Mixed Groups with Decomposition Bases and Global k-Groups

Except where reference is made to the work of others, the work described in this thesis is my own or was done in collaboration with my advisory committee. This thesis does not include proprietary or classified information.

Chad Mathews

Certificate of Approval:

Overtoun Jenda
Professor
Mathematics and Statistics

William Ullery, Chair
Professor
Mathematics and Statistics

H. Pat Goeters
Professor
Mathematics and Statistics

Peter Nylen
Professor
Mathematics and Statistics

Stephen L. McFarland
Dean
Graduate School

Mixed Groups with Decomposition Bases and Global k-Groups

Chad Mathews

A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
August 7, 2006

Mixed Groups with Decomposition Bases and Global k-Groups

Chad Mathews

Signature of Author

Date of Graduation

Vita

Michael Chad Mathews, son of Michael Brett and Tammy Elaine Mathews, was born on October 27, 1981 in LaGrange, Georgia. He is a 2000 graduate of Heard County Comprehensive High School in Franklin, Georgia. In the fall of that year, he entered the University of West Georgia and received the degree of Bachelor of Science in Mathematics on May 6, 2004. In August of 2004, he began his graduate study in the Department of Mathematics and Statistics at Auburn University.

Thesis Abstract

Mixed Groups with Decomposition Bases and Global k-Groups

Chad Mathews

Master of Science, August 7, 2006
(B.S., University of West Georgia, 2004)

38 Typed Pages

Directed by William Ullery

This thesis is devoted to proving assertions made without proof by Paul Hill and Charles Megibben in their fundamental papers regarding knice subgroups and the Axiom 3 characterization of global Warfield groups. The main theme throughout is the relationship between the notions of a global k-group and a group with a decomposition basis. Most of our results involve properties of the auxiliary notions of primitive element and $*$-valuated coproduct in both the mixed and torsion free settings.

Acknowledgments

The author would like to thank Dr. William Ullery for lending his algebraic knowledge to this project and for his unyielding patience and understanding. The author also wishes to thank his family for their love and support.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliography follows van Leunen's *A Handbook for Scholars.*

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

Table of Contents

Chapter 1

Introduction

Throughout this thesis, G will always denote an additively written abelian group and we shall only consider such groups. We do not exclude the possibility that G is a nonsplit mixed group. By this we mean that G may contain elements of both finite and infinite order and the torsion subgroup of G is not necessarily a summand.

Our main goal in this thesis is to provide the justification for many of the basic facts that were stated by P. Hill and C. Megibben in [7, 8] without proof. Indeed, unless explicitly stated to the contrary, most of our results appear there.

We conclude this brief introduction with an outline of the remainder of the paper. In chapter two, we state the definitions and provide the notation that will be used throughout. Chapter three consists of various properties of decomposition bases leading up to the proof that a group with a decomposition basis is a k-group. In chapter four, we discuss torsion free groups with decomposition bases and we show that a torsion free group is completely decomposable if and only if it has a decomposition basis. In chapter five, we show that every torsion free separable group is a k-group, and we use this fact to provide an example of a k-group without a decomposition basis.

Chapter 2

Preliminaries

This chapter is devoted to providing some basic results needed for the remaining chapters. The terminology and notation used here is due to Hill and Megibben [7, 8].

Let $\mathcal{O}_\infty$ denote the class of ordinals with the symbol ∞ adjoined as a maximal element with the convention that $\alpha < \infty$ for all $\alpha \in \mathcal{O}_\infty$. If $x \in G$, we write $|x|_p$ for the height of x in G at the prime p. So $|x|_p = \alpha$ if $x \in p^\alpha G$ and $x \notin p^{\alpha+1}G$, while $|x|_p = \infty$ if $x \in p^\alpha G$ for all $\alpha \in \mathcal{O}_\infty$. If $\mathbb{P}$ is the set of rational primes, a *height matrix* is a doubly infinite $\mathbb{P} \times \omega_0$ matrix $M = [m_{p,i}]$ where $m_{p,i} \in \mathcal{O}_\infty$ and $m_{p,i} < m_{p,i+1}$ for all $p \in \mathbb{P}$ and $i < \omega_0$. By a *height sequence*, we mean any sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega_0}$ where $\alpha_i \in \mathcal{O}_\infty$ and $\alpha_i < \alpha_{i+1}$ for all $i < \omega_0$. Thus the p-row $M_p = \{m_{p,i}\}_{i<\omega_0}$ of a height matrix M is a height sequence. We shall write $\|x\|$ for the height matrix of x in G; that is, $\|x\|$ is the doubly infinite matrix indexed by $\mathbb{P} \times \omega_0$ and having $|p^i x|_p$ as its (p,i) entry. Similarly, $\|x\|_p$ will denote the height sequence of x at p. We shall sometimes affix a superscript to p-heights and height matrices in order to emphasize the group in which the heights are computed.

