Thin-Type Dense Sets and Related Properties

by

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Abstract

Dense sets in topological spaces may be thought of as those which are ubiquitous. We discuss dense sets in product spaces which also have a thin-type property, making them in some sense rare or spread out. Thin-type properties include the previously studied properties thin, very thin, and slim. We construct examples showing that even in a separable space, there may be no countable very thin or slim dense set. We also define and discuss the properties $<\kappa$-thin, codimension 1 slim, and superslim. The definition of $<\kappa$-thin is between those of thin and very thin; the definition of superslim is between very thin and slim. Codimension 1 slim is slightly weaker than slim, in that since only some of the cross-sections are required to be nowhere dense, it is possible for a space to have a codimension 1 slim dense set but no slim dense set. We give some results about the existence of a $<\kappa$-thin dense set, in one case relating this to the existence of a very thin dense set. We show that a superslim dense set in a finite power of X is related to the existence of a certain type of collection of nowhere dense subsets of X.

The criteria (GC) and (NC), relating to collections of nowhere dense sets, are discussed. These were shown by Gruenhage, Natkaniec, and Piotrowski to imply the existence of a slim dense set in certain products. We consider when a space can satisfy (GC) with a collection of finite sets, and show that a collection witnessing (GC) cannot be uncountable if X is first countable and separable. We particularly consider ordered spaces, and characterize the linearly ordered and generalized ordered spaces which satisfy (GC), along with the linearly ordered spaces which satisfy (NC). The latter is connected to properties of ultrafilters. We also introduce a connection between a stronger version of (GC) and GN-separability.

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Chapter 1

Introduction

The concept of a dense set in a topological space is an elementary one: a set which meets every open set. Dense sets may be thought of as those that are ubiquitous in a space. In this dissertation, we look at the situations in which dense sets in product spaces may be well spread out. Thin, very thin, and so on are terms describing how sparsely a subset of a product space is distributed through the space. We shall be discussing the situations under which such special kinds of dense sets exist in product spaces, along with the characteristics of these sets. We will also consider some criteria which imply the existence of slim dense sets.

In Chapter 3, we will look at a new kind of thin dense set: a subset of a product of κ-many factors in which any two distinct points agree on less than κ coordinates. This is called $<\kappa$-thin, and the definition is between the definitions of thin and very thin. We will see that if each of κ^+ factor spaces has a dense set of size κ, there is a $<\kappa^+$-thin dense set in the product; whereas if each factor space has small cardinality relative to κ, the situation reduces to the existence of a very thin dense set. In particular, if X is separable, X^{ω_1} has a $<\omega_1$-thin dense set; but if $|X| < \omega$, X^{ω_1} has no $<\omega_1$-thin dense set. Using similar proof techniques, we address the question of the existence of very thin dense sets and slim dense sets of minimal cardinalities. Examples are constructed showing that a separable space with a dense set that is slim or very thin need not have a countable dense set of that type. These examples are subspaces of $2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of the continuum. We also prove (under MA+$\neg$CH) an extension of a theorem of Schröder [11] giving conditions on X which guarantee the existence of countable very thin dense set in $X^{\mathfrak{c}}$.

Chapter 4 examines a weakened form of slim which was proposed by Gruenhage, originally as an alternate definition of slim. We show that this property, called codimension 1 slim, is indeed weaker than slimness (in products of more than two factors) by constructing a space which has a codimension 1 slim dense set but no slim dense set. We then consider the property superslim (which, like "slim" and "codimension 1 slim", is based on cross-sections), and show an equivalence to a version of (NC), a criterion related to the existence slim dense sets.

(NC) is studied more extensively in Chapter 5, along with the related criterion (GC). (GC) and (NC) are specific conditions on the factor spaces which imply the existence of a slim dense set in a product. These are introduced in [6]. Both deal with a space having a certain type of collection of pairwise disjoint nowhere dense sets. We give some examples concerning the circumstances under which a collection satisfying (GC) can consist of finite or countable sets. We also discuss the possibility of such a collection being uncountable, giving results in the direction of impossibility. In particular, in any infinite T_2 space, one cannot have a collection witnessing (GC) which is uncountable and consists of finite sets.

In further characterizing spaces which satisfy (GC) or (NC), we prove that if X has a dense metrizable subspace, X satisfies (GC). This leads to some results on when a linearly ordered space (LOTS) satisfies (GC), as well as a characterization of generalized ordered spaces (GO-spaces) which satisfy (GC) as those which have a σ-disjoint π-base. We also construct a LOTS which does not satisfy (NC). The failure occurs because the space is the topological sum of two spaces with very different open sets; so we next consider when a sum $X \oplus Y$ may satisfy (NC). We show that if points have neighborhoods which are uniform in cardinality, the space will satisfy (NC). This is applied to a linearly ordered space to give a characterization of (NC) based on the intervals of the space. Also, (NC) satisfied by a sum of two strongly irresolvable spaces is characterized by the relationship of related ultrafilters under the Rudin-Keisler order on $\beta\omega$.

Finally, we consider how a space satisfying (GC) witnessed by a collection of finite sets is related to the space possessing the property of GN-separability, a selective separability property.

Chapter 2

Background

The concepts of "thin" and "very thin" sets were defined by Piotrowski [10]. "Slim" was defined by Gruenhage in [6].

Definition 2.1. *Let $X = \prod_{\alpha<\kappa} X_\alpha$ be a product space, and let $D \subseteq X$.*

1. *D is thin if whenever $x, y \in D$ with $x \neq y$, then $|\{\alpha < \kappa : x_\alpha \neq y_\alpha\}| > 1$.*

2. *D is very thin if whenever $x, y \in D$ with $x \neq y$, $x_\alpha \neq y_\alpha$ for all $\alpha < \kappa$.*

3. *D is slim if for every non-empty proper subset $K \subset \kappa$ and $\nu \in \prod_{\alpha \in K} X_\alpha$, the set $D \cap C(\nu)$ is nowhere dense in $C(\nu)$, where $C(\nu) = \{x \in X : x \restriction K = \nu\}$ is the cross-section of X at ν. We will call $D \cap C(\nu)$ the cross-section of D at ν.*

The definition we have given for very thin is stated in such a way as to show the connection with thin; very thin was defined in Piotrowski's paper [10] in the equivalent formulation: $\forall \alpha < \kappa$ and $p \in X_\alpha$, $|\{x \in D : x_\alpha = p\}| \leq 1$.

Geometrically, in $\mathbb{R}^3$, no two points of a thin set can lie on a line parallel to an axis. For a very thin set in $\mathbb{R}^3$, no two points can be on a line or in a plane parallel to an axis. It is clear that for X^2, the notions of thin and very thin coincide.

Piotrowski [10] showed that the product of 2^ω separable spaces has a countable thin dense subset, and that the product of 2^ω spaces, each of which has a countable π-base, has a very thin dense set. (Recall that a π-base is a collection $\mathcal{B}$ of non-empty open subsets of X such that each open set in X contains a member of $\mathcal{B}$.) He asked whether the second result could be weakened to "separable" instead of "countable π-base". Schröder [11] and Szeptycki [12] each constructed a separable space with no (very) thin dense set in its square. Schröder, in fact, constructed a class of counterexamples, and also discussed the number of topologies on a countable set which do not allow the square to have a thin dense set. He also weakened the π-base requirement slightly by only requiring a countable weak π-base; that is, the sets in the collection $\mathcal{B}$ are only required to be infinite, not open.

In [6], Gruenhage, Natkaniec, and Piotrowski prove a variety of results about the existence and nonexistence of thin, very thin, and slim dense sets in products of different spaces and in powers of a space.

The authors begin by presenting a generalization of Schröder's result about countable π-bases, plus an implication of the existence of a very thin dense set. This is based on some cardinal functions of X: $\Delta(X)$ is the least cardinal of a non-empty open subset of X; $d(X)$ is the least cardinal of a dense subset of X; and $\pi w(X)$ is the least cardinal of a π-base for X.

Proposition 2.2. *[6] Assume $X = \prod_{\alpha<\kappa} X_\alpha$, where all X_α are dense in themselves. Consider the following statements:*

(i) $\lambda = \inf_{\alpha<\kappa} \Delta(X_\alpha) \geq \sup_{\alpha<\kappa} \pi w(X_\alpha)$ *and* $2^\lambda \geq \kappa$.

(ii) *There is a very thin dense set in X.*

(iii) $\Delta(X_\alpha) \geq d(X_\beta)$ *for any* $\alpha, \beta < \kappa,\ \alpha \neq \beta$.

Then (i)⇒(ii)⇒(iii).

This is useful for demonstrating whether or not X has a very thin dense set. The authors also construct spaces which do not have certain types of dense sets, including:

- A metrizable space X for which X^n has no thin dense set for any $n \in \mathbb{N}$, and X^ω has no very thin dense set.

- Countably many dense-in-themselves spaces whose product does not have a thin dense set.

- Under (CH), a countable regular space X such that X^2 has a thin dense subset, but X^3 does not.

- Under(CH), a countable regular space X such that X^2 has no slim dense subset.

- A space X for which X^n has no slim dense set for any $n < \omega$ but X^ω does have a slim dense set.

- Under (CH), a countable regular space X such that X^2 has a very thin dense subset, X^3 has a thin dense set, and X^3 has no slim or very thin dense subset.

Another useful result on the existence of slim dense sets in infinite powers is the following, based on a sequence of dense sets in X.

Proposition 2.3. *[6] Suppose X admits a decreasing sequence D_n of dense sets such that $\bigcap_{n<\omega} D_n = \emptyset$. Then for any infinite cardinal κ, X^κ has a dense set which is both slim and thin.*

This is applied to show that under any of the following conditions on X, X^κ has a dense slim and thin set for any infinite cardinal κ:

- X is ω-resolvable;

- X has a meager dense subset;

- X is a separable Hausdorff space with no isolated points.

The last condition can be generalized to products of different spaces. However, Proposition 2.3 cannot be generalized to products of different spaces; an example is given to show that there is a sequence of countably many spaces with no isolated points, each having such a decreasing sequence of dense sets, whose product has no slim or thin dense set. The authors also show that, under $V = L$, every infinite power of a dense-in-itself space has a slim and thin dense set.

Gruenhage and Natkaniec [6] also introduce the criteria (GC) and (NC) on a space X, which imply that certain products have slim dense sets.

(NC) There is a pairwise disjoint collection $\mathcal{N}$ of nowhere dense sets in X such that, given any finite collection $\mathcal{U}$ of nonempty open sets in X, there is some $N \in \mathcal{N}$ which meets every $U \in \mathcal{U}$.

(GC) There is a pairwise disjoint collection $\mathcal{N}$ of nowhere dense sets in X such that every nonempty open set U meets all but finitely many $N \in \mathcal{N}$.

A weaker version of (NC) is also defined: for each $k < \omega$, (NC_k) is the statement obtained by requiring the collection $\mathcal{U}$ in the definition of (NC) to have cardinality $\leq k$.

(NC) and (GC) are studied in [6], with the following results. If X satisfies (NC), then every power of X has a slim dense set. If X_α, satisfies (GC) for all $\alpha < \kappa$, then there is a slim dense set in $\prod_{\alpha<\kappa} X_\alpha$. Several theorems about types of spaces which satisfy (NC) or (GC) are proved, and several spaces are constructed whose products do not have special dense sets of some kind. In particular, every metrizable dense-in-itself space, and every separable space with π-weight ω, satisfies (GC). While (GC) implies (NC), there is a space which satisfies (NC) but not (GC). (GC) is preserved by taking a topological sum, while (NC) is not. However, both (NC) and (GC) are productive.

It is not known whether there is a consistent example of a dense-in-itself space X such that X^ω does not have a slim dense set. However, such an example, if it exists, is shown in [6] to have a subspace which is strongly irresolvable and Baire. In seeking candidates for such an example, the authors show that a strongly irresolvable space X for which the ideal $\mathcal{N}$ of nowhere dense sets is selective cannot satisfy (NC). It is also shown that if a strongly irresolvable space satisfies (NC) by a collection $\mathcal{N}$, then the family of all $\mathcal{M} \subset \mathcal{N}$ such that $\mathcal{M}$ also witnesses (NC) is an ultrafilter on $\mathcal{N}$.

There is a connection between a stronger version of (GC) and a selective separability property GN-separable. Selective separability properties were studied in [1] by Bella, Bonanzinga, and Matveev. They discuss several stronger versions of separability, along with related covering properties, giving many implications involving the π-weight of a space and tightness properties, especially for subspaces of 2^κ.

Bella, Bonanzinga, and Matveev prove that X is GN-separable iff X is R-separable and every countable dense subset of X contains a groupable subset. Gruenhage observed that one could use techniques similar to those used in [6] for slim spaces to answer questions about selective separability [4]; there are also connections between the criterion (GC) and GN-separability, which we will look into.

Most of the results in the literature, and in this dissertation, involve X being dense-in-itself (having no isolated points); isolated points inhibit the formation of "nice" dense sets. This is especially true in the last section on (GC) and (NC).

We will denote the βth coordinate of a point $\overline{x} \in \prod_{\alpha<\kappa} X_\alpha$ by $\overline{x}_\beta$ or by $\overline{x}(\beta)$. All spaces are assumed to be Hausdorff unless otherwise noted. κ is a cardinal number, and κ^+ is the least cardinal greater than κ.

Chapter 3

Thin-type Properties

Two questions naturally arise when considering thin and very thin dense sets in product spaces. One is (if the product has very many factors at all) the gap between the two concepts. Distinct points in thin sets need only differ at more than one coordinate, while in very thin sets all coordinates must be different. There are obvious proposals for an intermediate definition, one of which we will consider here under the name $<\kappa$-thin. Another natural question is whether a product space which is separable, and has a special dense set, must have a special dense set witnessing the separability. We construct an example to show that this does not have to be the case.

3.1 $<\kappa$-thin Dense Sets

Definition 3.1. *Let D be a subset of $\prod_{\alpha<\lambda} X_\alpha$ and let κ be a cardinal less than or equal to λ. D is $<\kappa$-thin if for any $x, y \in D$, $x \neq y$,*

$$|\{\alpha < \lambda : x_\alpha = y_\alpha\}| < \kappa.$$

In the case that $\kappa = \lambda$ is regular, this is equivalent to: D is $<\kappa$-*thin* if for any $x, y \in D$, $x \neq y$, there is an $\alpha^* < \kappa$ such that $x_\alpha \neq y_\alpha$ for all $\alpha > \alpha^*$.

We will consider when products of κ-many spaces have $<\kappa$-thin dense sets, for an infinite cardinal κ. Note that if $|X| \geq \kappa$, then X^κ has a (more than) $<\kappa$-thin dense set by Theorem 2.4 in [6]. The dense set constructed there has the property that any distinct points differ on all but finitely many coordinates. We will now show that we also get a $<\kappa^+$-thin dense set in a product of κ^+ different spaces, provided each factor space has a dense subset of size κ.

First, we have a lemma, which has been established previously (see the reference to this fact in, for instance, [2], and in [16] for $\kappa^+ = \omega_1$), but is proved here for the convenience of the reader.

Lemma 3.2. *If κ is an infinite cardinal, there is a family $\{f_\xi : \xi < \kappa^+\}$ of functions from κ^+ to κ, with the property that if $\psi < \gamma < \kappa^+$, $|\{\alpha < \kappa^+ | f_\psi(\alpha) = f_\gamma(\alpha)\}| \leq \kappa$.*

Such a family is called almost disjoint or eventually different.

