On Continuously Urysohn Spaces

by

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A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 9, 2010

Keywords: Urysohn, Continuously Urysohn

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Abstract

We study the properties of weakly continuously Urysohn (denoted by wcU) and continuously Urysohn (denoted by cU) spaces. The class of continuously Urysohn spaces is known to contain the class of metrizable, submetrizable, and nonarchimedean spaces. In this work, by using the scattering process, we show that the class of proto-metrizable spaces is also contained in the class of continuously Urysohn spaces. We show that being a (weakly) continuously Urysohn space is not a multiplicative property, and that this property is not preserved under perfect maps. However, being a weakly continuously Urysohn space is preserved under perfect open maps. We give a proof that the topological sum of (weakly) continuously Urysohn spaces is also (weakly) continuously Urysohn and that any paracompact locally continuously Urysohn ordered space is also continuously Urysohn. We prove that a well-ordered space is continuously Urysohn if and only if it is hereditarily paracompact and we obtain a result which characterizes when the linear extension of a separable GO-space is continuously Urysohn.

Acknowledgments

I would like to express the deepest appreciation to my advisor, Dr. Gary Gruenhage, for his great patience, and excitement in regard to mentoring. Besides Mathematics, he taught me "not to give up", and "patience" are essential not only in research, but also in life. Without his guidance and persistent help this dissertation would not have been possible.

I would like to thank my committee members, Dr. Stewart Baldwin, Dr. Overtoun Jenda, and Dr. Thomas Pate, for their support. I also would like to thank Dr. Richard Sesek for serving as the outside reader.

My sincere appreciation also is extended to Dr. Michel Smith, and our lovely secretaries: Gwen, Carolyn, and Lori for creating an excellent work environment and for being supportive.

I am thankful to my friends: Suna, Yücel, Necibe, Ìlknur, Yasemin, Güven, Sean, and Erkan. During this journey, whenever I needed a shoulder, they were present. They showed me, good friends (like them) make things much easier, and they can be your family far from home.

I also would like to thank Dr. Ferhan Atıcı, Dr. Oya Özbakır, and Dr. Tom Richmond. They did not only start my United States journey, but also followed and supported me throughout each step on my way.

Last but not least thanks for my family. My father: MehmetAli Güldürdek, my mother: Ayşe Güldürdek, my sister: Ayşen Güldürdek, and my brother (in law): Serdar Samsun. During this work, my family was at the other side of the ocean. However, I never lived it this way. They were always as close as my heart. I do not know what would I do without their endless love and support. Because of my family, I always feel special, lucky, and loved.

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Chapter 1

Introduction

In 1993, Stepanova [16] defined a property, and showed that this property is equivalent to metrizability for paracompact p-spaces. Then, in 2002, Halbeisen and Hungerbühler [9] named the spaces which carry this property as continuously Urysohn spaces. Bennett and Lutzer [3] studied continuously Urysohn spaces in the class of ordered spaces. In 2007, Zenor [19] defined weakly continuously Urysohn spaces and gave the following characterization: A Hausdorff space is weakly continuously Urysohn if and only if continuous functions defined on the compact subsets can be continuously extended to the continuous functions defined on the space itself. Thus continuously Urysohn and weakly continuously Urysohn spaces have been shown to be interesting and useful classes of spaces. In this dissertation we continue their study.

A topological space X is said to be *continuously Urysohn(cU)* if there is a continuous function $\varphi : X^2 \setminus \Delta \to C(X)$ such that $\varphi(x,y)(x) \neq \varphi(x,y)(y)$, where $C(X)$ is the space of all bounded real-valued functions endowed with the norm topology, and $\Delta = \{(x,x) : x \in X\}$ is the diagonal. This notion first appeared in [16], and found its name in [9]. After this, in [19], the definition of a weakly continuously Urysohn space appeared. A space X is said to be *weakly continuously Urysohn(wcU)* if there is a continuous function $\theta : (X^2 \setminus \Delta) \times X \to \mathbb{R}$ such that $\theta(x,y,x) \neq \theta(x,y,y)$.

In [18], the class of continuously perfectly normal spaces, continuously normal spaces, and continuously completely regular spaces were defined, and studied by Phillip Zenor for the first time. When we look at these definitions, they carry the notion of continuity with the corresponding separation axiom. For instance, one may think that a continuously normal space is nothing different than a normal space, which has a continuous operator, separating the disjoint closed subsets. After this, in [7], some results concerning the local properties of continuously normal spaces, and continuously completely regular spaces are obtained.

Alexander V. Arhangel'skiĭ, in [1], introduced a new type of space, namely a *paracompact p-space*, and characterized this type as the class of preimages of metric spaces under perfect surjections. In [16] and [17], E.N. Stepanova called a mapping $\theta : X^2 \setminus \Delta \to C(X)$ *separating* if it satisfies the conditions for continuity, and $\theta(x,y)(x) \neq \theta(x,y)(y)$ for any point (x,y) of $X^2 \setminus \Delta$. She showed that the existence of a continuous separating mapping is a necessary and sufficient condition for a paracompact p-space to be metrizable. In [9], L. Halbeisen and N. Hungerbühler named the spaces carrying this property as continuously Urysohn spaces.

Following the hierarchy of separation axioms, naming a space with this property as continuously Urysohn space is not surprising. Recall that a space X is a *Urysohn* space if for any $(x,y) \in X^2 \setminus \Delta$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$. Also, it is easy to see that every continuously Urysohn space is Urysohn.

When we look at the definition of continuously Urysohn space, it is clear that metric spaces are examples of continuously Urysohn spaces. In [9], Halbeisen and Hungerbühler, examine the continuously Urysohn spaces with the additional property that the functions $\varphi(x,y)$ depend on just one parameter. They show that spaces with this property are spaces with a weaker metric topology, namely *submetrizable* spaces. It is clear that any submetrizable space is also continuously Urysohn.

After that, Bennett and Lutzer studied continuously Urysohn linearly ordered and generalized ordered spaces in [3]. They show that a continuously Urysohn generalized ordered space is hereditarily paracompact. Also, they proved that for a separable generalized ordered space, being continuously Urysohn, submetrizable, and having a G_δ-diagonal are equivalent.

In [19], a weakly continuously Urysohn space is defined, and the following theorem is proved. A space X is weakly continuously Urysohn if and only if there is a continuous function $e : \{f : f \in C(H),\ H \text{ is a compact subset of } X\} \to C(X)$, where $e(f)$ is a continuous extension of f and $\{f : f \in C(H),\ H$ is a compact subset of $X\}$ is endowed with Vietoris topology.

In [8], G. Gruenhage and P. Zenor worked on the properties of weakly continuously Urysohn spaces. They showed that every weakly continuously Urysohn wΔ-space has a base of countable order, and separable weakly continuously Urysohn spaces are submetrizable, hence continuously Urysohn. It is also shown that monotonically normal weakly continuously Urysohn spaces are hereditarily paracompact, and no linear extension of any uncountable subspace of the Sorgenfrey line is weakly continuously Urysohn. Finally, they showed that any nonarchimedean space is continuously Urysohn.

In this dissertation, we examine weakly continuously Urysohn and continuously Urysohn spaces in more detail.

We know that the class of nonarchimedean spaces are contained in the class of continuously Urysohn spaces. In chapter 3, by using the scattering process, we generalize this by showing that the class of proto-metrizable spaces is a subclass of continuously Urysohn spaces. We also prove that the topological sum of continuously Urysohn spaces and ultraparacompact, locally continuously Urysohn spaces are indeed continuously Urysohn.

In chapter 4, we prove that being a (weakly)continuously Urysohn space is not a multiplicative property. With a counterexample, we show that the image of a continuously Urysohn space under a perfect map is not always continuously Urysohn. On the other hand we prove that the perfect-open image of a weakly continuously Urysohn space is also weakly continuously Urysohn.

In chapter 5, ordered spaces are studied. From [3] we know that for separable generalized ordered spaces, being continuously Urysohn implies submetrizability. It is clear for a separable linearly ordered space, being continuously Urysohn is equivalent to metrizability. We answer the question: When is a linear extension of a separable generalized ordered space X continuously Urysohn? We also give a characterization of when separable generalized ordered spaces are weakly continuously Urysohn. For the well-ordered space case, we prove that the space is continuously Urysohn if and only if it is hereditarily paracompact. Finally, we give a proof which shows that a paracompact, locally continuously Urysohn linearly ordered space is indeed continuously Urysohn.

In the final chapter, we give a corollary to the free set lemma, and a theorem about the relation between a special partition of $X^2 \setminus \Delta$ and the function φ which witnesses continuously Urysohn on X. At the end, an example is provided.

Chapter 2

Notation and Background

$C(X)$ will denote the set of bounded real valued functions defined on the space X. τ_{norm} will denote the *norm topology* on $C(X)$.

For any $f \in C(X)$, the family $\{U_i(f)\}_{i=1}^{\infty}$, where

$U_i(f) = \{g \in C(X) :$ there exists an $a < 1/i$ such that$\mid f(x) - g(x) \mid < a$ for $x \in X\}$,

is a base at the point f for the space $C(X)$ endowed with norm topology.

Definition 2.1. *For any topological space X let $\mathcal{A}(X)$ denote a family of non-empty subsets of X. If $\{U_1, ..., U_n\}$ is a finite collection of open subsets of X, then the set*

$< U_1, ..., U_n >= \{H \in \mathcal{A}(X) : H \subset \bigcup_{i=1}^{n} U_i$, *and if* $1 \leq i \leq n$, *then* $H \cap U_i \neq \emptyset\}$ *is a basic open set for the* Vietoris topology *on* $\mathcal{A}(X)$.

For a topological space X, the *diagonal* is denoted by $\Delta = \{(x, x) : x \in X\}$.

Definition 2.2. *A subset A of the space X a* regular G_δ-set *if there is a sequence $\{U_n\}$ of open sets in X such that $A = \bigcap_{i=1}^{\infty} U_i = \bigcap_{i=1}^{\infty} \bar{U}_i$. Here, $\bar{U}$ denotes the closure of the set U. We will say that* X has a regular G_δ-diagonal *if Δ is a regular G_δ-set in X^2.*

Definition 2.3. *A topological space X is said to have a* zero-set diagonal *if there exists a continuous function $F : X^2 \to [0, 1]$ such that $F^{-1}(0) = \Delta$.*

Definition 2.4. *A space X is* submetrizable*(contractible onto a metric space) provided that there exists a one-to-one and continuous map from X onto a metric space.*

Definition 2.5. *A topological space X is said to be* separable *if there exists a countable $D \subset X$ such that $\bar{D} = X$.*

Theorem 2.6. *([12]) A separable space is submetrizable if and only if it has a zero-set diagonal.*

Definition 2.7. *([4]) A space X is a* <u>*$w\Delta$-space*</u> *provided that there is a sequence $\{W_i\}$ of open covers of X such that if, for each i, $x \in u_i \in W_i$ and $y_i \in u_i$, then $\{y_i\}$ has a cluster point.*

The notion of a *base of countable order* is due to Arhangel'skiĭ ([1]). Chaber, Čoban, and Nagami in [5] show that a space (X, τ) has a base of countable order if there is a function $\sigma : X^{<\omega} \to \tau$ satisfying:

(1) $\{\sigma(< x >) : x \in X\}$ covers X.

(2) If $x \in \sigma(< x_0, x_1, ..., x_n >)$, then $\sigma(< x_0, x_1, ..., x_n, x >) \subset \sigma(< x_0, x_1, ..., x_n >)$.

(3) $\{\sigma(< x_0, x_1, ..., x_n, x >) : x \in \sigma(< x_0, x_1, ..., x_n >)\}$ covers $\sigma(< x_0, x_1, ..., x_n >)$.

(4) If x and the sequence $x_0, x_1, ...$ satisfy $x, x_{n+1} \in \sigma(< x_0, x_1, ..., x_n >)$ for all n, then $\{\sigma(< x_0, x_1, ..., x_i >)\}_{i\in\omega}$ is a base at x. Here, $X^{<\omega} = \{< x_0, x_1, ..., x_n >: n < \omega, x_i \in X\}$

Definition 2.8. *A continuous mapping $f : X \to Y$ is* <u>*perfect*</u> *if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X.*

Searching for generalized metric spaces, Arhangel'skiĭ introduced in [1] a certain type of spaces, which could be characterized as the class of preimages of metric spaces under perfect surjections, and which are *paracompact p-spaces.*

Definition 2.9. *A space X is* <u>*nonarchimedean*</u> *if X has a base $\mathcal{B}$ which is a tree under reverse inclusion.*

Definition 2.10. *A set T is* <u>*transitive*</u> *if every element of T is a subset of T. A set is an* <u>*ordinal number*</u> *(an ordinal) if it is transitive and well-ordered by $\in$.*

If $\alpha = \beta + 1$, then α is a *successor ordinal.* If α is not a successor ordinal, then it is called a *limit ordinal.*

If the set of all ordinal numbers smaller than a limit ordinal α contains a subset A of type β such that for every $\xi < \alpha$ there exists a $\xi' \in A$ satisfying $\xi < \xi' < \alpha$, then we say that the ordinal β is *cofinal* with α.

An infinite ordinal α is an *initial ordinal* if α is the smallest among all ordinal numbers β satisfying $\mid \beta \mid = \mid \alpha \mid$. An initial ordinal α is *regular* if there is no $\beta < \alpha$ which is cofinal with α.

Definition 2.11. *An ordinal α is called a cardinal number (a cardinal) if $\mid \alpha \mid \neq \mid \beta \mid$ for all $\beta < \alpha$.*

For every cardinal number κ there exists an initial ordinal α such that $\mid \alpha \mid = \kappa$ and this α is unique. A cardinal number κ is *regular* if the initial ordinal α mentioned above is regular.