For two height matrices M and N, we write $N \leq M$ if $n_{p,i} \leq m_{p,i}$ for all primes p and $i < \omega_0$. We define the product kM of the positive integer k and the height matrix $M = [m_{p,i}]$ to be the height matrix having as its (p,i) entry $m_{p,j+i}$ where $j = |k|_p^{\mathbb{Z}}$. We say that M and N are *quasi-equivalent* and write $M \sim N$ if there are positive integers

k, l such that $N \leq kM$ and $M \leq lN$. Notice that $M \sim N$ implies that $M_q = N_q$ for all primes q for which $q \nmid k$ and $q \nmid l$.

Lemma 2.1. *For all $x \in G$ and positive integers k, $\|kx\| = k\|x\|$.*

Proof. We claim that $\|p^n x\| = p^n \|x\|$ for all primes p and positive integers n. Indeed,

$$\|p^n x\|_p = \{|p^n x|_p, |p^{n+1} x|_p, \ldots\} = p^n \{|x|_p, |px|_p, \ldots, |p^{n-1} x|_p, |p^n x|_p, \ldots\}$$
$$= p^n \|x\|_p = (p^n \|x\|)_p,$$

and if q is a prime different from p, $\|p^n x\|_q = \|x\|_q = p^n \|x\|_q$.

Now if $k = 1$, the result is clear. So suppose $k > 1$ where $k = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ for distinct primes p_i and positive integers n_i with $i \in \{1, 2, \ldots, r\}$. We proceed by induction on r. If $r = 1$, then we are done by what we have shown above. So suppose $r > 1$. By the induction hypothesis,

$$\|p_2^{n_2} \cdots p_r^{n_r} x\| = p_2^{n_2} \cdots p_r^{n_r} \|x\|.$$

Then again making use of the preceding paragraph, we have

$$\|kx\| = p_1^{n_1} \|p_2^{n_2} \cdots p_r^{n_r} x\| = p_1^{n_1} (p_2^{n_2} \cdots p_r^{n_r}) \|x\| = k\|x\|. \qquad \square$$

With each height matrix M and group G, we associate the fully invariant subgroups $G(M) = \{x \in G : \|x\| \geq M\}$ and $G(M^*) = \langle x \in G(M) : \|x\| \nsim M \rangle$. (We make the exception that if $M \sim \overline{\infty}$, where $\overline{\infty}$ is the height matrix with all entries ∞, then $G(M^*) = tG \cap G(M)$. Here tG denotes the torsion subgroup of G.) For each prime p and each height sequence $\overline{\alpha} = \{\alpha_i\}_{i<\omega_0}$, we define $G(\overline{\alpha}^*, p)$ to be the subgroup of G generated by those elements $x \in G$ such that $|p^i x|_p \geq \alpha_i$ for all i but $|p^i x|_p \neq \alpha_i$ for infinitely many i. Finally, we define the fully invariant subgroup $G(M^*, p)$ as $G(M^*, p) = G(M) \cap (G(M^*) + G(M_p^*, p))$.

Observe that if x is a generator of $G(M^*)$ (in the case $M \nsim \overline{\infty}$) or of $G(\overline{\alpha}^*, p)$, then so is mx for every nonzero integer m. Thus, for example, if $y \in G(M^*)$ and $M \nsim \overline{\infty}$, $y = x_1 + x_2 + \cdots + x_n$ with $\|x_i\| \geq M$ and $\|x_i\| \nsim M$ for all i.

