Evading Triangles without a Map

by

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Abstract

In this thesis we will solve the following shortest path problem. Let P be an arrangement of equilateral non-overlapping translated triangles in the plane and two points S and T so that the segment $\overline{ST}$ is parallel to a side of each of the triangles. Assume one needs to navigate from point S to point T by evading the triangular obstacles without any previous knowledge of the location of the obstacles. The navigator becomes aware of a triangle once it is contacted along the path. We will give an algorithm which enables the navigator to reach the target point T by a path of length at most $\sqrt{3}(d+\frac{2}{d})$, where d is the length of $\overline{ST}$.

Section 4 contains the proof, which is preceded by three sections reviewing some of the main results and methods of previously considered shortest path problems. In particular we will outline three papers concerning shortest path problems. First we will address the idea of permeability of a layer mentioned by J. Pach [2] in "On the Permeability Problem" using an integration technique. Then, we will show an improvement of permeability by G. Fejes Tóth [4] in his paper entitled "Evading Convex Discs" via existence of a path using the sweeping of a direction technique. Finally we will outline Chapter 3 of "Shortest Paths without a Map" by Papadimitriou and Yannakakis [3], where they show three simple heuristics of evading rectangles.

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Chapter 1

Introduction

The idea of evading convex discs was first addressed in the late 1960's by L. Fejes Tóth. By the 1980's, the topic of evading convex discs was being studied by mathematicians in a new mathematical field called computational geometry. Numerous publications centered around the idea of evading convex shapes in the plane.

The title "evading triangles without a map" refers to navigating through a plane toward a target point while evading unknown triangular obstacles. To visualize this problem, imagine walking through a thick forest with a compass and knowing in what direction and how far to travel. To restrict this forest we will assume there are mountains that are each translates of other, have bases of equilateral triangles, and are impassable. Finally we will also assume that the given direction is parallel to one side of the mountains' base. While moving the traveler cannot see through the thick forest, therefore he must rely on making decision once encountering a mountain.

Chapter 2

Background Research

In the 1970's several results were published about the notion of permeability of layers. A layer is defined as a parallel strip of a plane with width w containing non-overlapping open domains called obstacles. The permeability of a layer was defined by L. Fejes Tóth to be $\frac{w}{inf \ell}$, where ℓ is the length of a path from one edge of the layer to the other which evades all open domains of the layer. J. Pach [2] in "On the Permeability Problem" proved the following:

Theorem 2.1. *(J. Pach) The permeability of any layer of squares is at most $\frac{2}{3}$, and this bound can be achieved.*

Let $R = x_1x_2x_3x_4$ be a rectangle and $\psi_R = S_i|i \in I$ be a set of non-overlapping squares inside R (Figure 2.1). For any point x $\in x_1x_2$ we will define a path P_x to be a path from x to y, where y is the only point on x_3x_4 such that the segment $\overline{xy}$ is perpendicular to x_1x_2, which evades ψ_R. In order to evade ψ_R for every segment $\overline{u_iv_i} \in \overline{xy}$ which intersects a square $S \in \psi_R$ replace $\overline{u_iv_i}$ by the portion of the boundary between u_i and v_i on S_i which is the shortest; we will denote this as $l_i(x)$. Let $L(x)$ be the length of the path P_x

Lemma 2.1. $\int_{x_1}^{x_2} L(x)dx \leq \frac{2}{3}A(R)$, *where $A(R)$ is the area of the rectangle R.*

Proof of Lemma 2.1 will first show

$\int_{x_1}^{x_2}(l_i(x) - d(u_i(x)v_i(x)))dx \leq \frac{A(S_i)}{2}$.

If $\lambda_i(x)$ denotes the length of the shortest path of the boundary of S_i connecting $u_i(x)$ and

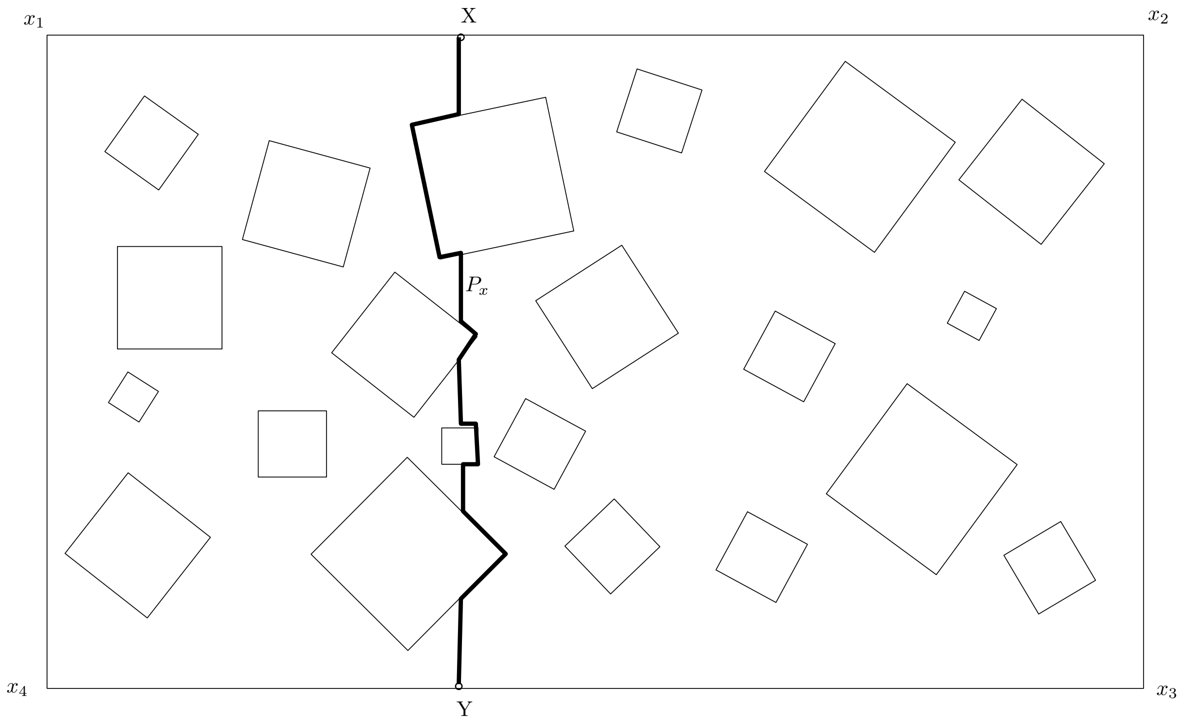

Figure 2.1: Path P_x in rectangle R

$v_i(x)$, we have

$$\int_{x_1}^{x_2} l_i(x) - d(u_i(x)v_i(x))dx = \int_{x_1}^{x_2} \lambda_i(x)dx - A(S_i)$$

$\int \lambda_i(x)dx$ is evaluated in the following way

If S_i has two of its sides parallel to $\overline{x_1x_2}$ the act of computing is trivially $\frac{2}{3}A(R)$, thus we will assume there exist a positive angle α_i between one of the sides of S_i and $\overline{x_1x_2}$ (Figure 2.1). We can also pick a side of S_i so that $0 < \alpha_i < \frac{\pi}{4}$. Now we will label the vertices s_1, s_2, s_3, and s_4 moving left to right with respect to $\overline{x_1x_2}$. We will define the side length of S_i to be d_i. We will now split S_i into three regions R_1, R_2, and R_3 by constructing lines perpendicular to $\overline{x_1x_2}$ through each of the vertices of S_i (Figure 2.2). For each region we will consider the leftmost vertical line to be it's y-axis and define r_i to be the interval spanned horizontally by region R_i.

