2$ we have

$$a(w_2 - w_1, w_2 - w_1) \ - \ \alpha(z_2 - z_1, \nabla\cdot w_2 - \nabla\cdot w_1) \ = \ -(1-\beta)\alpha < z_2 - z_1, w_2 - w_1 >,$$

$$Se(z_2 - z_1, z_2 - z_1) \; + \; \alpha(\nabla \cdot w_2 - \nabla \cdot w_1, z_2 - z_1) \; + \; h\frac{k}{\mu}(\nabla z_2, \nabla z_2 - \nabla z_1) \; =$$

$$h < h_{1_1}, z_2 - z_1 > \; + \; (1-\beta)\alpha < w_2 - w_1, z_2 - z_1 >,$$

$$a(w_1, w_2 - w_1) \; - \; \alpha(z_1, \nabla \cdot w_2 - \nabla \cdot w_1) \; = \; -(1-\beta)\alpha < z_1, w_2 - w_1 >,$$

$$Se(z_2 - z_1, z_1) \; + \; \alpha(\nabla \cdot w_2 - \nabla \cdot w_1, z_1) \; + \; h\frac{k}{\mu}(\nabla z_2, \nabla z_1) \; =$$

$$h < h_{1_1}, z_1 > \; + \; (1-\beta)\alpha < w_2 - w_1, z_1 >,$$

$$a(w_2, w_2 - w_1) \; - \; \alpha(z_2, \nabla \cdot w_2 - \nabla \cdot w_1) \; = \; -(1-\beta)\alpha < z_2, w_2 - w_1 >,$$

and

$$Se(z_2 - z_1, z_2) \; + \; \alpha(\nabla \cdot w_2 - \nabla \cdot w_1, z_2) \; + \; h\frac{k}{\mu}(\nabla z_2, \nabla z_2) \; =$$

$$h < h_{1_1}, z_2 > \; + \; (1-\beta)\alpha < w_2 - w_1, z_2 > .$$

If we add the previous six equations and use (3.159) and (3.160), we get

$$[[w_2 - w_1]]^2 \; + \; [[w_2]]^2 \; + \; Se||z_2 - z_1||^2 \; + \; Se||z_2||^2 \; + \; 2h\frac{k}{\mu}||\nabla z_2||_M^2$$
$$\leq [[w_1]]^2 \; + \; Se||z_1||^2 \; + \; 2h\eta||h_{1_2}||^2 \; + \; 2h\eta_1||z_2||_M^2.$$

Note that we are again using here the symmetry of the bilinear form $a(.,.)$ (3.15). Since $\eta_1 < \frac{k}{\mu}\delta$, then (by (3.162))

$$h\frac{k}{\mu}\delta||z_2||_M^2 \; - \; h\eta_1||z_2||_M^2 \; \geq \; 0,$$

therefore,

$$[[w_2 - w_1]]^2 + [[w_2]]^2 + Se||z_2 - z_1||^2 + Se||z_2||^2 \leq [[w_1]]^2 \ + \ Se||z_1||^2 \ + \ 2h\eta||h_{1_2}||^2.$$

The first and third terms in this inequality are positive, so we can write

$$[[w_2]]^2 \ + \ Se||z_2||^2 \ \leq \ [[w_1]]^2 \ + \ Se||z_1||^2 \ + \ 2h\eta||h_{1_2}||^2. \tag{3.165}$$

Repeating the same process, we get

$$[[w_j - w_{j-1}]]^2 + [[w_j]]^2 + Se||z_j - z_{j-1}||^2 + Se||z_j||^2$$

$$\leq \ [[w_{j-1}]]^2 + Se||z_{j-1}||^2 + 2h\eta||h_{1_j}||^2, \tag{3.166}$$

or

$$[[w_j]]^2 \ + \ Se||z_j||^2 \ \leq \ [[w_{j-1}]]^2 + Se||z_{j-1}||^2 + 2h\eta||h_{1_j}||^2. \tag{3.167}$$

That is we obtained

$[[w_1]]^2 \ + \ Se||z_1||^2 \ \leq \ h\eta||h_{1_1}||^2$

$[[w_2]]^2 \ + \ Se||z_2||^2 \ \leq \ [[w_1]]^2 + Se||z_1||^2 + 2h\eta||h_{1_2}||^2$

$\vdots$

$[[w_j]]^2 \ + \ Se||z_j||^2 \ \leq \ [[w_{j-1}]]^2 + Se||z_{j-1}||^2 + 2h\eta||h_{1_j}||^2$

Adding these to obtain

$$\begin{aligned}[[w_j]]^2 \ + \ Se||z_j||^2 &\leq h\eta||h_{1_1}||^2 + 2h\eta\Big[||h_{1_2}||^2 + \cdots + ||h_{1_j}||^2\Big] \\ &\leq 2h\eta\Big[||h_{1_1}||^2 + ||h_{1_2}||^2 + \cdots + ||h_{1_j}||^2\Big].\end{aligned}$$

Since $h_1(t) \in C^{0,1}(0,T;L^2(\Gamma_f))$, then there exists a constant d such that $||\frac{h_1(t+h)-h_1(t)}{h}|| \leq d$ for all t, $t+h \in I$, (see [10]). Then, $||h_1(t)||$ is a continuous function on I and so $||h_1(t)||$ attains a maximum on I, say $||\tilde{H}||$ i.e.,

$$\max_{t\in I} ||h_1(t)|| = ||\tilde{H}||.$$

Consequently,

$$\begin{aligned}
[[w_j]]^2 \ + \ Se||z_j||^2 &\leq 2h\eta j||\tilde{H}||^2 \\
&\leq 2\eta T||\tilde{H}||^2 \qquad (\text{since } h = \frac{T}{m}),
\end{aligned}$$

and

$$||w_j||_1 \ \leq \ ||\tilde{H}||\sqrt{\frac{\eta T}{G}} \qquad \text{and} \qquad ||z_j|| \ \leq \ ||\tilde{H}||\sqrt{\frac{2\eta T}{Se}}. \tag{3.168}$$

We have $h_1(t) \in C^{0,1}(0,T;L^2(\Gamma_f))$ and assume that $h_1(0) = 0$, then there exists a constant $C > 0$ such that $||h_1(\tau_2) - h_1(\tau_1)|| \ \leq \ C^2|\tau_2 - \tau_1|^2$.

Then, $||h_1(h)||^2 \ = \ ||h_1(h) - h_1(0)|| \ \leq \ C^2h^2$.

Inequalities (3.164)–(3.167) with $||h_{1_j}||^2 \leq C^2h^2$ yield

$[[w_1]]^2 \ + \ Se||z_1||^2 \ \leq \ h\eta||h_{1_1}||^2 \leq h^3\eta C^2$

$[[w_2]]^2 \ + \ Se||z_2||^2 \ \leq \ [[w_1]]^2 + Se||z_1||^2 + 2h^3\eta C^2$

$\vdots$

$[[w_{j-1}]]^2 + Se||z_{j-1}||^2 \ \leq \ [[w_{j-2}]]^2 + Se||z_{j-2}||^2 + 2h^3\eta C^2$

$[[w_j - w_{j-1}]]^2 + Se||z_j - z_{j-1}||^2 \ \leq \ [[w_{j-1}]]^2 + Se||z_{j-1}||^2 + 2h^3\eta C^2$

Adding these inequalities and simplifying, we get

$[[w_j - w_{j-1}]]^2 + Se||z_j - z_{j-1}||^2 \ \leq \ h^3\eta C^2 \ + \ 2(j-1)h^3\eta C^2$

Using $W_j = \frac{w_j - w_{j-1}}{h}$ and $Z_j = \frac{z_j - z_{j-1}}{h}$, the previous inequality becomes

$$
\begin{aligned}
[[W_j]]^2 + Se||Z_j||^2 &\leq h\eta C^2 \ + \ 2(j-1)h\eta C^2 \\
&\leq 2jh\eta C^2 \\
&\leq 2T\eta C^2 \qquad (\text{since } h = \frac{T}{m}),
\end{aligned}
$$

therefore,

$$||W_j||_1 \ \leq \ C\sqrt{\frac{\eta T}{G}} \qquad \text{and} \qquad ||Z_j|| \ \leq \ C\sqrt{\frac{2\eta T}{Se}}. \tag{3.169}$$

The estimates obtained in (3.168) and (3.169) are independent of h, thus remain valid for an arbitrary mesh d_n. That is, for every positive integer n and $j = 1, \cdots, m2^{n-1}$

$$||w_j^n||_1 \ \leq \ ||\tilde{H}||\sqrt{\frac{\eta T}{G}}, \qquad\qquad ||z_j^n|| \ \leq \ ||\tilde{H}||\sqrt{\frac{2\eta T}{Se}}, \tag{3.170}$$

$$||W_j^n||_1 \ \leq \ C\sqrt{\frac{\eta T}{G}}, \qquad \text{and} \qquad ||Z_j^n|| \ \leq \ C\sqrt{\frac{2\eta T}{Se}}. \tag{3.171}$$

These basic estimates can then be used to show existence and uniqueness of weak solutions as done for problem I (3.27)–(3.34).

3.3.6 Energy norm estimate for nonhomogeneous boundary condition

To find the energy norm for the poroelasticity problem with homogeneous initial condition and nonhomogeneous boundary condition, we follow the same steps done in Section 3.3.2 for homogeneous initial and boundary condition. Then we get

$$a(u, u_t) \ + \ Se(p_t, p) \ + \ \tfrac{k}{\mu}(\nabla p, \nabla p) \ = < h_1, p >,$$

that is,

$$\frac{1}{2}\frac{d}{dt}[[u]]^2 + \frac{Se}{2}\frac{d}{dt}||p||^2 + \frac{k}{\mu}||\nabla p||^2 \leq \eta_1||h_1||^2_{L^2(\Gamma_{tf})} + \eta_2||p||^2_M \tag{3.172}$$

with $\eta_2 < \frac{k}{\mu}c$. By Poincare's inequality, $||\nabla p||^2 \geq c||p||^2$, we have

$$\tfrac{k}{\mu}c||p||^2_M - \eta_2||p||^2_M \geq 0,$$

and so,

$$\frac{d}{dt}[[u]]^2 + Se\frac{d}{dt}||p||^2 \leq 2\eta_1||h_1||^2_{L^2(\Gamma_{tf})}.$$

Using the norm: $|||(u,p)||| = \Big([[u]]^2 + Se||p||^2\Big)^{\frac{1}{2}}$, we get

$$\frac{d}{dt}\Big(|||(u,p)|||^2\Big) \leq 2\eta_1||h_1||^2_{L^2(\Gamma_{tf})}.$$

