# Choice Numbers, Ohba Numbers and Hall Numbers of some complete k-partite graphs

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(Julian) Apelete D. Allagan

Certificate of Approval:

Chris Rodger
Professor
Mathematics and Statistics

Peter D. Johnson Jr. Chair
Professor
Mathematics and Statistics

Dean G. Hoffman
Professor
Mathematics and Statistics

George T. Flowers
Dean
Graduate School

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(Julian) Apelete D. Allagan

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(Julian) Apelete D. Allagan



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Signature of Author

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# Dissertation Abstract

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(Julian) Apelete D. Allagan



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The choice numbers of some complete $k-$partite graphs are found, after we resolved a dispute regarding the choice number of $K(4, 2, \ldots, 2)$ when $k$ is odd. Estimates of the choice numbers and the Ohba numbers of $K(m, n, 1, \ldots, 1)$ and $K(m, n, 2, \ldots, 2)$ are also discussed for various values of $1 \leq n \leq m$. Finally we close this research with the Hall numbers of $K(m, 2, \ldots, 2)$ when $m = 2, 4$.

## Acknowledgments

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Style manual or journal used Journal of Approximation Theory (together with the style known as “aums”). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

## Table of Contents

## List of Figures

## Chapter 1

## Introduction

Throughout this dissertation, the graph $G = (V, E)$ will be a finite *simple graph* with vertex set $V = V(G)$ and edge set $E = E(G)$.

### 1.1 Basic definitions

A *complete $k-$partite graph* $G$ is a graph with $k$ disjoint parts in which there is an edge between each pair of vertices of different parts and no other edges. When each part of $G$ has size exactly one, $G$ is said to be a *complete graph*. We use the notation $K(\underbrace{m_1, m_2, \ldots, m_k}_{k})$ to denote a complete $k-$partite graph ($k \geq 2$) in which the parts have sizes $m_1, m_2, \ldots, m_k$. The complete graph on $n$ vertices can be denoted by $K(1, \ldots, 1)$, but is usually denoted by $K_n$.

A collection of pairwise adjacent vertices forms a *clique*. An *independent set* (also known as a stable set) of a graph $G$ is a set of vertices of $G$ that are pairwise nonadjacent. The maximum size of such a set, denoted by $\alpha(G)$, is called the *independence number* of $G$.

A *subgraph* of the graph $G$ is a graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* $H$ of $G$ is the maximal subgraph of $G$ with vertex set $V(H)$. When we remove a vertex set, say $V_1 \subset V(G)$, we write $G - V_1$. For a single vertex, we write $G - v$.

The *line graph* $H$ of a graph $G$ is the graph, often denoted by $L(G)$, whose vertex set is the edge set of G; and two vertices are adjacent in $H$ if and only if their corresponding edges share an endpoint in $G$.

Let $G_1$ and $G_2$ be two graphs. The *join* of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph $H$ whose vertex set is $V(H) = V(G_1) \cup V(G_2)$, a disjoint union, and whose edge set is $E(G) = E(G_1) \cup E(G_2) \cup \{v_1 v_2 \mid v_1 \in V(G_1),\ v_2 \in V(G_2)$.

A *list assignment* to the graph $G$ is a function $L$ which assigns a finite set (list) $L(v)$ to each vertex $v \in V(G)$. A *proper* $L-$*coloring* of $G$ is a function $\psi : V(G) \to \bigcup_{v \in V(G)} L(v)$ satisfying, for every $u$, $v \in V(G)$,

(i) $\psi(v) \in L(v)$,

(ii) $uv \in E(G) \to \psi(v) \neq \psi(u)$.

The *choice number* or *list*$-$*chromatic number of* $G$, denoted by $ch(G)$, is the smallest integer $k$ such that there is always a proper $L-$coloring of $G$ if $L$ satisfies $|L(v)| \geq k$ for every $v \in V(G)$. We define $G$ to be $k-$*choosable* if it admits a proper $L-$coloring whenever $|L(v)| \geq k$ for all $v \in V(G)$; then $ch(G)$ is the smallest integer $k$ such that $G$ is $k-$choosable.

Since the chromatic number $\chi(G)$ is similarly defined with the restriction that the list assignment is to be constant, it is clear that for all $G$, $\chi(G) \leq ch(G)$. There are many graphs whose choice number exceeds (sometimes greatly) their chromatic number. Figure 1.1 depicts the smallest graph $G$ whose choice number exceeds its chromatic number.

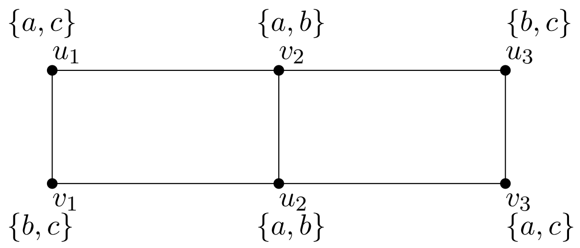


Figure 1.1: K(3,3) minus two independent edges with a list assignment $L$.

Notice that if we denote the parts of the bipartite graph $K(3,3)$ by $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ for $i = 1, 2, 3$, then $G \cong K(3,3) - (\{u_1v_3\} \cup \{v_1u_3\})$.

To see that $G$ is not properly $L-$colorable, suppose that $\psi$ is a proper $L-$coloring of $G$. Now, if $\psi(u_2) = b$ then $\psi(v_2) = a$ and $\psi(v_3) = c$. Hence, we cannot properly color $u_1$. On the other hand, if $\psi(u_2) = a$ then $\psi(v_2) = b$ and $\psi(v_3) = c$, and we cannot properly color $u_3$. Hence, the graph $G$ has no proper $L-$coloring and $|L(v)| \geq 2$ for all $v \in V(G)$.

Therefore, $G$ is not $2-$choosable, meaning $ch(G) > 2$. Further, since $G$ is connected, and neither a complete graph nor an odd cycle, by Brooks' theorem for the choice number [3], $ch(G) \leq \Delta(G) = 3$. Thus, $ch(G) = 3$.

Any graph $G$ for which the extremal case $\chi(G) = ch(G)$ holds is said to be *chromatic−choosable*. It is not hard to see that cycles, cliques and trees are all chromatic−choosable. (Well, the case of even cycles requires a little work. See [3].)

### 1.2 History

In graph theory, a vertex coloring problem is the coloring of the vertices of a graph under various constraints, with the aim of optimizing something, so that adjacent vertices receive different colors. The classical constraint is that colors are chosen from a fixed palette, and the aim is to minimize the number of colors available.

List colorings, as generalizations of the usual vertex coloring, were first introduced in the late 1970's by Vizing [13], and then independently by Erdös et al. [3] Erdös et al were inspired by Jeffrey Dinitz' problem: *if the cells of an $n \times n$ array are assigned sets of size $n$, can representatives of these sets necessarily be found for the cells so that no representative occurs more than once in any row or column of the array?* Dinitz' problem turned out to be a very important list coloring problem which can be restated as follows: *Is the line graph of the complete bipartite graph $K(n,n)$ chromatic−choosable?* The problem remained unsolved until 1995 when Frederick Galvin [4] proved that *the line graph of any bipartite multigraph is chromatic−choosable.* It is clear that Galvin's result is much stronger than the original Dinitz' problem. However, there remains one fundamental unanswered question about the original list coloring problem of Vizing (in Russian): *Is the line graph of any graph chromatic−choosable?* The affirmative answer to that open question is famously known as the (edge) list coloring conjecture. Because it is very difficult to find the choice number of any graph, some mathematicians have begun to doubt the validity of the conjecture. Nevertheless, in 1995, Gravier and Maffray [5], after proving that every $3-$chromatic claw−free perfect graph is chromatic−choosable, conjectured rashly that *every claw−free* *graph is chromatic−choosable*; a much stronger conjecture than the list coloring conjecture since every line graph is claw−free. Their statement brings us back into considering the list coloring conjecture once again. Surveys of choice numbers of graphs can be found in [14] and [15].

In 2002, Ohba conjectured [11] that *every graph* $G$ *with* $2\chi(G)+1$ *or fewer vertices is chromatic−choosable*. It is important to point out here that because every $k-$chromatic graph is a spanning subgraph of a complete $k-$partite graph, Ohba's conjecture is true if and only if it's true for every complete $k-$partite graph. Since then, several papers (see [8]) have been written specifically in attempts to find the choice numbers of some complete $k-$partite graphs which satisfy the hypothesis of Ohba's conjecture. Meanwhile, in 2005, Reed and Sudakov [12] proved that $G$ is *chromatic−choosable* when $|V(G)| \leq \frac{5}{3}\chi(G) - \frac{4}{5}$, a much stronger hypothesis than Ohba's.

### 1.3 Overview

We began this research with the list coloring conjecture in mind. We hoped to at least verify the conjecture for the line graph of $K(2,2,2)$. We were soon confronted by the grim difficulty of finding the choice number of any simple graph. But thanks to my advisor's flexibility, we simply decided to try to learn from what others have done. Soon we came across the papers written by Enomoto et al [2] and Xu, Yang[16]. There seems to be a contradiction between the results of Enomoto et al and Xu, Yang. In section 2.1, we resolve some of the issues related to both authors' results. Later, one of the authors' remarks in [2] lead us to investigate the choice numbers of some particular multipartite graphs. Our findings are presented in section 2.2. We close the chapter with some estimates of the choice numbers that we could not determine exactly. In the final chapter, we present some results on another list coloring graph parameter, the Hall number, closely related to the choice number.

# Chapter 2

## Choice numbers

In the first section of this chapter, we resolve a dispute over the choice number of the complete $k-$partite graph $K(4, 2, \ldots, 2)$ when $k$ is odd. Further, in the same section, we revise the proof in [2] about the choice number of $K(4, 2, \ldots, 2)$ when $k$ is odd. In the next section, estimates, and in some cases exact values, are obtained of the choice numbers of some complete multipartite graphs in which all parts except one are of sizes 1, 2 or 3. These results also estimate, and sometimes determine, the *Ohba number* of these graphs. The *Ohba number* of a (finite simple) graph is the smallest order of a clique such that the choice number and the chromatic number of the join of the graph with a clique of that order are equal.

Here is some background on the choice numbers of some complete multipartite graphs.

**Theorem A.** *(Erdös, Rubin and Taylor [3]) The complete $k-$partite graph $K(2, 2, \ldots, 2)$ is chromatic$-$choosable.*

Notice that this result establishes that every simple graph with independence number 2 is $k-$choosable. This is due to the fact that every $k-$chromatic graph is a spanning subgraph of some complete $k-$partite graph, in which the parts are the color classes from some proper $k-$coloring of the original graph.

**Theorem B.** *(Gravier and Maffray [5]) If $k > 2$, then the complete $k-$partite graph $K(3, 3, 2, \ldots, 2)$ is chromatic$-$choosable.*

This result does not hold for $k = 2$ since $K(3, 3)$ contains the subgraph in Figure 1.1, whose choice number is bigger than 2.

**Corollary B.** *The complete $k-$partite graph $K(3,2\ldots,2)$ is chromatic$-$choosable.*

Since $K(3,2\ldots,2)$ is a complete $k-$partite graph, $k=\chi(K(3,2\ldots,2))\leq ch(K(3,2\ldots,2))$. Further, $K(3,2\ldots,2)$ is a subgraph of the complete $k-$partite graph $K(3,3,2,\ldots,2)$. Therefore $ch(K(3,2\ldots,2))\leq k$ if $k>2$. Thus, $ch(K(3,2\ldots,2))=k$ if $k>2$. When $k=2$, we have $K(3,2)$, of which it is well known that the choice number is 2. See [9], for instance.

**Theorem C.** *(Kierstead [10]) Let $G$ denote the complete $k-$partite graph $K(3,3,3,\ldots,3)$. Then $ch(G)=\lceil\frac{(4k-1)}{3}\rceil$.*

Observe that this result implies that $ch(G)=k+1$ when $2\leq k\leq 4$ and $k+1<ch(G)<\frac{3k}{2}$ when $k\geq 5$.

## 2.1 Choice number $-$ A dispute resolved

### 2.1.1 Introduction

In 2002, Enomoto, Ohba, Ota, and Sakamoto published in [2] a proof that the choice number of the complete $k-$partite graph $G_k=K(4,2,\ldots,2)$ is $k$ if $k$ is odd and $k+1$ if $k$ is even. Recently Xu, Yang [16] claimed to detect errors in their proof, and that the choice number is $k+1$ in both cases. While Xu, Yang's proof is wrong, a certain amount of doubt has been cast on the proof of Enomoto, et al, which, while ingenious, is a proof by induction on $k$ with a complicated induction hypothesis, which should, and evidently does, arouse suspicion. While we judge the original proof to be valid, we offer here a lemma which we will use to simplify (but not shorten) the proof of Enomoto, et al, to establish beyond the shadow of a doubt that their result was correct in the first place. But first, here is an example of the choice number of a simple graph.

### 2.1.2 Example

Let $G$ denote a spanning subgraph of $K(2,2,2)$ depicted by Figure 2.1 with labeled edges. Let $H=L(G)$. Figure 2.2 is a drawing of $H$.

