ESTIMATING PROJECT VOLATILITY AND DEVELOPING DECISION SUPPORT SYSTEM IN REAL OPTIONS ANALYSIS

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ESTIMATING PROJECT VOLATILITY AND DEVELOPING DECISION SUPPORT SYSTEM IN REAL OPTIONS ANALYSIS

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Hyun Jin Han, son of Bong-Chul Han and Chun-Ja Park, was born on August 30, 1969 in Pusan, Korea. He graduated from Korea Military Academy, Seoul, Korea in March 1992 with a B.A. in Chinese. He was commissioned a second lieutenant in Republic of Korean Army as a graduate of KMA and started his military service as a platoon leader. As an officer, his responsibilities included supply control, resource allocation, and cost analysis & control for acquiring new weapon systems for Korean Army. He completed his M.S. in Operations Research from Korea National Defense University in January 2001. In August 2003, he joined Ph.D. program in Auburn University. He married Yun Kyung Lee, daughter of Young-Rea Lee and Dong-Soon Lim, on June 12, 1999. On January 22, 2001, their son, Ji Young (David), was born and on November 8, 2005, their daughter, Grace was born.

DISSERTATION ABSTRACT

ESTIMATING PROJECT VOLATILITY AND DEVELOPING DECISION SUPPORT SYSTEM IN REAL OPTIONS ANALYSIS

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Today's uncertain world requires firms to have a system in place that can analyze the flexibility of their projects. Real options are utilized frequently to quantify the benefits of taking a particular risk. The real options valuation process provides a methodology to measure the value of flexibility, and it assists the decision makers in making the optimal investment decision. The goal of this research is to develop the methodology for improving the real options application in actual capital investment decision making.

The Reverse Monte Carlo Simulation model (RMCS), which combines Monte Carlo simulation and the stochastic processes, is developed as a new volatility estimation

method for risky projects. Compared to previous simulation methods, RMCS results in more accurate volatility. Then a volatility revision processes based on the previous volatility estimation processes are proposed. A Bayesian revision process is suggested to estimate the new volatility when the initial volatility has been estimated by Monte Carlo simulation. Since specific cases that use typical types of Bayesian conjugate processes are hard to find in the real world, a Dirichlet conjugate process is applied to estimate posterior distributions of the future cash flows. After estimating the new distributions of the cash flows, the revised volatility can be computed using the RMCS approach.

Finally, a new early decision rule is developed in order to make real options more useful. This rule concentrates on maximizing the expected future project value. Under the new decision rule, an expected future value of the currently exercised option and the expected future option value are compared in order to determine the best exercise timing. An early decision map for "waiting," "early exercise," and "early divest" over the entire option life is developed to automate the decision in case some variables are revised in the future. The map indicates that increasing volatility enlarges the "waiting" area while decreasing volatility shrinks the "waiting" area. A simulation is applied to validate the newly developed decision rule by comparing the benefit of the early exercise rule and the volatility revision during the option life. The new decision rule is found to be useful in maximizing the expected profit of the delayed investment because the proposed decision model results in better than or equal to the current decisions model.

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CHAPTER 1

INTRODUCTION

Recently, real options analysis has been applied to the capital budgeting decisions under uncertainty. Real options analysis provides an opportunity to improve strategic investment decisions in an uncertain environment. However, the real options valuation concept requires some adjustments in order to be useful in management decisions. Although most managers understand that real options can address investment flexibility, they do not widely apply the real options valuation method in managing the uncertainty of their projects.

One of the commonly cited reasons for avoiding real options is that the volatility of the underlying project is too difficult to estimate. In financial options pricing, it is possible to estimate the volatility of the financial assets by reviewing the historical price of the underlying asset. However, in real options volatility is not directly estimated, so some restrictive assumptions are necessary to value the real options. This research develops a new method of estimating a risky project's volatility by comparing the project return distribution and the future project value distribution generated by Monte Carlo simulation.

One of the main advantages of real options is that it promotes taking time to observe the future market movements, thus decreasing the risk of a huge irreversible investment. The “wait and see” strategy of the traditional options framework means that a company will wait to proceed until the market is more favorable. The traditional options framework does not consider the information gathering activities that may take place during the option’s life. In other words, if a firm purchases a real option, it is assumed that only the final information for the project is available for their decision. However, the firm will continue to collect information to help it make the best possible decisions after they take the investment opportunity. In fact, the volatility of the underlying project, which is one of the most important variables, can change as time passes. In this study, a Bayesian revision process is employed to modify initial estimates of volatility.

Deciding the investment timing of the project is another important factor in real options valuation. However, the lack of research for determining an optimal investment timing of options has been a barrier to accepting real options as a useful tool in capital budgeting decisions. Recent research indicates that failing to exercise real options on time reduces the projects’ value much less than predicted, and the question of whether the real option holders exercise their options optimally has not been researched extensively. In this research, a new decision rule designed to maximize the future value of a project is introduced. It incorporates three different decision criteria in the defer options framework: 1) wait, 2) early exercise, and 3) early divest project; in contrast, the abandon options framework considers two decisions: 1) early exercise, and 2) wait.

Finally, two real options decision models are demonstrated to explain how to analyze the investment opportunity with the proposed methods.

1.1 Background

Before the real options framework was introduced to resolve the uncertainties of capital budgeting problems, two different approaches were popularly used: the Discounted Cash Flow analysis (DCF), which uses the risk-adjusted discount rate; and the Mean-Variance approach.

In the DCF approach, the most critical factor is deriving a correct risk-adjusted interest rate which represents the risk a company is taking. The interest rate is then used either to compute the Net Present Value (NPV) of a project, or it is considered as a Minimum Attractive Rate of Return (MARR) for decision making through Internal Rate-of-Return (IRR) criteria. However, accurately selecting the discount rate is one of the most difficult issues in a DCF analysis. The most commonly used rate in economics is the Capital Asset Pricing Model (CAPM). With CAPM, a market premium is presumed to be paid to shareholders for the risk associated with a particular industry. The Weighted Average Cost of Capital (WACC) is an alternative approach to estimating the risk-adjusted rate of return. With WACC, it is necessary to satisfy all the required returns the investor anticipated.

The real options valuation model is a different type of decision framework. While the above frameworks concentrate on the present "go" or "no-go" decision, the fundamental strategy of real options is "wait and see." Companies can apply real options to almost any situation where it is possible to estimate the NPV of a certain project without flexibility. It is then possible to analyze the project opportunity by considering the volatility of the project cash flows. Figure 1-1 shows a defer option framework, which is a basic type of real options.

Theoretically, the traditional decision making processes have difficulties in addressing the flexibility of investment decisions. In the present volatile market, the “wait and see” strategy is of the utmost importance because of the irreversibility of the capital budgeting decisions. Real options provides two important contributions to the capital budgeting decision. First, it provides a method for measuring the value of the project opportunity in several different circumstances: delaying the investment, abandoning the project, research and development project, or the potential growth of the investment during a certain time period. Second, real options are well-suited for deriving the price of tradable assets such as patents, licenses, or natural resource production projects.

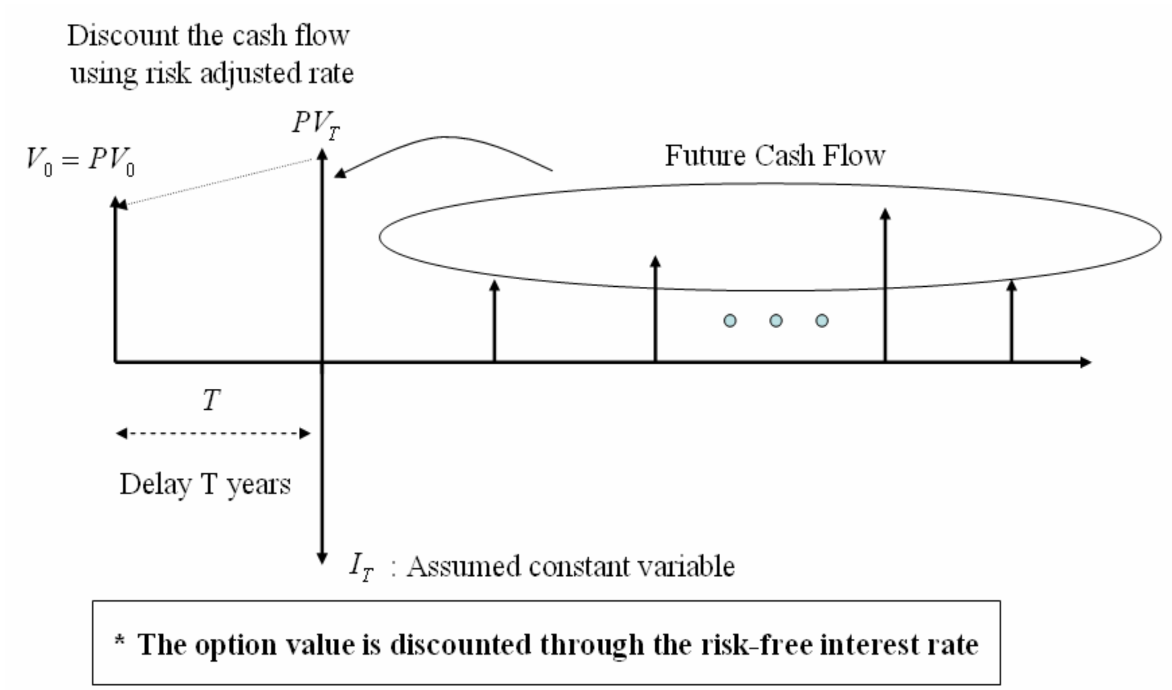


Figure 1-1. Example of defining real options parameters

1.2 Research Objectives

The primary objective of this study is to suggest ways that real options can be more practical in investment decisions. Although the purpose of applying the real options model is to address the uncertainty of the investment, it is not widely used to manage the risk of projects. The most significant problem that needs to be resolved is the development of a reasonable process to estimate the volatility of a risky project. The volatility of the underlying asset is one of the most important parameters of the options theory. In the financial options pricing, it is possible to estimate the volatility of the financial assets by reviewing the historical price of the underlying asset. However, the volatility of real options is not directly estimated; therefore some restrictive assumptions are required in order to value real options. In this research, a new method of Monte Carlo Simulation is applied to estimate the volatility of a project with an assumption that the DCF is possible to the project evaluation.

The second objective is the development of a method to revise the volatility of the real options in case new information is observed during the option's life. Since real options are rarely tradable, it is necessary to revise the volatility in order to support the irreversible investment decision in flexible market conditions. There has been very little research on this subject until now. This study develops a Bayesian conjugate process in case of general prior distributions and limited sample sizes. Dirichlet distribution is applied to revise predictions of the future cash flows, and a new volatility can be estimated by the simulation with the posterior random factors.

The third objective of this study is to determine the optimal exercise timing of the real options. Most investments in the project are irreversible, so choosing the right investment timing is critical in real options. Recent research has emphasized that failing to exercise real options on time reduces the projects' value much less than predicted, but the question of whether the real option holders exercise their options optimally has not yet been researched extensively. In this research, a decision map of the real options is developed by comparing the future value of waiting with the value of exercising. After simulating the values of the proposed method and the current decision model, a paired t-test is conducted to check the significance of the new decision rules.

Finally, two real options models demonstrate the application of the whole proposed process of volatility estimation, volatility revision, and decisions. A project's growth opportunity and a compound options framework are demonstrated to examine the changes on decisions.

1.3 Study Plan

Chapter 1 sets forth the background of and the motivation for this research. The introductory chapter also includes the research's objectives and the study plan.

Chapter 2 reviews previous studies related to the development of the real options valuation model and the application of it to various projects. Since the origins of the real options valuation model are found in financial options pricing, the latter model is briefly described. Then, some theories as to how to make real options more attractive to decision-makers are investigated. Finally, the research on applying real options in the real world is summarized.

Chapter 3 develops a volatility estimation method by combining Monte Carlo simulation and stochastic processes. The current methods of estimating volatility through

Monte Carlo simulation are examined along with a traditional concept of volatility in options theory. Then, a new way of estimating the volatility of real options is presented by considering the option life and the simulated project value followed by a numerical example of estimating volatility through Monte Carlo Simulation.

Chapter 4 suggests a volatility revision process based on the Bayesian revision process in case of a general prior distribution with very limited sample sizes. After demonstrating in a brief introduction how to estimate a cash flow distribution from a three-points estimation, Dirichlet conjugate processes are used to generate a posterior distribution of each parameter of the cash flows. The posterior volatility of the project is estimated by re-simulating the project value distribution.

Chapter 5 develops the concept of decision timing: when to exercise or divest the real options by comparing the future value of the project. I present a method for making such a decision by considering the investment opportunity cost concept. A simulation technique is used to demonstrate the benefit of the new exercise rule. A comparison between the expected profit of the decision rule set forth in this study and the traditional rule shows the advantages of the new approach. The fundamental of this simulation is generating a past dependant project values.

Chapter 6 describes two examples of the research being applied in specific investment opportunities. This investigation considers a project which has at least two investment phases. A growth options framework and a compound options framework are investigated to apply the methods suggested in previous chapters. In the growth options framework, a revised volatility and project value affect the early exercise decision while the compound options concentrate on go or no-go decision for the second investment.

Different decisions are presented in the two investment scenarios when new information is collected during the option life.

Chapter 7 presents a brief conclusion along with some suggestions for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

There are three main objectives in this research. The first objective is to develop a new method of volatility estimation that uses Monte Carlo Simulation. The second objective is to propose a Bayesian revision process to enhance decision making by updating the volatility of real options during the option life. The final objective is to develop early decision rules for the real options. I include a general decision map with a ratio between the project value and the investment cost with the goal of automating the investment decisions at each stage of the option.

In order to further investigate the fundamentals of real options valuation technique, it is necessary to understand the financial option pricing model because it is the origin of the real options valuation concept. So, a brief review of the financial option valuation models—the Black-Scholes model and the binomial lattice—will demonstrate the logic of the real options valuation model. After a brief comparison of real options valuation and financial options pricing, some conceptual meanings of real options from previous studies are reviewed in order to expand the use of real options in decision making. Then the various types of real options and their applications are examined along with the volatility estimation methods of the projects. After reviewing the research related to revising the volatility of the options, previous studies for improving the strategic decisions in real options valuation are summarized.

2.2 Background of Real Options

An option is a contract that gives its holder the right but not the obligation to take action at a predetermined price within a specified time period. There are two basic types of financial options: call options and put options. A call option gives the holder the right to buy an asset by a certain date for a predetermined price, while a put option gives the holder the right to sell an asset. The predetermined price of the asset for the actions is known as the exercise price, and the date is known as the expiration date. Five parameters determine the price of the options: 1) the current price of an asset (S_0), 2) the exercise price (K), 3) the expiration date (T), 4) the risk-free interest rate (r), and 5) the volatility of the asset (σ). There are also two types of options as determined by their exercise timing: European options, which can be exercised only on the expiration date; and American options, which can be exercised at any time during the option life.

The valuation of the option premium is of the utmost importance in financial options because the options are traded in a market such as the Chicago Board Options Exchange. The two most common option pricing models to appear in the literature are the Black-Scholes model (B-S model) and the binomial tree model. In 1973, Fischer Black, Myron Scholes, and Robert Merton presented a seminal paper in the pricing of stock options. The B-S model had a critical influence on option pricing and hedging options. Hull (2005) summarizes that they developed a theoretical option pricing model based on risk-free arbitrage with the following assumptions: 1) the percentage changes in the stock price in a short period of time follows approximate normal distribution, 2) the short period of time is independent, 3) the underlying stock does not pay any dividend, 4) the short-term interest rate is known and constant, 5) the investors can borrow or lend at the same risk-free interest rate, 6) there are no transaction costs in trading options, and 7) the options are European.

Cox, et al. (1979) present an algebra technique for pricing an option by constructing a binomial lattice. Since the binomial lattice model is simple and easy to understand, it is now a very popular way to explain the invisible logic of options. In the lattice model, the only assumption is that no arbitrage opportunities exist. Park (2006) summarizes three different concepts to price an option by binomial lattice: 1) the replicating-portfolio concept, 2) the risk-free financing concept, and 3) the risk-neutral concept. In spite of their differences, all three approaches yield the same result, so any of them can be selected to draw the lattice.

2.2.1 Real Options Valuations

Research indicates that managers can choose from seven or eight real option types to match their investment opportunity. Table 2-1 is a summary by Amram and Kulatilaka (1999) and Trigeorgis (2005). A defer option gives the holder the right to wait for a period of time for the project’s uncertainty to be resolved. In this framework, the company will have the flexibility to postpone a great portion of its irreversible investment by taking the option. The defer option is valuable in natural resources-related industries such as farming, oil extraction, and mining.

Table 2-1. Types of real options

Trigeorgis (2005)	Amram & Kulatilaka (1999)
Defer	Timing
Staging (Compound)	Growth
Expand	Staging (Compound)
Contract	Exit
Temporary Shut-down	Flexibility
Abandon	Operating
Switching	Learning
Growth	

A staging option considers a project which has a series of capital outlays over time. In the staging options point of view, each stage is considered an option for the investment of the next stage. R&D projects in the biotechnology and pharmaceutical industries, or the construction of an electric plant are examples of staging projects. The required outlay for the earlier stage of the project is considered an option premium for the right to invest in the later stage of the project.

Expand options and the growth options have the similar properties. Both consider the initial investment and follow-up investments in light of market conditions. According to Trigeorgis, the only difference in the two options is that with expand options, the initially invested project will be expanded if the market condition is good, while in the growth options framework the initial investment is the foundation for the other projects.

A contract option is the opposite of an expand option. In the contract options framework, if the market is not as favorable as expected, the manager is able to operate below capacity or reduce the scale of operations. Similarly, a temporary shut-down option indicates that the plant may cease to operate and then re-open according to market conditions. In the temporary shut-down options framework, the shifting costs are considered an option premium. In both the contract options and the temporary shut-down options framework, the cost of installing the flexible production system is considered an option premium. These options are useful in natural resource industries such as mining industry or fashion design and merchandising.

An abandon option gives its holder the right to sell the project for its salvage value if the project turns out not to be favorable. This option can be valued as an American put option pricing model. Finally, a switching option allows its holder to install a flexible system of inputs or outputs for a certain project. An example would be installing an energy converting system for a production line. The installment cost would be an option premium in such a case.

The premium involved in taking the investment opportunity is determined by five parameters similar to financial options pricing: 1) the present value of the project cash inflows (V_0), 2) the investment cost of the project (I), 3) the time to make the investment (T), 4) the risk-free interest rate (r), and 5) the uncertainty of the project cash flows (σ). Table 2-2 summarizes the input parameters of real options and its financial counterpart.

The valuation technique for real options is the same as for the financial options pricing model. The following example of a defer option explains the real options valuation.

Table 2-2. Financial and real options parameters

Financial Option		**Real Option**	
Stock Price	S_0	PV of the project cash inflows	V_0
Exercise Price	X	Investment cost of the project	I
Time to mature	T	Time to make investment decision	T
Stock Volatility	σ	Uncertainty of the project value cash flows	σ
Risk free rate	r	Risk-free interest risk	r

Assume that a firm has a project with a present cash flow value V_0 which can be invested in today with a requirement investment cost of I. Under the traditional decision rule, the positive $NPV=V_0 - I$ suggests that one should invest in the project today, and the negative NPV indicates a non-attractive project. If it is possible to defer the investment decision in the non-attractive project up to "T" years, real options play a role in valuing the delayed opportunity. The logic for valuing the opportunity appears below.

Assuming the firm takes the investment opportunity, it will invest in the project until time T if the value of the project is greater than I, otherwise the decision maker will decide not to invest. So the payoff at time T will be: $\max(V_T - I,0)$. Then the B-S model or binomial lattice approach will provide the option premium for taking the investment opportunity.

2.3 Volatility Estimation Methods in Real Options

Financial options derive their price from the value of their underlying financial assets, such as stocks. Option volatility can be estimated by either historical movements of asset market prices, or by calculating the implied volatility from the Black-Scholes model based on the market price of an option. Estimating the volatility of a real option is much more difficult because there are no historical returns or current market prices of the underlying projects. Research indicates that there are essentially six volatility estimation methods in the previous real options applications including simple assumptions.

2.3.1 Historical Volatility of the Underlying Asset

If the price of a natural resource determines the future cash flow of the project, then the historical volatility of that resource is considered the volatility of the project. Kelly (1998) used the binomial lattice approach to determine the investment timing for the initial public offering of a gold mine. The historical gold price was used to compute the future project value. Moel and Tufano (2002) also used gold price returns to analyze the optimal timing in which to close a mine in North America. When Cortazar and Casassus (1998) analyzed an investment project that expanded the production capacity of a copper mine, they considered the historical volatility of the price of copper as the uncertain factor. Cortazar, et al. (1998) also used the volatility of copper prices to evaluate the environmental investments for the copper production company. Kemna (1993), Smith and McCardle (1998) and Armstrong, et al. (2004) used the historical volatility of the price of oil for their decision analysis. Titman (1985) and Quigg (1993) used land price fluctuations in order to analyze an opportunity to develop the land.

2.3.2 Historical Volatility of the Compatible Asset

When the underlying asset is not tradable in the market, the volatility of the compatible asset is adapted to compute the project volatility. Benaroch and Kauffman

(1999) adapted the historical demand of point-of-scale transactions in California when they were estimating the volatility of a point-of-scale transaction project in the New England area. Insley and Rollins (2005) used the volatility of the price of a spruce-pine-fir 2’×4’ to evaluate the best timing for timber harvesting under the real options model.

2.3.3 Historical Volatility of the Company’s Stock Price

If the project volatility perfectly correlates with the stock price movement of the company, the volatility of the stock price of a company, this method is useful in the estimation process. Newton and Pearson (1994) assumed that the stock price of a R&D company is analogous with the expected NPV of the company’s project. Bollen (1999) applied a real options framework to value the opportunity of changing a project’s capacity by setting the demand for the product as a source of risk. He assumed that the traded asset span changes with demand. Herath and Park (1999) demonstrated that the stock price of a typical R&D company is perfectly correlated to the company’s R&D project. Miller and Park (2004 and 2005) did the same in a case involving a maintenance company.

2.3.4 Historical Volatility of the Industry Index

The historical volatility of the industrial group index has also been used to estimate the volatility of the project value, particularly in cases where there was a shortage of past data. Cassimon, et al. (2004) estimated the volatility of a real option project which values new drug applications and the R&D of pharmaceutical companies using the pharmaceutical industry’s standard deviation of equity. Jensen and Warren (2001) valued the research in the e-commerce project by using the average volatility of six e-commerce companies traded on NASDAQ as the uncertainty factor of their real options model. Teisberg (1994) analyzed a utility power plant construction project using an option pricing model by considering the six participating firms' historical returns as the volatility of the project value.

2.3.5 Monte Carlo Simulation

Monte Carlo simulation has been conducted to compute the volatility of the project itself by using the project's future cash flow, which is based on the DCF and plausible scenarios of future uncertainty. The historical data of the project or some assumptions are used for estimating the distribution of the input parameters in the simulation approach. Studies by Copeland and Antikarov (2003), Mun (2006), Herath and Park (2002), and Cobb & Charnes (2004) comprise the body of research that has used Monte Carlo simulation to estimate volatility.

Table 2-3 is the summary of volatility estimation methods of recent real options application papers.

The perspectives of the project will guide the choice of which volatility estimation method to use. Each method has its limitations and an alternative approach is therefore needed. Monte Carlo simulation comes the closest of any of the five methods to accurately estimate project volatility itself, but from a statistical point of view, current Monte Carlo simulation approaches are inadequate. So, a new volatility estimation approach uses Monte Carlo simulation to value the investment opportunity correctly is developed.

Table 2-3. Real options volatility estimation methods

Estimation method	Papers	Volatility factor
Historical volatility of the underlying asset	Kelly(1998)	Gold price
	Titman(1985)	Land
	Smith & McCardle(1998)	Oil
	Cortazar & Casassus(1998)	Copper
	Takizawa & Suzuki(2004)	Construction cost
	Cortazar et al.(1998)	Copper
	Moel & Tufano(2002)	Gold returns
	Davis (1998)	Precious metal
	Smit (2003)	Flight demand
	Kemna (1993)	Oil
	Quigg(1993)	Real estate price
	Armstrong et al.(2004)	Oil
Historical volatility of the compatible assets	Insley & Rollins(2005)	Lumber price,
	Benaroch & Kauffman(1999)	point-of-scale transactions
Historical volatility of the traded asset	Nembhard et al(2005)	Annualized NPV
	Bollen, Nicolas P. B.(1999)	Demand of the product.
	Newton and Pearson(1994)	Expected NPV
	Hemantha & Park(1999)	A stock price.
	Miller & Park(2004)	A stock price.
	Miller & Park(2005)	A stock price
Historical volatility of the industrial group index	Cassimon et al.(2004)	Pharmaceutical industry
	Jensen & Warren(2001)	6 e-commerce companies
	Teisberg(1994)	6 regulated firms
Monte Carlo simulation	Nembhard et al (2003)	
	Miller, Choi & Park.(2004)	
	Cobb & Charnes(2004)	
	Herath & Park(2002)	

2.4 Bayesian Revision Processes

One of the main advantages of real options is that it can lessen the risk of a huge irreversible investment by allowing decision makers to take the time to observe future market movements. Under the traditional options framework, the "wait-and-see" strategy allows a company to wait to proceed until the market is more favorable. However, in the real world companies take action to resolve uncertainty once the option is taken in order to improve their chances of making a profit. The information they obtain will effect the investment decision.

Bayesian statistics are widely used to revise prior beliefs after observing sample information. Miller and Park (2005) studied the impact of learning on a multi-staged investment scenario with an assumption of the B-S model. In their study, they used the normal conjugate distribution to compute the value of the acquired information. However, the real options valuation usually uses the binomial lattice with an assumption that the future outcome is discrete and not continuous. This is because real options has a kind of American options perspective. By using the binomial lattice, it is also possible to use any kind of distribution for option valuations. Therefore some kind of revision process for the general case is required for the learning real options framework.

In order to develop general processes of volatility revision in case the initial volatility of the options are estimated by Monte Carlo simulation, it is necessary to understand the characteristics of a special type of Bayesian revision process, Dirichlet distribution. Prueitt and Park (1992) presented a method for uncertainty resolution in generalized cases using Dirichlet revision process. The basis of their approach is to develop discrete approximations to continuous prior beliefs, record observed samples, and place the observations in discrete categories that correspond to distributions in a Bayesian framework.

2.5 Decision Timing in Real Options

After deciding to retain an option, the investors must decide the best timing of exercising or divesting the option in order to maximize their profit. Since the investment in the project has irreversible characteristics, deciding on the optimal investment timing is crucial. In financial put option theory, the timing to exercise is defined as the point at which the value of immediate exercise is higher than that of holding the option until its expiration date. Financial put option pricing holds that it is best to exercise when the current value of the project is higher than the expected value of the future flexibility. However, it is known and proven that early exercise is never optimal in the financial call option theory, which is applied to the major real options valuation model.

As Copeland and Tufano (2004) mentioned, defining the optimal exercise timing of the real option is a crucial factor in making real options actually work in the real world. They suggested that failing to exercise real options on time reduces the value of the projects much less than predicted. However, in spite of the importance of this issue, it has not been widely researched.

Brennan and Schwartz (1985) developed an evaluation model for deciding the optimal investment timing to continue or abandon a mining project by setting stochastic output prices. McDonald and Siegel (1986) studied a method to determine the optimal timing of investment in an irreversible project when the benefit and cost of the project follow Geometric Brownian Motion. By using the simulation technique, they indicated that for risk-averse investors, it is optimal to wait until benefits are twice the investment costs.

Yaksick (1996) suggested a method for computing the expected exercise timing of a perpetual American option, and Shackleton and Wojakowski (2002) developed a numerical expression for computing the expected return and for finding the optimal timing to exercise real options by using the risk-adjusted stopping time method, which is based on the actual probability distribution of payoff times. Rhys, Song, and Jindrichovska (2002) summarized recent developments in the topic of "The timing of real option exercise." They reported that only a few studies have been conducted to analyze this problem, but some progress is being made in the research.

Because capital investments are irreversible, deciding on the best timing of the investment is one of the most important factors in real options. Some researches have tried to define the decision timing, but most of their studies have been restricted to specific cases which can not be widely applied in real world decisions. The previous studies did not give helpful information which supports the decision of a company during the option life. A map which guides the rules of action for the real options is one of the most important information for decision makers, so new investment decision rules of the real options should be developed.

CHAPTER 3

ESTIMATING PROJECT VOLATILITY USING MONTE CARLO SIMULATION IN REAL OPTIONS

Abstract

Among the five general variables of real options, volatility is one of the most critical, and it is generally considered to be the only stochastic variable. However, estimating the volatility of the underlying project is rather problematical, and the difficulty involved makes some CEOs hesitate to use real options in their analysis. The five volatility estimation methods that have been used up to this point were discussed in Chapter 2.

