A-Stability For Two Species Competition Diffusion Systems

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DISSERTATION ABSTRACT

A-STABILITY FOR TWO SPECIES COMPETITION DIFFUSION SYSTEMS

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My dissertation research focuses on establishing the structural stability of the attractor ($\mathcal{A}$-stability) via Morse-Smale property for diffusive two-species competition systems

$$\begin{cases} \partial_t u = k_1 \Delta u + u f(x,u,v), \quad x \in \Omega, \\ \partial_t v = k_2 \Delta v + v g(x,u,v), \quad x \in \Omega, \\ Bu = Bv = 0, \quad x \in \partial\Omega, \end{cases} \tag{0.0.1}$$

on a C^∞ bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 1$, with either Dirichlet or Neumann boundary conditions. Here $u(t,x),\ v(t,x)$ are the densities of two competing species, $k_1,\ k_2$ are diffusive constants and $(f,g), \quad f,g : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ \ C^2$ functions satisfying

(**H1**) $f(x,0,0) > 0,\ g(x,0,0) > 0\ \forall x \in \bar{\Omega}$,

(**H2**) $\partial_u f(x,u,v),\ \partial_v f(x,u,v),\ \partial_u g(x,u,v),\ \partial_v g(x,u,v) < 0,\ \forall\ u,v \geq 0,\ \forall x \in \bar{\Omega}$,

(**H3**) $\sup_{x\in\bar{\Omega},\ v\geq 0} \limsup_{u\to\infty} f(x,u,v) < 0$,

(**H4**) $\sup_{x\in\bar{\Omega},\ u\geq 0} \limsup_{v\to\infty} g(x,u,v) < 0$.

These hypotheses describe key features of competition models, and since u and v are the densities of two species, we are only interested in nonnegative solutions (u, v). We therefore consider (1.0.1) in the positive cone of some appropriate phase space. Our main result states for the spatially one-dimensional case that if (0.0.1) is a Morse-Smale system on the positive cone, it is structurally stable. We also establish that the set of functions (f, g) for which (1.0.1) possess the Morse-Smale property, is open in the space of all pairs (f, g) satisfying (**H1**)–(**H4**) under the topology of C^2-convergence on compacta. Moreover, we show as a sufficient condition that if all critical elements of (1.0.1) are hyperbolic with one-dimensional unstable manifolds in case of equilibria (except 0) and two-dimensional unstable manifolds in case of periodic orbits, then system (1.0.1) has the Morse-Smale property. These results will have significant impact on the study of the asymptotic dynamics of various classes of discretizations of (0.0.1).

The proof of the openness of the set of functions for which (1.0.1) is a Morse-Smale system, is an adaption of an idea used in [23], to a positive cone setting. As for a sufficient condition under which the system has the Morse-Smale property, we are able to prove the transversality of unstable manifolds and local stable manifolds of critical elements under the hypotheses mentioned before.

The proof of the main result is quite technically difficult because we have to work in a positive cone setting. This proof can be broken down into a few main steps. First of all, since the long-term features of the dynamics of (1.0.1) are determined by the global attractor, which lies inside a sufficiently large ball, we can reduce (1.0.1) to a finite dimensional system by means of Chow, Lu & Sell's inertial manifold theorem (cf.[6]). Secondly, we prove that the finite-dimensional system obtained in first step is also a Morse-Smale system. Thirdly, we establish the $\mathcal{A}$-stability of the global attractor (in the positive cone) of the finite-dimensional system. The proof is an adaption of the main idea of a corresponding result in the monograph [13] by J. Hale, L. Magalhães, & W.Olivia. As mentioned before, their result does not apply to our problem because we work on a subset of the positive cone which cannot be considered to be a Banach manifold imbedded in a Banach space, the setting underlying the work of J. Hale, L. Magalhães, & W.Olivia. Since the attractor of the finite-dimensional system is the orthogonal projection of the global attractor of (1.0.1) (in the positive cone) to the phase space of the finite-dimensional system, the final step requires to derive the $\mathcal{A}$-stability of the global attractor of (1.0.1) (in the positive cone) from $\mathcal{A}$-stability of the global attractor of finite-dimensional system.

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Chapter 1

Introduction

A fundamental goal of theoretical ecology is to understand how the interactions of individual organisms with each other and with the environment affect the distribution and structure of populations. One way to achieve this goal is to use mathematical models. Among these models, reaction-diffusion models have received great attention from many experts in the field such as Fisher (1939), Skellam (1951), Kierstead and Slobodkin (1953), Fife (1979), Smoller (1982), Murray (1993), Grindrod (1996), Leung (1989), Hess (1991), Pao (1992), Hassel (1994), Cantrell and Cosner (1996),....

The reaction part of such models is derived from Lotka-Volterra models which in turn are frequently based on the logistic model of population growth. The latter was first suggested by Verhulst (1838) to describe the growth of human populations and was later derived independently by Pearl and Reed (1920) for modeling the population growth in the United States. Logistic models are based on the assumption that the growth of a population is determined by two factors, the reproduction rate (which is the difference of birth and death rates of individuals) and the limitation of the habitat's resources. The simplest logistic model of population growth leads to the following equation

$$\dot{u} = ru\left(1 - \frac{u}{C}\right),$$

where u stands for the population density, r is the growth rate and C (which is called carrying capacity) represents the maximum number of individuals that can be sustained by the resources of the habitat. It is obvious that over a long period of time, the size of the population will approach C. The first model of interacting species were introduced by Lotka (1925) and Volterra (1931). Derived from the logistic model of population growth, a Lotka-Volterra competition model for two species has the form

$$\begin{cases} \dot{u} = u(a_1 - b_1 u - c_1 v), \\ \dot{v} = v(a_2 - b_2 u - c_2 v), \end{cases}$$

where $u,\ v$ denotes the population densities of the two species. The extra terms $c_1 uv$, $b_2 uv$ added to the logistic equation of each species represent the negative effect which one species has on the other one due to competition for the habitat's resource. This type of model has also been studied thoroughly, and in general, the model predicts either competitive exclusion (extinction of the weaker competitor or the initially disadvantaged species) or stable coexistence of the two competing species. It appears that the predictions of the model correspond to biologically realistic situations although some of its assumptions are rather idealized, for example, the assumption of spatial homogeneity, the linear competition between the two species, ... However, this model has drawn enormous amount of empirical and theoretical research since its introduction because of its great value for ecological understanding,

In reality, individuals are not distributed homogeneously in their habitat and typically interact with both the physical environment and other individuals in their neighborhood. Therefore, there is the need for expanding Lotka-Volterra models by taking spatial dependence into account. It is a natural first attempt to assume that migration occurs from regions of higher population density towards regions of lower density, and Skellam (1951) has used an random walk approach to argue that, in fact, the dispersal of a population should be modeled by diffusion, which suggests the following extension of the above Lotka-Volterra competition model

$$\begin{cases} \partial_t u = k_1 \Delta u + u(a_1 - b_1 u - c_1 v), \ x \in \Omega, \\ \partial_t v = k_2 \Delta v + v(a_2 - b_2 u - c_2 v), \ x \in \Omega, \\ Bu = Bv = 0, \quad x \in \partial\Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a C^∞ bounded domain and B stands for either Dirichlet or Neumann boundary conditions which describe certain restrictions of the movement of the two species at the boundary of the analyzed region. E.g., homogeneous Neumann boundary conditions apply to isolated habitats with no migration through the boundary.

The long-term behavior of such systems is well understood. For example, a study by Brown (1980) for the Neumann case showed that if

$$a_1 c_2 - c_1 a_2 > 0 \ \text{ and } \ b_1 a_2 - a_1 b_2 > 0$$

then any solution with positive initial value converges over time to the spatially homogenous positive solution

$$\left(\frac{a_1 c_2 - c_1 a_2}{b_1 c_2 - c_1 b_2}, \frac{b_1 a_2 - a_1 b_2}{b_1 c_2 - c_1 b_2} \right)$$

which means the two species coexist. On the other hand, if

$$a_1 c_2 - c_1 a_2 > 0 \ \text{ but } \ b_1 a_2 - a_1 b_2 < 0$$

then solutions with positive initial value converges over time to the spatially homogenous positive solution

$$\left(\frac{a_1}{b_1}, 0\right)$$

which predicts the extinction of species 2. Clearly, reproduction rates, carrying capacities and competition rates may also vary throughout the habitat and in time (seasonal effects). This suggests to study more general reaction terms, space and possibly time dependent ones. Many interesting results have been obtained addressing such issues, and we mention Leung (1980), Pao (1981), Pao & Zhou (1982), Smith & Thieme (2001), Dancer & Zhang (2002),...

The starting point of my Ph.D. research work was the simulation of the long-term behavior of two competing species which populate overlapping, but different spatially heterogeneous habitats. How trustworthy are such findings considering that the issue is not the approximation of a solution of an initial value problem on a finite time interval, but the asymptotic behavior on an infinite time-interval. A paper [2] by S.M. Bruschi, A.N. Carvalho and J.G. Ruas-Filho addresses this issue for one-dimensional parabolic equation. The authors establish the dynamical equivalence of the flows on the attractor of the continuous problem and the attractor of the spatially discretized problem, respectively.

There are quite a few obstacles to extending this result. The most significant one which needs to be addressed first, is the question of structural attractor stability. In the case of a parabolic equation this issue is linked to the concept of gradient system, but this concept does not apply to parabolic systems in general, and in particular, not to systems under consideration.

My thesis research therefore focuses on establishing this structural stability of the attractor ($\mathcal{A}$-stability) for two-species competition systems with diffusion

$$\begin{cases} \partial_t u = k_1 \Delta u + u f(x,u,v), \quad x \in \Omega, \\ \partial_t v = k_2 \Delta v + v g(x,u,v), \quad x \in \Omega, \\ Bu = Bv = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.0.1}$$

on a C^∞ bounded domain $\Omega \subset \mathbb{R}^n$ with either Dirichlet or Neumann boundary conditions. Here $u(x,t)$, $v(x,t)$ are the densities of two competing species, $k_1,\ k_2$ are diffusion constants (called dispersal rates in the ecological literature), and $f,\ g$ are smooth functions on $\bar{\Omega}\times\mathbb{R}\times\mathbb{R}$ satisfying certain properties preserving features of competition models. Ecologically, one is only interested in nonnegative solutions (u,v).

It turns out that in order to understand the relationship between the dynamics of (1.0.1) and of certain classes of discretizations for (1.0.1), it is essential to investigate the structural stability of the global attractor of (1.0.1) on a positive cone. Obviously, one cannot expect that numerical approximations reflect the asymptotic behavior of a solution semi-flow, if the behavior is not "generically robust" against small perturbations.

The concept of structural stability was introduced by Andronov and Pontryagin (1937). Since then, a systematic theory of structural stability for diffeomorphisms and vector fields on manifolds has been well developed by M.M. Peixoto (1959), S. Smale (1967), D.V. Asonov (1967), C.Pugh (1967), J. Moser (1969), J. Palis (1969), J. Palis & S. Smale (1970), J.Robin (1971), C. Robinson (1976), S. T. Liao (1980), S. Newhouse (1980), J. Hale (1981), R.Mañé (1988), M. Hirsch (1990),... In 1984, the weaker notion of $A-$stability (attractor stability) which is more suitable in the infinite-dimensional case and for the numerical issue mentioned before, was introduced by J. Hale L. Magalhães & W.Olivia in [13].

The concept of Morse-Smale structure emerges as a sufficient condition for structural stability. Classically, Morse-Smale system refers to systems which have a finite number of critical elements, all of which are hyperbolic and satisfy a transversality condition when their stable and unstable manifolds intersect.

One of the celebrated results due to the J. Palis & S. Smale states that if a $C^r(r \geq 1)$ diffeomorphism on a compact C^∞ manifold without boundary is Morse-Smale, then it is structurally stable. In [13], J. Hale L. Magalhães & W.Olivia proved that any $f \in KC^r(B, B)$ which is Morse-Smale, is A-stable. Here, B is a Banach manifold imbedded in a Banach space E and the choice of the classes $KC^r(B, B)$ depends on the problems under consideration. Another important result in this context is due to Kennig Lu [22] who proved the "structural stability on a neighborhood of the attractor" of scalar parabolic equations.

Typically, Morse-Smale systems have been defined in the context of (Banach) manifolds with or without boundaries (see [26], [27], [24], [25],...), but positive cones do not fall into these categories. Therefore we first need to modify the classical concepts of Morse-Smale system and structural stability in such a way that they apply to positive cone settings. Since we are only interested in positive solutions, we will consider nonlinearities in the set of pairs (f, g), $\quad f, g : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \;\; C^2$ functions satisfying

(H1) $f(x, 0, 0) > 0, \;\; g(x, 0, 0) > 0 \;\forall x \in \bar{\Omega}$,

(H2) $\partial_u f(x, u, v), \;\; \partial_v f(x, u, v), \;\; \partial_u g(x, u, v), \;\; \partial_v g(x, u, v) < 0 \;\forall\; u, v \geq 0, \;\; \forall x \in \bar{\Omega}$,

(H3) $\sup_{x\in\bar{\Omega}, \; v\geq 0} \limsup_{u\to\infty} f(x, u, v) < 0$, (**H4**) $\sup_{x\in\bar{\Omega},\ u\geq 0} \limsup_{v\to\infty} g(x,u,v) < 0$.

Our main result states for the spatially one-dimensional case that if (1.0.1) is a Morse-Smale system on a positive cone, it is structurally stable. We also provide the sufficient condition under which the system has the Morse-Smale property. These results will have significant impact on the study of the asymptotic dynamics of various classes of discretizations of (1.0.1).

To obtain these results, we view (1.0.1) as an evolution equation on a suitable fractional power space. Under the conditions imposed on the nonlinearities, we have global existence of solutions and hence existence of a global attractor. Since the long-term features of the dynamics of the system are determined by the global attractor, which lies inside a sufficiently large ball, we first reduce (1.0.1) to a finite dimensional system by means of Chow, Lu & Sell's inertial manifold theorem [6]. Recall that an inertial manifold $\mathcal{I}$ is a subset of the phase space satisfying the following properties

1. $\mathcal{I}$ is a finite dimensional smooth manifold
2. $\mathcal{I}$ is invariant under the semi-flow generated by (1.0.1)
3. $\mathcal{I}$ is exponentially attracting solutions.

Next, we need to prove $\mathcal{A}$-stability in the (finite dimensional) inertial manifold setting. The proof adapts the main idea, J. Hale, L. Magalhães, & W.Olivia used in [13]. As mentioned before, their result cannot be applied to our problem because we work on a subset of the positive cone and not on a Banach manifold which is imbedded in another Banach space.

After having established the structural stability of (1.0.1) in this dissertation, we intend to investigate long-term aspects of various classes of numerical approximation schemes to (1.0.1) in the future work.

Having solved one problem in mathematics leads usually to an array of new questions. Let me just mention a few. Obviously, it will be an important task to address the same questions as considered here for spatially higher dimensional cases. The key obstacle arises from the fact that one cannot utilize C^1-inertial manifolds as a reduction tool since they rarely exist in higher dimensions.

Two important issues in connection with the original ecological problem arise: How does the size of the overlapping region of the two habitats affects coexistence and extinction. Mathematically speaking, one is lead to a peculiar bifurcation problem for the steady state system associated with (1.0.1). What is the impact of seasonal effects, a question, which in the general case leads to systems similar to (1.0.1), but with time-dependent reaction terms and on domains which vary in time.

Chapter 2

Definitions, Notations and Main Results

2.1 General Semiflows

Definition 2.1.1. *(local semi-flow, semi-flow)*

(1) Let (Y, d) be a metric space. A map $T: \mathcal{D}(T) \subset \mathbb{R}^+ \times Y \to Y$ is said to be a local semi-flow on Y if for each $y \in Y$, there is $\tau(y) > 0$ such that $[0, \tau(y)) \times \{y\} \subset \mathcal{D}(T)$, $(t, y) \notin \mathcal{D}(T)$ for any $t \geq \tau(y)$ (i.e. $[0, \tau(y))$ is the maximal interval of existence of the local semi-flow with initial condition y at $t = 0$), and the following hold,

(i) $T_0 = Id$,

(ii) Given $y \in Y$, and $t_1, t_2 \geq 0$, if $(t_1 + t_2, y) \in \mathcal{D}(T)$, then $(t_1, T_{t_2}(y)) \in \mathcal{D}(T)$ and $T_{t_1+t_2}(y) = T_{t_1} \circ T_{t_2}(y)$,

(iii) $T_t(x)$ is continuous in t, x for $(t, x) \in \mathcal{D}(T)$,

where $T_t(y) = T(t, y)$. Furthermore, if for each $y \in Y$, $\tau(y) = \infty$, then T is called a semi-flow on Y.

(2) Let T be a (local) semi-flow on Y and $r \in \mathbb{N}$. If Y is a Banach space and T satisfies the following additional property

(iv) $T_t(x)$ is continuous in t, x together with Fréchet derivatives in x up through order r for $(t, x) \in \mathcal{D}(T)$,

then T is called a (local) C^r semi-flow.

Definition 2.1.2. *(hyperbolicity)* Let (Y,d) be a metric space and $T : \mathcal{D}(T) \subset \mathbb{R}^+ \times Y \to Y$ be a local semi-flow on Y.

(i) $\alpha \in Y$ is a fixed point if $\tau(\alpha) = \infty$, $T_t(\alpha) = \alpha$ for all $t \geq 0$. A fixed point α of the semi-flow T is said to be hyperbolic if the spectrum $\sigma(DT_t(\alpha))$ of the Fréchet derivative $DT_t(\alpha)$ is disjoint from the unit circle of the complex plane for all $t > 0$.

(ii) $\alpha \subset Y$ is a periodic solution of period σ if for any $p \in \alpha$, $\tau(p) = \infty$, $T_{t+\sigma}(p) = T_t(p)$ for all $t \geq 0$, and $\{T_t(p) |\ t \in [0,\infty)\} = \alpha$. A periodic solution α of period σ of the semi-flow T is said to be hyperbolic if for any $p \in \alpha$, $\lambda = 1$ is a simple eigenvalue of $DT_\sigma(p)$ and the spectrum set $\sigma(DT_\sigma(p)) \setminus \{1\}$ of the Fréchet derivative $DT_\sigma(p)$ is disjoint from the unit circle of the complex plane.

(iii) A critical element of the semi-flow T is either a fixed point or a periodic solution.

Definition 2.1.3. Let (Y,d) be a metric space and $T : \mathcal{D}(T) \subset \mathbb{R}^+ \times Y \to Y$ be a local semi-flow on Y. We say that $y \in Y$ has a global backward extension with respect to T if there exists a continuous function $\varphi : (-\infty, \tau(y)) \to Y$ such that $\varphi(0) = y$ and $T_t(\varphi(s)) = \varphi(s+t)$ for all $t > 0$ and $t + s < \tau(y)$. If y has a global backward extension, we will write $T_{-t}(x)$ for $\varphi(-t)$, $t > 0$. The set $\bigcup_{\tau \in (-\infty, \tau(y))} \varphi(\tau)$ is called the *global orbit* of y with respect to T.

Definition 2.1.4. *(stable, local stable, unstable, local unstable manifolds)*

Let (Y,d) be a metric space and $T : \mathcal{D}(T) \subset \mathbb{R}^+ \times Y \to Y$ be a local semi-flow on Y. The stable, local stable, unstable, local unstable manifolds at a hyperbolic critical element α of the semi-flow T, denoted by $W^s(\alpha)$, $W^s_{\text{loc}}(\alpha)$, $W^u(\alpha)$, $W^u_{\text{loc}}(\alpha)$ are defined as follows

$$W^s(\alpha) = \{x \in Y \mid \tau(x) = \infty,\ d(T_t(x), \alpha) \to 0 \text{ as } t \to \infty\},$$

$$W^s_{\text{loc}}(\alpha) = \{x \in W^s(\alpha) \mid T_t(x) \in B\ \forall t \geq 0\},\ B \text{ is some neighborhood of } \alpha,$$

$$W^u(\alpha) = \{x \in Y \mid x \text{ has a global backward extension with respect to } T$$
$$\text{and } d(T_{-t}(x), \alpha) \to 0 \text{ as } t \to \infty\},$$
$$W^u_{\text{loc}}(\alpha) = \{x \in W^u(\alpha) \mid T_{-t}(x) \in B \ \forall t \geq 0\}, \ B \text{ is some neighborhood of } \alpha.$$

Definition 2.1.5. *(Tubular family for flows)* Let W be a finite dimensional Banach space and $r \in \mathbb{N}$. Consider a C^r semi-flow $\gamma : \mathcal{D}(\gamma) \subset \mathbb{R}^+ \times W \to W$ on W.