If X is a set of ordinals and $\alpha > 0$ is a limit ordinal then α is a *limit point* of X if $\sup(X \cap \alpha) = \alpha$.

Definition 2.12. *Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is a closed unbounded subset of κ if C is unbounded in κ and if it contains all its limit points less than κ.*

A set $S \subset \kappa$ is stationary if $S \cap C \neq \emptyset$ for every closed unbounded subset C of κ.

Definition 2.13. *Let X be a topological space. The function $\chi_A : X \to \{0,1\}$ defined for any subset A of X by setting $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$ is called characteristic function on A or characteristic of A.*

Note that, χ_A is a continuous function if and only if A is an open and closed (clopen) subset of X.

Chapter 3

Some Classes Which Are Continuously Urysohn

The journey of this dissertation started with the search of a space which is weakly continuously Urysohn but not continuously Urysohn. After Halbeisen and Hungerbühler's naming the spaces with a certain property as continuously Urysohn and Zenor's definition for weakly continuously Urysohn spaces, this search was natural. The definition of continuously Urysohn contains a continuous function from the off-diagonal of a space to the space of all continuous, bounded real valued functions on that space. On the other hand, the definition of weakly continuously Urysohn contains a continuous function from the off diagonal times the space itself to the real numbers.

Before providing any proofs, the following useful fact is presented. If X is continuously Urysohn space and φ is the witnessing function, then for every $(x,y) \in X^2 \setminus \Delta$, we can assume that $\varphi(x,y)(x) = 0$, and $\varphi(x,y)(y) = 1$.

Indeed, the mapping $\tilde{\varphi} : X^2 \setminus \Delta \to C(X)$, where $\tilde{\varphi}(x,y)$ is defined by:

$$\tilde{\varphi}(x,y)(t) = (\varphi(x,y)(t) - \varphi(x,y)(x))/(\varphi(x,y)(y) - \varphi(x,y)(x))$$

also witnesses for X being a continuously Urysohn space. Note that $\tilde{\varphi}(x,y)(x) = 0$, and $\tilde{\varphi}(x,y)(y) = 1$. There is a similar result for weakly continuously Urysohn spaces.

We will start with a known theorem. The class of metric spaces is a subclass of cU-spaces.

Theorem 3.1. *Any metric space X is also a cU-space.*

Proof. Suppose d is the metric function on space X. Then we can define the function $\varphi : X^2 \setminus \Delta \to C(X)$ as follows:

$\varphi(x,y)(z) = d(x,z)$ for all (x,y) in $X^2 \setminus \Delta$, and every z in X.

Claim 1: For each (x,y), $\varphi(x,y)$ is a continuous function.

Proof of Claim 1. For fixed (x,y) from $X^2 \setminus \Delta$, suppose $\epsilon > 0$ and $z \in X$ are given.

Note that $\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid d(x,z) - d(x,t) \mid$. Since d is a metric function, it is also continuous. We can set the neighborhood G of z such that $\mid d(x,z) - d(x,t) \mid < \epsilon$ for every $t \in G$.

Claim 2: φ is a continuous function.

Proof of Claim 2. Suppose $\epsilon > 0$ and $(x,y) \in X^2 \setminus \Delta$ are given. For any $z \in X$, $\mid \varphi(x,y)(z) - \varphi(u,v)(z) \mid = \mid d(x,z) - d(u,z) \mid <\mid d(x,u) \mid$. □

In [16], there is a stronger result.

Theorem 3.2. *A submetrizable space is continuously Urysohn.*

We also know that a space with a zero-set diagonal is weakly continuously Urysohn [17].

Definition 3.3. *([9]) If X is a continuously Urysohn space, then we call the corresponding family $\{\varphi(x,y) : (x,y) \in X^2 \setminus \Delta\}$ a continuous separating family for X.*

The following proposition characterizes topological spaces with continuous separating families that depend on only one of its parameters.

Proposition 3.4. *([9]) A space X admits a continuous separating family*

$$\{\varphi(x,y) : (x,y) \in X^2 \setminus \Delta\}$$

that depends on just one parameter if and only if X is submetrizable.

Recall that a space X is *nonarchimedean* if X has a base $\mathcal{B}$ which is a tree under reverse inclusion. In [8], Gruenhage and Zenor present a result related to nonarchimedean spaces.

Theorem 3.5. *([8]) Any nonarchimedean space is cU.*

This work continues with the investigation of other classes which are contained in the class of cU-spaces.

If $\mathcal{K}$ is a class of spaces, then define $\mathcal{S}(\mathcal{K})$ to be the class of spaces which are obtained by the following scattering process: take any space in $\mathcal{K}$, isolate all the points of some subset, replace each such point by a space in $\mathcal{K}$, and repeat transfinitely, taking some subspace of the inverse limit at limit ordinals. In [14], Nyikos has shown that the class of proto-metrizable spaces is precisely the class $\mathcal{S}(\mathcal{M})$ where $\mathcal{M}$ denotes the class of metrizable spaces.

We will give a proof to a theorem stating that the class of proto-metrizable spaces is a subclass of continuously Urysohn spaces. Before this proof, we present a theorem for an easier case.

Suppose X is a cU-space, P is a subset of X, and the space X' is defined as follows: Isolate the points of P, and replace each p in P by a cU-space X_p. The underlying set X' can be seen as: $X' = \bigcup\{\{p\} \times X_p : p \in P\} \cup (X \setminus P)$.

Let us define the function $\Pi : X' \to X$ as follows: $\Pi(x) = \begin{cases} \text{x}, & \text{if } x \in X \setminus P, \\ \text{p}, & \text{if } x \in \{p\} \times X_p, \text{ for some } p \in P. \end{cases}$

Note that if $x \in \{p\} \times X_p$ for some $p \in P$, then $x = (p, a)$ for some $a \in X_p$. In X' we declare an open neighborhood of a point $x = (p, a) \in X'$ to be $\{p\} \times U$ where U is an open neighborhood of a in X_p, and an open neighborhood of a point $x \in X \setminus P$ is $U' = \Pi^{-1}(U)$ where U is an open neighborhood of x in X.

Theorem 3.6. *Every space X' obtained from a cU-space (respectively, from a wcU-space) X by first isolating the points of some subset P, and then replacing these with cU-spaces(respectively, with wcU-spaces) is also a cU-space (respectively a wcU-space).*

Proof. Since X is a cU-space, there exists a continuous function $\theta : X^2 \setminus \Delta \to C(X)$ such that $\theta(x, y)(x) \neq \theta(x, y)(y)$. Also for every X_p there is a function θ_p witnessing cU on X_p. We define the function $\varphi : (X')^2 \setminus \Delta \to C(X')$ as follows:

$$\varphi(x,y)(z) = \begin{cases} \theta(\Pi(x),\Pi(y))(\Pi(z)), & \text{if } \Pi(x) \neq \Pi(y), \\ \theta_p(a,b)(c), & \text{if } p = \Pi(x) = \Pi(y) = \Pi(z) \\ & \text{where } x = (p,a), y = (p,b), and\ z = (p,c), \\ 0, & \text{if } p = \Pi(x) = \Pi(y) \neq \Pi(z). \end{cases}$$

It is clear that $\varphi(x,y)(x) \neq \varphi(x,y)(y)$.

First we are going to show that φ is a continuous function.

Claim 1: φ is a continuous function.

Proof of Claim 1.

Case 1: $\Pi(x) \neq \Pi(y)$.

Note that under Case 1 we have four subcases:

Subcase 1: $x, y \in X \setminus P$.

$x, y \in X$. Since X is a cU-space, for every $\epsilon > 0$ there exist U, V neighborhoods of x, y such that

$\forall (u,v) \in U \times V$ and $\forall z \in X$, $\mid \theta(x,y)(z) - \theta(u,v)(z) \mid < \epsilon$.

If we take $U' = \Pi^{-1}(U)$ and $V' = \Pi^{-1}(V)$ neighborhoods of x and y in the space X', then for all $(u', v') \in U' \times V'$ and $z \in X'$;

$\mid \varphi(x,y)(z) - \varphi(u',v')(z) \mid = \mid \theta(x,y)(\Pi(z)) - \theta(u',v')(\Pi(z)) \mid < \epsilon$.

Subcase 2: $x \in \{p\} \times X_p$ for some $p \in P$, $y \in \{q\} \times X_q$ for some $q \in P$, and $p \neq q$.

$x = (p,a)$, and $y = (q,b)$ where $a \in X_p$, and $b \in X_q$. Take any neighborhood U of a in X_p, and define $U' = \{p\} \times U$ neighborhood of x, and take any neighborhood V of b in X_q, similarly define $V' = \{q\} \times V$ neighborhood of y.

Let us look at $\varphi(u',v')$ for all $(u',v') \in U' \times V'$:

$\varphi(u',v')(z) = \theta(p,q)(\Pi(z))$, and $\varphi(x,y)(z) = \theta(p,q)(\Pi(z))$. That is,

$$\mid \varphi(x,y)(z) - \varphi(u',v')(z) \mid = 0.$$

Subcase 3: $x \in \{p\} \times X_p$ for some $p \in P$, and $y \in X \setminus P$.

$x = (p, a)$ for some $a \in X_p$. Since X is a cU-space, then there exists a neighborhood $U \times V$ of (p, y) such that:

$\forall (u, v) \in U \times V$ and $z \in X$, $\mid \theta(p, y)(z) - \theta(u, v)(z) \mid < \epsilon$. We set a neighborhood U' of x as $U' = \Pi^{-1}(U)$, and V' of y as $V' = \Pi^{-1}(V)$. For all $(u', v') \in U' \times V'$, and $z \in X'$

$\mid \varphi(x, y)(z) - \varphi(u', v')(z) \mid = \mid \theta(p, y)(\Pi(z)) - \theta(u, v)(\Pi(z)) \mid < \epsilon$.

Subcase 4: $x \in X \setminus P$, and $y \in \{p\} \times X_p$ for some $p \in P$.

This case can be proven similar to subcase 3.

Case 2: $\Pi(x) = \Pi(y) = p$.

$x = (p, a)$, and $y = (p, b)$ for some $p \in P$, where $a, b \in X_p$.

Since $a \neq b$, and X_p is a cU-space, there exists a $U \times V$ neighborhood of (a, b) such that for all $(u, v) \in U \times V$ and $c \in X_p$, $\mid \theta_p(a, b)(c) - \theta_p(u, v)(c) \mid < \epsilon$.

If we set $U' = \{p\} \times U$, as a neighborhood of x, and $V' = \{p\} \times V$ as a neighborhood of y, for all $(u', v') \in U' \times V'$, and $z \in X'$;

$$\mid \varphi(x, y)(z) - \varphi(u', v')(z) \mid = \begin{cases} \mid \theta_p(a, b)(c) - \theta_p(u, v)(c) \mid & \Pi(z) = p \\ \text{where } z = (p, c),\ u' = (p, u), and\ v' = (p, v), & \\ \mid 0 - 0 \mid & \Pi(z) \neq p. \end{cases}$$

Now we will show that for all (x, y), $\varphi(x, y)$ is a continuous function.

Claim 2: For every $(x, y) \in (X')^2 \setminus \Delta$, $\varphi(x, y)$ is a continuous function.

Proof of Claim 2.

Case 1: $\Pi(x) \neq \Pi(y)$.

Since $\theta(\Pi(x), \Pi(y))$ is a continuous function on X, for every $\epsilon > 0$, and every $\Pi(z) \in X$ there exists a neighborhood G of $\Pi(z)$ such that

$$\mid \theta(\Pi(x), \Pi(y))(\Pi(z)) - \theta(\Pi(x), \Pi(y))(p) \mid < \epsilon$$

for every $p \in G$.

So, for given any $\epsilon > 0$ and $z \in X'$ we take $G^{'} = \Pi^{-1}(G)$ open neighborhood of z, where G is the open neighborhood of $\Pi(z)$ as mentioned previously. Then, for every t in $G^{'}$ we have;

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid \theta(\Pi(x),\Pi(y))(\Pi(z)) - \theta(\Pi(x),\Pi(y))(\Pi(t)) \mid < \epsilon.$

Case 2: $\Pi(x) = \Pi(y)$.

Subcase 1: $\Pi(x) = \Pi(y) = \Pi(z) = p$.

Note that $\varphi(x,y)(z) = \theta_p(a,b)(c)$. Since $\theta_p(a,b)$ is a continuous function on X_p for any $\epsilon > 0$, and $c \in X_p$ there exists a neighborhood of G of c such that for every $d \in G$,

$$\mid \theta_p(a,b)(c) - \theta_p(a,b)(d) \mid < \epsilon.$$

With the help of this setting, for given any $\epsilon > 0$ and $z = (p,c) \in X'$ where;

$\Pi(x) = \Pi(y) = \Pi(z) = p$, we set $G^{'}$ neighborhood of z as $\{p\} \times G$, where G is the neighborhood of c mentioned before.

For every t in $G^{'}$, $t = (p,d)$ and $\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid \theta_p(a,b)(c) - \theta(a,b)(d) \mid < \epsilon$.

Subcase 2: $\Pi(x) = \Pi(y) \neq \Pi(z)$.

That means there exists a $p \in P$ such that $x, y \in \{p\} \times X_p$, and $\varphi(x,y)(z) = 0$.

If $z \in X \setminus P$, take a neighborhood G of z such that $p \notin G$, and define the neighborhood $G^{'}$ of z in $X^{'}$ as $G^{'} = \Pi^{-1}(G)$. If $z \in \{q\} \times X_q$, $z = (q,c)$ for a $q \in P$, then define the neighborhood of z as $G^{'} = \{q\} \times G$ where $c \in X_q$ and G is a neighborhood of c.