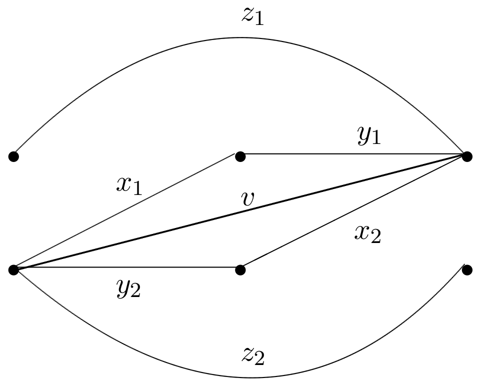


Figure 2.1: A spanning subgraph of K(2,2,2).

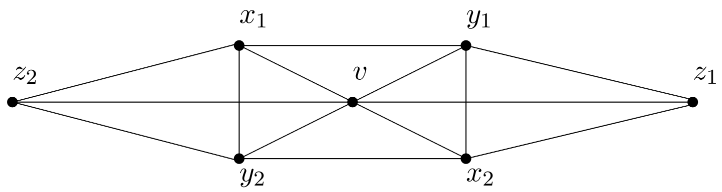


Figure 2.2: An induced subgraph of the line graph of K(2,2,2).

By defining the independent sets $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$, $V_3 = \{z_1, z_2\}$ and $V_4 = \{v\}$, we can deduce that $H$ is a subgraph of the complete $4-$partite graph $K(2, 2, 2, 2)$. Thus, $ch(H) \leq 4$ by Theorem A. Further, $4 = \chi(H) \leq ch(H)$. Hence $ch(H) = 4$.

### 2.1.3 The dispute resolution

Throughout this section we will assume the parts of $K(m, 2, \ldots, 2)$ to be $V_1, V_2, \ldots, V_k$ with $V_1 = \{x_1, x_2, \ldots, x_m\}$ and $V_i = \{u_i, v_i\}$, $i = 2, \ldots, k$. Also, in this section and beyond, in the remainder of this dissertation, we will frequently use the following notation: If $L$ is a list assignment to a graph $G$, and $S \subseteq V(G)$, then $L(S) = \bigcup\limits_{v \in S} L(v)$.

We first consider two important lemmas which will be used to simplify many of our arguments in this section.

**Lemma 2.1.** *(Enomoto, et al.[2]) Let $H$ denote the complete $r-$partite graph $K(m, 2, \ldots, 2)$ and $L$ a list assignment to $H$ such that*

i. $|L(w)| \geq r - 1$ for all $w \in V(H)$

ii. $|L(x)| \geq r$ for all $x \in V_1$

iii. $|L(w)| \geq r$ for at least one $w \in V_i$, $i = 2, \ldots, r$ and

iv. $L(x) \cap L(y) = \emptyset$ if $x, y \in V_i$, $x \neq y$, $1 \leq i \leq r$.

*Then $H$ is $L-$colorable.*

**Proof.**

Suppose there exists a nonempty set $S \subseteq V(H)$ such that $|L(S)| < |S|$. Let $t = |\{i \mid x_i \in S\}|$. By the assertions ii and iv, $rt \leq |L(S)|$. Then $rt \leq |L(S)| < |S| \leq 2(r-1) + t$. This implies that $(t-2)(r-1) < 0$. Hence $t \leq 1$. Therefore $|S| \leq 2(r-1) + 1 = 2r - 1$ and $|L(S)| \leq 2r - 2$. Thus, $S$ cannot contain both $u_i$ and $v_i$ for any $2 \leq i \leq r$. Hence $|S| \leq r$.

On the other hand, by the assertion i, $r - 1 \leq |L(S)|$ hence $r \leq |S|$. This implies that $t = 1$, meaning $S$ contains a vertex of $V_1$. Hence $r \leq |L(S)|$ and therefore $r < |S|$, a contradiction. Therefore $H$ is properly $L-$colorable, by Hall's theorem (see section 3.1); in fact, the complete graph on $V(H)$ is properly $L-$colorable.

□

**Lemma 2.2.** *(Allagan, Johnson [1] (2008)) Suppose $k \geq 2$. Suppose $A$, $B$ are disjoint $k-$sets of colors. Suppose $L$ is a list assignment to $G_k = K(4, 2, \ldots, 2)$ which assigns $A$ to each $u_i$, $B$ to each $v_i$ , $i = 2, \ldots, k$ and $|L(x_i)| \geq k$, $i = 1, 2, 3, 4$ . Suppose there is no proper $L-$coloring of $G_k$. Then $k$ is even, $|L(x_i)| = k$, $i = 1, 2, 3, 4$ and in fact $L$ is equivalent to $L_0$ which is of the form described by Enomoto et al (up to switching the roles* *of $A$ and $B$), as follows:*

1. $L_0(u_i) = A$ *and* $L_0(v_i) = B,$ *for every* $2 \leq i \leq k$ *and*
2. $L_0(x_1) = A_1 \cup A_3 \cup B_1,\ L_0(x_2) = A_1 \cup A_4 \cup B_2,$ $(**)$

   $L_0(x_3) = A_2 \cup A_4 \cup B_1$ *and* $L_0(x_4) = A_2 \cup A_3 \cup B_2.$

*where $B_1$, $B_2$ are $\frac{k}{2}$ sets partitioning $B$, and $A_1$, $A_2$, $A_3$, $A_4$ be disjoint sets of colors partitioning $A$ with $|A_1| = |A_2|$ and $|A_3| = |A_4|$.*

**Proof.**

Every proper $L-$coloring of $G_k - V_1 = K(2, \ldots, 2)$ uses $k-1$ elements of A and $k-1$ elements of B. They form a set $Q = C \bigcup D$, $C \subseteq A$, $D \subseteq B$ and $|C| = |D| = k-1$. There are $k^2$ such sets. Further, for each such set, there is a proper $L-$coloring of $G_k - V_1$ such that $Q$ is the set of colors appearing. Consider the pairs $(L(x_i), Q)$, $1 \leq i \leq 4$, such that $L(x_i) \subseteq Q$ and $Q$ is one of those $k^2$ sets. Since there is no proper $L-$coloring of $G_k$, each $Q$ appears at least once in such a pair. So there are at least $k^2$ such pairs. Now count the number of such $Q$ that $L(x_i)$ could be a subset of. Say $|L(x_i)| = y_i \geq k$ and $L(x_i)$ has $t$ elements from A, $y_i - t$ elements from B, $i = 1, 2, 3, 4$. (If $L(x_i)$ has an element not in $A \bigcup B$, then the number of $Q \subseteq A \bigcup B$ such that $L(x_i) \subseteq Q$ is zero.) Then $1 \leq t \leq\ k-1$, $1 \leq y - t \leq k-1$, because, otherwise, $A \subseteq L(x_i)$ or $B \subseteq L(x_i)$ and $L(x_i)$ is contained in no set $Q = C \bigcup D$. How many of the $Q = C \bigcup D$ is $L(x_i)$ contained in? Since $y = y_i \geq k$, the number is $(k-t)(k-(y-t)) = (k-t)(k+t-y) \leq (k-t)(k+t-k) = (k-t)t \leq k^2/4$ with equality in the last inequality iff $t = k/2$. The number of pairs is therefore at most $4(k^2/4) = k^2$. On the other hand, the number of pairs is at least $k^2$. Therefore the number of pairs is exactly $k^2$. Therefore, from the inequalities above, $k$ must be even, $t = k/2$ and $|L(x_i)| = k$, for each $i = 1, 2, 3, 4$. That is, each $L(x_i)$ is a $k-$set, with $k/2$ of its elements from A and $k/2$ of its elements from B. Further–and this will play a big role in deducing the form of the lists $L(x_i)$, $i = 1, 2, 3, 4$, each $Q = C \bigcup D$ contains one and only one of the

$L(x_i)$. In other words, for each $a \in A$, $b \in B$, thinking of $C = A \setminus \{a\}$, $D = B \setminus \{b\}$, exactly one of the $L(x_i)$ contains neither $a$ nor $b$.

We shall show that the $L(x_i)$ are of the required form after proving a short succession of claims.

**Claim 1** The intersection of any three $L(x_i)$, $i = 1, 2, 3, 4$ is empty.

**Proof.** Suppose without loss of generality that $a \bigcap L(x_1) \bigcap L(x_2) \bigcap L(x_3) \bigcap A$. Then for any $b \in B \bigcap L(x_4)$, there is no $L(x_i)$,containing neither $a$ nor $b$.

**Claim 2** No element of $A \bigcup B$ is in at most one of the $L(x_i)$, $i = 1, 2, 3, 4$.

**Proof.** Suppose $a \in A$ and $|\{i \mid a \in L(x_i)\}| \leq 1$. Without loss of generality, suppose $a \notin L(x_1) \bigcup L(x_2) \bigcup L(x_3)$. If $b \in B \bigcap L(x_4)$, then $b$ is not an element of at least two of $L(x_1)$ , $L(x_2)$, $L(x_3)$, otherwise b would be in the intersection of three $L(x_i)$'s. But then, there are two $L(x_i)$'s containing neither $a$ nor $b$.

**Claim 3** Every element of $A \bigcup B$ is in exactly two of the $L(x_i)$, $i = 1, 2, 3, 4$.

**Proof.** This result follows from the previous claims.

For $1 \leq i < j \leq 4$, let $A_{ij} = L(x_i) \bigcap L(x_j) \bigcap A$ and $B_{ij} = L(x_i) \bigcap L(x_j) \bigcap B$. By claim 1, the $A_{ij}$ are pairwise disjoint, and so are the $B_{ij}$. If $A_{ij} \neq \emptyset$ then $B_{ij} = \emptyset$, because, if $a \in A_{ij}$, $b \in B_{ij}$ then $a, b \notin L(x_t)$ for both $t \in \{1, 2, 3, 4\} \setminus \{i, j\}$, whereas we know that there is exactly one value of $t$ such that $a, b \notin L(x_t)$.

Consider $L(x_1) = A_{12} \bigcup A_{13} \bigcup A_{14} \bigcup B_{12} \bigcup B_{13} \bigcup B_{14}$.

Suppose $A_{12}, A_{13} \neq \emptyset$. Then $A_{14} = B_{12} = B_{13} = \emptyset$. This follows for $B_{12}, B_{13}$ from the remarks just above, and for $A_{14}$ because $B_{14}$ must be non-empty, since $L(x_1)$ contains $k/2$ elements of $B$. So, we can conclude that $|B_{14}| = k/2$ and $B_{24} = B_{34} = \emptyset$.

Similarly, if two of the $B_{1,j}$, $j \in \{2, 3, 4\}$, are non-empty, then two of the $A_{1,j}$ are empty and the other has $k/2$ elements. Without loss of generality, assume that $L(x_1) = A_{12} \bigcup A_{13} \bigcup B_{14}$ with $|B_{14}| = k/2 = A_{12} + A_{13}$; we are not assuming that both $A_{12}$ and $A_{13}$ are non-empty, but the fact that $|B_{14}| = k/2$ alone implies that $A_{14} = B_{12} = B_{13} = B_{24} =$

$B_{34} = \emptyset$, by previous remarks and the fact that all $k/2$ of $L(x_1)$ and $L(x_4)$'s elements from $B$ are in $B_{14}$.

**Case** 1: One of $A_{12}, A_{13}$ is empty. Say $A_{13} = \emptyset$. Then $|A_{12}| = k/2$ which implies that $A_{23} = A_{24} = \emptyset$. We have $L(x_1) = A_{12} \bigcup B_{14}$, $L(x_2) = A_{12} \bigcup B_{23}$, $L(x_3) = A_{34} \bigcup B_{23}$, $L(x_4) = A_{34} \bigcup B_{14}$. So $L$ is of the type described at the end of section1, with $A_{12}$, $A_{34}$ playing the roles of $A_1$,$A_2$ respectively, $A_3 = A_4 = \emptyset$, and $B_{14}$, $B_{23}$ playing the roles of $B_1$ and $B_2$.

**Case** 2: Suppose $A_{12} \neq \emptyset$ and $A_{13} \neq \emptyset$. Then $|A_{12}| + |A_{13}| = k/2$. Since, $L(x_2) = A_{12} \bigcup A_{23} \bigcup A_{24} \bigcup B_{23}$, $|B_{23}| = k/2$ and $|A_{23}| = \emptyset$. Hence $L(x_2) = A_{12} \bigcup A_{24} \bigcup B_{23}$, $L(x_3) = A_{13} \bigcup A_{34} \bigcup B_{23}$ and $L(x_4) = A_{24} \bigcup A_{34} \bigcup B_{14}$. Again we see that $L$ is equivalent to $L_0$ in $(**)$.

$\square$

**Theorem 2.1.** *( Enomoto et al. [2](2002)) Let $G_k$ denote the complete $k-$partite graph $K(4, 2, \ldots, 2)$. Then*

$$ch(G_k) = \begin{cases} k & if\ k\ is\ odd \\ k+1 & if\ k\ is\ even. \end{cases}$$

*Further, when $k$ is even, the only list assignments $L$ to $G_k$ satisfying $|L(v)| \geq k$ for each $v \in V(G_k)$ from which no proper coloring is possible are equivalent to $L_0$ previously described in $(**)$.*

Note that because $K(4, 2, \ldots, 2)$ is a subgraph of the complete $(k+1)-$partite graph $K(2, 2, \ldots, 2)$, it is clear that

$k = \chi(K(4, 2, \ldots, 2)) \leq ch(K(4, 2, \ldots, 2)) \leq ch(K(\underbrace{2, 2, \ldots, 2}_{k+1})) = k+1$, by Theorem A.