(1) Let α be a fixed point of γ. A tubular family of $W^s(\alpha)$, denoted by $\Gamma^s(\alpha)$, is a collection of disjoint C^r-submanifolds (called leaves) of W, denoted by $\{\Gamma^s_y\}$ or $\{\Gamma^s_y(\alpha)\}$ for clarity, indexed by y in an open neighborhood N of α in $W^u(\alpha)$ with the following properties

  a. $\Gamma^s(\alpha) = \bigcup_{y \in N} \Gamma^s_y$ is an open set of W containing $W^s(\alpha)$,

  b. $\Gamma^s_\alpha = W^s(\alpha)$,

  c. Γ^s_y intersects N transversally at y,

  d. The map $\Gamma^s(\alpha) \to N$, $\Gamma^s_y \mapsto y$ is continuous; the section s which sends $x \in \Gamma^s_y$ into the tangent space of Γ^s_y at x is a continuous map from $\Gamma^s(\alpha)$ into the Grassmann bundle over $\Gamma^s(\alpha)$.

(2) Let α be a periodic orbit of γ with period σ and S be a cross-section of α at $p \in \alpha$. S is called invariant if $\gamma_\sigma(U) \subset S$, where U is a neighborhood of $p \in S$. Let k be the restiction of γ_σ to U, $k := \gamma_{\sigma|_U} : U \to S$, and $\Gamma^s_y,\ y \in U$ be an invariant tubular family of $W^s(p)$ (with respect to k). The tubular family of $W^s(\alpha)$ is now defined by $\Gamma^s(\gamma_t(y)) = \gamma_t(\Gamma^s_y)$ for $y \in U$ and all $t \geq 0$.

(3) A system of tubular families of γ is a set of tubular families of fixed point(s) and periodic orbit(s). It is compatible if given $\Gamma^s_x(\alpha) \cap \Gamma^s_y(\beta) \neq \emptyset$ (where α, β are two different critical elements of γ), then one submanifold contains the other.

2.2 Semiflows Generated by Competition Models

Definition 2.2.1. Let Ω be a C^∞ bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $X \subset L^p(\Omega)$ $(p > n)$ be a fractional power space of $-\Delta : \mathcal{D}(\Delta) \to L^p(\Omega)$, see [15], satisfying $X \hookrightarrow C^1(\bar{\Omega})$, where $\mathcal{D}(\Delta) = \{u \in H^{2,p}(\Omega) |\ Bu = 0 \text{ on } \partial\Omega\}$. Here, $H^{2,p}(\Omega)$, $p \geq 1$ are the well-known Sobolev spaces (cf.[1]). Under the assumptions (**H1**)-(**H4**), (1.0.1) generates a (local) semi-flow on $X \times X$ (cf. [15]), we denote it by Π,

$$\Pi : \mathcal{D}(\Pi) \subset \mathbb{R}^+ \times X \times X \to X \times X$$

$$\Pi_t(u_0, v_0) := \Pi(t, u_0, v_0) = (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)), \quad t \in \tau(u_0, v_0),$$

where $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ is the solution of (1.0.1) with

$$(u(0, \cdot; u_0, v_0), v(0, \cdot; u_0, v_0)) = (u_0, v_0),$$

and $[0, \tau(u_0, v_0))$ is the maximal interval of existence of solution of (1.0.1) with initial condition (u_0, v_0) at $t = 0$. For clarity, we may write Π^{fg}_t instead of Π_t.

As mentioned in Chapter 1, due to the nature of (1.0.1), we are only interested in nonnegative solutions of (1.0.1). Therefore we introduce the positive cone $X_+ \times X_+$ of $X \times X$,

$$X_+ \times X_+ := \{(u, v) \in X \times X |\ u \geq 0,\ v \geq 0 \text{ on } \Omega\}.$$

A solution $(u(t,x), v(t,x))$ is called nonnegative if $(u(t,\cdot), v(t,\cdot)) \in X_+ \times X_+$ for $t \in [0, \tau(u(\cdot,0), v(\cdot,0))$. It can be proved that if $(u_0, v_0) \in X_+ \times X_+$, then $\tau(u_0, v_0) = \infty$ and $\Pi_t(u_0, v_0) \in X_+ \times X_+$ for all $t \geq 0$ (see Proposition 3.3.1). Hence Π_t $(t \geq 0)$, restricted to $X_+ \times X_+$ is a semi-flow.

The restriction to $X_+ \times X_+$ of stable, local stable, unstable, and local unstable manifolds of a critical element α of Π are denoted by

$$W^{u+}(\alpha) = W^u(\alpha) \cap (X_+ \times X_+),$$

$$W^{u+}_{\mathrm{loc}}(\alpha) = W^u_{\mathrm{loc}}(\alpha) \cap (X_+ \times X_+),$$

$$W^{s+}(\alpha) = W^s(\alpha) \cap (X_+ \times X_+),$$

$$W^{s+}_{\mathrm{loc}}(\alpha) = W^s_{\mathrm{loc}}(\alpha) \cap (X_+ \times X_+).$$

Definition 2.2.2. *(Global attractors and Non-wandering sets)*

Let X and Π^{fg} be defined as in definition 2.2.1.

a. We define the global attractor of Π^{fg}, denoted by $\mathcal{A}$ (or $\mathcal{A}(f,g)$), to be the set

$$\{(u,v) \in X_+ \times X_+ \mid (u,v) \text{ has a bounded global orbit}\}.$$

b. An element $(u,v) \in \mathcal{A}$ is called a non-wandering point if , for any neighborhood U of (u,v) in $\mathcal{A}$ and any $T > 0$, there exists $t_0 = t_0(U,T) > T$ and $(\tilde{u}, \tilde{v}) \in U$ such that $\Pi^{fg}_{t_0}(\tilde{u}, \tilde{v}) \in U$. The non-wandering set, denoted by $\Omega(f,g)$, is the set of non-wandering points.

Definition 2.2.3. *(competition order)* Given (u_1, v_1), $(u_2, v_2) \in X_+ \times X_+$, we write

$$(u_1, v_1) \leq_2 (u_2, v_2) \quad \text{if} \quad u_1 \leq u_2,\ v_2 \leq v_1,$$

$$(u_1, v_1) <_2 (u_2.v_2) \quad \text{if} \quad (u_1, v_1) \leq_2 (u_2, v_2),\ (u_1, v_1) \neq (u_2, v_2),$$

$$(u_1, v_1) \ll_2 (u_2, v_2) \quad \text{if} \quad (u_2 - u_1, v_1 - v_2) \in \text{Int}X_+ \times \text{Int}X_+.$$

Definition 2.2.4. A hyperbolic critical element α in $X_+ \times X_+$ of (1.0.1) is called a source if $W^{s+}(\alpha) \cap \mathcal{A} = \alpha$ and a sink if $W^{u+}(\alpha) = \alpha$, otherwise α is said to be a saddle.

Definition 2.2.5. *(fundamental domains and fundamental neighborhoods)*

Let $\alpha \in X_+ \times X_+$ be a hyperbolic critical element of (1.0.1).

a. If α is not a sink , a fundamental domain for $W^{u+}_{\text{loc}}(\alpha)$ is denoted by $G^{u+}(\alpha)$ and is defined as $G^{u+}(\alpha) = \partial B(\alpha)$ for some open disk $B(\alpha) \subset W^{u+}_{\text{loc}}(\alpha)$ (that is $B(\alpha)$ is a disk centered at α if α is a fixed point or a tubular neighborhood of α if α is a periodic orbit). Any neighborhood $N^{u+}(\alpha)$ in $\mathcal{A}$ of $G^{u+}(\alpha)$ such that $N^{u+}(\alpha) \cap W^{s+}(\alpha) = \emptyset$ is called fundamental neighborhood for $W^{u+}(\alpha)$.

b. If α is not a source , a fundamental domain for $W^{s+}_{\text{loc}}(\alpha)$ is denoted by $G^{s+}(\alpha)$ and is defined as $G^{s+}(\alpha) = \partial B(\alpha) \cap \mathcal{A}$ for some open disk $B(\alpha) \subset W^{u+}_{\text{loc}}(\alpha)$ (that is $B(\alpha)$ is a disk centered at α if α is a fixed point or a tubular neighborhood of α if α is a periodic orbit). Any neighborhood $N^{s+}(\alpha)$ in $\mathcal{A}$ of $G^{s+}(\alpha)$ such that $N^{s+}(\alpha) \cap W^{u+}(\alpha) = \emptyset$ is called fundamental neighborhood for $W^{s+}(\alpha)$.

Definition 2.2.6. *(Morse-Smale structure)* We say (1.0.1) has Morse-Smale structure *if*

a. the semi-flow $\Pi_t|_{X_+\times X_+}$ has a finite number of critical elements (i.e. fixed points or periodic solutions), all of which are hyperbolic and their union coincides with the non-wandering set $\Omega(f,g)$,

b. if α, β are two critical elements of $\Pi_t|_{X_+\times X_+}$, their unstable manifolds are finite-dimensional, and the global unstable manifold $W^{u+}(\alpha)$ and the local stable manifold $W^{s+}_{\text{loc}}(\beta)$ either do not intersect or they intersect transversally (i.e. $x \in W^{u+}(\alpha)\cap W^{s+}_{\text{loc}}(\beta)$ implies $T_xW^{u+}(\alpha)\oplus T_xW^{s+}_{\text{loc}}(\beta) = X\times X$).

Definition 2.2.7. Let

$$\mathcal{CP} := \{(f,g)|\ f,g : \bar{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\ C^2 \text{ functions},\ f,g \text{ satisfy } (\mathbf{H1})\text{-}(\mathbf{H4})\}.$$

We define a metric on $\mathcal{CP}$, denoted by $d_{\mathcal{CP}}$, as follows

$$\rho_N((f,g),(\tilde f,\tilde g)) := \|(f,g)-(\tilde f,\tilde g)\|_{C^2\left(\bar\Omega\times[-N,N]\times[-N,N]\right)},$$

$$d_{\mathcal{CP}}((f,g),(\tilde f,\tilde g)) := \sum_{N=1}^{\infty}\frac{(\rho_N((f,g),(\tilde f,\tilde g))}{2^N(1+\rho_N((f,g),(\tilde f,\tilde g)))}.$$

Definition 2.2.8. We define the Morse-Smale set as follows

$$\mathcal{MS} := \{(f,g)\in\mathcal{CP}|\ (1.0.1) \text{ has Morse-Smale structure}\}.$$

Definition 2.2.9. *($\mathcal{A}$-stability)* Given $(f_0,g_0)\in\mathcal{CP}$. System (1.0.1) is $\mathcal{A}$-stable if there exists an $\varepsilon_0>0$ such that for each $(f,g)\in\mathcal{CP}$, $d_{\mathcal{CP}}((f,g),(f_0,g_0))<\varepsilon_0$, there exists a homeomorphism

$$H:\ \mathcal{A}(f_0,g_0)\to\mathcal{A}(f,g)$$

which takes trajectories of $\mathcal{A}(f_0, g_0)$ to trajectories of $\mathcal{A}(f, g)$ and preserves the sense of direction in time.

2.3 Main Results

The following are the main results of the dissertation and are stated for the positive cone setting.

Theorem A. *The set $\mathcal{MS}$ (cf. Definition 2.2.8) is open (in $\mathcal{CP}$).*

Theorem B. *Given $(f, g) \in \mathcal{CP}$. Assume all the critical elements of* (1.0.1) *are hyperbolic and their union coincides with the non-wandering set $\Omega(f, g)$. Furthermore, suppose that the dimension of the unstable manifold of an equilibrium solution in $X_+ \times X_+ \setminus \{(0, 0)\}$ is at most one and the dimension of the unstable manifold of a periodic solution is at most two, then* (1.0.1) *has the Morse-Smale structure.*

Theorem C. *Let $\Omega = (0, 1)$. If $(f_0, g_0) \in \mathcal{MS}$, then* (1.0.1) *is $\mathcal{A}$-stable.*

Chapter 3

Preliminary results

3.1 General Semiflows

Definition 3.1.1. Let (Y, d) be a complete metric space, Λ be a metric space and $T^\lambda : \mathbb{R} \times Y \to Y$ be a continuous semi-flow for each $\lambda \in \Lambda$. The semi-flow T^λ is *asymptotically smooth* if, for any nonempty, closed, bounded set $B \subset Y$ for which $T_t^\lambda(B) \subset B$ $\forall t \geq 0$, there is a compact set $J_\lambda(B) \subset B$ such that $J_\lambda(B)$ attracts B under T_t^λ, that is, $d(T_t(B), J_\lambda(B)) \to 0$ as $t \to \infty$. We say the family of semi-flows $\{T^\lambda\}_{\lambda \in \Lambda}$, is *collectively asymptotically smooth* if $\overline{\bigcup_{\lambda \in \Lambda} J_\lambda(B)}$ is compact.

Theorem 3.1.1. Let (Y, d) be a complete metric space, Λ be a metric space and $T^\lambda : \mathbb{R} \times Y \to Y$, $t \geq 0$, is a semi-flow, for each $\lambda \in \Lambda$. Suppose that

(i) There is a bounded set $B \subset Y$ independent of λ such that B attracts points of Y under T_t^λ, that is, $d(T_t^\lambda(y), B) \to 0$ as $t \to \infty$, $\forall y \in Y$.

(ii) For any bounded set U, the set $V := \bigcup_{\lambda \in \Lambda} \bigcup_{t \geq 0} T_t^\lambda U$ is bounded,

(iii) The family of semi-flows is collectively asymptotically smooth.

Then the global attractor A_λ of T_λ is upper semicontinuous in λ.

Proof. This is Theorem 3.5.3 in [12]. □

Lemma 3.1.2. *(see [15])* Let A be a sectorial operator in a Banach space Y with $\operatorname{Re} \sigma(A) > \delta > 0$ and Y^α, Y^β $(0 \leq \alpha \leq \beta < 1)$ denote the fractional power spaces of Y and $f : Y^\alpha \to Y$

be locally Lipschitzian. Denoted by $w(t;t_0,w_0)$ the unique solution of

$$\begin{cases} \dot{w} + Aw = f(w), \ t > t_0, \\ w(t_0) = w_0 \in Y^\alpha. \end{cases} \tag{3.1.1}$$

Then there exist constants $C_1,\ C_2$ depending only on $\alpha,\ \beta$ such that

$$\begin{aligned} \|w(t;t_0,w_0)\|_{Y^\beta} \ \leq \ & C_1(t-t_0)^{-(\beta-\alpha)}e^{-\delta(t-t_0)}\|w_0\|_{Y^\alpha}+ \\ & C_2\int_{t_0}^t (t-s)^{-\beta}e^{-\delta(t-s)}\|f(w(s))\|_Y \ ds, \ \forall t_0 < t < \tau(t_0,w_0), \end{aligned}$$

where $[t_0,\tau(t_0,w_0))$ is the maximal interval of existence o solution of (3.1.1) with initial condition w_0 at $t=t_0$.

Proof. Using the variational representation of w, we have

$$\begin{aligned} w(t;t_0,w_0) &= e^{-A(t-t_0)}w_0 + \int_{t_0}^t e^{-A(t-s)}f(w(s)) \ ds \\ A^\beta w(t;t_0,w_0) &= A^\beta e^{-A(t-t_0)}w_0 + \int_{t_0}^t A^\beta e^{-A(t-s)}f(w(s)) \ ds \\ \|w(t;t_0,w_0)\|_{Y^\beta} &\leq \|A^{\beta-\alpha}e^{-A(t-t_0)}\|_Y\|A^\alpha w_0\|_Y + \int_{t_0}^t \|A^\beta e^{-A(t-s)}\|_Y\|f(w(s))\|_Y \ ds \\ &\leq C_1(t-t_0)^{-(\beta-\alpha)}e^{-\delta(t-t_0)}\|w_0\|_{Y^\alpha} + \\ & \quad C_2\int_{t_0}^t (t-s)^{-\beta}e^{-\delta(t-s)}\|f(w(s))\|_Y \ ds, \ \forall t_0 < t < \tau(t_0,w_0). \end{aligned}$$

□

3.2 Single Species Equation

Consider the boundary value problem

$$\begin{cases} \varphi_t = k\Delta\varphi + \varphi h(x,\varphi) \text{ on } \Omega, \\ B\varphi = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.2.1}$$

on a C^∞ bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with either Dirichlet or Neumann boundary conditions. Here $\varphi(x,t)$ is the density of certain species, k is diffusive constant, and h is a C^2 function $h : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfying

(**h1**) $h(x,0) > 0 \ \forall x \in \bar{\Omega}$,

(**h2**) $\partial_\varphi h(x,\varphi) < 0 \ \forall \ \varphi \geq 0, \ \forall x \in \bar{\Omega}$,

(**h3**) $\sup_{x\in\bar{\Omega}} \limsup_{\varphi\to\infty} h(x,\varphi) < 0$.

Definition 3.2.1. Let $\mathcal{L} := \{h : \bar{\Omega} \times \mathbb{R} \to \mathbb{R} \ C^2 \text{ function} \mid \ h \text{ satisfy } (\mathbf{h1})\text{-}(\mathbf{h3})\}$. We define a metric on $\mathcal{L}$, denoted by $d_{\mathcal{L}}$, as following

$$d_N(h,\tilde{h}) := \|h - \tilde{h}\|_{C^2(\bar{\Omega}\times[-N,N])},$$

$$d_{\mathcal{L}}(h,\tilde{h}) := \sum_{N=1}^{\infty} \frac{d_N(h,\tilde{h})}{2^N(1 + d_N(h,\tilde{h}))}.$$

Definition 3.2.2. Let $X \subset L^p(\Omega)$ $(p > n)$ be a fractional power space of $-\Delta : \mathcal{D}(\Delta) \to L^p(\Omega)$, see [15], satisfying $X \hookrightarrow C^1(\bar{\Omega})$, where $\mathcal{D}(\Delta) = \{\varphi \in H^{2,p}(\Omega) | \ B\varphi = 0 \text{ on } \partial\Omega\}$. Under the assumptions (**h1**)-(**h3**), (3.2.1) generates a (local) semi-flow on X (cf. [15]), we denote it by π,

$$\pi : D(\pi) \subset \mathbb{R}^+ \times X \to X$$

$$\pi_t(\varphi_0) := \pi(t, \varphi_0) = \varphi(t, \cdot; \varphi_0) \quad for \quad t \in [0, \tau(\varphi_0)),$$

where $\varphi(t, \cdot; \varphi_0)$ is the solution of (3.2.1) with $\varphi(0, \cdot; \varphi_0) = \varphi_0$ and $[0, \tau(\varphi_0))$ is the maximal interval of existence of solution of (3.2.1) with initial condition φ_0) at $t = 0$.

Proposition 3.2.1. For any $\varphi_0 \in X_+$, the solution $\pi_t(\varphi_0)$ of (3.2.1) with initial φ_0 exists and $\pi_t(\varphi_0) \in X_+$ for all $t \geq 0$.

Proof. For any $M > 0$, let $\tilde{\pi}_t(M)$ be the solution of the following ODE

$$\dot{\varphi} = \varphi \tilde{h}(\varphi)$$

here $\tilde{h}(\varphi) = \max_{x \in \bar{\Omega}} h(x, \varphi)$. By (**h1**) and (**h3**),

$$\tilde{h}(0) > 0, \ \tilde{h}(M) < 0 \quad for \quad M \gg 1.$$

Therefore $\tilde{\pi}_t(M)$ exists and $\tilde{\pi}_t(M) > 0$ for all $t > 0$.

For given $\varphi_0 \in X_+$, let $M_0 \gg 1$ be such that $\varphi_0(x) \leq M_0$ for all $x \in \bar{\Omega}$. Observe that $h(x, \varphi) \leq \tilde{h}(\varphi)$. Then by comparison principle for parabolic equations,

$$0 \leq \pi_t(\varphi_0) \leq \tilde{\pi}_t(M_0)$$

for all $t \in [0, \tau(\varphi_0))$. Since $\tilde{\pi}_t(M)$ exists for all $t > 0$, $\tau(\varphi_0) = \infty$.