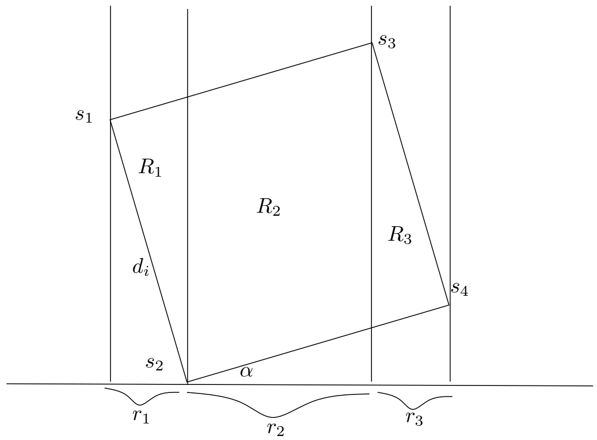

Figure 2.2: Square Regions

We will compute $\int \lambda_i(x)dx$ as $\int_{r_1} \lambda_i(x)dx + \int_{r_2} \lambda_i(x)dx + \int_{r_3} \lambda_i(x)dx$.

Now by symmetry $R_1 = R_3$, thus we need only compute for R_1 and R_2.

$\lambda_i = \frac{x}{d_i \sin(\alpha_i)}(d_i + d_i \tan(\alpha_i))$ inside R_1.(Figure 2.3)

Since the horizontal distance of the region is $d_i \sin(\alpha_i)$ we calculate $\int_{r_1} \lambda_i(x)dx$ as $\int_0^{d_i \sin(\alpha_i)} \frac{x}{d_i \sin(\alpha_i)}(d_i + d_i \tan(\alpha_i))dx = \frac{x^2(d_i+d_i \tan(\alpha_i))}{2(d_i \sin(\alpha_i))} \Big|_0^{d_i \sin(\alpha_i)} = \frac{(d_i \sin(\alpha_i))(d_i+d_i \tan(\alpha_i))}{2}$.

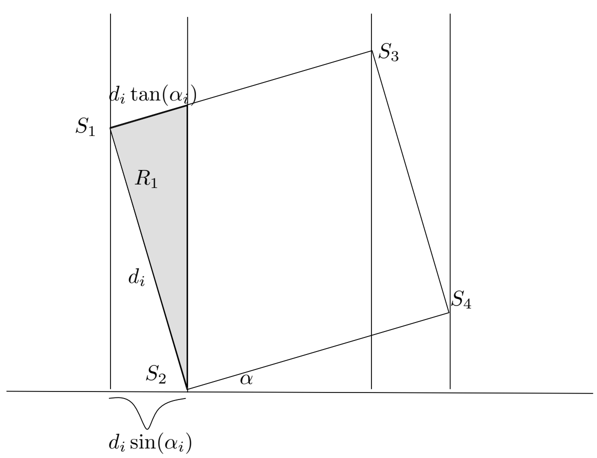

Figure 2.3: Region 1

Now for R_2 we calculate the integral over half of the region since it is symmetric. Therefore for the left half of the interval r_2, (Figure 2.4)

$\lambda_i = 2d_i - 2(\frac{x}{\cos(\alpha_i)})$ with the width of $r_2 = d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)$.

So $\int_{r_2} \lambda_i(x)dx = 2\int_0^{\frac{d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)}{2}} 2d_i - 2(\frac{x}{\cos(\alpha_i)})dx = 2(2d_ix - \frac{x^2}{\cos(\alpha_i)}|_0^{\frac{d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)}{2}}) =$
$d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)(d_i - \frac{d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)}{4\cos(\alpha_i)})$

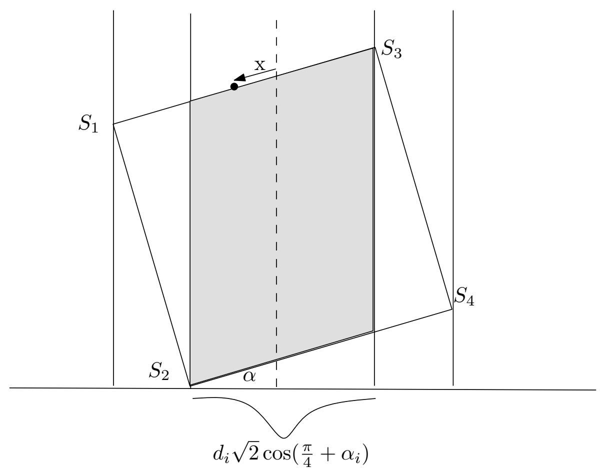

Figure 2.4: Region 2

Now by adding the two previous results we get the left half of S_i and by symmetry we multiply this sum by 2. Therefore $\int \lambda_i(x)dx = d_i\sin(\alpha_i)(d_i + d_i\tan(\alpha_i) + d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)(2d_i - \frac{d_i\sqrt{2}\cos(\frac{\pi}{4}+\alpha_i)}{2\cos(\alpha_i)} = \frac{3}{2}d_i^2\frac{1+2\cos^2(\alpha_i)}{3\cos(\alpha_i)} \leq \frac{3}{2}A(S_i)$.

Therefore $\int_{x_1}^{x_2} l_i(x) - d(u_i(x)v_i(x))dx = \int \lambda_i(x)dx - A(S_i) \leq \frac{3}{2}A(S_i) - A(S_i) = \frac{A(S_i)}{2}$

Thus $\int_{x_1}^{x_2} l_i(x)dx = \int_{x_1}^{x_2} l_i(x) - d(u_i(x)v_i(x)) + d(u_i(x)v_i(x))dx = \int_{x_1}^{x_2} l_i(x) - d(u_i(x)v_i(x))dx + \int_{x_1}^{x_2} d(u_i(x)v_i(x))dx \leq \frac{A(S_i)}{2} + A(S_i) = \frac{3A(S_i)}{2}$ $\square$.

The bound is achieved when ψ_R fills the layer (a tiling), thus $L(x)$ is just the sum of all $l_i(x)$ which produces the bound of $\frac{3}{2}$. $\square$

G. Fejes Tóth [4] improved this bound in his paper entitled "Evading Convex Discs".

Theorem 2.2. *(G. Fejes Tóth) Given a set of disjoint open squares with side-lengths not exceeding 1, any two points of the plane lying outside the squares at distance q from one another can be connected by a path evading the squares and having length at most* $\frac{3q+1}{2}$.

We let ψ be the set of open squares of side-length at most 1, as in the previous theorem. For labeling purposes we assume X is on the side X_1X_2. We define $\lambda(XY)$ to be the shortest path on the boundary of S from X to a point Y on S. Also define $\delta_d(XY) = \delta(XY)$ to be the distance gained by the vector $\overrightarrow{XY}$ in direction d. We define the path $\pi(P, d)$ to be an infinite path emanating from P in direction d. $\pi(P, d) = \pi$ is constructed by first traveling on a ray $\overrightarrow{R}$ in direction of d, then when contacting a square $S \in \psi$, evade S by traveling along the boundary of S from X to X_i such that $\frac{\delta(XX_i)}{\lambda(XX_i)}$ is maximum, and then continue in the direction of d from point X_i. Notice that $\pi(P, d)$ is not uniquely determined whenever for some $S \in \psi$ there exist more than one point X_i at which $\frac{\delta(XX_i)}{\lambda(XX_i)}$ is maximum.
Now for some point X on π we define $l(x)$ to be the length of the arc of π between P and X. To prove Theorem 2.2, first show

$$l(x) \leq \frac{3}{2}(\delta(PX) + 1)$$

To do so we need the following:

Lemma 2.2. *Let X be a boundary point on the square $S = X_1X_2X_3X_4$ with side length* 1, *such that the ray emanating from X in the direction d intersects S, then we have:*

$$\max_{1 \leq i \leq 4} \frac{\delta(XX_i)}{\lambda(XX_i)} \geq \frac{2}{3}$$

Proof of Lemma 2.2

For orientation purposes we assume that d is vertical and oriented downward. Also we can assume that π contacts S between X_1 and X_2. Since the case where the side X_1X_2 is perpendicular to d is trivial, we can also by symmetry assume that X_1 is the vertex which is vertically the highest. Now define a $= \delta(X_1X_2)$ and b $= \delta(X_1X_4)$ (Figure 2.5). Obviously $a^2+b^2=1$.

