The Chromatic Number of the Euclidean Plane

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Vita

Anna Elżbieta Borońska was born on March 2, 1975, in Mysłowice, Poland. She graduated from the mathematics and physics program of Maria Konopnicka General Lyceum in Katowice, in 1994. She then attended the Catholic University of Lublin for five years and graduated with a Master Degree of Arts in Law in May of 1999. Thereafter, she worked in Department of Legal Supervision, Silesian Voivodship Office in Katowice. From 1999 to 2003, she completed curriculum of the doctoral program in Law at the Catholic University of Lublin. She underwent legal training in the Office of the District Attorney in Katowice from 2001 to 2004. Subsequently, she passed Prosecutor's Examination at Court of Appeal in Katowice. She obtained a license of a legal adviser in Poland in 2005. She entered the Graduate Program at Auburn Univerity in August of 2007, as a collaborative student in the PhD program in Public Administration and Public Policy, and the MS program in Mathematics. She is married and has a son.

Thesis Abstract

The Chromatic Number of the Euclidean Plane

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We discuss the chromatic number of the plane problem. We provide a detailed history of its origins, along with some of the recent progress. Then we describe a proof, based on [1], of the following result.

Theorem 0.1 *Every distance excluding coloring of a locally finite plane tiling, with the property that the whole interior of any tile is colored by a single color, must employ at least six colors.*

The result was originally proved for polygonal tilings that have convex tiles, and the area of all tiles from the tiling bounded from below by a constant. It follows from our proof that the second condition is redundant. Moreover, using the fact that any polygon can be triangulated, Coulson's result is extended to tilings with non convex polygons.

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Chapter 1

History of the problem

The chromatic number of the plane problem asks for the minimum number of colors that are needed to paint all points in the plane, so that no two points in a given distance are colored alike. The question seems very natural and basic, but is yet to be fully answered. The problem goes back to 1950, when Edward Nelson, a graduate student at the University of Chicago at the time, created the problem working on the well known four-color problem. He passed the problem to other mathematicians at the University of Chicago, and soon the question about the chromatic number of the plane became well known in the mathematical community (see[6]). The problem sometimes is incorrectly credited to other mathematicians such as Paul Erdös, Martin Gardner, Hugo Hadwiger, or Leo Moser. Actually, the question was probably published for the first time in Martin's Gardner "*Mathematical Games*" column in "*Scientific American*" in 1960 [3]. Although in last nearly 60 years the chromatic number of the plane resisted all efforts aiming at an ultimate answer, a considerable amount of research discussing partial answers to this problem, or investigating related problems, accumulated in the mathematical literature. The following bounds on the chromatic number are well known:

$$4 \leq \text{chromatic number of the plane} \leq 7.$$

We shall explain how these bounds are obtained. At this point, however, let us mention that it has been recently shown by Saharon Shelah and Alexander Soifer [7], [8] that the answer to the problem may depend on the choice of axioms of set theory. Although in the present thesis we will consider only geometric aspects of the problem and will not explore those which belong to set theory, in this section for completeness sake, we shall briefly describe the results obtained by these authors.

Nelson was the first to offer a proof of the lower bound. It was published in [4], however credit was given to his peer John Isbell (see [6]). An independent proof for the lower bound comes from Leo Moser and William Moser, commonly known as the Moser Spindle [5]. We will describe the two examples from [4] and [5], starting with the Moser Spindle.

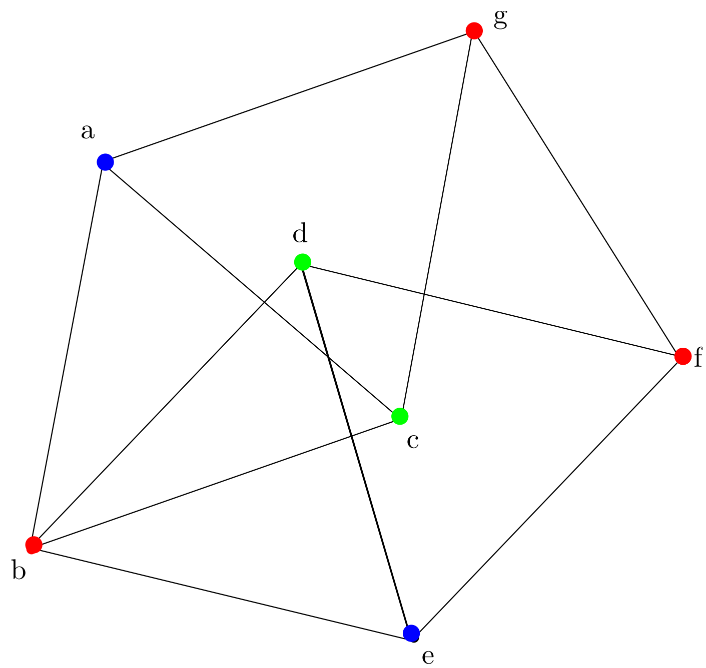

Figure 1.1: The Moser Spindle

Consider a graph exhibited in Figure 1.1. All edges are of unit length. Suppose this graph can be painted by three colors only, excluding unit distance. First consider the equilateral triangle abc. To exclude the distance 1, all three of these vertices must be of three different colors. Since the equilateral triangle acg shares the edge $[a, c]$, vertex g must be of the same color as b. On the other hand, in the triangle bed we must use the same two colors as for $[a, c]$ in order to color the edge $[e, d]$. Again, the triangle fed shares an edge $[e, d]$ with bed, and therefore f must be colored alike to b. Clearly, g and f are colored alike and are unit distance apart. This results in a contradiction.