Proposition 2.2. *The following results hold for all height matrices M, positive integers k, and primes p.*

(1) $G(kM) = kG(M)$

(2) $G((kM)^*) = kG(M^*)$

(3) $G((kM_p)^*, p) = kG(M_p^*, p)$

(4) $G((kM)^*, p) = kG(M^*, p)$

Proof. (1) First observe that for a given prime p and group G, we have that

$$G \supseteq pG \supseteq p^2 G \supseteq \cdots \supseteq p^\alpha G \supseteq \cdots .$$

That is, $p^\alpha G \supseteq p^{\alpha+1} G$ for every ordinal α. Since G is a set, there is a smallest ordinal λ such that $p^\lambda G = p^{\lambda+1} G$. Now, if $x \in G$, $|x|_p = \infty$ means that $x \in p^\alpha G$ for all $\alpha \geq \lambda$.

So if r is a positive integer and if $|x|_p = \infty$, then

$$x \in p^{\lambda+r}G = p^r(p^\lambda G)$$

which implies that there is a $y \in p^\lambda G$ such that $p^r y = x$. In particular, $|y|_p = \infty$.

We claim that $G(p^r M) \subseteq p^r G(M)$ for all primes p and positive integers r. We proceed by induction on r. For the case where $r = 1$, if $x \in G(pM)$, then $x = py$ with $|y|_p \geq m_{p,0}$ and $\|y\|_q = \|px\|_q$ for all primes $q \neq p$. But then $\|y\| \geq M$ so that $x \in pG(M)$. To finish the claim, note that

$$G(p^r M) = G(p(p^{r-1}M)) \subseteq pG(p^{r-1}M).$$

Then by induction, we have that $pG(p^{r-1}M) \subseteq p^r G(M)$. Hence, $G(p^r M) \subseteq p^r G(M)$ as claimed.

Now, if $k = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$ where p_i is a distinct prime and r_i is a positive integer for each i, then our argument above yields

$$G(kM) = G(p_i^{r_i}(k/p_i^{r_i})M) \subseteq p_i^{r_i}G((k/p_i^{r_i})M) \subseteq p_i^{r_i}G(M).$$

Therefore,

$$G(kM) \subseteq \bigcap_i p_i^{r_i}G(M) = (\prod_i p_i^{r_i})G(M) = kG(M).$$

Finally, if $x \in kG(M)$, then $x = ky$ for some $y \in G(M)$. But then Lemma 2.1 gives

$$\|x\| = \|ky\| = k\|y\| \geq kM.$$

That is, $x \in G(kM)$.

(2) We need to separately consider the cases where $M \sim \overline{\infty}$ and $M \nsim \overline{\infty}$.

Case 1. Suppose $M \sim \overline{\infty}$. Then, by definition, $k(G(M^*)) = k(G(M) \cap tG)$. Observe that $kM \sim \overline{\infty}$ and so

$$G((kM)^*) = G(kM) \cap tG = kG(M) \cap tG.$$

Now the fact that $k(G(M) \cap tG) \subseteq kG(M) \cap tG$ is clear. So suppose $x \in kG(M) \cap tG$. Then $x = ky$ for some $y \in G(M)$ and $nx = 0$ for some positive integer n. But then

$$nky = nx = 0$$

which implies that $y \in G(M) \cap tG$. Thus, $x \in k(G(M) \cap tG)$.

Case 2. Suppose $M \nsim \overline{\infty}$. If $x \in G((kM)^*)$, then

$$x = a_1 + a_2 + \cdots + a_n$$

where $a_i \in G(kM)$ and $\|a_i\| \nsim kM$ for $i = 1, 2, \ldots, n$. Then $a_i \in kG(M)$ and $a_i = kb_i$ where $b_i \in G(M)$ and $\|b_i\| \nsim M$. So $b_i \in G(M^*)$ which implies that $a_i \in kG(M^*)$.

Hence, $G((kM)^*) \subseteq kG(M^*)$. On the other hand, if $x \in kG(M^*)$, then $x = ky$ for some $y \in G(M^*)$. So

$$y = b_1 + b_2 + \cdots + b_n$$

where $b_i \in G(M)$ and $\|b_i\| \not\sim M$. But then

$$x = kb_1 + kb_2 + \cdots + kb_n$$

with $kb_i \in G(kM)$ and $\|kb_i\| \not\sim kM$. Therefore, $kb_i \in G((kM)^*)$ and $x \in G((kM)^*)$.