Integrating from 0 to t, where $t \in [0,T]$, and using the homogeneous initial conditions ($u(0) = p(0) = 0$), we obtain

$$|||(u(t),p(t))|||^2 \leq 2\eta_1 T||h_1||^2_{L^2(0,T;L^2(\Gamma_{tf}))}.$$

Therefore,

$$[[u]]^2 + Se||p||^2 \leq 2\eta_1 T||h_1||^2_{L^2(0,T;L^2(\Gamma_{tf}))},$$

from which

$$||p(t)|| \ \leq \ \sqrt{\frac{2\eta_1 T}{Se}}||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))}, \tag{3.173}$$

and

$$||u(t)||_1 \ \leq \ \sqrt{\frac{\eta_1 T}{G}}||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))}. \tag{3.174}$$

We can now obtain the energy norm estimates for the fully coupled poroelasticity system. We obtained the energy norm estimates for homogeneous boundary and initial conditions (3.112) and (3.113)

$$||u(t)||_1 \ \leq \ \frac{\sqrt{e^T}}{\sqrt{2G}\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))}, \quad ||p(t)|| \leq \frac{\sqrt{e^T}}{\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))}, \tag{3.175}$$

the energy norm estimates for nonhomogeneous initial conditions (3.147) and (3.148)

$$\begin{aligned} ||u(t)||_1 \ &\leq \ \sqrt{\frac{||v_0||^2}{2GSe} + \frac{T}{2\epsilon G}\left(\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right)^2}, \\ ||p(t)|| \ &\leq \ \sqrt{\frac{||v_0||^2}{Se^2} + \frac{T}{\epsilon Se}\left(\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right)^2}, \end{aligned} \tag{3.176}$$

and the energy norm estimates for nonhomogeneous boundary conditions (3.173) and (3.174)

$$||u(t)||_1 \leq \sqrt{\frac{\eta_1 T}{G}}||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))}, \qquad ||p(t)|| \leq \sqrt{\frac{2\eta_1 T}{Se}}||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))}. \tag{3.177}$$

Denote by

$$C_1 = \frac{\sqrt{e^T}}{\sqrt{2G}\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))},$$

$$C_2 = \frac{\sqrt{e^T}}{\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))},$$

$$C_3 = \sqrt{\frac{||v_0||^2}{2GSe} + \frac{T}{2\epsilon G}\left(\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right)^2},$$

$$C_4 = \sqrt{\frac{||v_0||^2}{Se^2} + \frac{T}{\epsilon Se}\left(\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right)^2},$$

$$C_5 = \sqrt{\frac{\eta_1 T}{G}}||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))},$$

and

$$C_6 = \sqrt{\frac{2\eta_1 T}{Se}}||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))}.$$

Hence the energy norm estimates for the fully coupled quasi-static poroelasticity problem is

$$||u(t)||_1 \leq C_1 + C_3 + C_5, \qquad \text{and} \qquad ||p(t)|| \leq C_2 + C_4 + C_6. \tag{3.178}$$

Energy norm estimates for the discrete poroelasticity problem

Let $V^h \subset V$ and $M^h \subset M$ be finite dimensional spaces. The weak formulation for the discrete problem with homogeneous initial and boundary conditions is: find $u^h \in V^h$ and $p^h \in M^h$ such that

$$a(u^h, v^h) - \alpha(p^h, \nabla \cdot v^h) \;=\; -(1-\beta)\alpha < p^h, v^h >, \qquad \forall v^h \in V^h,$$

$$Se(p_t^h, q^h) + \alpha(\nabla \cdot u_t^h, q^h) + \frac{k}{\mu}(\nabla p^h, \nabla q^h) \;=$$

$$(Q, q^h) + (1-\beta)\alpha < u_t^h, q^h >, \quad \forall q^h \in M^h.$$

Letting $v^h = u_t^h$ and $q^h = p^h$ and adding the two previous equation, we get

$$a(u^h, u_t^h) \ + \ Se(p_t^h, p^h) \ + \ \tfrac{k}{\mu}(\nabla p^h, \nabla p^h) \ = \ (Q, p^h),$$

that is,

$$\frac{1}{2}\frac{d}{dt}[[u^h]]^2 + \frac{Se}{2}\frac{d}{dt}||p^h||^2 \leq ||Q||\ ||p^h||.$$

From which it follows that

$$\begin{aligned}\frac{d}{dt}\left([[u^h]]^2 + ||p^h||^2\right) &\leq \frac{2||Q||}{\min(1, Se)}||p^h||,\\ &\leq \frac{||Q||^2}{(\min(1, Se))^2} + ||p^h||^2 + [[u]]^2.\end{aligned}$$

Applying Gronwall's inequality and the fact that $u^h(0) = p^h(0) = 0$ (from (3.70)), we obtain

$$||p^h(t)|| \ \leq \ \frac{\sqrt{e^T}}{\min(1, Se)}||Q||_{L^2(0,T;L^2(\Omega))}, \tag{3.179}$$

$$||u^h(t)||_1 \ \leq \ \frac{\sqrt{e^T}}{\sqrt{2G}\min(1, Se)}||Q||_{L^2(0,T;L^2(\Omega))}. \tag{3.180}$$

The estimates (3.179) and (3.179) are the same as (3.112) and (3.113) obtained for the semi-discrete problem with homogeneous initial and boundary conditions. Similarly, we can get the same energy estimates for the nonhomogeneous initial conditions and for the nonhomogeneous boundary conditions problems as done in sections 3.3.4 and 3.3.6. Therefore, we can obtain

$$||u^h(t)||_1 \leq C_1 + C_3 + C_5, \qquad \text{and} \qquad ||p^h(t)|| \leq C_2 + C_4 + C_6, \tag{3.181}$$

with $C_i, i = 1, \cdots, 6$ are as defined above.

3.4 Error estimates

In this section, we will obtain error estimates for the semi-discrete and for the fully discrete poroelasticity problem (3.19)–(3.26).

3.4.1 Error estimates for the semi-discrete problem

We will derive estimates of the difference between the solutions $u(t)$ and $p(t)$ and the approximations $u_n(t)$ and $p_n(t)$ by the method of discretization in time.

Since the poroelasticity problem (3.19)–(3.26) is linear, its error estimates can be written as the sum of error estimates for homogeneous initial conditions and error estimates for nonhomogeneous initial conditions.

Error estimates for homogeneous initial conditions

Consider the poroelasticity problem (3.19)–(3.26) with homogeneous initial conditions. Recall the definitions of the functions $u_n(t)$, $p_n(t)$, $\tilde{u}_n(t)$, $\tilde{p}_n(t)$, $U_n(t)$, and $P_n(t)$:

$$u_n(t) \;=\; w^n_{j-1} \;+\; (t - t^n_{j-1})W^n_j, \qquad t \in I^n_j = [t^n_{j-1}, t^n_j],$$

$$p_n(t) \;=\; z^n_{j-1} \;+\; (t - t^n_{j-1})Z^n_j, \qquad t \in I^n_j = [t^n_{j-1}, t^n_j],$$

$$\tilde{u}_n(0) = w^n_1$$

$$\tilde{u}_n(t) = w^n_j \qquad \text{for } t \in \tilde{I}^n_j = (t^n_{j-1}, t^n_j], \quad j = 1, \cdots, m2^{n-1},$$

$$\tilde{p}_n(0) = z_1^n$$

$$\tilde{p}_n(t) = z_j^n \qquad \text{for } t \in \tilde{I}_j^n = (t_{j-1}^n, t_j^n], \quad j = 1, \cdots, m2^{n-1},$$

$$U_n(0) = W_1^n$$

$$U_n(t) = W_j^n \qquad \text{for } t \in \tilde{I}_j^n := (t_{j-1}^n, t_j^n], \quad j = 1, \cdots, m2^{n-1},$$

and,

$$P_n(0) = Z_1^n$$

$$P_n(t) = Z_j^n \qquad \text{for } t \in \tilde{I}_j^n =: (t_{j-1}^n, t_j^n], \quad j = 1, \cdots, m2^{n-1}.$$

In section 3.3.1 (homogeneous initial and boundary conditions), we obtained (3.84)

$$||W_j^n||_1 \leq \frac{||Q||}{\sqrt{2GSe}} \quad \text{and} \quad ||Z_j^n|| \leq \frac{||Q||}{Se}, \tag{3.182}$$

and in section 3.3.5 (nonhomogeneous boundary conditions), we obtained (3.171)

$$||W_j^n||_1 \ \leq \ C\sqrt{\frac{\eta T}{G}}, \qquad \text{and} \qquad ||Z_j^n|| \ \leq \ C\sqrt{\frac{2\eta T}{Se}}. \tag{3.183}$$

Therefore since the poroelasticity system is linear, we have

$$||W_j^n||_1 \leq \frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}} \quad \text{and} \quad ||Z_j^n|| \leq \frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}}. \tag{3.184}$$

From which it follows that

$$||U_n(t)||_1 \leq \frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}}, \quad ||P_n(t)|| \leq \frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}} \ \forall t \in I = [0,T], \tag{3.185}$$

and

$$||\tilde{u}_n(0) - u_n(0)||_1 = ||w_1^n - w_0^n||_1 = ||h_n W_1^n||_1 \leq \left(\frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}} \right) h_n,$$

$$\begin{aligned} \tilde{u}_n(t) - u_n(t) &= w_j^n - w_{j-1}^n - (t - t_{j-1}^n) W_j^n \\ &= \left[h_n - (t - t_{j-1}^n) \right] W_j^n \qquad \forall t \in \tilde{I}_j^n. \end{aligned}$$

Similarly,

$$||\tilde{p}_n(0) - p_n(0)|| = ||z_1^n - z_0^n|| = ||Z_1^n h_n|| \leq \left(\frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}} \right) h_n,$$

$$\begin{aligned} \tilde{p}_n(t) - p_n(t) &= z_j^n - z_{j-1}^n - (t - t_{j-1}^n) Z_j^n \\ &= \left[h_n - (t - t_{j-1}^n) \right] Z_j^n \qquad \forall t \in \tilde{I}_j^n. \end{aligned}$$

We have in $\tilde{I}_j^n$: $0 < t - t_{j-1}^n \leq h_n$, then

$$||\tilde{u}_n(t) - u_n(t)||_1 \ \leq \ \left(\frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}} \right) h_n, \quad \forall \ t \in I = [0,T], \tag{3.186}$$

and

$$||\tilde{p}_n(t) - p_n(t)|| \ \leq \ \left(\frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}} \right) h_n, \quad \forall \ t \in I = [0,T]. \tag{3.187}$$