Proof. For each infinite ordinal $\alpha < \kappa^+$, let $g_\alpha : \alpha \to \kappa$ be a 1-1 function. Since $|\alpha| \leq \kappa$ for any $\alpha < \kappa^+$, this is possible. Then, for each ordinal ξ with $\omega \leq \xi < \kappa^+$, let f_ξ be defined by

$$f_\xi(\alpha) = \begin{cases} 0 & \alpha \leq \xi \\ g_\alpha(\xi) & \alpha > \xi \end{cases}$$

It is clear that $\{f_\xi : \omega \leq \xi < \kappa^+\}$ is a family of functions from κ^+ to κ.

Suppose $\omega < \psi < \xi < \kappa^+$. For any $\alpha > \xi$, $f_\psi(\alpha) = f_\xi(\alpha)$ implies that $g_\alpha(\psi) = g_\alpha(\xi)$, which is impossible since g is one-to-one and $\psi \neq \xi$. So

$$|\{\alpha < \kappa^+ | f_\psi(\alpha) = f_\xi(\alpha)\}| \leq |\xi| \leq \kappa.$$

This shows first that the functions are distinct for distinct $\xi < \kappa^+$; and also that the family is almost disjoint.

□

We can now prove:

Theorem 3.3. *For each $\alpha < \kappa^+$, let X_α have a dense subset of size κ. Then $\prod_{\alpha<\kappa^+} X_\alpha$ has a $<\kappa^+$-thin dense set.*

Proof. For each $\alpha < \kappa^+$, let $D^\alpha = \{d^\alpha_\xi : \xi < \kappa\}$ be the dense set in X_α. Let $\mathcal{F} = \{f_\beta : \beta < \kappa^+\}$ be the family of all functions f from a finite subset of κ^+ into κ. Let $\{g_\beta : \beta < \kappa^+\}$ be a family of κ^+ almost disjoint functions from κ^+ to κ. Define $e_\beta \in \prod_{\alpha<\kappa^+} X_\alpha$ as follows: $e_\beta(\alpha) = d^\alpha_{f_\beta(\alpha)}$ for all $\alpha \in dom f_\beta$ and $e_\beta(\alpha) = d^\alpha_{g_\beta(\alpha)}$ otherwise. Then the set $E = \{e_\beta : \beta < \kappa^+\}$ is $<\kappa^+$-thin dense: E is dense because, given any basic open set $\bigcap\limits_{i=1}^{n} \pi^{-1}_{\alpha_i}(U_i)$, there is a function $f_\beta \in \mathcal{F}$ with $dom f_\beta = \{\alpha_1, ..., \alpha_n\}$ and $d^{\alpha_i}_{f_\beta(\alpha_i)} \in U_i$. Given $e_\alpha, e_\beta \in E$, with $\alpha \neq \beta$, for all but finitely many $\gamma \in \kappa^+$, $e_\alpha(\gamma)$ is $d^\gamma_{g_\alpha(\gamma)}$ and $e_\beta(\gamma)$ is $d^\gamma_{g_\beta(\gamma)}$. Since g_α, g_β are almost disjoint functions on κ^+, they agree on at most $\kappa < \kappa^+$ points of κ^+. Thus, E is $<\kappa^+$-thin. □

In particular, Theorem 3.3 says that if X is separable, X^{ω_1} has a $<\omega_1$-thin dense set.

In the case of small (relative to the power) factor spaces, the situation relates directly to the existence of a very thin dense set.

Theorem 3.4. *If κ is an infinite regular cardinal and $|X|^+ < \kappa$, any $<\kappa$-thin set in X^κ has cardinality $\leq |X|$.*

Proof. Suppose E is a $<\kappa$-thin set in X^κ with $|E| > |X|$. Choose a set D consisting of $|X|^+$ distinct points from E; say $D = \{d^\alpha : \alpha < |X|^+\}$. For each (α, β) in $|X|^+ \times |X|^+$ with $\alpha < \beta$, let $\gamma_{(\alpha,\beta)}$ be the least element of κ with the property that $d^\alpha(\gamma) \neq d^\beta(\gamma)$ for all $\gamma > \gamma_{(\alpha,\beta)}$. This is possible because E is $<\kappa$-thin. Let $\gamma^* = sup\{\gamma_{(\alpha,\beta)} : \alpha, \beta < |X|^+,\ \alpha < \beta\}$. Note that since $|X|^+ \times |X|^+ < \kappa$, and κ is regular, $\gamma^* < \kappa$. Then, for all $\gamma > \gamma^*$ and $\alpha, \beta \in |X|^+$, $d^\alpha(\gamma) \neq d^\beta(\gamma)$. That is, $D' = \{(d^\alpha_\gamma)_{(\gamma>\gamma^*)} : d^\alpha \in D\}$ is very thin. Thus, $|D'| \leq |X|$, and $|D| = |D'|$. This contradicts our choice of $|D| > |X|$, so X has no $<\kappa$-thin subset of size greater than $|X|$. □

A specific consequence of this is:

Corollary 3.5. *If $|X_\alpha| < \omega$ for all $\alpha < \omega_1$, there is no $<\omega_1$-thin dense set in $\prod_{\alpha<\omega_1} X_\alpha$.*

Proof. Let $A \subset \omega_1$ be such that $|X_\alpha| = k$ for all $\alpha \in A$, where k is some finite number. Then, a $<\omega_1$-thin dense set in $\prod_{\alpha<\omega_1} X_\alpha$ will give a $<\omega_1$-thin dense set in $\prod_{\alpha\in A} X_\alpha$ by restricting the coordinates. $\prod_{\alpha\in A} X_\alpha$ is the same as k^{ω_1}, and by Theorem 3.4, a $<\omega_1$-thin set in k^{ω_1} has size $\leq k$, and thus cannot be dense. So there is no $<\omega_1$-thin dense set in $\prod_{\alpha<\omega_1} X_\alpha$. □

Theorem 3.4's effect on the existence of a $<\kappa$-thin dense set may now be clearly seen.

Corollary 3.6. *If κ is an infinite regular cardinal and $|X|^+ < \kappa$, X^κ has a $<\kappa$-thin dense set if and only if X^κ has a very thin dense set.*

Proof. ($\Leftarrow$) Clearly, a very thin set is $<\kappa$-thin.

($\Rightarrow$) Let D be a $<\kappa$-thin dense set in X^κ. Then, the construction in Theorem 3.4 gives a very thin set $D' \subset X^\kappa$ consisting of the tails of points of D. Since D was dense, so is D'. □

Corollary 3.7. *If X is countably infinite, then $X^{\mathfrak{c}^+}$ has no $<\mathfrak{c}^+$-thin dense set.*

Proof. Any $<\mathfrak{c}^+$-thin set in $X^{\mathfrak{c}^+}$ gives a very thin set in $X^{\mathfrak{c}^+}$, which must then be countable; but it is well-known that a product of $\mathfrak{c}^+$ Hausdorff spaces cannot be separable. □

Let us consider $<\mathfrak{c}$-thin dense sets in $X^\mathfrak{c}$. Schröder proved that a product of $\mathfrak{c}$-many spaces, each of which has a countable weak π-base, has a countable dense very thin subset. In Proposition 3.8 (below), we extend Schröder's result using Martin's Axiom to get the same result for $X^\mathfrak{c}$, where X is separable and has a π-base of size $< \mathfrak{c}$.

Proposition 3.8. *(MA+¬CH) If a separable space X with $\Delta(X) \geq \omega$ has a π-base of size $\kappa < \mathfrak{c}$, then there is a countable very thin dense set in $X^{\mathfrak{c}}$.*

Before proving Proposition 3.8, we borrow some notation from [15]. In constructing a dense set in $X^{\mathfrak{c}}$ from a dense set D in X, for a collection $\mathcal{I}$ of disjoint closed intervals with rational endpoints, where $|\mathcal{I}| = n$, and a set $\{d_1, ..., d_n\}$ of elements of D, we will denote by $p(\mathcal{I}; d_1, ..., d_n)$ the point with αth coordinate d_i if α is in the ith element of $\mathcal{I}$, and αth coordinate d_0 otherwise, where d_0 is a fixed element of D.

We also need to note that in Schröder's construction of a countable very thin dense set in a product of continuum-many spaces with countable weak π-bases, he constructs a weak π-base in the product from the weak π-bases on the factors. We will make this more explicit in our modification of Schröder's construction.

Proof of Proposition 3.8. Since X is separable, so is $X^{\mathfrak{c}}$; let E be a countable dense set in $X^{\mathfrak{c}}$ constructed from the countable dense set D in X using disjoint closed intervals with rational endpoints (as referenced above).

We will construct a weak π-base in $X^{\mathfrak{c}}$ from the π-base $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$. Let $\mathcal{I} = \{[r, s] \subset [0, 1] : r, s \in \mathbb{Q}\}$. Enumerate all finite pairwise disjoint collections of subsets of $\mathcal{I}$ as $\{\mathcal{F}_n : n < \omega\}$. For each $\mathcal{F}_n$, define a family of sets

$$\mathcal{V}_n = \mathcal{V}_{\mathcal{F}_n} = \Bigg\{ \prod_{\alpha<\omega_1} B_{\phi(\alpha)} \Big| \phi : [0, 1] \to \kappa,\ \phi(\alpha) = 0 \ \text{ if } \alpha \notin [r, s],$$

$$\text{for some } [r, s] \in \mathcal{F}_n,\ \phi(\alpha) = \beta_{[r,s]} \text{ if } \alpha \in [r, s] \in \mathcal{F}_n, \beta_{[r,s]} \in \kappa \Bigg\}.$$

Since each $\mathcal{F}_n$ is finite, and there are only κ many choices for each $\beta_{[r,s]}$, each $\mathcal{V}_n$ has size κ. So $\bigcup\{\mathcal{V}_n : n \in \omega\}$ also has size κ. Furthermore, each basic open set in $X^{\mathfrak{c}}$ contains a member of one of the $\mathcal{V}_n$'s: indeed, let $U = \prod_{\alpha<\mathfrak{c}} U_\alpha$, where each U_α is open in X and $U_\alpha = X$ for all but finitely many α, be a basic open set in $X^{\mathfrak{c}}$. Say $U_\alpha \neq X$ for $\alpha = \alpha_1, \alpha_2, ..., \alpha_n$. Since U_{α_i} is open for each $i = 1, ..., n$, it contains an element B_{β_i} of $\mathcal{B}$. Separate $\alpha_1, ..., \alpha_n$ by the disjoint intervals $[r_1, s_1], ..., [r_n, s_n] \in \mathcal{I}$. Define $\phi : [0,1] \to \kappa$ by $\phi([r_i, s_i]) = \beta_i$ and $\phi(\alpha) = 0$ if $\alpha \notin \cup_{i=1}^{n}[r_i, s_i]$. Then ϕ is the type of function that will generate a member of $\mathcal{V}_n$, and that member of $\mathcal{V}_n$ will be contained in U. Thus, $\bigcup\{\mathcal{V}_n : n \in \omega\}$ is a weak π-base for $X^{\mathfrak{c}}$ of size κ. Let $\mathcal{P} = \{P_\alpha : \alpha < \kappa\} = \bigcup\{\mathcal{V}_n : n \in \omega\}$.

Let $\mathbb{P} = \{f | f$ is a function from $n_f \in \omega$ to E and $\{f(i) : i < n_f\}$is very thin$\}$. Partially order $\mathbb{P}$ by function extension. Since E is countable, there are only countably many functions from elements of ω to E; so $\mathbb{P}$ is ccc.

For each $k < \omega$, define $D_k^1 = \{f \in \mathbb{P} | k \in dom(f)\}$. D_k^1 is dense for each k, since if $k \geq n_f = dom(f)$, we may extend f as follows: Let $f'(i) = f(i)$ for each $i < n_f$. For each i with $n_f \leq i \leq k$, choose by induction a point $f'(i) \in E$ with $\pi_\alpha(f'(i)) \neq \pi_\alpha(f'(j))$ for any $j < i$. This is possible, since at each step only finitely many $f'(j)$'s have been chosen, and $\pi_\alpha(f'(j)) \in \pi_\alpha(E)$, which, being dense in X, must be infinite. Then we have a function $f' : k+1 \to E$, so f' is an extension of f which is in D_k^1.

Now, for each $\alpha < \kappa$, define $D_\alpha^2 = \{f \in \mathbb{P} | \exists i < n_f \ (f(i) \in P_\alpha)\}$. We will show that every $f \in \mathbb{P}$ has an extension in D_α^2. For any $f \in \mathbb{P}$, $\{f(i) : i < n_f\}$ is finite, and we claim that $\pi_\beta(E \cap P_\alpha)$ is infinite for each $\beta < \mathfrak{c}$:

Let $\mathcal{I} = \{[a_0, a_1], ..., [a_{n-1}, a_n]\}$ be the intervals which were used to define P_α. This means that for each $k < n$, there is a β_k such that $\pi_x(P_\alpha) = B_{\beta_k}$ whenever $x \in [a_{k-1}, a_k]$. Consider the points $\{e \in E | e = p(\mathcal{I}; d_1, ..., d_n)$ for some $d_1, .., d_n \in D\}$. Since D is dense in X, $B_{\beta_k} \cap D$ is infinite for all β_k, $k = 1, ..., n$. Thus, there are infinitely many points of the form $p(\mathcal{I}; d_1, ..., d_n)$ where $d_k \in B_{\beta_k} \cap D$. These points will be in $P_\alpha \cap E$, so this intersection is infinite.

Thus, we will be able to find a point $\overline{x}$ in $E \cap P_\alpha$ so that $\{f(i) : i < n_f\} \cup \{\overline{x}\}$ is very thin. Then the function $f' = f \cup \{(n_f, \overline{x})\}$ is in D_α^2 and extends f, so D_α^2 is dense.

Let G be a filter in $\mathbb{P}$ which meets each of the κ dense sets in $\mathcal{D} := \{D^1_k : k < \omega\} \cup \{D^2_\alpha : \alpha < \kappa\}$. Since G is a filter and meets each D^1_k, $\cup G := f_G$ is a function from ω to E. Clearly, $F := \{f_G(i) : i < \omega\}$ is countable. To see that F is very thin, suppose $f_G(i), f_G(j) \in F$ are distinct. Without loss of generality, suppose $i < j$. Because G meets D^1_{j+1}, there is a function $h \in G$ with domain $n_h \geq j+1$, and $h \subset f_G$. So $f_G(i) = h(i)$ and $f_G(j) = h(j)$. Since the range of h is very thin, $h(i)$ and $h(j)$ cannot agree at any coordinate. Thus, F is very thin.

F is also dense in $X^{\mathfrak{c}}$: Suppose $U \subset X^{\mathfrak{c}}$ is open. Then U contains an element P_α of the weak $\pi-$base $\mathcal{P}$. G meets D^2_α, so there is $f \subset f_G$ with $f(i) \in P_\alpha$ for some $i < n_f$; that is, $f_G(i) \in P_\alpha \subset U$. So $F \cap U \neq \emptyset$. □

Suppose we have a space X which satisfies the conditions of Proposition 3.8. Then $X^{\mathfrak{c}}$ has a countable very thin dense set. If $|X| = \omega$ and $2^\omega > \omega_1$, then Corollary 3.6 shows that $X^{\mathfrak{c}}$ has a $<\mathfrak{c}$-thin dense set iff it has a very thin dense set. Since the very thin dense set must be countable, we see that our $<\mathfrak{c}$-thin dense set will be countable.

We have established:

Corollary 3.9. *(MA+$\neg$CH) Let X_α be a countable space with a countable weak π-base for each $\alpha < \mathfrak{c}$. Then $\prod_{\alpha<\mathfrak{c}} X_\alpha$ has a countable $<\mathfrak{c}$-thin dense set.*

We do not know if the converse is true.