(3) Let $x \in kG(M_p^*, p)$. Then $x = ky$ where $y \in G(M_p^*, p)$. That is,

$$y = a_1 + a_2 + \cdots + a_r$$

where for $j = 1, 2, \ldots, r$, $\|a_j\|_p \geq M_p$ and $|p^i a_j|_p \neq m_{p,i}$ for infinitely many $i < \omega_0$. Then,

$$x = ka_1 + ka_2 + \cdots + ka_r$$

where for $j = 1, 2, \ldots, r$, $\|ka_j\|_p = k\|a_j\|_p \geq kM_p$ and $|p^i ka_j|_p \neq m_{p,i+e}$ for infinitely many $i < \omega_0$ with $e = |k|_p^{\mathbb{Z}}$. Hence, $ka_j \in G((kM_p)^*, p)$ for each j, which gives that $x \in G((kM_p)^*, p)$.

For the reverse inclusion, let p^e be the largest power of p that divides k. That is, e is again equal to $|k|_p^{\mathbb{Z}}$. Observe that it is enough to show that $G((p^e M_p)^*, p) \subseteq p^e G(M_p^*, p)$. Let x be a generator of $G((p^e M_p)^*, p)$. Then $\|x\|_p \geq p^e M_p$ and $|p^i x|_p \neq m_{p,i+e}$ for infinitely many $i < \omega_0$. So we have that $|p^i x|_p \geq m_{p,i+e} \geq i + e$ for all i. It then follows that

$$x \in p^{i+e}G = p^e(p^i G)$$

and $x = p^e y$ for some $y \in p^i G$. But then $\|y\|_p \geq M_p$ and $|p^i y|_p \neq m_{p,i}$ for infinitely many i. Thus, $x \in p^e G(M_p^*, p)$.

(4) Let $x \in G((kM)^*, p)$. Then $x = a_1 + a_2$ where

$$a_1 \in G((kM)^*) = kG(M^*)$$

and

$$a_2 \in G((kM_p)^*, p) \cap G(kM) = k(G(M_p^*, p) \cap G(M)).$$

So $a_1 = kb_1$ where $b_1 \in G(M^*)$ and $a_2 = kb_2$ where $b_2 \in G(M_p^*, p) \cap G(M)$. But then

$$x = kb_1 + kb_2 = k(b_1 + b_2)$$

where $b_1 + b_2 \in G(M^*) + (G(M_p^*, p) \cap G(M))$. Hence, $x \in kG(M^*, p)$. Similarly, if $x \in kG(M^*, p)$, then $x = ky$ where $y \in G(M^*, p)$. So $y = a_1 + a_2$ where $a_1 \in G(M^*)$ and $a_2 \in G(M_p^*, p) \cap G(M)$. Then $x = ka_1 + ka_2$ where $ka_1 \in G((kM)^*)$ and $ka_2 \in G((kM_p)^*, p) \cap G(kM)$. Thus, $x \in G((kM)^*, p)$. $\square$

Definition 2.3. Call an element $x \in G$ *primitive* if for each height matrix M, prime p and positive integer n, $nx \in G(M^*, p)$ implies that either $\|x\| \not\sim M$ or $|p^i nx|_p \neq m_{p,i}$ for infinitely many $i < \omega_0$.

If $\{A_i\}_{i \in I}$ is a family of independent subgroups of the group G, then the direct sum $A = \bigoplus_{i \in I} A_i$ is said to be a *valuated coproduct* in G provided that if $a = \sum_{i \in I} a_i$ with $a_i \in A_i$, then $|a|_p = \bigwedge_{i \in I} |a_i|_p = min\{|a_i|_p\}_{i \in I}$ for all primes p. This concept can be equivalently written as $A \cap G(M) = \bigoplus_{i \in I}(A_i \cap G(M))$ for all height matrices M.