An alternative proof of the lower bound is provided by the example in [4]. Suppose only three colors are used to color the entire plane, with the unit distance excluded. Let a be a point in the plane. Consider a unit circle C around a. No point on C can be colored by the same color as a. Now consider also S, a circle centered at a and of radius $\sqrt{3}$. Let

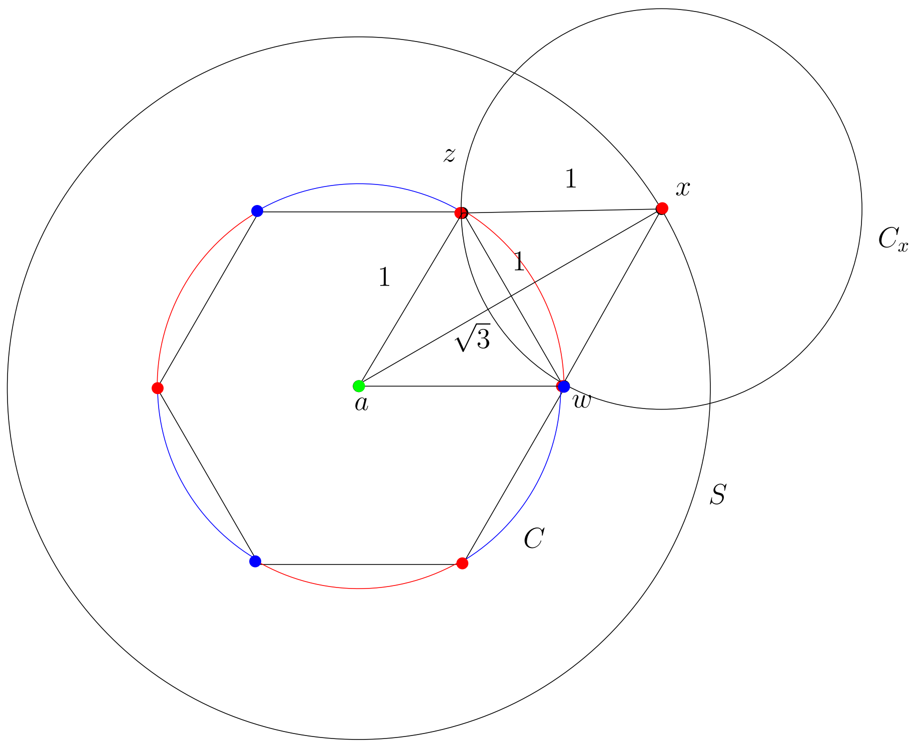

Figure 1.2: The example of Hadwiger

x be any point on S, and let C_x be a unit circle centered at x. Clearly C_x intersects C in two points, say w and z. Now notice that w, z, x, a are vertices of a unit rhombus. Since one of its diagonals is of length $\sqrt{3}$, the other one must be of unit length. This means that w and z are unit distance apart and therefore they must be colored by two different colors, both distinct from the one used to paint a. Consequently, since x is a unit distance apart from w and z, x must be painted alike to a. However, x was chosen arbitrarily on S, and therefore the entire S must be painted by the same color as a. This is a contradiction, since on S there are points a unit distance apart, and they can't be colored alike .

Isbell [4] is credited as first to estimate the upper bound using hexagonal coloring of the plane, with seven colors (see [6]). An alternative proof comes from L. Szekely employing tiling by squares [10]. We exhibit Isbell's idea in the following figure. The description is

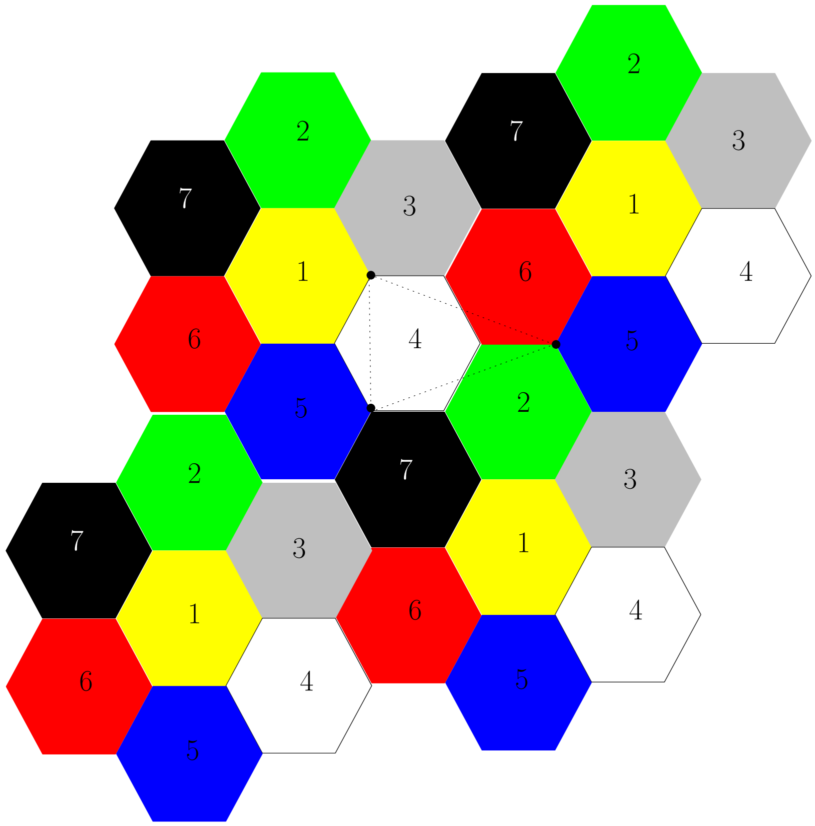

Figure 1.3: The hexagonal 7 coloring

based on [4]. The idea is to color a hexagonal tilling by seven different colors. The hexagons from the tiling are regular and of side length $\frac{2}{5}$. We choose a hexagon and color it by the first color. Then there are six hexagons each sharing an edge with this initial one. We assign one of the remaining six colors to each of them. Extend the coloring to the hexagonal tiling of the plane as shown in Figure 1.3. The boundary points on a given edge are colored arbitrarily by any of the two adjecent colors.