Rewriting the system (3.95) and (3.96) for n (instead of for n_k), then for almost every $t \in I$ we have

$$a(\tilde{u}_n, v) \;-\; \alpha(\tilde{p}_n, \nabla \cdot v) \;=\; -(1-\beta)\alpha < \tilde{p}_n, v >, \tag{3.188}$$

and

$$Se(P_n, q) \;+\; \alpha(\nabla \cdot U_n, q) \;+\; \frac{k}{\mu}(\nabla \tilde{p}_n, \nabla q) \;=$$

$$(Q, q) \;+\; < h_1, q > \;+\; (1-\beta)\alpha < U_n, q > . \tag{3.189}$$

Rewrite (3.188) and (3.189) for m

$$a(\tilde{u}_m, v) \;-\; \alpha(\tilde{p}_m, \nabla \cdot v) \;=\; -(1-\beta)\alpha < \tilde{p}_m, v >, \tag{3.190}$$

and

$$Se(P_m, q) \;+\; \alpha(\nabla \cdot U_m, q) \;+\; \frac{k}{\mu}(\nabla \tilde{p}_m, \nabla q) \;=$$

$$(Q, q) \;+\; < h_1, q > \;+\; (1-\beta)\alpha < U_m, q > . \tag{3.191}$$

Subtracting (3.190) from (3.188) and (3.191) from (3.189), we get

$$a(\tilde{u}_n - \tilde{u}_m, v) \;-\; \alpha(\tilde{p}_n - \tilde{p}_m, \nabla \cdot v) \;=\; -(1-\beta)\alpha < \tilde{p}_n - \tilde{p}_m, v >, \tag{3.192}$$

$$Se(P_n - P_m, q) \;+\; \alpha(\nabla \cdot U_n - \nabla \cdot U_m, q) \;+\; \frac{k}{\mu}(\nabla \tilde{p}_n - \nabla \tilde{p}_m, \nabla q) \;=$$

$$+\; (1-\beta)\alpha < U_n - U_m, q > . \tag{3.193}$$

Letting $v = U_n - U_m$, $q = \tilde{p}_n - \tilde{p}_m$, and adding (3.192) and (3.193), we obtain

$$a(\tilde{u}_n - \tilde{u}_m, U_n - U_m) + Se(P_n - P_m, \tilde{p}_n - \tilde{p}_m) + \frac{k}{\mu}(\nabla\tilde{p}_n - \nabla\tilde{p}_m, \nabla\tilde{p}_n - \nabla\tilde{p}_m) \;=\; 0. \quad (3.194)$$

That is

$$a(\tilde{u}_n - \tilde{u}_m - (u_n - u_m), U_n - U_m) + Se(P_n - P_m, \tilde{p}_n - \tilde{p}_m - (p_n - p_m))$$

$$+\frac{k}{\mu}(\nabla\tilde{p}_n - \nabla\tilde{p}_m, \nabla\tilde{p}_n - \nabla\tilde{p}_m) \;=\; -a(u_n - u_m, U_n - U_m) - Se(P_n - P_m, p_n - p_m). \quad (3.195)$$

Since $U_n - U_m$ and $P_n - P_m$ are the derivatives of $u_n - u_m$ and $p_n - p_m$, respectively, then

$$a(u_n - u_m, U_n - U_m) = \frac{1}{2}\frac{d}{dt}[[u_n - u_m]]^2$$

and

$$(p_n - p_m, P_n - P_m) = \frac{1}{2}\frac{d}{dt}||p_n - p_m||^2.$$

Furthermore, using the continuity of the bilinear form $a(\cdot,\cdot)$, (3.11) and denoting by $C_1 = \max\Big(2G, \frac{6G\nu}{1-2\nu}\Big)$, (3.195) becomes

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}[[u_n - u_m]]^2 &+ \frac{Se}{2}\frac{d}{dt}||p_n - p_m||^2 \\
&\le C_1\Big(||\tilde{u}_n - \tilde{u}_m - (u_n - u_m)||_1 ||U_n - U_m||_1\Big) \\
&\qquad + Se||\tilde{p}_n - \tilde{p}_m - (p_n - p_m)||\;||P_n - P_m|| \\
&\le C_1\Big(||\tilde{u}_n - u_n||_1 + ||\tilde{u}_m - u_m||_1\Big)\Big(||U_n||_1 + ||U_m||_1\Big) \\
&\qquad + Se\Big(||\tilde{p}_n - p_n|| + ||\tilde{p}_m - p_m||\Big)\;\Big(||P_n|| + ||P_m||\Big) \qquad (3.196)
\end{aligned}$$

From (3.185)–(3.187), (3.196) becomes

$$\frac{1}{2}\frac{d}{dt}[[u_n - u_m]]^2 + \frac{Se}{2}\frac{d}{dt}||p_n - p_m||^2$$
$$\leq 2C_1\left(\frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}}\right)^2 (h_n + h_m)$$
$$+2Se\left(\frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}}\right)^2 (h_n + h_m). \tag{3.197}$$

Let us denote by

$$C_2 = 2\left[C_1\left(\frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}}\right)^2 + Se\left(\frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}}\right)^2\right],$$

then (3.197) becomes

$$\frac{d}{dt}\left([[u_n - u_m]]^2 + Se||p_n - p_m||^2\right) \leq C_2(h_n + h_m). \tag{3.198}$$

Integrating (3.198) from 0 to T, we obtain

$$[[u_n - u_m]]^2 + Se||p_n - p_m||^2 \leq \int_0^T C_2(h_n + h_m)dt, \qquad \forall t \in I.$$

Thus

$$2G||u_n(t) - u_m(t)||_1^2 + Se||p_n(t) - p_m(t)||^2 \leq C_2T(h_n + h_m). \tag{3.199}$$

We have $u_m(t) \to u(t)$ in $H^1(\Omega)$ for almost every $t \in I$, $p_m(t) \to p(t)$ in $L^2(\Omega)$ for almost every $t \in I$, and $h_m \to 0$ for $m \to \infty$, thus

$$2G||u_n(t) - u(t)||_1^2 \leq C_2Th_n, \qquad \forall\, t \in I,$$

and

$$Se||p_n(t) - p(t)||^2 \leq C_2 T h_n, \qquad \forall\ t \in I.$$

Hence

$$||u_n(t) - u(t)||_1 \leq \sqrt{\frac{C_2 T h_n}{2G}}, \qquad \forall\ t \in I, \tag{3.200}$$

and

$$||p_n(t) - p(t)|| \leq \sqrt{\frac{C_2 T h_n}{Se}}, \qquad \forall\ t \in I, \tag{3.201}$$

where

$$C_2 = 2\left[C_1\left(\frac{||Q||}{\sqrt{2GSe}} + C\sqrt{\frac{\eta T}{G}}\right)^2 + Se\left(\frac{||Q||}{Se} + C\sqrt{\frac{2\eta T}{Se}}\right)^2\right],$$

and

$$C_1 = \max\left(2G, \frac{6G\nu}{1-2\nu}\right).$$

Error estimates for nonhomogeneous initial conditions

Consider now the poroelasticity system (3.19)–(3.26) with nonhomogeneous initial conditions. Then in section 3.3.3, we obtained (3.143) and (3.144)

$$||W_j^n||_1 \;\leq\; \frac{2k}{\sqrt{\epsilon\mu}}\;\frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}}\frac{||\nabla v_0||}{Se}$$

and

$$||Z_j^n|| \;\leq\; \frac{2k}{\sqrt{\epsilon\mu}}\;\frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}}\frac{||\nabla v_0||}{Se}.$$

Denote by

$$C = \frac{2k}{\sqrt{\epsilon\mu}}\;\frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}}\frac{||\nabla v_0||}{Se},$$

then

$$||W_j^n||_1 \;\leq\; C \qquad\qquad \text{and} \qquad\qquad ||Z_j^n|| \;\leq\; C.$$

The functions $u_n(t)$, $p_n(t)$, $\tilde{u}_n(t)$, $\tilde{p}_n(t)$, $U_n(t)$, and $P_n(t)$ are as defined above for the homogeneous initial condition case. Therefore,

$$||U_n(t)||_1 \;\leq\; C, \qquad\qquad ||P_n|| \;\leq\; C, \tag{3.202}$$

$$||\tilde{u}_n(0) - u_n(0)||_1 = ||w_1^n - w_0^n||_1 = ||h_n W_1^n||_1 \leq\; Ch_n,$$

and

$$\begin{aligned} \tilde{u}_n(t) - u_n(t) &= w_j^n - w_{j-1}^n - (t - t_{j-1}^n)W_j^n \\ &= \left[h_n - (t - t_{j-1}^n)\right] W_j^n \qquad\qquad \forall t \in \tilde{I}_j^n. \end{aligned}$$

Similarly,

$$||\tilde{p}_n(0) - p_n(0)|| = ||z_1^n - z_0^n|| = ||h_n Z_1^n|| \leq Ch_n,$$

and

$$\begin{aligned} \tilde{p}_n(t) - p_n(t) &= z_j^n - z_{j-1}^n - (t - t_{j-1}^n) Z_j^n \\ &= \left[h_n - (t - t_{j-1}^n)\right] Z_j^n \qquad\qquad \forall t \in \tilde{I}_j^n. \end{aligned}$$

Since in $\tilde{I}_j^n$: $0 < t - t_{j-1}^n \leq h_n$, then

$$||\tilde{u}_n(t) - u_n(t)||_1 \;\leq\; Ch_n, \quad \forall\, t \in I = [0, T], \tag{3.203}$$

and

$$||\tilde{p}_n(t) - p_n(t)|| \;\leq\; Ch_n, \quad \forall\, t \in I = [0, T]. \tag{3.204}$$

For almost every $t \in I$, the system (3.117) and (3.118), corresponding to mesh d_n, can be written as

$$a(\tilde{u}_n, v) \;-\; \alpha(\tilde{p}_n, \nabla \cdot v) \;=\; -(1-\beta)\alpha < \tilde{p}_n, v >, \tag{3.205}$$

$$Se(P_n, q) \;+\; \alpha(\nabla \cdot U_n, q) \;+\; \frac{k}{\mu}(\nabla \tilde{p}_n, \nabla q) \;=\; (1-\beta)\alpha < U_n, q > . \tag{3.206}$$

Rewriting (3.205) and (3.206) for m, we get

$$a(\tilde{u}_m, v) \;-\; \alpha(\tilde{p}_m, \nabla \cdot v) \;=\; -(1-\beta)\alpha < \tilde{p}_m, v >, \tag{3.207}$$

$$Se(P_m, q) \;+\; \alpha(\nabla \cdot U_m, q) \;+\; \frac{k}{\mu}(\nabla \tilde{p}_m, \nabla q) \;=\; (1-\beta)\alpha < U_m, q > . \tag{3.208}$$