Therefore for all $k > 1$, $k \leq ch(K(4, 2, \ldots, 2)) \leq k+1$. $(*)$

**Proof that $ch(G_k) = k+1$ if $k$ is even.**

From the remark $(*)$, when $k$ is even it suffices to prove that $ch(K(4,2,\ldots,2)) > k$.

When $k$ is even, Enomoto et al. give a list assignment to $K(4,2,\ldots,2)$ with all lists of length (cardinality) $k$ from which no proper coloring is possible. Actually when $k \geq 2$ and $k$ is even, they give $\lfloor \frac{k}{4} \rfloor + 1$ essentially different such list assignments, of the form of $L_0$ in $(**)$.

We will go through the proof that when $k$ is even, this list assignment to $K(4,2,\ldots,2)$ does not permit a proper coloring in order to contrast the situation with a challenge to the claim of Theorem 2.1 when $k$ is odd; See claim A, below.

Clearly $|L_0(v)| = k$ for all $v \in V(G_k)$. Any attempt to color the vertices of $K(4,2,\ldots,2)$ from the list assignment $L_0$ will require $k-1$ colors from $A$ for the $u_i$ and $k-1$ colors from $B$ for the $v_i$, $i = 2,\ldots,k$. Hence for the vertices in $V_1$ there remains only one color $a \in A$, and one color $b \in B$. Let's assume that $a \in A_1$ and $b \in B_1$; then $x_4$ cannot be colored. Similarly if $a \in A_1$ and $b \in B_2$, we cannot color $x_3$; and, similarly, for each of the remaining 6 cases $a \in A_i$, $b \in B_j$, some $x_t \in V_1$ cannot be colored. Thus, there is no proper $L_0-$coloring of $G_k$ and all lists are of length $k$. So, $ch(G_k) = k+1$ if $k$ is even by the remark $(*)$.

**Digression.**

**Claim A**( Xu, Yang [16](2007)) *$ch(K(4,2,\ldots,2)) = k+1$ for all $k > 1$.*

This claim contradicts the assertion of Theorem 2.1 when $k$ is odd.

In an attempt, albeit unsuccessful, to prove this claim when $k$ is odd, Xu, Yang defined the following:

*Let $A$ and $B$ be disjoint sets of colors with $|A| = k+1$ and $|B| = k-1$. Let $A_1, A_2, A_3, A_4$ be disjoint sets of colors partitioning $A$ such that $|A_1| = |A_2|$ and $|A_3| = |A_4|$. Let $B_1$, $B_2$ be $(k-1)/2$ sets partitioning $B$ and let $0$ be a color in $A$.* It is claimed that $G_k$ has no proper $L'-$coloring where $L'$ is a list assignment to $G_k$ defined as follows:

*1.* $L'(u_i) = A - \{0\}$ *and* $L'(v_i) = B \cup \{0\}$*, for every* $2 \le i \le k$ *and*

*2.* $L'(x_1) = A_1 \cup A_3 \cup B_1$*,* $L'(x_2) = A_1 \cup A_4 \cup B_2$*,*
$L'(x_3) = A_2 \cup A_4 \cup B_1$ *and* $L'(x_4) = A_2 \cup A_3 \cup B_2$*.*

It is clear that $|L'(u_i)| = |L'(v_i)| = k$ for every $2 \le i \le k$ and $|L'(x_i)| = \frac{(k+1)}{2} + \frac{(k-1)}{2} = k$ for all $1 \le i \le 4$. Without loss of generality we can assume that $0 \in A_1$.

We show that $G_k$ has a proper $L'-$coloring $\psi$, given any such list assignment $L'$.

Any proper coloring of the subgraph $K(2, \ldots, 2)$ of $G_k$ from the list assignment $L'$ will require $k-1$ colors from $A - \{0\}$ for the $u_i$ and $k-1$ colors from $B \cup \{0\}$ for the $v_i$, $i = 2, \ldots, k$. Color the $u_{i's}$ with a $k-1$ subset of the $k-$color set $A - \{0\}$ and color the $v_{i's}$ with the $k-1$ colors of the set $B$. There remain unused exactly one color of $A - \{0\}$, say $c$, and 0 (since 0 did not appear as a color on any of the $u_i$ and $v_i$, $i = 2, \ldots, k$) to color the vertices of $V_1$. By letting $c \in A_2$, we can have $\psi(x_3) = c = \psi(x_4)$ and $\psi(x_1) = 0 = \psi(x_2)$ (since $0 \in A_1$), giving a proper coloring of $G_k$.

Thus, the assertion of Xu, Yang about $ch(K(4, 2, \ldots, 2))$ , when $k$ is odd, is not proven by their list assignment, and the temptation is to dismiss it.

Theorem 3 of [2] reads: *Suppose that* $L$ *is a list assignment of* $G = K(4, 2, \ldots, 2)$ *such that* $|L(v)| \ge k$ *for each* $v \in V(G)$*. If* $G$ *is not* $L-$*colorable, then* $L$ *is essentially equivalent to* $L_0$ $(**)$*, namely, there exists a bijection* $\tau$ *of colors and automorphism* $\varphi$ *of* $G$ *satisfying* $\tau \circ L \circ \varphi = L_0$.

The proof in [2] is by induction on $k$, using the full conclusion of the theorem as the induction hypothesis. To put it another way, the theorem and the induction hypothesis could be: If $k \ge 1$ is odd, then $G_k$ is $k-$choosable, and if $k \ge 2$ is even then the only list assignments $L$ to $G_k$ such that $|L(w)| \ge k$ for all $w \in V(G_k)$, and there is no proper $L-$coloring of $G_k$, are equivalent to $L_0$ in $(**)$.

**Resumption of the proof of Theorem 2.1**

We go by induction on $k$, assuming as an induction hypothesis that $ch(G_k) = k$ when $k$ is odd. The case when $k = 1$ is trivial. Assume $k \geq 2$. Let $L$ be a list assignment to $G_k = K(4, 2, \ldots, 2)$ such that $|L(v)| \geq k$ for each $v \in V(G)$. Suppose that $G_k$ has no proper $L-$coloring. We proceed by induction on $k$, but our only induction hypothesis is that for $k$ odd, $ch(G_k) = k$. We will prove this and that when $k$ is even, $L(u_1) = \ldots = L(u_k) = A$, $L(v_1) = \ldots = L(v_k) = B$ (after renaming within each part, possibly), where $A$ and $B$ are disjoint $k-$sets. Then the conclusion for even $k$ follows from Lemma 2.2 .

**Claim 2.1.1.** $\bigcap\limits_{x \in V_1} L(x) = \emptyset$

Suppose that $c \in \bigcap\limits_{x \in V_1} L(x)$. Let $L' = L - \{c\}$. Color the vertices in $V_1$ with $c$. Then, $G_k - V_1 \cong K(2, 2, \ldots, 2)$ has a proper $L'-$coloring since $|L'(v)| \geq k - 1$ for every $v \in V(G_k - V_1)$. Thus, $G_k$ has a proper $L-$coloring, a contradiction.

We note here that the proof of Claim 2 in [2] (p.58) assumes as part of the induction hypothesis that when $k$ is even, only for $L$ of form $L_0$ is $G_k$ not properly $L-$colorable.

Our aim here is to give the proof without this as part of the induction hypothesis. Our proof will be longer, but more credible. What follows is Claim 2 in [2], with a different proof.

**Claim 2.1.2.** $L(u_i) \cap L(v_i) = \emptyset$ *for each* $i \geq 2$.

When $k = 2$ this follows easily from the non $L-$colorability of $G_k$. So, we assume $k \geq 3$. Suppose there is a color $c \in L(u_k) \cap L(v_k)$ . Color both $u_k$ and $v_k$ with $c$. Let $G_{k-1} = G_k - V_k$ and $L' = L - \{c\}$. Then the list assignment $L'$ satisfies the following assertions:

$(a)$ $|L'(v)| \geq k - 1$ since $|L(v)| \geq k$ for each $v \in V(G)$.

$(b)$ $G_{k-1}$ has no proper $L'-$coloring since $G_k$ has no proper $L-$coloring.

$(c)$ $L'(u_i) \cap L'(v_i) = \emptyset$ for each $2 \leq i \leq k - 1$: If $k - 1$ were odd, then, by $(a)$ and the induction hypothesis, $G_{k-1}$ would be properly $L'-$colorable, contradicting $(b)$. Therefore,

$k-1$ is even. If, say, $c' \in L'(u_{k-1}) \cap L'(v_{k-1})$, color $u_{k-1}$, $v_{k-1}$ with $c'$, set $L'' = L' - \{c'\}$ on $G_{k-2} = G_{k-1} - V_{k-1}$; but then $|L''(v)| \geq k-2$ for each $v \in V(G_{k-2})$, and $k-2$ is odd, so $G_{k-2}$ is properly $L''-$colorable by the inductive hypothesis. So $(b)$ is contradicted again.

$(d)$ $|L'(x_j)| \geq k$ for some $1 \leq j \leq 4$ since $c$ cannot appear on all lists in $V_1$ by Claim 2.1.1.

$(e)$ The intersection of any three lists on $V_1$ is empty.

Suppose $\tilde{c} \in L'(x_1) \cap L'(x_2) \cap L'(x_3)$. Then, we color $x_1, x_2, x_3$ with $\tilde{c}$ and the subgraph $G_{k-1} - \{x_1, x_2, x_3\}$ with the list assignment $L' - \{\tilde{c}\}$ satisfies the hypothesis of Lemma 2.1, by assertion $(c)$ and Claim 2.1.1. Therefore $G_{k-1}$ is $L'-$colorable, a contradiction.

$(f)$ $L'(x_i) \cap L'(x_j) \neq \emptyset$ for some $i \neq j$; Otherwise $L'(x_i) \cap L'(x_j) = \emptyset$ for each $i \neq j$. Then $G_{k-1}$ with $L'$ satisfies the hypothesis of Lemma 2.1, and so $G_k$ is properly $L'-$colorable, a contradiction.

$(g)$ If $a \in L'(x_1) \cap L'(x_2)$ then there exists $b \in L'(x_3) \cap L'(x_4)$.

Suppose $L'(x_3) \cap L'(x_4) = \emptyset$. Let $L'' = L' - a$ on $V_2 \cup \ldots \cup V_{k-1}$ and $L''(x_i) = L'(x_i)$, $i = 3, 4$. Then, the subgraph $G_{k-1} - \{x_1, x_2\}$ satisfies the hypothesis of Lemma 2.1 and so $G_{k-1} - \{x_1, x_2\}$ is properly $L''-$colorable, and thus $G_{k-1}$ is properly $L'-$colorable, a contradiction.

$(h)$ $L'(V_1) \subseteq L'(V_2 \cup \ldots \cup V_{k-1})$. Suppose there is a color $c \in L'(x_i)$ for some $i$ and $c \notin L'(V_2 \cup \ldots \cup V_{k-1})$. Color $x_i$ with $c$. Then $|L'(x_j)| \geq k-1$ for each $j \neq i$ and $|L'(v)| \geq k-1$ for each $v \in V_2 \cup \ldots \cup V_{k-1}$. By Corollary B, the subgraph $G - x_i \cong K(3, 2 \ldots, 2)$ has a proper $L'-$coloring, and therefore $G_{k-1}$ does also, a contradiction.

We use the previous assertions to prove the following subclaims.

**Subclaim 1:** If $\sigma \in L'(x_1) \cap L'(x_2)$ and $\tau \in L'(x_3) \cap L'(x_4)$ then $\{\sigma, \tau\} \subset B'$ for some $(k-1)-$set of colors $B'$, which is one of the lists on $V_i$ for each $i = 2, \ldots, k-1$.

**Proof.** Note that $\sigma \neq \tau$, by Claim 2.1.1.

Color $x_1, x_2$ with $\sigma$ and $x_3, x_4$ with $\tau$. Let $L'' = L' - \{\sigma, \tau\}$ and $H = G_{k-1} - V_1$. Then $|L''(v)| \geq k-3$ for each $v \in V(H)$. Because $L'(v_i) \cap L'(u_i) = \emptyset$, $i = 2, \ldots, k-1$, for each

$i \in \{2, \ldots, k-1\}$, either $|L''(u_i)|$, $|L''(v_i)| \geq k-2$, or one of $|L''(u_i)|$, $|L''(v_i)|$ is equal to $k-3$ and the other is greater than or equal to $k-1$. If, say, $|L''(u_2)|$, $|L''(v_2)| \geq k-2$, then we can apply Lemma 2.1 to the complete $(k-2)-$partite graph $H \cong K(2, \ldots, 2)$, with $V_2$ playing the role of the first part, to conclude that $H$ is properly $L''-$colorable, a contradiction. Therefore, in each of $V_2, \ldots, V_{k-1}$, the $L'$ list on one of the vertices , say $L'(v_j)$, contains $\sigma$ and $\tau$, and we have $|L''(v_i)| \geq k-3$, $|L''(u_i)| \geq k-1$.