This coloring is unit distance excluding. To see this, first focus on interior points of the hexagons. If we start with hexagons with a side length $\frac{2}{5}$ then no two points of the same color can be d distance apart, for any d between $\frac{4}{5}$ and $\frac{\sqrt{28}}{5}$. Clearly, no two points from the same hexagon are in a distance grater than $\frac{4}{5}$ apart (this is the length of the great diameter). To obtain the other number, consider the isosceles triangle given by dotted lines in the figure. Clearly, the side length of this triangle gives the minimal distance between two points colored alike, but from two different hexagons (in the figure the triangle refers to blue color, or 5). The base of this triangle is the smaller diameter of the hexagon, and therefore its length equals $\frac{2\sqrt{3}}{5}$. On the other hand, it is not hard to check that the height of this triangle is equal to the length of the great diameter, plus side length, minus the height of the isosceles triangle with sides of length $\frac{2}{5}$ and base $\frac{2\sqrt{3}}{5}$. Therefore the height of this dotted triangle is $\frac{4}{5} + \frac{2}{5} - \frac{1}{5} = 1$. Consequently the side length is $\sqrt{1^2 + (\frac{\sqrt{3}}{5})^2} = \frac{\sqrt{28}}{5}$. As far as boundary points are concerned, by the inequality on d, there is enough cushion so that it doesn't matter which of the two adjacent colors is assigned to each edge (see [9]).

Szekely's example is obtained by considering infinite rows of squares in the plane. Each square is of side length $\frac{1}{\sqrt{2}}$. We start with an arbitrary square and color it with the first color. Moving in the row from left to right we color squares with subsequent six colors, and then we start over again with the first one. We use a reversed order of colors when moving to the left in this row. Now, fill in the plane by identical copies of this initial row. Moving

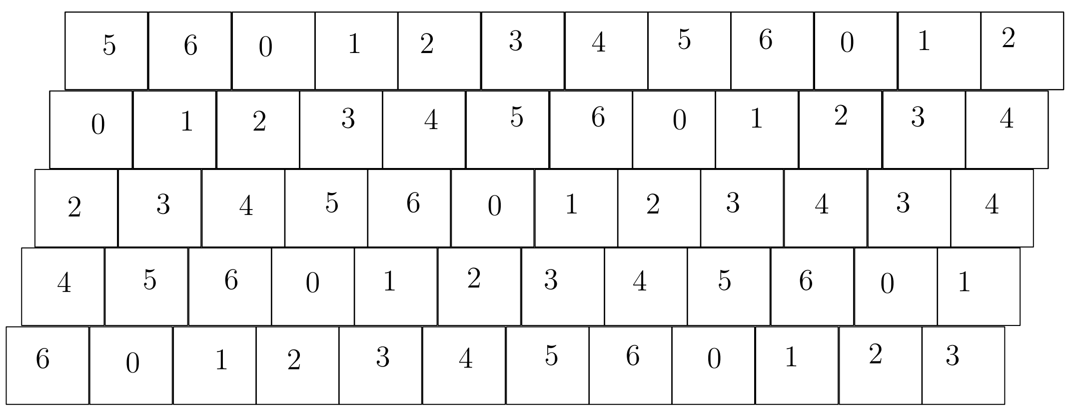

Figure 1.4: The example of Szekely

downward, let every subsequent row be shifted by $\frac{1}{\sqrt{2}} + 1.1$ units to the left. When moving upward shift by the same factor but to the right. Upper and right boundaries are included in the color of each square, except for the square's upper left and lower right corners (see [9]). It should be clear from the picture that this coloring of the squares, with seven different colors, is unit distance excluding. This is because the closest that two points colored alike, but from two different squares, can be is 1.1. On the other hand the diagonal of each of the squares is of length 1.

Now, let us move on to the results presented by Shelah and Soifer. The authors consider an equivalent formulation of the chromatic number of the plane problem. Namely, let U^2 be a graph on the set of all points of the plane as its vertex set, with two points adjacent iff they are 1 distance apart. The question now is: *What is the minimal number of colors that need to be employed in order to color every vertex of this graph, with no two points adjacent colored alike?* Finite subgraphs of of U^2 are called *finite unit distance plane graphs.* In 1951 it was shown by Erdös and Nicolaas de Bruijn [2] that the chromatic number of the plane is attained on some finite subgraphs. This fundamental result determined much of research to go in the direction of finite unit distance graphs, but at the same time, implicitly, was dependent on the axiom of choice. Motivated by this result, Shelah and Soifer consider how the above question may depend on the following axioms of set theory.

(AC)(Axiom of choice) Every family Φ of nonempty sets has a choice function; i.e., there is a function f such that $f(S) \in S$ for every S from Φ.

$(AC_{\aleph_0})$(Countable axiom of choice) Every countable family of nonempty sets has a choice function.

(DC)(Principle of dependent choices) If E is a binary relation on a nonempty set A, and for every $a \in A$ there is $b \in A$ with aEb, then there is a sequence $a_1, a_2, \ldots$ such that $a_n E a_{n+1}$ for every natural number n.

(LM) Every set of real numbers is Lebesgue measurable.

(ZF) Zermelo-Fraenkel system of axioms of set theory.

(ZFC) ZF with an addition of AC.

Axiom DC is a weak form of axiom AC, whereas axiom DC implies axiom $AC_{\aleph_0}$. In their first paper [7] Soifer and Shelah formulate the following theorem.

Theorem 1.1 *Assume that any finite unit distance plane graph has chromatic number not exceeding* 4. *Then:*

(i) In ZFC the chromatic number of the plane is 4.

(ii) In ZF+DC+LM the chromatic number of the plane is 5,6, or 7.

In the second paper [8] the authors extend the ideas from the first paper, giving two interesting examples that illustrate possible correlation of the above axioms of set theory and the chromatic number of the plane. Their first example is as follows.

Let G_2 be a graph with $\Re^2$ as the vertex set, and the set

$$\bigcup_{\epsilon \in F} \{(s,t) : s,t \in \Re, s - t - \epsilon \in Q^2\}$$

as the set of edges, where Q is the set of rationals, and

$$F = \{(\sqrt{2}, 0), (0, \sqrt{2}), (\sqrt{2}, \sqrt{2}), (-\sqrt{2}, \sqrt{2})\}.$$

Then

(i) In ZFC the chromatic number of G_2 is equal to 4.

(ii) In ZF+$AC_{\aleph_0}$+LM the chromatic number of G_2 is not equal to any positive integer n nor even to $\aleph_0$.