Note that same proof works for a wcU-space X, if one replaces the points of $P \subset X$ by wcU-spaces. For this case by using the function θ which witnesses X being a wcU-space, we define the function φ on $((X')^2 \setminus \Delta) \times X'$ as follows:

$$\varphi(x,y,z) = \begin{cases} \theta(\Pi(x),\Pi(y),\Pi(z)) & \text{if } \Pi(x) \neq \Pi(y), \\ \theta_p(a,b,c) & \text{if } p = \Pi(x) = \Pi(y) = \Pi(z) \\ & \text{where } x = (p,a),\ y = (p,b), and\ z = (p,c), \\ 0 & \text{if } p = \Pi(x) = \Pi(y) \neq \Pi(z). \end{cases}$$

□

Now that we have this theorem, we will use it to prove that the classes of continuously Urysohn and weakly continuously Urysohn spaces are closed under the scattering process.

Theorem 3.7. *If $c\mathcal{U}$ is the class of all continuously Urysohn spaces, then $S(c\mathcal{U})$ is a subclass of $c\mathcal{U}$. That is, the class of continuously Urysohn spaces is closed under the scattering process.*

Proof. Let γ be an ordinal and X_0 be a cU-space. We start the scattering process by isolating a subset of X_0, and replacing these points with cU-spaces. Let us name the resulting space X_1, and define the map $\sigma_{1,0} : X_1 \to X_0$ naturally. On the second step, we isolate a subset of X_1 and replace these points with cU-spaces. We name this space as X_2, and define the map $\sigma_{2,1} : X_2 \to X_1$ naturally. Similarly we can define X_n and $\sigma_{n,n-1}$ for all $n < \omega$. Also for any $m < n$ we define the map $\sigma_{n,m} : X_n \to X_m$ as $\sigma_{n,m} = \sigma_{k,m} \circ \sigma_{n,k}$ where $m < k < n$. By using the previous theorem we know that for every $n < \omega$, X_n is a cU-space.

At the first limit ordinal step, we define the space $X_\omega = \varprojlim X_n$ such that,

$\overrightarrow{x} =< x_0, x_1, ... >\in \varprojlim X_n$ if and only if $x_0 = \sigma_{1,0}(x_1)$, $x_1 = \sigma_{2,1}(x_2)$, ...

We also define the map $\sigma_{\omega,n}$ as the n^{th} projection map from X_ω into X_n. At the next step we define the space $X_{\omega+1}$ by repeating the scattering process. For all $n < \omega$, the maps $\sigma_{\omega+1,\omega}$ and $\sigma_{\omega+1,n}$ are defined similarly.

Finally, by replacing points with cU-spaces at successor ordinal steps, and by taking the limit space at the limit ordinal steps, we form a space $X_\gamma = \varprojlim X_\beta$. Note that for every $\beta < \gamma$, X_β is a cU-space with the witnessing continuous function φ_β. WLOG, we may suppose $\varphi_\beta(x, y)(x) = 0$, and $\varphi_\beta(x, y)(y) = 1$.

By using $\pi_\beta : \prod X_\alpha \to X_\beta$, β^{th} projection map, we define the continuous map

$\pi^*_\beta = \pi_\beta|_{X_\gamma} : X_\gamma \to X_\beta$, and $\{(\pi^*_\beta)^{-1}(U_\beta) : U_\beta$ is open subset of $X_\beta\}$ forms a base for the limit space X_γ.

Let us define the function $\varphi : X^2_\gamma \setminus \Delta \to C(X_\gamma)$ as follows:

$\varphi(\overrightarrow{x}, \overrightarrow{y})(\overrightarrow{z}) = \varphi_\alpha(x_\alpha, y_\alpha)(z_\alpha)$, where α is the least ordinal such that $x_\alpha \neq y_\alpha$. It is clear that $\varphi(\overrightarrow{x}, \overrightarrow{y})(\overrightarrow{x}) = 0$, $\varphi(\overrightarrow{x}, \overrightarrow{y})(\overrightarrow{y}) = 1$, and $\varphi(\overrightarrow{x}, \overrightarrow{y})$ is continuous for every $(\overrightarrow{x}, \overrightarrow{y})$ in $X^2_\gamma \setminus \Delta$. We only need to show the continuity of φ. Suppose $\epsilon > 0$ and $(\overrightarrow{x}, \overrightarrow{y})$ are given.

By examining the ordinal α, where $\overrightarrow{x}$ differs from $\overrightarrow{y}$ for the first time, we have 3 cases.

Case 1: $\alpha = 0$.

Since X_0 is a cU-space there exists a $U_0 \times V_0$ neighborhood of (x_0, y_0) such that;

$\mid \varphi_0(x_0, y_0)(z_0) - \varphi_0(u_0, v_0)(z_0) \mid < \epsilon$ for every (u_0, v_0) in $U_0 \times V_0$, and z_0 in X_0.

We set $U \times V = (\pi^*_0)^{-1}(U_0) \times (\pi^*_0)^{-1}(V_0)$. Then for every $(\overrightarrow{u}, \overrightarrow{v})$ in $U \times V$,

$\mid \varphi(\overrightarrow{x}, \overrightarrow{y})(\overrightarrow{z}) - \varphi(\overrightarrow{u}, \overrightarrow{v})(\overrightarrow{z}) \mid = \mid \varphi_0(x_0, y_0)(z_0) - \varphi_0(u_0, v_0)(z_0) \mid < \epsilon$.

Case 2: $\alpha > 0$ is a successor ordinal.

Note that $x_\alpha = (x_{\alpha-1}, a) \in X_\alpha$, and $y_\alpha = (y_{\alpha-1}, b) \in X_\alpha$ where a, b are in the cU-space which has been replaced with $x_{\alpha-1} = y_{\alpha-1} \in X_{\alpha-1}$. Since X_α is a cU-space, there exists $U_\alpha \times V_\alpha$ neighborhood of (x_α, y_α) such that,

$\mid \varphi_\alpha(x_\alpha, y_\alpha)(z_\alpha) - \varphi_\alpha(u_\alpha, v_\alpha)(z_\alpha) \mid < \epsilon$ for every (u_α, v_α) in $U_\alpha \times V_\alpha$, and z_α in X_α. WLOG, we can choose U_α, and V_α so that $\sigma_{\alpha,\alpha-1}(U_\alpha) = \sigma_{\alpha,\alpha-1}(V_\alpha) = x_{\alpha-1} = y_{\alpha-1}$.

We set $U \times V = (\pi^*_\alpha)^{-1}(U_\alpha) \times (\pi^*_\alpha)^{-1}(V_\alpha)$. Then for every $(\overrightarrow{u}, \overrightarrow{v})$ in $U \times V$,

$\mid \varphi(\overrightarrow{x}, \overrightarrow{y})(\overrightarrow{z}) - \varphi(\overrightarrow{u}, \overrightarrow{v})(\overrightarrow{z}) \mid = \mid \varphi_\alpha(x_\alpha, y_\alpha)(z_\alpha) - \varphi_\alpha(u_\alpha, v_\alpha)(z_\alpha) \mid < \epsilon$.

Case 3: $\alpha > 0$ is a limit ordinal.

Since α is a limit ordinal, for every $x_\alpha \in X_\alpha$, $x_\alpha = < x_\beta >_{\beta<\alpha}$. Also, $x_\alpha \neq y_\alpha$ means that there is a successor ordinal $\beta < \alpha$ such that $x_\beta \neq y_\beta$.

So, $\varphi(\overrightarrow{x}, \overrightarrow{y})(\overrightarrow{z}) = \varphi_\alpha(x_\alpha, y_\alpha)(z_\alpha) = \varphi_\beta(x_\beta, y_\beta)(z_\beta)$. We are back in case 2.

$\square$

Corollary 3.8. *Any proto-metrizable space is continuously Urysohn.*

After this result on proto-metrizable spaces, our attention was directed to spaces which are topological sum of continuously Urysohn spaces.

Recall that *topological sum* of the spaces $\{X_s\}_{s\in S}$ with $X_s \cap X_{s'} = \emptyset$ for $s \neq s'$ is a space with the underlying set $X = \bigcup\limits_{s\in S} X_s$, and the family $\mathcal{O}$ of open sets where $U \subset X$ is open if $U \cap X_s$ is open in X_s for every $s \in S$. Topological sum of the spaces $\{X_s\}_{s\in S}$ is denoted by $\bigoplus_{s\in S} X_s$, or by $X_1 \oplus X_2 \oplus ... \oplus X_k$ if $S = \{1, 2, ..., k\}$. Also for every $s \in S$, we know that X_s is a clopen subset of X.

Theorem 3.9. *The topological sum of continuously Urysohn spaces is also continuously Urysohn.*

Proof. Suppose $X = \bigoplus_{s\in S} X_s$, and for every $s \in S$, X_s is a cU-space with the witnessing function φ_s.

We define the function $\varphi : X^2 \setminus \Delta \to C(X)$ as follows:

$$\varphi(x,y)(z) = \begin{cases} 0, & \text{if } (\exists s \in S)\ (x, y \in X_s),\ \textit{and}\ (z \notin X_s), \\ \varphi_s(x,y)(z), & \text{if } (\exists s \in S)\ (x, y, z \in X_s), \\ \chi_{X_{s'}}(z), & \text{if } x \in X_s,\ \textit{and}\ y \in X_{s'}. \end{cases}$$

Claim 1: For every $(x, y) \in X^2 \setminus \Delta$, $\varphi(x,y)(x) \neq \varphi(x,y)(y)$.

Proof of Claim 1.

$$\varphi(x,y)(x) = \begin{cases} \varphi_s(x,y)(x) = 0, & \text{if } (\exists s \in S)\ (x, y \in X_s), \\ \chi_{X_{s'}}(x) = 0, & \text{if } x \in X_s,\ \textit{and}\ y \in X_{s'}. \end{cases}$$

On the other hand,

$$\varphi(x,y)(y) = \begin{cases} \varphi_s(x,y)(y) = 1, & \text{if } (\exists s \in S)\ (x,y \in X_s), \\ \chi_{X_{s'}}(y) = 1, & \text{if } x \in X_s,\ \textit{and } y \in X_{s'}. \end{cases}$$

Claim 2: For every $(x,y) \in X^2 \setminus \Delta$, $\varphi(x,y)$ is a continuous function.

Proof of Claim 2. Suppose $\epsilon > 0$, and $z \in X$ are given.

Case 1: There exists a $s \in S$ such that $x, y \in X_s$.

Subcase 1: $z \in X_s$.

Then $\varphi(x,y)(z) = \varphi_s(x,y)(z)$. Since $\varphi_s(x,y)$ is a continuous function there exists a neighborhood G_s of z in X_s such that;

$\mid \varphi_s(x,y)(z) - \varphi_s(x,y)(t') \mid < \epsilon$ for every $t' \in G_s$. We take $G = G_s$ open neighborhood of z in X, and for every $t \in G$,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid \varphi_s(x,y)(z) - \varphi_s(x,y)(t) \mid < \epsilon.$

Subcase 2: $z \notin X_s$.

Note that there exists a $p \in S$ such that $z \in X_p$, and $\varphi(x,y)(z) = 0$. We set $G = X_p$ as an open neighborhood of z. Then for every $t \in G$,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid 0 - 0 \mid.$

Case 2: $x \in X_s$ and $y \in X_{s'}$, $s \neq s'$.

Then $\varphi(x,y)(z) = \chi_{X_{s'}}(z)$. If $z \in X_{s'}$ we take $G = X_{s'}$, and if $z \notin X_{s'}$ we take $G = X_{s'}^c$ as a neighborhood of z. So, for every $t \in G$ we have;

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = 0.$

Claim 3: $\varphi : X^2 \setminus \Delta \to C(X)$ is a continuous function.

Proof of Claim 3.

Suppose $\epsilon > 0$, and $(x,y) \in X^2 \setminus \Delta$ are given.

Case 1: There exists a $s \in S$ such that $x, y \in X_s$.

Since X_s is a cU-space with the function φ_s there exists a neighborhood $U_s \times V_s$ of (x,y) in X_s so that for every $(u', v') \in U_s \times V_s$, and $z \in X_s$

$| \varphi_s(x,y)(z) - \varphi_s(u',v')(z) |< \epsilon.$

We set $U = U_s$, and $V = V_s$. Then for every $(u,v) \in U \times V$;

$$| \varphi(x,y)(z) - \varphi(u,v)(z) |= \begin{cases} | 0 - 0 |= 0, & \text{if } z \notin X_s, \\ | \varphi_s(x,y)(z) - \varphi_s(u,v)(z) |< \epsilon, & \text{if } z \in X_s. \end{cases}$$

Case 2: $x \in X_s$, and $y \in X_{s'}$, $s \neq s'$.

We take $U = X_s$, and $V = X_{s'}$. Then for every $(u,v) \in U \times V$;

$| \varphi(x,y)(z) - \varphi(u,v)(t) |=| \chi_{X_{s'}}(z) - \chi_{X_{s'}}(z) |= 0.$ □

By using a function defined similar to the function used in the proof of previous theorem, we have a result on ultraparacompact locally cU-spaces. Remember that a space X is *ultraparacompact* if for every open cover $\mathcal{U}$ of X, there exists a locally finite pairwise disjoint open refinement $\mathcal{V}$. Also, if every point in the space has an open neighborhood which is cU we say that the space is *locally cU-space.*

Corollary 3.10. *If X is a ultraparacompact and locally cU-space, then X is a cU-space.*

The analogues of Theorem 3.9, and corollary 3.10 also hold for wcU-spaces.

Chapter 4

Some Properties of Continuously Urysohn Spaces

The function witnessing being a cU-space appears stronger than the function witnessing being a wcU-space. It is clear that any cU-space is a wcU-space. We know that any submetrizable space is continuously Urysohn, and a zero-set diagonal implies being a weakly continuously Urysohn space. Furthermore, if a separable space is also a wcU-space, then it is also a cU-space.