Since $H$ has no proper $L''-$coloring, there exists a nonempty set $S \subseteq V(H)$ such that $|L''(S)| \leq |S| - 1$. (This is by Hall's theorem; see section 3.1.) Suppose $V_j \subseteq S$ for some $j \geq 2$.

Then $k-1+k-3 = 2(k-2) \leq |L''(u_j)| + |L''(v_j)| \leq |L''(S)| \leq |S| - 1 \leq 2(k-2) - 1$, a contradiction. So, $S$ contains at most one vertex of $V_j$ for each $j \geq 2$. Thus, $|S| \leq k-2$. Further, since $|L''(v)| \geq k-3$ for each $v \in V(H)$, $|S| = k-2$ and $|L''(S)| = k-3$. Thus, $S = \{v_2, \ldots, v_{k-1}\}$ and $L''(v_2) = \ldots = L''(v_{k-1})$. Further, since $L'' = L' - \{\sigma, \tau\}$, we can conclude that $L'(v_2) = \ldots = L'(v_{k-1}) = B'$ where $B'$ is a $(k-1)-$set of colors. As noted previously, $\sigma$, $\tau \in B'$.

**Corollary:** By the assertions $(f)$, $(g)$, and $(e)$, and Subclaim 1, there is such a set $B'$.

**Subclaim 2:** $L'(u_2) = \ldots = L'(u_{k-1}) = A'$, for some $(k-1)-$set of colors $A'$.

**Proof.**

Let $\sigma \in L'(x_1) \cap L'(x_2)$ and $B' = L'(v_i)$, $i = 2, \ldots, k-1$ be as in Subclaim 1 and its corollary, so $\sigma \in B'$ and $|B'| = k-1$. Then, $L'(x_3) \setminus B' \neq \emptyset$ and $L'(x_4) \setminus B' \neq \emptyset$ since $\sigma \notin L'(x_3) \cup L'(x_4)$ by the assertion $(e)$. Hence there exist colors $\alpha, \beta \notin B'$ such that $\alpha \in L'(x_3)$ and $\beta \in L'(x_4)$. Consider $x_1$ and $x_2$ to be colored with $\sigma$, $x_3$ with $\alpha$, and $x_4$ with $\beta$. Define $G' = G_{k-1} - V_1$ and $L'' = L' - \{\sigma, \alpha, \beta\}$. Since $G'$ is not $L''-$colorable, there exists a nonempty set $S \subseteq V(G')$ such that $|L''(S)| \leq |S| - 1$.

Suppose $V_j \subseteq S$ for some $j \geq 2$.

Then $(k-1) - 2 + (k-1) - 1 = 2k-5 \leq |L''(u_j)| + |L''(v_j)| \leq |L''(S)| \leq |S| - 1 \leq 2(k-2) - 1$. This implies that $|S| = 2(k-2)$ and $|L''(S)| = 2k-5$. It follows that

$|L''\{u_2, \ldots, u_{k-1}\}| = k - 3$ and thus that $L'(u_2) = \ldots = L'(u_{k-1}) = L''(u_i) \cup \{\alpha, \beta\}$. From there and the fact that $|L'(u_i)| \geq k - 1$, $i = 2, \ldots, k - 1$, it follows that $L'(u_i) = L''(u_i) \cup \{\alpha, \beta\} = A'$, $i = 2, \ldots, k - 1$, a $(k - 1)-$set.

On the other hand, if $S$ cannot contain both $u_j$ and $v_j$ for for any $j \geq 2$, then $(k-1)-2 \leq |L''(S)| \leq |S| - 1 \leq (k - 2) - 1$. This implies that $|S| = (k - 2)$ and $|L''(S)| = (k - 1) - 2 = |L''(u_j)|$ for every $j \geq 2$. We can once again conclude that $L'(u_2) = \ldots = L'(u_{k-1}) = A'$, for some $(k - 1)-$set of colors $A'$.

Because $|L'(x_i)| = k$ for some $i \in \{1, 2, 3, 4\}$, it now follows from Lemma 2.2, with $k$ there replaced by $k - 1$, that $G_{k-1}$ is properly $L'-$colorable after all, a contradiction. This establishes Claim 2.1.3, meaning $L(u_i) \cap L(v_i) = \emptyset$ for each $i = 2, \ldots, k$.

□

We proceed to prove the following sequences of claims which are very similar to the ones in [2]. They can also be easily derived as were the assertions $(e)$, $(f)$, $(g)$, $(h)$ in Claim 2.1.1 and the Subclaims 1 and 2 by letting $L$ and $k$ play the roles of $L'$ and $k-1$ respectively, and letting Claim 2.1.2 play the role played by $(c)$ in the proof of Claim 2.1.2.

**Claim 2.1.3.** *The intersection of any three lists in $V_1$ is empty.*

Suppose $c \in L(x_1) \cap L(x_2) \cap L(x_3)$. Then, we color $x_1, x_2, x_3$ with $c$ and the subgraph $G_k - \{x_1, x_2, x_3\}$ with $L' = L - \{c\}$ satisfies the hypothesis of Lemma 2.1, by Claim 2.1.2; therefore $G_k$ is properly $L-$colorable, a contradiction.

**Claim 2.1.4.** $L(x_i) \cap L(x_j) \neq \emptyset$ *for some* $i \neq j$.

Otherwise $L(x_i) \cap L(x_j) = \emptyset$ for each $i \neq j$. Thus, $G_k$ with $L$ satisfies the hypothesis of Lemma 2.1, and so $G_k$ is properly $L-$colorable, a contradiction.

**Claim 2.1.5.** *If* $a \in L(x_1) \cap L(x_2)$ *then there exists* $b \in L(x_3) \cap L(x_4)$.

Suppose $L(x_3) \cap L(x_4) = \emptyset$. Let $L' = L - a$ on $V_2 \cup \ldots \cup V_k$ and $L'(x_i) = L(x_i)$, $i = 3, 4$ with $L'$. Then, the subgraph $G_k - \{x_1, x_2\}$ satisfies the hypothesis of Lemma 2.1 and so $G_k - \{x_1, x_2\}$ is properly $L'-$colorable, and thus $G_k$ is properly $L-$colorable, a contradiction.

**Claim 2.1.6.** $L(V_1) \subseteq L(V_2 \cup \ldots \cup V_k)$.

Suppose there is a color $c \in L(x_i)$ for some $i$ and $c \notin L(V_2 \cup \ldots \cup V_k)$. Color $x_i$ with $c$. Then $|L(x_j)| \geq k$ for each $j \neq i$ and $|L(v)| \geq k$ for each $v \in V_2 \cup \ldots \cup V_k$. By corollary B, the subgraph $G - x_i \cong K(3, 2 \ldots, 2)$ has a proper $L-$coloring, a contradiction.

**Subclaim 3:** If $\sigma \in L(x_1) \cap L(x_2)$ and $\tau \in L(x_3) \cap L(x_4)$ then $\{\sigma, \tau\} \subset B$ for some $k-$set of colors $B$, which is one of the lists on $V_i$ for each $i = 2, \ldots, k$.

**Proof.**

Suppose $\sigma \in L(x_1) \cap L(x_2)$ and $\tau \in L(x_3) \cap L(x_4)$. Then color $x_1, x_2$ with $\sigma$ and $x_3, x_4$ with $\tau$. Let $L' = L - \{\sigma, \tau\}$ and $H = G_k - V_1$. Then $|L'(v)| \geq k - 2$ for each $v \in V(H)$. Because $L(v_i) \cap L(u_i) = \emptyset$, $i = 2, \ldots, k$, for each $i \in \{2, \ldots, k\}$, either $|L'(u_i)|$, $|L'(v_i)| \geq k - 1$, or one of $|L'(u_i)|$, $|L'(v_i)|$ is equal to $k - 2$ and the other is greater or equal to $k$. If, say, $|L'(u_2)|$, $|L'(v_2)| \geq k - 1$, then we can apply Lemma 2.1 to the complete $(k-1)-$partite graph $H \cong K(2, \ldots, 2)$, with $V_2$ playing the role of the first part, to conclude that $H$ is properly $L'-$colorable, a contradiction. Therefore, in each of $V_2, \ldots, V_k$, the $L$ list on one of the vertices , say $L(v_j)$, contains $\sigma$ and $\tau$, and we have $|L'(v_i)| \geq k - 2$, $|L'(u_i)| \geq k$, $i = 2, \ldots, k$.

Since $H$ has no proper $L'-$coloring, there exists a nonempty set $S \subseteq V(H)$ such that $|L'(S)| \leq |S| - 1$. Suppose $V_j \subseteq S$ for some $j \geq 2$.

Then $k + k - 2 = 2(k - 1) \leq |L'(u_j)| + |L'(v_j)| \leq |L'(S)| \leq |S| - 1 \leq 2(k - 1) - 1$, a contradiction. So, $S$ contains at most one vertex of $V_j$ for each $j \geq 2$. Thus, $|S| \leq k - 1$. Further, since $|L'(v)| \geq k - 2$ for each $v \in V(H)$, $|S| = k - 1$ and $|L'(S)| = k - 2$. Thus, $S = \{v_2, \ldots, v_k\}$ and $L'(v_2) = \ldots = L'(v_k)$. Further, since $L' = L - \{\sigma, \tau\}$, we can conclude that $L(v_2) = \ldots = L(v_k) = B$ where $B$ is a $k-$set of colors. As noted previously, $\sigma$, $\tau \in B$.

**Corollary:** By Claims 2.1.2 , 2.1.4, 2.1.5, and Subclaim 3, there is such a set $B$.

**Subclaim 4:** $L(u_2) = \ldots = L(u_k) = A$, for some $k-$set of colors $A$.

**Proof.**

Let $\sigma \in L(x_1) \cap L(x_2)$ and $B = L(v_i)$, $i = 2, \ldots, k$ be as in Subclaim 3 and its corollary, so $\sigma \in B$ and $|B| = k$. Then, $L(x_3) \setminus B \neq \emptyset$ and $L(x_4) \setminus B \neq \emptyset$ since $\sigma \notin L(x_3) \cup L(x_4)$ by Claim 2.1.3 . Hence there colors $\alpha, \beta \notin B$ such that $\alpha \in L(x_3)$ and $\beta \in L(x_4)$. Color $x_1$ and $x_2$ with $\sigma$, $x_3$ with $\alpha$, and $x_4$ with $\beta$. Define $G' = G_k - V_1$ and $L' = L - \{\sigma, \alpha, \beta\}$ as we color the vertices in $V_1$. Since $G'$ is not $L'-$colorable, there exists a nonempty set $S \subseteq V(G')$ such that $|L'(S)| \leq |S| - 1$.

Suppose $V_j \subseteq S$ for some $j \geq 2$.

Then $k-2+k-1 = 2k-3 \leq |L'(u_j)|+|L'(v_j)| \leq |L'(S)| \leq |S|-1 \leq 2(k-1)-1$. This implies that $|S| = 2(k-1)$ and $|L'(S)| = 2k-3$. It follows that $|L'\{u_2, \ldots, u_k\}| = k-2$. From there and the fact that $|L(u_i)| \geq k$, $i = 2, \ldots, k$, it follows that $L(u_i) = L'(u_i) \cup \{\alpha, \beta\} = A$, $i = 2, \ldots, k$, a $k-$set.

On the other hand, if $S$ cannot contain both $u_j$ and $v_j$ for for any $j \geq 2$, then $(k)-2 \leq |L'(S)| \leq |S|-1 \leq (k-1)-1$. This implies that $|S| = (k-1)$ and $|L'(S)| = k-2 = |L'(u_j)|$ for every $j \geq 2$. We can once again conclude that $L(u_2) = \ldots = L(u_k) = A$, for some $k-$set of colors $A$.

Thus, we have shown that $L(u_i) = A$, $L(v_i) = B$ for each $2 \leq i \leq k$, and by Claim 2.1.2, $A \cap B = \emptyset$. Thus, we established the hypothesis of Lemma 2.2 with the list assignment $L$ to $G_k$ satisfying that $|L(v)| \geq k$ for each $v \in V(G_k)$ and there is no proper $L-$coloring. By the conclusion of Lemma 2.2, $k$ must be even and $L$ must be equivalent to $L_0$ in (**). This concludes the proof of Theorem 2.1.

**Corollary 2.1.1.** *( Enomoto et al. [2](2002)) Let $G$ denote the complete $k-$partite graph $K(4, 2, \ldots, 2, 1)$. Then $ch(G) = k$.*

**Proof.**

When $k$ is odd, it is clear by Theorem 2.1 that $ch(G) = k$ since $k = \chi(K(4, 2, \ldots, 2, 1) \leq ch(K(4, 2, \ldots, 2, 1)) \leq ch(K(4, 2, \ldots, 2, 2)) = k$. When $k$ is even, the subgraph $G - v$ is $(k-1)-$choosable, where $v$ is the vertex of the part of size 1 . Hence $k = \chi(G) \leq ch(G) \leq k = k - 1 + 1$, again invoking Theorem 2.1. Therefore $ch(G) = k$ for all $k > 1$.

□

Later in [16]( p. 61), Xu, Yang concluded that $ch(K(4, 3, 2\ldots, 2)) = k+1$ for all $k > 1$. This conclusion was based on their erroneous Claim A.