□

Proposition 3.2.2. There is a unique positive stationary solution φ^h in $C^1(\bar{\Omega}) \cap C^2(\Omega)$ of (3.2.1) which satisfies $\|\varphi(t,\cdot;\varphi_0) - \varphi^h(\cdot)\|_X \to 0$ as $t \to \infty$ for any $\varphi_0 \in X_+$, $\varphi_0 \not\equiv 0$. Moreover, if $d_{\mathcal{L}}(h_n, h) \to 0$ as $n \to \infty$, $\|\varphi^{h_n} - \varphi^h\|_{C^1(\bar{\Omega})} \to 0$ as $n \to \infty$.

Proof. By (**h3**), there exist constants $M^h > 0$ and $\delta > 0$ such that $h(x,m) < -\delta$ for all $x \in \bar{\Omega}$ and for all $m \geq M^h$. Clearly, 0 is a lower solution of (3.2.1) and M^h is an upper solution of (3.2.1). By Theorem 3.4 in [29], (3.2.1) has a unique positive solution $\varphi^h \in C^\alpha(\bar{\Omega}) \cap C^2(\Omega)$ and $\varphi^h \leq M^h$. Since $\varphi^h \in C^\alpha(\bar{\Omega})$ and $h \in C^2(\bar{\Omega} \times \mathbb{R})$, the function $\varphi^h h(\cdot, \varphi^h) \in L^p(\Omega)$ for any $p \geq 1$. Therefore $\varphi^h = (-\Delta + Id)^{-1}(\varphi^h h(\cdot,\varphi^h) + \varphi^h) \in H^{2,p}(\Omega)$ for any $p \geq 1$. By the imbedding theorem in [1], we have $\varphi^h \in C^1(\bar{\Omega})$.

Now, we will prove that if $d_{\mathcal{L}}(h_n, h) \to 0$ as $n \to \infty$, $\|\varphi^{h_n} - \varphi^h\|_? \to 0$ as $n \to \infty$. Since $d_{\mathcal{L}}(h_n, h) \to 0$ as $n \to \infty$, there exists N such that $\|h_n - h\|_{C(\bar{\Omega},[0,M^h+1])} < \delta/2$, $\forall n > N$. This and the fact that $h(x,m) < -\delta < 0$ for all $x \in \bar{\Omega}$ and for all $m \geq M^h$ imply $h_n(x, M^h + 1) < -\delta/2 < 0$ for all $x \in \bar{\Omega}$ and for all $n > N$. By (**h2**), $h_n(x,m) \leq h_n(x, M^h + 1) < -\delta/2 < 0$ for all $x \in \bar{\Omega}$, for all $m \geq M^h + 1$ and for all $n > N$. Let $M := \max\{M^h + 1, M^{h_1}, \ldots, M^{h_N}\}$, we have $h(x,m),\ h_n(x,m) < 0$ for all $x \in \bar{\Omega}$, for all $m \geq M$ and for all $n \geq 1$. From the fact that $\varphi^{h_n} \leq M^{h_n}$ for all $n \geq 1$, we have $\varphi^{h_n} \leq M$ for all $n \geq 1$. Hence, $0 \leq \max_{\bar{\Omega}} \varphi^{h_n} h_n((\cdot, \varphi^{h_n}) + \varphi^{h_n}) \leq K$ for some $K > 0$. This and $\varphi^{h_n} = (-\Delta + Id)^{-1}(\varphi^{h_n} h_n((\cdot,\varphi^{h_n}) + \varphi^{h_n})) \in H^{2,p}(\Omega)$ for all $p \geq 1$ imply $\|\varphi^{h_n}\|_{H^{2,p}(\Omega)} < C$ for all $n \geq 1$ for some $C > 0$ because $(-\Delta + Id)^{-1}$ is a bounded linear operator. Therefore there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that $\varphi^{h_{n_k}}$ converges in $C^1(\bar{\Omega})$ to some u_0 (since the imbedding $H^{2,p} \hookrightarrow C^1(\bar{\Omega})$ is compact). Hence $-k\Delta\varphi^{h_{n_k}} = \varphi^{h_{n_k}} h_{n_k}(\cdot, \varphi^{h_{n_k}})$ converges to $u_0 h(\cdot, u_0)$ in $L^p(\Omega)$ for all $p \geq 1$. Because $-k\Delta$ (with boundary condition) is a closed operator on $L^p(\Omega)$, we have $u_0 \in \mathcal{D}(-k\Delta)$ and $-k\Delta\varphi^{h_{n_k}}$ converges to $-k\Delta u_0$.

Therefore, $-k\Delta u_0 = u_0 h(\cdot, u_0)$. Since $u_0 \in C^1(\bar{\Omega})$, we have $u_0 h(\cdot, u_0) + u_0 \in C^1(\Omega)$. Hence $u_0 = (-k\Delta + Id)^{-1}(u_0 h(\cdot, u_0) + u_0) \in C^{2,\alpha}(\Omega)$. This means u_0 is also a positive solution of (3.2.1). By uniqueness of solution of (3.2.1), $u_0 = \varphi^h$. Hence $\|\varphi^{h_n} - \varphi^h\|_{C^1(\bar{\Omega})} \to 0$ as $n \to \infty$. □

3.3 Two Species Competition Systems

Proposition 3.3.1. For any $t \geq 0$, we have $\Pi_t(X_+ \times \{0\}) \subset X_+ \times \{0\}$, $\Pi_t(\{0\} \times X_+) \subset \{0\} \times X_+$ and $\Pi_t(X_+ \times X_+) \subset X_+ \times X_+$.

Proof. Let $\pi_t(u_0)$ be the solution of (3.2.1) with initial $u_0 \in X_+$ and $h(x,u) = f(x,u,0)$ and $\pi_t(v_0)$ be the solution of (3.2.1) with initial $u_0 \in X_+$ and $h(x,u) = g(x,0,v)$. Then by (**H**2) and comparison principle for parabolic equations, we have

$$0 \leq u(t,\cdot;u_0,v_0) \leq \pi_t(u_0)$$

and

$$0 \leq v(t,\cdot;u_0,v_0) \leq \pi_t(v_0)$$

for $t \in [0, \tau(u_0,v_0))$. By Proposition 3.2.1, $\tau(u_0,v_0) = \infty$. □

Proposition 3.3.2. If (u_1,v_1), $(u_2,v_2) \in X_+ \times X_+$ and $(u_1,v_1) \leq_2 (u_2,v_2)$, then

$$\Pi_t(u_1,v_1) \leq_2 \Pi_t(u_2,v_2) \text{ for any } t \geq 0.$$

Moreover, if $(u_1,v_1) <_2 (u_2,v_2)$ and $(u_1,v_1) \notin X_+ \times \{0\}$, $(u_2,v_2) \notin \{0\} \times X_+$, then $\Pi_t(u_1,v_1) \ll_2 \Pi_t(u_2,v_2)$ for all $t > 0$.

Proof. See [21]. □

Chapter 4

Morse-Smale Structure and A Sufficient Condition

4.1 Morse-Smale Structure

In this section, we shall prove Theorem A, that is, the openness of Morse-Smale set $\mathcal{MS}$. We first show the upper semi-continuity of the global attractor $\mathcal{A}$.

Theorem 4.1.1. *Given $(f_0, g_0) \in \mathcal{CP}$. The global attractor $\mathcal{A}(f_0, g_0)$ of* (1.0.1) *is upper semi-continuous (in $X_+ \times X_+$).*

Proof. First, we will prove there exists a neighborhood Λ of (f, g) in $\mathcal{CP}$ and a bounded set $B \subset X_+ \times X_+$ independent of $(f, g) \in \Lambda$ such that B attracts points of $X_+ \times X_+$ under Π_t^{fg} for any $(f, g) \in \Lambda$. Let $u_0,\ v_0,\ \tilde{u},\ \tilde{v}$ be the unique positive solutions of (3.2.1) with $h = f_0(\cdot,\cdot,0),\ g_0(\cdot,0,\cdot),\ f(\cdot,\cdot,0),\ g(\cdot,0,\cdot)$ respectively. Repeating the argument used in proving Proposition 3.2.2, we can find a constant $M > 0$ such that

$$\|u_0\|_{C(\bar{\Omega})},\ \|v_0\|_{C(\bar{\Omega})},\ \|\tilde{u}\|_{C(\bar{\Omega})},\ \|\tilde{v}\|_{C(\bar{\Omega})} \leq M, \tag{4.1.1}$$

provided $d_{\mathcal{CP}}((f, g), (f_0, g_0))$ is sufficiently small. Let $\varepsilon_0 > 0$ be small enough so that

$$d_{\mathcal{CP}}((f, g), (f_0, g_0)) < \varepsilon_0 \Rightarrow \|(f, g) - (f_0, g_0)\|_{C(\bar{\Omega}\times[0,3M]\times[0,3M])\times C(\bar{\Omega}\times[0,3M]\times[0,3M])} \leq 1.$$

We then define

$$\Lambda := \{(f, g) \in \mathcal{CP} |\ (4.1.1) \text{ holds and } d_{\mathcal{CP}}((f, g), (f_0, g_0)) < \varepsilon_0\}.$$

For any $(u,v) \in X_+ \times X_+,\ (u,v) \neq (0,0)$, we have $(0,v) \leq_2 (u,v) \leq_2 (u,0)$. By lemma 3.3.2, we have

$$\Pi_t^{fg}(0,v) \leq_2 \Pi_t^{fg}(u,v) \leq_2 \Pi_t^{fg}(u,0),\ \forall t > 0,\ \forall (f,g) \in \Lambda. \tag{4.1.2}$$

Since $\Pi_t^{fg}(u,0) \to (\tilde{u},0)$ and $\Pi_t^{fg}(0,v) \to (0,\tilde{v})$ in $X \times X \hookrightarrow C(\bar{\Omega}) \times C(\bar{\Omega})$ as $t \to \infty$, there exists $t_0^{fg}(u,v) \geq 1$ such that

$$\|\Pi_t^{fg}(u,v)\|_{C(\bar{\Omega})\times C(\bar{\Omega})} \leq 3M,\ \forall t > t_0^{fg}(u,v),\ \forall (f,g) \in \Lambda. \tag{4.1.3}$$

Applying lemma 3.1.2 with $Y := L^p(\Omega),\ Y^\alpha = Y^\beta := X$, we have

$$\begin{aligned} \|\Pi_t^{fg}(u,v)\|_{X\times X} \leq\ & C_1 e^{-t}\|(u,v)\|_{X\times X} \\ & + C_2 \int_{t_0^{fg}(u,v)}^{t} (t-s)^{-\beta} e^{-(t-s)} \|H_{fg}(\Pi_s^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)}\, ds, \end{aligned} \tag{4.1.4}$$

for all $t > t_0^{fg}(u,v)$, where

$$\begin{aligned} H_{fg}:\ X \times X &\to L^p(\Omega) \times L^p(\Omega) \\ (u,v) &\mapsto [uf(.,u,v) + u, vg(.,u,v) + v], \end{aligned}$$

By (4.1.1), we have $\|H_{fg}(\Pi_t^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)} \leq K^{f_0 g_0}$ some constant $K^{f_0 g_0}$ depending on (f_0, g_0). Hence,

$$\begin{aligned} \|\Pi_t^{fg}(u,v)\|_{X\times X} \leq\ & C_1 e^{-t}\|(u,v)\|_{X\times X} \\ & + C_2 K^{f_0 g_0} \int_0^\infty (t-s)^{-\beta} e^{-(t-s)}\, ds,\ \forall t > t_0^{fg}(u,v), \end{aligned}$$

Since $\int_0^\infty (t-s)^{-\beta} e^{-(t-s)}\, ds < C_3$ for some constant C_3, we then have

$$\|\Pi_t^{fg}(u,v)\|_{X\times X} \quad \leq \quad C_1 e^{-t}\|(u,v)\|_{X\times X} + C_2 C_3 K^{f_0 g_0},\ \forall t > t_0^{fg}(u,v), \tag{4.1.5}$$

Let $B := \{(u,v) \in X_+ \times X_+ |\ \|(u,v)\|_{X_+\times X_+} \leq 2C_1 + C_2 C_3 K^{f_0 g_0}\}$. By (4.1.5), B attracts (u,v) under Π_t^{fg} for any $(f,g) \in \Lambda$.

Next, we prove that for any bounded set $U \subset X_+ \times X_+$, the set $\bigcup_{(f,g)\in\Lambda} \bigcup_{t\geq 0} \Pi_t^{fg}(U)$ is bounded. Because U is bounded in $X \times X \hookrightarrow C(\bar{\Omega}) \times C(\bar{\Omega})$, there exists a constant $K > M$ such that

$$(0,K) \leq_2 \Pi_t^{fg}(u,v) \leq_2 (K,0),\ \forall t \geq 0,\ \forall (u,v) \in U,\ \forall (f,g) \in \Lambda.$$

Hence

$$\|H_{fg}(\Pi_t^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)} < L(K,f,g),\ \forall t \geq 0,\ \forall (u,v) \in U. \tag{4.1.6}$$

where $L(K,f,g)$ is a constant depending on K, f, g. Using (4.1.6) and the fact that

$$d_{CP}((f,g),(f_0,g_0)) < \varepsilon_0 \Rightarrow \|(f,g)-(f_0,g_0)\|_{C^2(\bar{\Omega}\times[0,K]\times[0,K])\times C^2(\bar{\Omega}\times[0,K]\times[0,K])} < 2^K \varepsilon_0,$$

we have

$$\|H_{fg}(\Pi_t^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)} < \tilde{L}(K,f_0,g_0),\ \forall t \geq 0,\ \forall (u,v) \in U,\ \forall (f,g) \in \Lambda. \tag{4.1.7}$$

where $\tilde{L}(K, f_0, g_0)$ is a constant depending only on K, f_0 and g_0. Applying lemma 3.1.2 with $Y := L^p(\Omega)$, $Y^\alpha = Y^\beta := X$, we have

$$\begin{aligned} \|\Pi_t^{fg}(u,v)\|_{X\times X} &\leq C_1 e^{-t}\|(u,v)\|_{X\times X} \\ &+C_2\int_0^t (t-s)^{-\beta}e^{-(t-s)}\|H_{fg}(\Pi_s^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)}\, ds,\ \forall t>0. \end{aligned} \tag{4.1.8}$$

From (4.1.7) and (4.1.8), we have

$$\|\Pi_t^{fg}(u,v)\|_{X\times X} \leq C_1 e^{-t}\|(u,v)\|_{X\times X} + C_2\tilde{L}(K, f_0, g_0)\int_0^\infty (t-s)^{-\beta}e^{-(t-s)}\, ds \tag{4.1.9}$$

for all $t \geq 0$, for all $(u, v) \in U$ and for all $(f, g) \in \Lambda$. Because $\int_0^\infty (t-s)^{-\beta}e^{-(t-s)}\, ds < C_3$, we then have

$$\|\Pi_t^{fg}(u,v)\|_{X\times X} \leq C_1 e^{-t}\|(u,v)\|_{X\times X} + C_2C_3\tilde{L}(K, f_0, g_0),\ \forall t\geq 0,\ \forall (u,v)\in U,\ \forall (f,g)\in\Lambda. \tag{4.1.10}$$

It is clear from (4.1.10) that $\bigcup_{(f,g)\in\Lambda}\bigcup_{t\geq 0}\Pi_t^{fg}(U)$ is bounded in $X \times X$.

Finally, we prove that the family of semigroups $\{\Pi_t^{fg},\ t \geq 0\}$, $(f, g) \in \Lambda$ is collectively asymptotically smooth. Fix any $(f, g) \in \Lambda$. For any bounded, closed set $B \subset X_+ \times X_+$ for which $\Pi_t^{fg}(B) \subset B$, $\forall t \geq 0$, we define $J^{fg}(B) = \overline{\Pi_1^{fg}(B)}$ (the closure is taken in $X \times X$). Since B is closed, we have $J^{fg}(B) \subset B$. We also have $\Pi_t^{fg}(B) = \Pi_1(\Pi_{t-1}^{fg}(B)) \subset \Pi_1^{fg}(B) \subset J^{fg}(B)$, $\forall t > 1$. This means $J^{fg}(B)$ attracts B under Π_t^{fg}. Applying lemma 3.1.2 with $Y := L^p(\Omega)$, $Y^\alpha := X$, $Y^\beta := \tilde{X}$, where $\tilde{X}$ is another fractional power space of

$\Delta : \mathcal{D}(\Delta) \subset H^{2,p}(\Omega) \to L^p(\Omega)$ and $\beta > \alpha$ is chosen so that $\tilde{X} \hookrightarrow X$ is compact, we have

$$\begin{aligned} \|\Pi_1^{fg}(u,v)\|_{\tilde{X}\times\tilde{X}} &\leq C_1 e^{-1}\|(u,v)\|_{X\times X} \\ &\quad + C_2 \int_0^1 (1-s)^{-\beta} e^{-(1-s)} \|H_{fg}(\Pi_s^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)}\, ds. \end{aligned} \tag{4.1.11}$$

Because B is a bounded set in $X_+ \times X_+$, we can use same argument used for the bounded set and get

$$\|H_{fg}(\Pi_t^{fg}(u,v))\|_{L^p(\Omega)\times L^p(\Omega)} < \tilde{\tilde{L}}(B, f_0, g_0),\ \forall t \geq 0,\ \forall (u,v) \in B,\ \forall (f,g) \in \Lambda. \tag{4.1.12}$$

where $\tilde{L}(K, f_0, g_0)$ is a constant depending only on B, f_0 and g_0. From (4.1.12) and (4.1.12), we have

$$\|\Pi_1^{fg}(u,v)\|_{\tilde{X}\times\tilde{X}} \leq C_1 e^{-1}\|(u,v)\|_{X\times X} + C_2 \tilde{\tilde{L}}(B, f_0, g_0) \int_0^1 (1-s)^{-\beta} e^{-(1-s)}, \tag{4.1.13}$$

for all $(u,v) \in B$ and for all $(f,g) \in \Lambda$. (4.1.13) and the compact imbedding $\tilde{X} \hookrightarrow X$ imply $\overline{\bigcup_{(f,g)\in\Lambda} J^{fg}(B)}$ is a compact set in $X \times X$. Hence, the family of semigroups $\{\Pi_t^{fg},\ t \geq 0\}$, $(f,g) \in \Lambda$ is collectively asymptotically smooth. By lemma 3.1.1, $\mathcal{A}(f,g)$ is upper semi-continuous. □

Lemma 4.1.2. Let $e \in X \times X$ be an critical point of (1.0.1). Let $E^u(e)$ and $E^s(e)$ denote for the linear spaces spanned by eigenfunctions which correspond to eigenvalues with negative and positive real parts of the linearization of (1.0.1) at e. Define $E_r^u(e) = E^u(e) \cap B_r(e)$ and $E_r^s(e) = E^s(e) \cap B_r(e)$ where $B_r(e)$ is a sufficiently small neighborhood of e (in $X \times X$).

Then there are two C^1 maps

$$\begin{array}{rcl} \Phi_1: \ E_r^u(e) & \rightarrow & E_r^s(e) \\ (u,v) & \mapsto & (k(u,v), l(u,v)) \end{array}$$

and

$$\begin{array}{rcl} \Phi_2: \ E_r^s(e) & \rightarrow & E_r^u(e) \\ (u,v) & \mapsto & (m(u,v), n(u,v)). \end{array}$$

such that the local unstable and stable manifolds of e, $W^u_{\text{loc,r}}(e)$ and $W^s_{\text{loc,r}}(e)$, are the graphs of $\Phi_1 + e$ and $\Phi_2 + e$ respectively. Moreover, $\Phi_1(0) = 0$, $\Phi_2(0) = 0$, $D\Phi(0) \equiv 0$ and $D\Psi(0) \equiv 0$. Here, $D\Phi_1$ and $D\Phi_2$ are the Fretchet derivatives of Φ_1 and Φ_2.