Question 3.10. *If $X^{\mathfrak{c}}$ has a countable $<\mathfrak{c}$-thin dense set, must X have a countable weak π-base?*

In an effort to answer this question, the following result may turn out to be useful:

Lemma 3.11. *If X has a countable weak π-base, and D is a countable dense subset of X, there is a countable very thin dense set E in X^{ω_1} with the property that $D \subset \pi_\alpha(E)$ for all $\alpha < \omega_1$.*

Proof. This is a modification of the construction in [11], also used in Theorem 3.8, of a countable very thin dense set from a countable weak π base.

Let $\mathcal{W} = \{W_n : n < \omega\}$ be a countable weak π-base for X, where $W_0 = D$. (If D is not already in $\mathcal{W}$, we may add it and $\mathcal{W}$ will still be a weak π-base.) Say $D = \{d_n : n < \omega\}$.

Identify ω_1 with the closed unit interval and let $\mathcal{I} = \{[r,s] \subset [0,1] : r, s \in \mathbb{Q}\}$. Enumerate the collection of all finite pairwise disjoint collections from $\mathcal{I}$ as $\{\mathcal{F}_n : n < \omega\}$, in such a way that $\mathcal{F}_{2^k} = \left\{\left[\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} + \frac{1}{2^k}\right]\right\}$ for $k = 1, 2, 3, ...$ and $\mathcal{F}_{3^k}$ has the property that $\frac{1}{2} \notin \cup\mathcal{F}_{3^k}$. For each $\mathcal{F}_n$, define a family of sets

$$\mathcal{V}_n = \mathcal{V}_{\mathcal{F}_n} = \left\{ \prod_{\alpha<\omega_1} W_{\phi(\alpha)} | \phi : [0,1] \to \omega, \phi(\alpha) = 0 \text{ if } \alpha \notin [r,s] \text{ for some } [r,s] \in \mathcal{F}_n, \right.$$

$$\left. \phi(\alpha) = c_{[r,s]} \text{ if } \alpha \in [r,s] \in \mathcal{F}_n, \text{ for some } c_{[r,s]} \in \omega \right\}.$$

Since each $\mathcal{F}_n$ is finite, and there are only countably many choices for each $c_{[r,s]}$, each $\mathcal{V}_n$ is countable. So $\bigcup\{V_n : n \in \omega\}$ is also countable. Furthermore, each basic open set in X^{ω_1} contains a member of one of the $\mathcal{V}_n$'s: Indeed, let $U = \prod_{\alpha<\omega_1} U_\alpha$, where each U_α is open in X and $U_\alpha = X$ for all but finitely many α, be a basic open set in X^{ω_1}. Say $U_\alpha \neq X$ for $\alpha = \alpha_1, \alpha_2, ..., \alpha_n$. For each $i = 1, 2, ..., n$, U_{α_i} is open, so each U_{α_i} contains an element W_{n_i} of $\mathcal{W}$. Separate $\alpha_1, ..., \alpha_n$ by the disjoint intervals $[r_1, s_1], ..., [r_n, s_n] \in \mathcal{I}$. Define $\phi : [0,1] \to \omega$ by $\phi([r_i, s_i]) = n_i$ and $\phi(\alpha) = 0$ if $\alpha \notin \cup_{i=1}^n [r_i, s_i]$. Then ϕ is of the type of function that will generate a member of $\mathcal{V}_n$, and that member of $\mathcal{V}_n$ will be contained in U. Thus, $\bigcup\{V_n : n \in \omega\}$ is a countable weak π-base for X^{ω_1}.

We will define a very thin dense set in X^{ω_1} by carefully choosing one point from each V_n. Choose $x_0 \in V_0$ arbitrarily. If $x_0, x_1, ..., x_{n-1}$ have been selected, choose $x_n \in V_n$ such that:

(i) $\pi_\alpha(x_n) \neq \pi_\alpha(x_j)\ \forall 0 \leq j \leq n$

This is possible because there are only finitely many previously chosen points in each projection, and will make the set $\{x_n : n \in \omega\}$ very thin.

(ii) If $n = 2^k$ for some $k \in \mathbb{N}$ (so $V_n = V_{\{[\frac{1}{2}-\frac{1}{2^k}, \frac{1}{2}+\frac{1}{2^k}]\}}$), and $\alpha \notin \left[\frac{1}{2} - \frac{1}{2^k}, \frac{1}{2} + \frac{1}{2^k}\right]$, choose $\pi_\alpha(x_n)$ to be d_j, where $j = \min\{m < \omega : d_m \notin \pi_\alpha(x_l),\ 0 \leq l < n\}$.

(iii) If $n = 3^k$, choose $\pi_{1/2}(x_n)$ to be d_j, where $j = \min\{m < \omega : d_m \notin \pi_{1/2}(x_l),\ 0 \leq l < n\}$.

Since for every $\alpha \neq 1/2$, there exists a k_α such that $\alpha \notin \mathcal{F}_{2^k}$ for all $k > k_\alpha$, we see that $\pi_\alpha(\{x_n : n \in \omega\})$ will contain D for each n. Similarly, $\pi_{1/2}(\{x_n : n \in \omega\})$ will also contain D.

The set $\{x_n : n \in \omega\}$ will be dense (as in the proof of Theorem 3.8). $\square$

If it is possible to choose the countable weak π-base so that each member is contained in D, you may construct the very thin dense set so that all the projections are actually equal to D. For example, one may use this method to construct a very thin dense set in $\mathbb{Q}^{\omega_1}$ with the property that all coordinates are from the dyadic rationals.

3.2 Bounds on the Cardinalities of Special Dense Sets

As mentioned above, another question concerning these special dense sets relates to their cardinality. Must a product space with a very thin dense or slim dense set have such a special dense set which also witnesses the density of the space? For example, must a separable space which has a very thin dense set have a countable very thin dense set? We construct examples showing that the answer is negative both for very thin and for slim sets. We begin with a couple of facts:

Fact 3.12. *Let τ be a topology on ω generated by a maximal independent family. Then (ω, τ) embeds in $2^{\mathfrak{c}}$ as a dense subset.*

Proof. Let τ be a topology on ω generated by a maximal independent family $\{A_\alpha : \alpha < \mathfrak{c}\}$. Let $f : \omega \to 2^{\mathfrak{c}}$ be given by $f(n) = x_n$, where $x_n(\alpha) = 1$ if $n \in A_\alpha$ and $x_n(\alpha) = 0$ if $x \notin A_\alpha$.

Following the notation in [6], let basic open sets in (ω, τ) be denoted by $[\sigma] = \bigcap_{\alpha \in dom\sigma} A_\alpha^{\sigma(\alpha)}$ where σ is a function from a finite subset of $\mathfrak{c}$ into 2, $A_\alpha^1 = A_\alpha$, and $A_\alpha^0 = \omega \backslash A_\alpha$. Denote basic open sets in $2^{\mathfrak{c}}$ by $U_\sigma = \bigcap_{\alpha \in dom\sigma} \pi_\alpha^{-1}(U_\alpha^{\sigma(\alpha)})$, where σ is as above, $U_\alpha^1 = \{1\}$ and $U_\alpha^0 = \{0\}$. Since the only other open sets in 2 are $\emptyset$ and 2, which may be avoided in writing out a nonempty basic open set, this notation describes any proper nonempty subset of $2^{\mathfrak{c}}$. Then,

$$\begin{aligned} x_n \in U_\sigma \;&\Leftrightarrow\; x_n(\alpha) = \sigma(\alpha) \;\; \forall \alpha \in dom\sigma \\ &\Leftrightarrow\; \sigma(\alpha) = 1 \text{ iff } x_n(\alpha) = 1 \; \forall \alpha \in dom\sigma \\ &\Leftrightarrow\; \sigma(\alpha) = 1 \text{ iff } n \in A_\alpha \; \forall \alpha \in dom\sigma \\ &\Leftrightarrow\; n \in [\sigma] \end{aligned}$$

So the open sets in (ω, τ) correspond exactly under f to the open sets in $D = f(\omega) \subset 2^{\mathfrak{c}}$.

To see that D is dense in $2^{\mathfrak{c}}$, let U_σ be a nonempty basic open set in $2^{\mathfrak{c}}$. Then there is a $k \in \omega$ in $[\sigma] = \bigcap_{i=1}^n A_{\alpha_i}^{\sigma(\alpha_i)}$. The corresponding $x_k = f(k)$ will be in $U_\sigma \cap D$. $\square$

We use D to get a subspace of $2^{\mathfrak{c}}$ with a very thin dense set in its square.

Fact 3.13. *Let F be the set of all points $x \in 2^{\mathfrak{c}}$ for which the set $\{\alpha : x_\alpha = 1\}$ is finite. Let D be a dense subset of $2^{\mathfrak{c}}$ which is homeomorphic to (ω, τ), where (ω, τ) is as in Fact 3.12. If $X = D \cup F$, then X^2 has a very thin dense set.*

Proof. Let $X = D \cup F$. Since $X \subset 2^{\mathfrak{c}}$, $w(X) \leq w(2^{\mathfrak{c}}) = \mathfrak{c}$. (Recall that $w(X)$ is the minimum size of a base for X.) To see that $\Delta(X) \geq \mathfrak{c}$, note that any open set in X contains a basic open set of the form

$$U = (D \cup F) \cap U_\sigma,$$

where U_σ is basic open in $2^{\mathfrak{c}}$. Then the points x_γ, where

$$x_\gamma(\alpha) = \begin{cases} 1 & \sigma(\alpha) = 1 \\ 1 & \alpha = \gamma \\ 0 & \text{otherwise} \end{cases}$$

are in U for each $\gamma < \mathfrak{c}$ for which $\sigma(\gamma) \neq 0$; that is, all but finitely many γ's. So $|U| \geq \mathfrak{c}$; that is, $\Delta(X) \geq \mathfrak{c}$.

So, $\Delta(X) \geq \mathfrak{c} \geq w(X) \geq \pi w(X)$, which implies that X^2 has a very thin set by Proposition 2.1 in [6]. □

Now we will show that we can construct X in such a way that D cannot make a significant contribution to a special dense set.

Fact 3.14. *Let X be as in Fact 3.13. If $\mathcal{P}$ is a hereditary property (such as very thin), and any subset of D^2 with property $\mathcal{P}$ is nowhere dense in D^2, then X^2 cannot have a countable dense set with property $\mathcal{P}$.*

Proof. Suppose that $E \subset X^2$ is a countable dense set in X^2 that has the property $\mathcal{P}$. Then, by hypothesis, $E \cap D^2$ is nowhere dense in D^2. Since D^2 is dense in X^2, that means that $E \cap D^2$ is nowhere dense in X^2.

Since E is countable, the set $E_F := (\pi_1(E) \cap F) \cup (\pi_2(E) \cap F)$ is a countable subset of F, so there is an $\alpha < \mathfrak{c}$ with the property that for all $x \in E_F$, $x(\beta) = 0$ for all $\beta > \alpha$. (Specifically, we may take α to be $1 + sup\{\beta < \mathfrak{c} : x(\beta) = 1 \text{ for some } x \in E_F\}$.) Then the open set $V = \pi_\alpha^{-1}(\{1\}) \times \pi_\alpha^{-1}(\{1\})$ is open and nonempty in X^2 and does not contain any points with a coordinate in E_F. Thus, V does not meet the closure of the set of points from E which have at least one coordinate in F. But every point of E has either at least one coordinate in F or both coordinates in D. So $V \subset \overline{E \cap D^2}$, which contradicts that $E \cap D^2$ must be nowhere dense. □

We will now apply this to the properties $\mathcal{P}$ = very thin and $\mathcal{P}$ = slim. Observe that these properties are hereditary, in the sense that a subset of a very thin (resp. slim) set is also very thin (resp. slim).

Example 3.15. *There is a separable space X such that X^2 has a very thin dense subset, but X^2 does not have a countable very thin dense subset.*

Proof. Let $X = D \cup F$ as in Fact 3.13. Then X^2 has a countable dense set (D^2) and a very thin dense set. Since the topology on D is homeomorphic to a topology generated by a maximal independent family, D is strongly irresolvable. A very thin dense set in X^2 results in disjoint dense sets in X; for a strongly irresolvable space, this is impossible. In fact, if X is strongly irresolvable, any thin set in X^2 is nowhere dense. This follows from a result in [11] which says that if a space Y^2 has a thin dense set, there is a one-to-one map $\phi : Y \to Y$ with the property that for all nonempty $U \subseteq Y$, $\phi(U)$ is dense in Y. Clearly, if $A \subseteq X^2$ were somewhere dense, then we could find disjoint open sets $U_1, U_2 \subseteq Int(\overline{A})$, and apply $\phi|_{\overline{A}}$ to get disjoint dense sets in $\phi(Int(\overline{A}))$, contradicting that X is strongly irresolvable.

Thus, if E is a very thin subset of X^2, $E \cap D^2$ is also very thin, and so $E \cap D^2$ is nowhere dense in D^2. By Fact 3.14 with $\mathcal{P}$ = very thin, X^2 does not have a countable very thin dense set. □

Example 3.16. *(CH) There is a separable space X such that X^2 has a very thin dense subset, but X^2 does not have a countable slim dense subset.*

Proof. Consider the set $X = D \cup F$ where D is the space constructed in Example 3.4 in [6]. That is, D is the space (ω, τ) where τ is the topology generated by a particular maximal independent family. The family is constructed by induction in such a way that at stage $\gamma + 1$, a potentially slim dense subset E_γ of ω^2 is made not dense by adding to the independent family a set $T_{\gamma+1}$ that meets every open set from the preceding stage, but misses E_γ. The potential slim dense subsets each appear ω_1-many times.

One result of this construction is that any slim set in D^2 is nowhere dense. To see this, consider an open set $U \subset D^2$. There is some $\beta < \omega_1$ such that U is open in $(X, \tau_\gamma)^2$ for all $\gamma > \beta$. If E is slim dense in D^2, then E was slim dense in some stage of the construction. Thus, E would appear as E_α for some $\alpha > \beta$ in the construction. So, when $\tau_{\alpha+1}$ was defined, U was open, and thus contains one of the basic open sets which ${T_{\alpha+1}}^2$ was required to meet. However, ${T_{\alpha+1}}^2$ was constructed to miss E_α; so ${T_{\alpha+1}}^2 \cap \overline{E_\alpha} = \emptyset$. Thus, $U \not\subset \overline{E_\alpha}$. So E_α is nowhere dense in D^2.

Thus, by Fact 3.14 with $\mathcal{P} =$ slim, X^2 does not have a countable slim dense set. However, D^2 is countable and dense in X^2, and X^2 has a very thin (hence slim) dense set by Fact 3.13.

$\square$

Chapter 4

Slim-Like Properties

There are many ideas of smallness in topology: finite, $< \kappa$ for some relevant cardinal κ, nowhere dense, meager, etc. We have considered some of these already in the restrictions we have placed upon coordinates of dense sets; we now turn our attention to cross-sections of a set. The definition of a slim set explicitly places a restriction on the cross-sections; specifically that they be nowhere dense in the cross-section of the whole space. However, very thin also restricts cross-sections. One might consider a very thin set to be one for which the cross-sections are singletons. Indeed, suppose $D \subset \prod_{\alpha<\kappa} X_\alpha$ is very thin, and consider the cross-section of D at some $\nu \in \prod_{\alpha \in K} X_\alpha$. $D \cap C(\nu) = \{x \in D : x \upharpoonright K = \nu\}$; but this means that we have fixed at least one coordinate, say α; so $D \cap C(\nu) \subseteq \{x \in D | x(\alpha) = \nu(\alpha)\}$, which has only one point. So, once again, we observe a gap between two defined notions. We will define a superslim set which falls into this gap.

We will also consider what happens when we only place conditions on D's intersection with some cross-sections, rather than all; this will lead us to the concept of a codimension 1 slim set.

4.1 Codimension 1 Slim

We define the property codimension 1 slim, suggested by Gruenhage as an alternative to the definition of slim. We will show that it is slightly weaker than slim, in the sense that a space may have a codimension 1 slim dense set but no slim dense set.