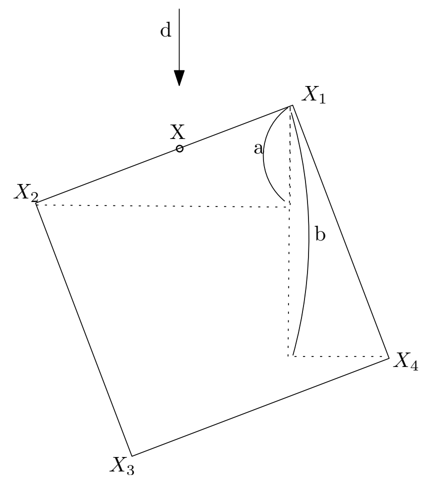

Figure 2.5: Labeling Square

Now we consider two cases

Case 1: a $\geq$ b

In this case $a \geq \frac{\sqrt{2}}{2}$, therefore $\frac{\delta(XX_2)}{\lambda(XX_2)} = a \geq \frac{\sqrt{2}}{2} > \frac{2}{3}$

Case 2: $0 < a < \frac{\sqrt{2}}{2} < b < 1$

In this case we show that either traveling to X_3 or X_4 produces $\frac{\delta}{\lambda} \geq \frac{2}{3}$.

Notice that $\frac{\delta(X_1X_4)}{\lambda(X_1X_4)} =$ b $> \frac{2}{3}$ and $\frac{\delta(X_2X_4)}{\lambda(X_2X_4)} = \frac{b-a}{2} < \frac{1}{2} < \frac{2}{3}$.

Also notice that as X moves from X_1 to X_2 the ratio $\frac{\delta(XX_4)}{\lambda(XX_4)}$ decreases, therefore there exists some point X_0 between X_1 and X_2 where $\frac{\delta(X_0X_4)}{\lambda(X_0X_4)} = \frac{2}{3}$. Obviously if $X \in X_0X_1$, then $\frac{\delta(XX_4)}{\lambda(XX_4)} \geq \frac{2}{3}$.

Now we assume $X \in X_0X_2$ and show that $\frac{\delta(XX_3)}{\lambda(XX_3)} > \frac{2}{3}$.

First notice that $\frac{\delta(XX_3)}{\lambda(XX_3)}$ increases as X approaches X_2, therefore it suffices to show that $\frac{\delta(X_0X_3)}{\lambda(X_0X_3)} \geq \frac{2}{3}$.

We denote the distance from X_0 to X_1 by x therefore producing:

$\delta(X_0X_3) = a + b - ax$ with $\lambda(X_0X_3) = 2 - x$,

$\delta(X_0X_4) = b - ax$ with $\lambda(X_0X_4) = 1 + x$.

Since $\frac{\delta(X_0X_4)}{\lambda(X_0X_4)} = \frac{2}{3} = \frac{b-ax}{1+x}$, we have 2(1 + x) = 3(b - ax)

We want to show that $2(2-x) < 3(a+b-ax)$. Since $b > 0$ this is equivalent to $3a^2-4a+1 < 0$, which is indeed true for $\frac{\sqrt{2}}{2} < a < 1$. Therefore we have shown Lemma 2.2 to be true.

Now returning to Theorem 2.2, if we begin π at a point P and end at a point X of π which does not lie on a boundary of some $S \in \psi$, then it is obvious that:

(1)

$$l(X) \leq \frac{3}{2}\delta(PX)$$

Thus we will look at the case where X lies on a boundary of a square which π evades. We will denote this square by S_i and let the first point where π contacts it be V_i and the last point where π contacts S_i be W_i. Since $l(V_i) \leq \frac{3}{2}\delta(PV_i)$, It is sufficient to prove the bound for $l(X) - l(V_i)$

We have two cases to consider.

Case 1: X and W_i lie on the same side of S_i

Notice $XW_i \leq 1$, therefore:

(2)

$$l(X) - l(V_i) = l(W_i) - l(V_i) - XW_i \leq \frac{3}{2}\delta(V_iW_i) - \delta(XW_i) =$$

$$\frac{3}{2}(\delta(V_iW_i) - \delta(XW_i)) + \frac{1}{2}\delta(XW_i) \leq \frac{3}{2}\delta(V_iX) + \frac{1}{2}$$

Case 2: X and V_i lie on the same side of S_i

In this case notice that $\frac{3}{2}\delta(V_iX) + \frac{1}{2} + l(V_i) - l(X)$ is a linear function of x. Thus it is enough to show that the endpoints satisfy the bound. If X is in fact a vertex of S, then this expression is positive, and if $X = V_i$, then as seen above this expression is $\frac{1}{2}$. Thus we know that for any point X on the path π, $l(X) \leq \frac{3}{2}(\delta(PX)+1)$.

Now for two points A and B we need to show that there exists a path π emanating from point A, travels through point B, and has a length of at most $\frac{3q+1}{2}$.

If there exist some direction d such that $\pi_{A,d}$ passes through B, then the above lemma shows existence. Therefore we shall assume there is no such direction.

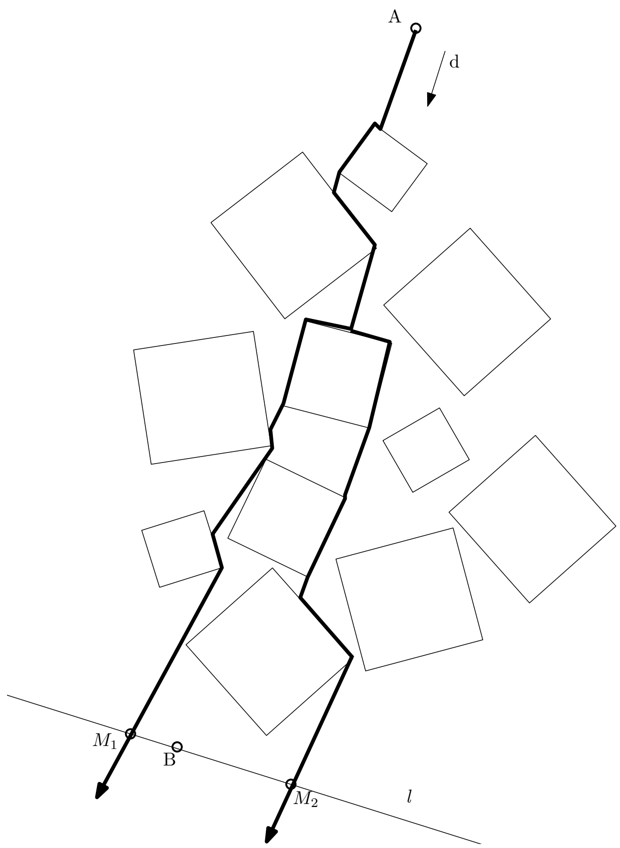

Figure 2.6: Multiple π paths

For every direction d we define a line l perpendicular to d passing through point B. As previously mentioned $\pi_{P,d}$ is not uniquely determined, thus we will label the intersection of $\pi_{A,d}$ and l by m_i, for $i \in I$ and order I from left to right on l. Consider d to be nearly perpendicular to $\overline{AB}$ pointing to the right and slightly downward; for such a d it is obvious that m_i is to the left of B. Similarly if d is to the left and slightly downward, m_i to the right of B. Now we sweep the direction d. Notice as we sweep d, if between two such directions π is uniquely determined, then m_i is continuous along l. Now if for d, there exist two vertices which satisfy Lemma 2.2, then we have two points which intersect l; one is the right most point of a continuous interval of l and the other is the left most point of a continuous interval of l. Since there is no d such that $B \in \pi_{A,d}$, then there must be some direction which has points m_i and m_j separated by B (Figure 2.6). We will consider this direction.