Proof. This is a corollary of theorem 6.1 in [31]. □

Lemma 4.1.3. Let $e \in X$ be a critical element of boundary value problem

$$\begin{cases} \varphi_t - k_1 \Delta \varphi = \varphi q(x, \varphi), & x \in \Omega \\ Bv = 0, & x \in \partial\Omega \end{cases} \tag{4.1.14}$$

where B is either Dirichlet or Neumann boundary condition, $q : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^2-function. Let $O^u(e)$ and $O^s(e)$ denote for the linear spaces spanned by eigenfunctions which correspond to eigenvalues with negative and positive real parts of the linearization of (4.1.14) at e. Define $O^u_r(e) = O^u(e) \cap Q_r(e)$ and $O^s_r(e) = O^s(e) \cap Q_r(e)$ where $Q_r(e)$ is a sufficiently small neighborhood of e (in X). Then there are two C^1 maps

$$\begin{array}{rcl} \tilde{h}_1: \ O_r^u(e) & \longrightarrow & O_r^s(e) \\ \varphi & \longmapsto & \tilde{h}_1(\varphi) \end{array}$$

and

$$\begin{aligned} \tilde{h}_2: \ O_r^s(e) &\longrightarrow O_r^u(e) \\ (u,v) &\longmapsto \tilde{h}_2(\varphi). \end{aligned}$$

such that the local unstable and stable manifolds of e, $W^u_{\text{loc,r}}(e)$ and $W^s_{\text{loc,r}}(e)$, are the graphs of $\tilde{h}_1 + e$ and $\tilde{h}_2 + e$ respectively. Moreover, $\tilde{h}_1(0) = 0$, $\tilde{h}_2(0) = 0$, $D\tilde{h}_1(0) \equiv 0$ and $D\tilde{h}_2(0) \equiv 0$. Here, $D\tilde{h}_1$ and $D\tilde{h}_2$ are the Fretchet derivatives of $\tilde{h}_1$ and $\tilde{h}_2$ respectively.

Proof. This is a corollary of theorem 6.1 in [31]. □

Proposition 4.1.4. *(representation of stable, unstable manifolds in positive cone)*

Let e be an critical point of (1.0.1) on $\partial(X_+ \times X_+)$. We have

$$W^{u+}_{\text{loc}}(e) = e + \{(u,v) + (k(u,v), l(u,v))|\ (u,v) \in E^u_r(e) \cap (X_+ \times X_+)\},$$

$$W^{s+}_{\text{loc}}(e) = e + \{(u,v) + (m(u,v), n(u,v))|\ (u,v) \in E^s_r(e) \cap (X_+ \times X_+)\},$$

where E^u_r, E^s_r and k, l, m, n are from lemma 4.1.2.

Proof. (1.0.1) has three fixed points on $\partial(X_+ \times X_+)$: $(0,0)$, $(u_0, 0)$ and $(0, v_0)$.

Case 1: $e = (0,0)$. By lemma 4.1.2 , we have

$$W^u_{\text{loc}}(e) = \{(u,v) + (k(u,v), l(u,v))|\ (u,v) \in E^u_r(e)\}.$$

We will prove

$$\begin{aligned} W^{u+}_{\text{loc}}(e) &:= W^u_{\text{loc}}(e) \cap (X_+ \times X_+) \\ &= \{(u,v) + (k(u,v), l(u,v))|\ (u,v) \in E^u_r(e) \cap (X_+ \times X_+)\}. \end{aligned}$$

By lemma 4.1.3, we can write the unstable manifold at 0 of the semi-flow generated by the boundary value problems

$$\begin{cases} u_t - k_1\Delta u = uf(x,u,0), & x \in \Omega, \\ Bu = 0, & x \in \partial\Omega. \end{cases}$$

and

$$\begin{cases} v_t - k_1\Delta v = vg(x,0,v), & x \in \Omega, \\ Bv = 0, & x \in \partial\Omega. \end{cases}$$

as

$$W^u_{\text{loc}}(0) = \{u + h_1(u) |\ u \in U^u_r(0)\} \text{ and } \widetilde{W}^u_{\text{loc}}(0) = \{v + \tilde{h}_1(v) |\ v \in V^u_r(0)\}.$$

For any $u \in U^u_r(0)$, we have $(u + h_1(u), 0) \in W^u_{\text{loc}}(e)$. Therefore, there exists $(\tilde{u}, \tilde{v}) \in E^u_r(e)$ such that $(\tilde{u}, \tilde{v}) + (k(\tilde{u}, \tilde{v}), l(\tilde{u}, \tilde{v})) = (u + h_1(u), 0)$. Hence

$$\begin{cases} \tilde{u} + k(\tilde{u}, \tilde{v}) = u + h_1(u) \\ \tilde{v} + l(\tilde{u}, \tilde{v}) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{u} - u = h_1(u) - k(\tilde{u}, \tilde{v}) \\ \tilde{v} = -l(\tilde{u}, \tilde{v}) \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{u} - u = h_1(u) - k(\tilde{u}, \tilde{v}) = 0 \\ \tilde{v} = -l(\tilde{u}, \tilde{v}) = 0 \end{cases}$$

because $(\tilde{u} - u, \tilde{v}) \in E^u_r(e)$, $(h_1(u) - k(\tilde{u}, \tilde{v}), -l(\tilde{u}, \tilde{v})) \in E^s_r(e)$ and $E^u_r(e) \cap E^s_r(e) = (0, 0)$. Hence $l(u, 0) = 0$ for all $u \in U^u_r(0)$. A similar argument yields $k(0, v) = 0$, $\forall v \in$

$V_r^u(0)$. For $(u,v) \in E_r^u(e)$, put $\Psi(\xi) = k(\xi u, v),\ 0 \le \xi \le 1$. We have

$$k(u,v) = \Psi(1) - \Psi(0) = \int_0^1 \Psi'(\xi) d\xi = \left(\int_0^1 \partial_1 k(\xi u, v) d\xi \right) u.$$

A similar argument yields

$$l(u,v) = \left(\int_0^1 \partial_2 l(u, \xi v) d\xi \right) v.$$

Therefore

$$(u,v) + (k(u,v), l(u,v)) = (u(K(u,v)+1), v(L(u,v)+1))$$

where

$$K(u,v) = \int_0^1 \partial_1 k(\xi u, v) d\xi \text{ and } L(u,v) = \int_0^1 \partial_2 l(u, \xi v) d\xi.$$

By lemma 4.1.2, we have $K(u,v), L(u,v) \ll 1$ for $u, v \ll 1$. Therefore, we have

$$\begin{aligned} W_{\text{loc}}^{u+}(e) &:= W_{\text{loc}}^u(e) \cap (X_+ \times X_+) \\ &= \{(u,v) + (k(u,v), l(u,v)) |\ (u,v) \in E_r^u(e) \cap (X_+ \times X_+)\}. \end{aligned}$$

Case 2: $e = (u_0, 0)$. By lemma 4.1.2 , we have

$$W_{\text{loc}}^u(e) = \{(u,v) + (k(u,v), l(u,v)) |\ (u,v) \in E_r^u(e)\}.$$

We will prove

$$\begin{aligned} W^{u+}_{\text{loc}}(e) \quad &:= W^u_{\text{loc}}(e) \cap (X_+ \times X_+) \\ &= \{(u,v) + (k(u,v), l(u,v))|\ (u,v) \in E^u_r(e) \cap (X_+ \times X_+)\}. \end{aligned}$$

By lemma 4.1.3, we can write the unstable manifold at u_0 of the semi-flow generated by the boundary value problem

$$\begin{cases} u_t - k_1 \Delta u = uf(x,u,0), \quad x \in \Omega, \\ Bu = 0, \quad x \in \partial\Omega. \end{cases}$$

as

$$W^u_{\text{loc}}(u_0) = \{u + h_1(u)|\ u \in U^u_r(u_0)\}.$$

For any $u \in U^u_r(u_0)$, we have $(u + h_1(u), 0) \in W^u_{\text{loc}}(e)$. Therefore, there exists $(\tilde{u}, \tilde{v}) \in E^u_r(e)$ such that $(\tilde{u}, \tilde{v}) + (k(\tilde{u}, \tilde{v}), l(\tilde{u}, \tilde{v})) = (u + h_1(u), 0)$. Hence

$$\begin{aligned} &\begin{cases} \tilde{u} + k(\tilde{u}, \tilde{v}) = u + h_1(u) \\ \tilde{v} + l(\tilde{u}, \tilde{v}) = 0 \end{cases} \\ \Rightarrow \ &\begin{cases} \tilde{u} - u = h_1(u) - k(\tilde{u}, \tilde{v}) \\ \tilde{v} = -l(\tilde{u}, \tilde{v}) \end{cases} \\ \Rightarrow \ &\begin{cases} \tilde{u} - u = h_1(u) - k(\tilde{u}, \tilde{v}) = 0 \\ \tilde{v} = -l(\tilde{u}, \tilde{v}) = 0 \end{cases} \end{aligned}$$

because $(\tilde{u} - u, \tilde{v}) \in E^u_r(e)$, $(h_1(u) - k(\tilde{u}, \tilde{v}), -l(\tilde{u}, \tilde{v})) \in E^s_r(e)$ and $E^u_r(e) \cap E^s_r(e) = (0,0)$. Hence $l(u,0) = 0$ for all $u \in U^u_r(u_0)$. For $(u,v) \in E^u_r(e)$, put $\Psi(\xi) =$

$l(u,\xi v),\ 0 \le \xi \le 1$. We have

$$l(u,v) = \Psi(1) - \Psi(0) = \int_0^1 \Psi'(\xi)d\xi = \left(\int_0^1 \partial_2 l(u,\xi v)d\xi\right)v.$$

Therefore

$$(u,v) + (k(u,v), l(u,v)) = (u + k(u,v), v(L(u,v)+1))$$

where

$$L(u,v) = \int_0^1 \partial_2 l(u,\xi v)d\xi.$$

By Lemma 4.1.2, we have $L(u,v) \ll 1$ for $u - u_0, v \ll 1$. Therefore, we have

$$\begin{aligned} W^{u+}_{\text{loc}}(e) &:= W^u_{\text{loc}}(e) \cap (X_+ \times X_+) \\ &= \{(u,v) + (k(u,v), l(u,v))|\ (u,v) \in E^u_r(e) \cap (X_+ \times X_+)\}. \end{aligned}$$

□

Proposition 4.1.5. Given $(f_0, g_0) \in \mathcal{CP}$. Let $e_0 \in X_+ \times X_+$ be a hyperbolic critical element of $\{\Pi_t^{f_0 g_0},\ t \ge 0\}$. Then there exist a neighborhood $\widetilde{\mathcal{O}}$ of e_0 in $(X_+ \times X_+)$ and a neighborhood $\mathcal{V}(f_0, g_0)$ of (f_0, g_0) (in $\mathcal{CP}$) such that given $(f,g) \in \mathcal{V}(f_0, g_0)$ there exists a unique homeomorphism

$$\rho := \rho(f,g) : e_0 \to \rho(e_0) =: e \in \widetilde{\mathcal{O}}$$

close to the inclusion $i : e_0 \to (X_+ \times X_+)$ in the C^0-topology, and $e \in X_+ \times X_+$ is a hyperbolic critical element of $\{\Pi_t^{fg},\ t \ge 0\}$. Moreover, the map $(f,g) \in \mathcal{V}(f_0, g_0) \mapsto \rho(f,g)$

is continuous, $W^{u+}_{\rm loc}(e)$, $W^{s+}_{\rm loc}(e)$ depend continuously on $(f,g) \in \mathcal{V}(f_0,g_0)$ which yields $\dim W^{u+}_{\rm loc}(e) = \dim W^{u+}_{\rm loc}(e_0)$ for all $(f,g) \in \mathcal{V}(f_0,g_0)$.

Proof. For any $l > 0$, we define

$$\begin{aligned} \mathcal{R}_l : \mathcal{CP} &\rightarrow C^2(\Omega \times [0,l] \times [0,l]) \times C^2(\Omega \times [0,l] \times [0,l]) \\ (f,g) &\mapsto (f_{|\bar{\Omega}\times[0,l]\times[0,l]}, g_{|\bar{\Omega}\times[0,l]\times[0,l]}). \end{aligned}$$

It is clear that $\mathcal{R}_l$ is a continuously linear map. By (**H3**) and (**H4**), there exist constants $K^{f_0g_0} > 0$ and $\delta > 0$ such that

$$f_0(x,m,n) < -\delta,\ g_0(x,m,n) < -\delta \text{ for all } x \in \bar{\Omega},\ \text{for all } m,\ n \geq K^{f_0g_0}. \tag{4.1.15}$$

Now, fix a neighborhood $U(f_0,g_0)$ of (f_0,g_0) in $\mathcal{CP}$ such that

$$\|(f,g)-(f_0,g_0)\|_{C^2(\bar{\Omega}\times[0,K^{f_0g_0}+1]\times[0,K^{f_0g_0}+1])\times C^2(\bar{\Omega}\times[0,K^{f_0g_0}+1]\times[0,K^{f_0g_0}+1])} < \delta/2. \tag{4.1.16}$$

From (4.1.15) and (4.1.16), we have

$$f(x,K^{f_0g_0}+1,K^{f_0g_0}+1),\ g(x,K^{f_0g_0}+1,K^{f_0g_0}+1) < -\delta/2,\ \forall x \in \bar{\Omega},\ \forall (f,g) \in U(f_0,g_0).$$

By (**H2**), we then have

$$f(x,m,n) < -\delta/2,\ g(x,m,n) < -\delta/2,\ \forall x \in \bar{\Omega},\ \forall m,n \geq K^{f_0g_0}+1,\ \forall (f,g) \in U(f_0,g_0).$$

Hence, all critical elements of $\{\Pi_t^{f_0g_0},\ t \geq 0\}$ and $\{\Pi_t^{fg},\ t \geq 0\}$ take values inside the contracting rectangle $[0, K^{f_0g_0}+1]\times[0, K^{f_0g_0}+1]$. Therefore, we can consider the restriction of the semi-flows $\{\Pi_t^{f_0g_0},\ t \geq 0\}$ and $\{\Pi_t^{fg},\ t \geq 0\}$ to the Banach manifold

$$\{(u,v) \in X \times X |\ u(\Omega), v(\Omega) \subset (0, K^{f_0g_0} + 1)\}.$$

If $e \in int(X_+ \times X_+)$, apply proposition 2.12 in [23] with $\mathcal{B} := \{(u,v) \in X \times X|\ u(\Omega), v(\Omega) \subset (0, K^{f_0g_0} + 1)\}$ and $\tilde{F} := C^2(\bar{\Omega} \times [0, K^{f_0g_0} + 1] \times [0, K^{f_0g_0} + 1]) \times C^2(\bar{\Omega} \times [0, K^{f_0g_0} + 1] \times [0, K^{f_0g_0} + 1])$; there exists a neighborhood $\mathcal{O}$ of e_0 in $\mathcal{B}$ and a neighborhood $\mathcal{Q}(f_0, g_0)$ of (f_0, g_0) in $\tilde{F}$ such that given $(f, g) \in \mathcal{Q}(f_0, g_0)$ there exists a unique homeomorphism

$$\rho := \rho(f, g) : e_0 \to \rho(e) =: e \in \mathcal{O}$$

close to the inclusion $i : e_0 \to \mathcal{B}$ in the C^0-topology and e is a hyperbolic critical element of Π_t^{fg}. Since ρ is close to the inclusion i, e must be in $\mathcal{B} \subset int(X_+ \times X_+)$ and therefore ρ is close to the inclusion $i : e_0 \to (X_+ \times X_+)$. Put $\widetilde{\mathcal{O}} = \mathcal{O} \cap (X_+ \times X_+)$. Then $e \in \mathcal{O}$. Because $W^{u+}_{\text{loc}}(e) := W^{u}_{\text{loc}}(e) \cap (X_+ \times X_+)$ and $W^{u}_{\text{loc}}(e)$ depends continuously on (f, g) (by proposition 2.12 in [23]), $W^{u+}_{\text{loc}}(e)$ also depends continuously on (f, g). Similar reason implies $W^{s+}_{\text{loc}}(e)$ depends continuously on (f, g). Because $R_{K_{f_0g_0}+1}$ is a continuous map, $R^{-1}_{K^{f_0g_0}+1}(\mathcal{Q}(f_0, g_0))$ is an open neighborhood of (f_0, g_0) in $\mathcal{CP}$. Then $\mathcal{V}(f_0, g_0) := U(f_0, g_0) \cap R^{-1}_{K^{f_0g_0}+1}(\mathcal{Q}(f_0, g_0))$ is the desired neighborhood of (f_0, g_0) in $\mathcal{CP}$.

If $e \in \partial(X_+ \times X_+)$, Due to the conditions (**H1**)-(**H4**), we have three cases $e = (0, 0)$, $e = (u_0, 0)$ and $e = (0, v_0)$ where $u_0,\ v_0$ are the unique stationary solutions of (3.2.1) with $h(\cdot, u) = f_0(\cdot, u, 0)$ and $h(\cdot, v) = g_0(\cdot, 0, v)$. Define $\rho(0, 0) = (0, 0)$, $\rho(u_0, 0) =$

$(\tilde{u}, 0)$, $\rho(0, v_0) = (0, \tilde{v})$ where $\tilde{u}$, $\tilde{v}$ are solutions of (3.2.1) with $h(\cdot, u) = f(\cdot, u, 0)$ and $h(\cdot, v) = g(\cdot, 0, v)$. By proposition 3.2.2 and lemma 4.1.2, we have $W^s(u_0, 0)$ and $W^s(\tilde{u}, 0)$ are C^1 close. Hence, $W^{s+}_{\text{loc}}(u_0, 0)$ and $W^{s+}_{\text{loc}}(\tilde{u}, 0)$ are C^1 close. Similar argument yields the conclusion for the $(0, v_0)$ and $(0, 0)$. □

Proposition 4.1.6. Given $(f_0, g_0) \in \mathcal{CP}$. Let $e_0 \in X_+ \times X_+$ be a critical element (e_0 is not a sink) of $\{\Pi_t^{f_0 g_0},\ t \geq 0\}$ and $N^{u+}(e_0)$ be a fundamental neighborhood of $W^{u+}(e_0)$. Then there exists a neighborhood $\mathcal{V}(f_0, g_0)$ of (f_0, g_0) in $\mathcal{CP}$ and a neighborhood $\tilde{\mathcal{O}}$ of e_0 in $X_+ \times X_+$ such that $N^{u+}(e_0)$ is also a fundamental neighborhood of $W^{u+}(e)$, where $e := \rho(e_0)$ is the unique hyperbolic critical element in $\tilde{\mathcal{O}}$ of $\{\Pi_t^{fg},\ t \geq 0\}$, $(f, g) \in \mathcal{V}(f_0, g_0)$. Here, ρ is the homeomorphism in proposition 4.1.5. Moreover, there exists a neighborhood $\tilde{B}$ of e_0 such that for any $(f, g) \in \mathcal{V}(f_0, g_0)$, we have $\tilde{B} \subset \bigcup_{t \geq 0} \Pi_{-t}^{fg}(N^{u+}(e_0)) \cup W^{s+}_{\text{loc}}(e)$.

Proof. If $e_0 \in int(X_+ \times X_+)$, we use the same argument used in proposition 4.1.5 and hence the result is direct from proposition 2.14 of [23]. We only need to consider the cases $e_0 \in \partial(X_+ \times X_+)$. By proposition 2.14 of [23], there exists a neighborhood $\mathcal{V}(f_0, g_0)$ of (f_0, g_0) in $\mathcal{CP}$ and a neighborhood $\mathcal{O}$ of e_0 in $X \times X$ such that $N^u(e_0)$ is also a fundamental neighborhood of $W^u(e)$, $e := \rho(e_0)$, is the unique hyperbolic critical element in $\mathcal{O}$ of $\{\Pi_t^{fg},\ t \geq 0\}$, $(f, g) \in \mathcal{V}(f_0, g_0)$. Here ρ is the homeomorphism in proposition 2.12 of [23]. By proposition 4.1.5, e is in $X_+ \times X_+$. Hence, e is the unique hyperbolic critical element in $\tilde{\mathcal{O}} := \mathcal{O} \cap (X_+ \times X_+)$ of $\{\Pi_t^{fg},\ t \geq 0\}$, $(f, g) \in \mathcal{V}(f_0, g_0)$. It is also obvious that $N^{u+}(e_0) := N^u(e_0) \cap (X_+ \times X_+)$ is also a fundamental neighborhood of $W^u(e) \cap (X_+ \times X_+) =: W^{u+}(e)$. Finally, we will prove the existence of $\tilde{B}$. By proposition

2.14 of [23], there exists a neighborhood B of e_0 such that

$$B \subset \bigcup_{t\geq 0} \Pi^{fg}_{-t}(N^u(e_0)) \cup W^s_{\text{loc}}(e).$$

Therefore,

$$B \cap (X_+ \times X_+) \subset \left[\bigcup_{t\geq 0} \Pi^{fg}_{-t}(N^u(e_0)) \cap (X_+ \times X_+) \right] \cup W^{s+}_{\text{loc}}(e).$$

Due to the fact that $X_+ \times X_+$ is invariant under the semi-flow $\{\Pi^{fg}_t,\ t \geq 0\}$, it can be verified that

$$\bigcup_{t\geq 0} \Pi^{fg}_{-t}(N^u(e_0)) \cap (X_+ \times X_+) = \bigcup_{t\geq 0} \Pi^{fg}_{-t}(N^u(e_0) \cap (X_+ \times X_+)) = \bigcup_{t\geq 0} \Pi^{fg}_{-t}(N^{u+}(e_0)).$$

Let $\tilde{B} := B \cap (X_+ \times X_+)$. Then we have

$$\tilde{B} \subset \bigcup_{t\geq 0} \Pi^{fg}_{-t}(N^{u+}(e_0)) \cup W^{s+}_{\text{loc}}(e).$$

For the cases $e = (0,0)$, $e = (u_0, 0)$ and $e = (0, v_0)$ where $u_0,\ v_0$ are the unique stationary solutions of (3.2.1) with $h(\cdot, u) = f_0(\cdot, u, 0)$ and $h(\cdot, v) = g_0(\cdot, 0, v)$, the conclusion is clear from the C^1 closeness of the local stable, unstable manifolds of $(u_0, 0)$ and $(\tilde{u}, 0)$ as well as the local stable, unstable manifolds of $(0, v_0)$ and $(0, \tilde{v})$. □

Proposition 4.1.7. The set of all critical hyperbolic elements of (1.0.1) in $X_+ \times X_+$ has a partial order structure $\leq_3$ defined by $e_1 \leq_3 e_2$ iff $W^{u+}(e_2) \cap W^{s+}_{\text{loc}}(e_1) \neq \emptyset$.