The second example is a modification of the first one. Namely, G_3 is defined exactly the same as G_2, with the only exception that now

$$F = \{(\sqrt{2}, 0), (0, \sqrt{2})\}.$$

Then

(i) In ZFC the chromatic number of G_3 is equal to 2.

(ii) In ZF+$AC_{\aleph_0}$+LM the chromatic number of G_3 is not equal to any positive integer n nor even to $\aleph_0$.

In the present thesis we will focus our attention on yet another partial result toward a final solution of the chromatic number problem. D. Coulson in [1] considered special type of distance excluding coloring of the plane, associated with certain types of polygonal tilings. He showed that such a coloring must employ at least six colors. In chapter 2 this result is stated in details in theorem 0.1, whereas the discussion of the proof will follow in chapter 3. This result was stated for colorings of polygonal tilings of the plane. However, the author assumed that two polygons from such a tiling meet either at a vertex, or along an entire common edge. Since any polygon can be triangulated without introducing new vertices, we will consider only colorings of triangulations of the plane; i.e., polygonal tilings where each tile is a triangle and the intersection of any two tiles is an edge of both tiles, a vertex, or empty.

Chapter 2

Preliminaries

We will use the following notation. $\Re^2$ denotes the Euclidean plane with the distance between two points $x, y \in \Re^2$ given by $||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. For a fixed $\epsilon > 0$ and $x \in \Re^2$, by $S_\epsilon(x)$ we will denote the circle of radius ϵ centered at x. A closed disk is the region $D_\epsilon(x)$ bounded by $S_\epsilon(x)$.

By a *tiling* of the plane $\Re^2$ we will understand a collection of polygons $\mathcal{P}$ such that

- $\bigcup \mathcal{P} = \Re^2$
- if $P_1, P_2 \in \mathcal{P}$ then $\mathrm{Int}P_1 \cap \mathrm{Int}P_2 = \emptyset$.

The *interior* $\mathrm{Int}P$ of a triangle P is the set of all points x for which there is a closed disk $G \subseteq P$ centered at x. An *interior point* of a triangulation is a point that is in the interior of one of the elements of $\mathcal{P}$. A *boundary point* is a point that is not an interior point. Note that a boundary point is either a vertex of a triangle from $\mathcal{P}$ or it lies on an edge of such a triangle. A polygon P is *convex* if for given $x, y \in P$ the straight segment $[x, y]$ joining x and y lies entirely in P; i.e. $[x, y] \subseteq P$. A tiling $\mathcal{P}$ is locally finite if for every point $x \in \Re^2$ there is a circle S centered at x that intersects only a finite number of elements from $\mathcal{P}$. A locally finite tiling $\mathcal{P}$ is a *triangulation* if every element of $\mathcal{P}$ is a triangle and the intersection of any two tiles is an edge of both tiles, a vertex, or the empty set.

A *coloring* of the plane is a surjective function $\Gamma : \Re^2 \to \Delta$, where Δ is the *set of colors*. We will denote these colors by greek letters such as $\alpha, \beta, \gamma, \delta, \chi$. Given a point x in the plane, and a color $\alpha \in \Delta$, we shall say that x is *α-colored* if $\Gamma(x) = \alpha$. We will consider only colorings with the following property:

- if P is an element of the tilling, $\alpha \in \Delta$ is a color, $x, y \in \mathrm{Int}P$, and x is α-colored, then y is α-colored as well;

i.e. any element of the tiling has its entire interior colored by the same color.

At this point it is important to address the following issue. We will not deal with the coloring of the boundary points at all. It is important to stress out that this does not mean that the coloring function is not defined for these points. Rather it means that the main result does not depend on how these boundary points are colored. In other words, a restriction of the coloring to interior points only already forces that the set of colors must consists of at least six elements.

We say that a coloring $\Gamma(x)$ of the plane is *distance excluding* if there is a distance d such that $||x - y|| = d$ implies that $\Gamma(x) \neq \Gamma(y)$; i.e. no two points of the same color can be d distance apart. From now on let such a distance d be set once and for all.

The main result that we are going to discuss is as follows.

Theorem: If $\Gamma : \Re^2 \to \Delta$ is a distance excluding coloring of a locally finite tiling $\mathcal{P}$, then Δ consists of at least six colors.

The above result was originally proved in [1] for locally finite polygonal tilings with convex tiles, and the area of all tiles from the tiling bounded from below by a constant. However, this last assumption is not needed to prove the above theorem, when we assume local finiteness, as we are going to exhibit. Alternatively, one could drop local finiteness of the triangulation, and assume the lower bound for the area of all triangles instead, and still prove the theorem. Also, as mentioned earlier, using the fact that any polygon can be triangulated allows us to deal only with triangulations, instead of polygonal tilings. This also allows for a generalization of the result by Coulson, since in such a case we do not need to state that the elements of the tilings are convex. We can start with any polygonal tiling $\mathcal{P}$, and after triangulating each of its elements we can consider a triangulation $\mathcal{P}'$. Since such a triangulation is still locally finite and every triangle is convex, the same arguments can be applied.

The rest of the present thesis is devoted to proving the above result. We shall present in detail a number of arguments that were indicated as true in [1], but an actual proof of these facts, not immediately apparent, was not given.

Chapter 3

Chromatic number of plane triangulations

From now on, let $\Gamma : \Re^2 \to \Delta$ be a distance excluding coloring of a triangulation $\mathcal{P}$ set once and for all, and let d be the excluded distance. We shall say that a vertex T is *tri-colored* if T belongs to three triangles from $\mathcal{P}$, each of which has its interior painted by a color different than the interiors of the other two. Similarly, a vertex is *bi-colored* if it belongs to two triangles that are not colored alike.

Lemma 3.1 *There is a tri-colored vertex T in the triangulation $\mathcal{P}$.*

Proof: Suppose to the contrary that there is no tri-colored vertex. Choose a triangle P and a subcollection $\mathcal{G}$ of the triangulation, that is maximal with respect to the following properties:

1. $P \in \mathcal{G}$,
2. if $\tilde{P} \in \mathcal{G}$ then $\mathrm{Int}P$ and $\mathrm{Int}\tilde{P}$ are colored alike,
3. $\bigcup \mathcal{G}$ is connected.