Now we will look at direction $-d$ and $\pi_{B,-d}$ noticing that this path must intersect one of the paths $\pi_{A,d}$. Let the first point where $\pi_{B,-d}$ intersect $\pi_{A,d}$be Z. Now Z will be on a boundary point of some $S_i \in \psi$. Thus by inequality (1) we know either $\pi_{A,d}$ or $\pi_{B,-d}$ is at most $\frac{3}{2}\delta$ and the other by (2) is at most $\frac{3}{2}\delta + \frac{1}{2}$. Now obviously $\delta_{-d}(BZ) = \delta_d(ZB)$ thus it follows that when connecting the appropriate π from A and B respectively there exists a path $P_{A,d}$ containing point B such that $l(B) \leq \frac{3}{2}AB + \frac{1}{2}$. □

Also G. Fejes Tóth uses a similar argument to show:

Theorem 2.3. *Given a set of disjoint open unit circles, any two points of the plane lying outside the circles at distance d from one another can be connected by a path evading the circles and having length at most* $\frac{2\pi}{\sqrt{27}}(d-2)+\pi$

Papadimitiou and Yannakakis [3] published "Shortest Paths without a Map" in November 1988, which is considered the first paper on this subject written by those working in the new field called "computational geometry". They introduce the concept of moving through the plane toward a target point while evading unknown obstacles in the section entitled "Obstacle Scenes". It seems natural to compare the algorithmic path to the shortest existing path, and show a bound for the ratio of their lengths. Surprisingly, in the case when the obstacles are rectangles with one side parallel to the segment having endpoints of the start (S) and target (T) such bound does not exist.

Theorem 2.4. *(Papadimitiou and Yannakakis) In the case of parallel rectangles there is no upper bound for the ratio of the algorithmic path to the shortest path.*

We prove Theorem 2.4 by constructing a configuration of rectangles which forces the ratio to be at least $\frac{\Omega(n^2)}{\Omega(n^{\frac{3}{2}})}$. We say that the path has order n, denoted by $\Omega(n)$, if the ratio is determined by some constant multiplied with n. In the construction, the start and target points are distance n apart. The configuration will consist of n rectangles each of which have a horizontal side length of ϵ and a vertical side length of n. The position of the rectangles will be revealed as we challenge the given algorithm. First we place a rectangle such that $\frac{n}{2}$ of the length of the rectangle is above the current position of the traveler and $\frac{n}{2}$ is below (Figure 2.7). Now continue this with n rectangles at each point where the traveler crosses the last vertical line shared which has a common point with the previously evaded obstacle. Thus the vertical distance traveled is at least $\frac{n}{2}$ for each of the rectangles and obviously a horizontal distance of n covered by the entire path, thus the path for the given algorithm is at least $\frac{n^2}{2} + n$, thus being $\Omega(n^2)$, and therefore the ratio of this path to the shortest path as n goes to infinity exceeds any constant bound.

Now the shortest path is found noticing that there exist a horizontal line at a distance $\leq$ n$\sqrt{n}$ from the segment $\overline{ST}$ which contacts less than $\sqrt{n}$ rectangles. For contradiction assume this line does not exist, thus assuming every line from $n\sqrt{n}$ above or below the segment $\overline{ST}$ contacts at least n rectangles. Conversely, if all lines within $n\sqrt{n}$ contact at least $\sqrt{n}$ rectangles, then sweeping a line from bottom to top of our range and integrating the total length of the common points of the horizontal line with the rectangles we conclude that the area covered by the intersection of all the lines and the rectangles is $2n^2\epsilon$. Since there are exactly n rectangles of length n and width ϵ, then there is exactly $n^2\epsilon$ area covered by the rectangles. Therefore there exists a horizontal line that contacts less than $\sqrt{n}$ rectangles, so we will use this line to traverse the layer by using the simple evade and return technique. Therefore traversing the layer on this line, with the evade and return heuristic, will have no more

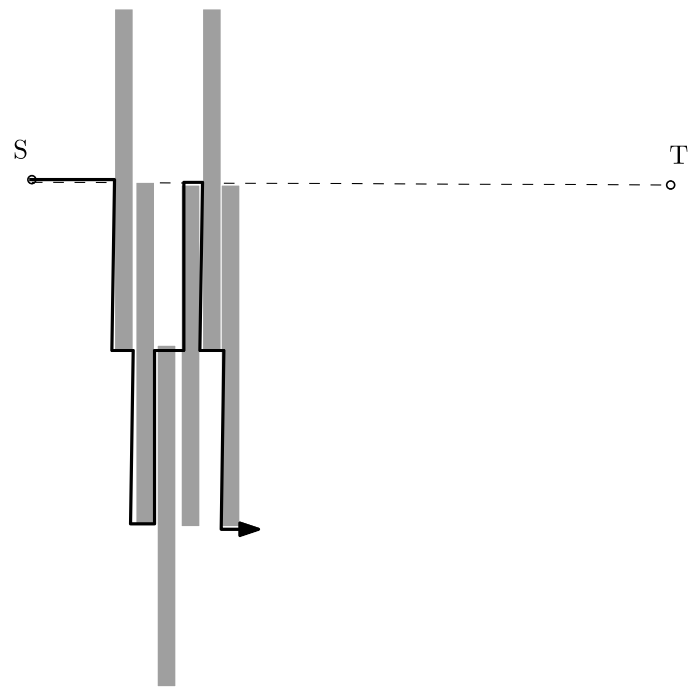

Figure 2.7: Rectangle configuration

than $n\sqrt{n}$ vertical distance and the distance of this line from $\overline{ST}$ is no more than $n\sqrt{n}$. So there exists a path from S to T that has a length of less than $3n\sqrt{n}$, thus being $\leq \Omega(n^{\frac{3}{2}})$. $\square$

The paper also addresses the issue of the obstacles being squares.

Definition 2.1. *The ratio of a given algorithm is defined ratio of the longest path created by the algorithm evading any configuration of obstacles in the plane to the length of the segment $\overline{ST}$ where s is the starting point and T is the target point of the path.*

Theorem 2.5. *No algorithm for evading square obstacles in a plane can produce a ratio better than $\frac{3}{2}$.*

To prove Theorem 2.5, following the exact same construction as before, an algorithm can produce a path length no shorter than $\Omega(\frac{3}{2}n)$. On the other hand, the same integration method shows a path of length $\Omega(n)$, thus algorithms cannot produces a ratio better than

$\frac{3}{2}$.

Finally Papadimitiou and Yannakakis give a heuristic that produces a ratio arbitrarily close to $\frac{3}{2}$ as n grows. The heuristic involves a bias which determines the direction the traveler evades a square. The heuristic allows for the path to evade a square by always choosing the corner of the square closest to segment $\overline{ST}$ provided it is no farther than $\frac{1}{2}+\beta$, where β is the bias, otherwise evade the square by going to the other corner. To determine the bias we will first define $\epsilon = \frac{1}{\sqrt{n}}$ for n the length of $\overline{ST}$, then begin with $\beta = \epsilon$. Now while moving through the plane, if the path requires you to move farther from $\overline{ST}$ then increase β by adding ϵ, but if you move closer to the segment $\overline{ST}$, then decrease β by subtracting ϵ.

In the heuristic it is obvious that β is never larger than $\frac{1}{2}$ therefore we never travel a vertical distance farther than $\frac{\frac{1}{2}}{\epsilon} = \Omega(\sqrt{n})$ from the segment $\overline{ST}$. Thus the only concern is when the path requires a vertical travel of more than $\frac{1}{2}$ to evade a square. In this case we notice that we never travel farther than $\frac{1}{2}+\beta$. Figure 2.8 is a possible graph of β.