Proof. Firstly, we have $e \leq_3 e$ because $W^{u+}(e) \cap W^{s+}_{\text{loc}}(e) = \{e\}$. Second, suppose $e_1 \leq_3 e_2$, $e_2 \leq_3 e_1$ and $e_1 \neq e_2$. Since $W^{u+}(e_2) \cap W^{s+}_{\text{loc}}(e_1) \neq \emptyset$ ($e_1 \leq_3 e_2$), we have $W^u(e_2) \cap W^s_{\text{loc}}(e_1) \neq \emptyset$. By proposition 3.4 of [23], there is a submanifold of $W^u(e_2)$ ε-C^1 close to $B(e_1) \cap W^u_{\text{loc}}(e_1)$ (ε is small enough and $B(e_1)$ is an appropriate open neighborhood of e_1). Since $W^{u+}(e_1) \cap W^{s+}_{\text{loc}}(e_2) \neq \emptyset$ ($e_2 \leq_3 e_1$), we can choose $p \in B(e_1) \cap W^{u+}_{\text{loc}}(e_1)$ such that $\Pi_{t_0}(p) \in W^{s+}_{\text{loc}}(e_2)$ for some $t_0 > 0$. Then for $q \in W^{u+}(e_2)$ ($q \neq e_2$) close enough to p, we must have $\Pi_{t_0}(q) \in W^{s+}_{\text{loc}}(e_2)$ (because of transversality). So, $q \in W^{u+}(e_2) \cap W^{s+}_{\text{loc}}(e_2)$ which is a contradiction. This implies $e_1 = e_2$. Thirdly, suppose $e_1 \leq_3 e_2$ and $e_2 \leq_3 e_3$. Using the similar argument as above, we have $W^{u+}(e_3) \cap W^{s+}_{\text{loc}}(e_1) \neq \emptyset$ which means $e_1 \leq_3 e_3$. Therefore, $\leq_3$ is partial order. □

Definition 4.1.1. Let $Crit(f, g)$ denote the set of all critical elements in $X_+ \times X_+$ of (1.0.1). For $e_1,\ e_n \in Crit(f, g)$, $e_1 \neq e_n$, the sequence $e_1 \leq_3 e_2 \leq_3 \ldots \leq_3 e_n$ (if exists) is called a chain from e_1 to e_n of length $n-1$. We also write $\text{beh}(e_1|e_n) = n-1$ if the maximum length of chains from e_1 to e_n is equal to $n-1$.

Proposition 4.1.8. *Let $(f_0, g_0) \in \mathcal{CP}$. Then there exist a neighborhood $V \subset X_+ \times X_+$ of the attractor $\mathcal{A}(f_0, g_0)$ such that if $d_{\mathcal{CP}}((f_0, g_0), (f, g))$ is small enough, we have*

$$\Omega(f, g) \cap V = \rho(Crit(f_0, g_0) \cap V),$$

where ρ is the homeomorphism in proposition (4.1.5).

Proof. We construct V by induction. Let e_{0_i} be any sink of $Crit(f_0, g_0)$ and $e_i := \rho(e_{0_i})$ where ρ is the homeomorphism in proposition (4.1.5). We have $e_i \in Crit(f, g)$ and e_{0_i}, e_i are close. Because e_{0_i}, e_i are close, there exists a neighborhood $V_0(e_{0_i}) \subset W^s_{\text{loc}}(e_{0_i})$ such that

$e_i \in V_0(e_{0_i}) \subset W^s_{\text{loc}}(e_i)$ where $B_{r(e_{0_i})}(f_0, g_0)$ is a sufficiently small neighborhood of (f_0, g_0) in $\mathcal{CP}$. Put $V_0 := \bigcup_i V_0(e_{0_i})$ and $r_0 := \min_i\{r(e_{0_i})\}$. We then have $\rho(Crit(f_0, g_0) \cap V_0) \subset \Omega(f, g) \cap V_0$. In the other hand, if $x' \in \Omega(f, g) \cap V_0$, then $x' \in V_0$ and $x' \in Crit(f, g)$. Since $x' \in V_0$, there exists $e_i := \rho(e_{0_i})$, $e_{0_i} \in Crit(f_0, g_0) \cap V_0$, such that $x' \in W^s_{\text{loc}}(e_i)$. Because $x' \in \Omega(f, g)$, x' must be e_i. Therefore, $x' \in \rho(Crit(f_0, g_0) \cap V_0)$. This implies $\Omega(f, g) \cap V_0 = \rho(Crit(f_0, g_0) \cap V_0)$ for all $(f, g) \in B_{r_0}(f_0, g_0)$.

Now, suppose we have constructed V_k, r_k corresponding to critical elements of Π^{fg}_t whose behaviors with respect to sinks are $\leq k$, that is,

$$\Omega(f, g) \cap V_k = \rho(Crit(f_0, g_0) \cap V_k),\ (f, g) \in B_{r_k}(f_0, g_0).$$

Let $e_{k+1} \in Crit(f_0, g_0)$ be a saddle of Π^{fg}_t whose behavior with respect to sinks is less than or equal $k+1$. For each $x \in G^{u+}(e_{k+1})$, we have $\Pi^{fg}_t(x) \to e$ for some $e \in Crit(f_0, g_0) \cap V_k$. Therefore, there exists $t_x > 0$ such that $\Pi^{fg}_{t_x}(x) \in V_k$. Let $\mathcal{O}_x \subset V_k$ be a neighborhood of $\Pi^{fg}_{t_x}(x)$ (in V_k). Then $U(x) := \Pi^{fg}_{-t_x}(\mathcal{O})$ is a neighborhood of x in $X_+ \times X_+$. $\mathcal{O}$ can be chosen small enough such that $U(x) \cap W^{s+}(e_{k+1}) = \emptyset$. Since $G^{u+}(e_{k+1})$ is compact, there exists $\{x_j\}^n_1$ such that $G^{u+}(e_{k+1}) \subset \bigcup_{j=1}^n U(x_j)$. Then $N^{u+}(e_{k+1}) := \bigcup_{j=1}^n U(x_j)$ is a fundamental neighborhood of $W^{u+}(e_{k+1})$. Put $t(e_{k+1}) = \max\{t_{x_j}\}^n_1$. Then for any $x \in N^{u+}(e_{k+1})$, we have $\Pi_t(x) \in V_k$ for some $t < t(e_{k+1})$. By proposition 4.1.6, there exist a neighborhood $B_{r(e_{k+1})}(f_0, g_0)$ of (f_0, g_0) in $\mathcal{C}$ and a neighborhood $\tilde{B}(e_{k+1})$ of e_{k+1} such that for any $(f, g) \in B_{r(e_{k+1})}(f_0, g_0)$, we have

$$\tilde{B}(e_{k+1}) \subset W^{s+}_{\text{loc}}(e'_{k+1}) \cup \left[\cup_{t \geq 0} \Pi^{f'g'}_{-t}(N^{u+}(e_{k+1}))\right], \tag{4.1.17}$$

where $e'_{k+1} := \rho(e_{k+1})$, ρ is the homeomorphism in proposition 4.1.5. Define $B_{k+1} := \bigcup \tilde{B}(e_{k+1})$, $r_{k+1} = \min\{r_k, \min\{r(e_{k+1})\}\}$, $t_{k+1} = \max\{t(e_{k+1})\}$ and

$$V_{k+1} = V_k \cup \left(\cup_{0 \le t \le t_{k+1}} \Pi_{-t}^{fg}(V_k) \right) \cup B_{k+1}.$$

We will prove for any $(f,g) \in B_{r_{k+1}}(f_0, g_0)$,

$$\Omega(f,g) \cap V_{k+1} = \rho(Crit(f_0,g_0) \cap V_{k+1}).$$

First, we prove $\rho(Crit(f_0,g_0) \cap V_{k+1}) \subset \Omega(f,g) \cap V_{k+1}$. If $x' \in \rho(Crit(f_0,g_0) \cap V_{k+1})$, then $x' = \rho(x)$, $x \in Crit(f_0,g_0) \cap V_{k+1}$. Because $x \in Crit(f_0,g_0)$, we have $x' \in Crit(f,g) \subset \Omega(f,g)$ by proposition (4.1.5). Since

$$Crit(f_0,g_0) \cap \left[V_k \cup \left(\cup_{0 \le t \le t_{k+1}} \Pi_{-t}^{fg}(V_k) \right) \right] = Crit(f_0,g_0) \cap V_k,$$

we have

$$Crit(f_0,g_0) \cap V_{k+1} = (Crit(f_0,g_0) \cap V_k) \cup (Crit(f_0,g_0) \cap B_{k+1}).$$

If $x \in Crit(f_0,g_0) \cap V_k$, then from the induction assumption, we have $x' \in \Omega(f,g) \cap V_k \subset \Omega(f,g) \cap V_{k+1}$. If $x \in Crit(f_0,g_0) \cap B_{k+1}$ then $x \in Crit(f_0,g_0) \cap \tilde{B}_{e_{k+1}}$ for some $e_{k+1} \in Crit(f_0,g_0)$. By (4.1.17), we then have $x \in W^{s+}_{\text{loc}}(e'_{k+1}) \cup \left[\cup_{t \ge 0} \Pi_{-t}^{fg}(N^{u+}(e_{k+1})) \right]$. Since x is a critical element, it can not be in $\cup_{t \ge 0} \Pi_{-t}^{fg}(N^{u+}(e_{k+1}))$. Therefore, $x \in W^{s+}_{\text{loc}}(e'_{k+1})$. Because $\rho(x)$ is a critical element of $\{\Pi_t^{fg},\ t \ge 0\}$, $\rho(x)$ must be $e'_{k+1} \in V_{k+1}$ which implies $x' := \rho(x) \in Crit(f,g) \cap V_{k+1}$. So, we have $\rho(Crit(f_0,g_0) \cap V_{k+1}) \subset \Omega(f,g) \cap V_{k+1}$.

Now we prove $\Omega(f,g)\cap V_{k+1} \subset \rho(Crit(f_0,g_0)\cap V_{k+1})$. If $x' \in \Omega(f,g)\cap V_{k+1}$, then $x' \in V_{k+1} := V_k \cup \left(\cup_{0<t\leq t_{k+1}} \Pi^{fg}_{-t}(V_k)\right) \cup B_{k+1}$. If $x' \in V_k$, then by the induction assumption, we have $x' \in \rho(Crit(f_0,g_0)\cap V_k) \subset \rho(Crit(f_0,g_0)\cap V_{k+1})$. So, we only need to consider the case $x' \in \cup \left(\cup_{0<t\leq t_{k+1}} \Pi^{fg}_{-t}(V_k)\right) \cup B_{k+1}$. Since $x' \in \Omega(f,g)$, x' can not be in $\cup_{0<t\leq t_{k+1}} \Pi^{fg}_{-t}(V_k)$. Hence, $x' \in B_{k+1}$ which means $x' \in \tilde{B}(e_{k+1})$ for some $e_{k+1} \in Crit(f_0,g_0)$. By (4.1.17), x' must be e'_{k+1}. This implies $\Omega(f,g)\cap V_{k+1} \subset \rho(f,g)(Crit(f_0,g_0)\cap V_{k+1})$. This completes the induction procedure. Finally we reach the sources and obtain a neighborhood V of $\mathcal{A}(f_0,g_0)$ satisfying $\Omega(f,g)\cap V = \rho(Crit(f_0,g_0)\cap V)$ provided $d_{\mathcal{CP}}((f_0,g_0),(f,g))$ is small enough. □

Now, we are ready to prove theorem A.

Theorem 4.1.9. The Morse-Smale set $\mathcal{MS}$ is open (in $\mathcal{CP}$).

Proof. For $(f_0,g_0) \in \mathcal{CP}$, let V be the neighborhood of $\mathcal{A}(f_0,g_0)$ in proposition (4.1.8). Since the map $(f_0,g_0) \mapsto \mathcal{A}(f_0,g_0)$ is upper semicontinuous (theorem 4.1.1), we have $\mathcal{A}(f,g) \subset V$ for all $(f,g) \in \mathcal{V}(f_0,g_0)$, where $\mathcal{V}(f_0,g_0)$ is a sufficiently small neighborhood of (f_0,g_0) in $\mathcal{CP}$. Applying proposition (4.1.8), we have

$$\Omega(f,g) = \Omega(f,g)\cap V = \rho[Crit(f_0,g_0)\cap V] = \rho(Crit(f_0,g_0)) \subset Crit(f,g).$$

But $Crit(f,g) \subset \Omega(f,g)$. Hence $Crit(f,g) = \Omega(f,g)$. So the map

$$\rho : Crit(f_0,g_0) \mapsto Crit(f,g)$$

is bijective. This implies the openness of $\mathcal{MS}$ (in $\mathcal{CP}$). □

4.2 A Sufficient Condition

In this section, we shall prove theorem B.

Theorem 4.2.1. If all the critical elements of (1.0.1) are hyperbolic (in $X_+ \times X_+$) and the dimension of the unstable manifold of any fixed point in $X_+ \times X_+ \setminus \{(0,0)\}$ is at most one and the dimension of the unstable manifold of a periodic solution is at most two, then (1.0.1) has Morse-Smale structure.

Proof. Case 1: Let e, $e' \in X_+ \times X_+$ be two fixed points of (1.0.1) with $e \neq (0,0)$. Suppose that $\exists\ \beta \in W^{u+}(e) \cap W^{s+}_{\text{loc}}(e')$. Since $\beta \in W^{u+}(e)$, there exist $t_0 > 0$ and $\xi \in W^{u+}_{\text{loc}}(e)$ such that $\beta = \Pi_{t_0}(\xi)$. Let $\omega(\tilde{e})$ be the normalized (in $X \times X$) principle eigenfunction of the linearization problem of system (1.0.1) at $\tilde{e} \neq (0,0)$. By the Krein-Rutman theorem (cf. [16]), we have $\omega(\tilde{e}) \gg_2 (0,0)$. By lemma 4.1.2, we have either

$$\lim_{\xi \to \tilde{e},\ \xi \in W^u(\tilde{e}),} \frac{\xi - \tilde{e}}{\|\xi - \tilde{e}\|_{X \times X}} = \omega(\tilde{e}) \tag{4.2.1}$$

or

$$\lim_{\xi \to \tilde{e},\ \xi \in W^u(\tilde{e})} \frac{\xi - \tilde{e}}{\|\xi - \tilde{e}\|_{X \times X}} = -\omega(\tilde{e}). \tag{4.2.2}$$

Applying (4.2.1) for $\tilde{e} = e$, there exists $\xi \in W^u_{\text{loc}}(e)$ such that

$$\xi - e = \|\xi - e\|_{X \times X} \left(\omega(e) + \frac{o(\|\xi - e\|_{X \times X})}{\|\xi - e\|_{X \times X}} \right) \gg_2 (0,0). \tag{4.2.3}$$

This implies

$$\Pi_t(\xi) - \Pi_t(e) = \Pi_t(\xi) - e \gg_2 (0,0),\ \forall\ t \geq 0. \tag{4.2.4}$$

Since $\beta \in W^{s+}_{\text{loc}}(e')$, we have

$$\Pi_t(\xi) \to e' \text{ (in } X \times X) \text{ as } t \to \infty. \tag{4.2.5}$$

Letting $t \to \infty$ in (4.2.4) and using (4.2.5), we have $e' \geq_2 e$. By proposition 3.3.2, we have $\Pi_t(e') \gg_2 \Pi_t(e)$, $\forall t > 0$ but this means $e' \gg_2 e$. Now, suppose $W^{u+}(e') \neq \{e'\}$. Applying (4.2.2) for $\tilde{e} := e'$, there exists $\eta \in W^u_{\text{loc}}(e')$ such that

$$\eta - e' = \|\eta - e'\|_{X\times X}\left(-\omega(e') + \frac{o(\|\eta - e'\|_{X\times X})}{\|\eta - e'\|_{X\times X}}\right) \ll_2 (0,0). \tag{4.2.6}$$

This implies

$$\Pi_t(\eta) - \Pi_t(e') = \Pi_t(\eta) - e' \ll_2 (0,0),\ \forall\, t \geq 0. \tag{4.2.7}$$

Since $e' \gg_2 e$, we can choose ξ and η closed enough to e and e' (respectively) such that $e \ll_2 \xi \ll_2 \eta \ll_2 e'$. This and $\Pi_t(\xi) \to e'$ (in $X \times X$) as $t \to \infty$ imply $\Pi_t(\eta) \to e'$ (in $C(\bar{\Omega}) \times C(\bar{\Omega})$) as $t \to \infty$. This and the fact that the ω-limit set of η is relative compact in $X \times X$ implies $\Pi_t(\eta) \to e'$ (in $X \times X$) as $t \to \infty$. Hence $\eta \in W^{s+}(e')$ which contradicts to the fact $\eta \in W^{u+}(e')$. Therefore $W^{u+}(e') = \{e'\}$. This implies $W^{s+}_{\text{loc}}(e') = X \times X$. Then, it is clear that we have $T_\beta W^{u+}(e) \oplus T_\beta W^{s+}_{\text{loc}}(e') = X \times X$.