There is no doubt that $ch(K(4, 3, 2, \ldots, 2)) \leq k + 1$ for all $k > 1$, since the $k-$partite graph $K(4, 3, 2, \ldots, 2)$ is a subgraph of the complete $(k+1)-$partite graph $K(4, 2, \ldots, 2, 1)$ the choice number of which is $k + 1$ by corollary 2.1.1.

Now, when $k$ is even, $k + 1 = ch(K(4, 2, \ldots, 2)) \leq ch(K(4, 3, 2, \ldots, 2)) \leq k + 1$. Thus, the assertion $ch(K(4, 3, 2\ldots, 2)) = k + 1$ is true when $k$ is even.

However, when $k$ is odd, it would have to be shown that $ch(K(4, 3, 2\ldots, 2)) > k$, something that Xu, Yang did not show, because their correction of Theorem 2.1 was invalid. We would have to provide a list assignment $L$ with $|L(v)| \geq k$, $v \in V(K(4, 3, 2\ldots, 2))$, for which there is no proper $L-$coloring of $K(4, 3, 2\ldots, 2)$. (I personally do not think there is such list assignment.)

**Theorem 2.2.** *(Enomoto et al.[2]) Let $G$ denote the complete $k-$partite graph $K(5, 2, \ldots, 2)$. Then $ch(G) = k + 1$.*

**Proof.**

$G$ is a subgraph of the complete $(k + 1)-$partite graph $K(3, 2, 2, \ldots, 2)$, the choice number of which is $k + 1$ by corollary B. Hence $ch(G) \leq k + 1$.

When $k$ is even, $k + 1 = ch(K(4, 2, \ldots, 2)) \leq ch(K(5, 2, \ldots, 2))$. Hence $ch(G) = k + 1$.

When $k$ is odd, Enomoto et al. gave the following list assignment:

Let $A$ and $B$ be disjoint $k - 1$ sets of colors. Let $A_1$, $A_2$ and $B_1$, $B_2$ be disjoint $\frac{k-1}{2}$ sets of colors partitioning $A$ and $B$ respectively. Further, let $C \subset A \cup B$ with $|C| = k$ and the color $0 \notin A \cup B$. Define an assignment $L$ of $G$ as follows:

1. $L(u_i) = A \cup \{0\}$ and $L(v_i) = B \cup \{0\}$, for every $2 \leq i \leq k$ and
2. $L(x_1) = A_1 \cup B_1 \cup \{0\}$, $L(x_2) = A_1 \cup B_2 \cup \{0\}$, $L(x_3) = A_2 \cup B_1 \cup \{0\}$, $L(x_4) = A_2 \cup B_2 \cup \{0\}$ and $L(x_5) = C$.

It is not hard to see that there is no proper $L-$coloring of $G$. Thus, $ch(G) > k$.

**Corollary 2.2.1.** *Let $G$ denote the complete $k-$partite graph $K(6, 2, \ldots, 2)$. Then $ch(G) = k + 1$.*

**Proof.**

Since $k + 1 = ch(K(5, 2, \ldots, 2)) \leq ch(K(6, 2, \ldots, 2))$, it is clear that $ch(G) \geq k + 1$. Further, $G$ is a subgraph of the complete $(k + 1)-$partite graph $K(3, 3, 2, \ldots, 2)$ which has choice number $k + 1$ by Theorem B. Thus, $ch(G) \leq k + 1$. So, $ch(G) = k + 1$.

## 2.2 Choice numbers and Ohba numbers

### 2.2.1 Introduction

In 2002, Ohba [11] proved that for any given graph $G$, there exists an integer $n_0$ such that for any $n \geq n_0$, the join $G \vee K_n$ satisfies $ch(G \vee K_n) = \chi(G \vee K_n)$.

The **Ohba number** of $G$ is the number $\phi(G)$ defined to be the smallest integer $n$ for which $ch(G \vee K_n) = \chi(G \vee K_n)$. In particular when $G$ is chromatic$-$choosable, $\phi(G) = 0$.

Observe that $|V(G \vee K_n)| \leq 2\chi(G \vee K_n) + 1$ if and only if $n \geq |V(G)| - 2\chi(G) - 1$. Now, Ohba's conjecture[11] states that *if $|V(G)| \leq 2\chi(G) + 1$, then $G$ is chromatic$-$choosable.* Thus, Ohba's conjecture would imply that

$\phi(G) \leq max(0, |V(G)| - 2\chi(G) - 1) \leq max(0, |V(G)| - 5)$ for every graph $G$ with an edge.

Conversely, if $\phi(G) \leq max(0, |V(G)| - 2\chi(G) - 1)$ for all $G$ then Ohba's conjecture is true. It is further clear that Ohba's conjecture is true for every graph of order at most 5, since Figure 1.1 is known to be the smallest graph that is not chromatic$-$choosable, and it is of order 6.

We present here findings of Ohba numbers of some complete $k-$partite graphs.

**Proposition 2.1.** *For any graph $G$, $\phi(G) \geq ch(G) - \chi(G)$.*

**Proof.**

If $G$ is chromatic$-$choosable, by the definition $\phi(G) = ch(G) - \chi(G) = 0$.

Suppose $G$ is not chromatic−choosable. Then $ch(G) > \chi(G)$. Let $s$ be the smallest positive integer such that $ch(G \vee K_s) = \chi(G \vee K_s)$. Since $\chi(G \vee K_s) = \chi(G) + s$, this implies that $s = ch(G \vee K_s) - \chi(G)$. Further, $ch(G) \leq ch(G \vee K_s)$ for all $s \geq 1$. So, $s \geq ch(G) - \chi(G)$. Thus, $\phi(G) \geq ch(G) - \chi(G)$.

□

**Proposition 2.2.** *Let $G$ denote the complete $k-$partite graph $K(4, 2, 2, \ldots, 2)$. Then*

$$\phi(G) = \begin{cases} 0 & if\ k\ is\ odd \\ 1 & otherwise. \end{cases}$$

**Proof.**

From Theorem 2.1, when $k$ is odd, $ch(G) = k = \chi(G)$. Thus, $\phi(G) = 0$ by definition. When $k$ is even, $ch(G) > k$. Further, $ch(G \vee K_1) = ch(K(4, 2, 2, \ldots, 2, 1)) = k + 1 = \chi(G \vee K_1)$ by corollary 2.1.1. Thus, $\phi(G) = 1$ when $k$ is even.

□

### 2.2.2 Choice numbers and Ohba numbers of K(m, n, 1,. . . , 1)

We present the choice numbers of the complete $k-$partite graphs $K(m, n, 1, \ldots, 1)$ for various values of $1 \leq n \leq m$ and their corresponding Ohba numbers. Pretty clearly, if $k-2 \leq \phi(K(m, n))$ then $\phi(K(m, n, 1, \ldots, 1)) = \phi(K(m, n)) - (k - 2)$. So, $\phi(K(m, n, 1, \ldots, 1)) = max\{0, \phi(K(m, n)) - (k - 2)\}$. Consequently, we just need $\phi(K(m, n))$.

Throughout this section, we denote the parts of the complete $k-$ partite graph $K(m, n, 1, \ldots, 1)$ by $V_1, V_2, \ldots, V_s$ where $V_1 = \{x_1, \ldots, x_m\}$, $V_2 = \{y_1, y_2, \ldots, y_n\}$ and $V_s = \{v_s\}$ for $s = 3, \ldots, k$.

**Theorem 2.3.** *Let $G$ denote the complete $k-$partite graph $K(m, n, 1, \ldots, 1)$. Then $ch(G) \leq n + k - 1$ for all $1 \leq n \leq m$.*

**Proof.**

When $k = 2$, it is shown in [9] that $ch(G) \le n+1$ for all $m \ge n$. The proof for arbitrary $k \ge 2$ will be similar. We denote $G' = G - V_1$, where $V_1$ is the part of $G$ of size $m$, and let $L$ be a list assignment to $G$ with $|L(v)| \ge n + k - 1$ for each $v \in V(G)$. Any proper $L-$coloring of $G'$ uses at most $n + k - 2$ distinct colors, say $\alpha_1, \dots, \alpha_{n+k-2}$. Thus, for each $v \in V_1$, $|L(v) - \{\alpha_1, \dots, \alpha_{n+k-2}\}| \ge 1$, and so $G$ is $L-$colorable. Hence $ch(G) \le n + k - 1$.

□

**Corollary 2.3.1.** *Let $G$ denote the complete $k-$partite graph $K(m, 1, 1, \dots, 1)$. Then $ch(G) = k$ for all $m \ge 1$.*

**Proof.**

From theorem 2.3 (when $n = 1$), $ch(K(m, 1, 1, \dots, 1) \le k + 1 - 1 = k$. Further, $k = \chi(K(m, 1, \dots, 1)) \le ch(K(m, 1, 1, \dots, 1))$, so we can conclude that $ch(K(m, 1, 1, \dots, 1)) = k$. It is fair to point out that this result also follows from the fact that $\chi(\overline{K}_m) = ch(\overline{K}_m) = 1$.

□

**Lemma 2.3.** *Let $H$ denote the complete $(k-1)-$partite graph $K(2, 1, \dots, 1)$ with parts $V_1 = \{y_1, y_2\}$, $V_s = \{v_s\}$, for each $s = 2, \dots, k-1$. Let $L$ be a list assignment to $H$ satisfying that $L(y_1) = A$ and $L(y_2) = B$ for some disjoint $k-$sets of colors $A$ and $B$, and $|L(w)| \ge k$ for each $w \in V(H)$. Then the number of different color sets arising from proper $L-$colorings of $H$ is at least $\dfrac{k^2 + 3k}{2}$.*

**Proof.**

Let $K_{k-2} \cong H - V_1$ and $\mathcal{C}_{i,j} = \{$ *color sets from proper $L-$colorings of $K_{k-2}$ with $i$ element(s) from $A$, $j$ element(s) from $B$* $\}$, with $0 \le i, j \le k-2$ and $i + j \le k - 2$.

Claim 1. $\displaystyle\sum_{\substack{0 \le i,j \le k-2 \\ i+j \le k-2}} c_{i,j} \ge \binom{k}{2}$ where $c_{i,j} = |\mathcal{C}_{i,j}|$.

The number of proper $L-$colorings of $K_{k-2}$ is at least $k(k-1)\dots(k-(k-3)) = k(k-1)\dots 3 = \dfrac{k!}{2}$. Further, since each color set appears at most $(k-2)!$ times, the number of distinct color sets arising from the proper $L-$colorings is at least $\dfrac{k!}{2(k-2)!}$ , meaning

$$\sum_{\substack{0 \le i,j \le k-2 \\ i+j \le k-2}} c_{i,j} \ge \frac{k!}{2(k-2)!} = \binom{k}{2}.$$

Define $\mathcal{D}_{p,q} = \{\, color\ sets\ from\ proper\ L{-}colorings\ of\ H\ with\ p\ element(s)\ from\ A,\ q\ element(s)\ from\ B\ \}$, with $1 \leq p, q \leq k$, $p + q \leq k$ and let $d_{p,q} = |\mathcal{D}_{p,q}|$. Then the total number of color sets from proper $L-$colorings of $H$ is $\sum_{\substack{1\leq p,q\leq k\\ p+q\leq k}} d_{p,q}$. Since any coloring of $H$ uses exactly one color from $A$ on $y_1$ and one color from $B$ on $y_2$, every color set in $\mathcal{D}_{p,q}$ is of the form $D = C \cup \{a, b\}$ for some $a \in A \setminus C$ and $b \in B \setminus C$ and $C \in \mathcal{C}_{p-1,q-1}$. For each $1 \leq p, q \leq k$, $p + q \leq k$, consider the bipartite graph with bipartition $\mathcal{D}_{p,q}$, $\mathcal{C}_{p-1,q-1}$ with $D \in \mathcal{D}_{p,q}$, $C \in \mathcal{C}_{p-1,q-1}$ adjacent if and only if $C \subseteq D$. Now each $C \in \mathcal{C}_{p-1,q-1}$ has degree $(k-(p-1))(k-(q-1))$ and each $D \in \mathcal{D}_{p,q}$ has degree at most $pq$ in this bipartite graph. Therefore $pqd_{p,q} \geq \sum_{D\in\mathcal{D}_{p,q}} deg(D) = \sum_{C\in\mathcal{C}_{p-1,q-1}} deg(C) = (k-p+1)(k-q+1)c_{p-1,q-1}$. Thus, the total number of proper $L-$coloring sets satisfies

$$\sum_{\substack{1\leq p,q\leq k\\ p+q\leq k}} d_{p,q} \geq \sum_{\substack{1\leq p,q\leq k\\ p+q\leq k}} \frac{(k-p+1)(k-q+1)}{pq} c_{p-1,q-1}. \tag{2.1}$$

Claim 2. $f(p, q) \geq \dfrac{(k+2)^2}{k^2}$ where $f(p, q) = \dfrac{(k-p+1)(k-q+1)}{pq}$, $1 \leq p, q \leq k$ and $p + q \leq k$.

Fix $s \in \{2, \ldots, k\}$ and consider values of $p$ and $q$ such that $p + q = s$. Then $p = s - q$, and $1 \leq q \leq s - 1$.