Among those five methods, Monte Carlo simulation is considered fundamental, and it is used for estimating the volatility of the future project opportunity. This research suggests a new way of volatility estimation called Reversed Monte Carlo Simulation (RMCS). A mathematical demonstration indicates that RMCS correctly estimates project volatility.

3.1 Introduction

There are mainly five volatility estimation methods in previous real options applications These are 1) considering the historical volatility of the underlying asset, 2) adapting the historical volatility of the compatible asset, 3) using the volatility of the company's stock price, 4) applying the volatility of the industrial group, and 5) the volatility generated by Monte Carlo simulation. Among the five methods, a Monte Carlo Simulation based volatility estimating method is considered as a fundamental method to estimate project volatility, and are used for estimating the volatility of the future project opportunity. However, with a statistical point of view, the current simulated volatility does not represent the correct volatility of the underlying project; generally it is over-estimated than the correct volatility. Therefore, we have adapted the simulation and the statistical analysis to develop a new volatility estimation method.

The remainder of this research is organized as follows. Section 2 reviews the previous applications of Monte Carlo simulation to real options. In section 3, the statistical meanings of the current Monte Carlo simulation-based volatility estimation methods are examined by a traditional concept of volatility in option theory. Section 4 presents a new way of estimating project volatility by considering the option life and the simulated project value. In the section a numerical example of estimating volatility through Monte Carlo simulation is demonstrated and compared to the current method. Finally a summary of the new method and some suggestions for future research are followed in section 5.

3.2 Monte Carlo Simulation in Real Options

3.2.1 Applications of Monte Carlo Simulation in Real Options

Monte Carlo simulation is used with the assumption that the DCF analysis is possible for the project because the simulation needs the future cash flow distribution of the project in order to estimate the volatility. The historical data of the project or some assumptions are used for estimating the distribution of the input parameters in this simulation approach.

Copeland and Antikarov (2003) and Mun (2006) suggested the same concept at the standard deviation of the rate of return distribution as the volatility of the project. Before running the simulation, they defined the annual return of the project as the ratio between the logarithmic value of PV_1 and PV_0. In this method, it is necessary to set the denominator PV_0 as a constant variable to simulate the volatility of the project return.

Herath and Park (2002) simulated the project volatility in a different way. They simulated both the denominator and the numerator with the assumption that the project cash flows are independent each other. Cobb & Charnes (2004) simulated the real options volatility in case of correlated cash flow by further developing Hearth and Park's model. Each of these methods has its own limitations that prevent it from serving as a general technique in the many different real options environments.

3.2.2 Volatility Estimation through Monte Carlo Simulation

In most cases, there are multiple uncertainties involved in the underlying project. Thus it is almost impossible to find the twin securities or tradable assets which would be necessary if a manager wanted to use the volatility of the financial assets to measure the volatility of the project. Therefore, a Monte Carlo approach can be useful in estimating the project's volatility in real options.

A few studies, such as Herath & Park (2002), Copeland & Antikarov (2003), Mun (2006), Miller, et al. (2004), Brandão et al (2005), and Godinho (2006), mentioned the Monte Carlo Simulation technique as a volatility estimation tool for real options. The fundamental assumption of the Monte Carlo Simulation approach is that it is possible to estimate future cash flows.

To explain the volatility of the project return, it is first necessary to define some notations. CF_t is the cash flow of the project during tth year of the total project life T, and PV_t is the present market value of the future cash flows from year t+1 to T. Then the present value of the project PV_t with a continuous compounding policy can be defined,

$$PV_t = \sum_{k=t+1}^{T} CF_k \cdot \exp\{-r \cdot (k-t)\} \tag{3-1}$$

Then the present worth of the project at time t, PW_t is defined as the sum of PV_t plus CF_t.

$$PW_t = PV_t + CF_t \tag{3-2}$$

Let z be a random variable that represents the continuous rate of return of project between time t and t+1. Then,

$$z = \ln\left[\frac{PW_{t+1}}{PW_t}\right] \tag{3-3}$$

The Copeland and Antikarov model (CA model) simulates the standard deviation of logarithm of the difference between PW_1 and PW_0 by considering the expected value of PW_0 as a fixed number. So, the volatility of the project is computed by the standard deviation of the expression

$$z = \ln\left[\frac{PW_1}{E[PW_0]}\right] = \ln\left[\frac{PW_1}{E[PV_0]}\right] \tag{3-4}$$

While the CA model considers the expected value of PV_0 as a fixed number, the Herath and Park model (HP model) considers PV_0 as a random variable. That indicates that both the denominators and the numerators are simulated simultaneously to estimate the volatility of the project return. Then the simulation processes are the same with the present project value and that of the first year.

Lately Godinho (2006) pointed out the problems that both of the above methods have a tendency to overestimate the project's volatility. He suggested an alternative simulation technique to estimate volatility: a two-level simulation model that correctly represents the amount of project risk. He uses an example to illustrate his claim.

3.3. CA vs. HP Volatility

In this section, the relation between CA and HP volatility is demonstrated. Then, a new volatility estimation method for the real option is suggested after discussing some conditions and the assumption for the CA model.

3.3.1 Defining Relations between CA Model and HP Model

In the CA model, the volatility of the project return during the first year is summarized to the standard deviation of $\ln(PV_0)$ distribution, while the HP model has $\sqrt{2}$ times higher volatility than the CA model according to the statistical procedures shown below. Following is the mathematical meaning of the simulated volatility, which is the standard deviation of the two models. In HP model, $\ln(PV_0)$ and $\ln(PV_0)'$ are independent identical distributions.

CA model

$$Var\left[\ln\left(\frac{PW_1}{E[PV_0]}\right)\right] = Var\left[\ln\left(PW_1\right) - \ln[E(PV_0)]\right]$$
$$= Var\left[\ln\left(e^r \cdot PV_0\right) - \ln[\left(E(PV_0)\right]\right]$$
$$= Var\left[\ln\left(e^r\right) + \ln(PV_0) - \ln[\left(E(PV_0)\right]\right]$$
$$= Var\left[\ln\left(PV_0\right)\right]$$

$$\therefore \quad \sigma = \sqrt{Var[\ln(PV_0)]}$$

HP model

$$Var\left[\ln\left(\frac{PW_1}{PV_0'}\right)\right] = Var\left[\ln\left(PW_1\right) - \ln(PV_0')\right]$$
$$= Var[\ln\left(PW_1\right)] + Var[\ln(PV_0')]$$
$$= Var[\ln(e^r \cdot PV_0)] + Var[\ln(PV_0')]$$
$$= Var[\ln(PV_0)] + Var[\ln(PV_0')]$$
$$= 2Var[\ln\left(PV_0\right)] \qquad \because PV_0 \text{ and } PV_0' \text{ are i.i.d.}$$

$$\therefore \quad \sigma = \sqrt{2 \cdot Var[\ln(PV_0)]}$$

The volatility derived from the HP model is always $\sqrt{2}$ times higher than the volatility of the CA model. Godinho (2006) reveals the computation results of the standard deviation of the HP and CA models using a numerical example. The review proves that the volatility of the HP model is nevertheless $\sqrt{2}$ times higher than the volatility of the CA model.

Because the relationship between the two models is explicit, from this point forward only the CA model will be considered in order to explain the meaning of the current volatility.

3.3.2 Statistical Interpretation of the Present Models

In real options valuation, we employ techniques which were initially developed for financial options pricing. So, it is necessary to understand a fundamental concept of the latter method in order to investigate the volatility of real options. In a financial option, volatility indicates "the flexibility of the stock price during a certain time period," and it does not represent the stock price distribution at certain point of time.

In order to confirm that the models satisfy the definition of financial options volatility, we first examined the volatility of the model changes by time changes. The definition of the volatility of the CA model does not include the value changes. It also does not reflect any time consideration for the volatility, which is theoretically known to be $\sigma\sqrt{t}$ by the financial options pricing model. In order to demonstrate the problem, a 2-year defer option was used. The simulated volatility of the option could be computed by CA model through,

$$Var\left[\ln\left(\frac{PW_1}{E[PV_0]}\right)\right] = Var\left[\ln\left(PV_0\right)\right] \tag{3-5}$$

Also, the flexibility during the two years can be computed,

$$\begin{aligned} Var\left[\ln\left(\frac{PW_2}{E[PV_0]}\right)\right] &= Var\left[\ln\left(PW_2\right) - \ln[E(PV_0)]\right] \\ &= Var\left[\ln\left(e^{2r}\cdot PV_0\right) - \ln[\left(E(PV_0\right)]\right] \\ &= Var\left[\ln\left(e^{2r}\right) + \ln(PV_0) - \ln[\left(E(PV_0\right)]\right] \\ &= Var\left[\ln\left(PV_0\right)\right] \end{aligned}$$

Since the calculated volatilities of above two cases are the same to the volatility of the one year return, the suggested method has some deficiencies that prevent its calculations from being accurate. The meaning of the volatility according to the CA model represents the standard deviate on project returns during the whole option life. As a result, the volatility estimated by the CA model is only correct when the option life is one year. Thus if the option life is longer than one year the volatility is overestimated, and if the option life is shorter than one year, it is underestimated.

The second checking point is the statistical meaning of the standard deviation of the model. The term $\ln(PV_0)$ does not mean the return during a certain time period. It is a natural log of a present value distribution. Therefore, the meaning of the standard deviation of CA simulation is the standard deviation of the log distribution of PV_0, not the return distribution. The logarithm of the standard deviation of the present value distribution is computed using the central limit theorem, and one can assume that the distribution of the project value PV_0 is normal with $N(\mu, \sigma^2)$. Then if we take the $\log PV_0$ distribution, the distribution has parameters of mean m and standard deviation s which is computed by (3-6), and (3-7). (Dixit and Pindyck)

$$m = \ln(\mu) - \frac{1}{2}\ln(\frac{\sigma^2}{\mu^2} + 1) \quad (3\text{-}6)$$

$$s = \sqrt{\ln(\frac{\sigma^2}{\mu^2} + 1)} \quad (3\text{-}7)$$

From the equation (3-7), it is clear that the volatility estimated by the CA model is simulating the log of the present value of the project.

In summary, Copeland's volatility is a transformed standard deviation of the present project value, and not of the project return. The model does not consider the time frame, a point which is critical for the option valuation model.

3.4 Developing a Reversed Monte Carlo Simulation

This section Dixit and Pindyck (1994) proposed the relationship between the volatility of the underlying asset (σ) and the volatility of future outcomes of the project (σ_T) by studying the perspectives of the normal and the lognormal distribution (see Figure 3-1). Moreover, following Ito's lemma and the general relationship between the parameters of lognormal and normal, they set the equations (3-8) and (3-9) to estimate

$$\mu_T = V_0 \cdot e^{rT} \tag{3-8}$$

$$\sigma_T^2 = V_0^{\,2} \cdot e^{2rT} \cdot (e^{\sigma^2 \cdot T} - 1) \tag{3-9}$$

In order to develop the new method of volatility estimation using Monte Carlo Simulation, the fundamental of options valuation is applied: the option premium is calculated based on the value of the project at the end of option life.

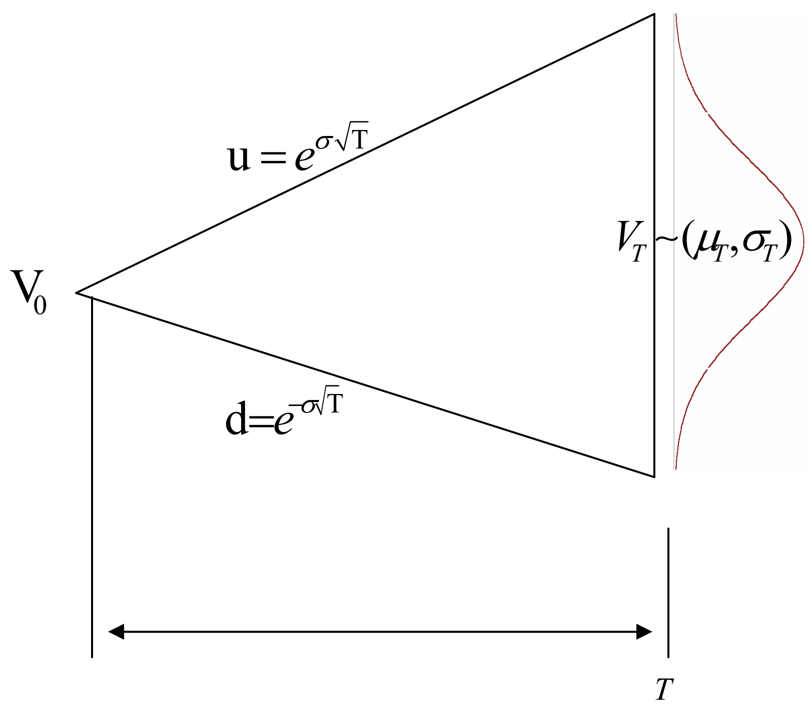


Figure 3-1. Relation between σ_T and σ

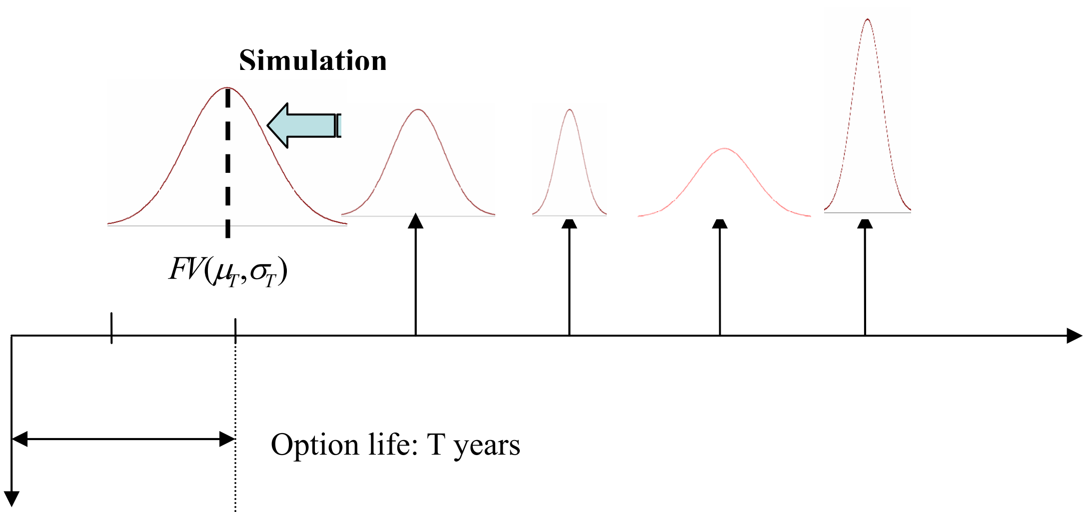


Figure 3-2. Future value simulation

Using the equation (3-8) and (3-9), it is possible to estimate the volatility of the project if we know the parameters μ_T and σ_T . In order to estimate the future value of the project in year T, Monte Carlo Simulation technique is used. Figure 3-2 illustrates the method to simulate the future worth of project cash flows with a risk-free discount rate. The future time T represents the end of the option life, not the end of the project life. If we let the parameters of future cash flow distribution, μ_T, and σ_T, respectively, then values of those two variables are computed by simulation.

Then the unknown variable σ can be calculated by the equation (3-10).

$$\sigma = \sqrt{\frac{\ln\left[\left(\frac{\sigma_T^2}{V_0^{\,2} e^{2rT}}\right) + 1 \right]}{T}} \tag{3-10}$$

3.4.1 Mathematical Proof for RMCS

The core of real options is taking extra time to make decisions concerning the irreversible investment. The real option premium is the price of taking the opportunity to defer decisions. In the real options valuation, generally the present value of the project is considered deterministic, and it will either increase or decrease according to its annual volatility σ. The terminal option premium is calculated by the estimated project value at the end of the option life. If it is possible to estimate the terminal project value, then it is also feasible to compute the annual volatility by Reverse Monte Carlo Simulation, as suggested in this research.

Consider a project which has a present value V_0. Assume the annual return of the project (r_i) is an identical independent normal distribution, and the known μ and σ^2 are the mean and variance of the return, respectively. Further assume that production will begin some years after the investment is made. The value of the project (PV_i) at the end of each year i can be expressed by the multiple formation of the project's present value and the annual return of the project, as below.

$$
\begin{aligned}
PV_1 &= V_0 e^{r_1} \\
PV_2 &= PV_1 \times e^{r_2} = V_0 e^{r_1} e^{r_2} \\
PV_3 &= PV_2 \times e^{r_3} = V_0 e^{r_1} e^{r_2} e^{r_3} \\
&\vdots
\end{aligned}
$$

By using simulation technique, the return of the project during the initial time and the designated timing is defined by

$$\ln\left(\frac{PV_i}{\mathrm{V}_0}\right) = \ln(PV_i) - \ln(\mathrm{V}_0) \qquad (3\text{-}11)$$

If we let μ_i' and σ_i' be the mean and the variance of the return during the period expressed as equation (3-11), then

At i = 1:

$$\mathrm{E}[\ln(\frac{PV_1}{\mathrm{V}_0})] = \mathrm{E}[\ln(PV_1) - \ln(\mathrm{V}_0)]$$

$$= \mathrm{E}[\ln(\mathrm{V}_0 e^{r_1}) - \ln(\mathrm{V}_0)]$$

μ_1':
$$= \mathrm{E}[\ln(\mathrm{V}_0) + \ln(e^{r_1}) - \ln(\mathrm{V}_0)]$$

$$= \mathrm{E}[\ln(\mathrm{V}_0)] + \mathrm{E}[r_1] - \mathrm{E}[\ln(\mathrm{V}_0)]$$

$$= \mu$$

$$\mathrm{Var}[\ln(\frac{PV_1}{\mathrm{V}_0})] = \mathrm{Var}[\ln(PV_1) - \ln(\mathrm{V}_0)]$$

$(\sigma_1')^2$:
$$= \mathrm{Var}[\ln(\mathrm{V}_0\, e^{r_1})]$$

$$= \mathrm{Var}[\ln(\mathrm{V}_0) + \ln(e^{r_1})]$$

$$= \sigma^2$$

At i = 2:

μ_2': $\mathrm{E}[\ln(\frac{PV_2}{\mathrm{V}_0})] = 2\mu$

$$\mathrm{Var}[\ln(\frac{PV_2}{\mathrm{V}_0})] = \mathrm{Var}[\ln(PV_2) - \ln(\mathrm{V}_0)]$$

$(\sigma_2')^2$:
$$= \mathrm{Var}[\ln(\mathrm{V}_0\, e^{r_1+r_2})]$$

$$= \mathrm{Var}[\ln(\mathrm{V}_0) + \ln(e^{r_1}) + \ln(e^{r_2})]$$

$$= 2\sigma^2$$

At i = 3:

μ_3': $\mathrm{E}[\ln(\frac{PV_3}{\mathrm{V}_0})] = 3\mu$

$$\mathrm{Var}[\ln(\frac{PV_3}{\mathrm{V}_0})] = \mathrm{Var}[\ln(PV_3) - \ln(\mathrm{V}_0)]$$

$(\sigma_3')^2$:
$$= \mathrm{Var}[\ln(\mathrm{V}_0\, e^{r_1+r_2+r_3})]$$

$$= \mathrm{Var}[\ln(\mathrm{V}_0) + \ln(e^{r_1}) + \ln(e^{r_2}) + \ln(e^{r_3})]$$

$$= 3\sigma^2$$

At i = t:

μ_t': $\quad E[\ln(\frac{PV_t}{V_0})] = t\mu$

$$\mathrm{Var}[\ln(\frac{PV_t}{V_0})] = \mathrm{Var}[\ln(PV_t) - \ln(V_0)]$$

$$(\sigma_t')^2: \quad = \mathrm{Var}[\ln(V_0\, e^{r_1+r_2+\cdots+r_t})]$$

$$= t\sigma^2$$

Thus the annual volatility of a project σ which bears cash inflow at the end of the t^th^ year will be;

$$\sigma = \frac{\sigma_t'}{\sqrt{t}} \tag{3-12}$$

The variance of the return distribution during the initial and the terminal time period $(\sigma_t')^2$ is computed by (3-9).

$$\sigma_t' = \sqrt{\ln(\frac{{\sigma_T}^2}{{V_0}^2 \cdot e^{2rT}} + 1)} = \sqrt{\ln(\frac{{\sigma_T}^2}{{\mu_T}^2} + 1)} \tag{3-13}$$

Since the equation (3-10) generated from the RMCS and the equation (3-13) from the mathematical definition are the same, the suggested RMCS model can be used as the volatility estimation tool of the project in case of “t” years option life.

3.4.2 Numerical Proof for RMCS

In order to show that the estimated volatility from RMCS correctly represents the risk of the project, it is good to present a simple project with a known return volatility. In our example, the future cash flows are estimated by the current point estimation of the project value and the random return distribution. All the iterations are recorded to compute the mean and the variance of future cash flows. The result of this simple project proved that the estimated volatility calculated through RMCS is the same as the pre-assigned volatility. This is an obvious improvement over the CA model, which consistently computes a higher volatility.

Now consider a project which has a 3-year defer option. The expected present value of the project is $1,000, and its annual volatile return follows a random distribution $N(0.1, 0.2^2)$. It is necessary to simulate the cash inflow at the end of the 3rd year because we deferred the project for 3 years.

The formula to compute the project value at the end of the 3rd year is

$$PV_3 = PV_2 \times e^{r_3} = V_0 e^{r_1} e^{r_2} e^{r_3} = 1{,}000 e^{r_1} e^{r_2} e^{r_3}$$

By using Monte Carlo Simulation, one finds the mean and variance of PV_3 which are the parameters for volatility estimation. Table 3-1 is the result of 1,000 iterations using Microsoft Excel. The simulated mean and standard deviation of PV_3 is 1415.56 and 504.46 respectively

Table 3-1. Sample example to simulate the PV_3 distribution

Iteration	r_1	r_2	r_3	PV_3
1	0.4348	0.0809	0.4621	2658.95
2	0.0289	-0.0469	-0.4868	603.56
…				
1,000	0.0585	0.3171	-0.0406	1397.9
Mean	**0.0901**	**0.1009**	**0.0961**	**1415.56**
Standard Deviation	**0.2019**	**0.2015**	**0.1951**	**504.46**

The volatility of the project is computed as 20% which is the same as pre-assigned annual volatility using the mean and variance shown in table 3-1 with RMCS.

$$\sigma = \sqrt{\frac{\ln(\frac{504.46^2}{1415.56^2}+1)}{3}} = 20\%$$

In contrast, the CA model calculated the simulated volatility as 34.58%.

We then borrowed the data in table 3-2 from Copeland and Antikarov (2003) and used a continuous (rather than annual) compound scenario to demonstrate the difference between the RMCS model and the CA model.

Table 3-2. Cash flows from Copeland and Antikarov (2003)

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Price/unit		10	10	9.5	9	8	7	6
Quantity		100	120	139	154	173	189	200
Variable cost/unit		6.0	6.0	5.7	5.4	4.8	4.2	3.6
Fixed costs		-20	-20	-20	-20	-20	-20	-20
Depreciation		-229	-229	-229	-229	-229	-229	-229
EBIT		**151**	**231**	**280**	**305**	**303**	**284**	**240**
Cash taxes		-61	-93	-112	-122	-121	-114	-96
Depreciation		229	229	229	229	229	229	229
Capital expenses	-1600	0	0	0	0	0	0	0
Working capital		-200	-40	-24	-13	0	13	24
Cash flows	**-1600**	**119**	**327**	**373**	**399**	**411**	**412**	**397**

The option to defer the project for two years is considered the uncertain investment environment. From the table, we derived the distribution of the present value of the project with a risk-adjusted interest rate. The simulated distribution has parameters $(\mu_0, \sigma_0^2) = (1464.28, 88758.15)$, and the present value of the project V_0 is 1464.28 when valuing the option. The CA volatility of the project is calculated as 20.8% by simulation and 20.2% by the equation (3-7).

In the RMCS framework, the project value will be recorded at the end of the 2nd year because the project will be deferred two years. With risk simulation, the distribution of the project value is recorded to $\mu_2 = 1,861.46$ and $\sigma_2{}^2 = 143,192.3$. The volatility of the project is found to be 14.7% by the equation (3-13).

3.5 Conclusion Remarks

The real options application presents an opportunity to improve strategic investment decisions in an uncertain environment. The volatility of the project value is one of the most critical and the only stochastic parameter in traditional real options; therefore, accurately estimating it is essential to valuing the future investment opportunity. However, estimating the volatility of a project is not an easy task because there is no exact historical data to use in the calculations.

Among the various methods of estimating volatility, Monte Carlo simulation alone captures the flexibility of the project itself. However, the statistical definition indicates that the current simulation-based volatility estimating models have some deficiencies that prevent them from accurately representing the true volatility of a project.

This research developed an alternative method of Monte Carlo Simulation, called Reversed Monte Carlo Simulation (RMCS), in order to enhance the use of real options valuation in strategic investment decisions. A mathematical demonstration indicated that RMCS correctly estimates project volatility.

In order to improve the reliability of the volatility derived from RMCS, generating a correct distribution of the future cash flows is the remaining challenge.

CHAPTER 4

BAYESIAN FRAMEWORK TO REVISE THE VOLATILITY OF REAL OPTIONS

Abstract

This paper studies a Bayesian process in order to apply the learning real options framework into the strategic investment decisions. In the learning real options framework, a firm works actively to improve its decision by resolving some uncertainty of the project during the option life, while the traditional options framework only consider a passive consequences of waiting. The volatility of the underlying project is estimated by the Monte Carlo Simulation with the random factors in the cash flow distributions. Re-simulating the revised cash flows with the acquired information of the random factors regenerates new volatility of the risky project. A Dirichlet conjugate processes are applied to estimate the posterior distributions of the random factors.

4.1 Introduction

Recently, real options valuation was developed as a method of capital budgeting decision making. Real options analysis gives a company an opportunity to improve its strategic investment decisions in an uncertain environment. Under the traditional options framework, the strategy of option, “wait-and-see,” means that a company will wait to proceed until the market is more favorable. However, a firm works actively to improve its decision by resolving some of the uncertainty of the project during the option life, while the traditional options framework only considers the passive consequences of waiting. The information obtained during the waiting period will effect investment decisions.

Bayesian statistics are widely used in the area of revising prior beliefs after observing samples. Miller and Park (2005) studied the impact of learning on a multi-staged investment through the B-S model. They consider the normal conjugate distribution in computing the value of the acquired information. However, the real options valuation usually uses the binomial lattice with an assumption that the outcome of the future is discrete. This is because the real options has an American options perspective. If we use the binomial lattice to value the real options, it is also possible to use any kind of distribution for option valuations. So some kind of revision process for the general case is required for the learning real options framework.

In this study, Bayesian processes are developed to revise the initial volatility of a risky project. It is also assumed that only three-point estimates of the random factors of the future cash flows are available to analyze the investment decision, because three-point estimates (an optimistic point, a pessimistic point, and a most likely point) are commonly used for measuring the risk of a project.

After briefly introducing the processes for converting the three-point estimate into a beta distribution of the project value, an initial volatility of the project is determined. Once new sets of information are acquired, a Dirichlet conjugate framework, is utilized to revise the random factors of the cash flows. Re-simulating with the posterior distribution then generates a new volatility of the risky project.

The remainder of this paper is organized as follows. Section 2 reviews the processes of continuous approximation from the three-point estimates. Section 3 frames the volatility revision processes through the Dirichlet conjugate process and Monte Carlo Simulation. Section 4 suggests a numerical example to show the whole developed processes. Finally, Section 5 provides concluding remarks and suggestions for future study.