While searching for an example which distinguishes wcU and cU, some new properties of these spaces were examined.

Firstly, we looked at the property of being wcU in product spaces. We discovered that being a wcU-space is not a multiplicative property. In order to have that result we proved the following lemma and two theorems.

Lemma 4.1. *If X is a space with a zero-set diagonal, then the product space $Y = X \times M$, where (M, d) is a metric space, also has a zero-set diagonal.*

Proof. Since X has a zero-set diagonal, there exists a continuous function $F : X^2 \to [0, 1]$ such that, $F^{-1}(0) = \Delta$. Let us now define the function $G : Y^2 \to \mathbb{R}$ as follows:

$$G((x, m), (y, n)) = F(x, y) + d(m, n).$$

Claim 1: G is continuous.

Proof of Claim 1. Given any $\epsilon > 0$ and $((x, m), (y, n)) \in Y^2$, we choose the open neighborhood $A = (U_1 \times V_1) \times (U_2 \times V_2)$. Here, $U_1 \times U_2$ is the neighborhood of (x, y) such that $\mid F(x, y) - F(z, t) \mid < \frac{\epsilon}{2}$ for every $(z, t) \in U_1 \times U_2$, and $V_1 \times V_2$ is the neighborhood of (m, n) such that $\mid d(m, n) - d(p, q) \mid < \frac{\epsilon}{2}$ for every $(p, q) \in V_1 \times V_2$.

Then

$| G((x,m),(y,n)) - G((z,p),(t,q)) | \leq | F(x,y) - F(z,t) | + | d(m,n) - d(p,q) | < \epsilon,$
for every $((z,p),(t,q)) \in A$.

Claim 2: $G^{-1}(0) = \Delta$.

Proof of Claim 2. Suppose, $G((x,m),(y,n)) = 0$. This is true if and only if $F(x,y) = 0$ and $d(m,n) = 0$, and this is true if and only if $x = y$ and $m = n$. □

Theorem 4.2. *Let X be a topological space, and $\omega+1 = [0,\omega]$ a convergent sequence. Then $Y = X \times [0,\omega]$ is a wcU-space if and only if X has a zero-set diagonal.*

Proof. First, suppose that X is a space with a zero-set diagonal. By Lemma 4.1, the product space $Y = X \times (\omega+1)$ has a zero-set diagonal. Since zero-set diagonal implies being a wcU-space, Y is a wcU-space. Now, suppose $Y = X \times (\omega+1)$ is a wcU-space. Then, there exists a continuous function $\Theta : (Y^2 \setminus \Delta) \times Y \to \mathbb{R}$ such that;

$$\Theta((x,n),(y,m),(x,n)) \neq \Theta((x,n),(y,m),(y,m)).$$

WLOG, we can assume that $\Theta((x,n),(y,m),(x,n)) = 0$, and $\Theta((x,n),(y,m),(y,m)) = 1$, and $0 \leq \Theta((x,n),(y,m),(z,p)) \leq 1$ for all $(x,n),(y,m),(z,p) \in Y$. For $n < \omega$, let's define the function $f_n : X^2 \to \mathbb{R}$ as follows:

$$f_n(x,y) = \Theta((x,n),(y,\omega),(y,n)).$$

Note that for all n, f_n is a continuous function. Finally we can define the function $F : X^2 \to \mathbb{R}$ such that; $F(x,y) = \sum_{n=1}^{\infty} \frac{f_n(x,y)}{2^n}$. Since each f_n is a continuous function, F is a continuous function. And also, $F^{-1}(0) = \Delta$. Indeed, if $x = y$, then for all n, $f_n(x,y) = \Theta((x,n),(x,\omega),(x,n)) = 0$. So, $F(x,y) = 0$. If $x \neq y$ and $F(x,y) = 0$, then $f_n(x,y) = 0$ for all n. That means, $\Theta((x,n),(y,\omega),(y,n)) = 0$. If we look at the sequence

$((x, n), (y, \omega), (y, n))$, it converges to the point $((x, \omega), (y, \omega), (y, \omega))$. Since Θ is a continuous function, $\Theta((x, n), (y, \omega), (y, n))$ converges to the point $\Theta((x, \omega), (y, \omega), (y, \omega))$. However, $\Theta((x, n), (y, \omega), (y, n)) = 0$ for all n, and $\Theta((x, \omega), (y, \omega), (y, \omega)) = 1$, so this is a contradiction. That is, $F^{-1}(0) = \Delta$, and the space X has a zero-set diagonal. □

Corollary 4.3. *If $X \times (\omega + 1)$ is wcU, then X has a regular G_δ-diagonal.*

Corollary 4.4. *If X has a non-G_δ point, then $X \times (\omega + 1)$ is not a wcU-space.*

Example 4.5. *There is cU-space X such that $X \times (\omega + 1)$ is not wcU.*

Proof. Let X be $L(\omega_1)$ where $L(\omega_1)$ is the one-point Lindelöfication of ω_1. That is, every $\alpha < \omega_1$ is isolated, and the neighborhoods of ω_1 are the sets with countable complements. Since $L(\omega_1)$ is a nonarchimedean space, by using Theorem 3.5 it is also a cU-space. On the other hand, by using Corollary 4.4 and the non-G_δ point ω_1 in $L(\omega_1)$, $L(\omega_1) \times (\omega + 1)$ is not a cU-space. □

Example 4.6. *Suppose X' is the topological sum of $L(\omega_1)$ and $\omega + 1$. Let us form the quotient space X of X' obtained by identifying ω_1 in $L(\omega_1)$ with ω in $\omega + 1$. Then X is not even a wcU-space.*

Proof. Suppose X is a wcU-space. Then there exists a continuous function

θ: $(X^2 \backslash \Delta) \times X \to \mathbb{R}$ such that $\theta(x, y, x) \neq \theta(x, y, y)$.

We can assume that $\theta(x, y, x) = 0$ and $\theta(x, y, y) = 1$. For every $n < \omega$, $\theta(p, n, p) = 0$, where $p = \omega = \omega_1$. So, for every $n < \omega$, there exists a neighborhood G_n of p such that $\theta(q, n, p) < 1/2$ for all $q \in G_n$. Note that $\bigcap G_n \neq \{p\}$. There exists a point $q_0 \in \bigcap G_n$ and $q_0 \neq p$. Then $\theta(q_0, n, p) < 1/2$ for all $n < \omega$. When we take the limit as $n \to \omega$; $\theta(q_0, n, p) \to \theta(q_0, p, p)$. Since $\theta(q_0, n, p) < 1/2$ and $\theta(q_0, p, p) = 1$ we have a contradiction. □

Corollary 4.7. *The cU-space and wcU-space properties are not always preserved under perfect maps.*

Proof. Let X and X' be as in Example 4.6. The quotient map $q : X' \to X$ is perfect, and X' is a cU-space. However, X is not even wcU. $\square$

We will show, however, that open-perfect maps preserve weakly continuously Urysohn spaces. Since we have an open-perfect map, we will use a theorem of Michael's from [13]. Before the statement of this theorem we will give some definitions that he mentions before that theorem.

Definition 4.8. *([13]) Let $f : X \to Y$ be onto, and $\mathcal{A}(X) = \{E \subset X : E \neq \emptyset\}$. We define*

(1) *$f^{-1*} : Y \to \mathcal{A}(X)$ by $f^{-1*}(y) = f^{-1}(y)$,*

(2) *$f^{-1**} : \mathcal{A}(Y) \to \mathcal{A}(X)$ by $f^{-1**}(E) = f^{-1}(E)$.*

Theorem 4.9. *([13]) Let X, Y be topological spaces, and $f : X \to Y$ be onto. Then with the Vietoris topology on $\mathcal{A}(X)$, and $\mathcal{A}(Y)$ we have:*

(1) *f^{-1*} is continuous if and only if f is open and closed,*

(2) *f^{-1**} is continuous if and only if f^{-1*} is continuous.*

Now, we can prove our theorem related to weakly continuously Urysohn spaces.

Theorem 4.10. *The perfect-open image of a weakly continuously Urysohn space is also weakly continuously Urysohn.*

Proof. Suppose $f : X \to Y$ is a perfect-open map from a wcU-space X onto a space Y.

If X admits a continuous separating function $\varphi : (X^2 \setminus \Delta) \times X \to \mathbb{R}$, by using Zenor's Lemma 2 from [19], there is a continuous function $\tilde{\varphi} : \mathcal{M}(X) \times X \to [0, 1]$ such that

(1) if $x \in H$, then $\tilde{\varphi}(H, K, x) = 0$, and if $x \in K$,then $\tilde{\varphi}(H, K, x) = 1$,

(2) if (H, K), $(H', K') \in \mathcal{M}(X)$ with $H \subset H'$ and $K' \subset K$, then $\tilde{\varphi}(H', K', x) \leq \tilde{\varphi}(H, K, x)$.

Here $\mathcal{M}(X) = \{(H, K) \in \mathcal{K}(X) \times \mathcal{K}(X) : H \cap K = \emptyset\}$, and $\mathcal{K}(X)$ denotes the space of compact subsets of X endowed with the Vietoris topology.

Suppose X is a wcU-space with the function φ, and let us define the function

$\theta : (Y^2 \setminus \Delta) \times Y \to \mathbb{R}$ by using the functions f, and $\tilde{\varphi}$.

$\theta(x, y, z) = max\{\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c) : c \in f^{-1}(z)\}$.

Since f is a perfect map, $f^{-1}(z)$ is a compact subset of X for every $z \in Y$. Also note that for every z from Y there exists a c_1 in $f^{-1}(z)$ such that;

$\theta(x, y, z) = max\{\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c) : c \in f^{-1}(z)\} = \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1)$.

Claim 1: $\theta(x, y, x) \neq \theta(x, y, y)$.

Proof of Claim 1. It is clear that $\theta(x, y, x) = 0$, and $\theta(x, y, y) = 1$ for every (x, y) from $Y^2 \setminus \Delta$.

Claim 2: $\theta : (Y^2 \setminus \Delta) \times Y \to \mathbb{R}$ is a continuous function.

Proof of Claim 2. Suppose $\epsilon > 0$, and (x, y, z) are given. Since $\tilde{\varphi}$ is a continuous function, for every $c \in f^{-1}(z)$ there are open neighborhoods H_c, K_c, and G_c of $f^{-1}(x)$, $f^{-1}(y)$, and c respectively, such that;

$\mid \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c) - \tilde{\varphi}(A, B, c') \mid < \epsilon$ for every $A \in H_c$, $B \in K_c$, and $c' \in G_c$.

By using the compactness of $f^{-1}(z)$ we can select a finite subset $\{c_1, c_2, ..., c_n\}$ of $f^{-1}(z)$ so that $\{G_{c_i} : i = 1, 2, ..., n\}$ covers $f^{-1}(z)$.

Since neighborhoods of $f^{-1}(x)$, and $f^{-1}(y)$ are open with respect to Vietoris topology, for every $1 \leq i \leq n$, we can assume that;

$H_{c_i} =< U_1, U_2, ..., U_r >$ is a neighborhood of $f^{-1}(x)$, and $K_{ci} =< V_1, V_2, ..., V_s >$ is a neighborhood of $f^{-1}(y)$.

So, we take $H = \bigcap_{i=1}^{n} H_{c_i}$ neighborhood of $f^{-1}(x)$, $K = \bigcap_{i=1}^{n} K_{c_i}$ neighborhood of $f^{-1}(y)$, and $< G_{c_1}, G_{c_2}, ..., G_{c_n} >$ neighborhood of $f^{-1}(z)$. Note that by using the definition of Vietoris topology, we can restate H, and K respectively as follows:

$H =< U_1, U_2, ..., U_m >$, and $K =< V_1, V_2, ..., V_k >$. Then there exist neighborhoods U of x, V of y, and W of z such that $f^{-1}(U) \subset \bigcup_{i=1}^{m} U_i$, $f^{-1}(V) \subset \bigcup_{i=1}^{k} V_i$, and $f^{-1}(W) \subset \bigcup_{i=1}^{n} G_{c_i}$.

Under this setting of neighborhoods, let us look at $\mid \theta(x, y, z) - \theta(u, v, w) \mid$ for every $u \in U$, $v \in V$, and $w \in W$.

Recall that $\theta(x, y, z) = max\{\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c) : c \in f^{-1}(z)\} = \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1)$.

Case 1: Suppose $\theta(u,v,w) = max\{\tilde{\varphi}(f^{-1}(u), f^{-1}(v), t) : t \in f^{-1}(w)\} = \tilde{\varphi}(f^{-1}(u), f^{-1}(v), t_1)$ where $t_1 \in G_{c_1}$. Then;

$\mid \theta(x,y,z) - \theta(u,v,w) \mid = \mid \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1) - \tilde{\varphi}(f^{-1}(u), f^{-1}(v), t_1) \mid$.

Since $f^{-1}(u) \in < U_1, U_2, ..., U_m > = \bigcap_{i=1}^{n} H_{c_i}$, and $f^{-1}(v) \in < V_1, V_2, ..., V_k > = \bigcap_{i=1}^{n} K_{c_i}$ we have $f^{-1}(u) \in H_{c_1}$, and $f^{-1}(v) \in K_{c_1}$. By using the fact that $\tilde{\varphi}$ is a continuous function and $t_1 \in G_{c_1}$ we conclude that:

$\mid \theta(x,y,z) - \theta(u,v,w) \mid < \epsilon$.