Now, $f(p, q) = f(s-q, q) = g(q) = \dfrac{(k+1-s+q)(k+1-q)}{(s-q)q}$. Also, we note that $g(1) = g(s-1) = \dfrac{k(k+2-s)}{(s-1)}$, and $g^{'}(q) = \dfrac{h(q)}{[(s-q)q]^2}$ where $h(q) = -(k+1)(k+1-s)[s-2q]$. Therefore, $g$ achieves a minimum on $[1, s-1]$ at $q = s/2$. We have for all $q \in [1, s-1]$, $f(s-q, q) \geq g(s/2) = f(s/2, s/2) = \dfrac{(k+1-s/2)^2}{s^2/4}$.

Clearly this minimum decreases as $s$ increases. Therefore, for all $p, q \in \{1, \ldots, k-1\}$, $p + q \leq k$, $f(p, q) \geq f(k/2, k/2) = \dfrac{(k/2+1)^2}{k^2/4} = \dfrac{(k+2)^2}{k^2}$.

From Claim 2 and the inequality 2.1,

$$\sum_{\substack{1\le p,q\le k\\ p+q\le k}} d_{p,q} \quad \ge \quad \frac{(k+2)^2}{k^2}\cdot \sum_{\substack{0\le i,j\le k-2\\ i+j\le k-2}} c_{i,j} \ge \quad \frac{(k+2)^2}{k^2}\cdot\frac{k!}{2(k-2)!} \quad = \quad \frac{k^2+3k}{2}-\frac{2}{k}.$$

Hence for all $k \ge 3$, the number of different color sets arising from proper $L-$colorings of $H$ is at least $\dfrac{k^2+3k}{2}$.

□

**Theorem 2.4.** *Let $G$ denote the complete $k-$partite graph $K(m,2,1,\ldots,1)$, $k\ge 3$. Then*

$$ch(G) = \begin{cases} k & if\ m < \dfrac{k^2+3k}{2} \\ k+1 & if\ m \ge k^2. \end{cases}$$

**Proof.**

Let $L$ be a list assignment to $G$ with $|L(v)| = k$ for each $v \in V(G)$. Suppose $G$ has no proper $L-$coloring.

Observe that $L(y_1)\cap L(y_2) = \emptyset$. Otherwise there is a color $c \in L(y_1)\cap L(y_2)$. Then we can color $y_1$ and $y_2$ with $c$ and the remaining subgraph $G - V_2 = K(m,1,\ldots,1)$ can be colored from $L - \{c\}$ because $ch(G - V_2) = k-1$.

Let $H = G - V_1$. Since $L(y_1)\cap L(y_2) = \emptyset$, by Lemma 2.3, the number of distinct sets arising from the proper $L-$colorings of the subgraph $H$ is at least $\dfrac{k^2+3k}{2}$.

Further, $G$ is not $L-$colorable if and only if the set of colors, which will be of size $k$, of each of the proper colorings of $H$ occurs as a list in $V_1$. Therefore for $m < \dfrac{k^2+3k}{2}$, $G$ is $L-$ colorable. Thus, if $m < \dfrac{k^2+3k}{2}$, $ch(G) \le k$. Also $k = \chi(G) \le ch(G)$, so $ch(G) = k$ if $m < \dfrac{k^2+3k}{2}$.

When $m = k^2$, we provide the following list assignment $L'$ to $V(G)$ for which there is no proper $L'-$coloring.

Let $A$ and $B$ be disjoint sets of colors of size $k$, say $A = \{\alpha_1,\ldots,\alpha_k\}$ and $B = \{\beta_1,\ldots,\beta_k\}$. Let $L'(y_1) = L'(v_3) = \ldots = L'(v_k) = A$, $L'(y_2) = B$.

Any coloring of $H = K(2, 1, \ldots, 1)$ requires exactly $k-1$ colors from $A$ and one color from $B$, and there are exactly $k^2$ color sets from such colorings. Let $m = k^2$ lists on $V_1$ be the $k^2$ different sets $(A \setminus \{\alpha_i\}) \cup \{\beta_i\}$, $1 \le i, j \le k$. Since each of the proper colorings of $H$ occurs as a list in $V_1$, $ch(K(m, 2, 1, \ldots, 1)) > k$ for $m = k^2$.

Further, by theorem 2.3, $ch(K(m, 2, 1, \ldots, 1)) \le k + 1$ for all $m$. This concludes the proof.

□

**Corollary 2.4.1.** *Ohba's conjecture holds for the complete $k-$partite graph $K(m, 2, 1, \ldots, 1)$ with $m \le k + 1$.*

**Proof.**

We denote $G = K(m, 2, 1, \ldots, 1)$. Observe that when $m \le k+1$, $|V(G)| \le (k+1) + 2 + (k-2) = 2k+1 = 2\chi(G)+1$. Thus, $G$ satisfies the hypothesis of Ohba's conjecture. Further, it is clear that $k + 1 \le \dfrac{k^2+3k}{2} - 1$ for all $k \ge 2$. Thus, by theorem 2.4, $G$ is chromatic$-$choosable when $m \le k + 1$.

**Corollary 2.4.2.** $\lfloor \sqrt{m} \rfloor - 1 \le \phi(K(m, 2)) \le \lceil \frac{-7+\sqrt{8m+17}}{2} \rceil$ *for* $m \ge 5$.

**Proof.**

If $k \le \lfloor \sqrt{m} \rfloor$, then $k^2 \le m$, so by Theorem 2.4 $k + 1 = ch(K(m, 2, 1 \ldots, 1)) > \chi(K(m, 2, 1 \ldots, 1)) = k$. Thus, if $k \le \lfloor \sqrt{m} \rfloor$, $\phi(K(m, 2)) \ge (k-2)+1 = k-1$. Consequently, $\phi(K(m, 2)) \ge \lfloor \sqrt{m} \rfloor - 1$, for all $m \ge 1$. Further, by Theorem 2.4, if $m \le \dfrac{k^2+3k}{2} - 1$ and $k \ge 3$, then $\phi(K(m, 2)) \le k-2$. The smallest positive value of $k$ for which $m \le \dfrac{k^2+3k}{2} - 1$ is the positive solution of $k^2+3k-2(m+1) = 0$, so the smallest integer value of $k$ satisfying that inequality is $k_0 = \lceil \frac{-3+\sqrt{8m+17}}{2} \rceil$; we have $\phi(K(m, 2)) \le k_0 - 2 = \lceil \frac{-7+\sqrt{8m+17}}{2} \rceil$. The requirement $m \ge 5$ ensures that $k_0 \ge 3$.

□

Remark:

$\phi(K(m, 2)) = 1$ for $4 \le m \le 8$, by Theorem 2.4 and the previously noted fact that $ch(K(m, 2)) = 3$ for all $m \ge 4$.

**Lemma 2.4.** *Let $G$ denote the complete $k-$partite graph $K(m, 3, 1, \ldots, 1)$, $L$ a list assignment to $G$ satisfying that $L(y_i) \cap L(y_j) \neq \emptyset$ for some $y_i \neq y_j \in V_2$ and $|L(v)| \geq k+1$ for each $v \in V(G)$. Then $G$ is $L-$colorable for all $m \geq 1$.*

**Proof.**

Suppose $c_1 \in L(y_1) \cap L(y_2)$ and say $c_2 \in L(y_3)$ with $c_1 \neq c_2$. We color the vertices in $V_2$ with $c_1$ and $c_2$. Let $G' = G - V_2$ and $L' = L - \{c_1, c_2\}$. Then $|L'(v)| \geq k+1-2 = k-1$ for each $v \in V(G')$. By Corollary 2.3.1, $G'$ has a proper $L'-$coloring for all $m \geq 1$. Thus, $G$ is properly $L-$colorable. In the case that $\bigcap_{y \in V_2} L(y) \neq \emptyset$, it is clear from the previous argument that $G$ is $L-$colorable for all $m \geq 1$.

□

**Theorem 2.5.** *Let $G$ denote the complete $k-$partite graph $K(m, n, 1, \ldots, 1)$, and $2 \leq n \leq m$. Then $ch(G) = n+k-1$ if $m \geq \binom{n+k-2}{k-1}(n+k-2)^{n-1}$.*

**Proof.**

Let $C_1, C_2, \ldots, C_n$ be disjoint $(n+k-2)-$sets of colors.

We provide the following list assignment $L$ to $G$, with $|L(v)| = n+k-2$ for each $v \in V(G)$ as follows: $L(y_1) = L(v_3) = \ldots = L(v_k) = C_1$ and $L(y_j) = C_j$ for each $2 \leq j \leq n$. $L$ on $V_1$ will be described shortly.

Any proper $L-$coloring of $G' = G - V_1 \cong K(n, 1, \ldots, 1)$ requires exactly $k-1$ colors from $C_1$ and exactly one color from each $C_j$ for $2 \leq j \leq n$ , giving $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ distinct sets of colors from proper $L-$colorings, each set of size $k$. Let $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ lists on $V_1$ be the $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ different sets of colors from such proper $L-$colorings of $G'$, and if $m > \binom{n+k-2}{k-1}(n+k-2)^{n-1}$, let the remaining vertices in $V_1$ be supplied with any lists whatever of size $n+k-2$. Since each of the $\binom{n+k-2}{k-1}(n+k-2)^{n-1}$ sets of proper colorings of $G'$ occurs as a list in $V_1$, $G$ cannot be properly $L-$colored, so $ch(K(m, n, 1, \ldots, 1)) > n+k-2$ for $m \geq$

$\binom{n+k-2}{k-1}(n+k-2)^{n-1}$. Further, from Theorem 2.3, $ch(G) \leq n+k-1$ for all $m \geq 2$. Thus, for $m \geq \binom{n+k-2}{k-1}(n+k-2)^{n-1}$, $ch(G) = n+k-1$.

**Corollary 2.5.1.** *With $G$, $m$, $n$ and $k$ as in the hypothesis of Theorem 2.5, if $2 \leq r \leq n-1$ and $m \geq \binom{r+k-2}{k-1}(r+k-2)^{r-1}$, then $ch(G) \geq r+k-1$.*

**Proof.**

When $m \geq \binom{r+k-2}{k-1}(r+k-2)^{r-1}$, $ch(K(m,r,1,\ldots,1)) = r+k-1$ by Theorem 2.5. Further, with $2 \leq r \leq n-1 < m$, $K(m,r,1,\ldots,1)$ is a subgraph of the graph $G = K(m,n,1,\ldots,1)$.

□

**Corollary 2.5.2.** *With $G$, $m$, $n$ and $k$ as in the hypothesis of Theorem 2.5, if $2 \leq r \leq n$ and $m \geq \binom{r+k-2}{k-1}(r+k-2)^{n-1}$, then $\phi(G) \geq r-1$.*

**Proof.**

By Proposition 2.1, $\phi(G) \geq ch(G) - \chi(G)$. Therefore $\phi(G) \geq r+k-1-(k) = r-1$.

□

## 2.3 An estimate of ch(K(m,2,. . . ,2))

### 2.3.1 Introduction

Throughout this section, $[n] = \{1,\ldots,n\}$ and $\binom{[n]}{t} = \{t-\textit{subsets of } [n]\}$.

For $n \geq m \geq t \geq 0$, **the covering number** $C(n,m,t)$ is defined by $C(n,m,t) = min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \binom{[n]}{m}$ and $\forall\ B \in \binom{[n]}{t}, \exists\ A \in \mathcal{F}$ *such that* $B \subseteq A\}$.

**Proposition 2.3.** *Let $G$ denote the complete $k-$partite graph $K(m,2,2,\ldots,2)$. Then $ch(G) \leq 2k-1$.*

**Proof.** Let $L$ be a list assignment to $G$ such that $|L(v)| \geq 2k-1$ for each $v \in V(G)$. Any proper $L-$coloring of $G - V_1 \cong K(2,\ldots,2)$ uses at most $2(k-1)$ distinct colors say

$\alpha_1, \ldots, \alpha_{2k-2}$. Thus, for each $v \in V_1$, $|L(v) - \{\alpha_1, \ldots, \alpha_{2k-2}\}| \geq 1$, and so $G$ is $L-$colorable. Hence $ch(G) \leq 2k - 1$.

□

**Lemma 2.5.** *$C(n, m, t)$ is also the smallest size of a collection $\mathcal{F}'$ of $n - m$ subsets of $[n]$ (or any other fixed $n-$set) such that for every $(n - t)-$set $B' \in \binom{[n]}{n-t}$, some $A' \in \mathcal{F}'$ is contained in $B'$.*

**Proof.**

Given $\mathcal{F}$, as in the original definition of $C(n, m, t)$, form $\mathcal{F}' = \{[n] \setminus A \mid A \in \mathcal{F}\}$, the collection of complements of sets in $\mathcal{F}$. Similarly, given $\mathcal{F}' \subseteq \binom{[n]}{n-m}$, form $\mathcal{F} = \{[n] \setminus A' \mid A' \in \mathcal{F}'\}$, the collection of complements of sets in $\mathcal{F}'$. Because $|\mathcal{F}| = |\mathcal{F}'|$, in each case, and because complementation reverses inclusion, verification of the lemma's claim is straightforward.