Notice that $\bigcup \mathcal{G}$ is bounded. Suppose it was not. Choose $x \in \mathrm{Int}P$ and consider the circle $S_d(x)$ of radius d centered at x. Since $\bigcup \mathcal{G}$ is not bounded by $S_d(x)$ there must be a point $y \in S_d(x) \cap \bigcup \mathcal{G}$. If y is an interior point then we obtain a contradiction, since x and y would be colored alike, and in distance d apart. If y is a boundary point to a triangle $\tilde{P}$ from $\mathcal{G}$, then choose an ϵ and a circle $S_\epsilon(x)$ around x small enough so that $S_\epsilon(x) \subseteq \mathrm{Int}P$. Also, choose $\epsilon_1 \leq \epsilon$ so that the circle $S_{\epsilon_1}(y)$ intersects the interior of $\tilde{P}$. Let $z \in S_{\epsilon_1}(y) \cap \mathrm{Int}\tilde{P}$ and let $S_d(z)$ be the circle centered at z of radius d. Clearly $S_d(z)$ intersects $S_{\epsilon_1}(x)$ in a point, say w. Since $S_{\epsilon_1}(x) \subseteq S_\epsilon(x)$ therefore w and z are colored alike. This is a contradiction, since

$||w-z|| = d$. We have obtained that $\bigcup \mathcal{G}$ is bounded. Consider the complement of $\bigcup \mathcal{G}$; i.e. consider $\Re^2 \setminus \bigcup \mathcal{G}$. There is an unbounded set U, that is a component of this complement. Notice that the boundary of U separates the plane (no point in the complement of U can be joined by a line segment with a point in U, without crossing this boundary) therefore this boundary must contain a closed loop, say L. L is a union of edges, each of which is shared by a triangle from $\mathcal{G}$ and by a triangle from $\mathcal{P} \setminus \mathcal{G}$.

Consider a subcollection $\mathcal{U}$ of all triangles R from $\mathcal{P} \setminus \mathcal{G}$ that intersect L. Notice that all elements from $\mathcal{U}$ are colored alike. Indeed, suppose the contrary. Pick a triangle P_1 from $\mathcal{U}$ with an edge contained in L. Suppose, there was a triangle P_2 in $\mathcal{U}$, colored not alike to P_1. Choose an edge $E_1 \subseteq L$ of P_1, and an edge $E_2 \subseteq L$ of P_2. Since L is connected, there is a path Y consisting of edges that joins E_1 with E_2 (with possible self-intersections). By the initial assumption there is no tri-colored vertex in this path, and therefore every vertex is bi-colored. Let α be the color of $\mathcal{G}$, β be the color of P_1, and γ the color of P_2. Walking from E_1 to E_2 along Y, let v_1 be the first bi-colored vertex where α-colored and γ-colored triangles meet. Backtrack to the previous vertex v_2. At this vertex α-colored and β-colored triangles meet. Consider the edge $[v_2, v_1]$. This edge is shared by exactly two triangles, one of which is a triangle $P_3 \in \mathcal{G}$, which in turn must be α-colored. Let P_4 be the triangle from $\mathcal{U}$ sharing $[v_2, v_1]$ with P_3. If P_4 is β-colored then v_1 is tri-colored. Otherwise v_2 is tri-colored. We obtained a clear contradiction, since we assumed there is no tri-colored vertex, and therefore all elements of $\mathcal{U}$ are colored alike.

Now, complete $\mathcal{U}$ so that $\bigcup \mathcal{U}$ is maximal with respect to two properties:

1. if $P_1, P_2 \in \mathcal{U}$ then $\mathrm{Int}P_1$ and $\mathrm{Int}P_2$ are colored alike,
2. $\bigcup \mathcal{U}$ is connected.

Choose an interior point $p \in \bigcup \mathcal{U}$. If the circle $S_d(p)$ of radius d centered at p intersects $\bigcup \mathcal{U}$, then we obtain a contradiction by the same arguments that were used to exhibit that $\bigcup \mathcal{G}$ is bounded. Otherwise, repeat the same reasoning as before replacing $\mathcal{G}$ with $\mathcal{G} \cup \mathcal{U}$. The local finiteness of the tilling assures that the region bounded by $\bigcup \mathcal{U}$ will be expanding (otherwise, the areas of triangles would need to tend to zero, and there would be an accumulation point, by Bolzano Weirstrass theorem-cf. proof of proposition 3.2) and, if neccesary, iterating the above procedure finitely many times we will obtain a contradiction.

Proposition 3.2 *Let C be a circle. Then there are only finitely many boundary points on C.*

Proof: Notice that by local finiteness of the tilling, C can have a nonempty intersection with only finite number of elements from the triangulation. A contrario, suppose C intersects infinitely many triangles from the triangulation. By Bolzano-Weirstrass theorem, applied to a circle, there must be an $x \in C$, such that for any given circle S around x there are points from infinitely many of these triangles in $S \cap C$. Therefore the triangulation is not locally finite at x. A contradiction.

Now, we will show that any tile can have only finitely many boundary points on C. Suppose to the contrary that there is a tile P that has an infinite number of boundary points on C.

First, infinitely many of these points must be interior points of edges of P, since P has only finitely many vertices. Second, for given edge E of P there are at most 2 points in common for E and C, since any straight arc intersects a circle in at most 2 points. This implies that if there are infinitely many boundary points of P on C, there must be also infinitely many edges of P. But P is a triangle. Contradiction.

Consequently, C has nonempty intersection with only finitely many tiles, each of which has only finitely many boundary points on C.