Case 2: Suppose $\theta(u,v,w) = max\{\tilde{\varphi}(f^{-1}(u), f^{-1}(v), t) : t \in f^{-1}(w)\} = \tilde{\varphi}(f^{-1}(u), f^{-1}(v), s_1)$ where $s_1 \notin G_{c_1}$.

Then there exists an $i \leq n$, such that $s_1 \in G_{c_i}$. By using the definition of θ, we know that $\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1) \geq \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_i)$. Since $\tilde{\varphi}$ is a continuous function,

$\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_i) - \epsilon < \tilde{\varphi}(f^{-1}(u), f^{-1}(v), s_1) < \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_i) + \epsilon$. That is;

$\tilde{\varphi}(f^{-1}(u), f^{-1}(v), s_1) < \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1) + \epsilon ...(*)$.

We also know that if $t \in G_{c_1} \cap f^{-1}(W)$, then $\tilde{\varphi}(f^{-1}(u), f^{-1}(v), t) \leq \tilde{\varphi}(f^{-1}(u), f^{-1}(v), s_1)$.

By using the continuity of $\tilde{\varphi}$ one more time we have;

$\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1) - \epsilon < \tilde{\varphi}(f^{-1}(u), f^{-1}(v), t) < \tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1) + \epsilon$. That is;

$\tilde{\varphi}(f^{-1}(x), f^{-1}(y), c_1) - \epsilon < \tilde{\varphi}(f^{-1}(u), f^{-1}(v), s_1) ...(**)$.

With the combination of $(*)$, and $(**)$ we finally have:

$\mid \theta(x,y,z) - \theta(u,v,w) \mid < \epsilon$. $\square$

Chapter 5

Continuously Urysohn Ordered Spaces

In this chapter we study weakly continuously Urysohn and continuously Urysohn properties on linearly ordered topological spaces(LOTS) and generalized ordered spaces(GO-spaces). Recall that a *LOTS* is a linearly ordered set with the usual open interval topology. A *GO-space* is a Hausdorff space with a linear order $<$ such that the topology of the space has a base consisting of convex sets. Equivalently, GO-spaces are subspaces of LOTS.

In [3], Bennett and Lutzer show that for a separable GO-space X, being continuously Urysohn, submetrizable, and having a G_δ-diagonal are equivalent properties. The behavior of the linear extensions of separable GO-spaces in that manner was not known. We have a theorem which characterizes exactly when a linear extension of a separable GO-spaces is continuously Urysohn. Let us start with the definition of linear extension of a GO-space.

Definition 5.1. *Let $X = (X, \tau, \leq)$ be a GO-space, where $(X, \leq)$ is a linearly ordered set and τ is a topology on X such that:*

(a) $\lambda(\leq) \subseteq \tau$, where $\lambda(\leq)$ is the open-interval topology of $\leq$,

(b) every point of X has a local τ-base consisting of (possibly degenerate) intervals of X.

Define a subset $X^* = (X, \tau, \leq)^*$ of $X \times \mathbb{Z}$ by

$X^* = (X \times \{0\}) \cup \{(x, n) : [x, \rightarrow [\in \tau \setminus \lambda$ *and* $n \leq 0\} \cup \{(x, m) :] \leftarrow, x] \in \tau \setminus \lambda$ *and* $m \geq 0\}$. *Then we order $X^* = (X, \tau, \leq)^*$ lexicographically, and it carries the usual open-interval topology of this lexicographic order.*

X^* is the smallest LOTS which contains X as a closed subspace.

Theorem 5.2. *Let X be a separable GO-space, X^* be any linear extension of X, and define $\mathcal{A} = \{x \in X | \{(y, x] : y < x\}$ base at $x\} \bigcup \{x \in X | \{[x, y) : x < y\}$ base at $x\}$. Then, $\mathcal{A}$ is countable if and only if X^* is continuously Urysohn.*

Proof. Let us first assume that X is a separable GO-space, X^* is a linear extension of X, and $\mathcal{A} = \{x | \{(y, x] : y < x\}$ base at $x\} \bigcup \{x | \{[x, y) : x < y\}$ base at $x\}$ is countable. We will show that X^* is a cU-space.

Case 1: $\mathcal{A}$ is countable, and X has no isolated points.

Claim: X is metrizable.

Proof of Claim. Since regular and second countable spaces are metrizable all we need to show is the second countability of X. By the definition of a GO-space without isolated points, the collection:

$\{] \leftarrow, x] : x \in L\} \cup \{[x, \rightarrow [: x \in R\} \cup \lambda(\leq)$, where L, R are disjoint subsets of X, and $\lambda(\leq)$ is the open interval topology on linearly ordered set X, is a subbase for X.

X is separable, so there exists a countable dense subset D of X. If we take the collection:

$\mathcal{S} = \{(\leftarrow, x] : x \in L\} \cup \{[x, \rightarrow) : x \in R\} \cup \lambda^{'}(\leq)$, where $\lambda'(\leq)$ is a subcollection of $\lambda(\leq)$, with intervals having their endpoints from the set D as the subbase, then the resulting base is a countable base and gives the same topology for X. That finishes the proof of claim.

We then conclude that metrizability of X implies the metrizability of X^*, and that implies cU for X^*.

Case 2: $\mathcal{A}$ is countable and X has isolated points:

First we construct a LOTS extension of X by assuming no isolated points. Let us call this space Y. In the previous case we showed that Y is a cU-space. Then we apply the scattering process to the space Y. Since Y is a cU-space, and the class of all cU-spaces is closed under scattering process X^* the linear extension of X is also a cU-space.

Let us now assume that X is a separable GO-space, and the cU hence wcU-space X^* is a linear extension of X.

Note that since for GO-spaces ccc implies hereditarily Lindelöfness, X is also a hereditarily Lindelöf space.

Claim: $\mathcal{A}$ is countable.

Proof of Claim. Suppose $\mathcal{A}$ is uncountable.

Case 1: $R = \{x \in X : [x, y) \text{ is a base at } x\ \}$ is uncountable.

By using hereditarily Lindelöfness of X, and removing countably many points if necessary we may assume that every $x \in R$ is a limit point of $(x, \rightarrow) \cap R$.

Given any r from R there must be a point $r' \in X^* \setminus R$ such that $r' < r$, and $s < r'$ for any s from R with $s < r$.

For $U \subseteq R$ let $U' = \{r' : r \in U\}$. Since being a wcU-space is hereditary $R \cup R'$ should also be a cU-space. Suppose φ witnesses that $R \cup R'$ is wcU.

Let $C \subset R$ be countable and dense in R. Note that each $r \in R$ has a local base of sets of the form $[r, c)$, where $c \in C$ and the interval computed in X^*.

WLOG, we may assume $\varphi(x, y, x) = 0$ and $\varphi(x, y, y) = 1$ for all $x \neq y \in R \cup R'$. Then for each $r \in R$ there is a point $c_r \in C$ such that;

$\varphi(r', [r, c_r)^2) > 3/4$.

Let $A_c = \{r \in R : c_r = c\}$. We choose a point $c \in C$ such that $\mid A_c \mid > \omega$. Since X is hereditarily Lindelöf space, and A_c is an uncountable subset of R there exist $x, r_n, r \in A_c$ such that $r_n \rightarrow x$ from the right and $r > r_n$ for all n.

Also $r_n' \rightarrow x$, and $(r, r_n) \subset [r_n, c)^2$. Thus $\varphi(r_n', r, r_n) > 3/4$, but $(r_n', r, r_n) \rightarrow (x, r, x)$ and $\varphi(x, r, x) = 0$.

So, we have a contradiction.

Case 2: $L = \{x : (y, x] \text{ is a base at } x\ \}$ is uncountable.

Proof of this case is identical to the proof of case 1. □

After this theorem we have a similar characterization for separable GO-spaces being weakly continuously Urysohn.

Theorem 5.3. *Let X be a separable GO-space. Then X is weakly continuously Urysohn if and only if $\mathcal{A}$ is countable, where $\mathcal{A} = \{x : x$ has an immediate successor or predecessor$\}$.*

Proof. First we suppose X is a separable, wcU, GO-space. Then there exists a continuous function $\theta : (X^2 \setminus \Delta) \times X \to \mathbb{R}$ such that:

$\theta(x, y, x) \neq \theta(x, y, y)$, and we can assume $\theta(x, y, x) = 0$, and $\theta(x, y, y) = 1$.

Suppose $\mathcal{A}$ is uncountable. WLOG we will assume that the set $R = \{r \in X :$ r has an immediate successor$\}$ is uncountable. So for every r from R there exists $r^{'}$ immediate successor of r. Let us define $R^{'} = \{r^{'} : r \in R\}$. Since being wcU-space is hereditary then $R \cup R^{'}$ is also wcU.

We have $\theta(r^{'}, r, r) = 1$. Then for every $r \in R$ there exists a basic neighborhood G_r of r such that, $\theta(r^{'}, G_r^2) > 3/4$.

As a result of the separability assumption on X there exists a countable dense subset D. So for every $r \in R$ there exists a $c_r \in G_r$. We can also assume that $(c_r, r] \subset G_r$. Note that there exists a c that repeats for uncountably many r.

Let us define $P = \{r : c \in G_r\}$ where $\mid P \mid > \omega$. Since X is hereditarily Lindelöf, P is also a Lindelöf space.

Then there exist $x, r_n, r \in P$ such that $r_n \to x$ and $c < r < r_n$ for every $n < \omega$. Note that $r_n^{'}$ converges to x.

We have $\theta(r_n^{'}, r, r_n) > 3/4$. However $(r_n^{'}, r, r_n) \to (x, r, x)$ and $\theta(x, r, x) = 0$ is a contradiction.

Now we suppose $\mathcal{A} = \{x : x$ has an immediate successor or predecessor$\}$ is countable.

Let us form a base $\mathcal{B} = \{\{x\} : x$ isolated in $X_\tau\} \cup \{ K : K$ convex-open with endpoints in $\mathcal{A} \cup D\}$; where X_τ is the topology on GO-space and D is the countable dense subset. It is clear that this is a countable base for a weaker order topology on X. So, X is submetrizable, hence cU. $\square$

In [3], Bennett and Lutzer prove that a stationary subset $\mathcal{S}$ of a regular uncountable cardinal κ is not continuously Urysohn, and for any monotonically normal space being continuously Urysohn implies being hereditarily paracompact. Also Gruenhage and Zenor in [8] show that any stationary subset of a regular uncountable cardinal cannot even be weakly continuously Urysohn. With the help of these results, we give a characterization for well-ordered spaces in the following theorem.

Theorem 5.4. *A well-ordered space is continuously Urysohn if and only if it does not contain a subset homeomorphic to a stationary subset of a regular uncountable cardinal. Equivalently, a well-ordered space is continuously Urysohn if and only if it is hereditarily paracompact.*

Proof. With the result of Bennett and Lutzer's mentioned above, we know that well-ordered cU-spaces cannot contain a stationary subset of a regular uncountable cardinal. All we need to show is that the cU property holds when a well-ordered set does not contain the subsets mentioned.

Since a subset of ω_1 contains a stationary subset if and only if it is nonmetrizable, claim is true for subspaces of ω_1.

Let α be any ordinal. Suppose the claim is true for the subsets of β for every $\beta < \alpha$. That is if $B \subset \beta$ does not contain a stationary subset of a regular uncountable cardinal, then B is a cU-space.

Claim : If $A \subset \alpha$ does not contain a stationary subset of a regular uncountable cardinal, then A is cU-space.

Proof of Claim.

Case 1: α is a limit ordinal.

Subcase 1: Cofinality of α is greater than ω. That is $cf\alpha > \omega$.

Suppose $cf\alpha = \kappa > \omega$. Note that κ is a regular cardinal. Also there exists a subset K of α, which is a copy of κ. We know that K is closed, order isomorphic to κ, and cofinal in α.

On the other hand; since A does not contain a stationary subset of any uncountable regular cardinal there exists a club subset C of K such that $C \cap A = \emptyset$. WLOG; we can assume $K = C$ and $K \cap A = \emptyset$. Since K^c is the union of disjoint convex subsets, $K^c = \bigcup_{\gamma} I_\gamma$. That is, K^c is the topological sum of cU-spaces. By using Theorem 3.9, we conclude that K^c is cU. Hence $A \subset K^c$ is cU.

Subcase 2: Cofinality of α is equal to ω. That is $cf\alpha = \omega$.

There exists a $K = \{k_n\}_{n<\omega}$ countable subset of α, where K is a copy of ω in α.

WLOG we can assume that, k_n is a successor ordinal for every n and we can write $\alpha = \bigcup_{n}[k_n, k_{n+1})$. Note that $I_n = [k_n, k_{n+1})$ is clopen and a cU-space with the witnessing function φ_n. Since, α is the topological sum of cU-spaces, it is cU.

Case 2: α is a successor ordinal. That is $\alpha = \gamma + 1$ for an ordinal γ.

WLOG, we can assume $A' = A \cup \{\gamma\}$, and we will show that A' is a cU-space. Otherwise, $A \subset \gamma < \alpha$ and A is a cU-space by assumption.

Subcase 1: Cofinality of γ is greater than or equal to ω. That is $cf\gamma \geq \omega$.

Let us set $cf\gamma = \tau \geq \omega$. There exists a cofinal subset T of γ which is a copy of τ. So $T \subset \gamma$ is closed and order isomorphic to τ. On the other hand since A does not contain a stationary subset of an uncountable regular cardinal there exists a club subset C of T such that $C \cap A = \emptyset$.

WLOG, we assume that $A = \gamma \setminus C$. Then $A = \bigcup_{\xi} I_\xi$, where $\{I_\xi\}_\xi$ is a pairwise disjoint collection of convex subsets of γ. For every ξ, we can define $I_\xi = (c_\xi, c_{\xi+1})$, where $c_\xi, c_{\xi+1} \in C$. Note that I_ξ is a cU-space with the witnessing function φ_ξ.