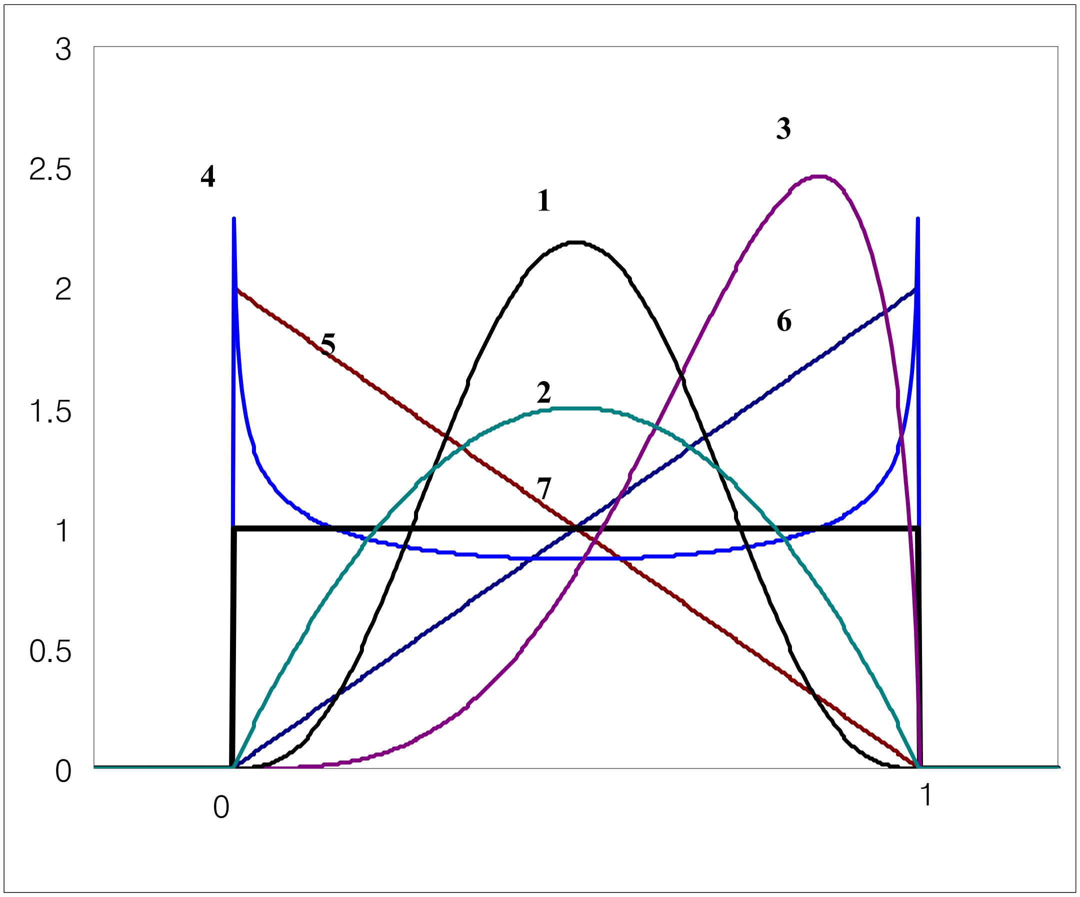
4.2 Estimating a Continuous Random Distribution from Three-point Estimates

Decision-makers begin their assessments by estimating the future aspects of the project. Cash flow, one of the most important of these aspects, is particularly difficult to estimate. When there is not a great deal of uncertainty surrounding a project, it is possible to use probability distributions of future cash flows in the estimation process. However, if the project is more uncertain, then only three possible outcomes can be estimated without probability information. If the project is extremely risky, only the outcome intervals can be estimated. This study assumes that it is possible to estimate at least two points of the random factors.

Park and Sharp-Bette (1990) stated that it is possible to express most of the random distributions when the outputs are bounded by beta distributions. Prueitt and Park (1992) applied the processes for the phase-capacity expansion project.

The standard beta distribution for random variable X is defined as

$$f(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\cdot\Gamma(\beta+1)} \cdot x^{\alpha} \cdot (1-x)^{\beta}, \quad 0 \le x \le 1 \tag{4-1}$$



Curve number	α	β
1	3	3
2	1	1
3	$2+\sqrt{2}$	$2-\sqrt{2}$
4	-0.2	-0.2
5	0	1
6	1	0
7	0	0

Figure 4-1. Shapes of Beta distribution in a specific parameters

The shape of beta distribution is determined by the parameters α and β. The mean, mode, and variance of beta distribution are also defined by these two parameters, and figure 4-1 demonstrates the shapes of the distribution for the different parameters.

$$mean = \frac{\alpha+1}{\alpha+\beta+2}$$

$$mode(M_o) = \frac{\alpha}{\alpha+\beta}$$

$$variance = \frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}$$

Since the standard beta distribution is bounded between 0 and 1, some adjustments are needed in order to transform a specific distribution which has the three points H, L, and M to the standard beta distribution. Equation (4-2) is the transformed beta distribution with the higher bound H, the lower bound L, and the mode M.

$$x = \frac{y-L}{H-L}$$

$$y = L + (H-L)x$$

$$f(y) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\cdot\Gamma(\beta+1)\cdot(H-L)^{\alpha+\beta+1}}\cdot(y-L)^{\alpha}\cdot(H-y)^{\beta}, \quad L \le x \le H \qquad (4\text{-}2)$$

The mean, variance, and mode of the transformed distribution are indicated as follows:

$$E[y] = L + \frac{(H-L)\cdot(\alpha+1)}{\alpha+\beta+2} \qquad (4\text{-}3)$$

$$Var[y] = \frac{(H-L)^2\cdot(\alpha+1)\cdot(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)} \qquad (4\text{-}4)$$

$$Mode[y] = \frac{L\beta + H\alpha}{\alpha + \beta} \tag{4-5}$$

The logic for determining a beta approximation from the three-point estimates uses standardized beta distribution. Since the only known value of a transformed distribution is the Mode ($M_o = \frac{M-L}{H-L}$), there is still not enough information to compute α and β. To solve this problem, an assumption is made that one-sixth of the range can be used as a rough estimate of standard deviation for the distribution. Then the equation (4-6) is found.

$$\alpha^3 + (7M_o - 36M_o^2 + 36M_o^3)\alpha^2 - 20M_o^2\alpha - 24M_o^3 = 0 \tag{4-6}$$

Finally, the parameter α and β, which determine the shape of the beta distribution, can be determined by solving the equation (4-6).

4.3 Bayesian Revision Processes

Bayesian revision is a method for developing a posterior probability distribution by integrating observations with a prior belief. Because it adds observed information to the estimation process, it can be very useful in resolving the uncertainty of a project. The basis of the revision process is the calculation of a posterior probability distribution based on the prior distribution and the observed information. There are two random variables to be considered in the revision process. One is related to the event for determining the prior probability. The other, called a likelihood function, is related to the samplings process. The simplified concept of the Bayesian theorem is demonstrated in figure 4-2.

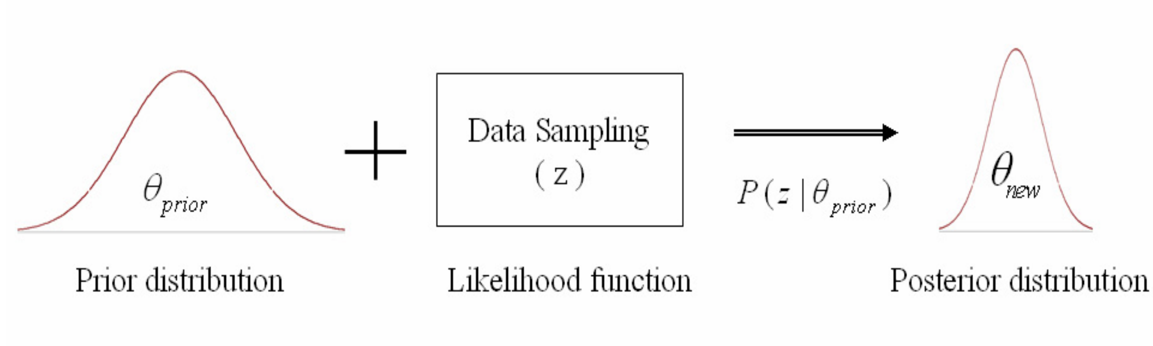


Figure 4-2. Bayesian revision processes

If the prior distribution is discrete, the process can be applied very simply. However, if the prior distribution is continuous, two cases are possible: discrete approximation or special types of likelihood function. In a case of continuous prior distribution, it is necessary that the prior distribution and the data-gathering process share a similar perspective. If a prior random distribution is conjugate with respect to the sampling process, the posterior distribution is the same as the prior distribution. There are three well known conjugate processes: 1) a normal prior distribution with a normal sampling process, 2) a beta prior distribution with a binomial data-gathering process, and 3) a gamma prior distribution with a Poisson sampling process.

Prueitt and Park (1992) presented a method for resolving uncertainty in generalized cases. They argued that one must first develop discrete approximations compared to continuous prior beliefs, then record the collected sample, and lastly, place the observations in discrete categories that correspond to distributions in a Bayesian framework. They stated that even though the nature of the NPV of a project's cash flow is a discrete random variable, the circumstances may preclude handling the cash flow as a continuous variable. Based on their research, three conditions for the application of the discrete approximation are presented below.

1. The uncertainty about the cash flows may not fit a particular probability density function as;
   (1-1) the prior distribution is based on a histogram that does not fit any particular distributions,
   (1-2) varying economic or environmental conditions may create multiple modes condition for the cash flow,
   (1-3) the cash flow may be developed on an incremental basis, and alternative reactions to specific economic conditions may lead to irregular distributions.
2. The prior beliefs and the sampling process may not form one of the natural conjugate distributions.
3. There may be a limited amount of available sample information, and it may not be appropriate to model them as a specific distribution.

In light of the above conditions, an adjustment is required in order to convert the continuous range of outcomes to a set of discrete intervals to be made before applying the Bayesian revision process.

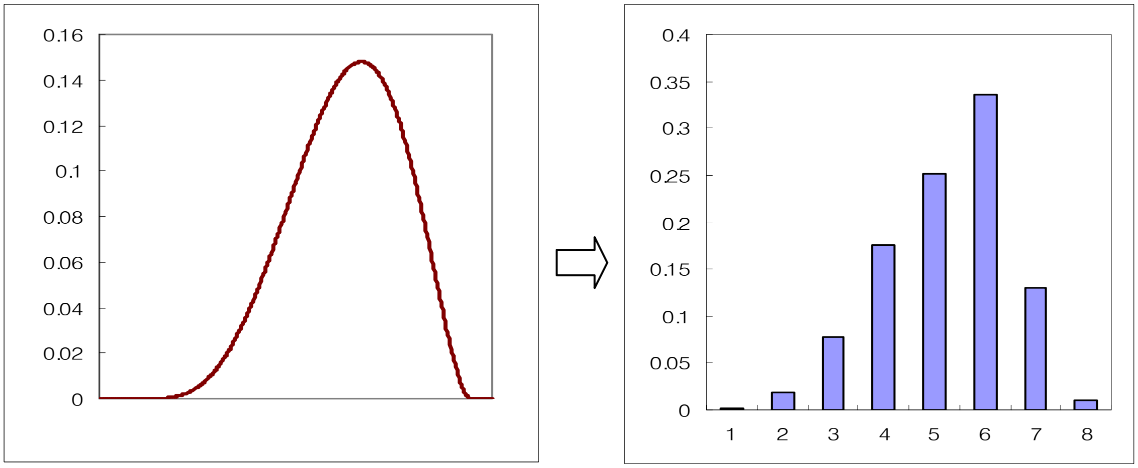
4.4 Volatility Revision Framework in Real Options

The uncertainty of the future is one of the main reasons for purchasing real options. The investors want to postpone their irreversible investment decision in such volatile circumstances by paying some premium to take the priority to act. Therefore, most of the real options applications concentrate on valuing the project opportunity by assuming that the volatility of a risky project is not changed. However, most of the practitioners are interested in the actions they can take after retaining the options, such as tracking the uncertainty resolutions and determining the timing of the irreversible investment. This new information will affect the investment decision. In order to apply Bayesian statistics for revising volatility, it is assumed that the initial volatility of an option is estimated through Monte Carlo simulation as demonstrated in chapter 3.

The present value (PV) of a given project is treated as a probability distribution with unknown parameters because of the uncertainty of the project. While the net present value analysis uses a risk-adjusted interest rate when discounting the future cash flows, the PV is defined as the sum of the discounted cash flows with a risk-free interest rate. Although the prior distributions are assumed to be beta by three-point estimation, the prior distributions of the random factors do not need to have a specific distribution in this case.

The distributions of the risk factors of the cash flows will be modeled with a discrete approximation so that the Dirichlet conjugate processes can be used to define the probability distribution. The Dirichlet distribution is a family of multinomial distributions parameterized by the vector α of nonnegative real numbers. It is a conjugate prior distribution of the multinomial in Bayesian revision statistics. The probability density function of the Dirichlet distribution which has k outcomes is:

$$f(x_1, x_{2,} \cdots, x_K; \alpha_1, \alpha_2, \cdots, \alpha_K) = \frac{1}{L(\alpha)} \prod_{k=1}^{K} x_k^{\alpha_k - 1}$$



Original distribution Discrete approximation

Figure 4-3. Developing a discrete approximation

$$\text{where, } \alpha_k \geq 0,\ x_k \geq 0,\ \sum_{k=1}^{K} x_k = 1,\ \textit{and}\ L(\alpha) = \frac{\prod_{k=1}^{K} \Gamma(\alpha_k)}{\Gamma\left(\sum_{k=1}^{K} \alpha_k\right)}$$

The first step in the Bayesian revision process is to transform the initial distribution, which is based on three-point estimation, by dividing the original distribution into discrete, non-overlapping intervals. The intervals are not required to be equal in length, but are presented as such for the sake of clarity. Once the intervals are defined, corresponding probabilities of the intervals, $\theta_k s$, are determined. Figure 4-3 illustrates the concept for the development of the discrete approximate distribution.

Once the initial interval probabilities are determined, the $\alpha_k s$ of the Dirichlet distribution are calculated by considering the information quality factor (IQF), which represents the amount of belief for the prior probability. For example, if the prior beliefs are considered to be twice the sample amount acquired through the study period, then the IQF = 2. So the $\alpha_k s$ are determined by:

$$\alpha_k = \theta_k \times \text{IQF} \quad \text{for} \quad for \;\; k = 1, \ldots, K$$

The next step is to determine the posterior distribution by conjugating the observation results, $x_k s$, to the prior probability distribution. In some cases, it is necessary to adjust the weight of the sample as well. Then the posterior probability $\theta''_k s$ are computed by:

$$\theta''_k = \frac{\alpha_k + x_k}{\sum_{k=1}^{K} \alpha_k + \sum_{k=1}^{K} x_k}$$

After the posterior probabilities of all the random factors are determined through the Bayesian conjugate processes, a new distribution of the present value of the project can be simulated. Then the posterior volatility of the project is estimated through the method suggested in chapter 3.

The processes demonstrate how to revise the volatility of real options if a new set of sample information is obtained. By revising the volatility of real options, the investors can more accurately determine the optimal timing of their irreversible investment in the project. The previous Bayesian real options framework just considers the value of learning by uncertainty resolution from the sampling. However, in the real world it is very possible that uncertainty will increase after the samples are taken, and the proposed methods can result in better decision making.

4.5 Numerical Example

In this section a numerical example demonstrates the proposed volatility revision processes. Since the initial volatility estimation is based on Monte Carlo Simulation, the uncertainty of the future cash flow of a project is assumed as three-point estimates.

Project descriptions

XYZ Chemical Company, a small manufacturer of a car bumper with carbohydrate-reinforced plastic, considers a new project for the next 5 years. Even though many uncertainties are expected in the future, the most possible cash flows are estimated in table 4-1. To develop the tables, the following assumptions are demonstrated.

Table 4-1. Cash flow estimation for XYZ Chemical Company

EOY	0	1	2	3	4	5
Income Statement						
Revenue						
Unit price		50	50	50	50	50
Demand		2,000	2,000	2,000	2,000	2000
Total revenue		100,000	100,000	100,000	100,000	100,000
Expenses						
Unit variable cost		15	15	15	15	15
Variable cost		30,000	30,000	30,000	30,000	30,000
Fixed cost		10,000	10,000	10,000	10,000	10,000
Depreciation		25,000	25,000	25,000	25,000	25,000
Taxable income		35,000	35,000	35,000	35,000	35,000
Income taxes(40%)		14,000	14,000	14,000	14,000	14,000
Net income		**21,000**	**21,000**	**21,000**	**21,000**	**21,000**
Cash Flow statement						
Operating Activity						
Net Income		21,000	21,000	21,000	21,000	21,000
Depreciation		25,000	25,000	25,000	25,000	25,000
Investment Activity						
Investment	160,000					
Net cash flow	**-160,000**	**46,000**	**46,000**	**46,000**	**46,000**	**46,000**

Table 4-2. Three-point estimates for the random factors

Random Factor	Pessimistic	Most Likely	Optimistic
Demand (EA)	1000	2000	4000
Unit Price ($)	40	50	55
Unit variable cost ($)	11	15	16
Fixed cost ($)	8,000	10,000	15,000

1. There will be no salvage value for the invested equipment at the end of the project.
2. The equipment will be fully depreciated using straight line deprecation method.
3. The risk-free interest rate 5% is assumed, and it does not change during the project.
4. Only the component described in table 4-1 is considered for the decision making.
5. The unit price, demand, unit variable cost, and fixed cost are the random variables, and it is possible to estimate their optimistic, pessimistic, and most likely point. Table 4-2 is the summary of their estimates.

Beta distribution models

The distributions of the random factors are defined by the following processes in order to convert the three-point estimates to the beta distributions. Table 4-3 shows the standardized mode, α, and β of the random factors of the project scenario, and Figures 4-4 – 4-7 illustrate the shapes of the distributions.

Table 4-3. Parameters of beta distributions

Random Factor	α	Mode	β
Demand (EA)	1.819	0.333	3.643
Unit Price ($)	3.643	0.667	1.819
Unit variable cost ($)	3.565	0.8	0.891
Fixed cost ($)	1.469	0.286	3.674

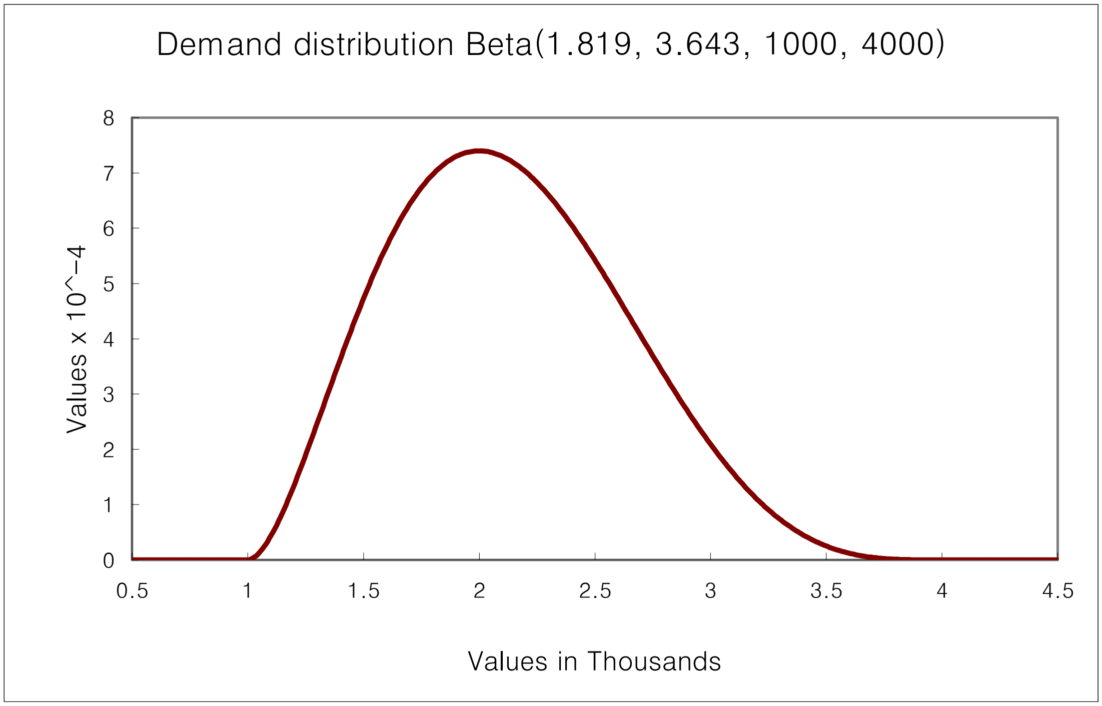


Figure 4-4. Shape of the demand distribution

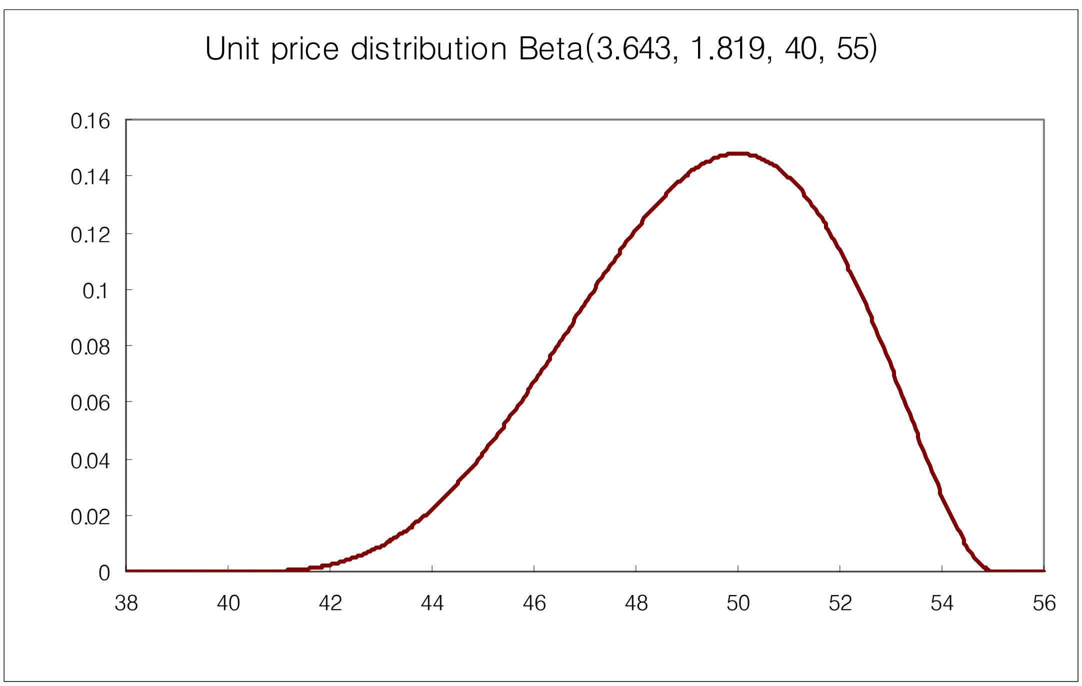


Figure 4-5. Shape of the unit price distribution

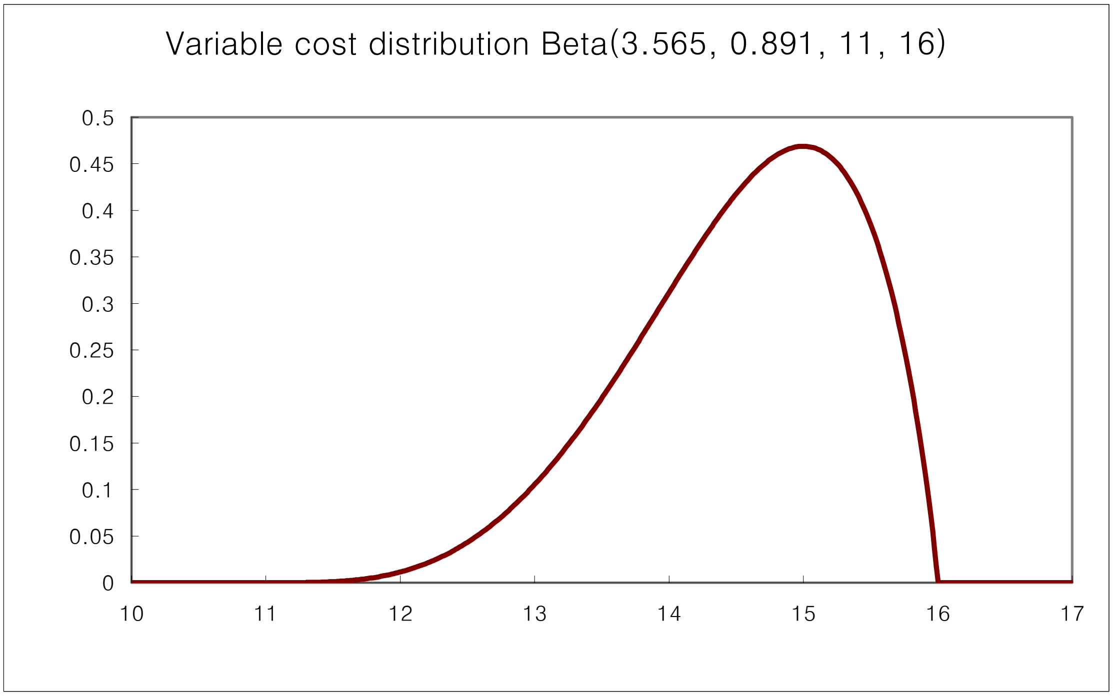


Figure 4-6. Shape of the unit variable cost distribution

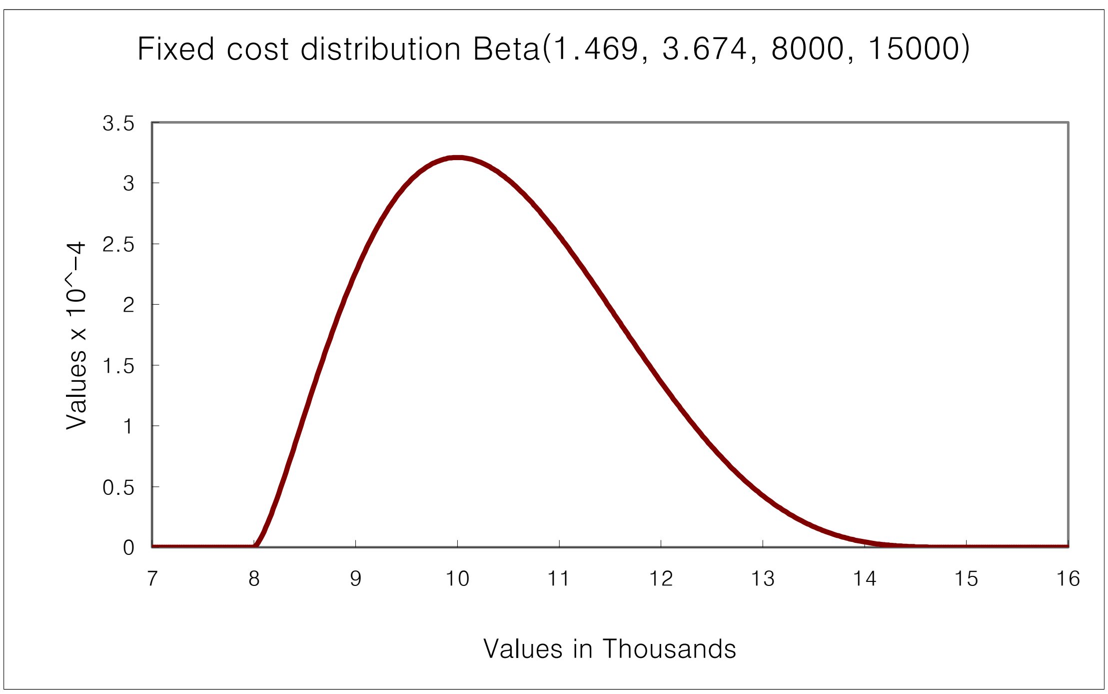


Figure 4-7. Shape of the fixed cost distribution

Volatility estimation

In this section, only the computation of the project volatility is considered. The mean and the variance of the project value is calculated by the @Risk simulation package. The shape of the PV distribution using the risk-free interest rate, which is assumed to be 5% in this case, is shown in Figure 4-8. Then the volatility of the project in case of "T" years of option life will be determined by RMCS. In this case, the volatility of the project is approximately 8% with an assumption that the company wants to defer the project in 2 years, and estimate the cost of taking the option position.