Definition 2.4. Given a family of independent subgroups $\{A_i\}_{i \in I}$ of the group G, we say that the direct sum $A = \bigoplus_{i \in I} A_i$ is a $*$-*valuated coproduct* in G if $A \cap F = \bigoplus_{i \in I}(A_i \cap F)$ for each fully invariant subgroup F of the form $G(M)$, $G(M^*)$, $G(M_p^*, p)$ or $G(M^*, p)$.

We call a group G *simply presented* if it can be presented by generators and relations where each relation is of the form $mx = y$ or $mx = 0$ with m a positive integer. By a *global Warfield group*, we mean a direct summand of a simply presented group. In the mixed setting, it is well known that a summand of a simply presented group is not necessarily simply presented.

A collection $\mathcal{C}$ of subgroups of G is called an *Axiom 3 system* if it satisfies the following conditions.

(0) $0 \in \mathcal{C}$.

(1) If $\{N_i\}_{i \in I} \subseteq \mathcal{C}$, then $\sum_{i \in I} N_i \in \mathcal{C}$.

(2) For each $N \in \mathcal{C}$ and countable subgroup A of G, there exists $M \in \mathcal{C}$ such that $N + A \subseteq M$ and M/N is countable.

Furthermore, we say that G satisfies *Griffith's version of Axiom* 3 if there exists a collection $\mathcal{C}$ of subgroups of G satisfying conditions (0) and (2) above with (1) replaced by the statement that $\mathcal{C}$ is closed under unions of ascending chains.

A subgroup N of G is a *nice subgroup* if for each prime p and ordinal α, the cokernel of the inclusion map $(p^\alpha G + N)/N \rightarrowtail p^\alpha(G/N)$ contains no element of order p.

Definition 2.5. A subgroup N of G is a *knice subgroup* if the following conditions are satisfied.

(1) N is nice in G.

(2) To each finite subset S of G, there corresponds a (possibly empty) finite set of primitive elements $\{x_1, x_2, \ldots, x_m\}$ such that $N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_m \rangle$ is a $*$-valuated coproduct that contains some positive multiple of $\langle S \rangle$.

A subset X of independent elements in a group G is said to be a *decomposition basis* if each $x \in X$ has infinite order, $G/\langle X \rangle$ is a torsion group, and $\langle X \rangle = \bigoplus_{x \in X} \langle x \rangle$ is a valuated coproduct in G.

The importance of the notions defined above is revealed by the following theorem.

Theorem 2.6 (Hill and Megibben [8])**.** *For an arbitrary group G, the following conditions are equivalent.*

(i) G satisfies Axiom 3 with respect to knice subgroups.

(ii) G satisfies Griffith's version of Axiom 3 *with respect to knice subgroups.*

(iii) G is the union of a smooth chain $(G_\alpha)_{\alpha<\tau}$ *of nice subgroups such that* $G_0 = 0$ *and, for each* α*, either* $G_{\alpha+1}/G_\alpha$ *is cyclic of prime order or else* $G_{\alpha+1} = G_\alpha \oplus \langle x_\alpha \rangle$ *is a valuated coproduct in G with* x_α *an element of infinite order.*

(iv) G is a direct summand of a simply presented group, and hence a global Warfield group.

(v) G has a decomposition basis and satisfies Axiom 3 *with respect to nice subgroups.*

We call a group G a *(global) k-group* if the trivial subgroup 0 is a knice subgroup. Since 0 is a nice subgroup of every group G, G is a k-group if and only if to each finite subset S of G, there corresponds a (possibly empty) finite set of primitive elements $\{x_1, x_2, \ldots, x_n\}$ such that $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$-valuated coproduct that contains some positive multiple of $\langle S \rangle$. Notice that it is immediate that every torsion group is a k-group. Also, by Theorem 2.6, every global Warfield group G satisfies Axiom 3 with respect to knice subgroups, and hence is a k-group.

Chapter 3

Mixed Groups with Decomposition Bases

In this chapter we show that a mixed group G with a decomposition basis X is a k-group (Theorem 3.5).