Let us define the function φ as follows:

If $x, y, z \in I_\xi$, then $\varphi(x, y)(z) = \varphi_\xi(x, y)(z)$.

If $x, y \in I_\xi$, and $z \notin I_\xi$, then $\varphi(x, y)(z) = 0$.

For the case when x, and y are not in the same convex subset:

If $x < y$, then there exists a I_μ such that $x \in I_\mu$, and $c_\mu < x < c_{\mu+1}$. We define $\varphi(x, y)(z) = 1 - \chi_{(c_\mu, c_{\mu+1}]}(z)$.

If $y < x$, then there exists a I_ν such that $y \in I_\nu$, and $c_\nu < y < c_{\nu+1}$. We define $\varphi(x,y)(z) = \chi_{(c_\nu, c_{\nu+1}]}(z)$.

Claim 1: $\varphi(x,y)(x) \neq \varphi(x,y)(y)$.

Proof of Claim 1.

Case 1: $x, y, z \in I_\xi$.

$\varphi(x,y)(x) = \varphi_\xi(x,y)(x) = 0$, and $\varphi(x,y)(y) = \varphi_\xi(x,y)(y) = 1$.

Case 2: Other cases.

If $x < y$, then $\varphi(x,y)(x) = 0$ and $\varphi(x,y)(y) = 1$. If $y < x$, then $\varphi(x,y)(x) = 0$ and $\varphi(x,y)(y) = 1$.

Claim 2: $\varphi(x,y)$ is a continuous function for every (x,y).

Proof of Claim 2.

Suppose $\epsilon > 0$, and a point z are given.

Case 1: $x, y, z \in I_\xi$.

Since $\varphi_\xi(x,y)$ is a continuous function, there is a neighborhood G_ξ of z in such that;

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid < \epsilon$, for every $t \in G_\xi$. We set $G = G_\xi$. Then,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid < \epsilon$, for every $t \in G$.

Case 2: $x, y \in I_\xi$, and $z \notin I_\xi$.

Subcase 1: $z \neq \gamma$. There exists a I_ς such that $z \in I_\varsigma$.

We set $G = I_\varsigma$. Then,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid < \epsilon$, for every $t \in G$.

Subcase 2: $z = \gamma$.

We set $G = (c_{\xi+1}, \gamma] \cap A'$. Then,

$\mid \varphi(x,y)(\gamma) - \varphi(x,y)(t) \mid < \epsilon$, for every $t \in G$.

Case 3: Other cases:

(1) $x < y$.

Subcase 1: $x, z \in I_\mu$.

We set $G = (c_\mu, c_{\mu+1}) \cap A'$. Then,

$| \varphi(x,y)(z) - \varphi(x,y)(t) | < \epsilon$, for every $t \in G$.

Subcase 2: $x \in I_\mu$, and $z \notin I_\mu$.

We set $G = [[c_\mu, c_{\mu+1}] \cap A']^c$.

$| \varphi(x,y)(z) - \varphi(x,y)(t) | < \epsilon$, for every $t \in G$.

(2) $y < x$

Subcase 1: $y, z \in I_\nu$.

We set $G = (c_\nu, c_{\nu+1}) \cap A'$.

$| \varphi(x,y)(z) - \varphi(x,y)(t) | < \epsilon$, for every $t \in G$.

Subcase 2: $y \in I_\nu$, and $z \notin I_\nu$.

We set $G = [[c_\nu, c_{\nu+1}] \cap A']^c$.

$| \varphi(x,y)(z) - \varphi(x,y)(t) | < \epsilon$, for every $t \in G$.

Claim 3: φ is a continuous function.

Proof of Claim 3.

Suppose an $\epsilon > 0$, and (x,y) are given.

Case 1: $x, y \in I_\xi$.

We define $\varphi_\xi(x,y)$ for (x,y). By using the continuity of φ_ξ we know that there is a neighborhood $U_\xi \times V_\xi$ of (x,y) in I_ξ such that

$| \varphi_\xi(x,y)(z) - \varphi_\xi(u,v)(z) |< \epsilon$ for every (u,v) from $U_\xi \times V_\xi$.

We take neighborhoods $U = U_\xi$ and $V = V_\xi$.

If $z \in I_\xi$, then $| \varphi(x,y)(z) - \varphi(u,v)(z) |=| \varphi_\xi(x,y)(z) - \varphi_\xi(u,v)(z) |< \epsilon$, and if $z \notin I_\xi$, then $\varphi(x,y)(z) = 0$ and $\varphi(u,v)(z) = 0$.

Case 2: Other cases:

(1) $x < y$.

We take $U = I_\mu$, and $V = (c_\mu, \gamma] \cap A'$. Note that $x \in I_\mu$. Then,

$\varphi(x,y)(z) = 1 - \chi_{(c_\mu, c_{\mu+1}]}(z)$, and $\varphi(u,v)(z) = 1 - \chi_{(c_\mu, c_{\mu+1}]}(z)$.

(2) $y < x$.

We take $U = (c_\nu, \gamma] \cap A'$ and $V = I_\nu$. Note that $y \in I_\nu$. Then,

$\varphi(x,y)(z) = \chi_{(c_\nu,c_{\nu+1}]}(z)$, and $\varphi(u,v)(z) = \chi_{(c_\nu,c_{\nu+1}]}(z)$.

Subcase 2: $\gamma = \beta + 1$, for some ordinal β. We define the set $A' = [0,\beta] \cap A \cup \{\gamma\}$. Notice that $[0,\beta]$ is cU, and γ is isolated. So, we have a topological sum of cU-spaces.

□

After these results we want to examine the locally continuously Urysohn property in ordered spaces. The question was: locally continuously Urysohn and what other assumption on the ordered spaces imply continuously Urysohn?.

It is known that for many cases, paracompactness and having a property locally imply this property for the space itself. We do not know if this is true for cU in general, but we prove that a locally continuously Urysohn paracompact LOTS is continuously Urysohn. Before proving this theorem, we will prove a useful lemma.

Lemma 5.5. *Assume X is a LOTS, $a, b \in X$, and $a < b$. If $(\leftarrow, b)$ and $(a, \rightarrow)$ are continuously Urysohn, then X is continuously Urysohn.*

Proof. Suppose that the functions which witness cU on $(\leftarrow, b)$, and $(a, \rightarrow)$ are φ_1, and φ_2 respectively. By using Proposition 2.1. from [3], we can assume that if $x < y$ then $\varphi_1(x,y)$ sends $(\leftarrow, x]$ to 0, and $[y, \rightarrow)$ to 1. Similarly, $\varphi_2(x,y)$ sends $(\leftarrow, x]$ to 0, and $[y, \rightarrow)$ to 1.

In addition to these assumptions normality gives a function $\theta : X \rightarrow [0,1]$ such that:

$\theta(x) = 0$ for $x \leq a$, and $\theta(x) = 1$ for $x \geq b$.

In order to prove this lemma, we will define a function $\varphi : X^2 \setminus \Delta \rightarrow C(X)$ as follows:

If $x < y$, define

$$\varphi(x,y)(z) = \begin{cases} \theta(z), & \text{if } z \leq x, \\ \theta(x).\varphi_2(x,y)(z) + (1-\theta(y)).\varphi_1(x,y)(z) + \theta(z), & \text{if } \ x < z < y, \\ \theta(x) + (1-\theta(y)) + \theta(z), & \text{if } y \leq z. \end{cases}$$

If $y < x$, define

$$\varphi(x,y)(z) = \begin{cases} (1-\theta(x)) + \theta(y) - \theta(z), & \text{if } z \leq y, \\ (1-\theta(x)).\varphi_1(x,y)(z) + \theta(y).\varphi_2(x,y)(z) - \theta(z), & \text{if } \ y < z < x, \\ -\theta(z), & \text{if } x \leq z. \end{cases}$$

Claim 1: $\varphi(x,y)(x) \neq \varphi(x,y)(y)$ for every $(x,y) \in X^2 \setminus \Delta$.

Proof of Claim 1. Suppose $x < y$. Then $\varphi(x,y)(x) = \theta(x)$, and $\varphi(x,y)(y) = \theta(x) + 1$. Similarly, if we assume $y < x$, then $\varphi(x,y)(x) = -\theta(x)$, and $\varphi(x,y)(y) = 1 - \theta(x)$.

Claim 2: $\varphi(x,y)$ is continuous for every $(x,y) \in X^2 \setminus \Delta$.

Proof of Claim 2. Suppose $\epsilon > 0$, $z \in X$ are given, and WLOG we may assume $x < y$.

Case 1: $z \leq x$. Then $\varphi(x,y)(z) = \theta(z)$.

Subcase 1: $z < x$.

Since θ is a continuous function there exists a neighborhood G' of z such that

$\mid \theta(z) - \theta(t) \mid < \epsilon$, for every $t \in G'$. We take $G = G' \cap (\leftarrow, x)$ neighborhood of z. Then for every $t \in G$,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid \theta(z) - \theta(t) \mid < \epsilon$.

Subcase 2: $z = x$, and $x = z < y \leq a$.

Since θ, and $\varphi_1(x,y)$ are continuous functions there exist neighborhoods G', and G_1 of $z = x$ such that $\mid \theta(t) \mid < \epsilon$ for every $t \in G'$, and $\mid \varphi_1(x,y)(t) \mid < \epsilon$ for every $t \in G_1$.

We set $G = (\leftarrow, y) \cap G' \cap G_1$.

If $t \in G$ also satisfies $t < x = z$, then $\mid \varphi(x,y)(x) - \varphi(x,y)(t) \mid = \mid 0 - 0 \mid = 0$.

If $t \in G$ also satisfies $t > x = z$, then $\mid \varphi(x,y)(x) - \varphi(x,y)(t) \mid = \mid \varphi_1(x,y)(t) \mid < \epsilon$.

Subcase 3: $z = x$, and $x = z \leq a < y < b$.

Since θ, and $\varphi_1(x, y)$ are continuous functions there exist neighborhoods G', and G_1 of $z = x$ such that $\mid \theta(t) \mid< \epsilon\backslash 2$ for every $t \in G'$, and $\mid \varphi_1(x, y)(t) \mid< \epsilon\backslash 2$ for every $t \in G_1$.

We set $G = (\leftarrow, y) \cap G' \cap G_1$.

If $t \in G$ also satisfies $t < x = z$, then $\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid=\mid 0 - 0 \mid= 0$.

If $t \in G$ also satisfies $t > x = z$, then

$\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid=\mid (1 - \theta(y))\varphi_1(x, y)(t) - \theta(t) \mid\leq$

$\mid (1 - \theta(y)) \mid . \mid \varphi_1(x, y)(t) \mid + \mid \theta(t) \mid< \epsilon$.

Subcase 4: $z = x$, and $x = z \leq a < b \leq y$.

Since θ is a continuous function, and $\theta(x) = 0$ there exists a neighborhood G' of $z = x$ such that $\mid \theta(t) \mid< \epsilon$ for every $t \in G'$.

We set $G = (\leftarrow, y) \cap G'$.

If $t \in G$ also satisfies $t < x = z$, then $\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid=\mid 0 - 0 \mid= 0$.

If $t \in G$ also satisfies $t > x = z$, then $\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid=\mid \theta(t) \mid< \epsilon$.

Subcase 5: $z = x$, and $a \leq x = z < y < b$.

Since θ, $\varphi_1(x, y)$, and $\varphi_2(x, y)$ are continuous functions there exist neighborhoods G', G_1, and G_2 of $z = x$ such that $\mid \theta(x) - \theta(t) \mid< \epsilon\backslash 3$ for every $t \in G'$, $\mid \varphi_1(x, y)(t) \mid< \epsilon\backslash 3$ for every $t \in G_1$, and $\mid \varphi_2(x, y)(t) \mid< \epsilon\backslash 3$ for every $t \in G_2$.

We set $G = (\leftarrow, y) \cap G' \cap G_1 \cap G_2$.

If $t \in G$ also satisfies $t < x = z$, then $\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid=\mid \theta(x) - \theta(t) \mid< \epsilon$.

If $t \in G$ also satisfies $t > x = z$, then

$\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid=\mid \theta(x) - \theta(x)\varphi_2(x, y)(t) - (1 - \theta(y))\varphi_1(x, y)(t) - \theta(t) \mid\leq$

$\mid \theta(x) - \theta(t) \mid + \mid \theta(x) \mid . \mid \varphi_2(x, y)(t) \mid + \mid (1 - \theta(y)) \mid . \mid \varphi_1(x, y)(t) \mid< \epsilon$.

Subcase 6: $z = x$, and $a \leq x = z < b < y$.

Since θ, and $\varphi_2(x, y)$ are continuous functions there exist neighborhoods G', and G_2 of $z = x$ such that $\mid \theta(x) - \theta(t) \mid< \epsilon\backslash 2$ for every $t \in G'$, and $\mid \varphi_2(x, y)(t) \mid< \epsilon\backslash 2$ for every $t \in G_2$.

We set $G = (\leftarrow, y) \cap G' \cap G_2$.

If $t \in G$ also satisfies $t < x = z$, then $\mid \varphi(x,y)(x) - \varphi(x,y)(t) \mid = \mid \theta(x) - \theta(t) \mid < \epsilon$.

If $t \in G$ also satisfies $t > x = z$, then

$\mid \varphi(x,y)(x) - \varphi(x,y)(t) \mid = \mid \theta(x) - \theta(x)\varphi_2(x,y)(t) - \theta(t) \mid \leq$

$\mid \theta(x) - \theta(t) \mid + \mid \theta(x) \mid . \mid \varphi_2(x,y)(t) \mid < \epsilon$.

Subcase 7: $z = x$, and $b \leq x = z < y$.