Definition 4.1. *Let D be a subset of $\prod_{\alpha<\lambda} X_\alpha$. D is codimension 1 slim if for every $\alpha \in \lambda$ and $x \in X_\alpha$, $D \cap C(x)$ is nowhere dense in $C(x)$, where $C(x) = \{f \in \prod_{\alpha<\lambda} X_\alpha : f(\alpha) = x\}$.*

The difference between codimension 1 slim and slim is that we only require the cross-sections of codimension 1 to be nowhere dense, not every cross-section. Obviously for a product of two spaces, the notions would coincide.

Example 4.2. *(CH) There is a space which has a codimension 1 slim dense set, but no slim dense set.*

Proof. We begin by constructing a codimension 1 slim dense set in $\mathbb{Q}^3$. Let $\{D_n : n < \omega\}$ be a partition of $\mathbb{Q}$ into dense sets. Let E be a very thin dense subset of ${D_0}^2$, such that for each $(q, r) \in E$, $q \neq r$. Note that ${D_0}^2$ has a very thin dense subset because $\Delta(D_0) = \pi w(D_0) = \omega$. Enumerate E by $\{(q_n, r_n) : n < \omega\}$.

For each $n \in \mathbb{N}$, define the sets A_n as follows:

$$A_n = \{(x, y, z) \in \mathbb{Q}^3 : (y, z) = (q_n, r_n) \in E, x \in D_n\}$$

Note that each A_n is a dense subset of a line parallel to the x axis. Let $A = \cup_{n=1}^{\infty} A_n$.

Claim 1: A is dense in $\mathbb{Q}^3$.

Let $U \times V \times W$ be open in $\mathbb{Q}^3$. Then, since E is dense in the dense subset ${D_0}^2$ of $\mathbb{Q}^2$, $V \times W$ must contain a point of E; say (q_m, r_m). Because D_m is dense in $\mathbb{Q}$, there is an $x_m \in U \cap D_m$. Then $(x_m, q_m, r_m) \in A_m \cap (U \times V \times W) \subset A \cap (U \times V \times W)$. So A is dense. $\triangle$

Claim 2: A is codimension 1 slim.

First, we consider a cross-section obtained by fixing the first coordinate at $x_0 \in \mathbb{Q}$. $C(x_0)$ is homeomorphic to $\mathbb{Q}^2$. $C(x_0) \cap A_n = \{(x, y, z) : x = x_0 \in D_n,\ (y, z) = (q_n, r_n) \in E\}$. Since the D_m's are disjoint, x_0 can only be in one D_m; if it is in D_m for some $m \in \mathbb{N}$, then this determines a unique point (q_m, r_m) for (y, z). So

$C(x_0) \cap A_n$ is either empty or one point; and it is nonempty for only a single n. Thus, $C(x_0) \cap (\cup_{n\in\mathbb{N}} A_n)$ is a single point.

Now, fix a point y_0 in the second coordinate and consider $A_n \cap C(y_0) = \{(x, y_0, z) : (y_0, z) = (q_n, r_n) \in E,\ x \in D_n\}$. Since E is very thin, there is at most one n for which $y_0 = q_n$, so the intersection $C(x_0) \cap (\cup_{n\in\mathbb{N}} A_n)$ is either empty or is the dense subset $D_n \times \{q_n\} \times \{r_n\}$ of the line $\{(x, q_n, r_n) : x \in \mathbb{Q}\}$.

Similarly, if we fix the third coordinate to be z_0, $C(z_0) \cap (\cup_{n\in\mathbb{N}} A_n)$ is either empty or is a dense subset of the line $\{(x, q_n, z_0) : x \in \mathbb{Q}\}$, specifically the dense subset for which q_n and $z_0 = r_n$ are fixed and x is any element of D_n.

Thus, we see that $C(a) \cap A$ is either empty, one point, or a dense subset of a line; this is nowhere dense in $\mathbb{Q}^2$. So A is codimension 1 slim in $\mathbb{Q}^3$. It should be noted that A fails to be slim; as we have seen, the intersection of A with certain lines is dense in those lines. This proves the claim. $\triangle$

We will now construct a finer topology on $\mathbb{Q}^3$, making all possible slim dense sets fail to be dense, while keeping A dense. This construction is patterned after Example 3.5 in [6].

Begin by letting τ_0 be the usual topology on $\mathbb{Q}$. Enumerate all subsets S of $\mathbb{Q}^3$ with the property that each $s \in S$ has distinct coordinates as $\{S_\alpha : 0 < \alpha < \omega_1\}$, with each appearing ω_1 times. We will add open sets U_α and $V_\alpha = \mathbb{Q} \backslash U_\alpha$ at each stage $\alpha > 0$ to construct a series of topologies $\{\tau_\alpha : \alpha < \omega_1\}$ with the properties:

1. $(\mathbb{Q}, \tau_\alpha)$ is regular and has no isolated points.

2. D_n is dense in $(\mathbb{Q}, \tau_\alpha)$ for each $n < \omega$.

3. A is dense in $(\mathbb{Q}, \tau_\alpha)^3$.

4. If S_α is slim dense in $(\mathbb{Q}, \tau_\alpha)^3$, $S_\alpha \cap {U_\alpha}^3 = \emptyset$.

We observe that τ_0 has these properties.

If τ_β has been defined for $\beta < \alpha$ satisfying (1)-(4), let τ_α be the topology generated by $\bigcup_{\beta<\alpha} \tau_\beta$. If S_α is not slim dense in τ_α, we will take $U_\alpha = \emptyset$. Suppose S_α is slim and dense in $(\mathbb{Q}, \tau_\alpha)^3$.

Since there is a countable base for τ_0, and thus we have only added countably many new open sets at each stage, there is a countable base $\mathcal{B}$ for ${\tau_\alpha}^3$. Let $\langle B_n, i_n \rangle$ be an enumeration of all pairs of open sets $B \in \mathcal{B}$ and $i < \omega$ so that each $\langle B, i \rangle$ occurs infinitely many times at an index which is $k \mod 8$, for each $k = 0, 1, ..., 7$. We will define finite sets $F_n, G_n \subset \mathbb{Q}$ and put $U = \bigcup F_n$ and $\bigcup G_n \subset \mathbb{Q} \backslash U$. Let $F_0 = G_0 = \emptyset$. Suppose F_m, G_m are defined for $m \leq n-1$. Look at $\langle B_n, i_n \rangle$, where $B_n = C_0 \times C_1 \times C_2$. Let $J_n = \{(q,r) \in E : [(\{q,r\} \cup F_{n-1} \cup G_{n-1})^3 \backslash (F_{n-1} \cup G_{n-1})^3] \cap S_\alpha \neq \emptyset\}$.

Recall that for each $(x_0, x_1, x_2) \in S_\alpha$, $x_0 \neq x_1 \neq x_2$, so that any point in $[(\{q,r\} \cup F_{n-1} \cup G_{n-1})^3 \backslash (F_{n-1} \cup G_{n-1})^3] \cap S_\alpha$ has a coordinate in $F_{n-1} \cup G_{n-1}$. Fixing one such coordinate gives a cross-section of S_α, which is slim; so any group of points (a,b) in J_n which belong to a particular cross-section will be nowhere dense in $\mathbb{Q}^2$. Since there are only finitely many choices for a point of $F_{n-1} \cup G_{n-1}$ and a coordinate in which to put it, we see that J_n is a finite union of nowhere dense sets. Thus J_n is nowhere dense; but E is dense in $C_1 \times C_2$, so we can pick a $(q_n, r_n) \in (C_1 \times C_2) \cap E$ which is not in J_n. $A_n \cap B_n$ will be a somewhere dense subset of the line $\{(x, q_n, r_n) : x \in \mathbb{Q}\}$, because D_n is dense in $(\mathbb{Q}, \tau_\alpha)$ and C_0 is open. Consider the set

$$H_n = \{m : [(\{m, q_n, r_n\} \cup F_{n-1} \cup G_{n-1})^3 \backslash (F_{n-1} \cup G_{n-1})^3] \cap S_\alpha \neq \emptyset\}$$

This set is nowhere dense, because if we fix two elements of $\{q_n, r_n\} \cup F_{n-1} \cup G_{n-1}$, we have a cross-section; and S_α is slim, so it will have nowhere dense intersection with the cross-section. Since there are only finitely many cross-sections to consider, the set H_n is nowhere dense. If $i_n = 0$, choose a point x_n from

$$\{q : \exists r \text{ such that } (r,q) \in E \text{ or } (q,r) \in E\} \backslash (H_n \cup (\cup_{m<n}(F_m \cup G_m))) \, .$$

Otherwise, choose a point $x_n \in D_{i_n} \setminus (H_n \cup (\cup_{m<n}(F_m \cup G_m)))$. This is possible because E and D_{i_n} are dense, H_n is nowhere dense, and F_m, G_m are finite.

Let $(x_n, q_n, r_n) = (a, b, c)$. We will add the points a, b, c to F_{n-1} and G_{n-1} to obtain F_n and G_n, respectively, according to the value of $n \pmod 8$:

$n \pmod 8$	F_n	G_n
0	a, b, c	
1	a, b	c
2	a, c	b
3	b, c	a
4	a	b, c
5	b	a, c
6	c	a, b
7		a, b, c

Note that if ${F_{n-1}}^3 \cap S_\alpha = \emptyset$, so does ${F_n}^3$: for, if $(x, y, z) \in ({F_n}^3 \setminus {F_{n-1}}^3) \cap S_\alpha$, at least one coordinate of (x, y, z) is in $\{x_n, q_n, r_n\}$. But, we defined J_n, H_n and (x_n, q_n, r_n) in such a way that no point with at least one of those three coordinates, and the others possibly in F_n, could be in S_α.

Now, let $U_\alpha = \bigcup_{n<\omega} F_n$. Define $\tau_{\alpha+1}$ to be the topology generated by τ_α along with U_α and $\mathbb{Q} \setminus U_\alpha$. Since at each stage we have added a set and its complement, the topology is regular, and we have that U_α and $\mathbb{Q} \setminus U_\alpha$ have infinite intersection with every open set because we chose $(q_n, r_n) \in C_1 \times C_2$, and C_1, C_2 ran over all elements of a basis. So there are no isolated points. We need to check that the D_n's are dense in $\tau_{\alpha+1}$, that A is dense in $(\mathbb{Q}, \tau_{\alpha+1})^3$, and that $(U_\alpha)^3$ misses S_α.

First, to see that D_n is dense, consider $U_\alpha \cap B$, where B is open in τ_α. Then $\pi_0^{-1}(B)$ is open in ${\tau_\alpha}^3$ and contains one of the B_m's, at a step where $m \equiv 0 \mod 8$ and $i_m = n$. Then the a chosen at that step is in $\pi_0(B_m) \cap D_n$, and was put into

$F_m \subset U_\alpha$. So $U_\alpha \cap B$ meets D_n. A similar argument with $m \equiv 7 \mod 8$ shows that $\mathbb{Q} \backslash U_\alpha \cap B$ meets D_n for any $B \in \tau_\alpha$.

Let $U^0 = U_\alpha$, $U^1 = \mathbb{Q} \backslash U_\alpha$. Fix a basic open set $B = B_0 \times B_1 \times B_2$ in $\mathbb{Q}^3$, and consider $B \cap A \cap (U^{k_0} \times U^{k_1} \times U^{k_2})$, where $k_i < 2$. B appeared as the first coordinate of a pair $\langle B_n, i_n \rangle$ infinitely times at an n which is congruent to any $k \mod 8$. So there is a point (in fact, infinitely many points) of $A \cap B$ for which the first coordinate is in U^{k_0}, the second is in U^{k_1}, and the third in U^{k_2}. Thus, A is dense in $(\mathbb{Q}, \tau_{\alpha+1})^3$.

Now, consider ${U_\alpha}^3 \cap S_\alpha$. Suppose $(a, b, c) \in {U_\alpha}^3$. Then, $(a, b, c) \in {F_n}^3 \cap S_\alpha$ for some n, since each of a, b, c must have been added at some stage, but this is impossible by the construction of F_n. So ${U_\alpha}^3 \cap S_\alpha = \emptyset$.

Once we have constructed the τ_α's as described, let $\tau = \cup_{\alpha<\omega_1} \tau_\alpha$, and let $X = (\mathbb{Q}, \tau)$. A will be dense in X^3, since it is dense at each stage. Suppose $S \subset X^3$ is slim and dense. Without loss of generality, each $s \in S$ has distinct coordinates, and since there are only countably many cross-sections to consider, S will appear as a slim dense S_α at some stage α. But then $U_{\alpha+1}$ misses S. So X^3 has no slim dense set. $\square$

Example 4.3. *A codimension 1 slim dense set in $\mathbb{Q}^3$, which fails to be slim in more ways.*

Let $\{D_n : n < \omega\}$ be a partition of $\mathbb{Q}$ into dense sets. For $i = 0, 1, 2$, let S_i be a very thin dense subset of ${D_i}^2$, such that for each $(q, r) \in S_i$, $q \neq r$. Enumerate S_i by $\{(q_n^i, r_n^i) : n < \omega\}$.

For each $i = 0, 1, 2$, $n < \omega$, define the sets A_n^i as follows:

$$A_n^0 = \{(x, y, z) \in \mathbb{Q}^3 : (y, z) = (q_n^0, r_n^0), x \in D_{3n+3}\}$$

$$A_n^1 = \{(x, y, z) \in \mathbb{Q}^3 : (x, z) = (q_n^1, r_n^1), x \in D_{3n+4}\}$$

$$A_n^2 = \{(x, y, z) \in \mathbb{Q}^3 : (x, y) = (q_n^2, r_n^2), x \in D_{3n+5}\}$$

Note that each A_n^i is a dense subset of a line parallel to a coordinate axis. Let

$$A = \bigcup_{n=0}^{\infty} (A_n^0 \cup A_n^1 \cup A_n^2)$$

A is dense in $\mathbb{Q}^3$ and is codimension 1 slim, essentially because A is the union of three sets which are the same as the A in the preceding example. $\square$

4.2 Superslim

Now, we will consider another cross-section based property which is, in a sense, between slim and very thin.

Definition 4.4. *A subset S of $\prod_{\alpha<\kappa} X_\alpha$ is superslim iff every cross-section of S is finite.*

Suppose S is superslim; since $\{s \in S : s(\alpha) = x\}$ is a cross-section for each $\alpha < \kappa$, $x \in X$, each point $x \in X$ can appear at most finitely many times in each coordinate. So $|S| \leq |X|$. On the other hand, suppose $T \subset X^\kappa$ is such that $T(x, \alpha) = \{t \in T : t(\alpha) = x\}$ is finite for each $\alpha < \kappa,\ t \in T$. Since any cross-section will involve fixing one or more coordinates, and thus be contained in a $T(x, \alpha)$, this condition implies that T is superslim. So superslim is equivalent to "each point of X appears only finitely many times in each coordinate." This is a difference with slim; in that case, since "nowhere dense" can be different in different dimensions, we had a space which had a codimension 1 slim dense set but not a slim dense set. Here, the key property is finiteness, which does not change when considering different powers of X.

We find that for a finite power, the existence of a superslim dense set is related to satisfying a strengthened version of the property (NC_k), which (as discussed above) is related to the existence of a slim dense set in X^k.

Proposition 4.5. *Let $k < \omega$. X satisfies (NC_k) witnessed by a collection of finite sets iff X^k has a countable superslim dense set.*

Proof. Both directions are modeled after proofs in [6]; the first after Proposition 4.1(2), and the second after Proposition 4.10.