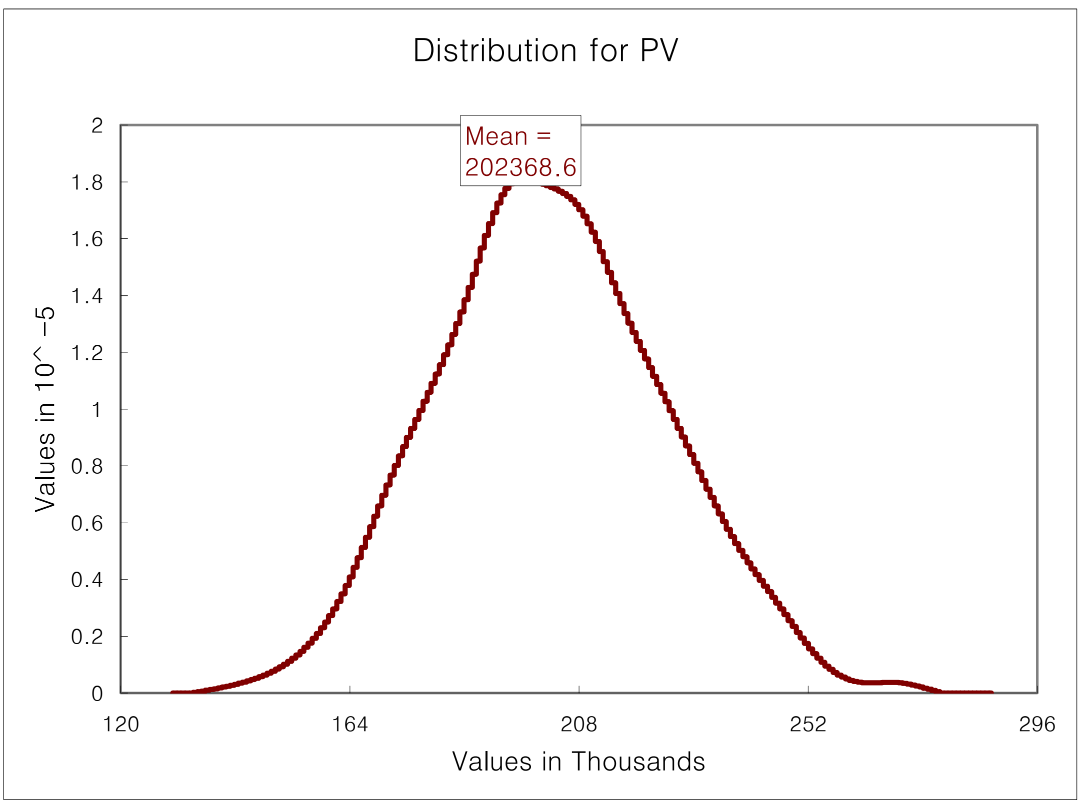


Figure 4-8. Simulated PV distribution

Table 4-4. New information with the initial three-point estimates

Random Factor	Initial belief			New Information
	Pessimistic	Most Likely	Optimistic	
Demand (EA)	1,000	2,000	4,000	2,400
Unit Price ($)	40	50	55	42
Unit variable cost ($)	11	15	16	14.6
Fixed cost ($)	8,000	10,000	15,000	12,000

Posterior volatility estimation

After taking the option for 1 year, the company collects new information for their project shown in table 4-4. From the survey of the management group of the company, the IQF of the initial belief is closer to 2. To make the computation simple, 10 equal length intervals of the discrete approximation are assumed for all the risky variables. For the demand distribution example, each interval has 300 events which transform it to the discrete distribution. Then the median of the intervals are considered as the event of the variable, and the probability of the intervals are recorded as a corresponding probability, $\theta_k s$, of the event. Then the α_k of the prior distribution is computed by $\alpha_k = \theta_k \times IQF = 2\theta_k$.

The next step is to determine the posterior distribution by conjugating the observation results, $x_k s$, to the prior belief. In this case study, it is assumed that the observation is considered as is. The corresponding probability $\theta''_k s$ of the posterior distribution is calculated after $\alpha''_k s$ is determined. Table 4-5 is the processes and the results of the Dirichlet conjugate distribution with the data based on @Risk simulation software. The @Risk simulation package delivered a slightly higher variance for the project. With the computation through the equation (3-10), the volatility 9% for the future option valuations is computed, which is 1% higher than the initial volatility.

Table 4-5. Revised discrete approximation for the risky variables

*** Demand**

k	θ_k	$\alpha_k = \theta_k \times IQF$	x_k	α''_k	θ''_k	Events
1	0.0287	0.0573		0.0573	0.0191	1150
2	0.1265	0.2529		0.2529	0.0843	1450
3	0.1974	0.3948		0.3948	0.1316	1750
4	0.2223	0.4447		0.4447	0.1482	2050
5	0.1866	0.3732	1	1.3732	0.4577	2350
6	0.1306	0.2613		0.2613	0.0871	2650
7	0.0736	0.1471		0.1471	0.0490	2950
8	0.0279	0.0558		0.0558	0.0186	3250
9	0.0061	0.0123		0.0123	0.0041	3550
10	0.0003	0.0006		0.0006	0.0002	3850
Sum	1	2	1	3	1	

*** Price**

k	θ_k	$\alpha_k = \theta_k \times IQF$	x_k	α''_k	θ''_k	Events
1	0.0003	0.0006		0.0006	0.0002	40.75
2	0.0060	0.0120	1	1.0120	0.3373	42.25
3	0.0287	0.0573		0.0573	0.0191	43.75
4	0.0714	0.1428		0.1428	0.0476	45.25
5	0.1309	0.2618		0.2618	0.0873	46.75
6	0.1866	0.3733		0.3733	0.1244	48.25
7	0.2217	0.4434		0.4434	0.1478	49.75
8	0.2018	0.4035		0.4035	0.1345	51.25
9	0.1230	0.2460		0.2460	0.0820	52.75
10	0.0297	0.0594		0.0594	0.0198	54.25
Sum	1	2	1	3	1	

Table 4-5. Revised discrete approximation for the risky variables (Con't)

*** Variable cost**

k	θ_k	$\alpha_k = \theta_k \times IQF$	x_k	α''_k	θ''_k	Events
1	0.0001	0.0003		0.0003	0.0001	11.25
2	0.0028	0.0056		0.0056	0.0019	11.75
3	0.0144	0.0287		0.0287	0.0096	12.25
4	0.0395	0.0789		0.0789	0.0263	12.75
5	0.0846	0.1692		0.1692	0.0564	13.25
6	0.1445	0.2890		0.2890	0.0963	13.75
7	0.2024	0.4048		0.4048	0.1349	14.25
8	0.2554	0.5108	1	1.5108	0.5036	14.75
9	0.2504	0.5009		0.5009	0.1670	15.25
10	0.0060	0.0120		0.0120	0.0040	15.75
Sum	1	2	1	3	1	

*** Fixed Cost**

k	θ_k	$\alpha_k = \theta_k \times IQF$	x_k	α''_k	θ''_k	Events
1	0.0528	0.1055		0.1055	0.0352	8350
2	0.1509	0.3017		0.3017	0.1006	9050
3	0.2173	0.4345		0.4345	0.1448	9750
4	0.2171	0.4342		0.4342	0.1447	10450
5	0.1707	0.3415		0.3415	0.1138	11150
6	0.1101	0.2203	1	1.2203	0.4068	11850
7	0.0563	0.1126		0.1126	0.0375	12550
8	0.0203	0.0406		0.0406	0.0135	13250
9	0.0043	0.0086		0.0086	0.0029	13950
10	0.0002	0.0004		0.0004	0.0001	14650
Sum	1	2	1	3	1	

4.6 Conclusion Remarks

Once the investment opportunity is undertaken, tracking the market information becomes one of the most important actions in determining the best timing for the follow-up strategic investment decisions. A Bayesian revision process is demonstrated in order to update the volatility of the real options in cases where the initial volatility was estimated through Monte Carlo Simulation. With an assumption of limited sample information, a probability conjugate process for the general distribution case is applied to estimate the posterior volatility of the project. The Dirichet conjugate distribution is applied for estimating the posterior volatility of the project by transforming the continuous prior distribution to the discrete distribution.

The previous Bayesian learning real options framework considers the value of learning by uncertainty resolution from the sampling. However, the specific methods to conjugate the sample information into the prior belief are not studied. This study demonstrates that Dirichlet revision processes can conjugate the general prior distribution with a very limited amount of sample information. Correct information on volatility and the future aspect of the project will make the decisions for irreversible investments more accurate.

More studies are still necessary to determine the importance of volatility in the decision making process, because very little research has been conducted regarding volatility revision after the options are purchased.

CHAPTER 5

DEVELOPING DECISION SUPPORT SYSTEM IN REAL OPTIONS

Abstract

One of the most critical issues in real options analysis is determining the optimal timing of the irreversible investment during the life of the option. Research indicates that failing to exercise real options on time reduces the projects' value much less than predicted. However, the question of whether real option holders exercise their options optimally has not been extensively researched. In this research, a new early decision rule for real options is developed, and a simulation technique validates this new rule. The result of the simulation indicates that the new decision rule gives higher profits than the traditional rule in most cases.

5. 1 Introduction

Recently, managers have addressed the uncertainty in various capital budgeting decisions by applying an options analysis to their evaluations of the project. This evaluation technique, real options analysis, provides an opportunity to improve strategic investment decisions in an uncertain environment, but the real options valuation concept requires some adjustments in order to be useful in management decisions. One of the most critical issues in real options is deciding the timing of investment or divestment in the project during the life of the option. Recent research shows that failing to exercise the real options on time reduces the projects' value much less than predicted. However, the question of whether real option holders exercise their options optimally has not been extensively researched. Therefore, the possibility of early action also needs to be considered in order to make real options more useful.

After deciding to retain the real option, the investors are required to decide the timing for exercising or divesting the option. Because of the irreversibility of the investments, deciding on the optimal investment timing is one of the most important factors in the real options valuation model. In the financial options model, the timing to exercise is defined as the point at which the value of immediate exercise is higher than that of holding the option to its expiration date. However, research has indicated that early exercise is never optimal on a non-dividend paying stock in the financial call option theory, which is applied to most real options valuation models (Hull, 2005).

Copeland and Tufano (2004) stated that defining the optimal exercise timing of the real option is essential in order for real options to work well in the real world. They suggested that failing to exercise real options on time reduces the value of the projects much less than predicted. However, in spite of its importance to decision makers, there has not been adequate research into selecting the optimal timing for real options. Brennan and Schwartz (1985) developed an evaluation model wherein they set stochastic output prices in order to decide the optimal investment timing for continuing or abandoning a mining project. McDonald and Siegel (1986) studied how to optimally time the investment in an irreversible project when the benefit and cost of the project follows Geometric Brownian Motion. They used the simulation technique to show that risk-averse investors should wait until benefits are twice the investment costs.

Yaksick (1996) suggested a method to compute an expected exercise timing of a perpetual American option. Shackleton and Wojakowski (2002) derived a numerical expression for computing the expected return and for finding the optimal timing for the exercise of real options by using the risk-adjusted stopping time method, which is based on the actual probability distribution of payoff times. Rhys, Song, and Jindrichovska (2002) summarized the recent developments in this area, reporting that only a few studies have been conducted to analyze the problem of timing, but concluding that some progress is being made in the research.

A new decision rule, based on the binomial lattice valuation model, is presented here to determine the optimal timing of irreversible investments. The expected profits under the new decision rule are compared to the traditional rule in order to demonstrate the advantages of this new approach, and a simulation technique is used to present the benefit of new early exercise rule in real options. The fundamental of the simulation for validating the new decision rule is generating path-dependant project values. After simulating the required amount of the project value path, the expected future values of the two cases are calculated. Then paired t-tests are conducted to observe the differences between the traditional decision rule and the new decision rule.

Section 2 reviews the general early exercise rule of options and discusses the new decision model. Section 3 substantiates the new approach by comparing the expected values of projects using the current options decision rule with those using the new decision rule. A simulation technique is used to generate the expected future project values. Section 4 applies the new decision model to a defer option and an abandon option. Finally, section 5 contains a summary of the research as well as its conclusions

5.2 Developing a New Decision Rule of Real Options

Although an early exercise of a financial call option of the non-dividend paying stock is known to be never optimal, the same theory does not apply to the real options because projects of a firm do not have the same characteristics as financial assets. One of the most important differences is that in the real options framework, the investment is irreversible once the project is undertaken, and the invested budget is generally not tradable. Therefore, deriving the optimal exercise timing of real options requires a unique approach. The logic of decision analysis is applied to find the optimal decision timing of the obtained options, because determining the exercise timing and the forfeit timing of real options is different from financial options. The process of developing the new decision rule concentrates on the total expected future profit by taking early actions.

5.2.1 Early Exercise in Financial Options

Before relating the details of the new decision model, it is necessary to review the early exercise decision of the financial option pricing. An American option can be exercised at any time during its life. The early exercise of the American option is decided by comparing the value of waiting to the payoff of early exercise. A binomial lattice valuation model that was originated by Cox, et al. (1979) is applied to calculate the early decision points.

Figure 5-1 illustrates the procedures for deciding the early exercise in node ① by a binomial lattice approach.

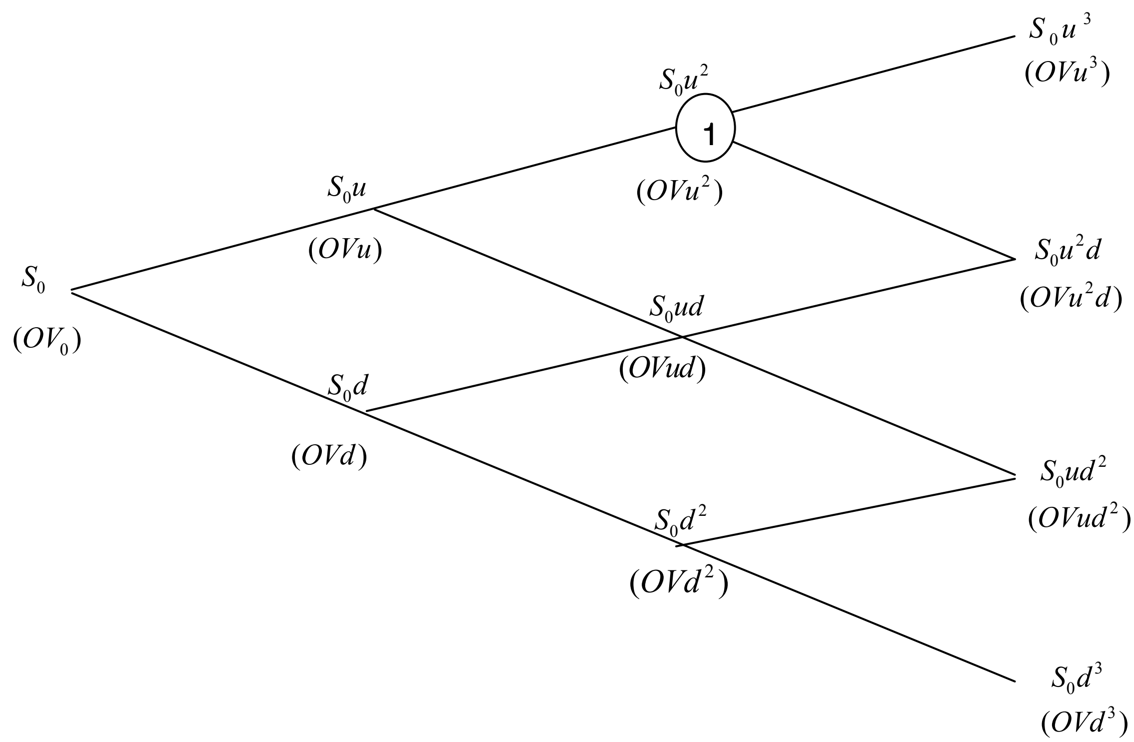


Figure 5-1. Binomial tree approach for early exercise decision

The initial stock price S_0 moves to one of two values, S_0u and S_0d, during the first time interval. The two values also will move to two possible directions, "up" or "down," during the next time interval, and so on. The parameter u represents an "up" movement and d a "down" movement during a time interval Δt. The other parameters in the lattice are p which represents the probability that the stock price takes an "up" movement, $1-p$ which is the probability that the stock price moves "down," and r the risk-free rate of the model. Then, the processes for deciding on early exercise at node ① are:

1. Compute the option value OVu^2 by waiting.

$$OVu^2 = [p(OVu^3) + (1-p)(OVu^2d)] \cdot e^{-r\Delta t}$$

2. Compute the immediate payoff.

$OVu^2 = \max(S_0 u^2 - K, 0)$, for a call option

$OVu^2 = \max(K - S_0 u^2, 0)$, for a put option

3. Select the highest value of step 1 and step 2 as the option value of the node ①. If the value computed by the immediate payoff option from step 2 is higher than that of waiting, which is defined from step 1, early exercise is preferred in the node. However, the empirical test demonstrates that early exercise is never optimal for American call options of the no-dividend paying stock, while early exercise may be possible in an American put option.

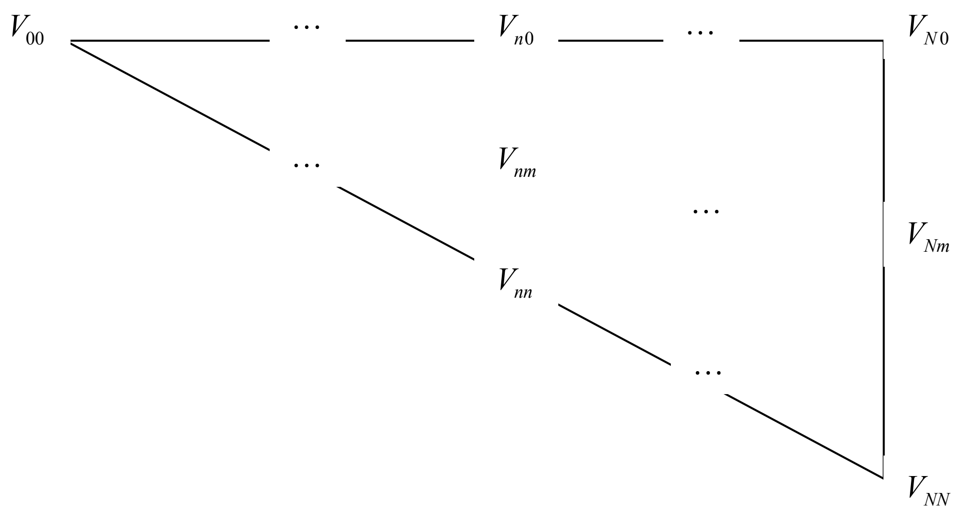
5.2.2 New Decision Model

The new idea for determining the early investment points of the real option is based on the opportunity cost concept, which is defined by comparing the expected future option value with the expected future profit earned by early action. Two important assumptions are necessary for the development of the new decision model.

- The first assumption is that once the option is exercised, the projected profit is immediately realized and is available for other investment purposes.
- The second assumption is that the investment in the other projects will earn the risk-adjusted rate of return of the company, compounded continuously. These two assumptions are widely used in the engineering economics analysis of the capital investment decision.

Before explaining the method of making decision in detail, it is necessary to define some notations.

Notations



- T : Option life, Δt : Length of the option period
- $\frac{T}{\Delta t} = N$: Number of time period during the option life
- n : Time node, $n = 0,1,2,...,N$
- m : Value states in each time node. $m = 0,1,2,...,n$
- V_{nm} : Estimated project value at m^{th} highest value of time n .
- OV_{nm} : Option value at node nm . For the call style option model, since we initially considered a defer option model, the option values at the end nodes are defined as $OV_{Nm} = V_{Nm} - I$ and the values at the other time nodes are defined as,

  $OV_{nm} = [OV_{(n+1)m} \cdot p + OV_{(n+1)(m+1)} \cdot (1-p)] \cdot e^{-r\Delta t}, n = 0,1,2,...,N-1, \ \ m = 0,1,2,...,n$

  where, p : risk-neutral probability

  For a put-style real option such as an abandonment option, the option values at the very last time nodes are determined by $OV_{Nm} = I - V_{Nm}$ and the value of the other time nodes are determined by,

$$OV_{nm} = [OV_{(n+1)m} \cdot p + OV_{(n+1)(m+1)} \cdot (1-p)] \cdot e^{-r\Delta t}, n = 0,1,2,...,N-1, \ \ m = 0,1,2,...,n$$

- PV_{nm} : Estimated early exercise profit at node nm.

  For the Call style $PV_{nm} = V_{nm} - I$, $n = 0,1,2,...,N-1$, $m = 0,1,2,...,n$

  For the Put style $PV_{Nm} = I - V_{Nm}$, $n = 0,1,2,...,N-1$, $m = 0,1,2,...,n$

Decision rule

The decision of early exercise is based on the option valuation model and the capital budgeting decision rule. In the new decision rule, the profit from the early investment will earn the risk-adjusted interest rate (R) which is generally higher than the risk-free interest rate. Three actions are determined by the new decision rule.

- Invest if the future value of the profit realized by the direct investment is higher than the expected option value.
- Divest when the expected option value reaches zero.
- Wait to observe the movements of the underlying project value.

Table 5-1 demonstrates the conditions for early action at time n after retaining the real options.

5.2.3 Early Exercise Points for Lower Volatility Projects

For the purposes of this study, three assumptions are made in the development of the early exercise conditions for real options: 1) the risk-adjusted interest rate is higher or equal to the risk-free interest rate, $\text{R} \geq \text{Risk-free rate}$, 2) the investment decision will be based on the expected value criteria, and 3) the decision on early investment is investigated once a year, $\Delta t = 1$.

The first step is to determine the condition of the early exercise points for a specific case. If all the terminal payoffs which are initiated from the current project value through the lattice approach are non-negative, which is common in projects of relatively lower volatility, then the conditions of the early exercise decision are derived as below with the remaining option life n. If the project value at time $(N-n)$ is higher than the right hand side of equation (5-1), the decision is for early exercise at that time. The value $V^*_{(N-n)m}$ represents the minimum project value for early exercise when n years are remaining. The process to derive the equation (5-1) is in Appendix 5-1.

$$V^*_{(N-n)m} = \frac{I(e^k - e^{-(n-1)r})}{e^k - e^r}, \quad n = 2, 3, \cdots N \tag{5-1}$$

Equation (5-1) implies that the project value for deciding the minimum investment conditions of the real option has no correlation with the risk factors of the project. Rather, the condition is a function of the remaining time, the risk-free rate, and the risk-adjusted interest rate of the company. The equation (5-1) is resulted in the low flexibility of the project; therefore, further studies are necessary for more risky projects.

Table 5-1. Three decision options

Course of Action	Conditions
Exercise the option	$0 < [OV_{(n+1)m} \cdot p + OV_{(n+1)(m+1)} \cdot (1-p)] \le PV_{nm} \cdot e^{MARR \cdot \Delta t}$,
Wait more time	$0 < PV_{nm} \cdot e^{MARR \cdot \Delta t} < [OV_{(n+1)m} \cdot p + OV_{(n+1)(m+1)} \cdot (1-p)]$,
Quit the option	$[OV_{(n+1)m} \cdot p + OV_{(n+1)(m+1)} \cdot (1-p)] = 0$,

5.2.4 General Rules of Early Exercise

Once the condition (5-1) is determined, the next step, which considers the existence of the negative PV, can be initiated. A new heuristic approach is developed in this section. The start of determining the general rules of the early exercise points is to examine d, a "down" movement, during a time interval Δt. The result of the examination is closely correlated to the volatility of the underlying project. For example, if $V^*_{(N-1)m} \geq \frac{I}{d}$ when only 1 year is remaining for the final decision, use the equation (5-1) because all the output at $n = N$ will be non-negative. Otherwise, the equation (5-1) is not applicable because the computed value is far less than the correct value. In this situation, we need an alternative approach for determining the minimum exercise points at each time node. The details of computations to develop the exercise points are in Appendix 5-2.

At $n = N - 1$

Step 1: Define the exercising point through the general approach and check the feasibility.

$$\begin{cases} V^*_{(N-1)m} \geq \dfrac{I}{d}, & \textit{Stop here, the number you get is the point} \\ V^*_{(N-1)m} < \dfrac{I}{d}, & \textit{Go to step two} \end{cases}$$

Step 2: Compute an alternative point by using the formula below.

$$V^{**}_{(N-1)m} \geq \frac{I(e^k - p)}{(e^k - u \cdot p)} \quad \text{where,} \quad I \leq V^*_{(N-1)m} < \frac{I}{d} \tag{5-2}$$

At $n = N-2$

Step 1: Define the exercising point through the general approach and check the feasibility.

$$\begin{cases} V^{*}_{(N-2)m} \geq \dfrac{I}{d^2}, & \text{Stop here, the number you get is the point} \\ V^{*}_{(N-2)m} < \dfrac{I}{d^2}, & \text{Go to step two} \end{cases}$$

Step 2: Compute an alternative point by using the formula below.

$$V^{**}_{(N-2)m} \geq \frac{I[e^k - pe^{-r}(2-p)]}{[e^k - p(u + e^{-r}(1-p))]} \quad \text{where,} \quad \frac{I}{d} \leq V^{*}_{(N-2)m} < \frac{I}{d^2} \quad (5\text{-}3)$$

At $n = N-3$

Step 1: Define the exercising point through the general approach and check the feasibility.

$$\begin{cases} V^{*}_{(N-3)m} \geq \dfrac{I}{d^3}, & \text{Stop here, the number you get is the point} \\ \dfrac{I}{d} \leq V^{*}_{(N-3)m} < \dfrac{I}{d^3}, & \text{Go to step two} \\ V^{*}_{(N-3)m} < \dfrac{I}{d}, & \text{Go to step three} \end{cases}$$

Step 2: Compute the first alternative point by using the formula below.

$$V^{**}_{(N-3)m} \geq \frac{I[e^k - pe^{-2r}(3(1-p) + p^2)]}{e^k - p[u + (1-p)e^{-r} + (1-p)^2 de^{-2r}]}, \text{ where} \frac{I}{d} \leq V^{*}_{(N-3)m} < \frac{I}{d^3} \quad (5\text{-}4)$$

Step 3: Compute the second alternative point using by the formula below

$$V^{***}_{(N-3)m} \geq \frac{I[e^k - p^2 e^{-2r}(2-p))]}{e^k - up^2 e^{-r}(u + e^{-r}(1-p))}, \text{ where } V^{*}_{(N-3)m} < \frac{I}{d} \quad (5\text{-}5)$$

From the processes suggested above and equations (5-2) through (5-5), determining the conditions of early investment is more complicated in the earlier stages of the option life. This is because it is more difficult to derive the mathematical replicating rules. This complexity could hinder the attempt to generalize the decision rules. However, the computational complexity can be resolved by a heuristic approach by developing computer software to calculate the minimum project value for the early exercise points. Initially Excel spreadsheet is used to demonstrate the basic rule, but C++ software is developed for the heuristic algorithm (See appendix 5-3).

5.2.5 Sensitivity Analysis for the New Decision Rules

This section demonstrates the sensitivity of the early decisions in cases of different volatility and different risk-adjusted interest rates. A ratio between the project value and the investment cost is used to define the general early decision points. It is anticipated that the decision points of early actions are closely correlated to the volatility of the project value. For example, if the volatility increases, the early investment point will increase, and the early divest point will decrease. If the volatility decreases, the minimum value for the early investment will decrease, and the early divest point will increase. Table 5-2 illustrates the early decision ratios with risk-adjusted rate 15%, risk-free rate 5%, $T = 7\ years$, and $\Delta t = 1\ year$.

In order to check the ratio's sensitivity to different risk-adjusted rates, the ratios of the four different interest rate cases are examined. If we increase the risk-adjusted rate, the ratio V/I for the early exercise points is decreased while the divest points remain the same. Table 5-3 demonstrates the early decision points of V/I ratio for four different volatility cases with the risk free rate of 5% and the option life of 7 years.