Since θ, and $\varphi_2(x,y)$ are continuous functions there exist neighborhoods G', and G_2 of $z = x$ such that

$\mid \theta(x) - \theta(t) \mid < \epsilon$ for every $t \in G'$, and $\mid \varphi_2(x,y)(t) \mid < \epsilon$ for every $t \in G_2$.

We set $G = (\leftarrow, y) \cap G' \cap G_2$.

If $t \in G$ also satisfies $t < x = z$, then $\mid \varphi(x,y)(x) - \varphi(x,y)(t) \mid = \mid \theta(x) - \theta(t) \mid < \epsilon$.

If $t \in G$ also satisfies $t > x = z$, then

$\mid \varphi(x,y)(x) - \varphi(x,y)(t) \mid = \mid \theta(x) \mid . \mid \varphi_2(x,y)(t) \mid < \epsilon$.

Case 2: $x < z < y$. Then $\varphi(x,y)(z) = \theta(x)\varphi_2(x,y)(z) + (1-\theta(y))\varphi_1(x,y)(z) + \theta(z)$.

Subcase 1: $x < y \leq a$.

Since $\varphi_1(x,y)$ is a continuous function there exists a neighborhood G_1 of z such that

$\mid \varphi_1(x,y)(z) - \varphi_1(x,y)(t) \mid < \epsilon$.

We set $G = (x,y) \cap G_1$. Then,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid \varphi_1(x,y)(z) - \varphi_1(x,y)(t) \mid < \epsilon$ for every $t \in G$.

Subcase 2: $x \leq a < y < b$.

Since $\varphi_1(x,y)$ is a continuous function there exists a neighborhood G_1 of z such that

$\mid \varphi_1(x,y)(z) - \varphi_1(x,y)(t) \mid < \epsilon$.

We set $G = (x,y) \cap G_1$. Then,

$\mid \varphi(x,y)(z) - \varphi(x,y)(t) \mid = \mid (1-\theta(y)) \mid . \mid \varphi_1(x,y)(z) - \varphi_1(x,y)(t) \mid < \epsilon$ for every $t \in G$.

Subcase 3: $x \leq a$, and $b \leq y$.

Since θ is a continuous function there exists a neighborhood G' of z such that

$\mid \theta(z) - \theta(t) \mid < \epsilon$.

We set $G = (x, y) \cap G'$. Then,

$\mid \varphi(x, y)(z) - \varphi(x, y)(t) \mid = \mid \theta(z) - \theta(t) \mid < \epsilon$ for every $t \in G$.

Subcase 4: $a < x < y < b$.

Since θ, $\varphi_1(x, y)$, and $\varphi_2(x, y)$ are continuous functions there exist neighborhoods G', G_1, and G_2 of z such that

$\mid \theta(z) - \theta(t) \mid < \epsilon \backslash 3$ for every $t \in G'$, $\mid \varphi_1(x, y)(z) - \varphi_1(x, y)(t) \mid < \epsilon \backslash 3$ for every $t \in G_1$, and $\mid \varphi_2(x, y)(z) - \varphi_2(x, y)(t) \mid < \epsilon \backslash 3$ for every $t \in G_2$.

We set $G = (x, y) \cap G' \cap G_1 \cap G_2$.

Then

$\mid \varphi(x, y)(x) - \varphi(x, y)(t) \mid \leq \mid \theta(x) \mid . \mid \varphi_2(x, y)(z) - \varphi_2(x, y)(t) \mid +$

$\mid 1 - \theta(y) \mid . \mid \varphi_1(x, y)(z) - \varphi_1(x, y)(t) \mid + \mid \theta(z) - \theta(t) \mid < \epsilon$.

Subcase 5: $a < x < b \leq y$.

Since θ, and $\varphi_2(x, y)$ are continuous functions there exist neighborhoods G', and G_2 of z such that

$\mid \theta(z) - \theta(t) \mid < \epsilon \backslash 2$ for every $t \in G'$, and $\mid \varphi_2(x, y)(t) \mid < \epsilon \backslash 2$ for every $t \in G_2$.

We set $G = (x, y) \cap G' \cap G_2$. Then,

$\mid \varphi(x, y)(z) - \varphi(x, y)(t) \mid \leq \mid \theta(x) \mid . \mid \varphi_2(x, y)(z) - \varphi_2(x, y)(t) \mid + \mid \theta(z) - \theta(t) \mid < \epsilon$.

Subcase 6: $b \leq x < y$.

Since $\varphi_2(x, y)$ is a continuous function there exists a neighborhood G_2 of z such that $\mid \varphi_2(x, y)(z) - \varphi_2(x, y)(t) \mid < \epsilon$ for every $t \in G_2$.

We set $G = (x, y) \cap G_2$. Then,

$\mid \varphi(x, y)(z) - \varphi(x, y)(t) \mid = \mid \varphi_2(x, y)(z) - \varphi_2(x, y)(t) \mid < \epsilon$ for every $t \in G$.

Case 3: $y \leq z$. Then $\varphi(x, y)(z) = \theta(x) + (1 - \theta(y)) + \theta(z)$.

The proof for this case is identical to the proof of case 1. And also the for the case $y < x$ the proof works similarly.

Claim 3: $\varphi : X^2 \setminus \Delta \to C(X)$ is a continuous function.

Proof of Claim. Suppose $\epsilon > 0$, and $(x, y) \in X^2 \setminus \Delta$ are given. And WLOG we assume $x < y$.

Since θ is a continuous function, there exists a neighborhoods U', and V' of x and y so that $\mid \theta(x) - \theta(u) \mid < \epsilon \backslash 4$ for every $u \in U'$, and $\mid \theta(y) - \theta(v) \mid < \epsilon \backslash 4$ for every $v \in V'$. Also note that $\mid \theta(x) \mid \leq 1$ for every $x \in X$.

If the location of (x, y) with respect to a, and b is examined, it is one of the following: only $\varphi_1(x, y)$ can be defined, only $\varphi_2(x, y)$ can be defined or they both can be defined.

Also we know that φ_1, and φ_2 are continuous functions.

Then there exists a neighborhood $U_1 \times V_1$ of (x, y) such that:

$\mid \varphi_1(x, y)(z) - \varphi_1(u, v)(z) \mid < \epsilon \backslash 4$ for every $(u, v) \in U_1 \times V_1$, and z.

Similarly, there exists a neighborhood $U_2 \times V_2$ of (x, y) such that:

$\mid \varphi_2(x, y)(z) - \varphi_2(u, v)(z) \mid < \epsilon \backslash 4$ for every $(u, v) \in U_2 \times V_2$, and z.

So, we set $U = U_1 \cap U_2 \cap U' \cap (\leftarrow, y)$, and $U = V_1 \cap V_2 \cap V' \cap (x, \rightarrow)$, depending on the location of (x, y). Note that WLOG, we may assume that $U_1 \cap V_1 = U_2 \cap V_2 = \emptyset$, and $u < v$ for every $u \in U$, and $v \in V$.

In addition to these, we assume that $\mid \varphi_i(x, y)(z) \mid \leq 1$ for every $(x, y) \in X^2 \setminus \Delta$, and $z \in X$, where $i = 1, 2$.

Case 1: $z \leq x$.

That is $\varphi(x, y)(z) = \theta(z)$. Under this case two subcases can occur.

Subcase 1: $z \leq u$. That is $\varphi(u, v)(z) = \theta(z)$, and we are done.

Subcase 2: $u < z < v$.

That is $\varphi(u, v)(z) = \theta(u).\varphi_2(u, v)(z) + (1 - \theta(v)).\varphi_1(u, v)(z) + \theta(z)$. Then,

$\mid \varphi(x, y)(z) - \varphi(u, v)(z) \mid = \mid -\theta(u)\varphi_2(u, v)(z) + (1 - \theta(v))\varphi_1(u, v)(z) \mid \leq$

$\mid \theta(u) \mid . \mid \varphi_2(u, v)(z) \pm \varphi_2(x, y)(z) \mid + \mid 1 - \theta(v) \mid . \mid \varphi_1(u, v)(z) \pm \varphi_1(x, y)(z) \mid < \epsilon.$

Case 2: $x < z < y$.

That is $\varphi(x, y)(z) = \theta(x).\varphi_2(x, y)(z) + (1 - \theta(y)).\varphi_1(x, y)(z) + \theta(z)$.

Subcase 1: $z \leq u$. That is $\varphi(u, v)(z) = \theta(z)$. Then,

$\mid \varphi(x,y)(z)-\varphi(u,v)(z) \mid = \mid -\theta(x)\varphi_2(x,y)(z)+(1-\theta(y))\varphi_1(x,y)(z) \mid \leq$

$\mid \theta(x) \mid . \mid \varphi_2(x,y)(z) \pm \varphi_2(u,v)(z) \mid + \mid 1-\theta(y) \mid . \mid \varphi_1(x,y)(z) \pm \varphi_1(u,v)(z) \mid \leq$

$\mid \varphi_2(x,y)(z)-\varphi_2(u,v)(z) \mid + \mid \varphi_1(x,y)(z)-\varphi_1(u,v)(z) \mid < \epsilon.$

Subcase 2: $u < z < v$.

That is $\varphi(u,v)(z) = \theta(u).\varphi_2(u,v)(z)+(1-\theta(v)).\varphi_1(u,v)(z)+\theta(z)$. Then,

$\mid \varphi(x,y)(z)-\varphi(u,v)(z) \mid = \mid \theta(x)\varphi_2(x,y)(z)+(1-\theta(y))\varphi_1(x,y)(z)-\theta(u)\varphi_2(u,v)(z)-$

$(1-\theta(v))\varphi_1(u,v)(z) \mid \leq$

$\mid \theta(x)\varphi_2(x,y)(z)-\theta(u)\varphi_2(u,v)(z) \pm \theta(x)\varphi_2(u,v)(z) \mid +$

$\mid \ (1-\theta(y))\varphi_1(x,y)(z)-(1-\theta(v))\varphi_1(u,v)(z) \pm (1-\theta(y))\varphi_1(u,v)(z) \mid \leq$

$\mid \theta(x) \mid . \mid \varphi_2(x,y)(z)-\varphi_2(u,v)(z) \mid + \mid \varphi_2(u,v)(z) \mid . \mid \theta(x)-\theta(u) \mid +$

$\mid 1-\theta(y) \mid . \mid \varphi_1(x,y)(z)-\varphi_1(u,v)(z) \mid + \mid \varphi_1(u,v)(z) \mid . \mid \theta(y)-\theta(v) \mid \leq$

$\mid \varphi_2(x,y)(z)-\varphi_2(u,v)(z) \mid + \mid \theta(x)-\theta(u) \mid + \mid \varphi_1(x,y)(z)-\varphi_1(u,v)(z) \mid +$

$\mid \theta(y)-\theta(v) \mid < \epsilon.$

Subcase 3: $v \leq z$. That is $\varphi(u,v)(z) = \theta(u)+(1-\theta(v))+\theta(z)$. Then,

$\mid \varphi(x,y)(z)-\varphi(u,v)(z) \mid = \mid \theta(x)\varphi_2(x,y)(z)+(1-\theta(y))\varphi_1(x,y)(z)-\theta(u)-(1-\theta(v)) \mid \leq$

$\mid \theta(x)\varphi_2(x,y)(z)-\theta(u) \pm \theta(x) \mid + \mid \ (1-\theta(y))\varphi_1(x,y)(z)-(1-\theta(v)) \pm (1-\theta(y)) \mid \leq$

$\mid \theta(x) \mid . \mid \varphi_2(x,y)(z)-1 \mid + \mid \theta(x)-\theta(u) \mid + \mid 1-\theta(y) \mid . \mid \varphi_1(x,y)(z)-1 \mid +$

$\mid \theta(y)-\theta(v) \mid \leq$

$\mid \varphi_2(x,y)(z)-1 \pm \varphi_2(u,v)(z) \mid + \mid \theta(x)-\theta(u) \mid + \mid \varphi_1(x,y)(z)-1 \pm \varphi_1(u,v)(z) \mid +$

$\mid \theta(y)-\theta(v) \mid \leq$

$\mid \varphi_2(x,y)(z)-1\varphi_2(u,v)(z) \mid + \mid \theta(x)-\theta(u) \mid + \mid \varphi_1(x,y)(z)-\varphi_1(u,v)(z) \mid +$

$\mid \theta(y)-\theta(v) \mid < \epsilon.$

Case 3: $y \leq z$.

That is $\varphi(x,y)(z) = \theta(x)+(1-\theta(y))+\theta(z)$.

Subcase 1: $v \leq z$. That is $\varphi(u,v)(z) = \theta(u)+(1-\theta(v))+\theta(z)$. Then,

$\mid \varphi(x,y)(z)-\varphi(u,v)(z) \mid = \mid \theta(x)-\theta(y)-\theta(u)+\theta(v) \mid \leq$

$\mid \theta(x)-\theta(u) \mid + \mid \theta(y)-\theta(v) \mid < \epsilon.$

Subcase 2: $u < z < v$.