($\Rightarrow$) Suppose X satisfies (NC_k), witnessed by the collection $\mathcal{N} = \{N_\alpha : \alpha < \lambda\}$, where for each α, $|N_\alpha| < \omega$. Let $D = \bigcup_{\alpha<\lambda} {N_\alpha}^k$. We will see that this set is superslim and dense.

Fix an element $d_0 \in D$, and consider the set $C(d_0, \alpha) = \{d \in D | d(\alpha) = d_0(\alpha)\}$. Since the members of $\mathcal{N}$ are pairwise disjoint, there is a unique $\beta < \lambda$ such that $d_0(\alpha) \in N_\beta$. But then, by construction of D, any $d \in C(d_0, \alpha)$ must be in ${N_\beta}^k$, which is a finite set. Thus, any point of X will appear only finitely many times in the αth coordinate; so D is superslim.

To see that D is dense, let $\prod_{i<k} U_i$ be a basic open set in X^k. Since $\mathcal{N}$ satisfies (NC_k), there is an $N_\alpha \in \mathcal{N}$ which meets each member of $\{U_0, ..., U_{k-1}\}$. Thus, ${N_\alpha}^k \cap \prod_{i<k} U_i \neq \emptyset$; so D is dense in X^k.

($\Leftarrow$) Conversely, suppose that X^k has a countable superslim dense set D. For each $d \in D$, let $c(d)$ be the set of coordinates of D, and let $c(D) = \cup\{c(d) : d \in D\} = \{x_n : n \in \omega\}$. Observe that $c(D)$ is indeed countable because D is, and each $c(d)$ is finite. Also, $c(D)$ is dense in X, so X is separable.

Define by induction a sequence of disjoint finite sets $\langle H_n : n < \omega\rangle$: Let $H_0 = \{x_0\}$. If H_n has been defined, let k_n be the least $k \in \omega$ such that $x_k \notin \cup_{i\leq n} H_i$. Let $H_{n+1} = \{x_{k_n}\} \cup \{c(d) | d \in D \wedge c(d) \cap H_n \neq \emptyset\} \setminus \cup_{i\leq n} H_i$. That is, we take x_{k_n} plus the remaining unused coordinates of each point with a coordinate in H_n.

It is clear from the construction that H_{n+1} is disjoint from H_i for $i \leq n$. Also, H_{n+1} is finite: indeed, suppose $x \in H_{n+1}$. Then $\{d \in D : d(i) = x\}$ is finite for each $i = 0, 1, ..., n-1$, because D is superslim. So when we consider the coordinates of all d in $\cup_{i\leq n}\{d \in D : d(i) = x\}$, we still have a finite set. So, as long as H_n is finite, so is H_{n+1}; and we see that H_0 is finite.

Now, for each infinite subset A of ω, enumerate A in an increasing fashion by $\{a_0, a_1, ...\}$. Define

$$\mathcal{N}_A = \{\cup_{i \le a_0} H_i,\ \cup_{i=a_0+1}^{a_1} H_i,\ \cup_{i=a_1+1}^{a_2} H_i, ...\}$$

We claim that for some $A \subseteq \omega$, $\mathcal{N}_A$ witnesses (NC$_k$).

Suppose not. Then, for every infinite $A \subseteq \omega$, we can find nonempty open sets $U(A,0), U(A,1)$,...,$U(A,k-1)$ with the property that no one member of $\mathcal{N}_A$ meets all of them. Let $\mathcal{A}$ be an uncountable almost disjoint family of subsets of ω. Since X is separable, X is (ccc); the collection $\{U(A,0) : A \in \mathcal{A}\}$ is uncountable, so there must be an uncountable subcollection $\mathcal{A}'$ for which $U(A,0) \cap U(B,0) \neq \emptyset$ for any $A, B \in \mathcal{A}'$. Then consider $\{U(A,1) : A \in \mathcal{A}'\}$; in the same way, we find that there is an uncountable subcollection $\mathcal{A}'' \subseteq \mathcal{A}'$ with $U(A,1) \cap U(B,1) \neq \emptyset$ for all $A, B \in \mathcal{A}''$. Continuing this process, we find that there must be sets $A, B \in \mathcal{A}$ such that $A \neq B$ but $U(A,i) \cap U(B,i) \neq \emptyset$ for each $i = 0, 1, ..., k-1$.

Consider the nonempty open subset of X^k given by $U = \prod_{i<k}[U(A,i) \cap U(B,i)]$. There is a $d \in D \cap U$, since D is dense. By the construction of the H_n's, there must be an m such that $c(d) \cap H_m \neq \emptyset$. Let n be the least such m.

By construction, $c(d) \subseteq H_n \cup H_{n+1}$: since n is the first such that $c(d) \cap H_n \neq \emptyset$, there are no coordinates of d in any previous H_k; and when we constructed H_{n+1}, we would have therefore have included all remaining coordinates of $c(d)$.

If $n \notin A$, then some $N \in \mathcal{N}_A$ contains both H_n and H_{n+1}. But then N meets each $U(A,i)$ (specifically, at $d(i)$). This contradicts that no member of $\mathcal{N}_A$ meets each $U(A,i)$; so $n \in A$. Similarly, $n \in B$.

Since A and B are members of an almost disjoint family, $A \cap B$ is finite. Each H_n is also finite; so $\cup_{n \in A \cap B} H_n$ is finite. But we have just shown that each $d \in D \cap U$ must have a coordinate in this finite set. Since D is superslim, each coordinate can appear only finitely many times; so $D \cap U$ is finite; but this contradicts denseness of D.

Therefore, some $\mathcal{N}_A$ must witness (NC_k) in X, and clearly the members of $\mathcal{N}_A$ are finite. $\square$

Question 4.6. *What is the relationship between "X^ω has a superslim dense set" and "X satisfies (NC) witnessed by a collection of finite sets"?*

This is more difficult than the question settled by Proposition 4.5; the above proof does not extend because the sets H_n as defined above will not be finite. This in turn is because $c(d)$ is not finite for most points $d \in X^\omega$. For the other direction, the slim dense set constructed as in the proof will not be finite if the power in question is ω; N_α^ω does not remain finite even if N_α was.

We observe that there are (nice) spaces which have no superslim dense set in their square.

Example 4.7. *There is a metrizable space X such that X^2 has no superslim dense set.*

Proof. Let $X = \mathfrak{c} \times \mathbb{Q}$, where $\mathfrak{c}$ has the discrete topology and $\mathbb{Q}$ has the usual topology. In [6], it is shown that a dense set in $X^2 \cong \mathfrak{c} \times \mathbb{Q}^2$ will have uncountable cross-sections of the type $\{(\alpha, q_1, q_2) : q_1, q_2 \in \mathbb{Q}\}$ for some fixed $(q_1, q_2) \in \mathbb{Q}^2$. The goal there is to show that X^2 has no very thin dense set; but since uncountable is more than finite as well as more than 1, this also shows that X^2 has no superslim dense set. $\square$

Since this example also has no very thin dense set, it remains to ask:

Question 4.8. *Is there a space such that X^2 has a superslim dense set but X^2 does not have a very thin dense set?*

Chapter 5

(GC) and (NC)

The properties (GC), (NC), and (NC$_k$) defined in [6] (and mentioned above, in Chapter 2) were defined and studied as criteria which guarantee the existence of slim dense sets in certain products. However, they are interesting for reasons other than the one which led to their establishment. We have already seen that (NC$_k$) witnessed by finite sets is applicable (even equivalent) to a property related to slim; we will eventually see that (GC) is related to a selective separability property on the factor space. This application does not even involve product spaces. Thus, we wished to study the criteria further; especially with the added condition that the sets in the collection be finite, we gain some interesting results.

5.1 Cardinalities of Collections Witnessing (GC) and (NC)

In [6], many of the results which show that certain types of spaces satisfy (GC) or (NC) result in the collection $\mathcal{N}$ consisting of finite sets. The next example shows that it is possible for a separable space which satisfies (GC) to have no collection of finite sets witnessing the property.

Example 5.1. *Let $X = D \cup F$ as in Fact 3.13. Then X is separable and satisfies (GC), but no collection of finite sets will witness (GC).*

Proof. A collection witnessing (GC) in X is $\mathcal{N} = \{N_k : k \in \omega\}$ where for each $k < \omega$, $N_k = \{x \in F : |\{\alpha < \mathfrak{c} : \pi_\alpha(x) = 1\}| = k\}$. It is clear that the elements of $\mathcal{N}$ are pairwise disjoint. To see that the N_k's are nowhere dense, let $x \in X \setminus N_k$. Then the number of coordinates of x which are 1 is either less than or equal to k, or more than k.

Suppose $\pi_{\alpha_i}(x) = 1$ for $i = 1, 2, ..., k+1$. Then $\bigcap_{i=1}^{k+1} \pi_{\alpha_i}^{-1}(1)$ is an open set separating x from N_k. If x has 1 in less than or equal to k coordinates, then $x \in \overline{N_k}$. In fact, $\overline{N_k} = \{x \in X : |\{\alpha < \mathfrak{c} : \pi_\alpha(x) = 1\} \leq k\}$. However, any open set in X may only be restricted on finitely many points; thus any open set U has points in it with arbitrarily many coordinates equal to 1 and cannot be contained in $\overline{N_k}$. Thus, $Int(\overline{N_k}) = \emptyset$ and the N_k's are nowhere dense.

Finally, suppose U is a basic open set in X. Then U is the restriction to X of a set of the form $\bigcap\limits_{i=1}^{n} \pi_{\alpha_i}^{-1}(\{1\}) \cap \bigcap\limits_{j=1}^{m} \pi_{\alpha_j}^{-1}(\{0\})$. Then, points in U have at least n ones, so U misses N_k for all $k < n$. However, U contains points with any finite number of ones greater than or equal to n, so $U \cap N_k \neq \emptyset$ for all $k \geq n$. Thus, $\mathcal{N}$ witnesses (GC).

Now, suppose that $\mathcal{N}$ is a collection of finite sets which witnesses (GC). If N_k, $k < \omega$, and M_k, $k < \omega$ are disjoint countable subcollections of $\mathcal{N}$, $\bigcup\limits_{k=1}^{\infty} N_k = \left(\bigcup\limits_{k=1}^{\infty}(N_k \cap F)\right) \cup \left(\bigcup\limits_{k=1}^{\infty}(N_k \cap D)\right)$ and $\bigcup\limits_{k=1}^{\infty} M_k = \left(\bigcup\limits_{k=1}^{\infty}(M_k \cap F)\right) \cup \left(\bigcup\limits_{k=1}^{\infty}(M_k \cap D)\right)$ must both be dense in X, and are disjoint. However, since the topology on D is generated by a maximal independent family, D is irresolvable. So one of $\bigcup(M_k \cap D)$ and $\bigcup(N_k \cap D)$ is not dense in D. Without loss of generality, suppose $\bigcup(M_k \cap D) := M_D$ is not dense in D. Then, there is an open set U in D which misses M_D.

That means $U \subset \overline{\cup M_k \cap F}$. However, since $\cup M_k \cap F$ consists of countably many points, each with ones in finitely many coordinates, there is a $\beta < \mathfrak{c}$ with $\pi_\alpha(M_k \cap F) = 0$ for all $\alpha > \beta$. Since for any point x with $x(\alpha) = 1$ for some $\alpha > \beta$, $x \in \pi_\alpha^{-1}(1)$ and $\pi_\alpha^{-1}(1) \cap (M_k \cap F) = \emptyset$, this means $\pi_\alpha(U) = 0$ for all $\alpha > \beta$. Suppose $U = U_\sigma \cap D$, where U_σ is a basic open set in X. $U_\sigma \subset X$ corresponds to the set $[\sigma]$ in ω. Because the topology on ω is generated by a maximal independent family, $[\sigma] \cap A_\alpha$ is nonempty, in fact infinite, for any α. So the n's in $[\sigma] \cap A_\alpha$ will correspond under the embedding map to points in $U_\sigma \cap D$ which have a one in the αth coordinate, in particular when $\alpha > \beta$. This is a contradiction. So there is no collection $\mathcal{N}$ of finite sets in X witnessing (GC). $\square$

We now see that in any non-separable space, no collection of finite or countable sets can witness (GC).

Proposition 5.2. *In any non-separable space satisfying (GC), the collection $\mathcal{N}$ witnessing this must contain at most finitely many finite or countable sets.*

Proof. Suppose $\mathcal{N}$ is a collection of pairwise disjoint nowhere dense sets in a non-separable space X, and there is a countably infinite subcollection $\mathcal{M}$ of $\mathcal{N}$ for which each $N \in \mathcal{M}$ is finite or countable. Any infinite subcollection of $\mathcal{N}$ must still witness (GC), so $\cup\mathcal{M}$ is a countable dense set in X. This is contradicts that X is not separable. $\square$

Since any infinite subcollection of a collection witnessing (GC) also witnesses (GC), we may assume that every member of a (GC) collection in a non-separable space is uncountable. We may apply this specifically to the lexicographic square, which does satisfy (GC).

Example 5.3. *There is a space satisfying (GC), witnessed by a countable collection, but no collection of finite or countable sets can witness (GC).*

Proof. Let $X = ([0,1]^2, \tau)$, where τ is the topology generated by the lexicographic order.

Since X is not separable, Proposition 5.2 implies that a collection witnessing (GC) cannot contain more than finitely many finite or countable sets.

Let us construct an $\mathcal{N}$ showing that X satisfies (GC). For a prime p, let $N_p = \{(x, a/p) : x \in \mathbb{R}, a \in \mathbb{N}, 1 \leq a < p\}$. Suppose $p_1 \neq p_2$; then if $(x,y) \in N_{p_1} \cap N_{p_2}$, $y = \frac{a_1}{p_1} = \frac{a_2}{p_2}$, which is impossible, since the fractions are both in lowest terms. Also, $Int(\overline{N_p}) = Int\left(N_p \cup ((0,1] \times \{0\}) \cup ([0,1) \times \{1\})\right) = \emptyset$, so the N_p's are nowhere dense. If $U \subset X$ is open, there is a basic open interval contained in U of the form $((x,a),(x,b))$. Then U must meet at least all N_p for which $1/p < (b-a)$; that is, all $p > \frac{1}{b-a}$ or all but finitely many p. So $\mathcal{N} = \{N_p : p \text{ prime}\}$ witnesses (GC) in X. $\square$

We now turn our attention to the cardinality of the collection witnessing (GC). The main question we will consider is whether such a collection may be uncountable. In [6] it is noted that the collection may be assumed to be countable; we want to know whether it is possible for it to be otherwise.

Our first result is that for separable first countable spaces, we cannot have an uncountable collection which witnesses (GC). We begin with a lemma.

Lemma 5.4. *If X is a first countable space which satisfies (GC) witnessed by an uncountable family $\mathcal{N} = \{N_\alpha : \alpha < \omega_1\}$, then for any $x \in X$, there is an $\alpha_x < \omega_1$ such that $x \in \overline{N_\alpha}$ for all $\alpha > \alpha_x$.*

Proof. Let $x \in X$, and let $\{U_n(x) : n \in \omega\}$ be a countable neighborhood base at x, consisting of open sets. Let $\mathcal{N} = \{N_\alpha : \alpha < \omega_1\}$ be a collection witnessing (GC) in X. For each n, $U_n(x)$ meets all but finitely many elements of $\mathcal{N}$; so let α_n be the least element of ω_1 such that $U_n(x) \cap N_\alpha \neq \emptyset$ for all $\alpha \geq \alpha_n$. Let $\alpha_x = \sup\{\alpha_n : n \in \mathbb{N}\} < \omega_1$. Let $\mathcal{N}' = \{N_\alpha : \alpha_x < \alpha < \omega_1\}$. Then, for each α with $\alpha_x < \alpha < \omega_1$, $N_\alpha \cap U_n(x) \neq \emptyset$ for all $n \in \mathbb{N}$; that is, $x \in \overline{N_\alpha}$ for all $N_\alpha \in \mathcal{N}'$. $\square$

Note that $\mathcal{N}'$, being an infinite subcollection of $\mathcal{N}$, still witnesses (GC).