Case 2: Let $e \neq (0,0)$ be an fixed point, e' be a periodic orbit with period σ and suppose that $\exists\, \beta \in W^{u+}(e) \cap W^{s+}_{\text{loc}}(e')$. Using the same argument as in the first part of case 1, there exists $\xi \in W^{u+}_{\text{loc}}(e)$ such that $\beta = \Pi_{t_0}(\xi)$ and

$$\Pi_t(\xi) \gg_2 e,\ \forall\, t \geq 0. \tag{4.2.8}$$

Since $\beta \in W^{s+}_{\text{loc}}(e')$, we have

$$\Pi_t(\xi) \to e' \text{ (in } X \times X) \text{ as } t \to \infty. \tag{4.2.9}$$

Let $p_0 \in e'$ be a limit point of $\{\Pi_t(\xi),\ t \geq 0\}$. We have an increasing sequence $t_n \to \infty$ as $n \to \infty$ such that $\Pi_{t_n}(\xi) \to p_0$. From (4.2.8), we have $\Pi_{t_n}(\xi) \gg_2 e,\ \forall\ t \geq 0$. Let $n \to \infty$, we have $p_0 \geq_2 e$. By proposition 3.3.2, we have $\Pi_\sigma(p_0) \gg_2 \Pi_\sigma(e)$ which is $p_0 \gg_2 e$ Now, suppose $W^{u+}(e') \neq \{e'\}$. This means $W^{u+}(e')$ is two-dimensional, $D\Pi_\sigma(p_0)$ has one eigenvalue with real part greater than 1. By Krein-Rutman theorem, this eigenvalue has a strongly positive eigenfunction $\omega(p_0)$. Let

$$W = \{\eta \in X \times X |\ \liminf_{n\to\infty}\ d_{X\times X}(\Pi^n_\sigma(\eta), e') > 0\}.$$

Then we have

$$\lim_{\|\eta - p_0\|_{X\times X}\to 0,\ \eta\in W} \frac{\eta - p_0}{\|\eta - p_0\|_{X\times X}} = -\omega(p_0). \tag{4.2.10}$$

This implies

$$\eta - p_0 = \|\eta - p_0\|_{X\times X}\left(-\omega(p_0) + \frac{o(\|\eta - p_0\|_{X\times X})}{\|\eta - p_0\|_{X\times X}}\right) \ll_2 (0,0). \tag{4.2.11}$$

Since $p_0 \gg_2 e$, we can choose ξ and η closed enough to e and p_0 (respectively) such that $e \ll_2 \xi \ll_2 \eta \ll_2 p_0$. Thus, $\Pi_{t_n}(\xi) \ll_2 \Pi_{t_n}(\eta) \ll_2 \Pi_{t_n}(p_0)$. The set $\{\Pi_{t_n}(p_0)\}$ is compact. So there exists a subsequence $t_{n_k}\}$ of $\{t_n\}$ and $p_1 \in e'$ such that $\Pi_{t_{n_k}}(p_0) \to p_1$ as $n \to \infty$. If $p_0 \neq p_1$, then we have $p_0 \leq_2 p_1$. Again, by proposition 3.3.2, we have $p_0 \ll_2 p_1$. By theorem 2.3 in [32], this can not happen. Therefore, $p_1 \equiv p_0$. Then we have $\Pi_{t_{n_k}}(\eta) \to p_0$

as $n \to \infty$ which is a contradiction to the fact $\eta \in W$. Hence, $W^{u+}(e') = \{e'\}$. This implies $W^{s+}_{\text{loc}}(e') = X \times X$. Then, it is clear that we have $T_\beta W^{u+}(e) \oplus T_\beta W^{s+}_{\text{loc}}(e') = X \times X$.

Case 3: Let e be a periodic orbit with period σ, e' be an equilibrium and suppose that $\exists\ \beta \in W^{u+}(e) \cap W^{s+}_{\text{loc}}(e')$. Since $\beta \in W^{u+}(e)$, there exist $p_0^* \in e$ and $\xi \in W^{u+}_{\text{loc}}(e)$ such that $\beta = \Pi_{t_0}(\xi)$ and $p_0^* \ll_2 \xi$. Since $\Pi_t(\xi) \to e'$ as $t \to \infty$, we have $p_0^* \leq_2 e'$. By proposition 3.3.2, we have $p_0^* \ll_2 e'$. Now, suppose $W^{u+}(e') \neq \{e'\}$. Applying (4.2.2) for $\tilde{e} := e'$, there exists $\eta \in W^u_{\text{loc}}(e')$ such that

$$\eta - e' = \|\eta - e'\|_{X\times X} \left(-\omega(e') + \frac{o(\|\eta - e'\|_{X\times X})}{\|\eta - e'\|_{X\times X}} \right) \ll_2 \tag{4.2.12}$$

This implies

$$\Pi_t(\eta) - \Pi_t(e') = \Pi_t(\eta) - e' \ll_2 (0,0),\ \forall\ t \geq 0. \tag{4.2.13}$$

Since $p_0^* \ll e'$, we can choose ξ and η closed enough to e' and p_0^* (respectively) such that $\xi \ll_2 \eta \ll_2 e'$. This and $\Pi_t(\xi) \to e'$ as $t \to \infty$ in $X \times X$ imply $\Pi_t(\eta) \to e'$ (in $C(\bar{\Omega}) \times C(\bar{\Omega})$) as $n \to \infty$. This and the fact that the ω-limit set of η is relative compact in $X \times X$ imply $\Pi_{t_n}(\eta) \to p_0$ (in $X \times X$) as $n \to \infty$. This is a contradiction to the fact $\eta \in W^{u+}(e')$. Therefore $W^{u+}(e') = \{e'\}$. This implies $W^{s+}_{\text{loc}}(e') = X \times X$. Then, it is clear that we have $T_\beta W^{u+}(e) \oplus T_\beta W^{s+}_{\text{loc}}(e') = X \times X$.

Case 4: Let e, e' be periodic orbits with periods σ, σ' respectively. Suppose that $\exists \beta \in W^{u+}(e) \cap W^{s+}_{\text{loc}}(e')$. Since $\beta \in W^{u+}(e)$, there exist $p_0^* \in e$ and $\xi \in W^{u+}_{\text{loc}}(e)$ such that $\beta = \Pi_{t_0}(\xi)$ and $p_0^* \ll_2 \xi$. Let $p_0 \in e'$ be a limit point of $\{\Pi_t(\xi),\ t \geq 0\}$. We have an increasing sequence $t_n \to \infty$ as $n \to \infty$ such that $\Pi_{t_n}(\xi) \to p_0$. Hence, $p_0 \gg_2 p_0^*$. Now, suppose $W^{u+}(e') \neq \{e'\}$. This means $W^{u+}(e')$ is two-dimensional, $D\Pi_\sigma(p_0)$ has one eigenvalue with real part greater than 1. By Krein-Rutman theorem, this eigenvalue has a strongly positive eigenfunction $\omega(p_0)$. Let

$$W = \{\eta \in X \times X| \liminf_{n\to\infty} \ d_{X\times X}(\Pi^n_\sigma(\eta), e') > 0\}.$$

Then we have

$$\lim_{\|\eta - p_0\|_{X\times X}\to 0,\ \eta\in W} \frac{\eta - p_0}{\|\eta - p_0\|_{X\times X}} = -\omega(p_0). \tag{4.2.14}$$

This implies

$$\eta - p_0 = \|\eta - p_0\|_{X\times X}\left(-\omega(p_0) + \frac{o(\|\eta - p_0\|_{X\times X})}{\|\eta - p_0\|_{X\times X}}\right) \ll_2 (0,0). \tag{4.2.15}$$

Since $p_0 \gg_2 p_0^*$, we can choose ξ and η closed enough to p_0^* and p_0 (respectively) such that $\xi \ll_2 \eta \ll_2 p_0$. Thus, $\Pi_{t_n}(\xi) \ll_2 \Pi_{t_n}(\eta) \ll_2 \Pi_{t_n}(p_0)$. The set $\{\Pi_{t_n}(p_0)\}$ is compact. So there exists a subsequence $t_{n_k}\}$ of $\{t_n\}$ and $p_1 \in e'$ such that $\Pi_{t_{n_k}}(p_0) \to p_1$ as $n \to \infty$. If $p_0 \neq p_1$, then we have $p_0 \leq_2 p_1$. Again, by proposition 3.3.2, we have $p_0 \ll_2 p_1$. By theorem 2.3 in [32], this can not happen. Therefore, $p_1 \equiv p_0$. Then we have $\Pi_{t_{n_k}}(\eta) \to p_0$ as $n \to \infty$ which is a contradiction to the fact $\eta \in W$. Hence, $W^{u+}(e') = \{e'\}$. This implies $W^{s+}_{\text{loc}}(e') = X \times X$. Then, it is clear that we have $T_\beta W^{u+}(e) \oplus T_\beta W^{s+}_{\text{loc}}(e') = X \times X$.

Case 5: Suppose $\exists\ \beta \in W^{u+}(0,0) \cap W^{s+}_{\text{loc}}(e),\ e \neq (0,0)$. If $W^{u+}(e) = \{e\}$, then the transverality is clear. Suppose $W^{u+}(e) \neq \{e\}$. Let $\omega(e)$ be the principle eigenfunction of the linearization problem of system (1.0.1) at $\tilde{e} \neq (0,0)$. We have $\omega(e) \gg_2 (0,0)$. Since $T_e W^{s+}(e)$ and $T_\beta W^{s+}(e)$ are close if $\beta \in W^{s+}_{\text{loc}}(e)$, we have $T_\beta W^{s+}(e) \oplus span\{\omega(\beta)\} = X \times X$ with $\omega(\beta)$ is close to $\omega(e)$. Let $\eta \in W^{u+}_{\text{loc}}(0,0)$ such that $\Pi_{t_0}(\eta) = \beta$ for some $t_0 > 0$. The linearization of (1.0.1) at $(0,0)$ is decoupled and we have two positive eigenfunctions ω^u and

ω^v. The linear space spanned by $\{\omega^u,\ \omega^v\}$ is in the tangent space $T_{(0,0)}W^{u+}(0,0)$. Since η is close to $(0,0)$, $T_\beta W^{u+}(0,0)$ and $T_{(0,0)}W^{u+}(0,0)$ are close. We can chose a strongly positive vector $\omega(\eta)$ in the linear space spanned by $\{\omega^u,\ \omega^v\}$ so that $\Pi_{t_0}(\omega(\eta)) = \omega(\beta)$. Since η is close to $(0,0)$, $T_\beta W^{u+}(0,0)$ and $T_{(0,0)}W^{u+}(0,0)$ are close. Hence $T_\beta W^{u+}(0,0) = span\{\omega(\beta)\}$. So, we have $T_\beta W^{s+}(e) \oplus T_\beta W^{u+}(0,0))\} = X \times X$. $\square$

Chapter 5

A-stability via Morse-Smale structure

5.1 Reduction to inertial manifold

We consider (1.0.1) for the one-dimensional case $\Omega = (0,1)$.

$$\begin{cases} u_t = k_1 u_{xx} + uf(x,u,v),\ x \in (0,1), \\ v_t = k_2 v_{xx} + vg(x,u,v),\ x \in (0,1), \\ Bu = Bv = 0. \end{cases} \tag{5.1.1}$$

Let

$$\begin{aligned} A_1 :\ \mathcal{D}(A_1) \subset L^2(\Omega) &\rightarrow L^2(\Omega) \\ (u,v) &\mapsto -k_1 u_{xx} + u, \end{aligned}$$

$$\begin{aligned} A_2 :\ \mathcal{D}(A_2) \subset L^2(\Omega) &\rightarrow L^2(\Omega) \\ (u,v) &\mapsto -k_2 v_{xx} + v, \end{aligned}$$

$$A = (A_1, A_2),\ \mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_1).$$

where $\mathcal{D}(A_1) = \mathcal{D}(A_2) = \{\varphi \in H^{2,2}(\Omega)|\ B\varphi = 0\}$, B is either Neumann or Dirichlet boundary condition. It is easy to see that the the fractional power spaces generated by $A_1^{1/2}$ and $A_2^{1/2}$ are same Hilbert space with different inner products $\left\langle A_1^{1/2}\cdot, A_1^{1/2}\bullet \right\rangle_{L^2(\Omega)\times L^2(\Omega)}$ and $\left\langle A_2^{1/2}\cdot, A_2^{1/2}\bullet \right\rangle_{L^2(\Omega)\times L^2(\Omega)}$, respectively. For simplicity, we denote both by X. The inner product in $X\times X$ is then $\langle\cdot,\bullet\rangle_{X\times X} = \left\langle A_1^{1/2}\cdot, A_1^{1/2}\bullet\right\rangle_{L^2(\Omega)\times L^2(\Omega)} + \left\langle A_2^{1/2}\cdot, A_2^{1/2}\bullet\right\rangle_{L^2(\Omega)\times L^2(\Omega)}$. It is implicitly understood that $\|\cdot\|_X = \|A_1^{1/2}\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_X = \|A_2^{1/2}\cdot\|_{L^2(\Omega)}$ depending on whether $\cdot$ is the first or second component of (u,v). The normalized (in $L^2(\Omega)\times L^2(\Omega)$) eigenfunctions of $A+Id$ which correspond to the eigenvalues $\{\lambda_i^u = 1+k_1(i\pi)^2,\ \lambda_j^v = 1+k_2(j\pi)^2 |\ i,j\in\mathbb{N}\cup\{0\}\ \}$ are $w_i^u(x) = (\sqrt{2}\cos(i\pi x), 0),\ w_i^v(x) = (0, \sqrt{2}\cos(j\pi x)),\ i,j\in \mathbb{N}\cup\{0\}$ if B is the Neumann boundary condition and are $w_i^u(x) = (\sqrt{2}\sin(i\pi x), 0),\ w_i^v(x) = (0,\sqrt{2}\sin(j\pi x)),\ i,j\in\mathbb{N}$ if B is Dirichlet boundary condition. We arrange the eigenvalues as an increasing sequence $\lambda_1 < \lambda_2 < \ldots <\ \lambda_N < \lambda_{N+1} < \ldots$. Let W_N be the linear space spanned by the $\{\lambda_i\}_1^N$ and $W_N^\perp$ be the orthogonal complement of W_N in $L^2(\Omega)\times L^2(\Omega)$. Let $\widetilde{F}_{fg}(u,v) = [uf(\cdot,u,v)+u, vg(\cdot,u,v)+v]$. Clearly, $\widetilde{F}_{fg}: X\times X \to H^{1,2}(\Omega)\times H^{1,2}(\Omega)$. Moreover, if B is the Dirichlet boundary condition, then $\widetilde{F}_{fg}: X\times X\to X\times X$ because $B(F(u,v)) = 0$. If B is the Neumann boundary condition, it is well-known that $X = H^{1,2}(\Omega)$. Note that $\mathcal{D}(-\Delta+Id)$ is dense in $H^{1,2}(\Omega)$ under the norm induced by the inner product $\langle\cdot,\bullet\rangle_{H^{1,2}(\Omega)\times H^{1,2}(\Omega)} = \langle -\Delta\cdot,\bullet\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle\cdot,\bullet\rangle_{L^2(\Omega)\times L^2(\Omega)}$. Therefore, we can consider $\widetilde{F}_{fg}: X\times X\to X\times X$ in both cases.

Lemma 5.1.1. Let $(f,g)\in\mathcal{CP}$. Assume $(u,v)\geq 0,\ (u,v)\in\mathcal{D}(A)$ and $\|(u,v)\|_{X\times X}\geq R$. If R is large enough, then there exists $m_0>0$ (depending on R and (f,g)) such that

$$\left\langle A(u,v) - \widetilde{F}_{fg}(u,v), (u,v)\right\rangle_{L^2(\Omega)\times L^2(\Omega)} \geq 2m_0.$$

Proof. We have

$$
\begin{aligned}
&\left\langle A(u,v) - \widetilde{F}_{fg}(u,v), (u,v) \right\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&= \langle -k_1 u_{xx} - u f(\cdot,u,v), u \rangle_{L^2(\Omega)} + \langle -k_2 v_{xx} - v g(\cdot,u,v), v \rangle_{L^2(\Omega)} \\
&= \int_\Omega [k_1 u_x^2 - u^2 f(\cdot,u,v)]dx + \int_\Omega [k_2 v_x^2 - v^2 g(\cdot,u,v)]dx \\
&= \left(k_1 \int_\Omega u_x^2 \ dx + k_2 \int_\Omega v_x^2 \ dx \right) - \int_\Omega u^2 f(\cdot,u,v) \ dx - \int_\Omega v^2 g(\cdot,u,v) \ dx.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|(u,v)\|_{X\times X}^2 &= \left\langle A_1^{1/2} u, A_1^{1/2} u \right\rangle_{L^2(\Omega)} + \left\langle A_2^{1/2} v, A_2^{1/2} v \right\rangle_{L^2(\Omega)} \\
&= \langle A_1 u, u \rangle_{L^2(\Omega)} + \langle A_2 v, v \rangle_{L^2(\Omega)} \\
&= k_1 \int_\Omega u_x^2 \ dx + k_2 \int_\Omega v_x^2 \ dx + \left(\int_\Omega u^2 \ dx + \int_\Omega v^2 \ dx \right) \\
&= k_1 \int_\Omega u_x^2 \ dx + k_2 \int_\Omega v_x^2 \ dx + \|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)}^2. \qquad (5.1.2)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left\langle A(u,v) - \widetilde{F}_{fg}(u,v), (u,v) \right\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&\geq \|(u,v)\|_{X\times X}^2 - \|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)}^2 - \int_\Omega u^2 f(\cdot,u,v) \ dx - \int_\Omega v^2 g(\cdot,u,v) dx. \qquad (5.1.3)
\end{aligned}
$$

From (**H3**) and (**H4**) there exists a positive constant R_0 such that

$$
\begin{aligned}
&f(\cdot,u,v) < -1/2 \quad \text{for all} \ \ v \geq 0, \ x \in (0,1), \ u \geq R_0, \\
&g(\cdot,u,v) < -1/2 \quad \text{for all} \ \ u \geq 0, \ x \in (0,1), \ v \geq R_0.
\end{aligned}
\qquad (5.1.4)
$$

Let

$$M_0 = \max\{ \sup_{\{0 \le u \le R_0\} \cap \{v \ge 0\}} f(\cdot, u, v), \sup_{\{0 \le v \le R_0\} \cap \{u \ge 0\}} g(\cdot, u, v)\}.$$

M_0 exists and finite because of (**H1**) and (**H2**). Choose R sufficiently large so that

$$2m_0 = \frac{1}{2}R^2 - R_0^2 - 2M_0R_0^2 > 0.$$

From (5.1.3) and (5.1.4), we have

$$\begin{aligned}
&\left\langle A(u,v) - \widetilde{F}_{fg}(u,v), (u,v) \right\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&\ge \frac{1}{2}\|(u,v)\|^2_{X\times X} - \frac{1}{2}\|(u,v)\|^2_{L^2(\Omega)\times L^2(\Omega)} - \int_\Omega u^2 f(\cdot,u,v)\ dx - \int_\Omega v^2 g(\cdot,u,v)dx \\
&\ge \frac{1}{2}R^2 - \frac{1}{2}\|(u,v)\|^2_{L^2(\Omega)\times L^2(\Omega)} - \int_{\{u\ge R_0\}} u^2 f(\cdot,u,v)\ dx - \int_{\{v \ge R_0\}} v^2 g(\cdot,u,v)\ dx \\
&\quad - \int_{\{0\le u<R_0\}} u^2 f(\cdot,u,v)\ dx - \int_{\{0 \le v < R_0\}} v^2 g(\cdot,u,v)\ dx \\
&\ge \frac{1}{2}R^2 - \frac{1}{2}\|(u,v)\|^2_{L^2(\Omega)\times L^2(\Omega)} + \frac{1}{2}\int_{\{u\ge R_0\}} u^2\ dx + \frac{1}{2}\int_{\{v\ge R_0\}} v^2\ dx \\
&\quad - \int_{\{0\le u<R_0\}} u^2 f(\cdot,u,v)\ dx - \int_{\{0 \le v < R_0\}} v^2 g(\cdot u,v)\ dx \\
&= \frac{1}{2}R^2 - \int_{\{0\le u<R_0\}} [u^2 f(\cdot,u,v) + \frac{1}{2}u^2]dx - \int_{\{0\le v<R_0\}} [v^2 g(\cdot,u,v) + \frac{1}{2}v^2]dx \\
&\ge \frac{1}{2}R^2 - R_0^2 - 2M_0R_0^2 = 2m_0.
\end{aligned}$$

□

Fix $(f_0, g_0) \in \mathcal{CP}$. Since $\mathcal{A}(f_0, g_0)$ is compact (in $X_+ \times X_+$), there exists $R > 0$ such that $\mathcal{A}(f_0, g_0) \subset \{(u,v) \in X_+ \times X_+ |\ \|(u,v)\|_{X\times X} < R\}$. By theorem 4.1.1, $\mathcal{A}(f_0, g_0)$ is upper semi-continuous in $X \times X$. Hence

$$\mathcal{A}(f,g) \subset \{(u,v) \in X_+ \times X_+ |\ \|(u,v)\|_{X\times X} < R\}, \tag{5.1.5}$$

provided $d_{\mathcal{CP}}((f,g),(f_0,g_0))$ is small enough. From now on, we fix $R > 0$ such that (5.1.5) holds and lemma 5.1.1 is true for (f_0, g_0). We define a C^∞ function $\varphi : [0,\infty]\times[0,\infty] \to [0,1]$ with the following properties

$$\begin{cases} \varphi \equiv 1 \text{ on } [0,R]\times[0,R], \\ \varphi \equiv 0 \text{ on } \{(s_1,s_2) \in \mathbb{R}^2 |\ \max\{s_1,s_2\} \geq 2R\}, \\ \sup_{s_1\geq 0} \|\varphi_{s_1}(s_1,s_2)\| \leq 1,\ \forall\ s_2 \geq 0, \\ \sup_{s_2\geq 0} \|\varphi_{s_2}(s_1,s_2)\| \leq 1,\ \forall\ s_1 \geq 0. \end{cases}$$

Let $F_{fg}(u,v) = \varphi(\|u\|_X, \|v\|_X)\widetilde{F}_{fg}(u,v)$. Since f, g are C^2 functions, F_{fg} has the following properties

1. $F : X \times X \to X \times X$ is bounded in C^1 norm. Note that the bound depends on (f,g),
2. $F(u,v) = \widetilde{F}(u,v)$ for all $\|(u,v)\|_{X\times X} \leq R$,
3. $F(u,v) = 0$ for all $\|(u,v)\|_{X\times X} \geq 2R$.