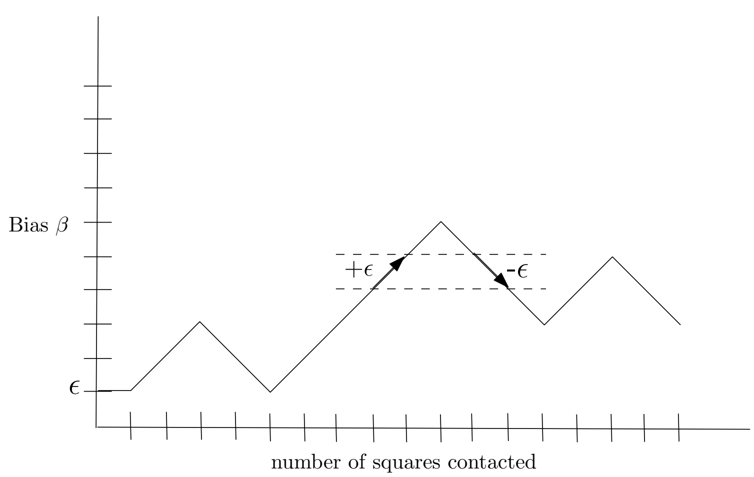

Figure 2.8: β graph

Notice that whenever the heuristic requires a travel of length greater than $\frac{1}{2}$ the value of β decreases. Now whenever this occurs there exists some evasion earlier, where β had been increased to the current value, therefore the vertical travel is equalized up to ϵ throughout the path. Since $n \cdot \epsilon = \sqrt{n}$ therefore as n grows, the heuristic produces a ratio arbitrarily close to $\frac{3}{2}$. $\square$.

Chapter 3

Problem and Heuristic

Many different shortest path problems can be formulated by simply changing the obstacles in the plane. The problem solved in this section was motivated by the results of A. Bezdek [1]. Our solution itself does not use the method of this paper, and we believe the result to be new. In this section we will solve the following:

Problem: We will assume we wish to traverse a plane toward a target point T from a starting point S while avoiding equilateral triangles of unit side length. The obstacles will be assumed to be non-overlapping translates with one side parallel to the segment $\overline{ST}$ and will be unknown until we come in contact with the triangle. Our goal is to create a heuristic which enables us to reach our target point along a path P that is shorter than the trivial path that is twice the length of the segment $\overline{ST}$.

The trivial Heuristic of evading each triangle returning to $\overline{ST}$ by following the boundary of the triangle obviously produces an upper bound of 2 (Figure 3.1).

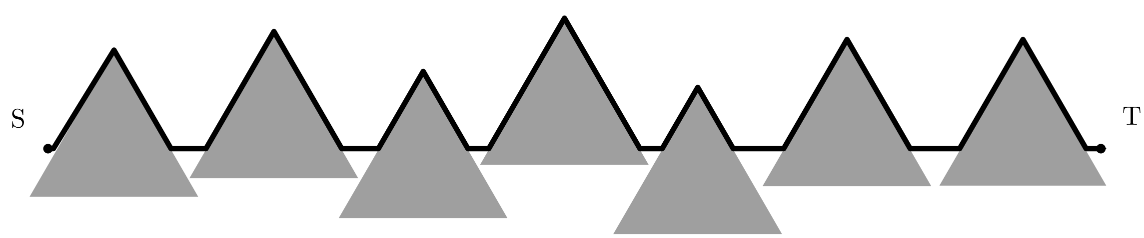

Figure 3.1: Path from S to T

A lower bound can be seen by the permeability of a layer discussed in Section 5 to be $\frac{1+\sqrt{3}}{2}$.

Triangle Heuristic: When navigating through the plane toward the target point T, follow the steps below. (For a better visual understanding we included Figure 3.2 of a concrete path created by this heuristic.

0. Start at S.

1. Travel along segment $\overline{ST}$ toward T.

   If T is reached, then go to 6.

   Else, next.

2. Travel along an edge of the contacted triangle toward the vertex not on the side parallel to the segment $\overline{ST}$.

   If you cross a line E forming a 30^o angle with $\overline{ST}$ passing through T, then go to 5.

   Else, next.

3. Take a 90^o turn toward T and travel on line L toward the segment $\overline{ST}$.

4. Travel along line L.

   a. If you reach $\overline{ST}$, repeat 1.

   b. Else, If you contact a triangle which overlaps the segment $\overline{ST}$, repeat 2.

   c. Else, If the contacted triangle does not overlap the segment $\overline{ST}$, then travel on the shortest path along the edges of the triangle back to line E and repeat 4.

5. Travel on line E toward T.

   If a triangle is contacted, avoid the triangular obstacle in the shortest path returning to line E and repeat 5.

   Else, T is reached, then go to 6.

6. Stop.

Theorem 3.1. *If the Triangle Heuristic is followed throughout the plane, then the ratio of the length of the path P to the length of the segment $\overline{ST}$ is smaller than $\sqrt{3}(1+\frac{2}{n^2}) \approx 1.732(1+\frac{2}{n^2})$, where n is the length of the segment $\overline{ST}$.*

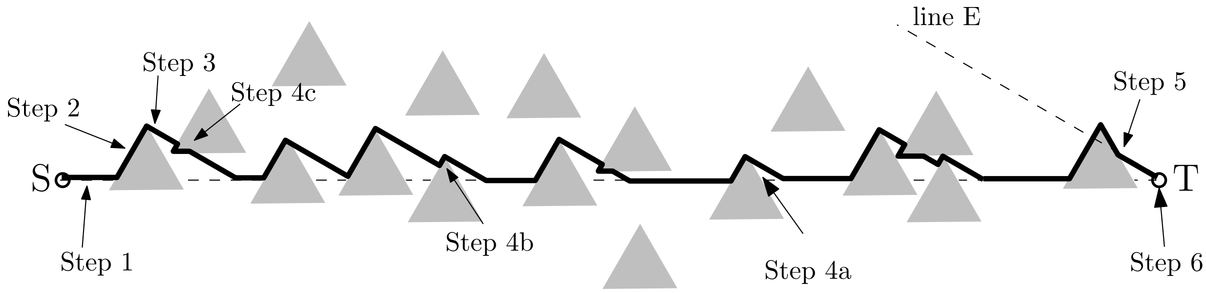

Figure 3.2: Path from S to T

Chapter 4

Proof of Theorem 3.1

For orientation purposes we assume that the segment $\overline{ST}$ is horizontal and all triangles point upward, meaning the side parallel to $\overline{ST}$ in each triangle is below the third vertex.We also use the same notation as introduced in section 3.

P $\bigcap$ $\overline{ST}$ is a collection of disjoint segments $\overline{a_i b_i}, i = 1, ..., k$. Let S $= a_1$ and assume the segments are labeled according to their order on $\overline{ST}$. We consider the subarcs $\phi_i \in P\ \overline{ST}$, where ϕ_i is the portion of P from b_i to a_{i+1}. Through the proof of Theorem 3.1 we refer to ϕ as one of these subarcs. Let L_ϕ be the length of ϕ and let D_ϕ be the length of the segment $\overline{b_i a_{i+1}}$. We will show that $\frac{L_\phi}{D_\phi} < \sqrt{3}$ for each ϕ of P.

A detailed analysis of the Triangle Heuristic reveals that there are only four different types of subarcs, ϕ, to consider:

Case 1: Subarc produced by three consecutive steps: 2 else, 3, 4a.

Case 2: Subarc produced by three consecutive steps: 2 else, 3, 4b.

Case 3: Subarc produced by three consecutive steps: 2 else, 3, 4c.

Case 4: Subarc produced by two consecutive steps: 2 if, 5.

Notice that each subarc begins by traveling to the vertex not on the side parallel to $\overline{ST}$. We will assign the variable t to the length between b_i and the before mentioned vertex. Then each time the heuristic requires a 90^o turn toward T (Figure 4.1).