Lemma 3.3 *The set of colors Δ must consists of at least five colors.*

Proof: Let C be a circle of radius d and centered at the tri-colored vertex T. By proposition 3.2 there are only finitely many boundary points on C. Let z be any point on C which is an interior point of a triangle Q. There is an $\epsilon > 0$ such that z is contained in Q with some closed disk $G_\epsilon(z)$ of radius ϵ centered at z. Clearly $G_\epsilon(z)$ is colored with the

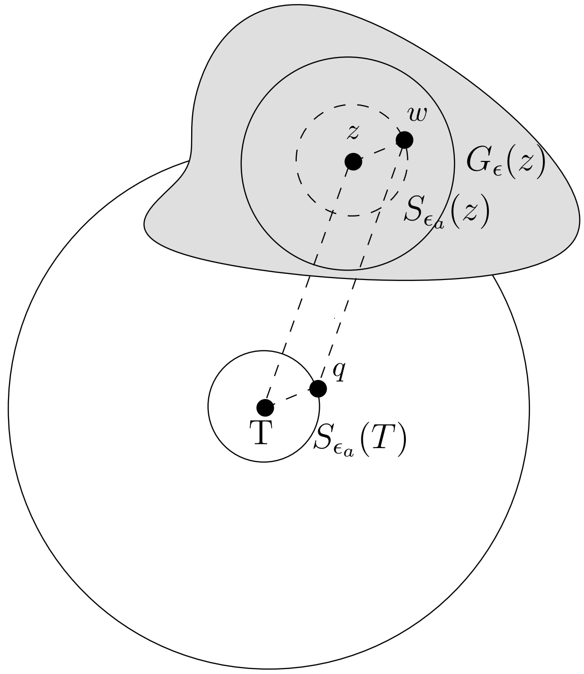

Figure 3.1: Five colors are needed.

same color as z. Let α, β, γ be the three colors meeting at T. Suppose z is colored by α and let $[z, T]$ be the straight segment of length d joining T with z. Arbitrarily close to T there are α-colored points. Therefore there is $\epsilon_\alpha \leq \epsilon$ such that there is an α-colored point q on the circle $S_{\epsilon_\alpha}(T)$ of radius ϵ_α centered at T. Consider the straight segment $[q, T]$ and along with the segment $[z, T]$ extend it to a parallelogram $[z, T, q, w]$. Then w is a point on the circle $S_{\epsilon_\alpha}(z)$ of radius ϵ_α centered at z. Since $S_{\epsilon_\alpha}(z) \subseteq G_\epsilon(z)$, w is α-colored, and in a distance d from α-colored point q. A contradiction.

By the same arguments it can be shown that z can be neither β-colored nor γ-colored. Now let v be a point on C in a distance d from z. By the same reasoning as above v cannot be either α-colored, or β-colored, or γ-colored. Since it is d distance apart from z, it cannot be of the same color as z, and therefore a fifth color is needed.

In Lemmas 3.4, 3.5, 3.6, 3.7 we assume that only five colors are used.

Lemma 3.4 *Let C be a circle of radius d and centered at the tri-colored vertex T. If $x, y \in C$, $||x - y|| = d$ and x is a boundary point of two distinct tiles colored by two different colors, then y cannot be an interior point of any tile.*

Proof: Suppose $\Delta = \{\alpha, \beta, \gamma, \delta, \chi\}$ is the set of colors, C is the circle of radius d centered at T and $x, y \in C$, $||x - y|| = d$. Again, let α, β, γ be the three colors meeting at T. Suppose

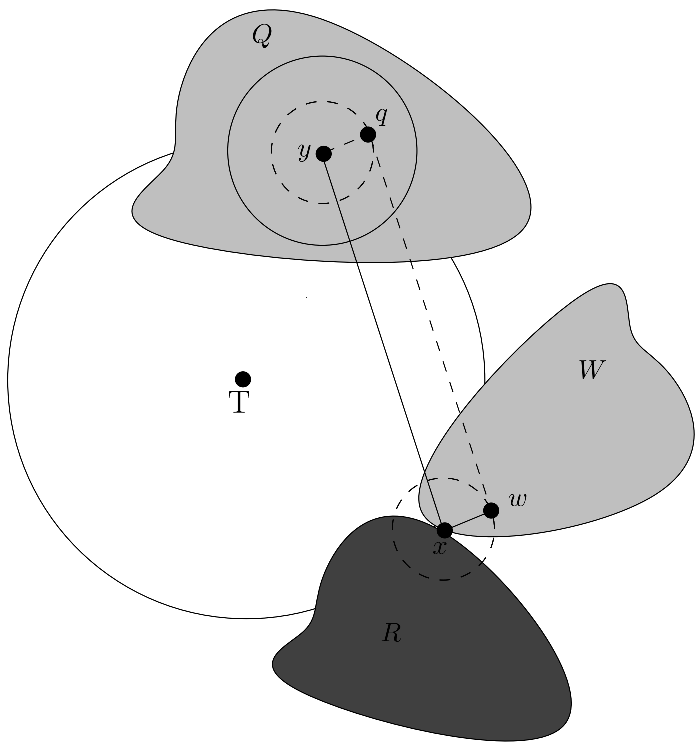

Figure 3.2: Proof of lemma 3.4.

that x is a boundary point of some two tiles, say W, R, colored by two different colors, but y is an interior point of a tile Q. By the reasoning in the proof of lemma 3.3, on the circle C there can be no interior point colored by α, β or γ. Therefore, Int$W$ and Int$R$ must be colored by δ or χ. Without loss of generality, suppose Int$W$ is δ-colored and Int$R$ is χ-colored. There is $\epsilon > 0$ such that y is contained in Q with a closed disk $G_\epsilon(y)$ of radius ϵ centered at y. Since we assume there are only 5 colors used, y must be either δ-colored or χ-colored. Suppose y is δ-colored. x is in the boundary of W and therefore there is $\epsilon_\delta \leq \epsilon$ such that there is a δ-colored point $w \in$ Int$W$ on the circle $S_{\epsilon_\delta}(x)$ of radius ϵ_δ centered at

x. Consider the straight segment $[x, w]$ and along with the straight segment $[y, x]$ extend it to a parallelogram $[y, x, w, q]$. Then q is a point on the circle $S_{\epsilon_\delta}(y)$ of radius ϵ_δ centered at y. Since $S_{\epsilon_\delta}(y) \subseteq G_\epsilon(y)$, q is in the interior of Q and therefore it is δ-colored, as y is. However, q is in a distance d from δ-colored point w. This gives a contradiction. The same arguments apply if x is χ-colored, and this concludes the proof.