Therefore, rather than studying system (5.1.1), we can study the modified one

$$\begin{cases} u_t = k_1 u_{xx} + \varphi(\|u\|_X, \|v\|_X) u f(x,u,v),\ x \in (0,1), \\ v_t = k_2 v_{xx} + \varphi(\|u\|_X, \|v\|_X) v g(x,u,v),\ x \in (0,1), \\ Bu = Bv = 0. \end{cases} \tag{5.1.6}$$

For simplicity, we will sometimes write F instead of F_{fg}. Let $\Theta = (u, v)$, we can rewrite system (5.1.6) as

$$\Theta_t + A\Theta = F(\Theta),\ \Theta \in \mathcal{D}(A). \tag{5.1.7}$$

Lemma 5.1.2. Given $(f_0, g_0) \in \mathcal{CP}$ and $\varepsilon_0 > 0$. Define

$$\Lambda := \{(f, g) \in \mathcal{CP}|\ d_{\mathcal{CP}}((f, g), (f_0, g_0)) < \varepsilon_0\}.$$

Then there exists positive constants $M_0,\ M_1$ depending on $\varepsilon_0,\ f_0,\ g_0$ such that

$$\|F_{fg}(u, v)\|_{X\times X} \le M_0,\ \forall (u, v) \in X \times X,\ \forall (f, g) \in \Lambda, \tag{5.1.8}$$

$$\|DF_{fg}(u_0, v_0)(\xi, \eta)\|_{X\times X} \le M_1\|(\xi, \eta)\|_{X\times X},\ \forall\ (u_0, v_0),\ (\xi, \eta) \in X \times X,\ \forall (f, g) \in \Lambda, \tag{5.1.9}$$

and

$$\text{Supp } F_{fg} \subset \{(u, v) \in X \times X :\ \|(u, v)\|_{X\times X} \le 2R\},\ \forall (f, g) \in \Lambda. \tag{5.1.10}$$

Proof. It is clear from the definition of F_{fg} that

$$\text{Supp } F_{fg} \subset \{(u, v) \in X \times X :\ \|(u, v)\|_{X\times X} \le 2R\},\ \forall (f, g) \in \Lambda.$$

For (5.1.8) and (5.1.9), the existence of the constant M_0 and M_1 is guaranteed by (5.1.2) and the fact that

$$\{(u, v) \in X \times X|\ \|(u, v)\|_{X\times X} \le 2R\} \subset \{(u, v) \in X \times X|\ \|(u, v)\|_{C(\bar{\Omega})\times C(\bar{\Omega})} \le 2RC\},$$

where C is the embedding constant $X \hookrightarrow C(\bar{\Omega})$. Therefore , we can choose a sufficiently small neighborhood Λ of (f_0, g_0) such that

$$\|(f,g)-(f_0,g_0)\|_{C^2(\bar{\Omega}\times[-2RC,2RC]\times[-2RC,2RC])} < 1, \ \forall (f,g) \in \Lambda.$$

□

Lemma 5.1.3. *(Gap condition)* Let $K_0 = 16M_1^2,\ K_1 = 4M_1$ where M_1 is the positive constant in proposition 5.1.2. Then there exists $N \in \mathbb{N}$ such that

$$\lambda_N > K_0 \text{ and } \lambda_{N+1} - \lambda_N > 2K_1.$$

Proof. Since the eigenvalues of A can be rearranged as a strictly increasing sequence converging to ∞, we always have $\lambda_N > K_0$. Because $\lambda_{N+1} - \lambda_N = O(N)$, it is clear that $\lambda_{N+1} - \lambda_N > 2K_1$ if N is sufficiently large. □

Proposition 5.1.4. Given $(f_0, g_0) \in \mathcal{CP}$. Let Λ be defined as in proposition 5.1.2. Choose a natural number N such that the gap condition in lemma 5.1.3 holds. Let $W := W_N$. Then, for each $(f,g) \in \Lambda$, there exists a C^1 map $\Phi_{fg} : W \to W^\perp \cap (X \times X),\ \Phi_{fg} = (\Phi_{fg}^u, \Phi_{fg}^v)$ with the following properties

(i) Supp $\Phi_{fg} \subset \{(u,v) \in X \times X :\ \|(u,v)\|_{X\times X} \leq 2R\},\ \forall (f,g) \in \Lambda$,

(ii) $\|\Phi_{fg}(w)\|_{X\times X} \leq L_0,\ \forall w \in W,\ \forall (f,g) \in \Lambda$ where $L_0 = \frac{M_0 e^{-1/2}}{2\lambda_{N+1}}$ (M_0 is the positive constant in proposition 5.1.2),

(iii) The manifold $\mathcal{I}_{fg}$:=Graph Φ_{fg} is invariant under the flow generated by (5.1.7) and attracts all solutions of (5.1.7) exponentially,

(iv) $\|D\Phi_{fg}(w)\|_{\mathcal{L}(W,X\times X)} \leq L_1,\ \forall w \in W,\ \forall (f,g) \in \Lambda$, where $L_1 \leq \frac{2M_1}{\lambda_{N+1}-\lambda_N}$ (M_1 is the positive constant proposition 5.1.2),

(v) $\Phi_{fg},\ D\Phi_{fg}$ are continuous in (f,g).

Proof. Apply theorem 2.1 in [27] with $\alpha = \beta = 1/2,\ k = 1,\ C_0 = M_0,\ C_1 = M_1$. For simplicity, we will write Φ_{fg} as Φ when no confusion should arises. □

From now on, let W be the linear space defined in proposition 5.1.4 with N chosen large enough such that

$$L_0^2 - \|F_{f_0g_0}\|_0 L_0 < m_0,\ (R + \|F\|_0)L_0L_1 < m_0/2 \tag{5.1.11}$$

where $\|F_{f_0g_0}\|_0 = \sup_{(u,v)\in X\times X} \|F(u,v)\|_{X\times X}$. Let P be the orthogonal projection of $L^2(\Omega) \times L^2(\Omega)$ to W and $Q = Id - P$. By applying P and Q to (5.1.7), we obtain the system

$$w_t + Aw = (P \circ F)(w + w^\perp), \tag{5.1.12}$$

$$w_t^\perp + Aw^\perp = (Q \circ F)(w + w^\perp),\ w \in W. \tag{5.1.13}$$

By proposition 5.1.4, we can write (5.1.12) as

$$w_t + Aw = (P \circ F)(w + \Phi(w)),\ w \in W. \tag{5.1.14}$$

Lemma 5.1.5. Given $(f_0, g_0) \in \mathcal{CP}$. Let

$$\Gamma_R = \{w \in W |\ \|w + \Phi_{f_0g_0}(w)\|_{X\times X} = R,\ w + \Phi(w) \geq 0\}.$$

Then

$$\langle Aw - (P \circ F_{f_0 g_0})(w + \Phi_{f_0 g_0}(w)), \nu(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} \geq m_0 > 0, \ \forall w \in \Gamma_R, \tag{5.1.15}$$

where $\nu(w)$ is the outer normal vector at w and m_0 (depending on f_0, g_0) is the positive constant defined in proposition 5.1.1.

Proof. For simplicity, we will write $\Phi, \ F$ instead of $\Phi_{f_0 g_0}, \ F_{f_0 g_0}$. By lemma 5.1.1, we have

$$\langle A(w + \Phi(w)) - F(w + \Phi(w)), w + \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} \geq 2m_0, \ \forall w \in \Gamma_R. \tag{5.1.16}$$

On the other hand, we have

$$\begin{aligned}
&\langle A(w + \Phi(w)) - F(w + \Phi(w)), w + \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} \\
&= \langle Aw + A\Phi(w) - (P \circ F)(w + \Phi(w)) - (Q \circ F)(w + \Phi(w)), w + \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} \\
&= \langle Aw - (P \circ F)(w + \Phi(w)), w \rangle_{L^2(\Omega) \times L^2(\Omega)} \\
&\quad + \langle A\Phi(w) - (Q \circ F)(w + \Phi(w)), \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)}, \ \forall w \in W.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle Aw - (P \circ F)(w + \Phi(w)), w \rangle_{L^2(\Omega) \times L^2(\Omega)} \qquad (5.1.17) \\
&\geq 2m_0 - \langle A\Phi(w) - (Q \circ F)(w + \Phi(w)), \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} \\
&= 2m_0 - \langle A\Phi(w), \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} + \langle (Q \circ F)(w + \Phi(w)), \Phi(w) \rangle_{L^2(\Omega) \times L^2(\Omega)} \\
&\geq 2m_0 - \|\Phi(w)\|^2_{X \times X} - \|(Q \circ F)(w + \Phi(w))\|_{L^2(\Omega) \times L^2(\Omega)} \|\Phi(w)\|_{L^2(\Omega) \times L^2(\Omega)} \\
&\geq 2m_0 - \|\Phi(w)\|^2_{X \times X} - \|F(w + \Phi(w))\|_{L^2(\Omega) \times L^2(\Omega)} \|\Phi(w)\|_{L^2(\Omega) \times L^2(\Omega)}, \ \forall w \in \Gamma_R.
\end{aligned}$$

Using property (ii) in proposition 5.1.4 and (5.1.11), we have

$$\begin{aligned}
&\langle Aw - (P \circ F)(w + \Phi(w)), w\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&\geq 2m_0 - \|\Phi(w)\|^2_{X\times X} - \|F\|_0 \|\Phi(w)\|_{X\times X} \\
&\geq 2m_0 - L_0^2 - \|F\|_0 L_0 \geq m_0, \ \forall w \in \Gamma_R. \qquad (5.1.18)
\end{aligned}$$

Define

$$\begin{aligned}
H : W &\rightarrow \mathbb{R} \\
p &\mapsto \langle D\Phi(w)(p), \Phi(w)\rangle_{X\times X}.
\end{aligned}$$

It is clear that H is a continous linear functional on $(W, \|\cdot\|_{X\times X})$. By the Riesz representation (cf. [37]), there exists $w^* \in W$ such that $\langle w^*, w\rangle = H(w), \ \forall w \in W$ and

$$\|w^*\|_{X\times X} = \|H\|_{\mathcal{L}(W,\mathbb{R})} \leq \|D\Phi(w)\|_{\mathcal{L}(W,X\times X)} \|\Phi(w)\|_{X\times X} \leq L_0 L_1 < 1, \qquad (5.1.19)$$

where $L_0, \ L_1$ are constants in proposition (5.1.4). The outer normal vector $\nu(w)$ at $w \in \Gamma_R$ then has the representation $\nu(w) = 2w + 2w^*$. Therefore,

$$\begin{aligned}
&\langle Aw - (P \circ F)(w + \Phi(w)), \ \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&= \langle Aw - (P \circ F)(w + \Phi(w)), 2w + 2w^*\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&= \langle Aw - (P \circ F)(w + \Phi(w)), 2w\rangle_{L^2(\Omega)\times L^2(\Omega)} + \\
&\quad \langle Aw - (P \circ F)(w + \Phi(w)), 2w^*\rangle_{L^2(\Omega)\times L^2(\Omega)}, \ \forall w \in \Gamma_R. \qquad (5.1.20)
\end{aligned}$$

From (5.1.18) and (5.1.20), we have

$$
\begin{aligned}
&\langle Aw - (P \circ F)(w + \Phi(w)),\ \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)} \\
&\geq 2m_0 - 2\|Aw\|_{L^2(\Omega)\times L^2(\Omega)}\|w^*\|_{L^2(\Omega)\times L^2(\Omega)} \\
&\quad -2\|(P \circ F)(w + \Phi(w))\|_{L^2(\Omega)\times L^2(\Omega)}\|w^*\|_{L^2(\Omega)\times L^2(\Omega)} \\
&\geq 2m_0 - 2\|Aw\|_{L^2(\Omega)\times L^2(\Omega)}\|w^*\|_{L^2(\Omega)\times L^2(\Omega)} \qquad (5.1.21)\\
&\ -2\|F\|_0\|w^*\|_{L^2(\Omega)\times L^2(\Omega)},\ \forall w \in \Gamma_R.
\end{aligned}
$$

We have $\|Aw\|_{L^2(\Omega)\times L^2(\Omega)} = \sqrt{\sum_1^N c_i^2\lambda_i^2}$ for $w \in W_N$, $w = \sum_1^N c_i w_i$. If $w \in \Gamma_R$, we have $\|w\|_{X\times X} \leq \|w + \Phi(w)\|_{X\times X} = R$. But $\|w\|_{X\times X} = \sqrt{\sum_1^N c_i^2\lambda_i}$. Therefore,

$$
\|Aw\|_{L^2(\Omega)\times L^2(\Omega)} \leq \sqrt{\lambda_N \Sigma_1^N c_i^2\lambda_i} \leq \sqrt{\lambda_N}\ R. \qquad (5.1.22)
$$

From (5.1.11), (5.1.19), (5.1.21) and (5.1.22), we have

$$
\langle Aw - (P \circ F)(w + \Phi(w)),\ \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)} \geq 2m_0 - 2(R + \|F\|_0)L_0L_1 \geq m_0.
$$

□

Lemma 5.1.6. Given $(f_0, g_0) \in \mathcal{CP}$ and $\varepsilon_0 > 0$. If $d_{\mathcal{CP}}((f,g),(f_0,g_0))$, $(f,g) \in \mathcal{CP}$, is small enough so that $\|F_{fg} - F_{f_0g_0}\|_{X\times X} < \frac{m_0}{4(R+C)}$ where C is the constant from the embedding $X \hookrightarrow C(\bar{\Omega})$. Then

$$
\langle Aw - (P \circ F_{fg})(w + \Phi_{f_0g_0}(w)), \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)} \geq m_0/2 > 0,\ \forall w \in \Gamma_R, \qquad (5.1.23)
$$

where $\nu(w)$ is the outer normal vector at w.

Proof. We have

$$\begin{aligned}
&\langle Aw - (P \circ F_{fg})(w + \Phi_{f_0g_0}(w)), \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)} \qquad (5.1.24)\\
&= \langle Aw - (P \circ F_{f_0g_0})(w + \Phi_{f_0g_0}(w)), \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)}\\
&\quad + \langle (P \circ F_{fg})(w + \Phi_{f_0g_0}(w)) - (P \circ F_{f_0g_0})(w + \Phi_{f_0g_0}(w)), \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)}, \ \forall w \in W.
\end{aligned}$$

From (5.1.15) and (5.1.24), we have

$$\begin{aligned}
&\langle Aw - (P \circ F_{fg})(w + \Phi_{f_0g_0}(w)), \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)}\\
&\geq m_0 + \langle (P \circ F_{fg})(w + \Phi_{f_0g_0}(w)) - (P \circ F_{f_0g_0})(w + \Phi_{f_0g_0}(w)), \nu(w)\rangle_{L^2(\Omega)\times L^2(\Omega)}\\
&\geq m_0 - \|(F_{fg} - F_{f_0g_0})(w + \Phi_{f_0g_0}(w))\|_{L^2(\Omega)\times L^2(\Omega)} \|\nu(w)\|_{L^2(\Omega)\times L^2(\Omega)}\\
&\geq m_0 - \|F_{fg} - F_{f_0g_0}\|_0 (2\|w\|_{L^2(\Omega)\times L^2(\Omega)} + 2\|w^*\|_{L^2(\Omega)\times L^2(\Omega)})\\
&\geq m_0 - \|F_{fg} - F_{f_0g_0}\|_0 (2R + 2C) \geq m_0 - \frac{m_0}{2} = m_0/2, \ \forall w \in \Gamma_R.
\end{aligned}$$

□

Let $\mathcal{W}_R = \{w \in W | \ \|w + \Phi_{f_0g_0}(w)\|_{X\times X} \leq R, \ w + \Phi_{f_0g_0}(w) \geq 0\}$. The boundary of $\mathcal{W}_{\mathcal{R}}$ (in W) includes Γ_R, Γ_R^u and Γ_R^v where

$$\Gamma_R^u := \{w \in W | \ u + \Phi^u_{f_0g_0}(u, v) \geq 0, \ v + \Phi^v_{f_0g_0}(u, v) = 0\},$$

$$\Gamma_R^v := \{w \in W | \ u + \Phi^u_{f_0g_0}(u, v) = 0, \ v + \Phi^v_{f_0g_0}(u, v) \geq 0\}.$$

Proposition 5.1.7. Given $(f_0, g_0) \in \mathcal{CP}$. Let $\Lambda(f_0, g_0)$ be a sufficiently small neighborhood of (f_0, g_0) in $\mathcal{CP}$. The semi-flows $\{\Pi_t^{fg}{}_{|W},\ t \geq 0\},\ (f,g) \in \Lambda(f_0, g_0)$ are invariant on $\mathcal{W}_R$.

Proof. Clearly, the semi-flows $\{\Pi_t^{fg}{}_{|W},\ t \geq 0\},\ (f,g) \in \Lambda(f_0, g_0)$ are invariant on $\Gamma_R^u,\ \Gamma_R^v$. On Γ_R, by lemma 5.1.6 guarantees inward flows. □

Proposition 5.1.8. If $(f, g) \in \mathcal{MS}$ then (5.1.14) has Morse Smale property (in $\mathcal{W}_R$).