Subtracting now (3.207) from (3.205) and (3.208) from (3.206), we get

$$a(\tilde{u}_n - \tilde{u}_m, v) \ - \ \alpha(\tilde{p}_n - \tilde{p}_m, \nabla \cdot v) \ = \ -(1-\beta)\alpha < \tilde{p}_n - \tilde{p}_m, v >, \tag{3.209}$$

$$Se(P_n - P_m, q) \ + \ \alpha(\nabla \cdot U_n - \nabla \cdot U_m, q) \ + \ \frac{k}{\mu}(\nabla \tilde{p}_n - \nabla \tilde{p}_m, \nabla q) \ =$$

$$(1-\beta)\alpha < U_n - U_m, q > . \tag{3.210}$$

Adding (3.209) and (3.210) with $v = U_n - U_m$ and $q = \tilde{p}_n - \tilde{p}_m$, we get

$$a(\tilde{u}_n - \tilde{u}_m, U_n - U_m) + Se(P_n - P_m, \tilde{p}_n - \tilde{p}_m) + \frac{k}{\mu}(\nabla \tilde{p}_n - \nabla \tilde{p}_m, \nabla \tilde{p}_n - \nabla \tilde{p}_m) \ = \ 0,$$

which is exactly (3.194). Hence we can obtain (3.196)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[[u_n - u_m]]^2 + \frac{Se}{2}\frac{d}{dt}||p_n - p_m||^2 & \\ \leq C_1\Big(||\tilde{u}_n - u_n||_1 + ||\tilde{u}_m - u_m||_1\Big)\Big(||U_n||_1 + ||U_m||_1\Big) & \\ +Se\Big(||\tilde{p}_n - p_n|| + ||\tilde{p}_m - p_m||\Big)\ \Big(||P_n|| + ||P_m||\Big), & \end{aligned} \tag{3.211}$$

where $C_1 = \max\left(2G, \frac{6G\nu}{1-2\nu}\right)$ is the continuity constant for the bilinear form $a(\cdot,\cdot)$.
Using now (3.202)–(3.204), (3.211) becomes

$$\frac{d}{dt}\Big([[u_n - u_m]]^2 + ||p_n - p_m||^2\Big) \leq 2\frac{C^2}{\min(1, Se)}(C_1 + Se)(h_n + h_m). \tag{3.212}$$

Integrating (3.212) from 0 to T, we get

$$2G||u_n(t) - u_m(t)||_1^2 + ||p_n(t) - p_m(t)||^2 \leq 2\frac{C^2}{\min(1, Se)}(C_1 + Se)T(h_n + h_m).$$

We have $u_m(t) \to u(t)$ in $H^1(\Omega)$ for almost every $t \in I$, $p_m(t) \to p(t)$ in $L^2(\Omega)$ for almost every $t \in I$, and $h_m \to 0$ for $m \to \infty$, hence

$$||u_n(t) - u(t)||_1 \leq C\sqrt{\frac{(C_1 + Se)}{G\min(1, Se)}Th_n}, \qquad \forall\, t \in I, \tag{3.213}$$

and

$$||p_n(t) - p(t)|| \leq C\sqrt{\frac{2(C_1 + Se)}{\min(1, Se)}Th_n}, \qquad \forall\, t \in I, \tag{3.214}$$

with

$$C = \frac{2k}{\sqrt{\epsilon\mu}}\ \frac{\max(C_1, Se)}{(\min(2G, Se))^{\frac{3}{2}}}\frac{||\nabla v_0||}{Se} \qquad \text{and} \quad C_1 = \max\left(2G, \frac{6G\nu}{1 - 2\nu}\right).$$

3.4.2 Error estimates for the fully discrete problem

The weak formulation of the quasi-static poroelasticity problem is: find $u \in V$ and $p \in M$ such that

$$a(u, v) - \alpha(p, \nabla \cdot v) = (F, v) - (1-\beta)\alpha < p, v >, \qquad \forall v \in V, \tag{3.215}$$

$$Se(p_t, q) + \alpha(\nabla \cdot u_t, q) + \frac{k}{\mu}(\nabla p, \nabla q) = (Q, q) + < h_1, q > + (1-\beta)\alpha < u_t, q >, \quad \forall q \in M, \tag{3.216}$$

for almost every $t \in I$. Using backward time discretization, we get

$$a(u_i, v) - \alpha(p_i, \nabla \cdot v) = (F_i, v) - (1-\beta)\alpha < p_i, v >, \qquad \forall v \in V, \tag{3.217}$$

$$Se(p_i - p_{i-1}, q) + \alpha(\nabla \cdot u_i - \nabla \cdot u_{i-1}, q) + h\frac{k}{\mu}(\nabla p_i, \nabla q) = h(Q_i, q) + h < h_{1_i}, q > + (1-\beta)\alpha < u_i - u_{i-1}, q >, \quad \forall q \in M. \tag{3.218}$$

Let $V^h \subset V$ and $M^h \subset M$ be finite dimensional spaces. The weak formulation for the discrete problem is: find $u^h \in V^h$ and $p^h \in M^h$ such that

$$a(u_i^h, v^h) - \alpha(p_i^h, \nabla \cdot v^h) = (F_i, v^h) - (1-\beta)\alpha < p_i^h, v^h >, \qquad \forall v^h \in V^h, \tag{3.219}$$

$$Se(p_i^h - p_{i-1}^h, q^h) + \alpha(\nabla \cdot u_i^h - \nabla \cdot u_{i-1}^h, q^h) + h\frac{k}{\mu}(\nabla p_i^h, \nabla q^h) = h(Q_i, q^h) + h < h_{1_i}, q^h > + (1-\beta)\alpha < u_i^h - u_{i-1}^h, q^h >, \quad \forall q^h \in M^h. \tag{3.220}$$

Since $V^h \subset V$ and $M^h \subset M$, we have

$$a(u_i, v^h) - \alpha(p_i, \nabla \cdot v^h) \;=\; (F_i, v^h) - (1-\beta)\alpha < p_i, v^h >, \qquad \forall v^h \in V^h, \tag{3.221}$$

$$Se(p_i - p_{i-1}, q^h) + \alpha(\nabla \cdot u_i - \nabla \cdot u_{i-1}, q^h) + h\frac{k}{\mu}(\nabla p_i, \nabla q^h) \;=$$

$$h(Q_i, q^h) + h < h_{1_i}, q^h > +(1-\beta)\alpha < u_i - u_{i-1}, q^h >, \quad \forall q^h \in M^h. \tag{3.222}$$

Subtracting (3.221) from (3.219) and (3.222) from (3.220), we obtain

$$a(u_i^h - u_i, v^h) - \alpha(p_i^h - p_i, \nabla \cdot v^h) \;=\; -(1-\beta)\alpha < p_i^h - p_i, v^h >, \tag{3.223}$$

$$Se(p_i^h - p_i, q^h) + \alpha(\nabla \cdot u_i^h - \nabla \cdot u_i, q^h) + h\frac{k}{\mu}(\nabla p_i^h - \nabla p_i, \nabla q^h) \;=\; Se(p_{i-1}^h - p_{i-1}, q^h)$$

$$+\alpha(\nabla \cdot u_{i-1}^h - \nabla \cdot u_{i-1}, q^h) + (1-\beta)\alpha < u_i^h - u_i, q^h > -(1-\beta)\alpha < u_{i-1}^h - u_{i-1}, q^h > . \tag{3.224}$$

Letting $v^h = u_i^h - w^h$ and $q^h = p_i^h - z^h$ and using
$a(u_i^h - u_i, u_i^h - w^h) = a(u_i^h - u_i, u_i^h - u_i) + a(u_i^h - u_i, u_i - w^h)$, we get

$$a(u_i^h - u_i, u_i^h - u_i) - \alpha(p_i^h - p_i, \nabla \cdot u_i^h - \nabla \cdot w^h) \;=$$

$$-(1-\beta)\alpha < p_i^h - p_i, u_i^h - w^h > -a(u_i^h - u_i, u_i - w^h), \tag{3.225}$$

and

$$Se(p_i^h - p_i, p_i^h - p_i) + \alpha(\nabla \cdot u_i^h - \nabla \cdot u_i, p_i^h - z^h) + h\frac{k}{\mu}(\nabla p_i^h - \nabla p_i, \nabla p_i^h - \nabla p_i) =$$

$$Se(p_{i-1}^h - p_{i-1}, p_i^h - z^h) + \alpha(\nabla \cdot u_{i-1}^h - \nabla \cdot u_{i-1}, p_i^h - z^h) + (1-\beta)\alpha < u_i^h - u_i, p_i^h - z^h >$$

$$-(1-\beta)\alpha < u_{i-1}^h - u_{i-1}, p_i^h - z^h > - Se(p_i^h - p_i, p_i - z^h) - h\frac{k}{\mu}(\nabla p_i^h - \nabla p_i, \nabla p_i - \nabla z^h). \quad (3.226)$$

Adding (3.225) and (3.226) and using the coercivity and the continuity of the bilinear form $a(\cdot,\cdot)$ (with C is the continuity constant), we have

$$2G||u_i - u_i^h||_1^2 + Se||p_i - p_i^h||^2 + h\frac{k}{\mu}||\nabla p_i - \nabla p_i^h||^2$$

$$\leq \alpha||p_i - p_i^h||\,||\nabla \cdot u_i^h - \nabla \cdot w^h|| + \alpha||\nabla \cdot u_i - \nabla \cdot u_i^h||\,||p_i^h - z^h|| + C||u_i - u_i^h||_1||u_i - w^h||_1$$

$$+Se||p_i - p_i^h||\,||p_i - z^h|| + h\frac{k}{\mu}||\nabla p_i - \nabla p_i^h||\,||\nabla p_i - \nabla z^h|| + Se||p_{i-1} - p_{i-1}^h||\,||p_i^h - z^h||$$

$$+\alpha||\nabla \cdot u_{i-1} - \nabla \cdot u_{i-1}^h||\,||p_i^h - z^h|| + (1-\beta)\alpha||p_i - p_i^h||_{\Gamma_{tf}}||u_i^h - w^h||_{\Gamma_{tf}}$$

$$+(1-\beta)\alpha||u_i - u_i^h||_{\Gamma_{tf}}||p_i^h - z^h||_{\Gamma_{tf}} + (1-\beta)\alpha||u_{i-1}^h - u_{i-1}||_{\Gamma_{tf}}||p_i^h - z^h||_{\Gamma_{tf}}. \quad (3.227)$$