Proposition 5.5. *A first countable, separable space X cannot satisfy (GC) witnessed by an uncountable collection of sets.*

Proof. Let $D = \{d_n : n < \omega\}$ be a dense subset of X and let $\{N_\alpha : \alpha < \omega_1\}$ witness (GC) in X. By Lemma 5.4, for each d_n, there is an element α_{d_n} of ω_1 with the property that $d_n \in \overline{N_\alpha}$ for all $\alpha > \alpha_{d_n}$. Let $\alpha^* = \sup\{\alpha_{d_n} : n \in \omega\} < \omega_1$. Then for all $\alpha > \alpha^*$, $n \in \omega$, $d_n \in \overline{N_\alpha}$. That is, $D \subset \overline{N_\alpha}$ for all $\alpha > \alpha^*$. But this contradicts the fact that N_α must be nowhere dense. $\square$

Suppose a space X satisfies (GC), witnessed by a collection (of arbitrary size) of which infinitely many of the members are countable or finite sets. Then, that space is separable, since if we have a countably infinite collection of countable or finite sets which witnesses (GC), their union is a countable dense set. So, Proposition 5.5 means that: if a first countable space satisfies (GC) witnessed by a collection $\mathcal{N}$, either all but finitely members of $\mathcal{N}$ are uncountable, or $\mathcal{N}$ is countable (or both). This is a partial answer to the question of whether a collection witnessing (GC) may be uncountable.

Question 5.6. *Is there a first countable space which satisfies (GC), witnessed by an uncountable collection (of which all but finitely members must be uncountable)?*

Even in the general case, an uncountable family witnessing (GC) cannot substantially consist of finite sets.

Proposition 5.7. *Suppose $|X| \geq \omega$. If a collection $\mathcal{N}$ witnesses (GC) or (NC) in X, and $\mathcal{N}$ is composed of finite sets, there is no $k < \omega$ such that $|N| = k$ for all $N \in \mathcal{N}$.*

Proof. Suppose to the contrary that $\mathcal{N}$ is a collection witnessing (GC) or (NC) in X, and there is a $k \in \mathbb{N}$ such that for every $N \in \mathcal{N}$, $|N| = k$. Let $\mathcal{U} = \{U_0, U_1, ..., U_k\}$ be a pairwise disjoint collection of $k+1$ open sets. If $\mathcal{N}$ witnesses (GC), it also witnesses (NC) (see [6], remarks after the definitions of (GC) and (NC)). So there is an $N \in \mathcal{N}$ which meets every member of $\mathcal{U}$; say $x_i \in N \cap U_i$ for $i \in k+1$. Since $\mathcal{U}$ is a pairwise disjoint collection, $x_i \neq x_j$ for any $i \neq j$. This contradicts that $|N| = k$; so the collection $\mathcal{N}$ cannot exist as described. □

Corollary 5.8. *No uncountable family of finite sets can witness (GC).*

Proof. If $\mathcal{N}$ were such an uncountable family, for some $k \in \omega$ there would be a countable subcollection $\mathcal{N}_k$ consisting of all elements of size k. This would still witness (GC); but this contradicts Proposition 5.7. So, in any collection witnessing (GC) in X, at most countably many sets can be finite. □

This raises the question:

Question 5.9. *Can an uncountable family of closed sets witness (GC)?*

5.2 Ordered Spaces and (GC)

We now turn our attention to examining the conditions under which ordered spaces satisfy (GC). First, we look at linearly ordered spaces (LOTS), and then at generalized ordered spaces (GO-spaces).

Lemma 5.10. *If a space X contains a dense subspace Y which satisfies (GC), then X satisfies (GC).*

Proof. Let $\mathcal{N}$ be the collection witnessing (GC) in Y. We claim that $\mathcal{N}$ also witnesses (GC) in X.

Let $N \in \mathcal{N}$. If there is an open set $U \subset cl_X(N)$, then $U \cap Y \subset cl_X(N) \cap Y = cl_Y(N)$. Since N is nowhere dense in Y, $U \cap Y$ must be empty. But $U \cap N \subset U \cap Y = \emptyset \Rightarrow U = \emptyset$. Thus N is nowhere dense in X.

If U is an open set in X, then $U \cap Y$ is nonempty (since Y is dense) and open in Y, so $U \cap Y$ meets all but finitely many members of $\mathcal{N}$. Thus, U meets all but finitely many members of $\mathcal{N}$.

So, $\mathcal{N}$ is a pairwise disjoint collection of nowhere dense sets in X, and each open set in X meets all but finitely many members of $\mathcal{N}$; that is, $\mathcal{N}$ witnesses (GC) in X. □

Proposition 5.11. *If X has a dense metrizable subspace, then X satisfies (GC).*

Proof. By Proposition 4.2 in [6], every metrizable dense-in-itself space satisfies (GC), so this follows directly from Lemma 5.10. □

We now characterize the ordered spaces which satisfy (GC).

Lemma 5.12. *If a LOTS satisfies (GC), it has a σ-disjoint π-base.*

Proof. Suppose $\mathcal{N} = \{N_k : k < \omega\}$ is a collection witnessing (GC) in a linearly ordered space X. For each $k < \omega$, define $U_k = X \backslash (\overline{N_0} \cup \overline{N_1} \cup ... \cup \overline{N_k})$. U_k is open, so it may be written as a collection $\mathcal{I}_k$ of disjoint open intervals. Let $\mathcal{B} = \cup_{k<\omega} \mathcal{I}_k$. Obviously $\mathcal{B}$ is σ-disjoint. We claim that each nonempty interval $(a, b) \subset X$ contains a member of $\mathcal{B}$:

Since (a, b) is open, it must meet all but finitely many members of $\mathcal{N}$. In particular, there are $k, l < \omega$ such that N_k and N_l both meet (a, b), say at c_k and c_l. Without loss of generality, $c_k < c_l$. Then, for any $m > \max\{k, l\}$, c_k and c_l are in $M := \overline{N_0} \cup \overline{N_1} \cup ... \cup \overline{N_m}$. M is nowhere dense in X, so it is nowhere dense in (c_k, c_l). Thus, $\mathcal{I}_m$ contains an interval which is contained in (c_k, c_l), say I. Then $I \subset (a, b)$ and $I \in \mathcal{B}$. So $\mathcal{B}$ is a σ-disjoint π-base for X. $\square$

If X is a Baire LOTS, we see that the converse of Proposition 5.11 is true.

Theorem 5.13. *Let X be a linearly ordered space with no isolated points. If X is Baire and satisfies (GC), X has a dense metrizable subspace.*

Proof. Let X be a Baire LOTS which satisfies (GC), witnessed by a collection $\mathcal{N} = \{N_k : k < \omega\}$. For each $k < \omega$, define $U_k = X \backslash (\overline{N_0} \cup \overline{N_1} \cup ... \cup \overline{N_k})$. Since the N_k's are nowhere dense, each U_k is dense and open; so $Y = \bigcap_{k<\omega} U_k$ is dense in the Baire space X. We will show that Y is metrizable by constructing a σ-locally finite base for Y.

Consider the σ-disjoint π-base $\bigcup_{k<\omega} \mathcal{I}_k$ for X given by Lemma 5.12. Note that for each k, $\bigcup \mathcal{I}_k = U_k$. Let $\mathcal{B}$ be the restriction of this π-base to Y. If $x \in Y$, then $x \in \bigcup \mathcal{I}_k$ for all $k < \omega$. Since $\mathcal{I}_k$ is a collection of disjoint open intervals, this means that the interval $J \in \mathcal{I}_k$ containing x is a neighborhood of x which does not meet any other members of $\mathcal{I}_k$. Thus, $J \cap Y$ witnesses that $\{I \cap Y : I \in \mathcal{I}_k\}$ is locally finite.

To see that $\mathcal{B}$ forms a base for Y, let $x \in (a, b) \cap Y$, where (a, b) is a basic open set in X. Since x is in $Y = \bigcap_{k<\omega} U_k = \bigcap_{k<\omega} (\bigcup \mathcal{I}_k)$, there is an I_k for each $k < \omega$

such that $x \in I_k \subset \mathcal{I}_k$. Suppose there is no k such that $I_k \subset (a,b)$. Then either $(a,b) \subset I_k$ for all $k < \omega$, or one of a, b is in I_k for all $k < \omega$.

If $(a,b) \subset I_k$ for all k, then $(a,b) \subset X \backslash \overline{N_k}$ for all k, which contradicts that $\mathcal{N}$ witnesses (GC).

If one of a, b, say a, is in I_k for all k, $(a,x) \subset I_k$ for all k because $x \in I_k$ and I_k is an interval. This again gives a contradiction with $\mathcal{N}$ witnessing (GC), because then (a,x) would miss infinitely many N_k's.

So, for some k, $I_k \subset (a,b)$, and thus $I_k \cap Y \subset (a,b) \cap Y$. This shows that $\mathcal{B}$ is a σ-locally finite base for Y. Y is T_3, being a subspace of a LOTS, and this means that Y is metrizable. □

Combining Proposition 5.11 and Theorem 5.13 shows that a Baire dense-in-itself LOTS satisfies (GC) iff it has a dense metrizable subspace. For non-Baire spaces, this is not true, as the following example shows.

Example 5.14. *There is a LOTS satisfying (GC) which does not have a dense metrizable subspace.*

Proof. The space is Gruenhage and Lutzer's example in [5] of a LOTS that is Volterra but not Baire. In their paper, they show that Volterra=Baire in any space with a dense metrizable subspace; so this space clearly does not have such a subspace.

Let X be the set of all functions f from ω_1 to the integers with the property that $f(\alpha) = 0$ for all but finitely many α; give X the topology generated by the lexicographic order. A neighborhood base at each $f \in X$ is $\{B(f,\alpha) : \alpha < \omega_1\}$, where $B(f,\alpha) = \{g \in X : \forall \beta \leq \alpha,\ g(\beta) = f(\beta)\}$.

For each $k < \omega$, let $N_k = \{f \in X : |\{\alpha < \omega_1 : f(\alpha) \neq 0\}| = k\}$. The collection $\mathcal{N} = \{N_k : k < \omega\}$ is clearly pairwise disjoint. If $f \in X$ is nonzero for at least $k+1$ coordinates, then there is $\alpha < \omega_1$ such that $f(x) \neq 0$ for at least $k+1$ coordinates less than α; so $B(f,\alpha) \cap N_k = \emptyset$. Thus, $\overline{N_k} \subseteq \{f \in X : |\{\alpha < \omega_1 : f(\alpha) \neq 0\}| \leq k\}$.

This cannot contain any $B(g,\beta)$, since these open sets contain points with arbitrarily many nonzero coordinates. So each N_k is nowhere dense.

Now, let $U \subset X$ be open and nonempty. Then there exist f and α such that $B(f,\alpha) \subset U$. Let $K = |\{\beta : f(\beta) \neq 0 \text{ and } \beta \leq \alpha\}|$. Then, since every $g \in B(f,\alpha)$ must agree with f up to the αth coordinate, every g in $B(f,\alpha)$ has at least K nonzero coordinates. In fact, for every $n > K$, there is a $g \in B(f,\alpha)$ with n nonzero coordinates. So $B(f,\alpha) \cap N_k$ for all $k > K$.

Thus, $\mathcal{N}$ witnesses (GC) in X. $\square$

We will now build on our work with LOTS to obtain a more general result: the characterization of all GO-spaces which satisfy (GC).

Theorem 5.15. *A GO-space with no isolated points satisfies (GC) iff it has a σ-disjoint π-base.*

Proof. ($\Rightarrow$) Suppose a GO-space X satisfies (GC). X may be considered the dense subspace of a LOTS L. By Lemma 5.10, L satisfies (GC), which means that L has a σ-disjoint π-base (Lemma 5.12). Restricting the members of this π-base to X will give a σ-disjoint π-base for X.

($\Leftarrow$) Suppose X is a GO-space with no isolated points, and $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_n$ is a σ-disjoint π-base for X, such that each $\mathcal{B}_n$ is a disjoint collection. Without loss of generality, we may assume that the elements of $\mathcal{B}$ are convex open sets.

For each n, consider $\bigcup \mathcal{B}_n$. If this is not dense in X, then $X \setminus \overline{\bigcup \mathcal{B}_n}$ is a nonempty open set; write it as a collection of disjoint convex open sets $\{U_\alpha : \alpha < A\}$. Let $\mathcal{C}_n = \mathcal{B}_n \cup \{U_\alpha : \alpha \in A\}$. Note that $\bigcup \mathcal{C}_n$ is dense in X, and $\mathcal{C}_n$ is a collection of disjoint convex open sets.

We will define $\mathcal{I}_n$, $n < \omega$, by induction. Let $\mathcal{I}_0 = \mathcal{C}_0$. For each $n > 0$, let $\mathcal{I}_n = \{U \cap V : U \in \mathcal{C}_n,\ V \in \mathcal{I}_{n-1}\}$. Note that $\mathcal{I}_n$ is a disjoint collection of convex open sets: If $x \in (U_1 \cap V_1) \cap (U_2 \cap V_2)$ for some $U_1 \cap V_1, U_2 \cap V_2 \in \mathcal{I}'_n$, then $x \in U_1 \cap U_2 \Rightarrow U_1 = U_2$, since the elements of $\mathcal{C}_n$ are pairwise disjoint. Similarly, $x \in V_1 \cap V_2$ and $V_1, V_2 \in \mathcal{I}_{n-1}$ implies that $V_1 = V_2$. So $U_1 \cap V_1 = U_2 \cap V_2$.

Also note that $\bigcup \mathcal{I}_n$ is still dense in X. Moreover, $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ is a π-base, since $\bigcup_{n\in\omega} \mathcal{C}_n$ contains a π-base and every member of $\mathcal{C}_n$ contains some member of an $\mathcal{I}_k$.

Now, we will use $\mathcal{I}$ to define, by induction, the collection $\mathcal{N}$ witnessing (GC). Pick an element $x_0^I \in I$ from each $I \in \mathcal{I}_0$. Set $N_0 = \{x_0^I | I \in \mathcal{I}_0\}$. If N_k has been defined, for each $I \in \mathcal{I}_{k+1}$, consider the set $F_I = \{x_n^J | I \subseteq J \text{ and } J \in \mathcal{I}_n,\ n \leq k\}$. Observe that there are only finitely many $\mathcal{I}_n$'s with $n \leq k$. By the construction, the members of $\mathcal{I}_n$ are pairwise disjoint; so there can only be one J in each $\mathcal{I}_n$ with $I \subseteq J$. So F_I is a finite set.

Now, for each $I \in \mathcal{I}_{k+1}$, chose an element $x_{k+1}^I \in I \backslash F_I$. Since X is a T_2 space without isolated points, and each $I \in \mathcal{I}_{k+1}$ is open, I is infinite; so this is always possible. Let $N_{k+1} = \{x_{k+1}^I | I \in \mathcal{I}_{k+1}\}$.

Since X has no isolated points, $I \backslash F_I$ is open; it is also dense in I. Thus, since each $\cup \mathcal{I}_k$ is open and dense in X, so is each $U_k = \bigcup \{I \backslash F_I : I \in \mathcal{I}_k\}$. Thus, $X \backslash U_k$ is nowhere dense; in particular, $N_{k-1} \subseteq X \backslash U_k$ is nowhere dense. (Recall that F_I contains the points x_{k-1}^I.) Also, by construction, $N_k \cap N_l = \emptyset$ if $k \neq l$; so we see that $\mathcal{N} = \{N_k | k < \omega\}$ is a pairwise disjoint collection of nowhere dense sets.