Proof. For simplicity, we write $\mathcal{I}$ and Π_t for $\mathcal{I}_{fg}$ and Π_t^{fg}. Let $e_0 \in X_+ \times X_+$ be a critical element of (5.1.7). By applying P to (5.1.7), we have $P(e_0)$ is a critical element of (5.1.14) in $\mathcal{W}_R$. Now, let $w \in \mathcal{W}_R$ be a critical element of (5.1.14). Because $\mathcal{I}$ is invariant under the semi-flow $\{\Pi_t,\ t \geq 0\}$, we have $\Pi_t(w(0) + \Phi(w(0))) \in \mathcal{I},\ \forall t \geq 0$. Therefore, there exists $\tilde{w}(t),\ t \geq 0$ such that $\Pi_t(w(0) + \Phi(w(0))) = \tilde{w}(t) + \Phi(\tilde{w}(t)), \forall t \geq 0$. Since $\Pi_t(w(0) + \Phi(w(0))$ is the solution of (5.1.7) with initial $w(0) + \Phi(w(0))$, $P(\Pi_t(w(0) + \Phi(w(0)))) = P(\tilde{w}(t) + \Phi(\tilde{w}(t))) = \tilde{w}(t)$ is a solution of (5.1.14) with initial $w(0)$. Because of unique solvability of (5.1.14), we have $\tilde{w} = w$. Hence, $w + \Phi(w) \in X_+ \times X_+$ is a critical element of (5.1.7). Therefore, if $(f, g) \in \mathcal{MS}$, then (5.1.14) has finitely many critical elements in $\mathcal{W}_R$. Next, suppose $w_0 \in \mathcal{W}_R$ is a critical element of (5.1.14), we have $w_0 + \Phi(w_0) \in X_+ \times X_+$ is a critical element of (5.1.7). Let $\mathcal{I}_{fg}^{R+} = \mathcal{I}_{fg} \cap \{(u,v) \in X_+ \times X_+ |\ \|(u,v)\|_{X \times X} \leq R\}$ and P_{fg} be the restriction of the orthogonal projection P to $\mathcal{I}_{fg}^{R+}$. Clearly, P_{fg} is a homeomorphism from $\mathcal{I}_{fg}^{R+}$ to $\mathcal{W}_R$ and $\gamma_t = P_{fg} \circ \Pi_t \circ P_{fg}^{-1}\ \forall t \geq 0$ on $\mathcal{W}_R$, where $\{\gamma_t,\ t \geq 0\}$ is the semi-flows generated by (5.1.14). Therefore, $(D\gamma_t)(w) = (P_{fg} \circ D\Pi_t \circ P_{fg}^{-1})(w)$ which implies that λ is an eigenvalue of $(D\gamma_t)(w)$ with eigenfunction φ if and only if it is an eigenvalue of $D\Pi_t(w + \Phi(w))$ with eigenfunction $\varphi + \Phi(\varphi)$. Since $(f, g) \in \mathcal{MS}$, $w_0 + \Phi(w_0)$ is a hyperbolic critical element of (5.1.7). Hence, w_0 is a hyperbolic critical element of (5.1.14). Finally, we prove the transversal intersection of stable and unstable manifolds of critical elements (in $\mathcal{W}_R$) of (5.1.14). Given two critical elements w_1, w_2 of (5.1.14). Suppose $\tilde{W}^u(w_1) \cap \tilde{W}^s_{\text{loc}}(w_2) \neq \emptyset$ where $\tilde{W}^u(w_1)$, $\tilde{W}^s_{\text{loc}}(w_2)$ are unstable and local stable manifolds of w_1 and w_2. Let $e_i = w_i + \Phi(w_i)$, $i = 1, 2$. Clearly, $\tilde{W}^u(w_1) = \{w \in W | \ w + \Phi(w) \in W^{u+}(e_1)\}$. Let $\beta \in \tilde{W}^u(w_1) \cap \tilde{W}^s_{\text{loc}}(w_2)$. We now prove $T_\beta \tilde{W}^u(w_1) \oplus T_\beta \tilde{W}^s_{\text{loc}}(w_2) = W$. Since $\eta = \beta + \Phi(\beta) \in W^{u+}(e_1) \cap W^{s+}(e_2)$, we have $T_\eta W^{u+}(e_1) \oplus T_\eta W^{s+}(e_2) = X \times X$ (because $(f, g) \in \mathcal{MS}$). By the invariant foliation theory in [5], there exists a unique C^1 leaf $\mathcal{J}$ with codimension N which passes through η and is transversal to the inertial manifold $\mathcal{I}$. Let $W^{s*} := \{w + \Phi(w) | \ w \in \tilde{W}^s(w_2)\}$. Then we have that $T_\eta W^{s+}(e_2) = T_\eta W^{s*} \oplus T_\eta \mathcal{J}$. Hence $W^{u+}(e_1)$ is transversal to W^{s*} and the dimension of $T_\eta W^{u+}(e_1) \oplus T_\eta W^{s*}$ is N. Since $\|\Phi\|_{\mathcal{L}(W, W^\perp \cap X \times X)} < 1$, the map $w \mapsto w + \Phi(w)$ is a diffeomorphism from W to $\mathcal{I}$ and it maps $\tilde{W}^u(w_1)$ to $W^{u+}(e_1)$, $\tilde{W}^s(w_2)$ to W^{s*}. Hence, $\tilde{W}^s(w_1)$ is transversal to $\tilde{W}^s(w_2)$ because $T_\eta W^{u+}(e_1) \oplus T_\eta W^{s*}$. □

5.2 Tubular Family Theorem

Lemma 5.2.1. Given an ordered chain $\alpha_1 \leq_3 \alpha_2 \leq_3 \ldots \leq_3 \alpha_n$. Let $G^u(\alpha_n)$ be a fundamental domain of $W^u(\alpha_n)$ and $W^s(\alpha_i)$, $1 \leq i \leq n$, be the stable manifold of α_i. Then we have

$$\partial W^s(\alpha_1) \cap G^u(\alpha_n) \subset \bigcup_{2 \leq i \leq n-1} W^s(\alpha_i) \cap G^u(\alpha_n).$$

Proof. Let $x \in \partial W^s(\alpha_1) \cap G^u(\alpha_n)$. There exists a sequence $\{x_j\} \subset W^s(\alpha_1)$, $x_j \to x$ as $j \to \infty$. We also have $\|\Pi^{fg}_t(x_j) - \Pi^{fg}_t(x)\| \leq \|\Pi^{fg}_t\| \|x_j - x\|$ for any $t > 0$. Notice that $\Pi^{fg}_t(x_j) \in W^s(\alpha_1)$ for all j and for all $t > 0$. Thus, for any given $\varepsilon > 0$, $\Pi^{fg}_t(x) \in B_\varepsilon(W^s(\alpha_1))$ for all $t > 0$. Suppose $\Pi^{fg}_t(x) \to \beta$, then β must be one of the α_i, $2 \leq i \leq n-1$. So $x \in \bigcup_{2 \leq i \leq n-1} W^s(\alpha_i) \cap G^u(\alpha_n)$. □

Theorem 5.2.2. *(Tubular family Theorem)* Let $(f,g) \in \mathcal{MS}$. There exists a compatible, invariant system of tubular families $\{\Gamma^i\}$, each Γ^i being a tubular family of $W^s(\alpha_i)$, where α_i is a critical element of γ_t^{fg}, where γ_t^{fg} is the flow introduced in the proof of proposition (5.1.8).

Proof. Define $L_0 = \{\alpha \mid \alpha \text{ is a sink}\}$, $L_1 = \{\alpha \mid beh(\beta|\alpha) = 1,\ \beta \in L_1\}$,

$L_k = \{\alpha \mid beh(\alpha|\beta) = k,\ \beta \in L_1\}$. For each $\alpha \in L_1$, define $\Gamma^\alpha = W^s(\alpha)$. In this proof, $\Gamma^\bullet$ denotes a tubular family of $\bullet$

Assume that Γ^α has been constructed for all $\alpha \in \bigcup_{0\le i\le k-1} L_i$. We will construct Γ^α, $\alpha \in L_k$. Let $\alpha \in L_k$ be a periodic orbit, $p \in \alpha$, $G^u(p)$ be the fundamental domain of $W^u_{loc}(p)$ and $\beta \in L_{k-1}$, $W^u(\alpha) \bigcap W^s_{loc}(\beta) \neq \emptyset$. From lemma 5.2.1, we have $G^u(p) \bigcap \partial W^s(\beta) = \emptyset$. Here, the boundary is relative with respect to $W^s(\beta)$. Now we apply lemma 2.5 in [27] with $C = G^u(p)$, $W = W^u(p)$, $U_0 = \emptyset$, $U = U_\beta = \Gamma^\beta \cap N^u(p)$ ($N^u(p)$ is a properly chosen fundamental neighborhood of $W^u_{loc}(p)$), we get a continuous retraction $r_\beta : U_\beta \to W^u(p)$ which satisfies $(i), (ii), (iii)$ of lemma 2.5 in [27]. Since Γ^β are mutually separated, we can define $r_{k-1} = \bigcup_{\beta \in L_{k-1},\ W^u(\alpha)\cap W^s_{loc}(\beta)\neq\emptyset} r_\beta$. That is,

$$r_{k-1} : (\bigcup_{\beta \in L_{k-1},\ W^u(\alpha)\cap W^s_{loc}(\beta)\neq\emptyset} \Gamma^\beta) \cap N^u(p) \to W^u(p)$$

Next , we will extend r_{k-1} to $\bigcup_{\xi \in L_{k-2},\ W^u(\alpha)\cap W^s_{loc}(\xi)\neq\emptyset} \Gamma^\xi$. Let

$$\xi \in L_{k-2},\ W^u(\alpha) \bigcap W^s_{loc}(\xi) \neq \emptyset$$

From lemma 2.1.5, we have

$$\partial W^s(\xi) \cap G^u(p) \subset \bigcup_{\beta \in L_{k-1}, W^u(\alpha)\cap W^s_{loc}(\beta)\neq\emptyset} W^s(\beta) \cap G^u(p) \subset \bigcup_{\beta \in L_{k-1}, W^u(\alpha)\cap W^s_{loc}(\beta)\neq\emptyset} \Gamma^\beta \cap N^u(p)$$

Applying lemma 2.5 in [27] with $C = G^u(p)$, $W = W^u(p)$, $r_0 = r_{k-1}$, $U = U_\xi = \Gamma^\beta \cap N^u(p)$, $U_0 = \bigcup_{\beta \in L_{k-1},\ W^u(\alpha)\cap W^s_{loc}(\beta)\neq\emptyset} \Gamma^\beta \cap N^u(p)$, we get a continuous retraction $r_\xi : U_\xi \to W^u(p)$ which satisfies $(i), (ii), (iii)$ of lemma 2.5 in [27]. Define $r_{k-2} = \bigcup_{\xi \in L_{k-1},\ W^u(\alpha)\cap W^s_{loc}(\xi)\neq\emptyset} r_\xi$, that is,

$$r_{k-2} : (\bigcup_{\xi \in L_{k-2},\ W^u(\alpha)\cap W^s_{loc}(\xi)\neq\emptyset} \Gamma^\beta) \cap N^u(p) \to W^u(p).$$

Continuing this process through $L_{k-3},\ L_{k-4}, ...,\ L_1$, we have a continuous retraction $r : N^u(p) \to W^u(p)$. Now, let N be an open neighborhood of $p \in \alpha$ in $W^u(p)$ with $\partial N = G^u(p)$. For each $y \in N$ there is a unique x_y and a unique $t_{x_y} \in G^u(p)$ such that $y = \gamma^{fg}_{-t_{x_y}}(x_y)$. We define $T^\alpha_y = \gamma^{fg}_{-t_{x_y}}(r^{-1}(x_y))$ and $T^\alpha_p = W^s(p)$. Then $\{T^\alpha_y\}_{y\in N}$ is a invariant tubular family of $W^s(p)$ under $k = \xi_t(\pi, .)_{|N}$ (π is the period of α). The tubular family of $W^s(\alpha)$ is now defined as in definition (2.1.5). □

5.3 A-stability

We shall prove Theorem C in this section.

Theorem 5.3.1. Let $(f_0, g_0) \in \mathcal{CP}$. The semi-flows $\{\gamma_t^{f_0 g_0}{}_{|\mathcal{W}_R},\ t \geq 0\}$ is $\mathcal{A}$-stable (the notion of γ_t was introduced in the proof of proposition 5.1.8).

Proof. Let p_2 be a critical element of $\{\gamma_t^{f_0 g_0}{}_{|\mathcal{W}_R},\ t \geq 0\}$ with behavior ≤ 1 with respect to sources. Consider source p_1 such that $\mathrm{beh}(p_1|p_2) = 1$. Put $\bar{p}_i = \rho(p_i)$ where $\rho = \rho(f, g),\ (f, g) \in \mathcal{CP}$, is the homeomorphism in Proposition (4.1.5). Since $\tilde{W}^u(p_1)$ is C^1 close to $\tilde{W}^u(\bar{p}_1)$ on compact sets, $\tilde{W}^u(p_1) \cap \tilde{G}^s(p_1)$ is compact, there exists a diffeomorphism $h_2 : \tilde{W}^u(p_1) \cap \tilde{G}^s(p_2) \to \tilde{W}^u(\bar{p}_2) \cap \tilde{W}^s_{\text{loc}}(\bar{p}_2)$, h_2 is close to the identity map. For any $y \in \tilde{W}^u(p_1) \cap \tilde{W}^s_{\text{loc}}(p_2)$, there exists a unique positive (or negative) time $\tau = \tau(y)$ such that $\gamma^{fg}_\tau(y) \in \tilde{W}^u(p_1) \cap \tilde{G}^s(p_2)$. Therefore we can extend h_2 to $\tilde{W}^u(p_1) \cap \tilde{W}^s(p_2)$ by using the flows

$$h_2 = \gamma^{fg}_{-\tau} \circ h_2 \circ \gamma^{f_0 g_0}_\tau : \tilde{W}^u(p_1) \cap \tilde{W}^s(p_2) \to \tilde{W}^u(\bar{p}_1) \cap \tilde{W}^s(\bar{p}_2).$$

We do the same for all other sources p_1 of which $\text{beh}(p_1|p_2) = 1$. Then we can extend h_2 to $\mathcal{A}^0_{\mathcal{W}_R} \cap \tilde{W}^s(p_2)$ where $\mathcal{A}^0_{\mathcal{W}_R}$ is the attractor of $\{\gamma^{f_0 g_0}_t{}_{|\mathcal{W}_R},\ t \geq 0\}$. Note that $\mathcal{A}^0_{\mathcal{W}_R} = P(\mathcal{A}(f_0, g_0))$. Repeat the same procedure for other p_2 with behavior ≤ 1 with respect to the sources to extend h_2 to $\bigcup_{beh(p|sources)=1}(\mathcal{A}^0_{\mathcal{W}_R} \cap \tilde{W}^s(p))$.

The next step is to consider p_3 with behavior ≤ 2. For the sources of behavior 1 with respect to p_3, the procedure is similar to the one we just use above. Let p_1 be a source with $\text{beh}(p_1|p_3) = 2$. There exists at least a sequence $p_3 \leq_3 p_2 \leq_3 p_1$ such that $\text{beh}(p_1|p_2)$ $=\text{beh}(p_2|p_3) = 1$. Since $\text{beh}(p_2|p_3) = 1$, we can define a diffeomorphism $\bar{h}_3$ on $\tilde{W}^u(p_2) \cap \tilde{G}^s(p_3)$ similarly to the way we define h_2. The existence of compatible system of unstable foliations guarantees that $\tilde{W}^u(p_1)$ intersects $\tilde{W}^s_{\text{loc}}(p_3)$. For each leaf of $\tilde{W}^u(p_1) \cap \tilde{G}^s(p_3)$ which is near $\tilde{W}^u(p_2) \cap \tilde{G}^s(p_3)$, there corresponds a unique point $y \in \tilde{W}^u(p_1) \cap \tilde{W}^s_{\text{loc}}(p_2)$ near p_2 such that $\gamma_{t_0}(y)$ belongs to that leaf for some $t_0 > 0$. We index these leaves as $\mathcal{J}_y,\ y \in \tilde{W}^u(p_1) \cap \tilde{W}^s_{\text{loc}}(p_2)$. Since $\mathcal{J}_y$ is near $\tilde{W}^u(p_2) \cap \tilde{G}^s(p_3)$, there exists a diffeomorphism $i_y : \mathcal{J}_y \to \tilde{W}^u(p_2) \cap \tilde{W}^s_{\text{loc}}(p_3)$. The same happens for the perturbed system, so we also have a diffeomorphism $i_{h_2(y)} : \mathcal{J}_{h_2(y)} \to \tilde{W}^u(\bar{p}_2) \cap \tilde{W}^s_{\text{loc}}(\bar{p}_3)$. Both i_y and $i_{h_2(y)}$ are close to identity map. The composition map $\bar{\bar{h}}_{3,y} = i^{-1}_{h_2(y)} \circ \bar{h}_3 \circ i_y$ is a diffeomorphism from $\mathcal{J}_y$ to $\mathcal{J}_{h_2(y)}$. Using $\bar{\bar{h}}_{3,y},\ y \in \tilde{W}^u(p_1) \cap \tilde{W}^s_{\text{loc}}(p_2)$, we can extend $\bar{h}$ to a small neighborhood

U of $\tilde{W}^u(p_2)\cap\tilde{G}^s(p_3)$ in $\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_2)\Big)\cap\tilde{G}^s(p_3)$ as follows

$$\bar{\bar{h}}_3(x)=\tilde{h}_{3,y}(x),\ x\in\mathcal{J}_y.$$

To extend $\bar{\bar{h}}_3$ to $\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_2)\Big)\cap\tilde{G}^s(p_3)$, we apply the Isotopy Extension Theorem (cf. [13]). Let N be the relative boundary in $\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_2)\Big)\cap\tilde{G}^s(p_3)$ of a small neighborhood $U_1\subset U$ in the domain of $\bar{\bar{h}}_3$, $M=\gamma^{f_0g_0}_{(-\epsilon,\epsilon)}(\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_2)\Big)\cap\tilde{G}^s(p_3)\setminus U_2)$, where $U_2\subset U_1$. Since $\tilde{W}^u(p_1)\cap\tilde{G}^s(p_3)$ and $\tilde{W}^u(\bar{p}_2)\cap\tilde{W}^s_{\text{loc}}(\bar{p}_3)$ are C^1 close, there exists a diffeomorsphism d which maps $\tilde{W}^u(p_1)\cap\tilde{G}^s(p_3)$ to $\tilde{W}^u(\bar{p}_2)\cap\tilde{W}^s_{\text{loc}}(\bar{p}_3)$. Let $j=d^{-1}\circ\bar{\bar{h}}:N\to M$. Applying the Isotopy Extension Theorem for j, we obtain an extension $\varphi:M\to M$ of j such that $\varphi_{|N}=j$, $\varphi(x)=x$ for all x outside a neighborhood V of $j(N)$. Let $\bar{\bar{\bar{h}}}_3=d\circ\varphi$. Define $h_3(x)=\bar{\bar{h}}_3(x)$ if $x\in U_1$ and $h_3(x)=\bar{\bar{\bar{h}}}_3(x)$ if $x\in\Big[\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_1)\Big)\cap\tilde{G}^s(p_3)\Big]\setminus U_1$. Then h_3 is an extension of $\bar{\bar{h}}_3$ to $\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_1)\Big)\cap\tilde{G}^s(p_3)$, $h_3:\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_1)\Big)\cap\tilde{G}^s(p_3)\to\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_1)\Big)\cap\tilde{W}^s_{\text{loc}}(\bar{p}_3)$. Using again the flows to extend h_3 to $\Big(\tilde{W}^u(p_1)\cup\tilde{W}^u(p_2)\Big)\cap\tilde{W}^s(p_3)$. For other possible sequence(s) $p_3\leq_3 p_2\leq_3\bar{p}_1$, we repeat the same procedure to extend h_3 to $\mathcal{A}^0_{\mathcal{W}_R}\cap\tilde{W}^s(p_3)$. The last step shows the full induction procedure. Because there are only finitely many critical elements, the induction is completed when we reach the sinks. Since $\cup_{p-\text{crit. element}}\mathcal{A}^0_{\mathcal{W}_R}\cap\tilde{W}^s(p)=\mathcal{A}^0_{\mathcal{W}_R}$, and $(\mathcal{A}^0_{\mathcal{W}_R}\cap\tilde{W}^s(p_i))\cap(\mathcal{A}^0_{\mathcal{W}_R}\cap\tilde{W}^s(p_j))=\emptyset,\ i\neq j$, we can define

$$h:\mathcal{A}^0_{\mathcal{W}_R}(f,g)\longrightarrow\mathcal{A}^0_{\mathcal{W}_R}(f',g')$$

by $h=h_1\cup h_2\cup h_3\cup\ldots$. The final step is to check the continuity of h. For any $x\in\mathcal{A}^0_{\mathcal{W}_R}(f,g)$, there exists p_i such that $x\in\mathcal{A}^0_{\mathcal{W}_R}\cap\tilde{W}^s(p_i)$. Notice that it is sufficient to prove the continuity of h at those $x \in \mathcal{A}^0_{\mathcal{W}_R} \cap \tilde{W}^s_{loc}(p_i)$. If p_i is a source or a sink, the continuity is trivial because $\mathcal{A}^0_{\mathcal{W}_R} \cap \tilde{W}^s_{loc}(p_i)$ is either p_i or $\mathcal{A}^0_{\mathcal{W}_R}$. Assume p_i is a saddle. Let $x_n \to x,\ x_n \in \mathcal{J}^u_{x_n}(p_i)$. We have $h(x_n) \in \mathcal{J}^u_{h_i(x_n)}(p'_i)$. By a property of tubular families, $\mathcal{J}^u_{h_i(x_n)}(p'_i)$ converges to $\mathcal{J}^u_{h_i(x)}(p'_i)$. Therefore the set of accumulation points of $\{h(x_n)\}$ is contained in $\tilde{W}^s_{\mathrm{loc}}(p'_i) \cap \mathcal{J}^u_{h_i(x)}(p'_i) = \{h_i(x)\}$. Since the set of accumulation points of $\{h(x_n)\}$ has only one single element $h_i(x)$, $h(x_n)$ must converge to $h_i(x) = h(x)$. Thus h is continuous. Hence h is a homeomorphism. □

Theorem 5.3.2. If $(f_0, g_0) \in \mathcal{MS}$, then $\{\Pi^{f_0g_0}_t,\ t \geq 0\}$ is $\mathcal{A}$-stable.

Proof. Let $h : \mathcal{A}^0_{\mathcal{W}_R} \to \mathcal{A}_{\mathcal{W}_R}$ be the homeomorphism in theorem 5.3.1. Define $H = P^{-1}_{fg} \circ h \circ P_{f_0g_0}$. Then $H : \mathcal{A}(f_0, g_0) \to \mathcal{A}(f, g)$ is a homeomorphism taking trajectories of $\mathcal{A}(f_0, g_0)$ to trajectories of $\mathcal{A}(f, g)$ and preserves the sense of direction in time. □

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