Table 5-2. Ratios (V/I) for early decisions in different volatilities

Volatility	Decisions	1	2	3	4	5	6
20%	Exercise	3.4792	3.1033	2.7233	2.3242	1.9050	1.4642
	Quit	0.3008	0.3675	0.4492	0.5492	0.6700	0.8183
25%	Exercise	3.4792	3.1192	2.7233	2.3242	1.9050	1.4642
	Quit	0.2233	0.2867	0.3675	0.4725	0.6067	0.7792
30%	Exercise	3.5058	3.1650	2.7950	2.3633	1.9050	1.4642
	Quit	0.1650	0.2233	0.3008	0.4067	0.5492	0.7408
35%	Exercise	3.6175	3.2208	2.9042	2.4742	1.9625	1.4642
	Quit	0.1225	0.1742	0.2467	0.3500	0.4967	0.7050
40%	Exercise	3.7800	3.2958	2.9800	2.5983	2.0733	1.4858
	Quit	0.0908	0.1333	0.2017	0.3008	0.4492	0.6700
45%	Exercise	3.9600	3.4975	3.0867	2.7333	2.1925	1.5433
	Quit	0.0675	0.1050	0.1650	0.2592	0.4067	0.6375
50%	Exercise	4.1517	3.7142	3.2000	2.8767	2.3200	1.6042
	Quit	0.0500	0.0825	0.1350	0.2233	0.3675	0.6067
55%	Exercise	4.3533	3.9442	3.3192	3.0275	2.4533	1.6675
	Quit	0.0367	0.0642	0.1108	0.1917	0.3325	0.5767
60%	Exercise	4.5625	4.1850	3.4425	3.1842	2.5942	1.7325
	Quit	0.0275	0.0500	0.0908	0.1650	0.3008	0.5492
65%	Exercise	4.7767	4.4342	3.6108	3.3475	2.7417	1.8008
	Quit	0.0200	0.0392	0.0742	0.1425	0.2725	0.5217
70%	Exercise	4.9950	4.6900	4.1000	3.5150	2.8950	1.8708
	Quit	0.0150	0.0300	0.0608	0.1225	0.2467	0.4967
75%	Exercise	5.2142	4.9500	4.0825	3.6858	3.0550	1.9433
	Quit	0.0108	0.0233	0.0500	0.1050	0.2233	0.4725
80%	Exercise	5.4342	5.2125	4.3308	3.8608	3.2200	2.0192
	Quit	0.0083	0.0183	0.0408	0.0908	0.2017	0.4492
85%	Exercise	5.6533	5.4758	4.5858	4.0383	3.3908	2.0967
	Quit	0.0058	0.0142	0.0333	0.0783	0.1825	0.4275
90%	Exercise	5.9183	5.7383	4.8467	4.2175	3.5667	2.1767
	Quit	0.0042	0.0108	0.0275	0.0675	0.1650	0.4067
95%	Exercise	6.2333	5.9975	5.1117	4.3975	3.7467	2.2600
	Quit	0.0033	0.0083	0.0225	0.0575	0.1492	0.3867

Table 5-3. Ratios (V/I) for early decisions in different risk-adjusted rates

Volatility	R	Decisions	1	2	3	4	5	6
30%	15%	Exercise	3.5058	3.1650	2.7950	2.3633	1.9050	1.4642
		Quit	0.1650	0.2233	0.3008	0.4067	0.5492	0.7408
	20%	Exercise	2.7100	2.4359	2.2172	1.9823	1.6874	1.3344
		Quit	0.1650	0.2233	0.3008	0.4067	0.5492	0.7408
	25%	Exercise	2.2895	2.1259	1.9095	1.7595	1.5672	1.2993
		Quit	0.1650	0.2233	0.3008	0.4067	0.5492	0.7408
40%	15%	Exercise	3.7800	3.2958	2.9800	2.5983	2.0733	1.4858
		Quit	0.0908	0.1333	0.2017	0.3008	0.4492	0.6700
	20%	Exercise	2.9142	2.6806	2.3408	2.1498	1.8554	1.4306
		Quit	0.0908	0.1333	0.2017	0.3008	0.4492	0.6700
	25%	Exercise	2.4419	2.3113	2.0687	1.8878	1.7048	1.3851
		Quit	0.0908	0.1333	0.2017	0.3008	0.4492	0.6700
50%	15%	Exercise	4.1517	3.7142	3.2000	2.8767	2.3200	1.6042
		Quit	0.0500	0.0825	0.1350	0.2233	0.3675	0.6067
	20%	Exercise	3.1544	2.9670	2.5700	2.3396	2.0439	1.5346
		Quit	0.0500	0.0642	0.1108	0.1917	0.3325	0.5767
	25%	Exercise	2.6422	2.5255	2.2844	2.0298	1.8559	1.4769
		Quit	0.0500	0.0500	0.0908	0.1650	0.3008	0.5492
60%	15%	Exercise	4.5625	4.1850	3.4425	3.1842	2.5942	1.7325
		Quit	0.0275	0.0500	0.0908	0.1650	0.3008	0.5492
	20%	Exercise	3.4095	3.2785	2.8656	2.5444	2.2491	1.6458
		Quit	0.0275	0.0500	0.0908	0.1650	0.3008	0.5492
	25%	Exercise	2.9217	2.7583	2.5158	2.1807	2.0175	1.5748
		Quit	0.0275	0.0500	0.0908	0.1650	0.3008	0.5492

The general trend of the early decision rule of the different project volatility can be developed from the data in Table 5-2. The ratio of the project value to the investment cost is used to develop the general rule. Once the V/I ratios are determined for the early decision, the minimum value of the early decision point can be computed by multiplying the investment cost by the ratio.

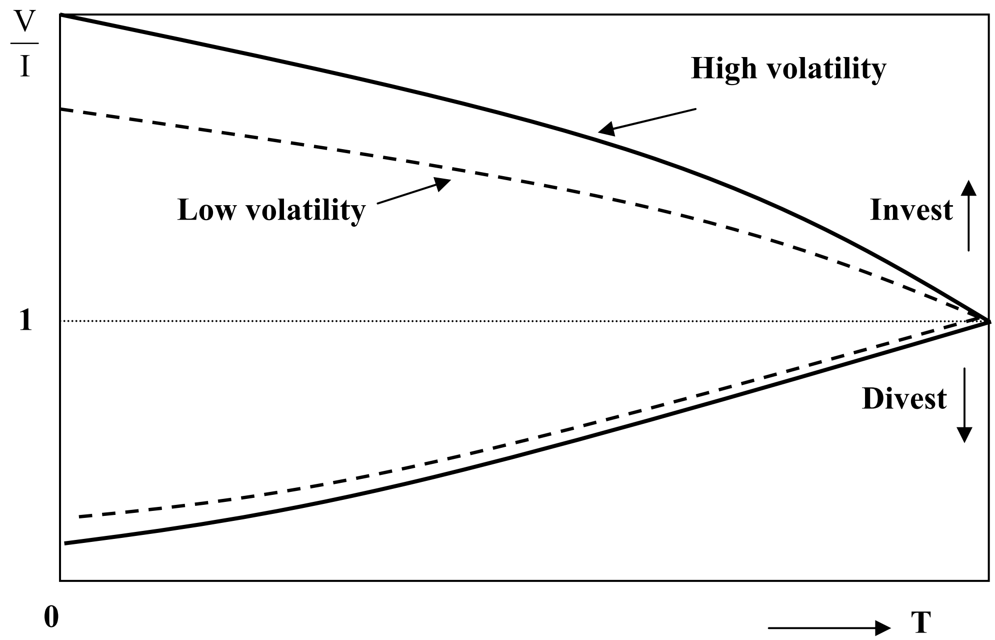


Figure 5-2. Decision changes in case of different volatilities

Figure 5-2 demonstrates the changes of the ratio V/I of the decision map in cases of different volatilities. The figure demonstrates that the higher volatility project has a larger waiting area than the lower volatility project. This trend indicates that the decision-maker has more time to react to changes in market conditions.

Figure 5-3 illustrates the trend of the early decision point in cases of different risk-adjusted interest rates. The figure shows that the higher the risk-adjusted interest rate, the lower the early investment points. However, the early divest point does not change even when the interest rate is different.

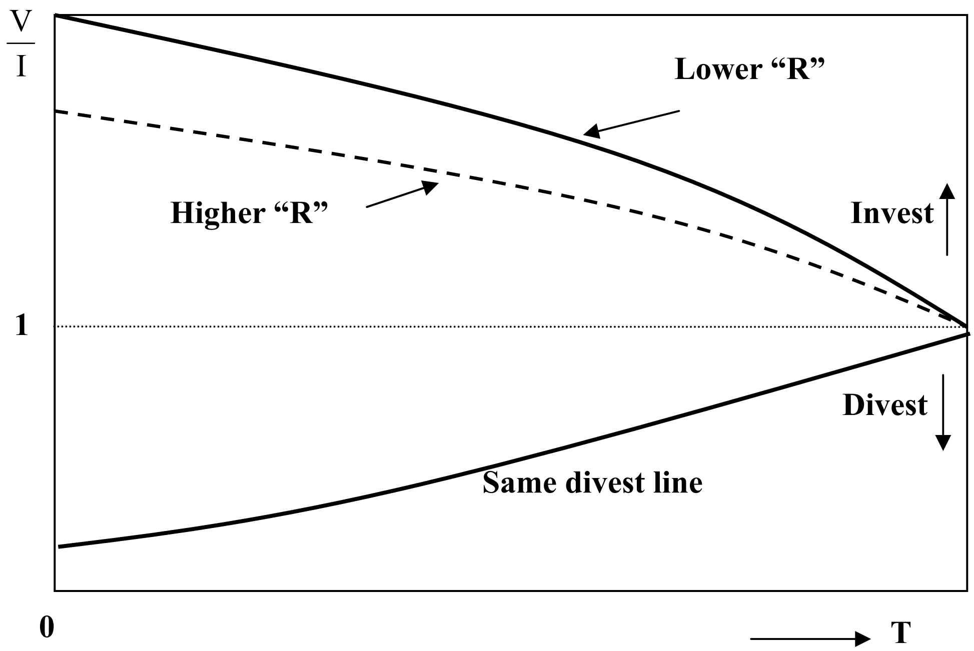


Figure 5-3. Decision changes in case of different risk-adjusted interest rates

5.3 Validation of the New Decision Model through Simulation

A simulation technique is used to demonstrate the benefits of a new decision rule that considers early actions in real options. The expected profit earned under this new decision rule, as developed in this study, and the profit made under the traditional rule are compared to determine the advantages of the new approach.

The fundamental feature of this simulation is the generation of a path-dependant project value of every year. This means that the project value during the next period depends on its previous value. Figure 5-4 is an illustration of the path-dependent simulation of a future project value. After simulating a path of the future project value, the expected future values and the standard deviation of the two cases are calculated. Finally, paired t-tests are conducted to determine if there is a statistical significance between the two decision rules.

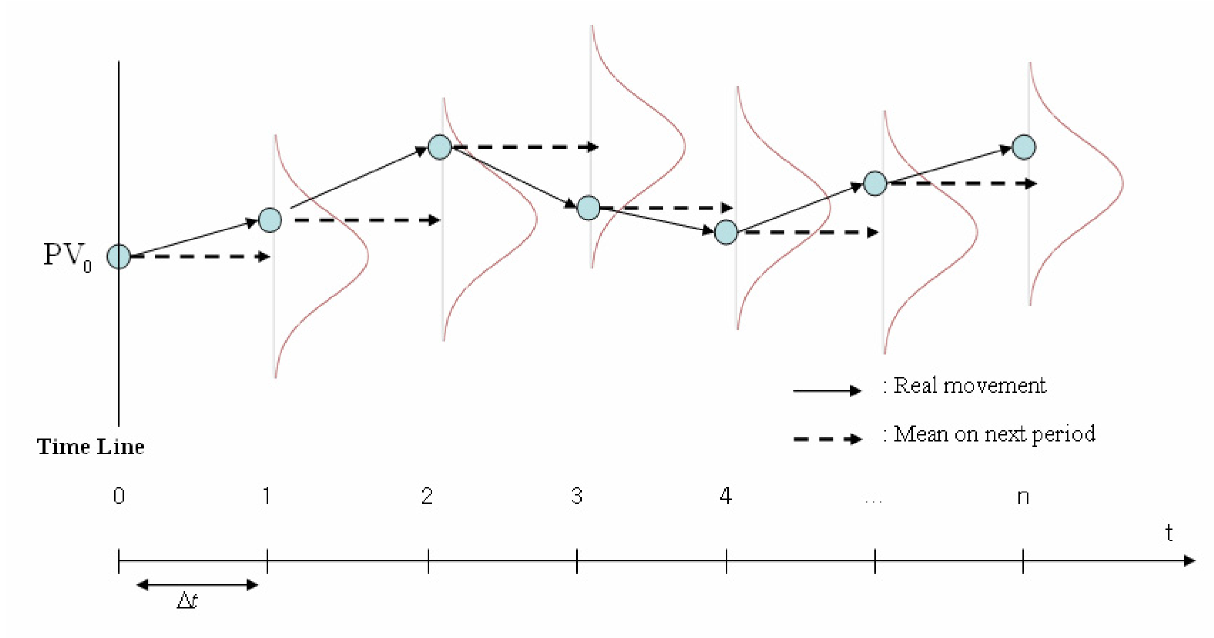


Figure 5-4. Illustration of simulating a project value

5.3.1 Defer Option Framework

To develop the simulation model, we assume that the present value of the project PV_0 is deterministic, and if a generated project value meets an early decision rule, the decision is made at that time. If not, wait another period to make the decision..

The procedures for the simulation are listed below:

Step 1: Simulate a project value path for the pre-assigned iterations.

Step 2: Apply the decision rule without volatility revision. If the decision indicates "Early invest" or "Early divest," record the profit at the time of exercise. If the result is "Wait," then go to the next time period and recheck the decision rule. Figure 5 shows the "Early invest" and "Early divest" cases.

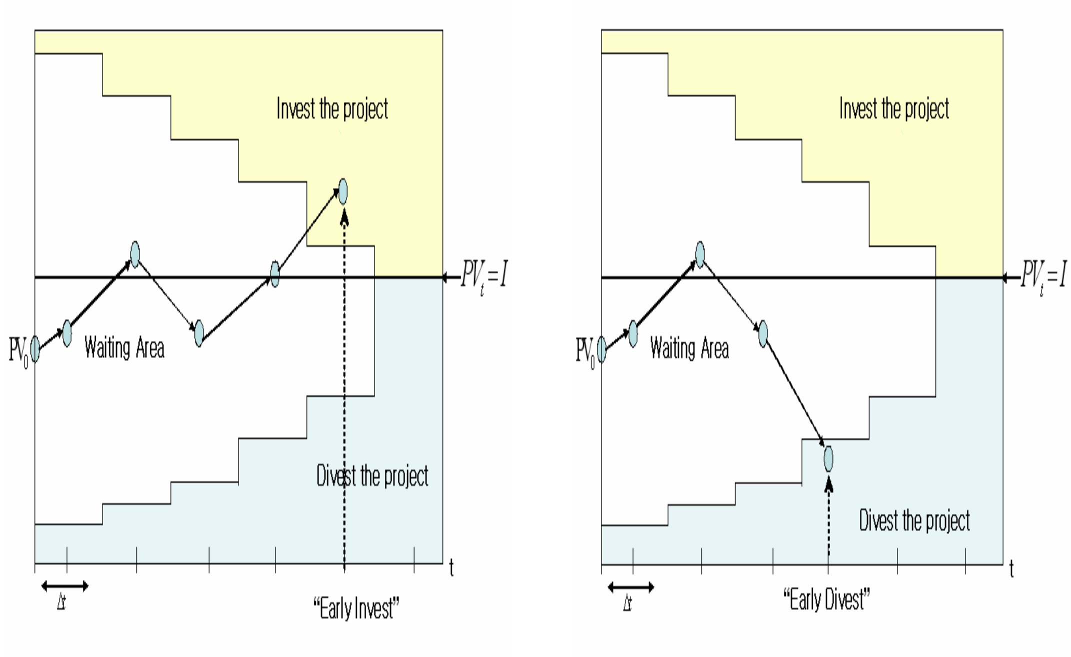


Figure 5-5. Illustrations for early decisions: Invest (Left), Divest (Right)

Step 3: If the decision is "Early invest" or "Early divest," the future value of the payoff at the exercise timing will be recorded.

Step 4: The two final decisions, "Invest" or "Divest," will be determined at the terminal node.

Step 5: Compute the expected future value in case of no early exercise rule and the new decision rule.

Step 6: Conduct the paired-t test to check the statistical significance of the results.

The simulation process originates from Table 5-2, which shows the early decision points of the defer option. The simulation has been conducted for the different volatilities with an initial project value of \$10, an investment cost of \$12, a risk-adjusted interest rate of 15%, a risk-free interest rate of 5%, and an option life of 7 years with $\Delta t = 1\ year$. With @Risk the project value of each year is calculated by conducting 1,500 iterations for generating the annual return. Then the effects of the two decision rules on the expected value and the standard deviation of the project value are examined. To check the consistency of the decision making processes, 16 different volatility cases are compared.

According to the simulation results shown in Table 5-4, the early decision rule developed in this study gives a higher expected future value than the traditional rule in most of the cases with 95% confidence intervals. However, if the volatility continues to increase, the choice of rule makes little statistical difference. Table 5-4 shows the profit at the end of the 7th year as calculated by two simulations, indicating that the proposed decision is quite effective for a wide range of volatility examined.

Table 5-4. Simulation results for a defer option

Volatility	Traditional rule (A)		New rule (B)		Differences (B-A)		Z value (1.96)	Result
	mean	std	mean	std	mean	std		
20%	**1.48**	**3.56**	**1.61**	**3.86**	**0.14**	**1.59**	**3.41**	**New**
25%	**2.09**	**4.96**	**2.46**	**5.48**	**0.37**	**3**	**4.78**	**New**
30%	**2.71**	**7.02**	**3.35**	**7.34**	**0.64**	**4.92**	**5.04**	**New**
35%	**3.35**	**8.79**	**4.05**	**8.86**	**0.7**	**6.16**	**4.40**	**New**
40%	**3.69**	**10.38**	**4.69**	**10.56**	**1.00**	**7.96**	**4.87**	**New**
45%	**4.63**	**14.93**	**5.43**	**12.38**	**0.79**	**11.03**	**2.77**	**New**
50%	**5.85**	**19.15**	**6.88**	**15.54**	**1.03**	**14.82**	**2.69**	**New**
55%	**5.53**	**18.33**	**6.96**	**15.77**	**1.42**	**15.53**	**3.54**	**New**
60%	**6.25**	**22.64**	**8.40**	**19.91**	**2.16**	**18.87**	**4.43**	**New**
65%	**7.62**	**29.17**	**9.44**	**21.72**	**1.82**	**24.39**	**2.89**	**New**
70%	**8.7**	**33.95**	**10.7**	**24.53**	**1.99**	**29.24**	**2.64**	**New**
75%	**8.52**	**37.79**	**10.95**	**26.41**	**2.43**	**30.06**	**3.13**	**New**
80%	10.29	44.12	11.95	29.40	1.65	37.07	1.72	Indifference
85%	11.51	61.22	13.28	32.89	1.77	55.7	1.23	Indifference
90%	11.75	81.715	14.1	34.91	2.35	73.59	1.24	Indifference
95%	14.07	69.74	15.98	38.3	1.91	62.52	1.18	Indifference

5.3.2 Abandon Option Framework

For the abandon option, only two actions, “early abandonment” or “waiting,” are available for strategic decisions at any point in the option life. Since the American options framework permits early exercise when the project value is profitable, the payoff by traditional American option pricing has the same perspectives as the new decision rule.

The procedures for the abandonment simulation are the same as for the defer option simulation, with the exception of the decision criteria. The revised processes are listed below:

Step 1: Simulate a project value path for the pre-assigned iterations. The first step is the same as in the defer option simulation.

Step 2: Apply decision rule. If the decision indicates “Early action,” record the profit and cancel the remaining schedule. If the results indicate “Wait,” then go to the next time period and reevaluate the decision rule. Continue the process until it reaches the terminal node.

Step 3: The final decision, “Abandon” or “Do not abandon,” will be determined at the terminal node. Compute the expected future value under the traditional early action rule and the new decision rule.

Step 4: Conduct the paired-t test to check the statistical significance of the results.

Table 5-5 is the result of the simulation which has an initial project value of $10, an investment cost of $7, a risk-adjusted interest rate of 15%, a risk-free interest rate of 5%, and an option life of 7 years with $\Delta t = 1\ year$. The simulation results of abandon option demonstrate the opposite results with the defer options cases. The expected future values under the new decision rule and the traditional rule are examined, and the results also demonstrate that the new rule is equally effective when compared with the traditional rule.

Table 5-5. Simulation results for an abandon option

Volatility	Traditional rule (A)		New rule (B)		Differences (B-A)		Z value (1.96)	Decision
	mean	std	mean	std	mean	Std		
20%	1.00	1.57	0.98	1.51	-0.02	0.40	-1.72	Indifference
25%	1.60	2.21	1.58	2.13	-0.03	0.58	-1.75	Indifference
30%	2.34	2.92	2.35	2.83	0.01	0.76	0.48	Indifference
35%	3.00	3.51	3.00	3.40	0.00	0.88	0.08	Indifference
40%	3.77	4.07	3.79	3.96	0.02	0.96	0.82	Indifference
45%	4.52	4.49	4.57	4.38	0.05	1.03	1.92	Indifference
50%	**5.24**	**4.86**	**5.29**	**4.74**	**0.07**	**1.18**	**2.18**	**New**
55%	**5.88**	**5.19**	**5.93**	**5.07**	**0.07**	**1.21**	**2.24**	**New**
60%	**6.61**	**5.30**	**6.70**	**5.15**	**0.09**	**1.30**	**2.61**	**New**
65%	**7.11**	**5.44**	**7.23**	**5.34**	**0.11**	**1.21**	**3.62**	**New**
70%	**7.58**	**5.55**	**7.66**	**5.44**	**0.08**	**1.15**	**2.66**	**New**
75%	**7.58**	**5.54**	**8.21**	**5.41**	**0.08**	**1.31**	**2.35**	**New**
80%	**8.35**	**5.58**	**8.50**	**5.48**	**0.15**	**1.14**	**4.94**	**New**
85%	**8.80**	**5.56**	**8.94**	**5.44**	**0.14**	**1.14**	**4.73**	**New**
90%	**9.29**	**5.43**	**9.43**	**5.33**	**0.14**	**1.15**	**4.66**	**New**
95%	**9.47**	**5.53**	**9.57**	**5.45**	**0.10**	**1.05**	**3.79**	**New**

5.4 Comprehensive Illustrating Example

In order to demonstrate the methods of determining the decision points, two real options decision examples are used. A defer option scenario is shows the whole process of selecting the decision timing of the real option. A decision map which shows the decision points of the real options is also demonstrated for the defer options scenario, and a method of revising the decision map is included. It satisfies the initial belief that early investment and forfeiting decisions are closely correlated to the amount of the project volatility. The decision rule for an abandon option is demonstrated in a similar scenario.

5.4.1 Case 1: A Defer Option

<u>Defer option scenario</u>

Project value (V_{00}) = \$10, Investment cost = \$12

Risk-adjusted interest rate (R)= 15%, Risk-free interest rate (r) = 5%

Volatility of the project (σ) = 30%, $T = 7\ years$, $\Delta t = 1\ year$

Derived information: $u = 1.3499$, $d = 0.7408$, $p = 0.5097$, $1 - p = 0.4903$

<u>Processes to develop the decision map</u>

Step 1: Compute the monetary value (V_{nm})

m \ n	0	1	2	3	4	5	6	7
0	10.00	13.50	18.22	24.60	33.20	44.82	60.50	81.66
1		7.41	10.00	13.50	18.22	24.60	33.20	44.82
2			5.49	7.41	10.00	13.50	18.22	24.60
3				4.07	5.49	7.41	10.00	13.50
4					3.01	4.07	5.49	7.41
5						2.23	3.01	4.07
6							1.65	2.23
7								1.22

Step 2: Compute the option value (OV_{nm})

m \ n	0	1	2	3	4	5	6	7
0	3.67	5.98	9.58	14.99	22.87	33.96	49.08	69.66
1		1.64	2.87	4.95	8.36	13.74	21.79	32.82
2			0.53	1.01	1.93	3.64	6.81	12.60
3				0.08	0.17	0.35	0.73	1.50
4					0.00	0.00	0.00	0.00
5						0.00	0.00	0.00
6							0.00	0.00
7								0.00

Step 3: Define the profit of immediate exercise (PV_{nm})

m \ n	0	1	2	3	4	5	6	7
0	0.00	1.50	6.22	12.60	21.20	32.82	48.50	69.66
1		0.00	0.00	1.50	6.22	12.60	21.20	32.82
2			0.00	0.00	0.00	1.50	6.22	12.60
3				0.00	0.00	0.00	0.00	1.50
4					0.00	0.00	0.00	0.00
5						0.00	0.00	0.00
6							0.00	0.00
7								0.00

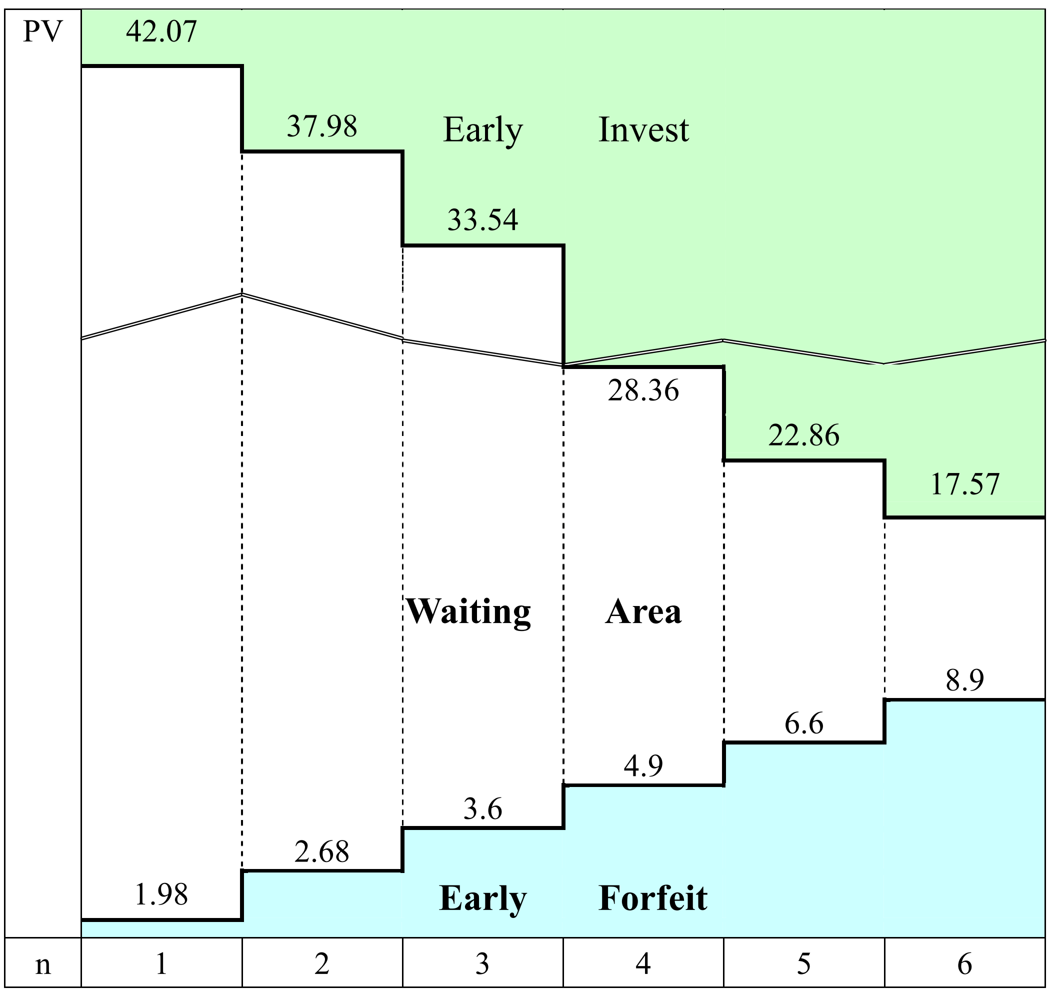
Step 4: Compute the future expected profit ($PV_{nm} \cdot e^{R \cdot \Delta t}$)

m \ n	0	1	2	3	4	5	6	7
0	0.00	1.74	7.23	14.63	24.63	38.13	56.34	
1		0.00	0.00	1.74	7.23	14.63	24.63	
2			0.00	0.00	0.00	1.74	7.23	
3				0.00	0.00	0.00	0.00	
4					0.00	0.00	0.00	
5						0.00	0.00	
6							0.00	
7								

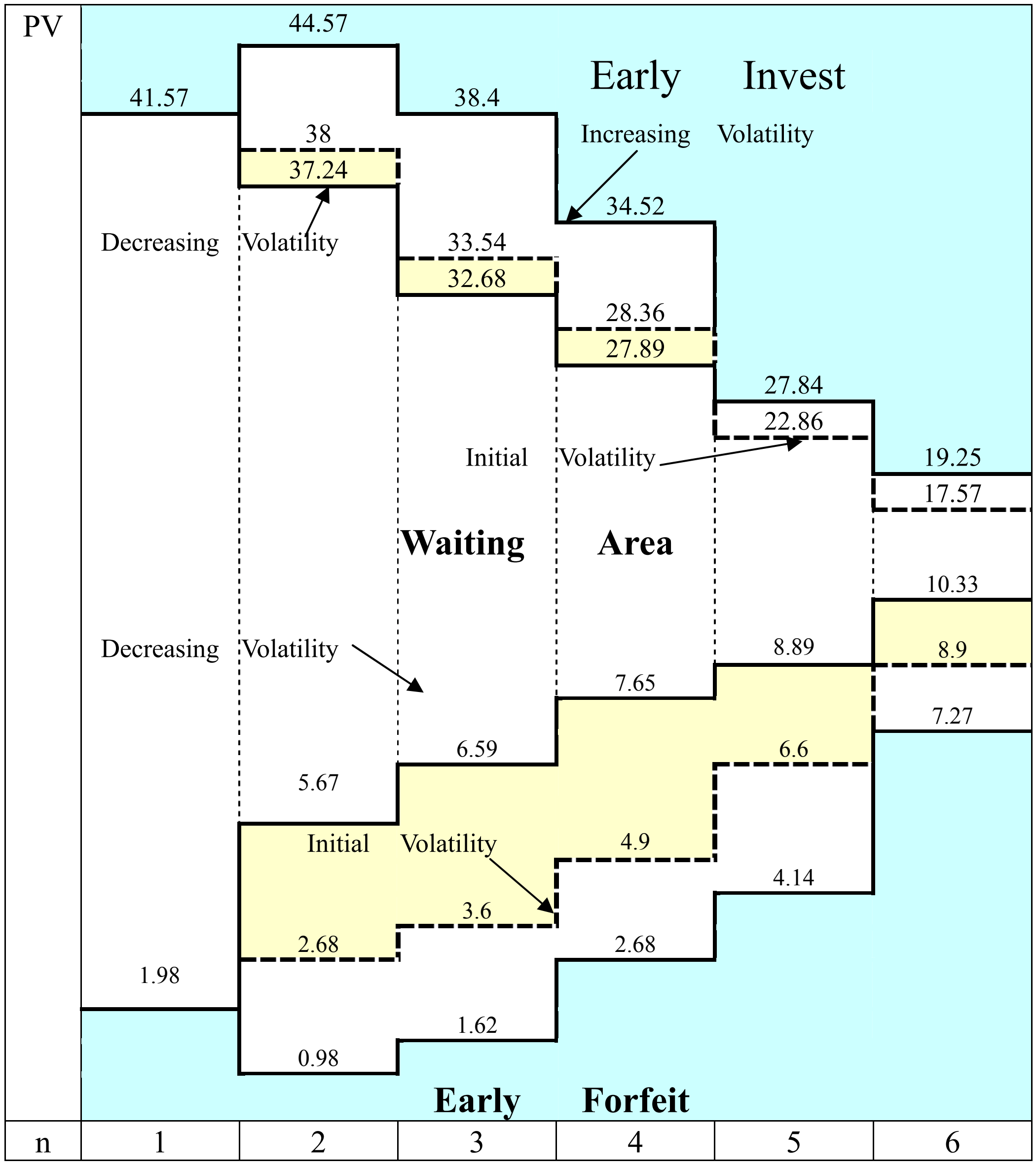
Step 5: Calculate the future value of waiting ($OV_{(n+1)m} \cdot p + OV_{(n+1)(m+1)} \cdot (1-p)$)

m \ n	0	1	2	3	4	5	6	7
0	3.86	6.29	10.07	15.76	24.05	35.70	51.60	
1		1.73	3.02	5.21	8.79	14.44	22.90	
2			0.56	1.07	2.03	3.83	7.16	
3				0.09	0.18	0.37	0.76	
4					0.00	0.00	0.00	
5						0.00	0.00	
6							0.00	
7								

Step 6: Developing a decision map



Step 7: Revised decision map: In case of decreasing volatility to 15% and increasing volatility to 50% at the end of year 2.