Using now $||\nabla \cdot u|| \leq \sqrt{3}||\nabla u|| \leq \sqrt{3}||u||_1$, $||p_i - p_i^h||_{\Gamma_{tf}} \leq ||p_i - p_i^h||_M \leq ||p_i - p_i^h||_1$, and Young's inequality ($ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$), (3.227) becomes

$$2G||u_i - u_i^h||_1^2 + Se||p_i - p_i^h||^2 + h\frac{k}{\mu}||\nabla p_i - \nabla p_i^h||^2$$

$$\le \frac{1}{2\epsilon_1}\left(\alpha\sqrt{3}||u_i^h - w^h||_1 + Se||p_i - z^h||\right)^2 + \frac{\epsilon_1}{2}||p_i - p_i^h||^2$$

$$+\frac{1}{2\epsilon_2}\left(\alpha\sqrt{3}||p_i^h - z^h|| + C||u_i - w^h||_1\right)^2 + \frac{\epsilon_2}{2}||u_i - u_i^h||_1^2 + h\frac{k}{\mu}\frac{\epsilon_3}{2}||\nabla p_i - \nabla p_i^h||^2$$

$$+h\frac{k}{\mu}\frac{1}{2\epsilon_3}||\nabla p_i - \nabla z^h||^2 + Se||p_{i-1} - p_{i-1}^h||\,||p_i^h - z^h|| + \alpha\sqrt{3}||u_{i-1} - u_{i-1}^h||_1||p_i^h - z^h||$$

$$+(1-\beta)\alpha\frac{\epsilon_4}{2}||p_i - p_i^h||_1^2 + (1-\beta)\alpha\frac{1}{2\epsilon_4}||u_i - w^h||_1^2 + (1-\beta)\alpha\frac{\epsilon_5}{2}||u_i - u_i^h||_1^2$$

$$+(1-\beta)\alpha\frac{1}{2\epsilon_5}||p_i^h - z^h||_1^2 + (1-\beta)\alpha||u_{i-1}^h - u_{i-1}||_1||p_i^h - z^h||_1. \tag{3.228}$$

By Poincare's inequality $||\nabla p_i - \nabla p_i^h||^2 \ge \delta||p_i - p_i^h||_1^2$, for some $\delta > 0$. Choose ϵ_3 and ϵ_4 sufficiently small such that $2h\frac{k}{\mu}\delta - (h\frac{k}{\mu}\epsilon_3 - (1-\beta)\alpha\epsilon_4) > 0$, choose ϵ_2 and ϵ_5 small enough so that $4G - (\epsilon_2 + (1-\beta)\alpha\epsilon_5) > 0$, and choose ϵ_1 sufficiently small so that $(2Se - \epsilon_1) > 0$. Furthermore, $||p_i - z^h|| \le h||p_i||_{H^2(\Omega)}$ and $||p_i^h - z^h|| \le h||p_i||_{H^2(\Omega)}$. Hence (3.228) becomes

$$(4G - (\epsilon_2 + (1-\beta)\alpha\epsilon_5))||u_i - u_i^h||_1^2 + (2Se - \epsilon_1)||p_i - p_i^h||^2$$

$$\le \frac{h^2}{\epsilon_1}\left(\alpha\sqrt{3}||u_i||_{H^2(\Omega)} + Se||p_i||_{H^2(\Omega)}\right)^2 + \frac{h^2}{\epsilon_2}\left(\alpha\sqrt{3}||p_i||_{H^2(\Omega)} + C||u_i||_{H^2(\Omega)}\right)^2$$

$$+h^2\frac{k}{\mu\epsilon_3}||p_i||_{H^2(\Omega)}^2 + 2hSe||p_{i-1} - p_{i-1}^h||\,||p_i||_{H^2(\Omega)}$$

$$+2h\alpha\sqrt{3}||u_{i-1} - u_{i-1}^h||_1||p_i||_{H^2(\Omega)} + h^2(1-\beta)\frac{\alpha}{\epsilon_4}||u_i||_{H^2(\Omega)}^2$$

$$+h^2(1-\beta)\frac{\alpha}{\epsilon_5}||p_i||_{H^2(\Omega)}^2 + 2h(1-\beta)\alpha||u_{i-1}^h - u_{i-1}||_1||p_i||_{H^2(\Omega)}. \tag{3.229}$$

Using again Young's inequality, we obtain

$$(4G-(\epsilon_2+(1-\beta)\alpha\epsilon_5))||u_i-u_i^h||_1^2+(2Se-\epsilon_1)||p_i-p_i^h||^2$$

$$\leq \frac{h^2}{\epsilon_1}\left(\alpha\sqrt{3}||u_i||_{H^2(\Omega)}+Se||p_i||_{H^2(\Omega)}\right)^2+\frac{h^2}{\epsilon_2}\left(\alpha\sqrt{3}||p_i||_{H^2(\Omega)}+C||u_i||_{H^2(\Omega)}\right)^2$$
$$+h^2||p_i||^2_{H^2(\Omega)}\left[\frac{k}{\mu\epsilon_3}+Se^2||p_{i-1}-p_{i-1}^h||^2+\alpha^2(3+(1-\beta)^2)||u_{i-1}-u_{i-1}^h||^2\right.$$
$$\left.+(1-\beta)\frac{\alpha}{\epsilon_5}\right]+h^2(1-\beta)\frac{\alpha}{\epsilon_4}||u_i||^2_{H^2(\Omega)}. \tag{3.230}$$

In section 3.3 we derived energy norm estimates (3.178) and (3.181)

$$||u(t)||_1 \leq C_1+C_3+C_5, \qquad ||p(t)|| \leq C_2+C_4+C_6,$$

and

$$||u^h(t)||_1 \leq C_1+C_3+C_5, \qquad ||p^h(t)|| \leq C_2+C_4+C_6,$$

where

$$C_1 = \frac{\sqrt{e^T}}{\sqrt{2G}\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))},$$

$$C_2 = \frac{\sqrt{e^T}}{\min(1,Se)}||Q||_{L^2(0,T;L^2(\Omega))},$$

$$C_3 = \sqrt{\frac{||v_0||^2}{2GSe}+\frac{T}{2\epsilon G}\left(\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right)^2},$$

$$C_4 = \sqrt{\frac{||v_0||^2}{Se^2}+\frac{T}{\epsilon Se}\left(\frac{k}{\mu}\frac{||\nabla v_0||}{Se}\right)^2},$$

$$C_5 = \sqrt{\frac{\eta_1 T}{G}} ||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))},$$

and

$$C_6 = \sqrt{\frac{2\eta_1 T}{Se}} ||h_1||_{L^2(0,T;L^2(\Gamma_{tf}))}.$$

Therefore,

$$\begin{aligned} ||u_{i-1} - u_{i-1}^h||^2 &\le \Big(||u_{i-1}|| + ||u_{i-1}^h|| \Big)^2 \\ &\le 4\left(C_1 + C_3 + C_5\right)^2 \end{aligned} \tag{3.231}$$

and

$$\begin{aligned} ||p_{i-1} - p_{i-1}^h||^2 &\le \Big(||p_{i-1}|| + ||p_{i-1}^h|| \Big)^2 \\ &\le 4\left(C_2 + C_4 + C_6\right)^2. \end{aligned} \tag{3.232}$$

Hence (3.230) becomes

$$\begin{aligned} &(4G-(\epsilon_2+(1-\beta)\alpha\epsilon_5))||u_i-u_i^h||_1^2+(2Se-\epsilon_1)||p_i-p_i^h||^2 \\ &\quad \le \frac{h^2}{\epsilon_1}\Big(\alpha\sqrt{3}||u_i||_{H^2(\Omega)} + Se||p_i||_{H^2(\Omega)}\Big)^2 + \frac{h^2}{\epsilon_2}\Big(\alpha\sqrt{3}||p_i||_{H^2(\Omega)} + C||u_i||_{H^2(\Omega)}\Big)^2 \\ &\qquad + h^2||p_i||_{H^2(\Omega)}^2 \Bigg[\frac{k}{\mu\epsilon_3} + 4Se^2\left(C_2 + C_4 + C_6\right)^2 + 4\alpha^2(3 + (1-\beta)^2)\left(C_1 + C_3 + C_5\right)^2 \\ &\qquad\qquad + (1-\beta)\frac{\alpha}{\epsilon_5}\Bigg] + h^2(1-\beta)\frac{\alpha}{\epsilon_4}||u_i||_{H^2(\Omega)}^2. \end{aligned} \tag{3.233}$$

Denoting by K the right hand side of (3.233), we get

$$||u_i - u_i^h||_1 \;\leq\; \sqrt{\frac{K}{4G - (\epsilon_2 + (1-\beta)\alpha\epsilon_5)}} \tag{3.234}$$

and

$$||p_i - p_i^h||_1 \;\leq\; \sqrt{\frac{K}{2Se - \epsilon_1}} \tag{3.235}$$

Chapter 4

Numerical methods

Our objective is to approximate concurrently solutions of the system of partial differential equations

$$-G\nabla^2 u - \frac{G}{1-2\nu}\nabla(\nabla \cdot u) + \alpha \nabla p = F, \qquad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$\frac{\partial}{\partial t}(Se\, p + \alpha \nabla \cdot u) - \nabla \cdot (\frac{k}{\mu}\nabla p) = Q, \qquad \text{in } \Omega \times (0,T), \tag{4.2}$$

for both the solid displacement u (a vector field) and the fluid pressure p (a scalar field). To this end, we developed several algorithms: 2dpflow, 3dpflow, and 3dupfem. These algorithms approximate the solution of each equation separately, approximating the displacement u in equation (4.1) assuming that the pressure p is given or approximating p in equation (4.2) assuming that u is given. For the fully coupled system, we first developed a segregated algorithm (it3dupfem) then a coupled algorithm (c3dupfem).

4.1 2-D algorithm for the diffusion equation: 2dpflow

A 2-dimension finite element method (2dpflow) was used to approximate the solution of the diffusion equation:

$$(\frac{\partial}{\partial t}(Se \cdot p + \alpha \nabla \cdot u) - \nabla \cdot (\frac{k}{\mu}\nabla p) \ = \ Q$$

for the fluid pressure p, assuming that the vector displacement u is known. The domain considered was a box with Dirichlet boundary condition on the top and homogeneous Neumann boundary condition on the left/right side and bottom.

A brief description of the numerical method:

Let $V = \{v : \nabla v$ is a piecewise continuous on Ω and $v|_\Gamma = 0\}$.

We start with finite element discretization in space: we multiply the diffusion equation by a test function v and integrate over the domain Ω to obtain

$$\int_\Omega Se\ p_t \cdot v\ d\Omega - \int_\Omega \frac{k}{\mu}(\triangle p) \cdot v\ d\Omega \ = \ \int_\Omega f \cdot v\ d\Omega.$$

Here f is our right hand side consisting of the two terms: the source term Q and the term containing the known vector displacement u.

Applying Green's formula and using the boundary condition, we get

$$\int_\Omega Sep_t \cdot v\ d\Omega + \int_\Omega \frac{k}{\mu}(\nabla p) \cdot (\nabla v)\ d\Omega \ = \ \int_\Omega f \cdot v\ d\Omega.$$

To approximate a solution on $\Omega \times (0, T)$, divide $(0, T)$ into n subintervals, each of length $\tau = \frac{T}{n}$, and $p(x, n\tau) \approx p^n(x)$.

Using finite difference backward time discretization, we obtain

$$\int_\Omega Se\frac{p^{n+1} - p^n}{\tau} \cdot v\ d\Omega + \int_\Omega \frac{k}{\mu}(\nabla p^n) \cdot (\nabla v)\ d\Omega \ = \ \int_\Omega f^n \cdot v\ d\Omega.$$

The superscript n denotes the discrete time level at which the function is evaluated and τ is the time step.