Figure 4.1: ϕ_1

Now we examine each of the four types of subarcs by its direction through the triangle heuristic starting at step 2.

Case 1: Subarc produced by three consecutive steps: 2 else, 3, 4a.

We return to the segment $\overline{ST}$ without contacting another triangle(Figure 4.2).

Thus ϕ forms the legs of a right triangle with $\overline{b_i a_{i+1}}$ the hypotenuse of this triangle.

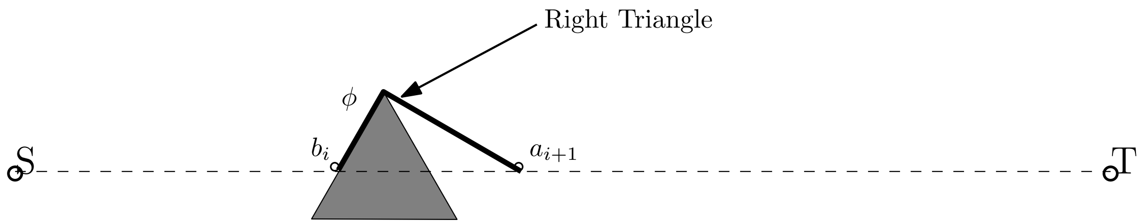

Figure 4.2: Case 1

Since the obstacles are equilateral triangle translates, this right triangle has two angles being 60 and 30 degrees respectively. Thus it is obvious that the ratio of $\frac{L_\phi}{D_\phi}$ is $\frac{2+\sqrt{3}}{4} \approx 1.366$.

Case 2: Subarc produced by three consecutive steps: 2 else, 3, 4b.

We contact a second triangle which overlaps the segment $\overline{ST}$.

In this case we let ϕ have an end point where the path reaches this second triangle; the next ϕ will begin there. Note that starting a ϕ on the edge of the triangle only reduces the ratio since the edge of the triangle is in a ratio of 2:1 with respect to $\frac{L_\phi}{D_\phi}$. We distinguish two subcases depending on the position of the second triangle. Thus the next ϕ can be considered as one of our 3 types of subarcs.

Part A: The first triangle contacted by ϕ is above the second triangle contacted (Figure 4.3).

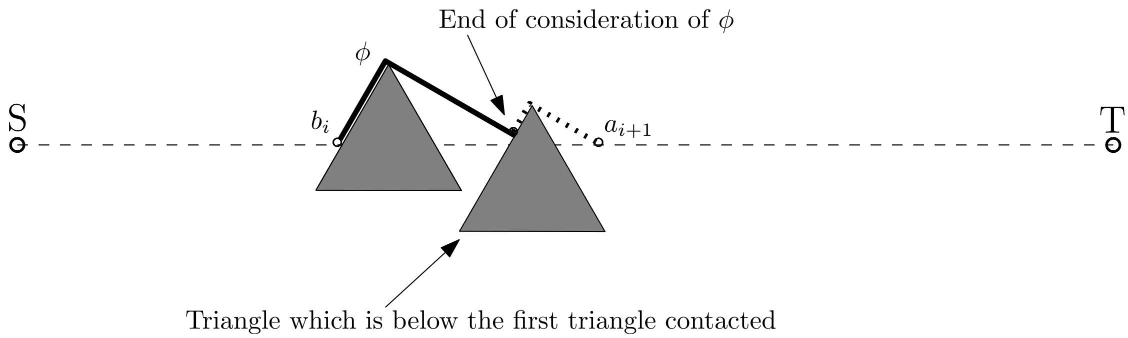

Figure 4.3: Case 2-A

The greatest ratio is found when the two triangles are touching (Figure 4.4). Here we find that the ratio of $\frac{L_\phi}{D_\phi}$ is equal to $\frac{t+\frac{\sqrt{3}}{2}}{(\frac{t}{2}+\frac{3}{4})}$ which is maximized as t approaches 1, producing a ratio approximately equal to 1.4928, being smaller than the needed bound.

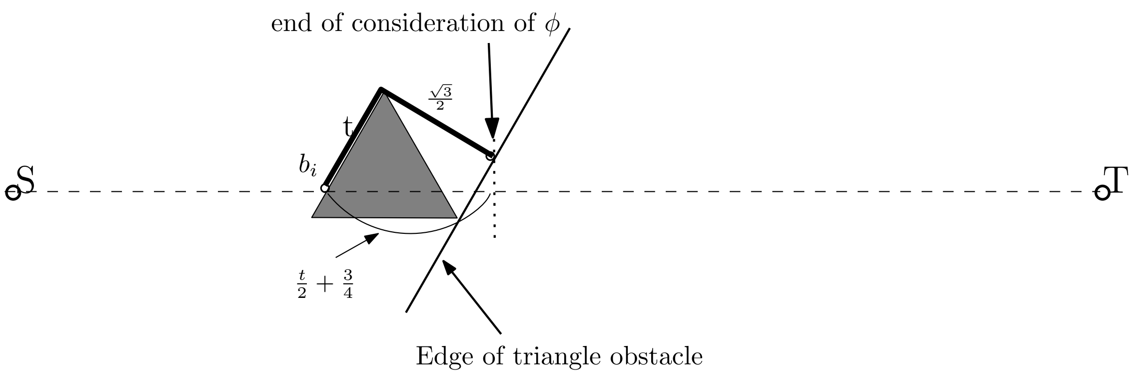

Figure 4.4: Maximizing Case 2-A

Part B: The first triangle contacted by ϕ is below the second triangle contacted (Figure 4.5).

Again it is obvious that the maximum ratio occurs when the two triangles are touching. It is also easy to see that the ratio increases as the second triangle slides along the first one so that its base gets closer to $\overline{ST}$. Indeed, when the second triangle moves toward the

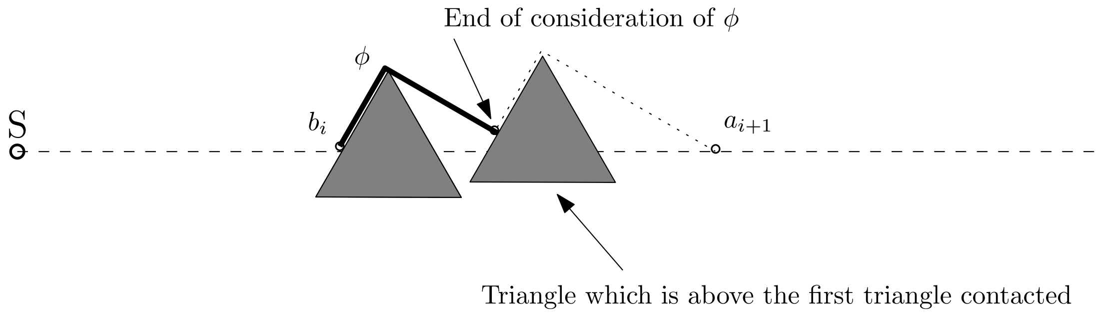

Figure 4.5: Case 2-B

segment $\overline{ST}$ along the edge of the first triangle the part of ϕ which is not following the side of the first triangle decreases. This is significant since the portion of ϕ which is following the side of the first triangles travels with a ratio of 2:1 with respect to $\frac{L_\phi}{D_\phi}$ whereas the second part of ϕ travels with a ratio of $\sqrt{3}$:1 with respect to $\frac{L_\phi}{D_\phi}$. Therefore as the second part of ϕ decreases, the ratio of $\frac{L_\phi}{D_\phi}$ increases. Thus the position of the triangles which maximizes the ratio of $\frac{\phi}{D_\phi}$ is as in Figure 4.6.