Step 8: Develop early decision points of the option by considering the early exercise point and the divest point in specific volatility range: In this example, a volatility range from 20% to 95% is determined and shown in Table 5-6. Note that the value of each column is identical with the value of multiplying Table 5-2 by the investment cost of $12.

Table 5-6. Decision points of the defer option example

Volatility	Decisions	1	2	3	4	5	6
20%	Exercise	41.75	37.24	32.68	27.89	22.86	17.57
	Quit	3.61	4.41	5.39	6.59	8.04	9.82
25%	Exercise	41.75	37.43	32.68	27.89	22.86	17.57
	Quit	2.68	3.44	4.41	5.67	7.28	9.35
30%	Exercise	42.07	37.98	33.54	28.36	22.86	17.57
	Quit	1.98	2.68	3.61	4.88	6.59	8.89
35%	Exercise	43.41	38.65	34.85	29.69	23.55	17.57
	Quit	1.47	2.09	2.96	4.2	5.96	8.46
40%	Exercise	45.36	39.55	35.76	31.18	24.88	17.83
	Quit	1.09	1.6	2.42	3.61	5.39	8.04
45%	Exercise	47.52	41.97	37.04	32.8	26.31	18.52
	Quit	0.81	1.26	1.98	3.11	4.88	7.65
50%	Exercise	49.82	44.57	38.4	34.52	27.84	19.25
	Quit	0.6	0.99	1.62	2.68	4.41	7.28
55%	Exercise	52.24	47.33	39.83	36.33	29.44	20.01
	Quit	0.44	0.77	1.33	2.3	3.99	6.92
60%	Exercise	54.75	50.22	41.31	38.21	31.13	20.79
	Quit	0.33	0.6	1.09	1.98	3.61	6.59
65%	Exercise	57.32	53.21	43.33	40.17	32.9	21.61
	Quit	0.24	0.47	0.89	1.71	3.27	6.26
70%	Exercise	59.94	56.28	46.11	42.18	34.74	22.45
	Quit	0.18	0.36	0.73	1.47	2.96	5.96
75%	Exercise	62.57	59.4	48.99	44.23	36.66	23.32
	Quit	0.13	0.28	0.6	1.26	2.68	5.67
80%	Exercise	65.21	62.55	51.97	46.33	38.64	24.23
	Quit	0.1	0.22	0.49	1.09	2.42	5.39
85%	Exercise	67.84	65.71	55.03	48.46	40.69	25.16
	Quit	0.07	0.17	0.4	0.94	2.19	5.13
90%	Exercise	71.02	68.86	58.16	50.61	42.8	26.12
	Quit	0.05	0.13	0.33	0.81	1.98	4.88
95%	Exercise	74.8	71.97	61.34	52.77	44.96	27.12
	Quit	0.04	0.1	0.27	0.69	1.79	4.64

5.4.2 Case 2: An Abandon Option

An option to abandon for salvage value gives a firm with an on-going project the right to sell its facility at any time during the option period, thus downsizing to lessen its risk. In the abandon options framework, only two actions are available during the option life: early abandonment and waiting until the end of the period. The processes for valuing the abandon option are similar to the financial put option model. Therefore an early exercise decision in the financial put options model is considered an early abandonment point for the underlying on-going project. In contrast to the defer options model, two courses of action have to be investigated because there is no early investment alternative. The valuation and decision concerning the abandon option is based on the binomial lattice model, as was the defer options scenario. The detailed processes are also the same as in the defer options example.

Example data

Project value (V_{00}) = \$10, Sell out price = \$7

Risk-adjusted interest rate (R) = 15%, Risk-free interest rate(r)= 5%

Volatility of the project (σ) = 30%, $T = 7$ *years*, $\Delta t = 1$ *year*

Derived option parameters: $u = 1.3499$, $d = 0.7408$, $p = 0.5097$, $N = \frac{T}{\Delta t} = 7$

Two alternatives are considered for investment decisions in abandon options. The first alternative is selling the facility to realize \$7 when the project value is decreasing to \$7, but to guarantee the minimum payoff. The other alternative is holding the project until the end of the option if the market is favorable.

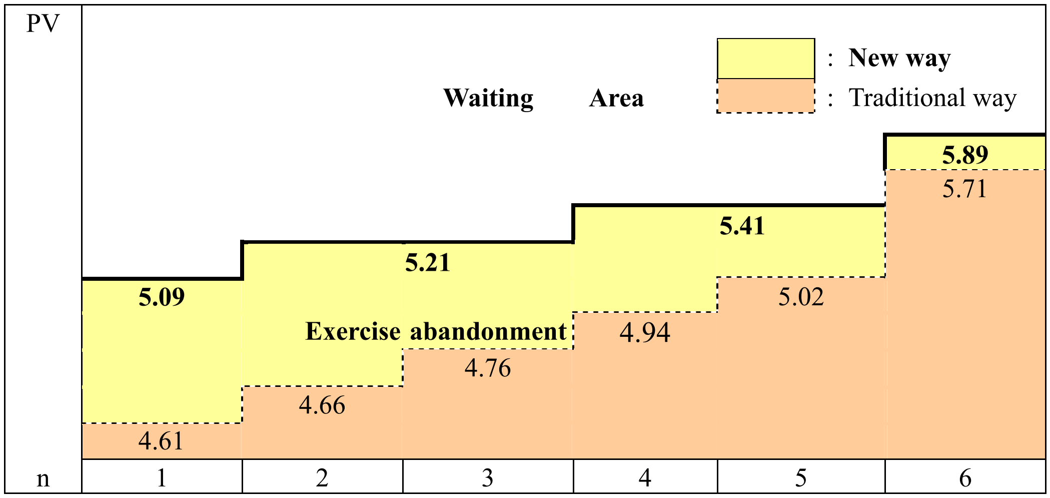


Figure 5-6. Decision map for an abandon option

Figure 5-6 is the decision map for this specific example, and it illustrates that the new early decision points of the option are slightly higher than that of the traditional American put option valuation model.

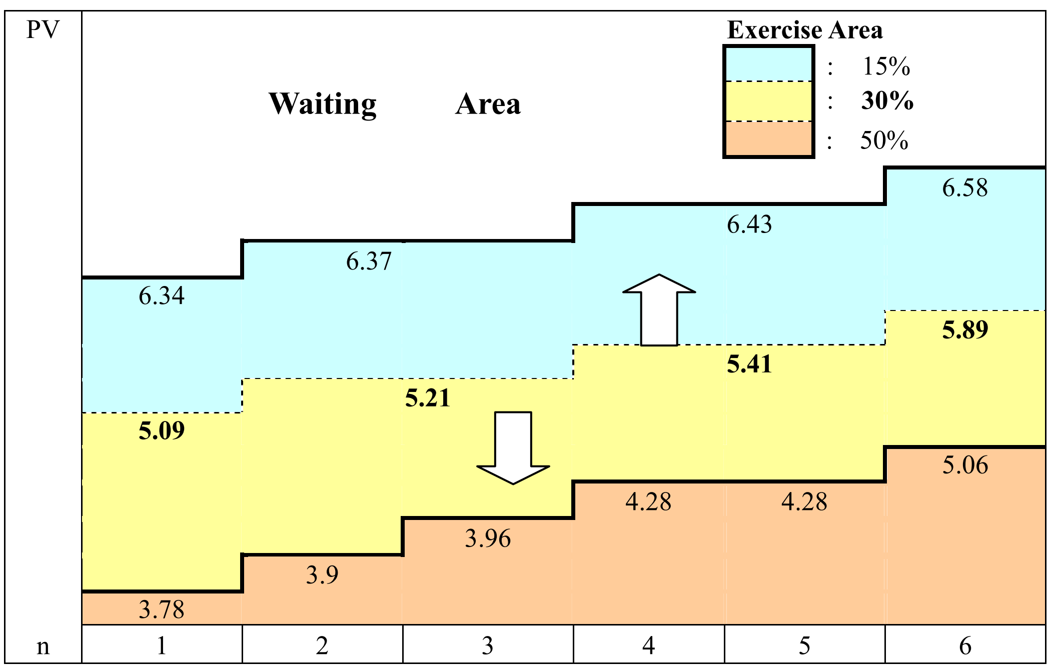


Figure 5-7. Decision map for abandon option in case of volatility change

Table 5-7. Decision points of the abandon option example

Volatility	Decisions	1	2	3	4	5	6
20%	Traditional	5.66	5.73	5.73	5.87	5.87	6.24
	New	5.91	5.97	5.97	6.08	6.08	6.34
25%	Traditional	5.12	5.18	5.23	5.43	5.44	5.97
	New	5.49	5.58	5.58	5.74	5.74	6.11
30%	Traditional	4.61	4.66	4.76	4.94	5.02	5.71
	New	5.09	5.21	5.21	5.41	5.41	5.89
35%	Traditional	4.15	4.19	4.32	4.48	4.64	5.45
	New	4.72	4.84	4.86	5.1	5.1	5.67
40%	Traditional	3.73	3.76	3.92	4.07	4.28	5.21
	New	4.37	4.5	4.53	4.81	4.81	5.46
45%	Traditional	3.35	3.38	3.56	3.69	3.95	4.98
	New	4.06	4.19	4.23	4.54	4.54	5.26
50%	Traditional	3.02	3.04	3.23	3.35	3.65	4.76
	New	3.78	3.9	3.96	4.28	4.28	4.06
55%	Traditional	2.73	2.74	2.94	3.05	3.37	4.54
	New	3.52	3.64	3.71	4.05	4.05	4.87
60%	Traditional	2.46	2.47	2.68	2.77	3.11	4.34
	New	3.28	3.41	3.47	3.82	3.83	4.70
65%	Traditional	2.23	2.24	2.45	2.53	2.88	4.15
	New	3.07	3.19	3.26	3.61	3.62	4.52
70%	Traditional	2.03	2.04	2.24	2.32	2.66	3.96
	New	2.88	3	3.07	3.41	3.43	4.36
75%	Traditional	1.85	1.86	2.05	2.12	2.47	3.79
	New	2.71	2.83	2.89	3.24	3.25	4.20
80%	Traditional	1.7	1.7	1.88	1.95	2.29	3.62
	New	2.55	2.67	2.73	3.07	3.09	4.05
85%	Traditional	1.56	1.56	1.73	1.8	2.13	3.46
	New	2.41	2.53	2.59	2.92	2.94	3.91
90%	Traditional	1.43	1.44	1.6	1.67	1.98	3.31
	New	2.29	2.4	2.45	2.78	2.8	3.77
95%	Traditional	1.33	1.33	1.48	1.54	1.84	3.17
	New	2.17	2.28	2.33	2.66	2.67	3.64

In order to examine the effect of the volatility revision in the abandon options framework, the early abandonment points are investigated in cases of different volatility. In contrast to the defer options framework, the early action points decrease when the volatility increases, and increase when the volatility decreases. In order to see the specific movement of the map, two volatility revising scenarios are used: volatility decreasing to 15% and increasing to 50% (see Figure 5-7). Finally the early decision points for the abandon option with the volatility ranges from 20% to 95% are demonstrated in Table 5-7.

5.5 Conclusion Remarks

In this paper, a new decision rule for real options is proposed by maximizing the expected future project value. The expected future value when the option is exercised now and the expected future value of the option are compared to determine when to exercise. A decision map for waiting, early exercise, and early divesting during the entire option life is also developed to automate the decisions when some variables of the options changes over the option life. The results correspond to the early belief that "increasing volatility enlarges the waiting area, and decreasing volatility shrinks the waiting area."

In order to validate the advantages of the proposed decision rule, a simulation technique is used to determine the benefits of adopting the new early exercise rule during the option life. The expected profits under new decision rule and the traditional rule are compared in order to demonstrate the advantages of the new approach. The results of the numerical examples indicate that the new decision rule is quite effective to maximize the expected profit in a wide range of scenarios examined.

Appendix 5.1 Processes to develop equation 5-1

- **At** $n = N-1$

$$PV_{(N-1)m} \cdot e^{k \cdot \Delta t} \geq [OV_{Nm} \cdot p + OV_{N(m+1)} \cdot (1-p)]$$

$$PV_{(N-1)m} \cdot e^{k} \geq [PV_{Nm} \cdot p + PV_{N(m+1)} \cdot (1-p)]$$

$$(\because) \quad OV_{Nm} = PV_{Nm} = \max(V_{Nm} - I, 0)$$

$$(V_{(N-1)m} - I) \cdot e^{k} \geq (V_{(N-1)m} \cdot u - I) \cdot p + (V_{(N-1)m} \cdot d - I) \cdot (1-p)$$

$$(\because) \quad \begin{aligned} PV_{(N-1)m} &= \max(V_{(N-1)m} - I, 0) \\ PV_{Nm} &= \max(V_{(N-1)m} \cdot u - I, 0) \\ PV_{N(m+1)} &= \max(V_{(N-1)m} \cdot d - I, 0) \end{aligned}$$

$$V_{(N-1)m} \cdot \{e^{k} - up - d(1-p)\} \geq -pI - (1-p)I + Ie^{k}$$

$$V_{(N-1)m} \cdot \{e^{k} - d - (u-d)p\} \geq I(e^{k} - 1)$$

$$V_{(N-1)m} \cdot \{e^{k} - d - (u-d) \cdot \frac{e^{r} - d}{u-d}\} \geq I(e^{k} - 1)$$

$$V_{(N-1)m}(e^{k} - e^{r}) \geq I(e^{k} - 1)$$

$$V_{(N-1)m} \geq \frac{I(e^{k} - 1)}{(e^{k} - e^{r})}$$

- **At** $n = N-2$

$$PV_{(N-2)m} \cdot e^{k} \geq [OV_{(N-1)m} \cdot p + OV_{(N-1)(m+1)} \cdot (1-p)]$$

$$PV_{(N-2)m} \cdot e^{k} \geq (V_{(N-1)m} - Ie^{-r}) \cdot p + (V_{(N-1)(m+1)} - Ie^{-r}) \cdot (1-p)$$

$(\because)$

$$
\begin{aligned}
OV_{(N-1)m} &= [OV_{Nm} \cdot p + OV_{N(m+1)} \cdot (1-p)] \cdot e^{-r} \\
&= [PV_{Nm} \cdot p + PV_{N(m+1)} \cdot (1-p)] \cdot e^{-r} \\
&= [(V_{Nm} - I) \cdot p + (V_{N(m+1)} - I) \cdot (1-p)] \cdot e^{-r} \\
&= [(V_{(N-1)m} \cdot u - I) \cdot p + (V_{(N-1)m} \cdot d - I) \cdot (1-p)] \cdot e^{-r} \\
&= [V_{(N-1)m} \cdot up - Ip + V_{(N-1)m} \cdot d - I - V_{(N-1)m} \cdot dp + Ip] \cdot e^{-r} \\
&= [V_{(N-1)m}(u-d)p + V_{(N-1)m} \cdot d - I] \cdot e^{-r} \\
&= [V_{(N-1)m}(u-d)\frac{(e^{r}-d)}{(u-d)} + V_{(N-1)m} \cdot d - I] \cdot e^{-r} \\
&= [V_{(N-1)m} \cdot e^{r} - I] \cdot e^{-r} \\
&= V_{(N-1)m} - Ie^{-r}
\end{aligned}
$$

$$
\begin{aligned}
OV_{(N-1)(m+1)} &= [OV_{N(m+1)} \cdot p + OV_{N(m+2)} \cdot (1-p)] \cdot e^{-r} \\
&= [PV_{N(m+1)} \cdot p + PV_{N(m+2)} \cdot (1-p)] \cdot e^{-r} \\
&= [(V_{N(m+1)} - I) \cdot p + (V_{N(m+2)} - I) \cdot (1-p)] \cdot e^{-r} \\
&= [(V_{(N-1)(m+1)} \cdot u - I) \cdot p + (V_{(N-1)(m+1)} \cdot d - I) \cdot (1-p)] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)} \cdot up - Ip + V_{(N-1)(m+1)} \cdot d - I - V_{(N-1)(m+1)} \cdot dp + Ip] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)}(u-d)p + V_{(N-1)(m+1)} \cdot d - I] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)}(u-d)\frac{(e^{r}-d)}{(u-d)} + V_{(N-1)(m+1)} \cdot d - I] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)} \cdot e^{r} - I] \cdot e^{-r} \\
&= V_{(N-1)(m+1)} - Ie^{-r}
\end{aligned}
$$

$$
\begin{aligned}
(V_{(N-2)m} - I) \cdot e^{k} &\geq V_{(N-1)m} \cdot p + V_{(N-1)(m+1)} \cdot (1-p) - Ie^{-r} \\
&\geq V_{(N-2)m}up + V_{(N-2)m}d(1-p) - Ie^{-r} \\
&\geq V_{(N-2)m}(u-d)p + V_{(N-2)m}d - Ie^{-r} \\
&\geq V_{(N-2)m} \cdot e^{r} - Ie^{-r}
\end{aligned}
$$

$$V_{(N-2)m}(e^{k} - e^{r}) \geq Ie^{k} - Ie^{-r}$$

$$\therefore \quad V_{(N-2)m} \geq \frac{I(e^{k} - e^{-r})}{e^{k} - e^{r}}$$

- **At** $n = N-3$

$$PV_{(N-3)m} \cdot e^{k} \geq [OV_{(N-2)m} \cdot p + OV_{(N-2)(m+1)} \cdot (1-p)]$$

$$PV_{(N-3)m} \cdot e^{k} \geq (V_{(N-2)m} - Ie^{-2r}) \cdot p + (V_{(N-2)(m+1)} - Ie^{-2r}) \cdot (1-p)$$

$(\because)$

$$OV_{(N-2)m} = [OV_{(N-1)m} \cdot p + OV_{(N-1)(m+1)} \cdot (1-p)] \cdot e^{-r}$$

$$\begin{aligned} OV_{(N-2)m} &= [(V_{(N-1)m} - Ie^{-r}) \cdot p + (V_{(N-1)(m+1)} - Ie^{-r})(1-p)] \cdot e^{-r} \\ &= [(V_{(N-2)m}u - Ie^{-r}) \cdot p + (V_{(N-2)m}d - Ie^{-r})(1-p)] \cdot e^{-r} \\ &= [(V_{(N-2)m}(u-d)p + V_{(N-2)m}d - Ie^{-r}] \cdot e^{-r} \\ &= [(V_{(N-2)m}(u-d)\frac{(e^{r}-d)}{(u-d)} + V_{(N-2)m}d - Ie^{-r}] \cdot e^{-r} \\ &= [(V_{(N-2)m}e^{r} - Ie^{-r}] \cdot e^{-r} \\ &= V_{(N-2)m} - Ie^{-2r} \end{aligned}$$

$(\because)$

$$\begin{aligned} OV_{(N-1)m} &= [OV_{Nm} \cdot p + OV_{N(m+1)} \cdot (1-p)] \cdot e^{-r} \\ &= V_{(N-1)m} - Ie^{-r} \end{aligned}$$

$$\begin{aligned} OV_{(N-1)(m+1)} &= [OV_{N(m+1)} \cdot p + OV_{N(m+2)} \cdot (1-p)] \cdot e^{-r} \\ &= V_{(N-1)(m+1)} - Ie^{-r} \end{aligned}$$

$$OV_{(N-2)(m+1)} = [OV_{(N-1)(m+1)} \cdot p + OV_{(N-1)(m+2)} \cdot (1-p)] \cdot e^{-r}$$

$$\begin{aligned} OV_{(N-2)(m+1)} &= [(V_{(N-1)(m+1)} - Ie^{-r}) \cdot p + (V_{(N-1)(m+2)} - Ie^{-r})(1-p)] \cdot e^{-r} \\ &= V_{(N-2)(m+1)} - Ie^{-2r} \end{aligned}$$

$(\because)$
$$\begin{aligned} OV_{(N-1)(m+1)} &= [OV_{N(m+1)} \cdot p + OV_{N(m+2)} \cdot (1-p)] \cdot e^{-r} \\ &= V_{(N-1)(m+1)} - Ie^{-r} \end{aligned}$$

$$\begin{aligned} OV_{(N-1)(m+2)} &= [OV_{N(m+2)} \cdot p + OV_{N(m+3)} \cdot (1-p)] \cdot e^{-r} \\ &= V_{(N-1)(m+2)} - Ie^{-r} \end{aligned}$$

$$PV_{(N-3)m} \cdot e^{k} \geq (V_{(N-3)m}u - Ie^{-2r}) \cdot p + (V_{(N-3)m}d - Ie^{-2r})(1-p)$$

$$\begin{aligned}(V_{(N-3)m} - I) \cdot e^{k} &\geq (V_{(N-3)m}up + V_{(N-3)m}d(1-p) - Ie^{-2r} \\ &\geq V_{(N-3)m}e^{r} - Ie^{-2r}\end{aligned}$$

$$\therefore \quad V_{(N-3)m} \geq \frac{I(e^{k} - e^{-2r})}{e^{k} - e^{r}}$$

Appendix 5.2 Processes to decide the early exercise points

- **At** $n = N-1$

Step 1: Define the exercising point through the general approach and check the feasibility.

$$\begin{cases} V^*_{(N-1)m} \geq \frac{I}{d}, & \textit{Stop here, the number you get is the point} \\ V^*_{(N-1)m} < \frac{I}{d}, & \textit{Go to step two} \end{cases}$$

Step 2: Compute an alternative point by using the formula below.

$$(V_{(N-1)m} - I) \cdot e^k \geq (OV_{Nm} - I) \cdot p + (OV_{N(m+1)} - I) \cdot (1-p)$$

$$(V_{(N-1)m} - I) \cdot e^k \geq (V_{(N-1)m} \cdot u - I) \cdot p$$

$$(\because) \quad PV_{N(m+1)} = \max(V_{(N-1)m} \cdot d - I, 0) = 0$$

$$V_{(N-1)m} \cdot (e^k - up) \geq Ie^k - pI$$

$$V^{**}_{(N-1)m} \geq \frac{I(e^k - p)}{(e^k - u \cdot p)} \quad \text{where,} \quad I \leq V^*_{(N-1)m} < \frac{I}{d} \tag{5-2}$$

- **At** $n = N-2$

Step 1: Define the exercising point through the general approach and check the feasibility.

$$\begin{cases} V^*_{(N-2)m} \geq \frac{I}{d^2}, & \textit{Stop here, the number you get is the point} \\ V^*_{(N-2)m} < \frac{I}{d^2}, & \textit{Go to step two} \end{cases}$$

Step 2: Compute an alternative point by using the formula below.

$$PV_{(N-2)m} \cdot e^k \geq [OV_{(N-1)m} \cdot p + OV_{(N-1)(m+1)} \cdot (1-p)]$$

$$PV_{(N-2)m} \cdot e^k \geq (V_{(N-1)m} - Ie^{-r}) \cdot p + (V_{(N-1)(m+1)} - Ie^{-r}) \cdot (1-p)$$

$$(\because) \quad \begin{aligned} OV_{(N-1)m} &= [OV_{Nm} \cdot p + OV_{N(m+1)} \cdot (1-p)] \cdot e^{-r} \\ &= V_{(N-1)m} - Ie^{-r} \end{aligned}$$

$$OV_{(N-1)(m+1)} = [OV_{N(m+1)} \cdot p + OV_{N(m+2)} \cdot (1-p)] \cdot e^{-r}$$
$$= [(V_{N(m+1)} - I) \cdot p] \cdot e^{-r}$$
$$= [V_{(N-1)m} \cdot dp - Ip] \cdot e^{-r}$$

$$(V_{(N-2)m} - I) \cdot e^{k} \geq (V_{(N-1)m} - Ie^{-r}) \cdot p + (V_{(N-1)m} \cdot d - I)pe^{-r}(1-p)$$
$$\geq [V_{(N-1)m}(p + dpe^{-r}(1-p)] - Ipe^{-r} - Ipe^{-r}(1-p)$$
$$\geq V_{(N-2)m}u[p + dpe^{-r}(1-p)] - Ipe^{-r}(2-p)$$
$$\geq V_{(N-2)m}p[u + e^{-r}(1-p)] - Ipe^{-r}(2-p)$$

$$\therefore \quad V^{**}{}_{(N-2)m} \geq \frac{I[e^{k} - pe^{-r}(2-p)]}{[e^{k} - p(u + e^{-r}(1-p))]} \quad \text{where, } \frac{I}{d} \leq V^{*}{}_{(N-2)m} < \frac{I}{d^2} \qquad (5\text{-}3)$$

- **At** $n = N-3$

Step 1: Define the exercising point through the general approach and check the feasibility.

$$\begin{cases} V^{*}{}_{(N-3)m} \geq \dfrac{I}{d^3}, & \textit{Stop here, the number you get is the point} \\ \dfrac{I}{d} \leq V^{*}{}_{(N-3)m} < \dfrac{I}{d^3}, & \textit{Go to step two} \\ V^{*}{}_{(N-3)m} < \dfrac{I}{d}, & \textit{Go to step three} \end{cases}$$

Step 2: Compute the first alternative point by using the formula below.

$$PV_{(N-3)m} \cdot e^{k} \geq [OV_{(N-2)m} \cdot p + OV_{(N-2)(m+1)} \cdot (1-p)]$$

$$PV_{(N-3)m} \cdot e^{k} \geq (V_{(N-2)m} - Ie^{-2r}) \cdot p$$
$$+ [V_{(N-2)(m+1)} \cdot p(u + e^{-r}(1-p)) - Ipe^{-r}(2-p)]e^{-r}(1-p)$$

$$(\because)\, OV_{(N-2)m} = V_{(N-2)m} - Ie^{-2r}$$

$$
\begin{aligned}
OV_{(N-2)(m+1)} &= [OV_{(N-1)(m+1)} \cdot p + OV_{(N-1)(m+2)} \cdot (1-p)] \cdot e^{-r} \\
&= [(V_{(N-1)(m+1)} - Ie^{-r}) \cdot p + (V_{(N-1)(m+1)} d - I) \cdot pe^{-r}(1-p)] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)} \cdot p - Ipe^{-r} + (V_{(N-1)(m+1)} d - I) \cdot pe^{-r}(1-p)] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)} \cdot p - Ipe^{-r} \\
&\quad + V_{(N-1)(m+1)} dpe^{-r} - Ipe^{-r} - V_{(N-1)(m+1)} dp^2 e^{-r} + Ip^2 e^{-r})] \cdot e^{-r} \\
&= [V_{(N-1)(m+1)} \cdot p(1 + de^{-r} - dpe^{-r}) - 2Ipe^{-r} + Ip^2 e^{-r}] e^{-r} \\
&= [V_{(N-1)(m+1)} \cdot p(1 + de^{-r} - dpe^{-r}) - Ipe^{-r}(2-p)] e^{-r} \\
&= [V_{(N-2)(m+1)} \cdot up(1 + de^{-r} - dpe^{-r}) - Ipe^{-r}(2-p)] e^{-r} \\
&= [V_{(N-2)(m+1)} \cdot p(u + e^{-r} - pe^{-r}) - Ipe^{-r}(2-p)] e^{-r} \\
&= [V_{(N-2)(m+1)} \cdot p(u + e^{-r}(1-p)) - Ipe^{-r}(2-p)] e^{-r}
\end{aligned}
$$

$$
(\because)\, OV_{(N-1)(m+1)} = V_{(N-1)(m+1)} - Ie^{-r}
$$

$$
\begin{aligned}
OV_{(N-1)(m+2)} &= [(V_{N(m+2)} - I) \cdot p] \cdot e^{-r} \\
&= (V_{(N-1)(m+1)} d - I) \cdot pe^{-r}
\end{aligned}
$$

$$
\begin{aligned}
PV_{(N-3)m} \cdot e^{k} &\geq (V_{(N-2)m} - Ie^{-2r}) \cdot p \\
&\quad + [V_{(N-2)(m+1)} \cdot p(u + e^{-r}(1-p)) - Ipe^{-r}(2-p)] e^{-r}(1-p) \\
&\geq V_{(N-2)m} p - Ipe^{-2r} \\
&\quad + V_{(N-2)(m+1)} \cdot p(u + e^{-r}(1-p)) e^{-r}(1-p) - Ipe^{-2r}(2-p)(1-p) \\
&\geq V_{(N-3)m} up + V_{(N-3)m} dp(u + e^{-r}(1-p)) e^{-r}(1-p) - Ipe^{-2r}(3 - 3p + p^2) \\
&\geq V_{(N-3)m} up + V_{(N-3)m} p(1 + de^{-r}(1-p)) e^{-r}(1-p) - Ipe^{-2r}(3 - 3p + p^2) \\
&\geq V_{(N-3)m} p(u + (1-p)e^{-r} + (1-p)^2 de^{-2r} - Ipe^{-2r}(3(1-p) + p^2)
\end{aligned}
$$

$$
(V_{(N-3)m} - I) \cdot e^{k} \geq V_{(N-3)m} p[u + (1-p)e^{-r} + (1-p)^2 de^{-2r}] - Ipe^{-2r}(3(1-p) + p^2)
$$

$$
\therefore\ V^{**}{}_{(N-3)m} \geq \frac{I[e^{k} - pe^{-2r}(3(1-p) + p^2)]}{e^{k} - p[u + (1-p)e^{-r} + (1-p)^2 de^{-2r}]}, \text{ where } \frac{I}{d} \leq V^{*}{}_{(N-3)m} < \frac{I}{d^3} \quad (5\text{-}4)
$$