Rearranging the previous equation and assuming that Se, k, and μ are constants, then we have

$$\int_\Omega p^{n+1} \cdot v \; d\Omega \;=\; \int_\Omega p^n \cdot v \; d\Omega - \frac{\tau}{Se}\frac{k}{\mu}\int_\Omega (\nabla p^n)\cdot(\nabla v)\; d\Omega + \frac{\tau}{Se}\int_\Omega f^n \cdot v \; d\Omega. \qquad (4.3)$$

Construct $V^h \subset V$, where V^h is a finite dimensional space (the set of all functions which are linear on each subinterval and continuous on Ω). Construct a basis for V^h, choose $\varphi_j \in V^h$, $1 \leq j \leq n$, with

$$\varphi_j(x_i) = \begin{cases} 1 & \text{if } i = j, \quad 1 \leq i,j \leq n, \\ 0 & \text{if } i \neq j, \quad 1 \leq i,j \leq n. \end{cases}$$

Equation (4.3) is a system of linear algebraic equations of the form

$$M\; p^{n+1} \;=\; M\; p^n - C_1\; A\; p^n + C_2\; b,$$

$$M\; p^{n+1} \;=\; (M - C_1\; A)p^n + C_2\; b, \qquad (4.4)$$

where $C_1 = \frac{\tau}{Se}\frac{k}{\mu}$, $C_2 = \frac{\tau}{Se}$, $M = \int_\Omega \varphi(i)\cdot\varphi(j)$, $A = \int_\Omega (\nabla\varphi(i)\cdot\nabla\varphi(j))$, and $b = \int_\Omega f\varphi(j)$. Now, instead of expressing the right hand side of (4.4) entirely at time n, it is averaged at n and $n+1$. This is called the Crank-Nicolson method, the result is as follows

$$M\; p^{n+1} - M\; p^n \;=\; -\frac{C_1}{2}\; A\; p^{n+1} - \frac{C_1}{2}\; A\; p^n + C_2\; b.$$

Or equivalently,

$$(M + \frac{C_1}{2}\; A)p^{n+1} \;=\; (M - \frac{C_1}{2}\; A)p^n + C_2\; b.$$

We then approximate this system of equations for the scalar pore pressure p.

To test this program, data for which the exact solution is known are generated and compared to the approximate solution obtained from the developed algorithm.

The graph of 2Dpflow from MATLAB comparing the exact solution and the approximate solution is shown below:

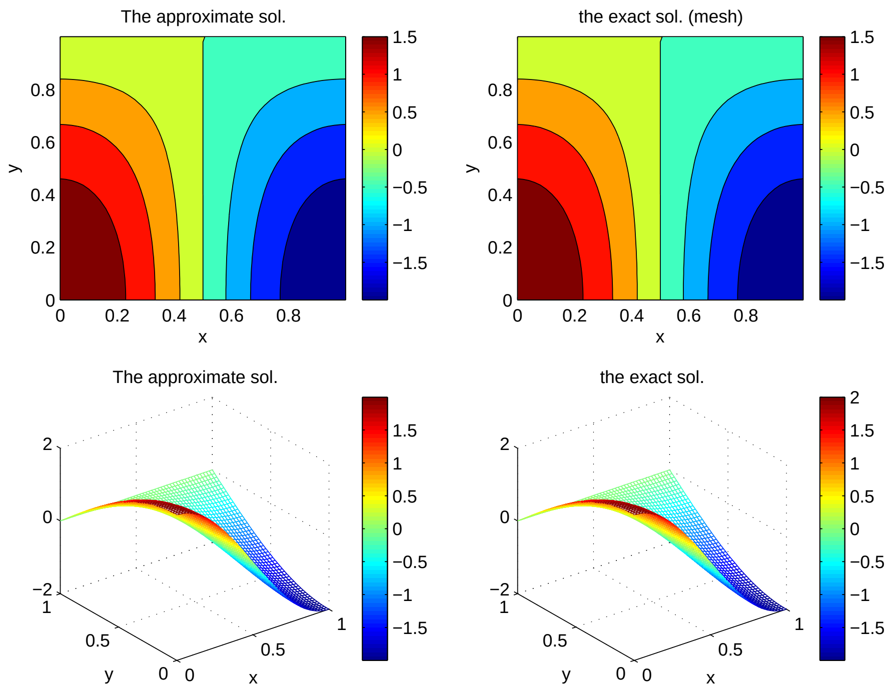

Figure 4.1: Comparing approximate solution p^n and exact solution p from 2dpflow at the last time step

Figure 4.1 depicts the fluid pressure in the square (0, 1)x(0, 1) at the last time step $(n = 1)$ for the approximate solution p^n and the exact solution p. As we can see from the graph, the approximate solution p^n on the left hand side looks exactly the same as the exact solution p on the right hand side.

4.2 3-D algorithm for the diffusion equation: 3dpflow

The same equation - the diffusion equation - is solved for the pressure p assuming u is given using a 3-dimensional finite element discretization in space and second order Crank Nicolson discretization in time. We consider the equation posed on a cube with homogeneous Dirichlet boundary conditions. The (MATLAB) code is 3dpflow.

Again the approximate solution and the exact solution (we solve the equation for data for which the exact solution is known) are compared to test and validate the program.

The plot for the approximate solution and the exact solution at the last time step is shown below.

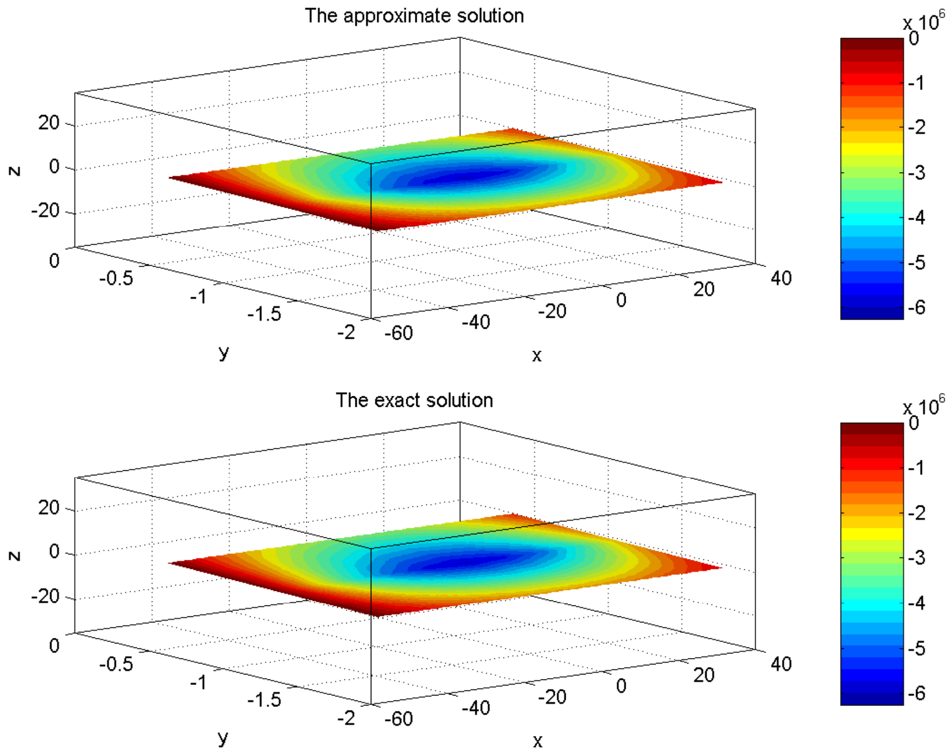

Figure 4.2: Comparing approximate solution p^n and exact solution p from 3dpflow at the last time step

From Figure 4.2, we clearly see that the approximate solution p^n and the exact solution p are similar which is evidence for the validity of our code 3dpflow.

4.3 3-D algorithm for the elasticity equation: 3dfem

The program 3dfem uses a 3-dimensional finite element method to approximate the displacements u in the elasticity equation:

$$-G\nabla^2 u - \frac{G}{1-2\nu}\nabla(\nabla \cdot u) + \alpha\nabla p \;=\; F,$$

assuming that the pore pressure p is given. The equation was approximated in a box with homogeneous Dirichlet boundary conditions.

This program was tested the same way by approximating the displacement u^n and comparing it to the exact solution u. So, the exact solution and the approximate solution are compared in the following plot.

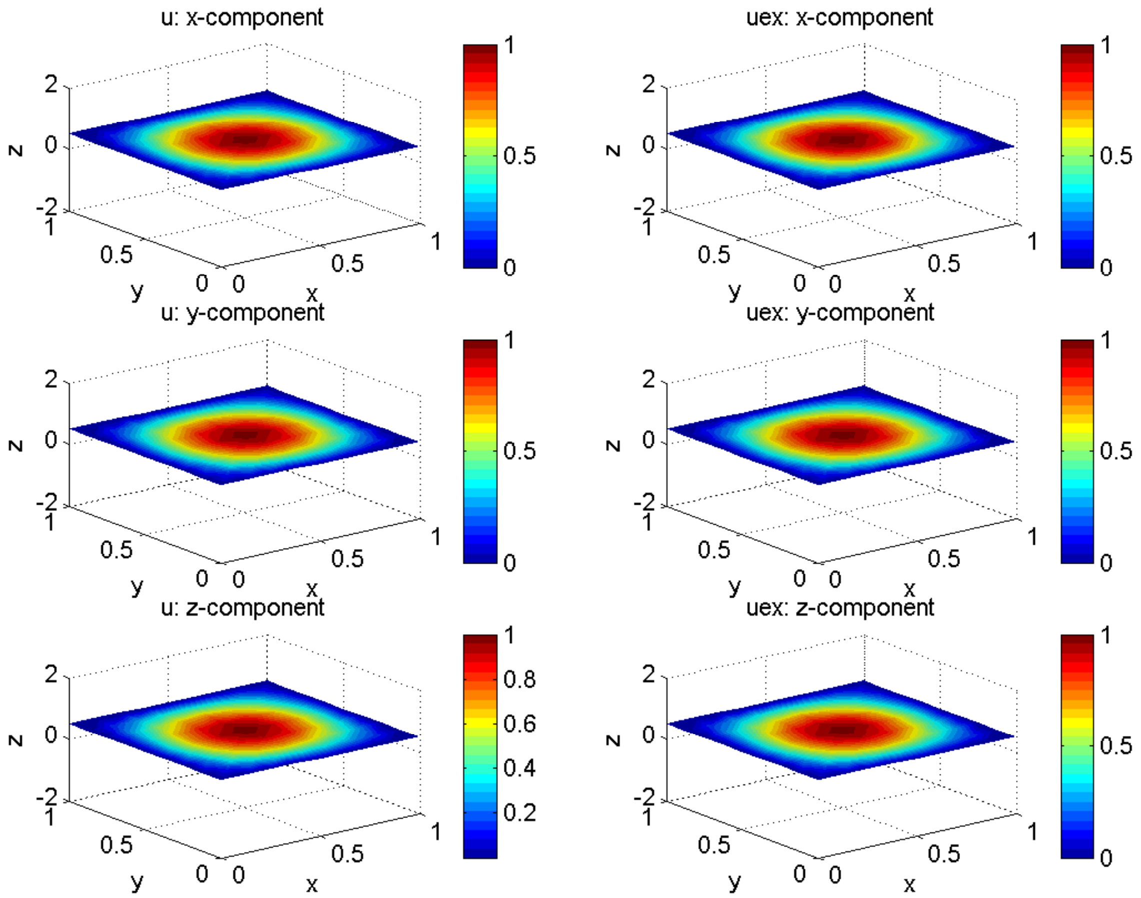

Figure 4.3: Comparing approximate solution u^n and exact solution u from 3dfem

Here we are plotting the vector displacement u in the box (0,1)x(0,1)x(0,1). The first, second, and third row correspond to the x-component, y-component, and z-component of the displacement u respectively. The left column is for the approximate solution u^n, and the right column is for the exact solution u providing evidence for the validity of 3dfem (since the graphs for the exact solution look exactly the same as the ones for the approximate solution as shown in Figure 4.3).