□

**Theorem 2.6.** *Let $G$ denote the complete $k-$partite graph $K(m, 2, \ldots, 2)$ and $k \leq r \leq 2k - 2$. If $m \geq C(r, \lceil r/2 \rceil, r - k + 1).C(r, \lfloor r/2 \rfloor, r - k + 1)$ then $ch(K(m, 2, \ldots, 2)) \geq r + 1$.*

**Proof.**

Let $A$, $B$ be disjoint $r-$sets. Denote by $V_1, V_2, \ldots, V_k$ the parts of $G$, with $V_1 = \{x_1, \ldots, x_m\}$, $V_i = \{u_i, v_i\}$, $i = 2, \ldots, k$. Start defining a list assignment to $G$ by assigning $A$ to each $u_i$ and $B$ to each $v_i$. By Lemma 2.4, we can find a family $\mathcal{F}_1$ of $r - \lfloor r/2 \rfloor = \lceil r/2 \rceil-$subsets of $A$ and a family $\mathcal{F}_2$ of $r - \lceil r/2 \rceil = \lfloor r/2 \rfloor-$subsets of $B$ such that every $r - (r - k + 1) = (k - 1)-$subset of A contains some set in $\mathcal{F}_1$, and every $(k - 1)-$subset of $B$ contains some set in $\mathcal{F}_2$, and $|\mathcal{F}_1| = C(r, \lceil r/2 \rceil, r - k + 1)$, $|\mathcal{F}_2| = C(r, \lfloor r/2 \rfloor, r - k + 1)$. Make $|\mathcal{F}_1|.|\mathcal{F}_2|$ lists of length $r$ by forming the unions $F_1 \cup F_2$, $F_1 \subseteq \mathcal{F}_1$, $F_2 \subseteq \mathcal{F}_2$. If $m \geq |\mathcal{F}_1|.|\mathcal{F}_2|$ then we can endow $V_1$ with these lists. Then for every proper coloring of $G \setminus V_1$, some list on $V_1$ is in the set of colors used. Hence $ch(K(m, 2, \ldots, 2)) > r$ for $m \geq C(r, \lceil r/2 \rceil, r - k + 1).C(r, \lfloor r/2 \rfloor, r - k + 1)$.

□

**Corollary 2.6.1.** *If* $m \geq \binom{2k-2}{k-1}^2$ *then* $ch(K(m, 2, \ldots, 2)) = 2k - 1$.

**Proof.** For $r = 2(k-1)$, if $m \geq C(2k-2, k-1, k-1)^2 = \binom{2k-2}{k-1}^2$ then $ch(K(m, 2, \ldots, 2)) \geq 2k - 1$ by Theorem 2.6. Further, using Proposition 2.3, we establish that $ch(K(m, 2, \ldots, 2)) = 2k - 1$.

□

## Chapter 3

## Hall numbers

### 3.1 Some necessary conditions

**Theorem 3.1.** *(P.Hall [6]). Suppose $A_1, A_2, \ldots, A_n$ are (not necessarily distinct) finite sets. There exist distinct elements $a_1, a_2, \ldots, a_n$ such that $a_i \in A_i$, $i = 1, 2, \ldots, n$ if and only if for each $J \subseteq \{1, 2, \ldots, n\}$, $|\bigcup\limits_{j \in J} A_j| \geq |J|$.*

A proper $L-$coloring of a complete graph $K_n$ is certainly a system of distinct representatives of the finite list $L(v)$, and any list $A_1, A_2, \ldots, A_n$ of sets can be regarded as lists assigned to $K_n$. Therefore, as noted in [7], Hall's theorem can be restated as:

**Theorem 3.2.** *(Hall's theorem restated). Suppose that $L$ is a list assignment to $K_n$. There is a proper $L-$coloring of $K_n$ if and only if, for all $U \subseteq V(K_n)$, $|L(U)| = |\bigcup\limits_{u \in U} L(u)| \geq |U|$.*

Let $L$ be a list assignment to a simple graph $G$, $H$ a subgraph of $G$ and $\mathcal{P} = \{1, 2, \ldots\}$ a set of colors. If $\psi : V(G) \to \mathcal{P}$ is a proper $L-$coloring of $G$, then for any subgraph $H \subset G$, $\psi$ restricted to $V(H)$ is a proper $L-$coloring of $H$.

For any $\sigma \in \mathcal{P}$, let $H(\sigma, L) = <\{v \in V(H) \mid \sigma \in L(v)\}>$ denote the subgraph of $H$ induced by the support set $\{v \in V(H) \mid \sigma \in L(v)\}$. For convenience, we sometimes simply write $H_\sigma$.

For each $\sigma \in \mathcal{P}$, $\psi^{-1}(\sigma) = \{v \in V(H) \mid \psi(v) = \sigma\} \subseteq V(G_\sigma)$ and $\psi^{-1}(\sigma)$ is an independent set. Further, $\psi^{-1}(\sigma) \cap H \subseteq V(H_\sigma)$. So, $|\psi^{-1}(\sigma) \cap H| \leq \alpha(H_\sigma)$ where $\alpha$ is the vertex independence number. This implies that

$$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) \geq \sum_{\sigma \in \mathcal{P}} |\psi^{-1}(\sigma) \cap H| = |V(H)| \text{ for all } H \subseteq G.$$

When $G$ and $L$ satisfy the inequality

$$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) \geq |V(H)| \qquad (3.1)$$

for each subgraph $H$ of $G$, they are said to satisfy ***Hall's Condition***, a necessary condition for a proper $L-$coloring of $G$. Because removing edges does not diminish the vertex independence number, for $G$ and $L$ to satisfy Hall's Condition it suffices that (3.1) holds for all induced subgraphs $H$ of $G$.

Hall's Condition is sufficient for a proper coloring when $G = K_n$, because if $H$ is an induced subgraph of $K_n$ then

$$\alpha(H_\sigma) = \begin{cases} 1 & if\ \sigma \in \bigcup_{v \in V(H)} L(v),\ for\ each\ \sigma \in \mathcal{P}, \\ 0, & otherwise. \end{cases}$$

So

$$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) = |\bigcup_{v \in V(H)} L(v)|;$$

therefore Hall's Condition, that

$$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) \geq |V(H)|$$

for every such $H$, is just a restatement of the condition in Theorem 3.2. (It is necessary to point out here that if $\sigma \notin L(v)$ for some $v \in V(H)$ then $H_\sigma$ is the null graph, and $\alpha(H_\sigma) = 0$.) Consequently, Hall's Theorem may be restated: For complete graphs, Hall's Condition on the graph and a list assignment suffices for a proper coloring.

The temptation to think that there are many graphs for which Hall's Condition is sufficient can be easily dismissed. Figure 3.1 is the smallest graph with a list assignment $L_0$ for which Hall's Condition holds, and yet $G$ has no proper $L_0-$coloring.

**Remark.**

It is clear that if $H$ is an induced subgraph of $G$ and $H \neq G$, then $H \subseteq G - v$ for some $v \in V(G)$. So, if $G - v$ has a proper $L-$coloring, then $H \subseteq G - v$ must satisfy (by necessity) (3.1). Thus, in practice, in order to show that $G$ and $L$ satisfy Hall's Condition, it suffices to verify that $G - v$ is properly $L-$colorable for each $v \in V(G)$ and that $G$ itself satisfies the inequality (3.1) .

Denoted by $h(G)$, the *Hall number* of a graph $G$ is the smallest positive integer $k$ such that there is a proper $L-$coloring of $G$, whenever $G$ and $L$ satisfy Hall's Condition and $|L(v)| \geq k$ for each $v \in V(G)$. In [7] the following facts are shown:

**1.** If $|L(v)| \geq \chi(G)$ for every $v \in V(G)$ then $G$ and $L$ satisfy Hall's Condition.

**2.** $h(G) \leq ch(G)$ for every $G$.

**3.** If $ch(G) > \chi(G)$ then $h(G) = ch(G)$.

**4.** If $h(G) \leq \chi(G)$ then $\chi(G) = ch(G)$.

These facts underline our findings in the next section.

### 3.2 Hall numbers of some complete multipartite graphs

Throughout this section, $G$ is a simple graph and $L$ is a list assignment to $V(G)$ such that $L(v) \subset \{1, 2, \ldots, \} = \mathcal{P}$, an integer set of symbols. If $\sigma \notin L(v)$ for some $v \in V(G)$, then $G_\sigma$ is the null graph. Further, we denote by $\psi(v)$, any attempted proper coloring of some $v \in V(G)$.

#### 3.2.1 Example

Consider the complete bipartite graph $K(2, 2)$ in Figure 3.1 with parts $V_i = \{u_i, v_i\}$, $i = 1, 2$ and $L_0$ the list assignment indicated.

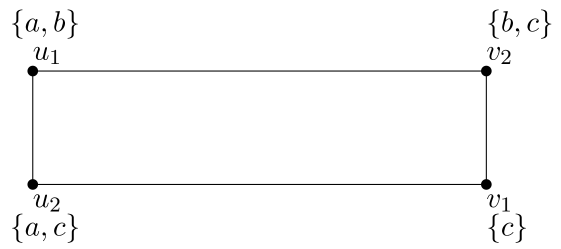


Figure 3.1: A list assignment to K(2,2).

If $v_1$ is colored $c$, as it must be, then $u_2$ must be colored $a$ and $v_2$ must be colored $b$ in a proper coloring, so $u_1$ cannot be properly colored.

However, we will show that $G$ and $L_0$ satisfy Hall's Condition using the argument described in the previous remark. First, for each $v \in V(G)$, it is easy to see that $G - v$ is properly $L_0-$colorable, meaning every proper induced subgraph $H \subset G$ satisfies the inequality (3.1) in Hall's Condition. We now proceed to verify the inequality (3.1) for $G$ itself.

Now, $\alpha(G_c) = 2$ and $\alpha(G_b) = \alpha(G_a) = 1$. So, $4 = \sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) \geq |V(G)| = 4$ where $\mathcal{P} = \{a, b, c, \ldots\}$. Thus, $G$ and $L_0$ satisfy Hall's Condition and yet $G$ has no proper $L_0-$coloring. Thus, $1 < h(G) \leq 2$ by Fact **2** and Theorem A. Therefore, $h(G) = 2$.

### 3.2.2 Some Hall numbers

**Theorem 3.3.** *$h(K(2, \ldots, 2)) = k$ when $k \geq 2$.*

**Proof.**

Let the partite sets of the complete $k-$partite graph $G = K(2, \ldots, 2)$ be $V_1, \ldots, V_k$ with $V_i = \{u_i, v_i\}$, for $i = 1, 2, \ldots, k$.

In Example 3.2.1, we showed that $h(G) = k$ when $k = 2$. So, to complete the proof, we suppose $k \geq 3$.

Let $A$ be a nonempty set of colors with $|A| = k - 2$ and $a, b, c$ be distinct colors not in A. We define $L$ a list assignment to $G$ as follows:

1. $L(u_1) = A \cup \{a, b\}$, $L(u_2) = L(u_3) = \ldots = L(u_{k-1}) = A \cup \{a\}$, $L(u_k) = A \cup \{c\}$ and

2. $L(v_1) = A \cup \{b, c\}$, $L(v_2) = L(v_3) = \ldots = L(v_k) = A \cup \{b\}$.

Observe that $|L(v)| \geq k - 1$ for every $v \in V(G)$.

**Claim 3.3.1.** *The graph $G$ is not properly $L-$colorable.*

**Proof.**

In the following cases, we consider all possible distinct ways to properly color the vertices of some part of $G$, say $V_1$ . We then conclude that the remaining subgraph $H = G - V_1$ is not proper $L'$-colorable where $L' = L - \{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_2\} \in \bigcup_{v \in V_1} L(v)$. ($\alpha_1, \alpha_2$ are not necessarily distinct colors; they are the colors on $V_1$.) Let $\psi$ denote the attempted proper coloring.

**Case 1:** $\psi(u_1) = b$ or $\psi(v_1) = b$.

Let $S =< \{v_2, \ldots, v_k\} >$, an induced subgraph of $H$. Then $k-2 = |A| = |\bigcup_{v \in V(S)} L'(v)| < |V(S)| = k - 1$. Since the subgraph $S$ is a clique, we cannot properly color $S$ from $L'$.

**Case 2:** $\psi(u_1) = a$ and $\psi(v_1) = c$.

Similarly as described in case 1, by letting $S =< \{u_2, \ldots, u_k\} >$, it's clear that we cannot properly color $S$, from $L'$.

**Case 3:** $\psi(u_1) = \gamma$ or $\psi(v_1) = \gamma$ for some color $\gamma \in A$.

Similarly as in case 1, $k - 2 = |\bigcup_{v \in V(S)} L'(v)| < |V(S)| = k - 1$. Hence we cannot properly color $H$ from $L'$.

□

**Claim 3.3.2.** $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) \geq |V(G)|$.

**Proof.**

It is clear that $\alpha(G_a) = \alpha(G_c) = 1, \alpha(G_b) = 2$; further, $\alpha(G_\sigma) = 2(k - 2)$ for every $\sigma \in A$. Hence $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) = 2k = |V(G)|$.