Lemma 3.5 *Let C be a circle of radius d and centered at the tri-colored vertex T. Then there is a regular hexagon H inscribed into C, each vertex of which is a boundary point of some two tiles colored by two different colors.*

Proof: Let x be any point on the circle C of radius d around T, where two tiles whose

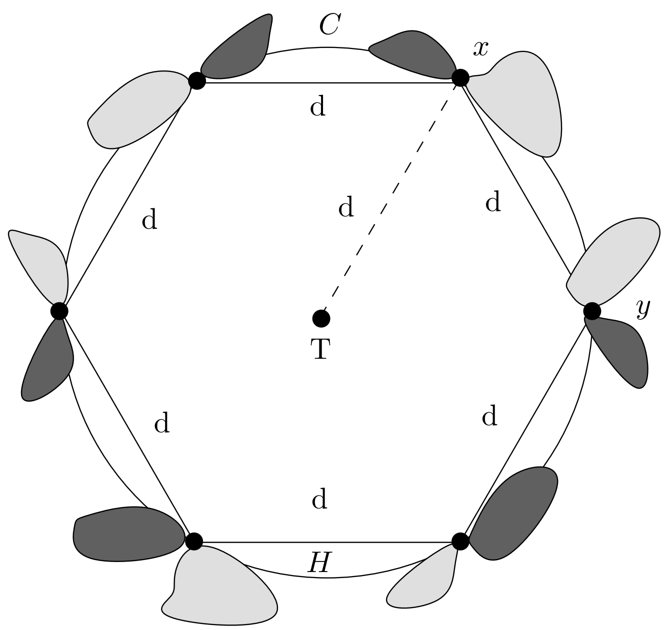

Figure 3.3: The regular hexagon H around the tri-colored vertex T.

interiors are of two different colors meet. By lemma 3.4 if y is on on the circle C in the distance d apart from x, then y is not an interior point. This process can be repeated in order to obtain four more different points with the same property as x and y, since in a circle of radius d one can inscribe a regular hexagon of side length d. Therefore these six points span a regular hexagon H inscribed into C and the proof is complete.

Let A be a circular subarc of the circle C. We will write $A =< v, w >$ to indicate the shorter of the two subarcs of C, with endpoints in v and w. We say that A invades the interior of P if $A \cap \mathrm{Int}P \neq \emptyset$.

Lemma 3.6 *Assume that the boundary of C is colored with two colors only. Let H be the regular hexagon described in lemma 3.5. Let v, w be two vertices of H sharing an edge $[v, w]$. Let $A =< v, w >$. By definition of H there are two triangles meeting at v one δ-colored, and the other χ-colored. Suppose P_v is one of the two with v in its boundary and such that the arc A invades $\mathrm{Int}P_v$. Let P_w be defined analogously for w. Then $\Gamma(\mathrm{Int}P_v) = \Gamma(\mathrm{Int}P_w)$; i.e. the interiors of P_v and P_w are of the same color.*

Proof: Let v, w, P_v, P_w be as described above. Suppose by contradiction the interiors of P_v

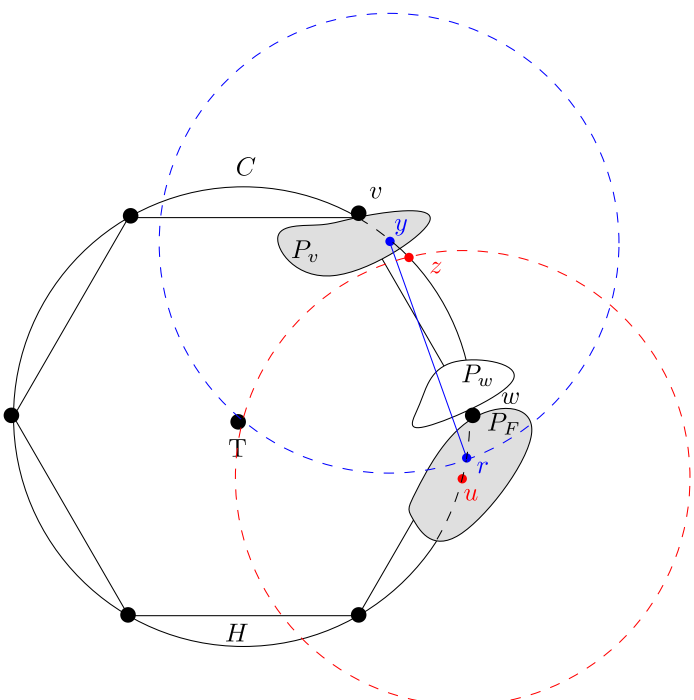

Figure 3.4: Proof of lemma 3.6.

and P_w are not of the same color; i.e. let $\mathrm{Int}P_v$ be δ-colored and $\mathrm{Int}P_w$ be χ-colored. Let F be the subarc of C such that $F = C \setminus A$, and let P_F be a δ-colored triangle from the triangulation that is invaded by F and meets P_w in w. The existence of such a triangle is guaranteed by definition of H; since we assume that P_w is χ-colored therefore P_F must be δ-colored. F invades the interior of P_F and there exists a point $u \in \mathrm{Int}P_F \cap F$. The entire subarc $< w, u >$ of F is δ-colored (with a possible exception for w). Let z be a point on A in the distance d from u. Since arc A invades $\mathrm{Int}P_v$ there must be points from P_v on the arc $< v, z >$. Choose $y \in \mathrm{Int}P_v \cap < v, z >$. Clearly y is of distance d from a point r on $< w, u >$. But this is a contradiction, since both z and y are δ-colored.