4.4 Segregated algorithm: it3dupfem

This program approximates solutions of the system of the two partial differential equations: the elasticity equation and the diffusion equation with an iterative method using 3-dimensional finite element method. That is, approximating the elasticity equation

$$-G\nabla^2 u - \frac{G}{1-2\nu}\nabla(\nabla\cdot u) + \alpha\nabla p \;=\; F,$$

for the displacements u. The body force per unit bulk volume F is set to be the gravity force, and the pressure p is initialized using the program 3D-DEF (Gomberg and Ellis [7]). Then the displacement is used in the diffusion equation:

$$\frac{\partial}{\partial t}(S_e p + \alpha\nabla\cdot u) - \nabla\cdot(\frac{k}{\mu}\nabla p) \;=\; Q,$$

to solve for the pore pressure p.

In this equation - the diffusion equation - the initial pressure is set as before using 3D-DEF.

The system solved at each time step is:

$$-G\triangle(u_{(i)}^{n+1}) - \frac{G}{1-2\nu}\nabla(\nabla\cdot(u_{(i)}^{n+1})) + \alpha\nabla(p_{(i)}^{n+1}) \;=\; F^{n+1}, \tag{4.5}$$

$$S\frac{p_{(i+1)}^{n+1} - p_{(i)}^{n}}{\tau} + \alpha\frac{\nabla\cdot(u_{(i)}^{n+1}) - \nabla\cdot(u_{(i)}^{n})}{\tau} - \frac{k}{\mu}\triangle p_{(i+1)}^{n+1} \;=\; Q^n. \tag{4.6}$$

The superscript n denotes the discrete time level at which the function is evaluated and the subscript i denoted the inner iteration (counter).

Giving u^n and p^n and guessing $p_{(0)}^{n+1}$, we first calculate $u_{(0)}^{n+1}$ using equation (4.5) then we substitute $u_{(0)}^{n+1}$ into equation (4.6) to find $p_{(1)}^{n+1}$.

The iteration yields $u^{n+1} = u_{(i)}^{n+1}$ and $p^{n+1} = p_{(i+1)}^{n+1}$. The process is repeated several times until convergence, then the solution at the next time step is computed in a similar manner. This algorithm did converge, i.e., the difference between the previous calculated displacement and the next calculated displacement is less than or equal to some tolerance, similarly the difference between the previous calculated pressure p and the next calculated pressure p is less than or equal to some tolerance.

4.5 Coupled algorithm: c3dupfem

A coupled algorithm (with 3-D finite element method) is used to approximate the solution of the system of the two coupled partial differential equations.

$$-G\nabla^2 u - \frac{G}{1-2\nu}\nabla(\nabla \cdot u) + \alpha\nabla p \;=\; F,$$

$$\frac{\partial}{\partial t}(Sp + \alpha\nabla \cdot u) - \nabla \cdot (\frac{k}{\mu}\nabla p) \;=\; Q.$$

In other words, after discretization using finite elements in space and second order Crank-Nicolson in time, the system of linear algebraic equations to be solved has the form:

$$\begin{bmatrix} A & B \\ B^T & -C \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} \widetilde{F} \\ \widetilde{Q} \end{bmatrix}$$

The system is solved for the vector displacement u and pore pressure p.

The (MATLAB) code c3dupfem approximated solutions of the system in the box (0,1)x(0,1)x(0,1) with homogeneous Dirichlet boundary condition. In this program, data for which the exact solution is known are generated and compared to the approximate solution obtained from the developed program. The following graph compares the approximate solution and the exact solution for the pore pressure at the last time step.

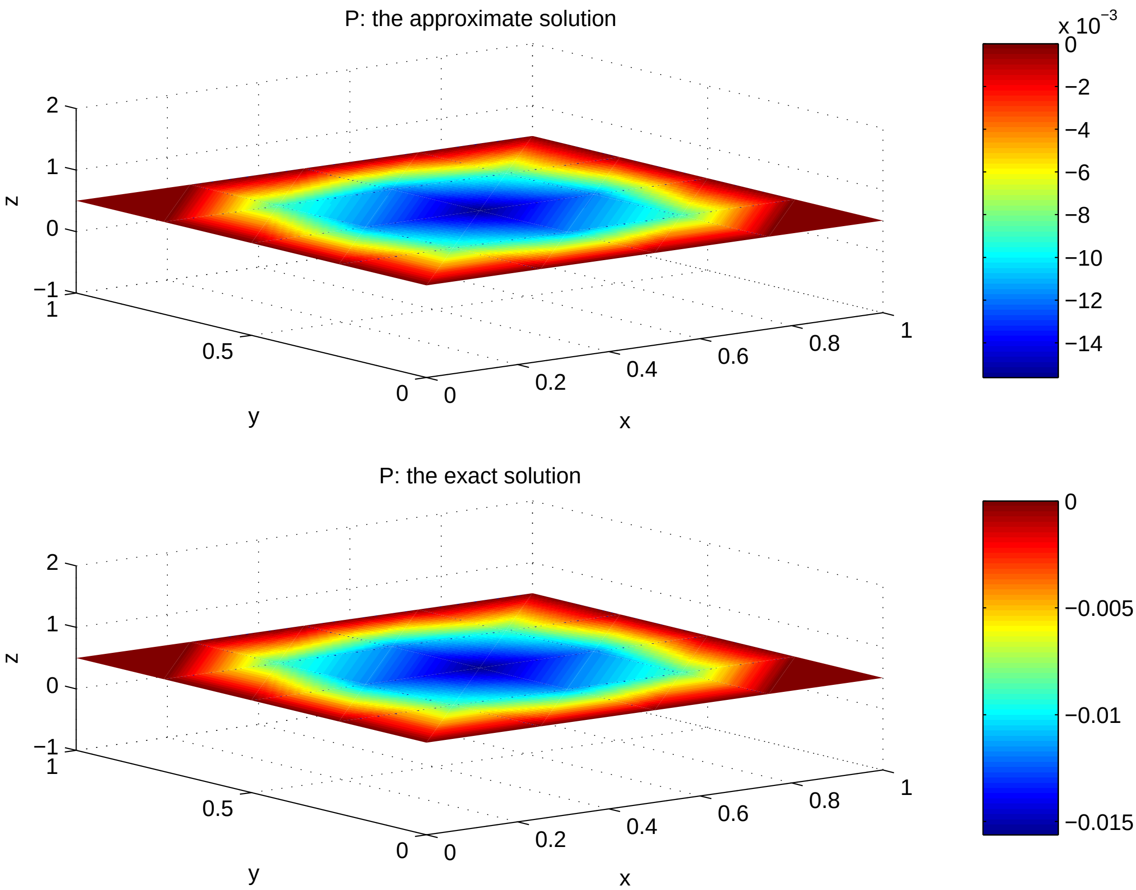

Figure 4.4: Comparing approximate solution p^n and exact solution p from c3dupfem at the last time step

The graph below compares the approximate solution and the exact solution for the vector displacement at the last time step.

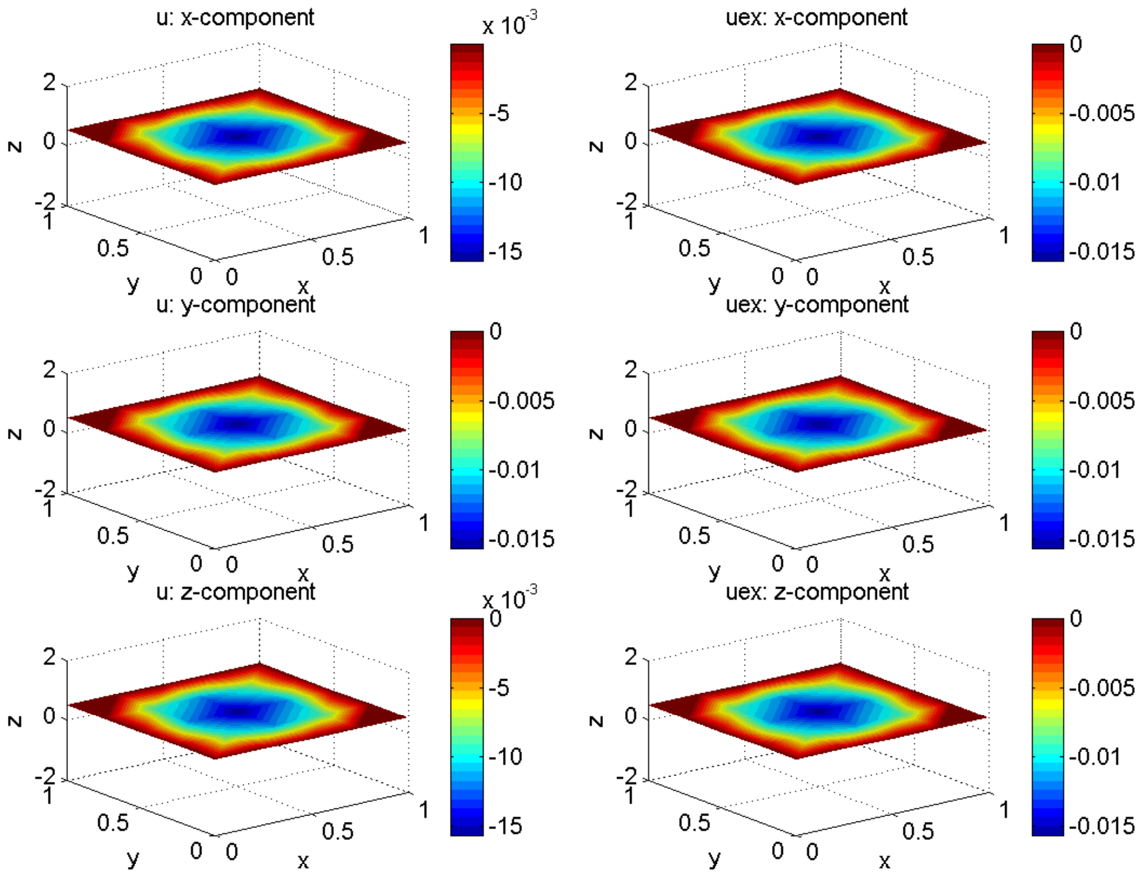

Figure 4.5: Comparing approximate solution u^n and exact solution u from c3dupfem at the last time step

Figure 4.4 clearly shows that the approximate solution is the same as the exact solution for the pressure, and in Figure 4.5 the approximate solution for the x-component, y-component, and z-component of the displacement on the left hand side look exactly the same as the ones of the exact solutions on the right hand side which is evidence for the validity of c3dupfem code.