It remains to see that $\mathcal{N}$ witnesses (GC) in X. Let U be a nonempty open subset of X. Then, U contains a member of the π-base $\mathcal{I}$. Say $I_0 \in \mathcal{I}_k$ is contained in U. Then $x_k^{I_0} \in I_0 \subseteq U$ implies that $U \cap N_k \neq \emptyset$. Now, by the construction of the $\mathcal{I}_n$'s, there is an $I_1 \in \mathcal{I}_{k+1}$, with $I_1 \subseteq I_0$. Since $x_{k+1}^{I_1} \in I_1 \cap N_{k+1}$, and $I_1 \subseteq U$, $U \cap N_{k+1} \neq \emptyset$. Continuing in this way, we can find $I_n \in \mathcal{I}_{k+n}$ which is contained in U; the $x_{k+n}^{I_n}$ chosen to be in I_n at stage $k+n$ of the induction will be in $N_{k+n} \cap U$. So U meets all but finitely members of $\mathcal{N}$. $\square$

This result leads easily to the following example:

Example 5.16. *A Souslin line cannot satisfy (GC).*

Proof. Recall that a Souslin line is a dense linearly ordered space which is (ccc) but not separable. Let S be a Souslin line with no isolated points. (Note that spaces with isolated points cannot satisfy (GC).) Suppose S satisfies (GC); then there is a σ-disjoint π-base in S. But, S is ccc; so the π-base will be countable, and choosing a point from each member of the π-base gives a countable dense set in S, which is impossible. So S cannot satisfy (GC). $\square$

5.3 Ordered Spaces and (NC)

We consider next the conditions under which an ordered space satisfies (NC).

Lemma 5.17. *If a space X satisfies (NC), then there is a collection of cardinality at most $\Delta(X)$ which witnesses (NC).*

Proof. Suppose $\mathcal{N}$ witnesses (NC) in X, and that $U \subset X$ is open, with $|U| = \Delta(X)$. Let $V = X \backslash \overline{U}$. Define $\mathcal{N}' = \{N \in \mathcal{N} : N \cap U \neq \emptyset\}$. The members of $\mathcal{N}'$ are clearly pairwise disjoint, and since each meets U, there can be at most $|U| = \Delta(X)$ of them. We claim that $\mathcal{N}'$ witnesses (NC). Given a finite pairwise disjoint collection $\mathcal{U}$ of open sets in X, either at least one of them meets U, or none of them do.

Suppose no element of $\mathcal{U}$ meets U; then $\mathcal{U} \cup \{U\}$ is a pairwise disjoint, finite collection of open sets in X. So there is an $N \in \mathcal{N}$ which meets each; but this N meets U, so it is in $\mathcal{N}'$.

Now, suppose that at least one element of $\mathcal{U}$ meets U. Consider the refinement of $\mathcal{U}$ given by

$$\mathcal{U}' = (\{A \cap U : A \in \mathcal{U}\} \cup \{A \cap V : A \in \mathcal{U}\}) \backslash \{\emptyset\}$$

This is a finite pairwise disjoint collection of nonempty open sets in X; so there is an $N \in \mathcal{N}$ which meets each member of $\mathcal{U}'$. Since there was an $A \in \mathcal{U}$ for which $A \cap U \neq \emptyset$, this N must meet U. So $N \in \mathcal{N}'$; but since each $A \in \mathcal{U}$ must meet either U or V, N meets each member of $\mathcal{U}$ (the original collection) as well.

Thus, $\mathcal{N}'$ witnesses (NC). $\square$

Note that this shows also that whenever $\mathcal{N}$ witnesses (NC) in a space X, and $U \subset X$ is open, the subcollection $\mathcal{N}_U = \{N \in \mathcal{N} : U \cap N \neq \emptyset\}$ also witnesses (NC) in X.

Example 5.18. *There is a LOTS which does not satisfy (NC).*

Proof. The space is $X = \mathbb{Q} \oplus Y$, where $\mathbb{Q}$ is the rationals with the usual topology and Y is the set $\mathbb{Z}^{\omega_1}$ with the lexicographic order topology.

We will show that no countable union of nowhere dense sets is dense in Y. This will be sufficient because the union of any collection witnessing (NC) must be dense in X, and thus in Y. However, $\Delta(X) = \omega$ (since $\mathbb{Q}$ is an open subspace), so Proposition 5.17 shows that if X satisfies (NC), there is a countable collection witnessing (NC).

Now, let us see that, if $\{N_k : k < \omega\}$ is a collection of nowhere dense sets in Y, $\cup_{k<\omega} N_k$ is not dense in Y. Let U be a nonempty open set in Y. Since N_0 is nowhere dense, there is a nonempty open set $U_0 \subset U \backslash \overline{N_0}$. Since a LOTS is regular and the intervals form a base, we may choose an interval $I_0 = (f_0, g_0') \subset \overline{I_0} \subset U_0$. Let $\alpha_0 = min\{\alpha < \omega_1 : f_0(\alpha) \neq g_0'(\alpha)\}$. Define

$$g_0(\alpha) = \begin{cases} g_0'(\alpha) & \alpha \notin \{\alpha_0, \alpha_0 + 1\} \\ f_0(\alpha) & \alpha = \alpha_0 \\ f_0(\alpha) + 2 & \alpha = \alpha_0 + 1 \end{cases}$$

Observe the following about g_0:

- $f_0 < g_0$ because $\alpha_0 + 1$ is the first α at which $f_0(\alpha) \neq g_0(\alpha)$, and $f_0(\alpha_0 + 1) < f_0(\alpha_0 + 1) + 2 = g_0(\alpha_0 + 1)$.

- $g_0 < g_0'$ because g_0, f_0, and g_0' agree until α_0, and then $g_0(\alpha_0) = f_0(\alpha_0)$, which is less than $g_0'(\alpha_0)$ because $f_0 < g_0'$.

Thus, (f_0, g_0) is an open interval which, being contained in I_0, misses $\overline{N_0}$. Since Y has no adjacent points, the interval is nonempty.

Now, suppose $\alpha_k, (f_k, g_k)$ have been defined for $k < n$, satisfying:

- $(f_0, g_0) \supset (f_1, g_1) \supset ... \supset (f_{n-1}, g_{n-1})$
- $\alpha_0 < \alpha_1 < ... < \alpha_{n-1}$
- $f_k(\alpha) = g_k(\alpha)$ for all $\alpha \leq \alpha_k$
- $g_k(\alpha_k + 1) = f_k(\alpha_k + 1) + 2$
- $g_k(\alpha_{k-1} + 1) = f_k(\alpha_{k-1} + 1) = f_{k-1}(\alpha_{k-1} + 1) + 1$

Define α_n and (f_n, g_n) as follows:

Since $\overline{N_0} \cup ... \cup \overline{N_n}$ is nowhere dense, choose an interval $(f_n, g'_n) \subset [f_n, g'_n] \subset (f_{n-1}, g_{n-1})$ which misses $\overline{N_0} \cup ... \cup \overline{N_n}$, with the property that $f_n(\alpha_{n-1}+1) = g'_n(\alpha_{n-1}+1) = f_{n-1}(\alpha_{n-1}+1)+1$. This is possible because any function which is equal to f_{n-1} and g_{n-1} up to α_{n-1}, with $\alpha_{n-1}+1$st coordinate so defined, is between f_{n-1} and g_{n-1}, since $f_{n-1}(\alpha_{n-1}+1) < f_{n-1}(\alpha_{n-1}+1)+1 < f_{n-1}(\alpha_{n-1}+1)+2 = g_{n-1}(\alpha_{n-1}+1)$. Choosing any two such functions to be endpoints of an open interval would give an interval J properly contained in (f_{n-1}, g_{n-1}), with the property that every function in J agrees with f_{n-1} and g_{n-1} up to the point where they split, and is directly between them at the next coordinate. We may then take a subinterval of J which misses $\overline{N_0} \cup ... \cup \overline{N_n}$. Let $\alpha_n = min\{\alpha < \omega_1 : f_n(\alpha) \neq g'_n(\alpha)\}$. Note that, since $f_{n-1} < f_n < g'_n < g_{n-1}$, $\alpha_n > \alpha_{n-1}$. Define

$$g_n(\alpha) = \begin{cases} g'_n(\alpha) & \alpha \notin \{\alpha_n, \alpha_n + 1\} \\ f_n(\alpha) & \alpha = \alpha_n \\ f_n(\alpha) + 2 & \alpha = \alpha_n + 1 \end{cases}$$

As before, (f_n, g_n) is a nonempty open interval contained in (f_n, g'_n), thus contained in (f_{n-1}, g_{n-1}) and missing $\cup_{i \le n} \overline{N_n}$.

Continuing in this way, we get a sequence of intervals $(f_0, g_0) \supset (f_1, g_1) \supset (f_2, g_2) \supset \ldots$ with the property that $f_{n-1} < f_n < g_n < g_{n-1}$, $f_n(\alpha) = g_n(\alpha)$ for all $\alpha \le \alpha_n$, $f_n(\alpha_n + 1) + 2 = g_n(\alpha_n + 1)$, and $f_n(\alpha_{n-1} + 1) = g_n(\alpha_{n-1} + 1) = f_{n-1}(\alpha_{n-1} + 1) + 1 < g_{n-1}(\alpha_{n-1} + 1)$.

Consider $\bigcap_{n \in \mathbb{N}} I_n$. Suppose there is an $h \in \mathbb{Z}^{\omega_1}$ such that $h > f_n$ for all n and $h < g_n$ for all n; in this case, h is an upper bound for $\{f_n\}_{n \in \mathbb{N}}$; but we claim $f_n \not\to h$. To see this, for each n, we let $\alpha_n < \omega_1$ be the least such that $f_n(\alpha_n) \ne h(\alpha_n)$. Then $\beta = \sup\{\alpha_n : n \in \mathbb{N}\} < \omega_1$ because the set is countable. Let h^- be any function which is identical to h for all $\alpha \le \beta$, and $h^-(\beta + 1) = h(\beta + 1) - 1$, and h^+ be any function which is greater than h. $I_f = (h^-, h^+)$ is an interval containing h which misses every element of $\{f_n\}$, because at the point α_n, $f_n(\alpha_n) < h(\alpha_n) = h^-(\alpha_n)$, and for all $\alpha < \alpha_n$, $f(\alpha) = h(\alpha) = h^-(\alpha)$.

Similarly, we may find an interval $I_g = (p^-, p^+)$, where p^- is any function less than h and p^+ is a function that agrees with h until after the point at which the g_n's have differed from h. This interval contains h but misses every element of $\{g_n\}$. Then, (h^-, p^+) is a nonempty interval with endpoints that are less than every element of $\{g_n\}$ and greater than every element of $\{f_n\}$. So $(h^-, p^+) \subset \cap_{n \in \mathbb{N}} I_n$. But, any point in $\cap_{n \in \mathbb{N}} I_n$ must miss N_k for every $k < \omega$; so $\cup_{k < \omega} N_k$ is not dense in Y.

We now show that there is indeed an h which is greater than f_n and less than g_n for each $n \in \mathbb{N}$. Define

$$h(\alpha) = \begin{cases} f_0(\alpha) & \alpha < \alpha_0 + 1 \\ f_0(\alpha) + 1 = f_1(\alpha_0 + 1) & \alpha = \alpha_0 + 1 \\ f_1(\alpha) & \alpha_0 + 1 < \alpha < \alpha_1 + 1 \\ f_1(\alpha) + 1 = f_2(\alpha_1 + 1) & \alpha = \alpha_1 + 1 \\ ... & \\ f_n(\alpha) & \alpha_{n-1} + 1 < \alpha < \alpha_n + 1 \\ f_n(\alpha) + 1 = f_{n+1}(\alpha_n + 1) & \alpha = \alpha_n + 1 \\ f_{n+1}(\alpha) & \alpha_n + 1 < \alpha < \alpha_{n+1} + 1 \\ ... & \\ 0 & \alpha \geq sup\{\alpha_n + 1 : n \in \omega\} \end{cases}$$

with the understanding that a line is omitted if there is no α in the prescribed range (ie, if $\alpha_k = \alpha_{k-1} + 1$).

It is clear that $h : \omega_1 \to \mathbb{Z}$, so $h \in Y$. We claim that $f_n < h < g_n$ for all $n \in \omega$.

By definition of the functions f_n, g_n, for any $k < n$, $\alpha < \alpha_k$, $f_n(\alpha) = g_n(\alpha) = f_k(\alpha) = g_k(\alpha)$. So $\min\{\alpha : h(\alpha) \neq f_n(\alpha)\} = \min\{\alpha : h(\alpha) \neq g_n(\alpha)\} = \alpha_n + 1$. And $h(\alpha_n+1) = f_n(\alpha_n+1)+1$, which is between $f_n(\alpha_n+1)$ and $g_n(\alpha_n+1) = f_n(\alpha_n+1)+2$. So $f_n < h < g_n$. Thus, $h \in \cap_{n<\omega}(f_n, g_n)$. □

This example failed to satisfy (NC) because it contained disjoint subspaces on which the cardinality and structure of the open sets varied widely. We now examine some conditions under which the topological sum $X \oplus Y$ will satisfy (NC), given that each of X and Y do.

Proposition 5.19. *Let X be a space with the property that each point has a neighborhood of cardinality $\Delta(X)$, and a local π-base of cardinality $\Delta(X)$. Then X satisfies (NC).*

Proof. Fix a maximal pairwise disjoint collection $\mathcal{C} = \{C_\alpha : \alpha < |X|\}$ of open subsets of X, each of size $\lambda = \Delta(X)$. For each α, there is a π-base $\mathcal{B}^\alpha$ for C_α of cardinality λ. (For instance, we could take $\mathcal{B}^\alpha$ to be the union of the local π-bases of size λ for each of the λ-many points of C_α.) Index $\mathcal{B}^\alpha$ as $\{B^\alpha_\beta : \beta < \lambda\}$. Since $\Delta(X)$ is the minimum cardinality of an open set, and each B^α_β is contained in the open set C_α which has size λ, $|B^\alpha_\beta| = \lambda$ for each α, β. Index the collection of all finite subsets of λ as $\{F_\gamma : \gamma < \lambda\}$. For each $\gamma < \lambda$, let $A_\gamma = \{B^\alpha_\beta : \alpha < |X|, \beta \in F_\gamma\}$. That is, A_γ is the collection of all π-base elements whose index is in F_γ, across all the C_α's. Choose sets N_γ by induction so that:

- N_γ contains one point from each element of A_γ
- $N_{\gamma'} \cap N_\gamma = \emptyset$ for all $\gamma' < \gamma$

  This is possible because $|B^\alpha_\beta| = \lambda > \gamma$ for all α, β.

Note that since each N_γ has finite intersection with each C_α, the N_γ's are nowhere dense.

To see that $\mathcal{N} = \{N_\gamma : \gamma < \lambda\}$ witnesses (NC) in X, let $U_1, ..., U_n$ be a pairwise disjoint collection of open sets in X. For each U_i, choose a $C_{\alpha_i} \in \mathcal{C}$ such that $U_i \cap C_{\alpha_i} \neq \emptyset$. For each i, $U_i \cap C_{\alpha_i}$ is an open subset of C_{α_i}, so it contains an element $B^{\alpha_i}_{\beta_i} \in \mathcal{B}^{\alpha_i}$. The collection of indices $\{\beta_1, \beta_2, ..., \beta_n\}$ is a finite subset of λ, so it was one of the F_γ's. Then, N_γ contains a point from $B^\alpha_{\beta_i}$ for each $i = 1, .., n$ and for every $\alpha < |X|$, in particular for $\alpha_1, ..., \alpha_n$. Thus, $N_\gamma \cap U_i \supset N_\gamma \cap B^{\alpha_i}_{\beta_i} \neq \emptyset$ for each $i = 1, 2, ..., n$. □

In particular, we may apply this proposition to an ordered space. The fact that the intervals are a base gives us the uniformity we need, as long as we have one interval for each point that matches a standard.