Lemma 3.7 *Let H be the regular hexagon described in lemma 3.5, and v, w its two consecutive vertices. Let $[v, w]$ be an edge of H, and R_v be a ray starting at v, that makes a right angle with $[v, w]$. Then there is an $\epsilon > 0$ and a disk $D_\epsilon(v)$ around v, such that any point on $R_v \cap D_\epsilon(v)$ is neither χ-colored nor δ-colored (χ, δ are colors meeting at at v).*

Proof: Let $[u, v]$ be the edge of H meeting $[v, w]$ at v. By the reasoning of lemma 3.6

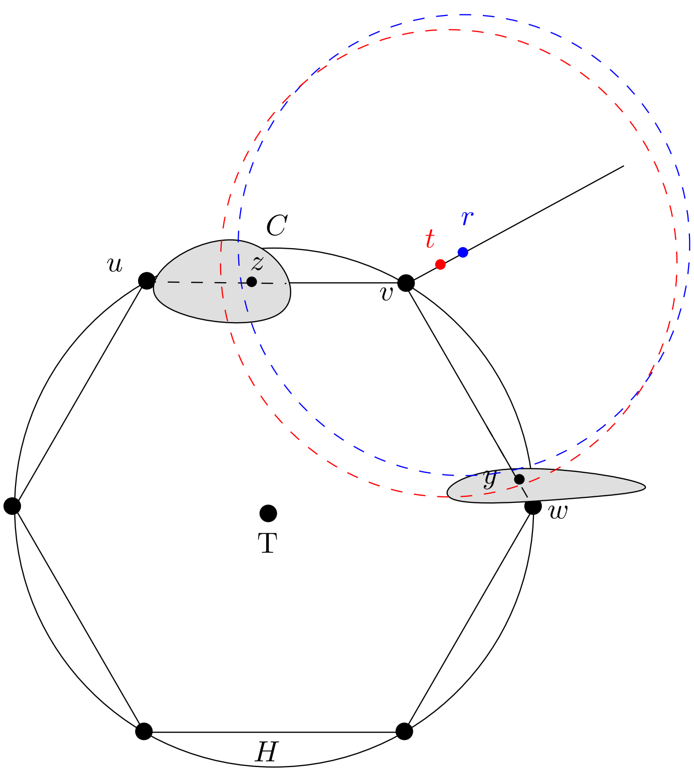

Figure 3.5: Proof of lemma 3.7.

without loss of generality we can assume that there is $z \in [u, w]$ (close to u) such that (u, z)

is colored by a single color. Similarly, we can assume that there is $y \in [v, w]$ (close to w) such that (y, w) is colored by a single color. Then (u, z) and (y, w) are not colored alike, and assume that (u, z) is χ-colored, but (y, w) is δ-colored. Now, there is a point $t \in R_v$ close enough to v so that $S_d(t)$ intersects (y, w). Clearly t cannot be δ-colored, nor any other point on the straight arc $[v, t]$. Similarly, there is $r \in R_v$ close enough to v so that $S_d(r)$ intersects (u, z). Clearly r cannot be δ-colored, nor any other point on the straight arc $[v, r]$. The lemma follows with $\epsilon = \min\{|r - v|, |t - v|\}$.

Theorem: If $\Gamma : \Re^2 \to \Delta$ is a distance excluding coloring of a locally finite tiling $\mathcal{P}$, then Δ consists of at least six colors.

Proof: Suppose to the contrary, that we can use 5 colors only and exclude the distance

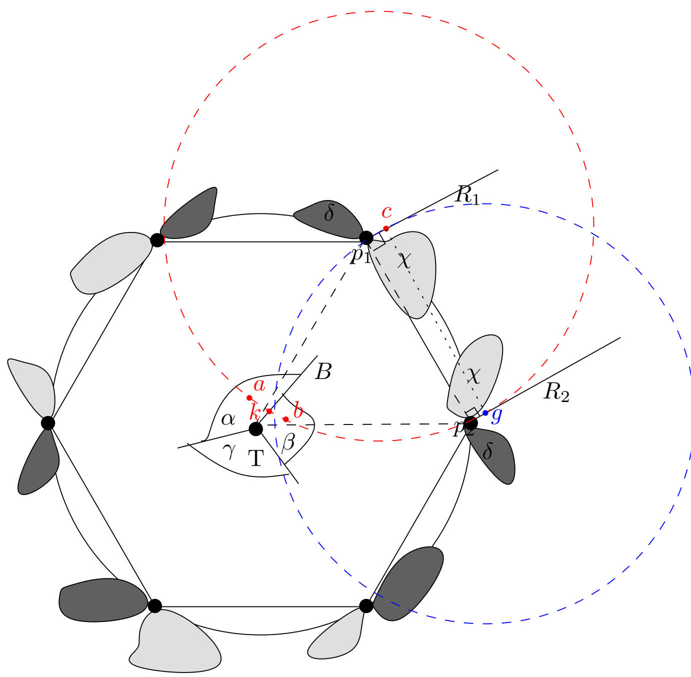

Figure 3.6: Six colors are needed.

d. Consider an edge B with a vertex at T, that belongs to two triangles P_1 and P_2 such that $T \in P_1 \cap P_2$ and IntP_1 is α-colored and IntP_2 is β-colored. The ray starting at T and containing B intersects the hexagon H in a point, say p. Let p_1 and p_2 be two vertices of the edge of the hexagon H that contains p (possibly $p = p_1$ or $p = p_2$). Notice that both $[p_1, T]$ and $[p_2, T]$ make acute angles with B.

Consider the ray R_1 starting at p_1 that makes a right angle with $[p_1, p_2]$, and the ray R_2 starting at p_2 that makes a right angle with $[p_1, p_2]$. Close to p_1 and p_2, R_1 and R_2 are neither δ-colored nor χ-colored.

Choose a point c on R_1 and g on R_2 close enough to p_1 and p_2 respectively, so that the circles $S_d(c)$,$S_d(g)$ of radius d centered at c and g respectively, both intersect the edge B in its interior. One can assure that $||c - p_1|| = ||g - p_2||$.

Let k be in $S_d(c) \cap B$. Observe that there is a circular arc $[a, b] \subseteq S_d(c)$ containing k such that the circular segment (a, k) is α-colored and the circular segment (k, b) is β-colored, by definition of B. Therefore c cannot be either α-colored or β-colored. Consequently c must be γ-colored. By the same arguments, assuming that only five colors are to be used, g must also be γ-colored. However, $||c - g|| = d$, since both R_1 and R_2 make right angle with $[p_1, p_2]$ (which is of length d) and the straight segment $[c, g]$ is parallel to $[p_1, p_2]$. This contradiction implies that a sixth color, say κ is needed, which completes the proof.

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