Chapter 5

Conclusion and future work

In this work, we considered the interaction between fluid pressure changes and the deformation of a porous elastic material. Starting from the force equilibrium equation and the linear constitutive equations, we formulated the equations describing the coupled processes of elastic deformation and the pore fluid pressure in a porous medium. The fully coupled system of equations does not in general yield closed form solutions. The algorithm 3D-DEF (Gomberg and Ellis [7]) approximates Biot's system for the displacement from which the strain ϵ and the stress σ can be calculated. In order to calculate pore pressure changes, the 3P-Flow (see [9]) algorithm uses the above calculated strain ϵ and stress σ. In other words, 3D-DEF approximates the quasi-static elasticity equation for the vector displacement u. Using these results, 3P-Flow then approximates the pressure in the diffusion equation. Thus the two algorithms together do not approximate the fully coupled system of the two partial differential equations. Our main objective in this work was to derive numerical algorithms for approximating solutions to the fully coupled system by concurrently approximating solutions for the vector displacement u and the scalar pressure p. This objective was attained. Our numerical algorithms were extensively tested. After numerically approximating the fully coupled system, we considered the problem of existence and uniqueness of solutions.

In [14] Showalter showed existence and uniqueness of strong and weak solutions using abstract theory. This work proposed a constructive approach based on Babuska-Brezzi theory and Rothe's method to show existence and uniqueness of weak solutions for the quasi-static poroelasticity system. Our approach suggested numerical methods which were used to approximate solutions of the quasi-static poroelasticity system. Moreover, error estimates were derived.

In the numerical experiment for the fully coupled system (c3dupfem), all the coefficients in the equilibrium equation for momentum conservation and the diffusion equation for Darcy flow were set to one except Poisson's ratio ν that was set to 1/3. If we use the physical coefficients, then the matrix

$$M = \begin{bmatrix} A & B \\ B^T & -C \end{bmatrix}$$

has high condition number since this matrix M is "close" to being singular. Our future work is to construct and solve the system with approximate Schur complement. In other words, we compute the Schur complement of the matrix M and precondition it with its diagonal. That is, we solve instead the following problem

$$D^{-1} \begin{bmatrix} A & B \\ 0 & -C - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = D^{-1} \begin{bmatrix} \widetilde{F} \\ \widetilde{Q} - B^T A^{-1} \widetilde{F} \end{bmatrix}$$

Here D is the diagonal matrix whose diagonal is

$$diag \begin{bmatrix} A & B \\ 0 & -C - B^T A^{-1} B \end{bmatrix}$$

The condition number of the matrix

$$D^{-1} \begin{bmatrix} A & B \\ 0 & -C - B^T A^{-1} B \end{bmatrix}$$

is of order 1. It seems now that we can obtain accurate approximate solutions since the matrix is far from being singular (this will be our future work).

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APPENDICES

Appendix A

Notation and Inequalities

A.1 Notation for derivatives

Let U be a subset of $\mathbb{R}^n$. Assume that $u : U \to \mathbb{R}$, $x \in U$.

- Gradient vector: $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n})$.
- Laplacian of u: $\triangle u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

Vector-valued function

If $m > 1$ and $u : U \to \mathbb{R}^m, u = (u_1, u_2, \cdots, u_m)$, then

- Gradient matrix:

$$\nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_m}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_n} \end{pmatrix}$$

- If $m = n$, then divergence of u is

$$\nabla \cdot u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$$

Multi-index notation

- $\partial_i u = \frac{\partial u}{\partial x_i}$, $i = 1, \cdots, n$.
- $\partial_i^m u = \underbrace{\partial_i \cdots \partial_i}_{\text{m times}} u$, $i = 1, \cdots, n$, $m \in \mathbb{Z}_+$, $(\partial_i^0 u = u)$.

- Let $\alpha \in \mathbb{Z}_+^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$

  $D^\alpha u = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u$. The order of this derivative is the order of α, i.e. $|\alpha| := \sum_{i=1}^n \alpha_i$.

A.2 Spaces of continuous and differentiable functions

- $C(U)$: the set of all continuous functions $u : U \to \mathbb{R}$.

- $C^k(U)$, $k \in \mathbb{N}$: the set of all continuous functions $u : U \to \mathbb{R}$ with continuous partial derivatives up to and including k.

- $C^0(U) = C(U)$ and $C^\infty(U) = \cap_{m \in \mathbb{Z}_+} C^k(U)$.

- $L^p(U)$: the set of all functions $u : U \to \mathbb{R}$ such that u is Lebesgue measurable, $||u||_{L^p(U)} < \infty$, where $||u||_{L^p(U)} = \left(\int_U |f|^p dx\right)^{\frac{1}{p}} \quad (1 \leq p < \infty)$.

- $L^\infty(U)$ the set of all functions $u : U \to \mathbb{R}$ such that u is Lebesgue measurable, $||u||_{L^\infty(U)} < \infty$.

- $H^1(U)$: space of all functions $u \in L^2(U)$ whose first derivatives are square integrable.

- $H^2(U)$: space of all functions $u \in L^2(U)$ whose first and second derivatives are square integrable.

- $H_0^1(U)$: space of all functions $u \in H^1(U)$ such that $u|_{\partial U} = 0$.

- $W^{m,p}(U)$: the set of all functions $u \in L^p(U)$ that have weak derivatives $D^\alpha u \in L^p(U)$ for all $\alpha \in \mathbb{Z}_+^N$ with $|\alpha| \leq m$. Also,
  $||u||_{W^{m,p}(U)} := \left(\sum_{\alpha \in \mathbb{Z}_+^N{}_{|\alpha| \leq m}} ||D^\alpha u||_{L^p(U)}^p\right)^{\frac{1}{p}}$ if $p < \infty$, and
  $||u||_{W^{m,\infty}(U)} := \max ||D^\alpha u||_{L^p(U)} | \alpha \in \mathbb{Z}_+^N, |\alpha| \leq m$.

- $W_0^{m,p}(U)$: we say that a function u is in $W_0^{m,p}(U)$ if u is the limit in $W^{m,p}(U)$, of a sequence of C^m-functions with compact support in U.

Appendix B

Preliminaries

Definition 3 *(see [10]): Let $I = [0, T]$ and let H be a Hilbert space. A mapping $y(t) : I \rightarrow H$ is called an abstract function from I into H.*

The set of all abstract functions continuous in I, equipped with the norm

$$||y||_{C(I,H)} = \max_{t \in I} ||y(t)||_H$$

is called the space $C(I, H)$.

Definition 4 *(see [10]): A simple function is an abstract function which attains, on I, only a finite number of "values" $f_1, \cdots, f_m \in H$, on (Lebesgue) measurable sets $N_1, \cdots, N_m$ with measures $\mu_1, \cdots, \mu_m$, respectively.*

The Bochner integral of a simple function is defined by

$$\int_I y(t)dt \ = \ \sum_{i=1}^{m} f_i \mu_i.$$

Measurable functions in the Bochner sense (see [10]) are functions which can be approximated, to arbitrary accuracy, by simple functions.
The space $L^2(I, H)$ is the space of functions which are square integrable in the Bochner sense, i.e. Bochner integrable and satisfying

$$\int_I ||y(t)||_H^2 dt \ < \ \infty$$

with the scalar product

$$(y_1, y_2)_{L^2(I,H)} = \int_I (y_1(t), y_2(t))_H dt$$

and the norm

$$||y||^2_{L^2(I,H)} = \int_I ||y(t)||^2_H dt. \tag{B.1}$$

Convergence $y_n \to y$ in $L^2(I, H)$ means that

$$\lim_{n\to\infty} \int_I ||y - y_n||^2_H dt = 0.$$

By the Riesz theorem, a primitive function $Y(t)$ is defined by

$$(Y(t), f)_H = \int_0^t (y(\tau), f)_H d\tau \qquad \forall f \in H.$$

Then

$$Y \in C(I, H)$$

(that is, $Y(t)$ is continuous abstract function in the interval I (see [10])) and

$$Y \in AC(I, H)$$

(that is, $Y(t)$ is absolutely continuous (see [10])) for every $y \in L^2(I, H)$.

The derivative which is $Y'(t) = y(t)$ in $L^2(I, H)$, exists almost everywhere.

Theorem B.1 *There exists a constant $c > 0$ depending only on the domain G, such that for every function $u \in W_2^{(1)}(G)$ we have*

$$||u||_{L^2(\Gamma)} \leq c||u||_{W_2^{(1)}(G)}$$

Consider the boundary value problem (bvp):

$$-\triangle u = f \qquad \text{in } D,$$

$$u = 0 \qquad \text{on } \partial D,$$

where $D \subset \mathbb{R}^N$ is a bounded domain, and $f : D \to \mathbb{R}$ is given.

Strong solution of (bvp): Given $p \in (1, \infty)$, then $u \in W^{2,p}(D) \cap W_0^{2,p}(D)$ satisfying the partial differential equation $-\triangle u = f$ in the sense of weak derivatives is the strong solution of (bvp).

Weak solution of (bvp): Given $p \in (1, \infty)$, then $u \in W_0^{1,p}(D)$ satisfying $\int_D (\nabla u) \cdot (\nabla v) = \int_D f \ v$ for all $v \in W_0^{1,p'}(D)$ is the weak solution of (bvp).

Poincare inequality (see [19]): Let Ω be bounded and $l = 1, 2, \cdots$. Then there exists a constant c dependent only on the diameter of Ω, such that for all $\phi \in W_0^{2,1}(\Omega)$

$$||\phi||_l^2 \leq c \sum_{|s|=l} \int_\Omega |D^s \phi(x)|^2 dx.$$

Holder's inequality (see [5]): Assume $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U)$, $v \in L^q(U)$, we have

$$\int_U |uv| dx \ \leq \ ||u||_{L^p(U)} ||v||_{L^q(U)}$$

.