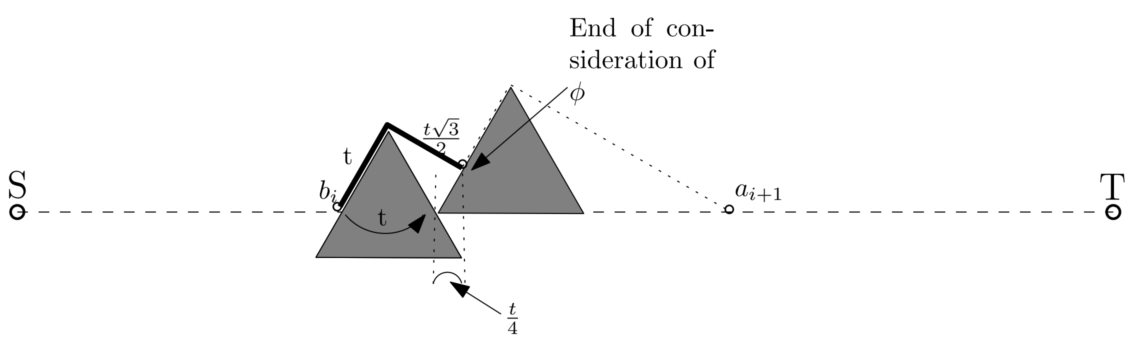

Figure 4.6: Maximizing Case 2-B

Now $\frac{L_\phi}{D_\phi} = \frac{t+\frac{\sqrt{3}t}{2}}{t+\frac{t}{4}} = \frac{4+2\sqrt{3}}{5} \approx 1.493 < \sqrt{3}$.

Case 3: Subarc produced by three consecutive steps: 2 else, 3, 4c.

We contact a triangle which does not overlap the segment $\overline{ST}$ (Figure 4.7).

Part A: We either repeat step 4 as 4a or 4c. As in the heuristic, we will avoid the

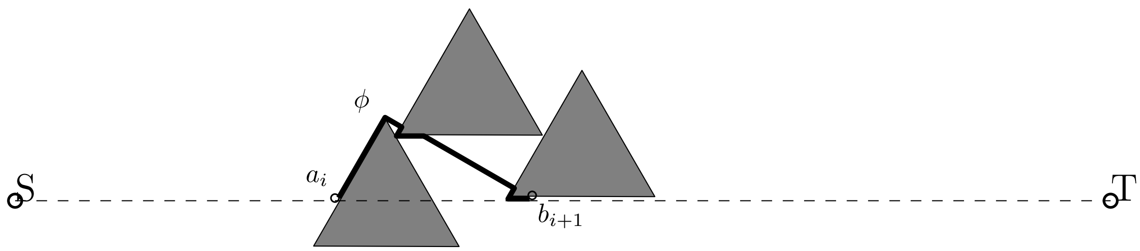

Figure 4.7: Case 3 - A

triangle by following its perimeter in the shortest direction back to $\overline{ST}$. By the assumptions of the triangles the configuration that maximizes $\frac{L_\phi}{D_\phi}$ is when the second triangle's perimeter and ϕ have the longest part, i.e. when the second triangle is contacting the first triangle on the segment $\overline{ST}$. Therefore the configuration which maximizes $\frac{L_\phi}{D\phi}$ approaches the same configuration as in Case 2 - Part B. Since the triangle is strictly above $\overline{ST}$, then ϕ ends once it has completely evaded the second triangle (Figure 4.8).

We will also make the observation that only two triangles that do not overlap the segment $\overline{ST}$ can be in contact with any particular ϕ. In this case the detours caused by two triangles are shorter than the one which can be generated by one triangle.

The ratio associated with the extreme case depicted on Figure 4.8 is $\frac{L_\phi}{D_\phi} = \frac{3+2\sqrt{3}}{2+\sqrt{3}} = \sqrt{3}$.

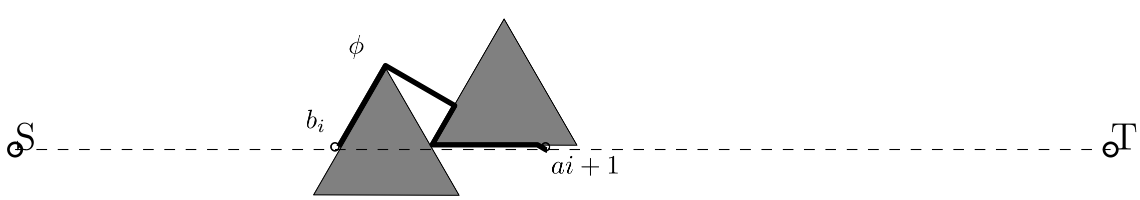

Figure 4.8: Maximizing Case 3 - A

Part B: We repeat step 4 as 4b.

We then contact a triangle which does overlap the segment $\overline{ST}$(Figure 4.9).

Notice that the triangle which overlaps the segment $\overline{ST}$ must be below the first triangle contacted as in Case 2 - Part A. Otherwise the middle triangle cannot be involved in ϕ. Therefore this case is similar to the one shown on Figure 4.9.

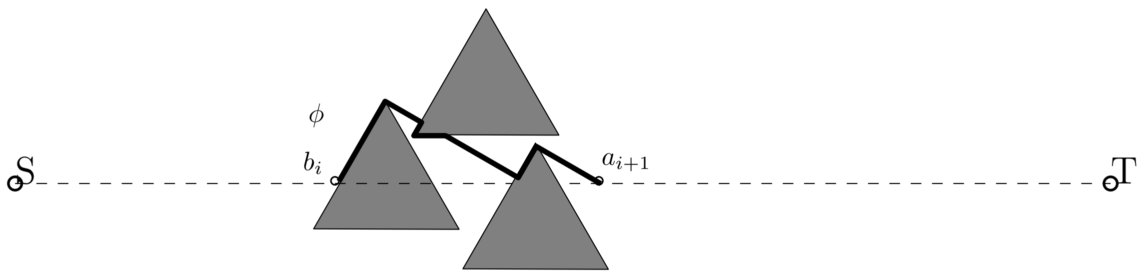

Figure 4.9: Case 3 - B

Now we find the configuration which maximizes $\frac{L_\phi}{D_\phi}$. Just as in Case 3 we can assume that the triangle which does not overlap segment $\overline{ST}$ is touching the first triangle of ϕ. Also without loss of generality, we can assume the triangle overlapping segment $\overline{ST}$ is touching the before mentioned triangle or as in Case 3 it could be moved downward to be in contact. For now we calculate $\frac{L_\phi}{D_\phi}$ as if ϕ ends when it contacts the triangle which overlaps $\overline{ST}$ as in Case 3. Finally, notice that as we lower both of these triangles while keeping them in contact, D_ϕ stays the same and obviously as in Case 3 the length of ϕ increases thus the maximum configuration is as in Figure 4.10.

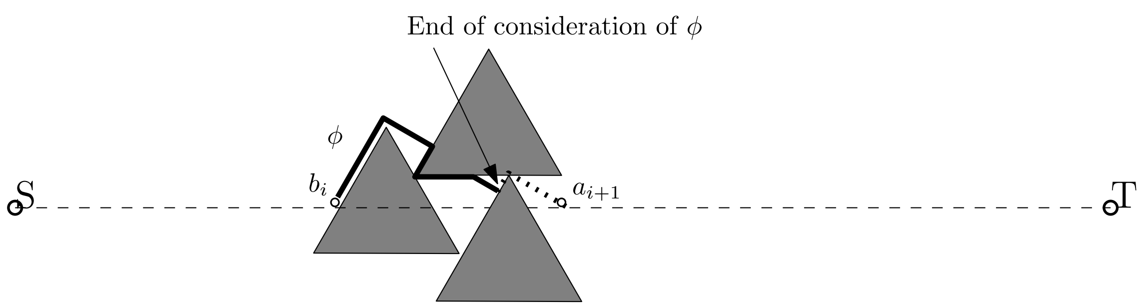

Figure 4.10: Maximizing Case 3 - B

Now for such a ϕ, $\frac{L_\phi}{D_\phi}$ depends on t. As seen above $\frac{\phi}{D_\phi}$ is $\frac{4t+3+\sqrt{3}}{2t+3} = 2 + \frac{-3+\sqrt{3}}{2t+3}$ which is obviously maximized when t = 1, thus $\leq \frac{7+\sqrt{3}}{5} \approx 1.7564$.