Step 3: Compute the second alternative point using by the formula below

$$
PV_{(N-3)m} \cdot e^{k} \geq [OV_{(N-2)m} \cdot p + OV_{(N-2)(m+1)} \cdot (1-p)]
$$

$$
PV_{(N-3)m} \cdot e^{k} \geq OV_{(N-2)m} \cdot p
$$

$$PV_{(N-3)m} \cdot e^k \geq (V_{(N-2)m} \cdot pe^{-r}(u + e^{-r}(1-p)) - Ipe^{-2r}(2-p)) \cdot p$$
$$\geq V_{(N-2)m} \cdot p^2 e^{-r}(u + e^{-r}(1-p)) - Ip^2 e^{-2r}(2-p)$$
$$\geq V_{(N-3)m} \cdot up^2 e^{-r}(u + e^{-r}(1-p)) - Ip^2 e^{-2r}(2-p)$$

$$(\because)\ OV_{(N-2)(m+1)} = 0$$

$$OV_{(N-2)m} = [OV_{(N-1)m} \cdot p + OV_{(N-1)(m+1)} \cdot (1-p)] \cdot e^{-r}$$
$$= [V_{(N-1)m} \cdot p - Ipe^{-r} + (V_{(N-1)m} d - I) \cdot pe^{-r}(1-p)] \cdot e^{-r}$$
$$= [V_{(N-1)m} p(1 + de^{-r}(1-p)) - Ipe^{-r}(2-p)] \cdot e^{-r}$$
$$= V_{(N-2)m} \cdot pe^{-r}(u + e^{-r}(1-p)) - Ipe^{-2r}(2-p)$$

$$(\because)\ OV_{(N-1)(m+1)} = V_{(N-1)(m+1)} - Ie^{-r}$$

$$OV_{(N-1)(m+1)} = [(V_{N(m+1)} - I) \cdot p] \cdot e^{-r}$$
$$= (V_{(N-1)m} d - I) \cdot pe^{-r}$$

$$(V_{(N-3)m} - I) \cdot e^k \geq V_{(N-3)m} \cdot up^2 e^{-r}(u + e^{-r}(1-p)) - Ip^2 e^{-2r}(2-p)$$

$$\therefore\ V^{***}{}_{(N-3)m} \geq \frac{I[e^k - p^2 e^{-2r}(2-p))]}{e^k - up^2 e^{-r}(u + e^{-r}(1-p))}, \text{ where } V^{*}{}_{(N-3)m} < \frac{I}{d} \quad (5\text{-}5)$$

Appendix 5.3

```
// This is for determining the optimal timing of exercising the defer options //

# include <iostream.h>
# include <iomanip.h>
# include <fstream.h>
# include <cmath>
# include <stdlib.h>

const int NumNode = 100;
double Max(double,double);

void main()
{
        double value;
        double invest;
        double sigma;
        double risk_free;
        double marr;
        double t;

        cout << " Enter Initial Project Value: "<< endl;
        cin >> value;
        cout << " Enter Investment cost: "<< endl;
        cin >> invest;
        cout << " Enter the volatility: "<< endl;
        cin >> sigma;
        cout << " Enter the risk free rate: "<< endl;
        cin >> risk_free;
        cout << " Enter the MARR of the company: "<< endl;
        cin >> marr;
        cout << " Enter the remaining option life: "<< endl;
        cin >> t;
        cout << " Enter a delta of the binomial tree: "<< endl;
        cin >> delta;

        double u = exp( sigma * sqrt(delta));
        double d = exp( -1 * sigma * sqrt(delta));
        double p = ( exp(risk_free * delta) - d) / ( u - d );
        double q = 1 - p ;
        double Project_V[NumNode][NumNode];
        double Immediate_Profit[NumNode][NumNode];
        double Option_V[NumNode][NumNode];
        double Exp_Option[NumNode][NumNode];
        double Future_Value[NumNode][NumNode];
```

```
// Developing a monetary tree //
 int N = int((t/delta)+1);
 Project_V[0][0] = value;
 for ( int k = 0; k < 10000; k++){
        double function = -10;
        Project_V[0][0] = Project_V[0][0] + 0.01;
        for ( int i = 1; i < N; i++){
                Project_V[0][i] = 0;
                Project_V[i][0] = Project_V[i-1][0] * u;
        }
        for ( i = 1; i < N; i++ ){
                for ( int j = 1; j < N ; j++){
                        if (i < j)
                                Project_V[i][j] = 0.0F;
                        else
                                Project_V[i][j] = Project_V[i-1][j-1] * d;
                }
        }
 // Compute the profit by immediate exercise//
        for ( i = 0; i < N; i++ ){
                for ( int j = 0; j < N ; j++){
                        if (i < j)
                            Immediate_Profit[i][j] = 0.0F;
                        else
                            Immediate_Profit[i][j] = Project_V[i][j] - invest;
                }
        }
 // Compute the option premium of the end node//
        for ( i = 0; i < N; i++ ){
                for ( int j = 0; j < N ; j++){
                        Option_V[i][j] = 0;
                }
        }

        for ( int j = 0; j < N; j++ ){
                Option_V[N-1][j] = Max(0.0F,Immediate_Profit[N-1][j]);
        }
 // Compute the option premium of each node//
        for ( i = N-2; i >= 0; i-- ){
                for (int j = 0; j < N; j++ ){
                        if( i < j)
                            continue;
                        else
                            Option_V[i][j] = exp(- risk_free * delta)
*((Option_V[i+1][j] * p) + (Option_V[i+1][j+1] * q));
```

```
                }
            }
        // Compare two numbers in each node//
            // Compute the expect option value at the next period //
            for ( i = 0; i < N; i++ ){
                for ( int j = 0; j < N ; j++){
                    Exp_Option[i][j] = 0;
                }
            }
            for ( i = N-2; i >= 0; i-- ){
                for (int j = 0; j < N; j++ ){
                    if( i < j)
                        continue;
                    else
                         Exp_Option[i][j]  =  (  OptionV[i+1][j]  *  p  +
  Option_V[i+1][j+1] * q );
                }
            }
            // Compute the future value of current period with MARR//
            for ( i = 0; i < N; i++ ){
                for (int j = 0; j < N; j++ ){
                   Future_Value[i][j] = Immediate_Profit[i][j]*exp(marr * delta);
                }
            }
            // minimize the difference between two values//
            function = Exp_Option[0][0] - Future_Value[0][0];
            if (function <= 0.00001)
                break;
        }
            cout << "Minimum project value is:" << Project_V[0][0] << endl;
}

double Max(double x , double y)
{
        double temp;
        if ( x <= y){
            temp = x;
            x = y;
            y = temp;
        }
        return x;
}
```

CHAPTER 6

COMPREHENSIVE APPLICATIONS OF NEW REAL OPTIONS MODEL

6.1. Introduction

In order to examine the impact of the last three chapters' proposals on decision making, two real options decision models are presented. The application of the new volatility estimation method; the volatility revision process; and the new decision rules for two specific growth project opportunities, a growth option and a compound option, are demonstrated. It is assumed in the examples that a growth option occurs when a company needs to make an initial investment in order to support follow-on investment. Projects that require phased-expansions or a web-based technology investment are examples of projects that have growth options perspectives. In the growth options framework, investing in the initial small scale investment provides the option for the follow-on large scale project opportunities, and the amount of loss from the initial investments represents the option premium.

Compound options are similar to growth options, but while growth options concentrate on phase-expansions of the already developed project, compound options apply to projects that have more than a two-phase investment. A good example of a compound option is a R&D project for a new medicine, because the investment decision in a pharmaceutical project depends on its future revenue at the end of multiple stages.

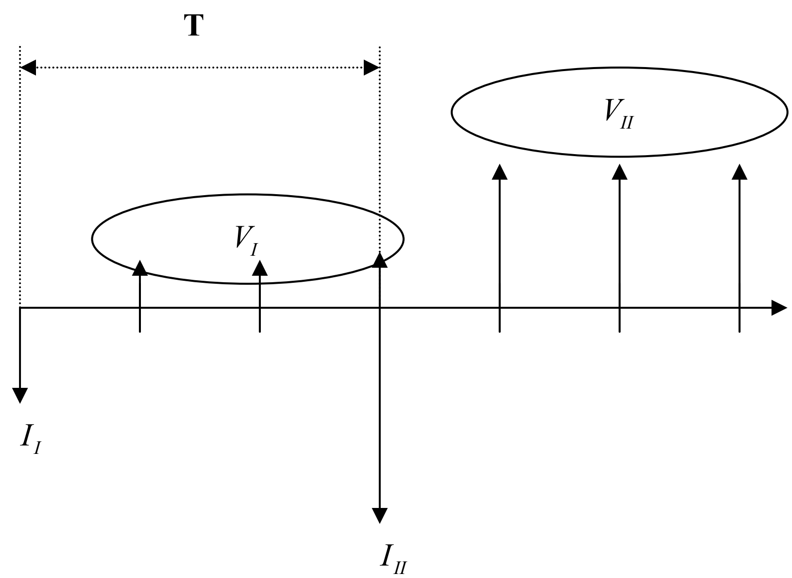


Figure 6-1. Project scenario of a growth option

6.2. Growth Options Framework

Figure 6-1 demonstrates a brief of a growth option scenario. In the growth options framework, the loss from the initial investment $NPV(I_I - V_I)$ is considered as an option premium to take the opportunity to expand the project into the next phase of production. It is assumed that the follow-on investment is available at any time during the option life. The option parameters of the growth option are defined as follows: the present value of the estimated project value V_{II} represents V_0, the required large-scale investment cost I_{II} is I, time to make the investment (T), risk-free interest rate (r), and the uncertainty of the project cash flows (σ). The valuation process for growth options is the same as for the defer options valuation model. Since real options permits the early exercise option, the general binomial lattice model can be applied for the option valuation. In the growth options framework, when the computed option premium (OP) through the valuation model is higher than $NPV(I_I - V_I)$, the required cost to take the large-scale investment opportunity, the investment should be initiated.

The following example is provided in order to apply the processes demonstrated in the previous three chapters into the growth options framework. Consider a project of XYZ Chemical with an opportunity to expand the production level at 3 years from the initial investment. The expected cash flows are summarized in table 6-1. All the values and costs in the table are the expected values.

Table 6-1. Cash flow estimation for the growth opportunity

EOY	0	1	2	3	4	5	6
Income Statement							
Revenue							
Unit price		20	20	20	24	24	24
Demand		1,000	1,000	1,000	2,500	2,500	2,500
Total revenue		20,000	20,000	20,000	60,833	60,833	60,833
Expenses							
Unit variable cost		10	10	10	15	15	15
Variable cost		10,000	10,000	10,000	36,339	36,339	36,339
Fixed cost		6,000	6,000	6,000	11,728	11,728	11,728
Depreciation		3,333	3,333	3,333	7,500	7,500	7,500
Taxable income		667	667	667	5,266	5,266	5,266
Income taxes(40%)		267	267	267	2,107	2,107	2,107
Net income		**400**	**400**	**400**	**3,160**	**3,160**	**3,160**
Cash Flow statement							
Operating Activity							
Net Income		400	400	400	3,160	3,160	3,160
Depreciation		3,333	3,333	3,333	7,500	7,500	7,500
Investment Activity							
Investment	-10,000			-30,000			
Net cash flow	**-10,000**	**3,733**	**3,733**	**-26,267**	**10,660**	**10,660**	**10,660**

Table 6-2. Prior belief of the random factors for the 2nd phase investment

Random Factor	Pessimistic	Most Likely	Optimistic
Demand (EA)	1000	Unknown	4000
Unit Price ($)	15	25	30
Unit variable cost ($)	11	15	16
Fixed cost ($)	10,000	11,430	15,000

It is also estimated that there are four uncertainty factors among the elements of the cash flow table: 1) the unit price, 2) demand, 3) unit variable cost, and 4) fixed cost. Because of the high degree of uncertainty involved in estimating the random distribution, only three-point estimates are demonstrated. Typically, it is very difficult to estimate the demand of the 2nd phase, so only the possible outcome ranges are suggested. The prior beliefs of the random variables are shown in table 6-2.

6.2.1 Initial Option Valuation

6.2.1.1 Initial Volatility Estimation

The distributions of the random factors are defined by the processes in chapter 4, which explains how to convert the three-point estimates to beta distributions. The shapes of the risky variables are defined by three parameters: 1) the standardized mode, 2) α, and 3) β. Note that the demand distribution is converted to Uniform random distribution since only the upper and the lower bounds are known. Table 6-3 shows the standardized mode, α, and, β of the scenario's random factors.

Table 6-3. Parameters of beta distribution

Random Factor	α	Mode	β
Demand (EA)	0	-	0
Unit Price ($)	3.643	0.667	1.819
Unit variable cost ($)	3.565	0.8	0.891
Fixed cost ($)	1.469	0.286	3.674

Once the distributions of the risky elements in the cash flow table are determined, the volatility estimation processes becomes simpler through the use of the model suggested in Chapter 3. The @Risk simulation package calculates the mean and the variance of the 2^{nd} investment at the end of the 3^{rd} year as $28,521 and $17,621 respectively. Since the expected present value with the risk-free interest rate of 6% is $23,823, the volatility of the project is derived to 33% by the following formula:

$$\sigma = \sqrt{\frac{\ln\left[\left(\frac{\sigma_0^{\ 2}}{\mu_0^{\ 2} e^{2rT}}\right) + 1 \right]}{T}} = \sqrt{\frac{\ln\left[\left(\frac{17621^2}{23823^2 \cdot e^{0.36}}\right) + 1 \right]}{3}} = 33\%$$

6.2.1.2 Option Premium and Investment Decision

Once the volatility of the real option is defined, the next step is to evaluate the project opportunity. From the NPV approach, the project is not favorable because of the negative NPV -4.17K.

$$NPV(initial) = -10K + 3.7K(P/A,12\%,3) = -1.11K$$

$$NPV(2nd) = [-30K + 10.7K(P/A,12\%,3)](P/F,12\%,3) = -3.06K$$

$$NPV(Total) = -1.11K - 3.06K = -4.17K$$

However, the options framework gives a different result than the NPV approach. Since the calculated growth option premium is 2.11K through the binomial lattice model with the parameters $V_0 = 18.2K$, $I = 30K$, $T = 3$ years, $\sigma = 0.33$, $r = 0.06$, and $\Delta t = 1$, the strategic net present value by taking the investment opportunity is computed to 1K.

SNPV = -1.11K + 2.11K = 1K

Finally, the decision will be to take the 2nd phase investment opportunity by investing in the initial stage of the project even if the NPV of the initial investment is negative. Figure 6-2 is an overall cash flow of the XYZ project for a growth options framework with exercise time T.

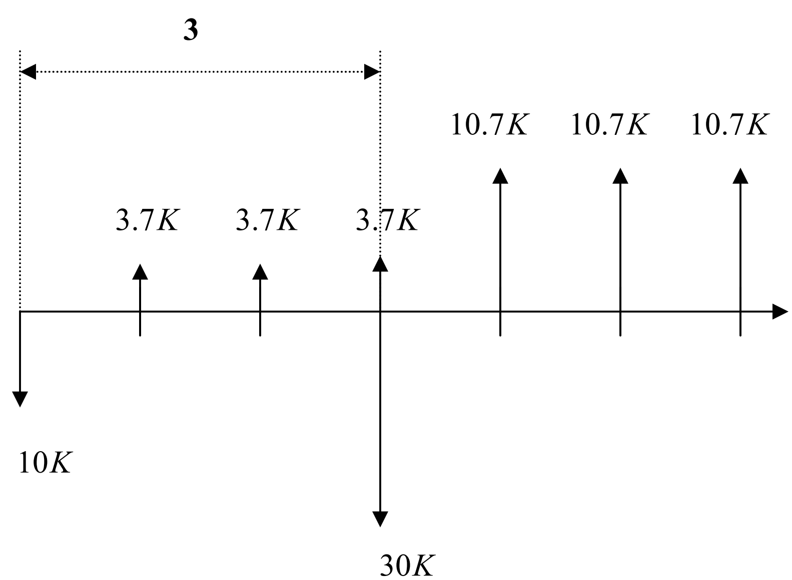


Figure 6-2. Cash flow diagram of the growth opportunity

6.2.2 Volatility Revision

One year after initiating the project, a new set of information is collected so the company needs to reevaluate the project opportunity by combining the observed information to their initial belief. The observation indicates a new demand of 2,400EA, a unit price of \$24, a unit variable cost of \$14.6, and a fixed cost of 12,000. From the survey to the company's management group, the IQF of the initial belief is closer to 3. To make the computation simple, 10 equal length intervals of the discrete approximation are assumed for the risky variables. For the demand distribution example, each interval has 300 events when it is transformed to the discrete distribution. The median of the intervals is considered as the event of the variable, and the probability of the intervals is recorded as a corresponding probability, $\theta_k s$, of the event. Then the α_k of the prior distribution is computed by $\alpha_k = \theta_k \times IQF = 3\theta_k$.

The next step is to determine the posterior distribution by conjugating the observation results, $x_k s$, to the prior belief. In this case study, the observation is assumed as is. The $\theta''_k s$ of the posterior probability are calculated after $\alpha''_k s$ are determined. Table 6-4 is the result of the Bayesian conjugate distribution with the data based on the @Risk simulation software. The simulation shows that the standard deviation of the cash flow at the end of the 3rd year is \$15,082, and the expected value of the project at the end of the 1st year is \$25,029. The new volatility is computed to 35% by equation (3-10), and the expected value of the project at the end of year 1, \$20.1K, is still less than its investment cost. Therefore, it is advisable to wait until the end of the option life. The logic behind the decision to wait will follow in section 6.4.

Table 6-4. Revised discrete approximation of the growth option

Variables	k	θ_k	$\alpha_k = \theta_k \times IQF$	x_k	α''_k	θ''_k	Events
Demand	1	0.10	0.30		0.30	0.075	1150
	2	0.10	0.30		0.30	0.075	1450
	3	0.10	0.30		0.30	0.075	1750
	4	0.10	0.30		0.30	0.075	2050
	5	0.10	0.30	1	1.30	0.325	2350
	6	0.10	0.30		0.30	0.075	2650
	7	0.10	0.30		0.30	0.075	2950
	8	0.10	0.30		0.30	0.075	3250
	9	0.10	0.30		0.30	0.075	3550
	10	0.10	0.30		0.30	0.075	3850
	Sum	1	3	1	4	1	
Price	1	0.0003	0.0009		0.0009	0.0002	15.75
	2	0.0060	0.0180		0.0180	0.0005	17.25
	3	0.0287	0.0861		0.0861	0.0215	18.75
	4	0.0714	0.2142		0.2142	0.0536	20.25
	5	0.1309	0.3927		0.3927	0.0982	21.75
	6	0.1866	0.5598		0.5598	0.1340	23.25
	7	0.2217	0.6651	1	1.6651	0.4163	24.75
	8	0.2018	0.6054		0.6054	0.1514	26.25
	9	0.1230	0.3690		0.3690	0.0923	27.75
	10	0.0297	0.0891		0.0891	0.0223	29.25
	Sum	1	3	1	4	1	
Variable Cost	1	0.0001	0.0003		0.0003	0.0001	11.25
	2	0.0028	0.0084		0.0084	0.0021	11.75
	3	0.0144	0.0432		0.0432	0.0108	12.25
	4	0.0395	0.1185		0.1185	0.0296	12.75
	5	0.0846	0.2538		0.2538	0.0635	13.25
	6	0.1445	0.4335		0.4335	0.1084	13.75
	7	0.2024	0.6072		0.6072	0.1518	14.25
	8	0.2554	0.7662	1	1.7662	0.4416	14.75
	9	0.2504	0.7512		0.7512	0.1878	15.25
	10	0.0060	0.0180		0.0180	0.0045	15.75
	Sum	1	3	1	4	1	
Fixed Cost	1	0.0528	0.1584		0.1584	0.0396	10250
	2	0.1509	0.4527		0.4527	0.1132	10750
	3	0.2173	0.6519		0.6519	0.1630	11250
	4	0.2171	0.6513		0.6513	0.1628	11750
	5	0.1707	0.5121		0.5121	0.1280	12250
	6	0.1101	0.3303	1	1.3303	0.3326	12750
	7	0.0563	0.1689		0.1689	0.0422	13250
	8	0.0203	0.0609		0.0609	0.0152	13750
	9	0.0043	0.0129		0.0129	0.0032	14250
	10	0.0002	0.0006		0.0006	0.0002	14750
	Sum	1	3	1	4	1	

6.2.3 Decisions on Growth Options

It is necessary to summarize the option parameters before automating the processes for calculating the points of the early exercise at the end of each year. In this case study, the initial option parameters are shown below.

• Project value (V_{00}) = \$18.2K, Investment cost = \$30K

• Risk-adjusted interest rate (R) = 12%, Risk-free interest rate = 6%

• Volatility of the project (σ) = 33%, T = 3 years, $\Delta t = 1$ *year*

Table 6-5 is the summary of the suggested decision criteria at the end of years 1, 2, and 3. The table indicates that the decision at the end of the 1st year is "wait" if the expected value of the 2nd phase project is more than \$15.51K and less than \$84.86K. The decision will be "invest" if the estimated project value is higher than \$84.84K, and "divest" if the value is lower than \$15.51K.

In the case of volatility revision in the growth options framework, no further investment decision remains because the investment was already made. This means that only the investment decision map will be revised through the volatility changes, since the initial investment to select the 2nd project opportunity is considered as a sunk cost during the option life because of its irreversibility.

Table 6-5. Comparison of the decision points in case of volatility change

EOY	Exercise		Forfeit	
	Initial σ	Revised σ	Initial σ	Revised σ
1	84.86↑	84.86↑	15.51↓	14.90↓
2	58.26↑	58.26↑	21.57↓	21.14↓
3	30↑		30↓	

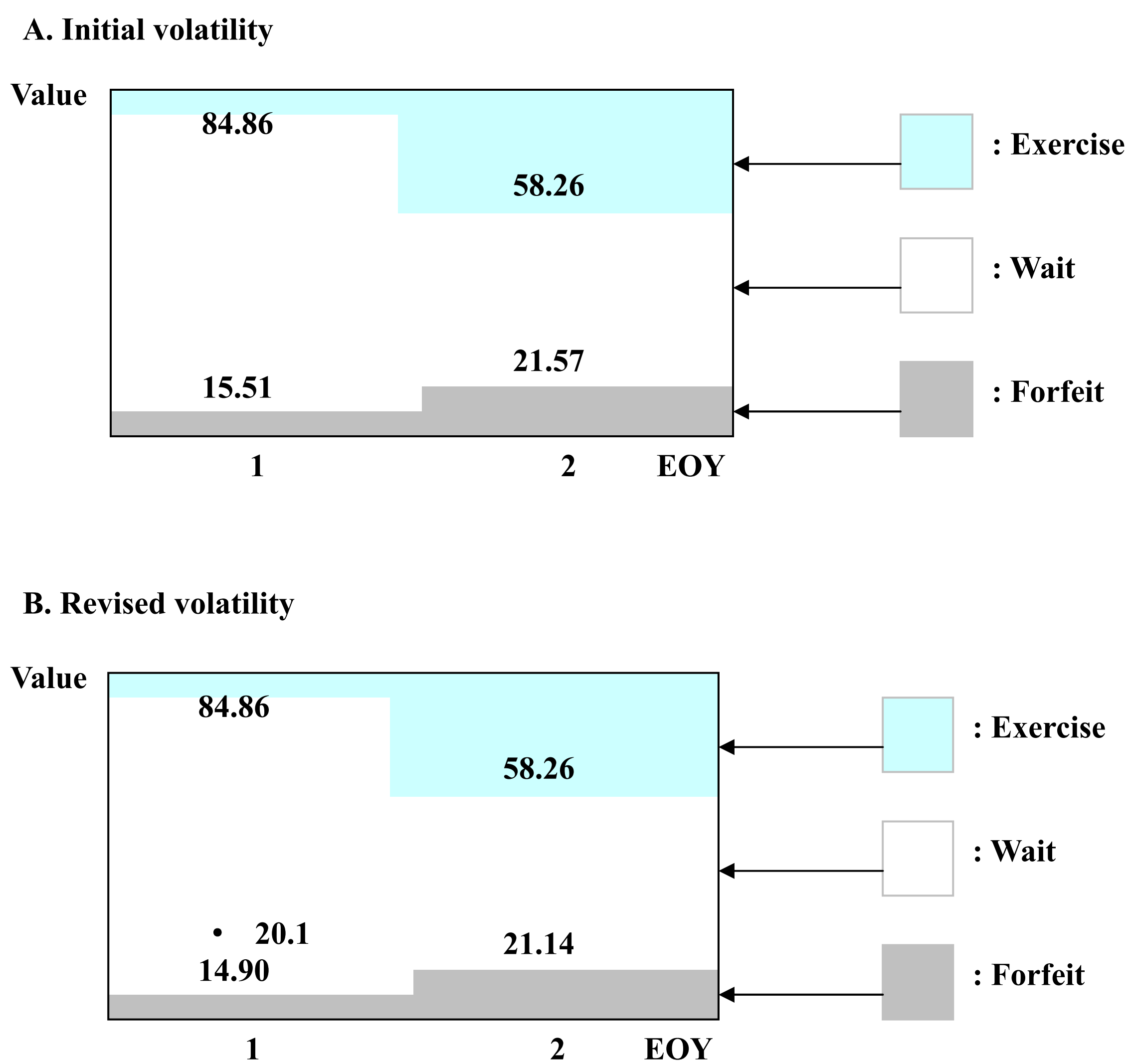


Figure 6-3. Decision framework of the growth option

The only change to take place through collecting more information is the early exercise decision, as demonstrated in chapter 5. Figure 6-3 demonstrates how decisions change according to volatility.

In this example, the volatility increases approximately 2% by RMCS with the revised cash flows. With the new parameters, it is calculated that the early exercise points have not changed because the value of $\frac{I}{d}$ has not increased over the initial exercise point. However, the early forfeit points are moved down. As the revised investment decision map show, if the expected present value of the project is estimated as $20.1K at the end of the first year, the decision should be to wait a little longer to see the future characteristics of the project.

6.3 Compound Options Framework

A sequential investment is considered through the compound options framework. Since there are multiple investment stages in the project, early investment for the latter phase is not permitted before the completion of the previous stages. This is true even if the future is estimated to be favorable. Figure 6-4 illustrates a simplified compound options framework.