□

**Claim 3.3.3.** *Every proper induced subgraph $H$ of $G$ is properly $L-$colorable.*

**Proof.**

In the following cases we provide a (not necessarily unique) proper coloring for each induced subgraph $H$ of $G$, of the form $G - v$, $v \in V(G)$.

**Case** 1: $H = G - u_1$.

Let $\psi(v_1) = c$ and color the $2(k-2)$ vertices of the subgraph $G - (V_1 \cup V_2)$ with the colors from $A$ (by coloring vertices of the same part with the same color). Then let $\psi(u_2) = a$ and $\psi(v_2) = b$.

**Case** 2: $H = G - v_1$.

Let $\psi(u_1) = a$ and color the $2(k-2)$ vertices of the subgraph $G - (V_1 \cup V_k)$ with the colors from $A$ with the same color appearing on $u_i$ and $v_i$, $i = 2, \ldots, k-1$. Then, let $\psi(u_k) = c$ and $\psi(v_k) = b$.

**Case** 3: $H = G - u_i$, for some $2 \leq i \leq k$.

Let $\psi(v_i) = b$ and color the remaining $2(k-2)$ vertices of the subgraph $G - (V_i \cup V_1)$ with the colors from $A$. Then, let $\psi(u_1) = a$ and $\psi(v_1) = c$.

**Case** 4: $H = G - v_i$, for some $2 \leq i \leq k-1$.

Let $\psi(u_i) = a$ and color the remaining $2(k-2)$ vertices of the subgraph $G - (V_i \cup V_1)$ with the colors from $A$. Then, let $\psi(u_1) = \psi(v_1) = b$.

**Case** 5: $H = G - v_k$.

Let $\psi(u_k) = c$ and color the $2(k-2)$ vertices of the subgraph $G - (V_1 \cup V_k)$ with the colors from $A$. Finally, let $\psi(u_1) = \psi(v_1) = b$.

From the previous claims, we can conclude that $h(G) > k-1$. Thus, by Theorem A and Fact 2, $h(G) = k$. This concludes the proof.

□

**Theorem 3.4.** *Let $G$ denote the complete $k-$partite graph $K(4,2,\ldots,2)$ with $k \geq 2$. Then*

$$h(G) = \begin{cases} k & if\ k\ is\ odd \\ k+1 & if\ k\ is\ even. \end{cases}$$

**Proof.**

When $k$ is even, from Chapter 2 we have that $k = \chi(G) < ch(G) = k+1$. Thus, from Fact 3, it is clear that $h(G) = ch(G) = k+1$ for all even $k \geq 2$.

Suppose $k \geq 3$ is odd.

Let the partite sets, or parts, $V_1, V_2, \ldots, V_k$ of the complete $k-$partite graph $G$ be $V_1 = \{x_1, x_2, x_3, x_4\}$ and $V_i = \{u_i, v_i\}$, $i = 2, \ldots, k$, $k \geq 2$.

Let $C_1$ and $C_2$ be disjoint $k-2$ sets of colors and 0 an object not in $C_1 \cup C_2$. Let $A = C_1 \bigcup \{0\}$, $B = C_2 \cup \{0\}$. Let $A_1$, $A_2$ and $B_1$, $B_2$ be disjoint $(k-1)/2$ sets of colors partitioning $A$ and $B$ respectively. Without loss of generality, let $0 \in A_2 \cap B_2$. Let $a$,$b$ be distinct objects not in $A \cup B$. Define a list assignment $L$ to $G$ as follows:

1. $L(u_2) = A$, $L(v_2) = B$, $L(u_i) = C_1 \cup \{a\}$ and $L(v_i) = C_2 \cup \{b\}$, for every $3 \leq i \leq k$ and

2. $L(x_1) = A_1 \cup B_1$, $L(x_2) = A_1 \bigcup B_2$, $L(x_3) = A_2 \cup B_1$ and $L(x_4) = A_2 \cup B_2 \cup \{a\}$

Notice that $|L(v)| = k-1$ for every $v \in V(G)$.

**Claim 3.4.1.** *$G$ is not properly $L-$colorable.*

Every proper $L-$coloring of $G - V_1 = K(2,\ldots,2)$ uses $k-1$ elements of $C_1 \cup \{0, a\}$ and $k-1$ elements of $C_2 \cup \{0, b\}$. We proceed by exhausting the possible cases in attempts to proper $L-$color $G$.

**Case** 1: suppose $\psi(u_2) \neq 0 \neq \psi(v_2)$. Then all of the colors of $C_1 \cup C_2 \cup \{a, b\}$ will be used to color $G - V_1$. Hence we cannot color $x_1$ (since $A_1 \cup B_1 \subset C_1 \cup C_2$).

**Case** 2: suppose $\psi(u_2) = \psi(v_2) = 0$

**Case** 2.1: $\psi(u_i) \neq a$ and $\psi(v_i) \neq b$ for every $3 \leq i \leq k$.

Then all of the colors of $C_1 \cup C_2$ will be used to color $G - (V_1 \cup V_2)$. Once again we cannot color $x_1$.

**Case** 2.2: $\psi(u_i) = a$ and $\psi(v_j) = b$ for some $i, j \neq 2$.

Then there remains exactly one color, say $c_1 \in C_1$ and exactly one color, say $c_2 \in C_2$. If $c_1 \in A_1$ and $c_2 \in B_1$, then we cannot color $x_4$. Likewise if $c_1 \in A_1$ and $c_2 \in B_2$, then we cannot color $x_3$. Also if $c_1 \in A_2$ and $c_2 \in B_1$, $x_2$ cannot be colored and if $c_1 \in A_2$, $c_2 \in B_2$, $x_1$ cannot be colored as well.

**Case** 2.3: $\psi(u_i) \neq a$ for all $i \neq 2$ and $\psi(v_j) = b$ for some $j \geq 3$. Then there remains exactly one color, say $c_2 \in C_2$ and none of $C_1$. As in the previous case, If $c_2 \in B_1$, then we cannot color $x_2$. Likewise if $c_2 \in B_2$, then we cannot color both $x_1$ and $x_3$.

**Case** 2.4: $\psi(u_i) = a$ for some $i \geq 3$ and $\psi(v_j) \neq b$ for all $j \geq 3$. Then there remains exactly one color, say $c_1 \in C_1$ and none of $C_2$. As before, if $c_1 \in A_1$, then we cannot color either of $x_3$ and $x_4$. Likewise if $c_1 \in A_2$, then we cannot color either of $x_1$ and $x_2$.

**Case** 2.5: $\psi(u_i) \neq a$ and $\psi(v_j) \neq b$ for all $3 \leq i, j \leq k$. Clearly the coloring cannot be properly extended to any of $x_1, x_2, x_3$.

Notice that we can skip the case where $\psi(u_2) = 0$ and $\psi(v_2) \neq 0$ (or vice versa), since if there is a proper $L-$coloring with one of $u_2$, $v_2$ colored with 0, then there is a proper $L-$coloring with both colored 0.

From the previous cases we can conclude that $G$ is not properly $L-$ colorable.

**Claim 3.4.2.** $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) \geq |V(G)|$.

**Proof.**

Notice that $\alpha(G_\sigma) = 2$ for every $\sigma \in C_1 \cup C_2$. Also $\alpha(G_0) = 3$ and $\alpha(G_a) = \alpha(G_b) = 1$. Hence $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) = 2(2(k-2)) + 5 = 4k - 3 \geq 2k + 2 = |V(G)|$ for every $k \geq 3$.

**Claim 3.4.3.** *If $k \geq 5$, then every proper induced subgraph $H$ of $G$ is properly $L-$colorable.*

**Proof.**

We proceed by considering the possible subgraphs of $G$ obtained by deleting a single vertex.

**Case** 1: $H = G - u_i$, for some $i$.

Let $\psi(x_2) = \psi(x_3) = \psi(x_4) = 0$. Color $G - V_1$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ (colors $a, b$ included). Hence there remains exactly one unused color of $C_1$, say $c_1$, and arrange that $c_1 \in A_1$. Let $\psi(x_1) = c_1$.

**Case** 2: $H = G - v_i$, for some $i$. Following the coloring argument in the previous case, there remains exactly one unused color of $C_2$, say $c_2$, and arrange that $c_2 \in B_1$. Let $\psi(x_1) = c_2$.

**Case** 3: $H = G - x_1$. Let $\psi(x_2) = \psi(x_3) = \psi(x_4) = 0$. It is easy to see that we can color the remaining subgraph $G - V_1$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ ($a, b$ included).

**Case** 4: $H = G - x_2$. Let $\psi(u_2) = \psi(v_2) = 0$, and $\psi(x_4) = a$. Color the vertices of $G - (V_1 \cup V_2)$ with the colors from $C_1 \cup C_2 \cup \{b\}$ ($b$ included). Then there remains exactly one unused color of $C_2$, say $c_2$, and arrange that $c_2 \in B_1$. Let $\psi(x_1) = \psi(x_3) = c_2$.

**Case** 5: $H = G - x_4$. Let $\psi(u_2) = \psi(v_2) = 0$. Color the vertices of $G - (V_1 \cup V_2)$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ ($a, b$ included). Then there remains exactly one unused color of $C_1$, say $c_1$, and arrange that $c_1 \in A_1$, and exactly one unused color of $C_2$, say $c_2$, and arrange that $c_2 \in B_1$. Let $\psi(x_1) = c_1 = \psi(x_2)$ and $\psi(x_3) = c_2$.

**Case** 6: $H = G - x_3$. Let $\psi(u_2) = \psi(v_2) = 0$. Color the vertices of $G - (V_1 \cup V_2)$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ ($a, b$ included). Then there remains exactly one unused color of $C_1$, say $c_1$, and arrange that $c_1 \in A_1$, and exactly one unused color of $C_2$, say $c_2$, and arrange that $c_2 \in B_2$. Let $\psi(x_1) = c_1 = \psi(x_2)$ and $\psi(x_4) = c_2$.

Notice here that when $k = 3$, $A_2 = B_2 = \{0\}$. Therefore, the attempted coloring of $H = G - x_3$ in case 6 fails, and, in fact $H$ is not properly $L-$ colorable. However, $H = G - x_3$ with the given list assignment $L$ satisfies the inequality (3.1). We can safely end the proof here when $k = 3$.

Still, there follows a list assignment specifically for the case when $k = 3$, which we hope will be of interest.

We define a list assignment $L$ to $G = K(4,2,2)$ as follows:

1. $L(u_2) = \{1,0\}$, $L(v_2) = \{2,0,c\}$, $L(u_3) = \{1,a\}$, $L(v_3) = \{2,b\}$ and
2. $L(x_1) = \{1,2\}$, $L(x_2) = \{1,0\}$, $L(x_3) = \{0,a\}$ and $L(x_4) = \{b,c\}$

It is easy to verify that $G$ satisfies claims 3.3.1 and 3.3.2. We proceed therefore to verify only claim 3.3.3 for the subgraphs $H$ of $K(4,2,2)$ in the following cases.

**Case**1: $H = G - u_2$.

Let $\psi(v_2) = 2, \psi(u_3) = a, \psi(v_3) = b$. Also $\psi(x_2) = 0 = \psi(x_3), \psi(x_1) = 1$ and $\psi(x_4) = c$.

**Case**2: $H = G - v_2$.

Let $\psi(u_2) = 1, \psi(u_3) = a, \psi(v_3) = b$. Also $\psi(x_2) = 0 = \psi(x_3), \psi(x_1) = 2$ and $\psi(x_4) = c$.

**Case**3: $H = G - u_3$.

Let $\psi(u_2) = \psi(v_2) = 0, \psi(v_3) = b$. Also $\psi(x_1) = 1 = \psi(x_2), \psi(x_3) = a$ and $\psi(x_4) = c$

**Case**4: $H = G - v_3$.

Let $\psi(u_2) = 1, \psi(v_2) = c, \psi(u_3) = a$. Also $\psi(x_1) = 2, \psi(x_2) = 0 = \psi(x_3)$ and $\psi(x_4) = b$.

**Case**5: $H = G - x_1$.

Let $\psi(u_2) = 1$, $\psi(v_2) = 2$, $\psi(u_3) = a$ and$\psi(v_3) = b$ . Also let $\psi(x_1) = 0 = \psi(x_2)$ and $\psi(x_4) = c$.

**Case**6: $H = G - x_2$.

Let $\psi(u_2) = 0 = \psi(v_2), \psi(u_3) = 1$ and$\psi(v_3) = b$ . Also $\psi(x_1) = 2, \psi(x_3) = a$ and $\psi(x_4) = c$.

**Case**7: $H = G - x_3$.

Let $\psi(u_2) = 0 = \psi(v_2), \psi(u_3) = a$ and$\psi(v_3) = b$ . Also $\psi(x_1) = 1 = \psi(x_2)$ and $\psi(x_4) = c$.

**Case**8: $H = G - x_4$.

Let $\psi(u_2) = 1, \psi(v_2) = c, \psi(u_3) = a$ and $\psi(v_3) = b$ . Also $\psi(x_1) = 2$ and $\psi(x_2) = 0 = \psi(x_3)$.

We conclude that $G$ and $L$ satisfy Hall's Condition. So, $k \leq h(G) \leq ch(G)$ by Fact 2. Therefore, $h(G) = k$ for all $k \geq 3$ odd.

$\square$

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