Currently this case can produce a ratio larger than the claim of $\sqrt{3}$ but if we have such a ϕ we will have limitation on the remaining portion of ϕ. Therefore, we will now show that these two consecutive portions produce a $\frac{L_\phi}{D_\phi}$ that is smaller than $\sqrt{3}$. So for this situation we will have the first ϕ maximum value as $4t + 3 + \sqrt{3}$, then we will look at the second ϕ. As the first ϕ increases it approaches the restriction that the second ϕ has a length t which approaches 0. Thus whichever configuration the second portion of ϕ must navigate, it no longer includes a large portion that follows the edge of a triangle having a ratio of 2:1 with respect to $\frac{L_\phi}{D_\phi}$. Without calculation we can see that all other portions of any possible ϕ are significantly smaller than the ratio of the theorem because all cases include the edge having a ratio of 2:1, but when all parts of ϕ are calculated the total of $\frac{L_\phi}{D_\phi}$ is smaller than the ratio of the theorem.

Case 4: Subarc produced by two consecutive steps: 2 if, 5.

We reach the line E which forms a 30^o angle with $\overline{ST}$ passing through T. This is the only place where the triangle heuristic does not guarantee to produce $\frac{L_\phi}{D_\phi}$ smaller than the claim. This occurs when a triangle is placed closely in front of the target point requiring the path P to evade this triangle by a ϕ which follows two edges of the triangle, thus $\frac{L_\phi}{D_\phi}$ approaches 2. Thus as $\overline{ST}$ gets larger this ϕ will have less of an impact on the overall ratio of L_ϕ to D_ϕ contributes the additional $\frac{2}{n}$ term to the bound.

Chapter 5

Lower Bound for any Heuristic and Open Problems

A lower bound for the problem given in Section 3 is easily seen to be $\frac{4}{3}$ by computing the path through the densest packing of equilateral triangular translates. If $\overline{ST}$ passes through the side of a triangle then it is easy to see that the shortest path is $\frac{4}{3}d$, where d is the length of $\overline{ST}$ (Figure 5.1), yet we will use the technique from Section 2 to improve this lower bound.

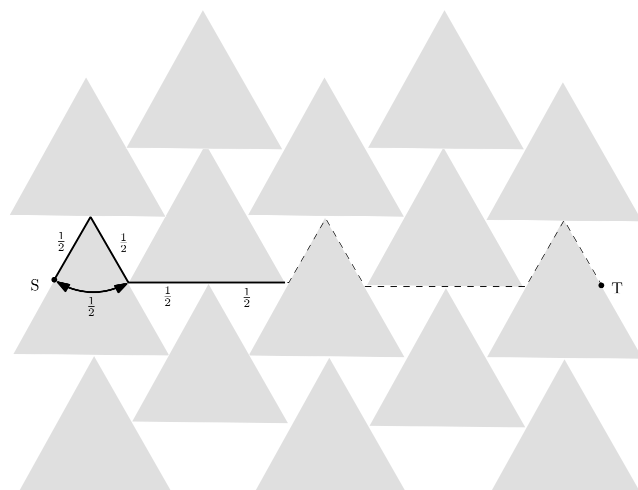

Figure 5.1: Path through densest packing

Theorem 5.1. $\frac{1+\sqrt{3}}{2}$ *is a lower bound for traversing a plane while evading triangular obstacles.*

We will challenge a heuristic by placing triangles once the navigator has reached a line extending 60^o from $\overline{ST}$, in a way similar to the technique used by Papadimitriou and Yannakakis [3]. First let S = S_1,then extend a line L_i, forming 60^o with $\overline{ST}$ through point S_i. Now place an equilateral triangle which shares a side with L_i such that S_i is the midpoint of this side. Now the triangle that has been placed will determine a layer between L_i and L_{i+1}. The vertex which is not on line L_i will now define a line L_{i+1} which goes through this vertex and forms a 60^o angle with $\overline{ST}$. Now the path created by any heuristic must intersect this line, thus consider the intersection point of the path and L_{i+1} to be S_{i+1} and repeat this process of placing triangles. Since the size of the triangles can be determined as necessary, we can form a series of layers, in this fashion, which connect S to T (Figure 5.2). For each

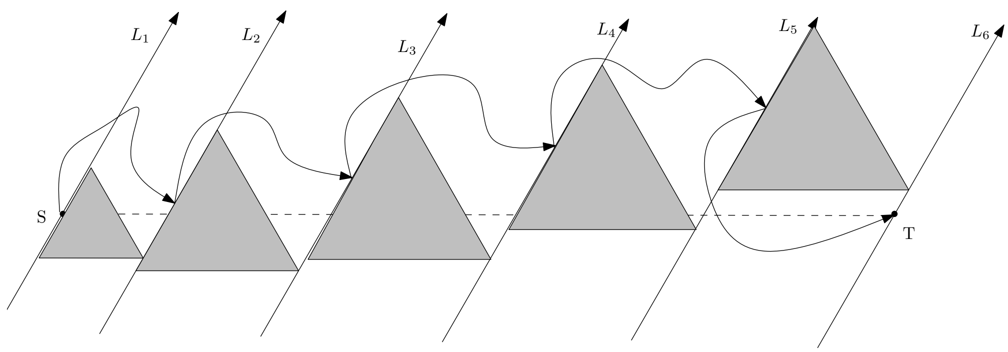

Figure 5.2: Layers between S and T

layer we will show that, with the triangle placed in this manor, the navigator will travel in a ratio of $\frac{1+\sqrt{3}}{2}$ to the distance traveled with respect to the $\overline{ST}$. We can assume the triangle has side length 1 thus forcing the navigator to travel a length of at least $\frac{1}{2} + \frac{\sqrt{3}}{2}$ to traverse the layer, and the length of $\overline{ST} \bigcap$ the layer is 1 (Figure 5.3).

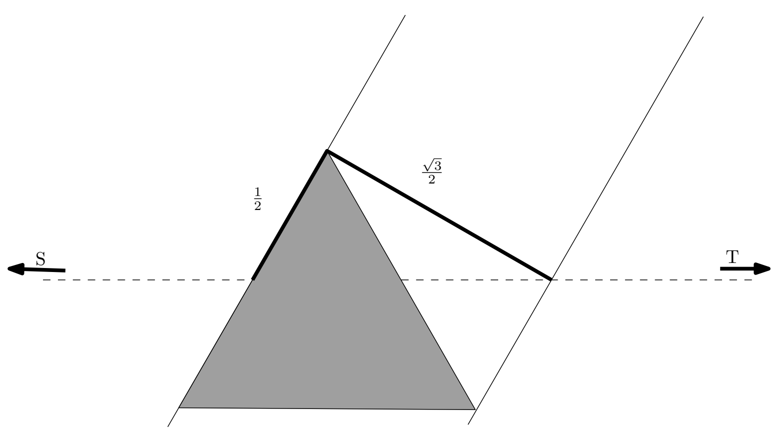

Figure 5.3: Navigating a Layer

Open Problems:

From the results of Theorem 3.1 it seems natural to investigate the possibility of changing the restraints of the obstacles. Such changes involve allowing one or more of the following:

1. The triangular obstacles need not be of the same side length.
2. The triangular obstacles can be of any orientation in the plane.
3. The triangular obstacles need not be equilateral.

From the conclusion of these variations it might be possible to generalize the results in order to traverse 3-space with obstacles being regular tetrahedra.

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