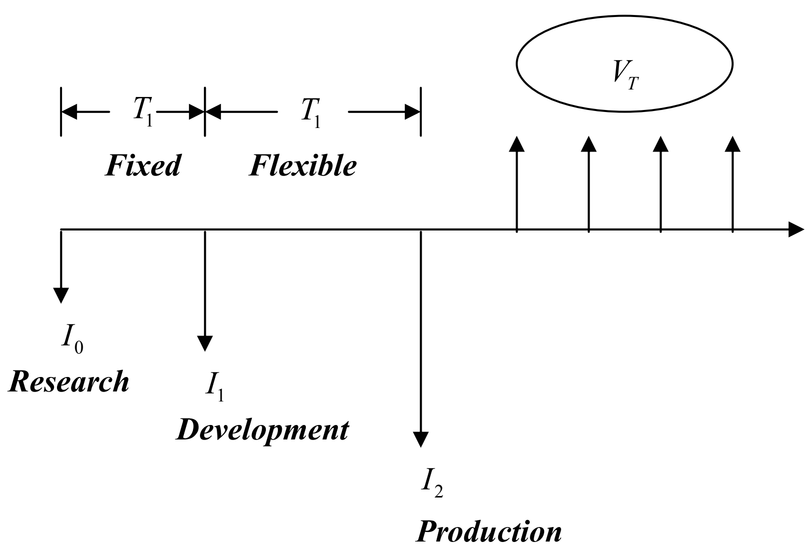


Figure 6-4. Project scenario of a compound option

From the figure, the initial investment cost, I_0, is viewed as an option premium for acquiring the right to invest in the proceeding projects. With the options framework, the present value of the future cash inflows, V_T, is considered as the current project value, while the investment cost of each stage is the exercise price. Therefore a binomial lattice approach, which is a slightly different process than the simple option valuation model, should be applied to value the stage-based investment opportunity.

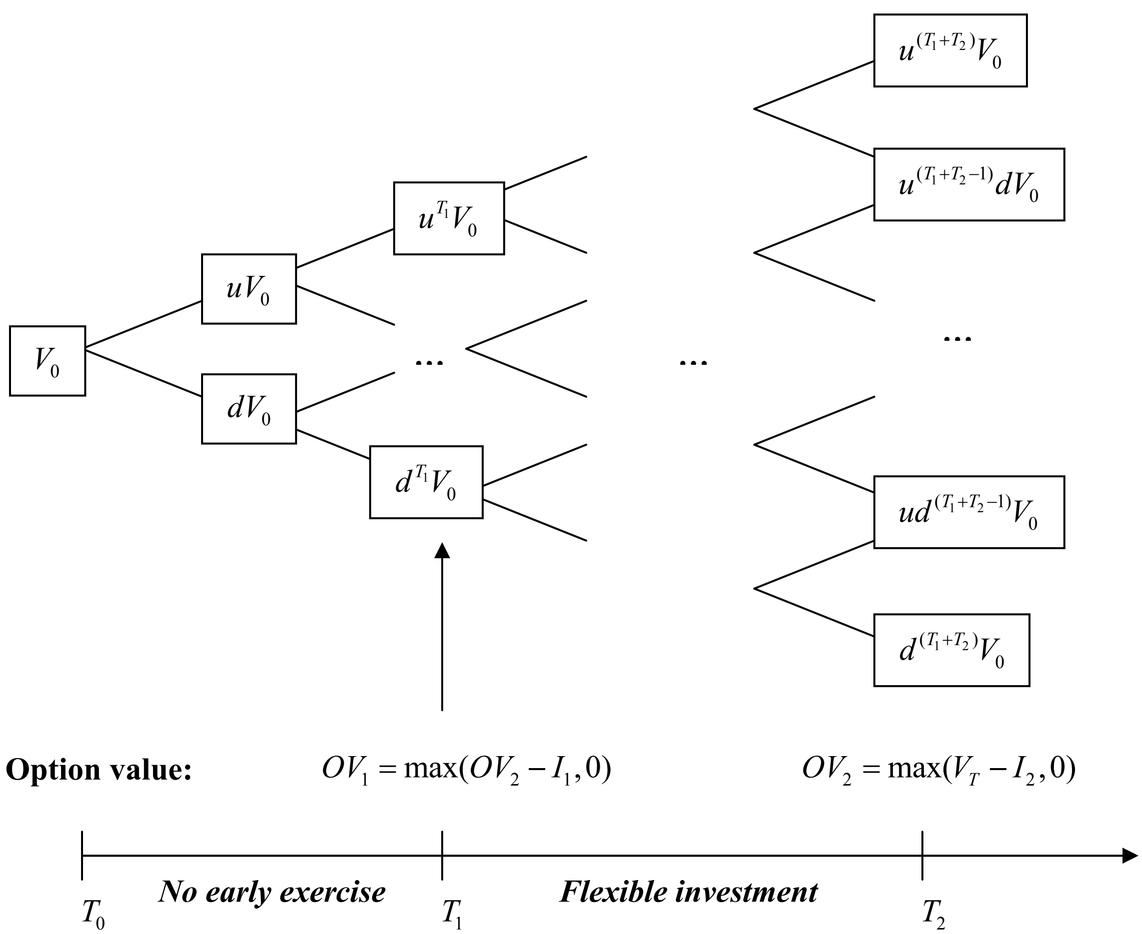


Figure 6-5. Decisions in compound options framework

Figure 6-5 represents the binomial lattice used in order to value the compound option premium. Compared to the general lattice model, there is one more decision node. In the figure, the additional decision immediately follows the first stage of the project. Since the binomial lattice approach uses the dynamic programming skill, the decision is based on the option premium of the initial computation.

Below are the logical processes for handling a project evaluation that has a compound option perspective, as suggested by Park (2006):

Step 1: Calculate the initial underlying asset value by calculating the up and down factors and determining the present value of the future cash flow V_0 over the planning horizon.

Step 2: Calculate the longer-term option value and the shorter-term option value, because the value of a sequential option is based on the earlier option.

Step 3: Calculate the combined-option premium.

6.3.1. Compound Options Scenario

A pharmaceutical firm is considering developing a new drug. It is known that the development process has 6 stages excepting the additional post-marketing test: 1) discovery, 2) pre-clinical testing, 3) clinical phase I, 4) clinical phase II, 5) clinical phase III, and 6) FDA approval. However, the process can be shortened to 3 stages: 1) research, 2) development, and 3) production. In this example, the three stages are considered in order to collect the information for the project.

It will take 2 years to complete the research stage, which requires a \$0.5M investment cost including labor and initial laboratory preparations. Upon completion of the research, \$3M in pre-clinical and clinical test costs must be expended in order to proceed to the next stage. The estimated testing times are 3 years after the success of the research. Including the government approval fee and the cost of building the mass production system, $7M is necessary for the investment. The expected cash flows of the project are summarized in the Table 6-6, and the firms' risk-adjusted interest rate is 15%. Consider the risk-free interest rate as constant at 5% during the project.

Table 6-6. Cash flow estimation for the compound option

EOY	0	1	2	3	4	5	6	7	8	9	10
Income Statement											
Revenue											
Unit price							0.08	0.08	0.08	0.08	0.08
Demand							200	200	200	200	200
Total revenue							15.5	15.5	15.5	15.5	15.5
Expenses											
Unit variable cost							0.04	0.04	0.04	0.04	0.04
Variable cost							7.5	7.5	7.5	7.5	7.5
Fixed cost							2	2	2	2	2
Taxable income							5.98	5.98	5.98	5.98	5.98
Income taxes(40%)							2.39	2.39	2.39	2.39	2.39
Net income							**3.59**	**3.59**	**3.59**	**3.59**	**3.59**
Cash Flow statement											
Operating Activity											
Net Income							3.59	3.59	3.59	3.59	3.59
Investment Activity											
Investment	-0.5		-3			-7					
Net cash flow	**-0.5**		**-3**			**-7**	**3.59**	**3.59**	**3.59**	**3.59**	**3.59**

Table 6-7. Prior belief for the random factors

Random Factor	Low bound	Most Likely	High bound
Demand (EA)	50	Unknown	350
Unit Price (M$)	0.02	0.08	0.12
Unit variable cost (M$)	0.01	0.03	0.10
Fixed cost (M$)	0	2	4

Since there are so many uncertainties in the future market, the survey of the project management team suggests the three-point scenario estimates for the four uncertain variables: 1) demand, 2) unit price, 3) unit variable cost, and 4) fixed cost. Table 6-7 expresses the current beliefs as to the random variables with a pessimistic, optimistic, and most likely point.

6.3.2 Initial Option Valuation

6.3.2.1 Initial Volatility Estimation

Before deciding to take the project opportunity, it is necessary to determine the project's flexibility. The distributions of the random factors are converted to the beta distributions through the beta approximation processes shown in chapter 4. Note that the demand distribution is converted to Uniform because only the upper and the lower bounds are known information. Table 6-8 represents the standardized mode, α, and β of the project scenario's random factors.

Table 6-8. Parameters of beta distribution

Random Factor	α	Mode	β
Demand (EA)	0	-	0
Unit Price (M$)	3.4805	0.6	2.3203
Unit variable cost (M$)	1.0303	0.778	3.6108
Fixed cost (M$)	2.2982	0.5	2.2982

Once the distributions of the uncertain factors are determined, the volatility estimation processes are simplified by using the model suggested in Chapter 3. The @Risk simulation package calculates the mean and the variance of the cash inflows at the end of the 5^{th} year with a risk-free interest rate of 5% as \$15.55M and \$15.59M respectively. Since the expected present value of the cash flows is \$12.18M, the volatility of the project is computed to 37%.

6.3.2.2 Option Premium and Investment Decision

With the NPV approach, the project is not acceptable because of the negative NPV -0.26M.

$$NPV(research) = -0.5M$$

$$NPV(Development) = -3M(P/F,15\%,2) = -2.27M$$

$$NPV(Production) = [-7M + 3.59M(P/A,15\%,5)](P/F,15\%,5) = 2.51M$$

$$NPV(Total) = -0.5M - 2.27M + 2.51M = -0.26M$$

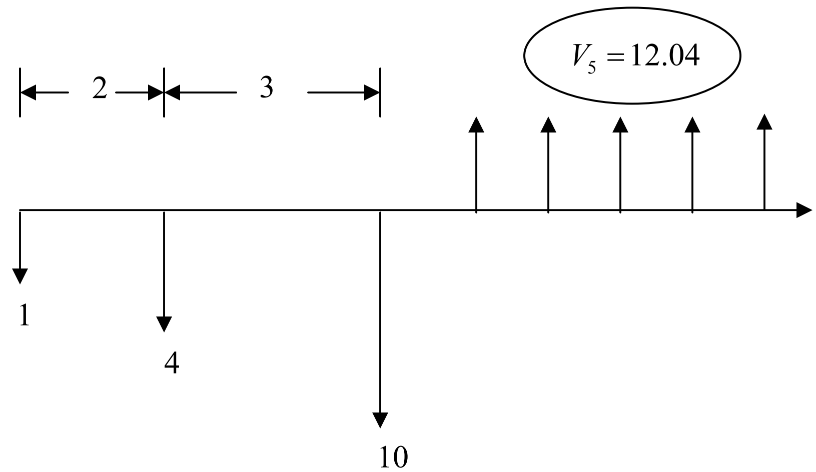


Figure 6-6. Project cash flow of the R&D project

To apply the compound options framework, it is necessary to define the parameters of the compound options. The cash flow diagram of the pharmaceutical project (Figure 6-6) determines the option parameters found in Table 6-9.

In the compound options framework, the initial investment of $0.5M is considered a premium for taking the opportunity to invest in the follow-on projects. The option premium for taking the following investment opportunities is computed through the binomial lattice valuation method as $0.52M, with $\Delta t = 1\ year$. Since $0.5M is cheaper than the option premium, investment in the research phase of the project is recommended.

Table 6-9. Compound options parameters

Parameters	V_0	T_1	T_2	I_1	I_2	σ	r
Value	5.98M	2	3	3M	7M	37%	5%

6.3.3 Volatility Revision

One year after initiating the project, a new set of information is collected so the company needs to reevaluate the project opportunity by combining the observed information to their initial belief. The sample indicates a new demand of 205EA, a unit price of $0.1M, a unit variable cost of $0.05M, and a fixed cost of $2.75M. According to the company's management group survey, the IQF of the initial belief is closer to 3. Ten equal-length intervals of the discrete approximation are assumed for the risky variables in order to simplify the computation. The median values of the intervals are considered as the events of the variables, and the probabilities of the intervals are recorded as a corresponding probability, $\theta_k s$, of the events. Then the α_k of the prior distribution is computed by $\alpha_k = \theta_k \times IQF = 3\theta_k$.

The next step is to determine the posterior distribution by conjugating the observation results, $x_k s$, to the prior belief. In this case study, it is assumed that the observation is considered as is. The $\theta''_k s$ of the posterior probability are calculated after the $\alpha''_k s$ are determined. Table 6-10 is the result of the Bayesian conjugate distribution with the data based on the @Risk simulation software. The simulation result shows that the standard deviation of the cash flow at the end of the 5th year is $16.09M and the expected value of the project at the end of the 1st year is $14.11M. Therefore, the new volatility is computed to 40% by equation (3-10). The expected value of the project at the end of year 1 is estimated to $7.59M. In this particular case, the volatility of the project is slightly increased by conjugating new information during the waiting period.

Table 6-10. Revised discrete approximation of the compound option

Variables	k	θ_k	$\alpha_k = \theta_k \times IQF$	x_k	α''_k	θ''_k	Events
Demand	1	0.10	0.30		0.30	0.075	65
	2	0.10	0.30		0.30	0.075	95
	3	0.10	0.30		0.30	0.075	125
	4	0.10	0.30		0.30	0.075	155
	5	0.10	0.30		0.30	0.075	185
	6	0.10	0.30	1	1.30	0.325	215
	7	0.10	0.30		0.30	0.075	245
	8	0.10	0.30		0.30	0.075	275
	9	0.10	0.30		0.30	0.075	305
	10	0.10	0.30		0.30	0.075	335
	Sum	1	3	1	4	1	
Price	1	0.0006	0.0018		0.0018	0.0004	0.025
	2	0.0112	0.0336		0.0336	0.0084	0.035
	3	0.0466	0.1398		0.1398	0.0350	0.045
	4	0.1022	0.3066		0.3066	0.0767	0.055
	5	0.1696	0.5088		0.5088	0.1272	0.065
	6	0.2137	0.6411		0.6411	0.1603	0.075
	7	0.2070	0.6210		0.6210	0.1552	0.085
	8	0.1609	0.4827		0.4827	0.1207	0.095
	9	0.0767	0.2301	1	1.2301	0.3075	0.105
	10	0.0115	0.0345		0.0345	0.0086	0.115
	Sum	1	3	1	4	1	
Variable Cost	1	0.0981	0.2942		0.2942	0.0735	0.0145
	2	0.2100	0.6299		0.6299	0.1575	0.0235
	3	0.2222	0.6665		0.6665	0.1666	0.0325
	4	0.1934	0.5801		0.5801	0.1450	0.0415
	5	0.1376	0.4128	1	1.4128	0.3532	0.0505
	6	0.0823	0.2470		0.2470	0.0617	0.0595
	7	0.0398	0.1193		0.1193	0.0298	0.0685
	8	0.0138	0.0413		0.0413	0.0103	0.0775
	9	0.0029	0.0086		0.0086	0.0021	0.0865
	10	0.0001	0.0004		0.0004	0.0001	0.0955
	Sum	1	3	1	4	1	
Fixed Cost	1	0.0027	0.0082		0.0082	0.0021	0.2
	2	0.0308	0.0925		0.0925	0.0231	0.6
	3	0.0929	0.2788		0.2788	0.0697	1
	4	0.1637	0.4911		0.4911	0.1228	1.4
	5	0.2098	0.6294		0.6294	0.1573	1.8
	6	0.2098	0.6294		0.6294	0.1573	2.2
	7	0.1637	0.4911	1	1.4911	0.3728	2.6
	8	0.0929	0.2788		0.2788	0.0697	3
	9	0.0308	0.0925		0.0925	0.0231	3.4
	10	0.0027	0.0082		0.0082	0.0021	3.8
	Sum	1	3	1	4	1	

However, the acquired information in the compound options framework plays a different role in the decision compared to simple options. Since there are two phases for valuing the option premium, two decision scenarios should be considered for selecting the investment timings. The first scenario is that the volatility is updated before the development phase investment, and the second scenario is that the volatility is changed after the development phase investment. In other words, if the new information is collected before the 2^{nd} investment, then it affects the decision on the next investment. However, when the information is collected after the last investment decision is made, it is necessary to find the early exercise points.

6.3.4 Effect of Revising Volatility on Decision Making

In the compound options framework, two decisions remain following the purchase of the research option. Since it is assumed that the early exercise option is not available during the research phase, the collected information will affect the decision whether or not to invest in the development phase. In this case, simply follow the option valuation model with the posterior volatility and the project value. The other cases consider the change in volatility during the development phase of the project. In this case, it is assumed that early exercise is possible.

6.3.4.1 Volatility Change before the 2^{nd} Investment

Assume that a new set of information is observed to make possible a revision of the original decision one year after the research phase investment. The new set of information is assumed to be the same as the scenario which was introduced in section 6.3.3. In this case, since early actions are not available, the decision is delayed until the end of the 2^{nd} year, and the decision then would be whether or not to take another option.

The required cost to invest in the development phase is viewed an option premium for obtaining the right to invest in the production phase, while the previous investment cost is considered a sunk cost.

In order to support decision making at this point, the relative volatility and the project value graph, which justify further investment, are derived in Figure 6-7. The bold line in the figure represents the border of two investment decisions: 1) invest, or 2) do not invest. By reducing the volatility because of the new information, the expected project value should be increased to stay in the investment decision region. Figure 6-8 demonstrates what would happen if the volatility changes from 37% to 40%. In this case, if the expected value of the future project is less than 7.98M and higher than 7.80M, "do not invest" was the better option without the volatility revision. However, it is a better decision to invest in the project if the real volatility is 40%.

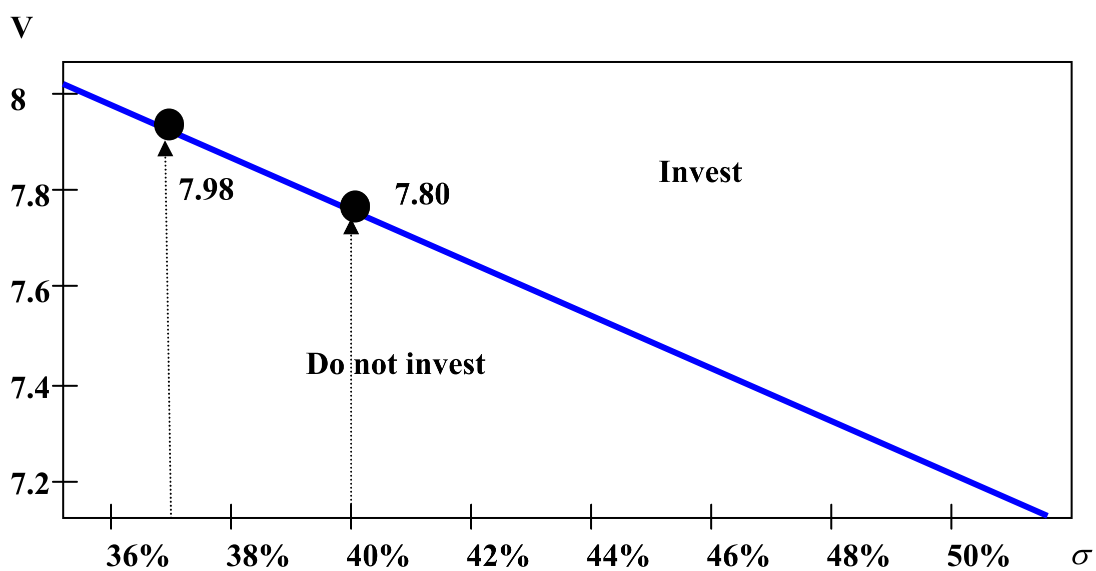


Figure 6-7. The decision map for the 2nd phase investment

6.3.4.2 Volatility Change after the 2nd Investment

In case of volatility revision after the 2nd phase investment, only the final decision remains. The two previous investments to obtain the 3rd phase investment opportunity are considered a sunk cost because of its irreversibility. The only changes in this stage will affect the early exercise decisions under the rule shown in chapter 5.

Assume that the development phase investment was already undertaken with the initial volatility, and that a new set of information is collected one year from the development investment. The new volatility is assumed to be 20% using the processes demonstrated in section 6.3.3. To automate the processes of defining the points of early actions at the end of each year, it is necessary to apply the processes demonstrated in section 5.2. Here, we use the option parameters for this case study: the risk-adjusted interest rate (R) of 15%, the risk-free interest rate (r) of 5%, and the option life (T) of 3 years. Table 6-11 is the summary of the suggested decision criteria at the end of years 3, 4, and 5. This table shows that if the expected present value of the project one year after the 2nd investment is made is $14M, then the decision should be “Exercise” in light of the revised volatility, while it was “Wait” under the initial volatility. Figure 6-8 demonstrates how decisions change according to volatility.

Table 6-11. Comparison of the decision points in case of volatility change

EOY	Exercise		Forfeit	
	Initial σ	Revised σ	Initial σ	Revised σ
3	14.04↑	13.33↑	3.34↓	4.69↓
4	10.25↑	10.25↑	4.84↓	5.73↓
5	7↑		7↓	

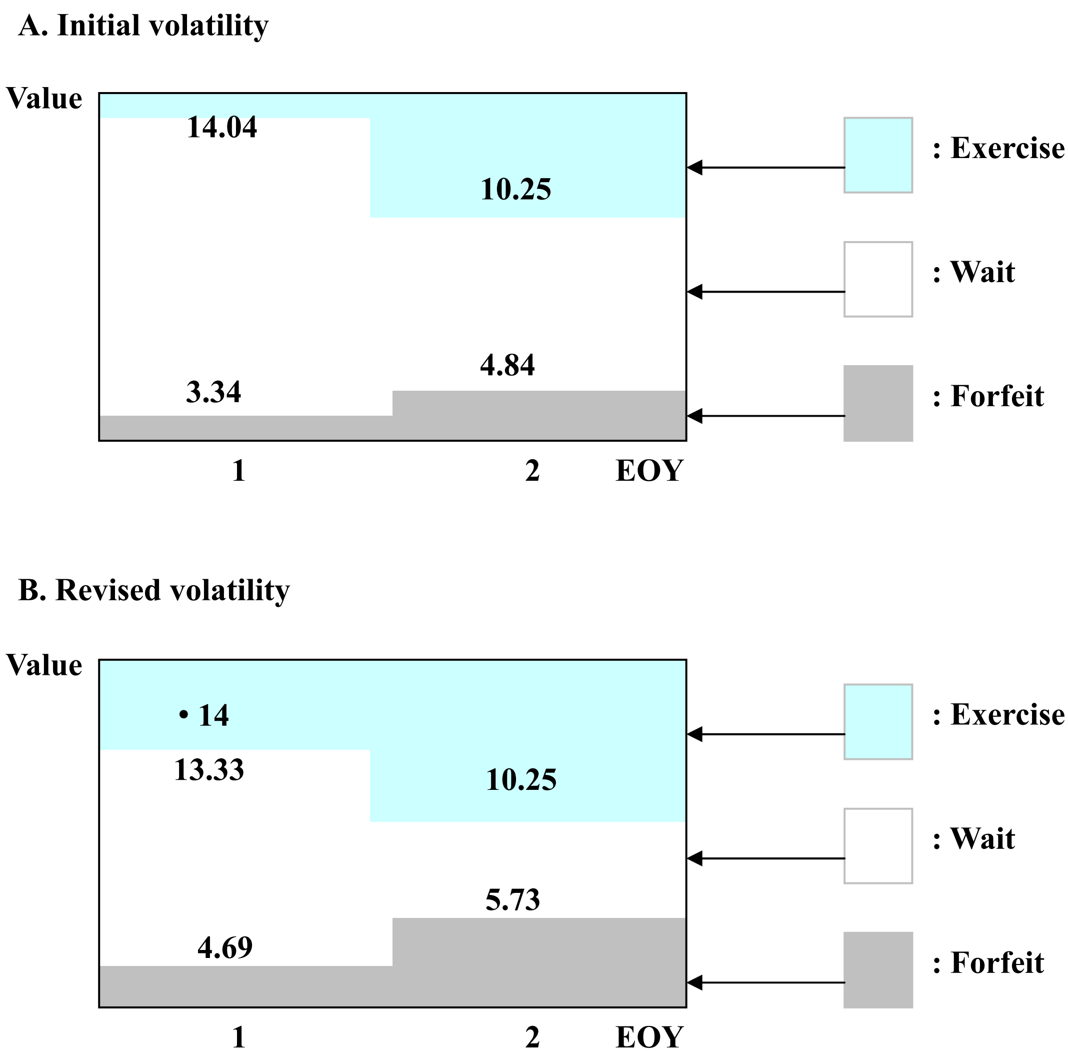


Figure 6-8. Decision framework of the compound option

6.4 Conclusion Remarks

This chapter presents all of the sequences, from estimating the project cash flows to the early action rule of a real option, needed to apply real options to capital budgeting decisions. A Dirichlet conjugate distribution is applied to revise the volatility of the project when new information is acquired. A new decision map is developed in case the volatility changes. Since one of the most important purposes of the real options valuation model is deriving a value of the investment opportunity, examples of intuitive growth options and compound options are presented.

Since the only investment decision involved in buying the option is already made even in a growth option, the remaining decision is related to the irreversible large investment, which is the same as the pure investment in a defer option. In the growth options framework, only information collected during the option life will affect the early exercise decision. The decision-making process for compound options is different because of the future decisions that remain. There can be two possible schedules for volatility revisions in such cases: before or after the 2^{nd} investment. The new volatility will impact the decision on the 2^{nd} investment when such new information is collected before the 2^{nd} investment, while the early exercise rule is applied when the information arrives after the investment.

CHAPTER 7

CONCLUSIONS

Today's uncertain world requires firms to have a system in place that can analyze the flexibility of their projects. Real options are utilized frequently to quantify the benefits of taking a particular risk. The real options valuation process provides a methodology to measure the value of flexibility, and it assists the decision makers in making the optimal investment decision. The goal of this research is to develop the methodology for improving the real options application in actual capital investment decision making. In order to accomplish the goal, three proposals are suggested and studied. However, as demonstrated in the text, the three objectives are quite complementary.

The initial proposal to enhance the use of real options is developing a DCF-based volatility estimation. Most researchers argue that determination of the correct project volatility is the most important parameter of real options valuation, but it is very difficult to estimate the true volatility of a project. Since the DCF analysis is popular in today's capital budgeting decisions, it is reasonable to estimate the flexibility of the project by using the already existing information. So, in this study, the Reverse Monte Carlo Simulation model (RMCS), which combines Monte Carlo simulation and the stochastic processes, is developed as a new volatility estimation method for risky projects.

Secondly, volatility revision processes based on the previous volatility estimation processes are proposed. Even though real options gives a firm an opportunity to improve its strategic investment decisions in an uncertain environment, the traditional options framework did not consider the data gathering activity that usually takes place in the real world. In this study, a Bayesian revision process is suggested to estimate the new volatility when the initial volatility has been estimated by Monte Carlo simulation. Since specific cases that use typical types of Bayesian conjugate processes are hard to find in the real world, a Dirichlet conjugate process is applied to estimate posterior distributions of the future cash flows. After estimating the new distributions of the cash flows, the revised volatility can be computed using the RMCS approach.

Finally, this study proposes a new early decision rule in order to make real options more useful. This rule concentrates on maximizing the expected future project value. Under the new decision rule, an expected future value of the currently exercised option and the expected future option value are compared in order to determine the best exercise timing. An early decision map for "waiting," "early exercise," and "early divest" over the entire option life is developed to automate the decision in case some variables are revised in the future. The map indicates that increasing volatility enlarges the "waiting" area while decreasing volatility shrinks the "waiting" area.

A simulation is applied to validate the newly developed decision rule by comparing the benefit of the early exercise rule and the volatility revision during the option life. Then the expected profits under the new decision rule and the traditional rule are compared in order to demonstrate the advantages of the new approach. The new decision rule is found to be useful in maximizing the expected profit of the delayed investment because the proposed decision model results are better than or equal to the current decisions model.

Two real options decision models are demonstrated in order to apply the proposed processes to specific investment opportunities. In the growth options framework, the revised volatility and the project value affect the early exercise decision, while the compound options framework concentrates on go or no-go decisions for the next phase investments. The information obtained during the option life has an effect on the early exercise decision in the growth options framework. The new volatility will impact the decision on the 2^{nd} investment when such new information is gathered before the 2^{nd} investment, while the early exercise rule is applied when the information arrives after the 2^{nd} investment.

Our research on volatility estimation through RMCS provides a practical volatility of real options, especially when an estimate of the project's future cash flows is available. We also researched revising the volatility through a Bayesian revision process in case the initial volatility is based on RMCS. Then an early decision map of the real options is developed to automate the decisions. Below are several future research opportunities in real options volatility and decision making.

- Estimating a correct distribution of the cash flows in order to improve the reliability of the volatility obtained through RMCS
- Valuing an investment opportunity in case the estimate of the future project value is dependent on the past value.
- Developing an algorithm for implementing the Dirichlet conjugate process in volatility estimation so that the volatility revision process will be easier

- Application of the new decision rule to various investment opportunities in order to verify that the rule works in real world projects

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