Corollary 5.20. *A LOTS X with the property that for every point, there is an interval of cardinality $\Delta(X)$ containing that point, satisfies (NC).*

Proof. If I is such an interval, a local π-base of cardinality $\Delta(X)$ is given by $\{(a,b) : a < b; a, b \in I\}$. $\square$

In [6], the authors show that if a collection $\mathcal{N}$ witnesses (NC_2) in a strongly irresolvable space X, the family $\mathcal{F}$ of all $\mathcal{M} \subset \mathcal{N}$ such that $\mathcal{M}$ also witnesses (NC_2) in X is an ultrafilter on $\mathcal{N}$. They also show that in a strongly irresolvable space, a collection $\mathcal{N}$ witnesses (NC_2) if and only if it witnesses (NC). So in strongly irresolvable spaces, we have similar ultrafilters on the collections which witness (NC). These ultrafilters may be used to characterize the conditions under which $X \oplus Y$ satisfies (NC), given that X and Y are strongly irresolvable and satisfy (NC).

Theorem 5.21. *Let X and Y be strongly irresolvable spaces which satisfy (NC). $X \oplus Y$ satisfies (NC) if and only if there are collections $\mathcal{N}_X$, $\mathcal{N}_Y$ witnessing (NC) in X and Y respectively, and a function $f : \mathcal{N}_X \to \mathcal{N}_Y$ such that $f(\mathcal{M}) \in \mathcal{F}_Y$ for all $\mathcal{M} \in \mathcal{F}_X$ and $f^{-1}(\mathcal{M}) \in \mathcal{F}_X$ for all $\mathcal{M} \in \mathcal{F}_Y$ (where $\mathcal{F}_X$, $\mathcal{F}_Y$ are the ultrafilters on $\mathcal{N}_X$ and $\mathcal{N}_Y$ respectively, discussed above).*

Proof. ($\Leftarrow$) Suppose we have such a function from $\mathcal{N}_X$ to $\mathcal{N}_Y$, and for $N \in \mathcal{N}_Y$, consider $f^{-1}(N) \subset \mathcal{N}_X$. Since $\mathcal{F}_Y$ is an ultrafilter on $\mathcal{N}$ and $\{N\}$ does not witness (NC) and thus is not in $\mathcal{F}_Y$, $\mathcal{N}_Y \backslash \{N\} \in \mathcal{F}_Y$. So $f^{-1}(\mathcal{N}_Y \backslash \{N\}) = \mathcal{N}_X \backslash f^{-1}(N) \in \mathcal{F}_X$. So $\bigcup \mathcal{N}_X \backslash f^{-1}(N)$ is dense in X. Since X is strongly irresolvable, this means that $\bigcup f^{-1}(N)$ must be nowhere dense.

Define $\mathcal{N}_{X\oplus Y}$ to be $\{N \cup (\bigcup f^{-1}(N)) : N \in \mathcal{N}_Y\}$. Since f is a function, this is a pairwise disjoint collection of sets in $X \oplus Y$. Each $N \cup (\bigcup f^{-1}(N))$ is nowhere dense, since $N \in \mathcal{N}_Y$ and we have shown that $f^{-1}(N)$ is nowhere dense.

To see that $\mathcal{N}_{X\oplus Y}$ witnesses (NC), it is enough to show that if $U_1, .., U_n$ is a collection of pairwise disjoint open sets in X and $V_1, .., V_k$ is a collection of pairwise disjoint open sets in Y, there is an element of $\mathcal{N}_{X\oplus Y}$ which meets each of $U_1, ..., U_n, V_1, ..., V_k$. For an open set $U \subset X$, define $\mathcal{M}_U$ to be the set of elements of $\mathcal{N}_X$ which meet U. By the remark after Lemma 5.17, $\mathcal{M}_U \in \mathcal{F}_X$. Since $\mathcal{F}_X$ is a filter, $\mathcal{M} := \mathcal{M}_{U_1} \cap \mathcal{M}_{U_2} \cap ... \cap \mathcal{M}_{U_n} \in \mathcal{F}_X$. Consider $f(\mathcal{M}) \in \mathcal{F}_Y$. $f(\mathcal{M})$ witnesses (NC), so there is an $N_Y \in f(\mathcal{M})$ which meets every member of $\{V_1, ..., V_k\}$. Then, $f^{-1}(N_Y) \subset \mathcal{M}$, so $\cup f^{-1}(N_Y)$ meets each of the U's. Thus, $N_Y \cup (\cup f^{-1}(N_Y)) \in \mathcal{N}_{X\oplus Y}$ and meets each of the U's and V's.

($\Rightarrow$) Suppose $X \oplus Y$ satisfies (NC), witnessed by a collection $\mathcal{N}_{X\oplus Y}$. Without loss of generality, we may assume that each member of $\mathcal{N}_{X\oplus Y}$ meets both X and Y. Indeed, since X and Y are each open in $X \oplus Y$, $\mathcal{M}_X = \{N \in \mathcal{N}_{X\oplus Y} : X \cap N \neq \emptyset\}$ and $\mathcal{M}_Y = \{N \in \mathcal{N}_{X\oplus Y} : Y \cap N \neq \emptyset\}$ are in $\mathcal{F}_{X\oplus Y}$. So $\mathcal{M}_X \cap \mathcal{M}_Y \in \mathcal{F}_{X\oplus Y}$, and if necessary we will use $\mathcal{M}_X \cap \mathcal{M}_Y$ in place of $\mathcal{N}_{X\oplus Y}$.

Define

$$\mathcal{N}_X = \{N \cap X : N \in \mathcal{N}_{X\oplus Y}\}$$

$$\mathcal{N}_Y = \{N \cap Y : N \in \mathcal{N}_{X\oplus Y}\}$$

$$f : \mathcal{N}_X \to \mathcal{N}_Y : N \cap X \mapsto N \cap Y$$

It is clear that $\mathcal{N}_X$ witnesses (NC) in X and $\mathcal{N}_Y$ witnesses (NC) in Y. Let $\mathcal{F}_X$, $\mathcal{F}_Y$ be the ultrafilters associated with $\mathcal{N}_X$ and $\mathcal{N}_Y$, respectively.

Claim: Let $\mathcal{M} \in \mathcal{F}_X$. $\tilde{\mathcal{M}} := \{N \in \mathcal{N}_{X\oplus Y} : N \cap X \in \mathcal{M}\} \in \mathcal{F}_{X\oplus Y}$

It is enough to show that $\cup\tilde{\mathcal{M}}$ is dense in $X \oplus Y$ (see proof of Lemma 4.13 in [6]). Suppose that there is an open set $U \subset X \oplus Y$ which does not meet $\bigcup \tilde{\mathcal{M}}$, and consider $\mathcal{M}_U = \{N \in \mathcal{N}_{X\oplus Y} : N \cap U \neq \emptyset\} \in \mathcal{F}_{X\oplus Y}$. $\mathcal{M}_U \cap \tilde{\mathcal{M}} = \emptyset$, since every member of $\mathcal{M}_U$ meets U and no member of $\tilde{\mathcal{M}}$ meets U. Since these are both subcollections of the pairwise disjoint collection $\mathcal{N}_{X\oplus Y}$, $\cup\mathcal{M}_U$ and $\cup\tilde{\mathcal{M}}$ are disjoint. But $\cup\mathcal{M}_U$ is dense in $X \oplus Y$, hence in X; and $\left(\bigcup \tilde{\mathcal{M}}\right) \cap X = \bigcup \mathcal{M}$ is also dense in X. This contradicts that X is strongly irresolvable. So $\bigcup \tilde{\mathcal{M}}$ must be dense in $X \oplus Y$, and $\tilde{\mathcal{M}} \in \mathcal{F}_{X\oplus Y}$.

Now, for $\mathcal{M} \in \mathcal{F}_X$,

$$\begin{aligned} f(\mathcal{M}) &= \{Y \cap N : N \cap X \in \mathcal{M}\} \\ &= \{Y \cap N : N \in \widetilde{\mathcal{M}}\} \end{aligned}$$

Since $\widetilde{\mathcal{M}}$ witnesses (NC) in $X \oplus Y$, $f(\mathcal{M})$ witnesses (NC) in Y. So $f(\mathcal{M}) \in \mathcal{F}_Y$.

Similarly, $f^{-1}(\mathcal{M}) \in \mathcal{F}_X$ for each $\mathcal{M} \in \mathcal{F}_Y$. □

When $\mathcal{N}_X$ and $\mathcal{N}_Y$ are countable, the ultrafilters $\mathcal{F}_X$ and $\mathcal{F}_Y$ may be regarded as members of $\beta\omega$. In this case, the function condition described above is the statement that $\mathcal{F}_X \preceq \mathcal{F}_Y$, where $\preceq$ is the Rudin-Keisler order on $\beta\omega$. Recall that when $p, q \in \beta\omega$, $p \preceq q$ iff there is a function $f : \omega \to \omega$ such that $\beta f(q) = p$, where βf is the Stone extension of f. It is well-known (see, for instance, [13]) that $\beta f(q) = p$ is equivalent to $\forall Q \in q(f(Q) \in p)$, which is equivalent to $\forall P \in p(f^{-1}(P) \in q)$. Thus, we have:

Corollary 5.22. *If X and Y are strongly irresolvable spaces which satisfy (NC), witnessed by countable collections $\mathcal{N}_X$ and $\mathcal{N}_Y$, respectively, $X \oplus Y$ satisfies (NC) if and only if $\mathcal{F}_X$ is Rudin-Keisler equivalent to $\mathcal{F}_Y$.*

Proof. $\mathcal{F}_X \preceq \mathcal{F}_Y \Leftrightarrow X \oplus Y$ satisfies (NC) $\Leftrightarrow Y \oplus X$ satisfies (NC) $\Leftrightarrow \mathcal{F}_Y \preceq \mathcal{F}_X$ □

5.4 (GC) and GN-separability

The definitions of (GC) and GN-separable are very similar. We want to investigate the relationship between these two properties. We will use the following definitions from [1]:

Definition 5.23. *A dense set D is groupable if it can be partitioned as $D = \bigcup\{A_n : n \in \omega\}$ where each A_n is nonempty and finite, and every nonempty open set in X intersects all but finitely many A_n.*

Definition 5.24. *A space X is GN-separable if for every sequence $\langle D_n : n < \omega \rangle$ of dense subsets of X there are $d_n \in D_n$ such that $D = \{d_n : n < \omega\}$ is groupable.*

Definition 5.25. *A space X is R-separable if for every sequence $\langle D_n : n < \omega \rangle$ of dense subsets of X, one can pick $d_n \in D_n$ such that $\{d_n : n < \omega\}$ is dense in X.*

It is clear from the definitions that X has a groupable dense set iff X satisfies (GC) witnessed by a collection of finite sets. Therefore if X is GN-separable, since one then has a groupable dense set, then X satisfies (GC) witnessed by a collection of finite sets. We now wish to know if "(GC) witnessed by a collection of finite sets" implies GN-separable. We have the following partial result, which addresses one of the key ideas: going from the existence of one groupable dense set to having every dense set be groupable.

Fact 5.26. *If X is countable and satisfies (GC) witnessed by a collection of finite sets, then every dense set in X is groupable.*

Proof. Let $\mathcal{N} = \{N_k | k < \omega\}$ be a collection witnessing (GC) in X, where each N_k is finite. ($\mathcal{N}$ must be countable because X is.) Let $X' = X \setminus \bigcup \mathcal{N}$. If $X' \neq \emptyset$, $|X'| \leq \omega$; enumerate X' by $\{x_k : k < |X'|\}$. Then $\mathcal{N}' = \{N_k \cup \{x_k\} | k < |X'|\} \cup \{N_k | k \geq |X'|\}$ is also a collection witnessing (GC) in X. $\mathcal{N}'$ which covers X, and each member of $\mathcal{N}'$ is still finite. (This step is based on the proof of Proposition 73 in [1].)

Let $D \subset X$ be dense. For each $n < \omega$, define $A_n = \{D \cap N_n\}$. Since $|D| = \omega$ and the N_n's are finite and pairwise disjoint, countably many of the A_n's will be nonempty; take these nonempty A_n's to form the collection which witnesses that D is groupable. □

This proof relied on the fact that X was countable. If X is merely separable, the situation becomes more complicated. GN-separability relies in part on the ability to pick a countable dense set from within any set that is dense in X; when X is countable, any subset of X is already countable. If X is larger, though, it is possible that although X may be separable, there will be a dense set without a countable subset which is also dense. We can use the cardinal function $\delta(X)$ to describe this situation:

$$\delta(X) = \max\{\min\{|Y| : Y \subset D,\ Y \text{ is dense in } X\} : D \subseteq X \text{ dense}\}.$$

If $\delta(X) > \omega$, it is clear that X cannot be GN-separable. For then, if Y is a dense subset of X with no countable dense subset, the collection $\langle D_n : n < \omega \rangle$ with $D_n = Y$ for all n has no dense set of the form $\{d_n : n < \omega, d_n \in D_n\}$, much less a groupable one. In the terminology of [1], $\delta(X) > \omega \Rightarrow X$ is not R-separable. GN-separability implies R-separability ([1] Proposition 72), so a non-R-separable space is not GN-separable.

However, it is (consistently) possible for a space with $\delta(X) > \omega$ to satisfy (GC) witnessed by a collection of finite sets, as we show in the following example. Recall that $\mathfrak{p}$ is the least cardinal of a family of infinite subsets of ω with every finite intersection infinite, such that there is no infinite set almost contained in every member of the collection.

Example 5.27. *Assume $\mathfrak{p} > \omega_1$. There is a space which satisfies (GC), witnessed by a collection of finite sets, which is separable but not GN-separable.*

Proof. Let $X = 2^{\omega_1}$. By Proposition 4.5 in [6], if $\omega_1 < \mathfrak{p}$, then 2^{ω_1} satisfies (GC) witnessed by a collection of finite sets. Also, 2^{ω_1} is separable.

However, X is not GN-separable. Consider the set

$$Y = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x_\alpha = 1\}| < \omega\}.$$

Y is dense in X: if $U_\sigma \subseteq X$ is open, where $U_\sigma = \{f | f : \omega_1 \to 2 \ \wedge \ f|_{dom \sigma} = \sigma\}$ for a function σ defined on a finite subset of ω_1, $\sigma(\alpha) = 1$ for only finitely many α; letting $x_\alpha = \sigma(\alpha)$ if $\alpha \in dom(\sigma)$ and 0 otherwise, we see that $x \in U_\sigma \cap Y$.

But, no countable subset of Y is dense in X. Suppose to the contrary that $Y' \subseteq Y$ is countable. Say $Y' = \{y_n | n \in \omega\}$. For each $n < \omega$, there is a $\gamma_n < \omega_1$ such that $y_n(\beta) = 0$ for all $\beta > \gamma_n$. Let $\gamma^* = sup\{\gamma_n : n < \omega\} < \omega_1$. Then $\pi^{-1}_{\gamma^*+1}(\{1\})$ is open and misses Y'. $\square$

Therefore, it will be useful to turn our attention to spaces with $\delta(X) = \omega$, particularly countable spaces.

Question 5.28. *Is there a countable space X which satisfies (GC) witnessed by a collection of finite sets, but is not GN-separable?*

R-separability along with "every countable dense subset contains a groupable dense subset" is equivalent to GN-separability, so we could answer this question by answering:

Question 5.29. *Is there a countable space X which satisfies (GC) witnessed by a collection of finite sets, but is not R-separable?*

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