That is $\varphi(u, v)(z) = \theta(u).\varphi_2(u, v)(z) + (1 - \theta(v)).\varphi_1(u, v)(z) + \theta(z)$. Then,

$\mid \varphi(x, y)(z) - \varphi(u, v)(z) \mid = \mid \theta(x) + (1 - \theta(y)) - \theta(u)\varphi_2(u, v)(z) - (1 - \theta(v))\varphi_1(u, v)(z) \mid \leq$

$\mid \theta(x) - \theta(u)\varphi_2(u, v)(z) \pm \theta(u) \mid + \mid 1 - \theta(y) - (1 - \theta(v))\varphi_1(u, v)(z) \pm (1 - \theta(v)) \mid \leq$

$\mid \theta(x) - \theta(u) \mid + \mid \theta(u) \mid . \mid 1 - \theta_2(u, v)(z) \mid + \mid 1 - \theta(v) \mid . \mid 1 - \varphi_1(u, v)(z) \mid +$

$\mid \theta(y) - \theta(v) \mid \leq$

$\mid \theta(x) - \theta(u) \mid + \mid 1 - \varphi_2(u, v)(z) \pm \varphi_2(x, y)(z) \mid + \mid 1 - \varphi_1(u, v)(z) \pm \varphi_1(x, y)(z) \mid +$

$\mid \theta(t) - \theta(v) \mid \leq$

$\mid \theta(x) - \theta(u) \mid + \mid \varphi_2(x, y)(z) - \varphi_2(u, v)(z) \mid + \mid \varphi_1(x, y)(z) - \varphi_1(u, v)(z) \mid +$

$\mid \theta(y) - \theta(v) \mid < \epsilon$.

□

After this lemma, we are going to prove that a paracompact, locally continuously Urysohn LOTS is continuously Urysohn. Here a space X is *locally continuously Urysohn* provided that every $x \in X$ has an open neighborhood which is also a continuously Urysohn space.

Theorem 5.6. *If X is a paracompact, locally continuously Urysohn LOTS, then X is continuously Urysohn.*

Proof. Suppose X is a paracompact, locally cU LOTS.

Since X is locally cU, then for every $x \in X$ there exists a neighborhood U_x which is also a cU-space. The collection $\mathcal{U} = \{U_x : x \in X\}$ is an open cover for the space.

By using paracompactness, we find $\mathcal{V} = \{I_\alpha : \alpha \in A\}$ locally finite open refinement of $\mathcal{U}$ such that for every $\alpha \in A$:

(1) $I_\alpha = (a_\alpha, b_\alpha)$,

(2) $\bar{I_\alpha}$ is cU,

(3) $|\{\beta \in A : I_\alpha \cap I_\beta \neq \emptyset\}| \leq 2$.

Let us define an equivalence relation $\approx$ as follows:

$I_\alpha \approx I_\beta$ if and only if there exists a finite chain link between I_α, and I_β. If we examine the equivalence classes $[I_\alpha]$, it is clear that $|[I_\alpha]| \leq \omega$. Suppose $\widetilde{I_\alpha} = \bigcup_{I_\beta \in [I_\alpha]} I_\beta$. Then for every $\alpha, \beta \in A$:

(i) $\widetilde{I_\alpha}$ is clopen,

(ii) $\widetilde{I_\alpha}$ is convex, and

(iii) $\widetilde{I_\alpha} \cap \widetilde{I_\beta} = \emptyset$ for $\alpha \neq \beta$.

Claim : $\widetilde{I_\alpha}$ is cU for every $\alpha \in A$.

Proof of Claim.

Case 1: $[I_\alpha]$ is finite.

Then by using the previous lemma, we are done.

Case 2: $[I_\alpha]$ is countably infinite.

Let us denote this connected collection as $[I_\alpha] = \{I_0, I_1, ...\}$, and pick $x, y \in \widetilde{I_\alpha}$ such that $x < y$. We will show that $[x, \rightarrow)$, and $(\leftarrow, y]$ are cU.

These proofs work identically, so we will only prove that $[x, \rightarrow)$ is cU. Note that $\widetilde{I_\alpha}$ can be densely embedded into a compact LOTS $\hat{X}$. Let us say $X^* = \hat{X} \setminus \{lastpoint\}$.

Subcase 1: $X^* \setminus \widetilde{I_\alpha}$ is cofinal in X^*. That means for every $a \in X^*$ there exists a $b \in X^* \setminus \widetilde{I_\alpha}$ such that $a < b$. Then, if $I_0 = (a_0, b_0)$, there exists an $x_0 \in X^* \setminus \widetilde{I_\alpha}$ so that $b_0 < x_0$, if $I_1 = (a_1, b_1)$, there exists an $x_1 \in X^* \setminus \widetilde{I_\alpha}$ so that $b_1 < x_1$, and by following this idea we have a sequence $x_0, x_1, ...$ Note that $(\leftarrow, x_0] \cap [x, \rightarrow)$ is either empty or can be covered by finitely many of I_α's which implies cU. Similarly, $[x_n, x_{n+1}] \cap [x, \rightarrow)$ is either empty or cU for every $n < \omega$. That tells us $[x, \rightarrow)$ is cU.

Subcase 2: $X^* \setminus \widetilde{I_\alpha}$ is not cofinal in X^*. That means there exists a $p \in X^*$ such that $q < p$ for every $q \in X^* \setminus \widetilde{I_\alpha}$. So WLOG, we assume $x < p$, and pick $a \in X^*$ such that $p < a$. Then, $[x, \rightarrow) = [x, \rightarrow) \cap \{[x, a) \cup (p, \rightarrow)$.

So far we we showed that $[x, \rightarrow)$ is cU for the countable and connected collection $\widetilde{I_\alpha}$, and $x \in \widetilde{I_\alpha}$. With a similar proof one can show that $(\leftarrow, y]$ is also cU. Combining these together we say $\widetilde{I_\alpha}$ is cU. Since $\widetilde{I_\alpha}$'s make a partition of X, we have a topological sum of cU spaces, which is also cU.

Note also that if X has a last point m, in addition to this proof we only need to look at the collection $\widetilde{I_\alpha}$ which contains m. Pick $x \in \widetilde{I_\alpha}$. By revisiting subcase 1 and subcase 2, $[x, m]$ can be covered by finitely many I_α's. So it is cU. $\square$

Chapter 6

Miscellaneous

A set $S \subset X$ is said to be *free* for a set function F if $x, y \in S$ with $x \neq y$ imply $x \notin F(y)$.

Free Set Lemma ([11]) If $F : X \to [X]^{<\lambda}$ where $\lambda < |X|$ then there is a free set $S \subset X$ for F, with $|S| = |X|$. Here $[X]^{<\lambda}$ is the collection of all subsets with cardinality less than λ.

Corollary 6.1. *$X = L(\omega_2)$, the one-point Lindelöfication of ω_2, is not a cU-space.*

Proof. Suppose X is a cU space with the witnessing function φ. WLOG we can assume $\varphi(\alpha, \beta)(\alpha) = 0$, and $\varphi(\alpha, \beta)(\beta) = 1$ for every $(\alpha, \beta) \in X^2 \setminus \Delta$.

For every $\alpha < \omega_2$, $\varphi(\omega_2, \alpha)(\omega_2) = 0$, and $\varphi(\alpha, \omega_2)(\omega_2) = 1$.

Since ω_2 is a P-point there exists a neighborhood G'_α of ω_2 such that:

$\varphi(\beta, \alpha)(\gamma) = 0$ for all $\beta, \gamma \in G'_\alpha$.

Similarly, there exists a neighborhood G''_α of ω_2 such that:

$\varphi(\alpha, \beta)(\gamma) = 1$ for all $\beta, \gamma \in G''_\alpha$.

We set $G_\alpha = G'_\alpha \cap G''_\alpha$. Then $\varphi(\beta, \alpha)(\gamma) = 0$, and $\varphi(\alpha, \beta)(\gamma) = 1$ for every $\alpha < \omega$ and for all $\beta, \gamma \in G_\alpha$.

Let us say $C_\alpha = G^c_\alpha$. Note that $|C_\alpha| < \omega$ for all $\alpha < \omega_2$.

If we define the set function $F : X \to [X]^{<\omega}$ as follows: $F(\alpha) = C_\alpha$ by using the free set lemma, there exists a free set $S \subset X$ such that;

$|S| = |X| = \omega_2$ and $\alpha, \gamma \in S$ and $\alpha \neq \gamma$ implies $\alpha \notin F(\gamma) = C_\gamma$, and $\gamma \notin F(\alpha) = C_\alpha$.

Let's take $\alpha \neq \gamma \in S$. Then $\alpha \notin C_\gamma$ implies $\alpha \in G_\gamma$, and $\gamma \notin C_\alpha$ implies $\gamma \in G_\alpha$.

That is $\varphi(\alpha, \gamma)(\omega_2) = 0$, and $\varphi(\alpha, \gamma)(\omega_2) = 1$.

$\square$

Theorem 6.2. *$X^2 \setminus \Delta$ admits a partition $\mathbb{P}$ into clopen sets such that for every $P \in \mathbb{P}$ there exist A, B clopen subsets of X satisfying $P \subset A \times B \subset X^2 \setminus \Delta$ if and only if X is a continuously Urysohn space with the witnessing function $\varphi : X^2 \setminus \Delta \to C(X)$ such that $\varphi(x, y)(z) = 0$ or $\varphi(x, y)(z) = 1$ for every $(x, y) \in X^2 \setminus \Delta$, and $z \in X$.*

Proof. Suppose that there exists a partition with the mentioned properties. For every $(x, y) \in X^2 \setminus \Delta$, there is a $P \in \mathbb{P}$ such that $(x, y) \in P$.

Let us define $\varphi : X^2 \setminus \Delta \to C(X)$ as follows:

$\varphi(x, y)(z) = \chi_B(z)$, where $P \subset A \times B \subset X^2 \setminus \Delta$. Then,

(1) $\varphi(x, y)(x) = 0$, and $\varphi(x, y)(y) = 1$.

(2) $\varphi(x, y)$ is a continuous function for every $(x, y) \in X^2 \setminus \Delta$. Indeed,

Suppose $\epsilon > 0$, and $z \in X$ are given. For fixed (x, y);

If $(x, z) \in A \times B$ we set $G = B$. Then,

$\mid \varphi(x, y)(z) - \varphi(x, y)(t) \mid = \mid \chi_B(z) - \chi_B(t) \mid = 0$ for every $t \in G$.

If $(x, z) \notin A \times B$ we set $G = B^c$. Then,

$\mid \varphi(x, y)(z) - \varphi(x, y)(t) \mid = \mid \chi_B(z) - \chi_B(t) \mid = 0$ for every $t \in G$.

(3) φ is a continuous function. Indeed,

Suppose $\epsilon > 0$, and $(x, y) \in X^2 \setminus \Delta$ are given. We will set $U = \Pi_1(P) \subset A$, and $V = \Pi_2(P) \subset B$, where Π_1 is the first projection map from X^2 onto X, and Π_2 is the second projection map from X^2 onto X. Then,

$\mid \varphi(x, y)(z) - \varphi(u, v)(z) \mid = \mid \chi_B(z) - \chi_B(z) \mid = 0$.

We now suppose that X is a cU-space with the function φ, and $\varphi(x, y)(z) = 0$ or $\varphi(x, y)(z) = 1$ for every (x, y), and z. WLOG we may assume $\varphi(x, y)(x) = 0$, and $\varphi(x, y)(y) = 1$.

Let us fix $\epsilon = 1\backslash 2$, and pick $(x, y) \in X^2 \setminus \Delta$. By using the continuity of φ, there exists a neighborhood $U'_{xy} \times V'_{xy}$ of (x, y) such that

$\mid \varphi(x, y)(z) - \varphi(u, v)(z) \mid < 1\backslash 2$ for every $(u, v) \in U'_{xy} \times V'_{xy}$, and $z \in X$.

Since φ is a function which carries every $(x,y) \in X^2 \setminus \Delta$ to a function which is always 0 or 1 then $\varphi(x,y) = \varphi(u,v)$ for every $(u,v) \in U'_{xy} \times V'_{xy}$.

For a fixed $(x,y) \in X^2 \setminus \Delta$ we define the set $P = \{(u,v) \in X^2 \setminus \Delta : \varphi(u,v) = \varphi(x,y)\}$. Since $\varphi(x,y)$ and $\varphi(u,v)$ are continuous functions, P is closed. In addition to this, by using the fact that φ is a locally constant function, P is also open. It is clear that for any two sets P_1 and P_2 which are defined this way, either $P_1 = P_2$ or $P_1 \cap P_2 = \emptyset$. So, we have our partition $\mathbb{P}$.

Finally, for a point (x,y), the sets $(\varphi(x,y))^{-1}(0)$, and $(\varphi(x,y))^{-1}(1)$ are clopen sets. Also, $P = \{(u,v) \in X^2 \setminus \Delta : \varphi(u,v) = \varphi(x,y)\} \subset (\varphi(x,y))^{-1}(0) \times (\varphi(x,y))^{-1}(1)$.

$\square$

Although we do not have an example which is weakly continuously Urysohn, but not continuously Urysohn there is a candidate. The next example, due to Reed [15]. In [2] it is verified that X is a continuously symmetrizable space. That implies a zero-set diagonal, which implies wcU.

In [2] it is also proved that X is not submetrizable. Since submetrizability implies cU, there is still hope about this space not being cU.

Example 6.3. *Let* $X = X_0 \cup X_1 \cup U$*, where* $X_0 = \mathbb{R} \times \{0\}$*,* $X_1 = \mathbb{R} \times \{-1\}$*, and* $U = \mathbb{R} \times (0, \infty)$*. If* $x = (a,0) \in X_0$*, then* x' *denotes* $(a,-1) \in X_1$*. For* $n \in \mathbb{N}$*, and* $x = (a,0) \in X_0$ *we let* $V_n(x) = \{x\} \cup \{(s,t) \in U : (t = s - a) \wedge (0 < t < \frac{1}{n})\}$*, and* $V_n(x') = \{x'\} \cup \{(s,t) \in U : (t = a - s) \wedge (0 < t < \frac{1}{n})\}$*.*

The topology τ *on* X *is induced by isolating all elements of* U*, and using the collections* $\{V_n(x) : n \in \omega, n \geq 1\}$ *and* $\{V_n(x') : n \in \omega, n \geq 1\}$ *as bases of the topology at* x *and* x'*, respectively.*

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