Proposition 3.1. *If G has a decomposition basis X, then each element of X is primitive.*

Proof. Suppose that $x \in X$ and that $nx \in G(M^*, p)$ for some positive integer n, height matrix M and prime p. Assuming that $\|x\| \sim M$, we need to show that $|p^i nx|_p \neq m_{p,i}$ for infinitely many $i < \omega_0$. Because $G(M^*, p) = G(M) \cap (G(M^*) + G(M_p^*, p))$, we can write

$$nx = a_1 + a_2 + \cdots + a_r + b_1 + b_2 + \cdots + b_s$$

where, for $l = 1, 2, \ldots, r$, $a_l \in G(M)$ with $\|a_l\| \nsim M$ and, for $j = 1, 2, \ldots, s$, $\|b_j\|_p \geq M_p$ with $|p^i b_j|_p \neq m_{p,i}$ for infinitely many i.

Select a positive integer k so that all ka_l and kb_j are in $\langle X \rangle$. Then, for $l = 1, 2, \ldots, r$,

$$ka_l = c_l x + c_{l,1} x_1 + \cdots + c_{l,t} x_t$$

and, for $j = 1, 2, \ldots, s$,

$$kb_j = d_j x + d_{j,1} x_1 + \cdots + d_{j,t} x_t$$

where $x, x_1, \ldots, x_t$ are distinct elements of X and, for all l and j, $c_l, c_{l,1}, \ldots, c_{l,t}$ and $d_j, d_{j,1}, \ldots, d_{j,t}$ are contained in $\mathbb{Z}$. Since x has infinite order and $x, x_1, \ldots, x_t$ are $\mathbb{Z}$-independent elements of G,

$$knx = ka_1 + ka_2 + \cdots + ka_r + kb_1 + kb_2 + \cdots + kb_s$$

implies that

$$\sum_{l=1}^{r} c_l + \sum_{j=1}^{s} d_j = kn. \tag{†}$$

In particular, there is at least one c_l or d_j that is not 0.

We claim that $c_l = 0$ for all l. Indeed, if $c_l \neq 0$ for some l, then

$$ka_l = c_l x + c_{l,1}x_1 + \cdots + c_{l,t}x_t$$

and the fact that $\langle x \rangle \oplus \langle x_1 \rangle \oplus \cdots \oplus \langle x_t \rangle$ is a valuated coproduct imply that $\|ka_l\| \leq \|c_l x\|$. Recall that we are operating under the assumption that $\|x\| \sim M$. So, if we select a positive integer m such that $\|x\| \leq mM$, then

$$\|a_l\| \leq \|ka_l\| \leq \|c_l x\| = (|c_l|)\|x\| \leq (|c_l|m)M$$

with $|c_l|m > 0$. Moreover, we know that $M \leq \|a_l\|$ and we obtain the contradiction that $\|a_l\| \sim M$. Therefore, $c_l = 0$ for all l, as claimed.

We now know that $\sum_{l=1}^{r} c_l = 0$ and can conclude from condition (†) that

$$\sum_{j=1}^{s} d_j = kn.$$

Let p^e be the largest power of p that divides kn. Then, there is some d_j that is not divisible by p^{e+1}. After reindexing if necessary, we may assume that d_1 is not divisible by p^{e+1}. Since

$$p^i knb_1 = d_1 p^i nx + d_{1,1} p^i nx_1 + \cdots + d_{1,t} p^i nx_t$$

for all $i < \omega_0$, and since $\langle x \rangle \oplus \langle x_1 \rangle \oplus \cdots \oplus \langle x_t \rangle$ is a valuated coproduct, we have that

$$|p^{e+i} b_1|_p = |p^i knb_1|_p \leq |d_1 p^i nx|_p \leq |p^{e+i} nx|_p.$$

Because $|p^{e+i} b_1|_p \geq m_{p,e+i}$ for all i, and $|p^{e+i} b_1|_p \neq m_{p,e+i}$ for infinitely many values of i, we conclude that $|p^{e+i} nx|_p \neq m_{p,e+i}$ for infinitely many values of i. Therefore, $|p^i nx|_p \neq m_{p,i}$ for infinitely many i, and the proof is complete. □

Lemma 3.2. *$tG \cap G(M) \subseteq G(M^*)$ for every height matrix M.*

Proof. We may assume that $M \nsim \overline{\infty}$, since otherwise $tG \cap G(M) = G(M^*)$ by definition. Now, if $x \in tG \cap G(M)$, then $x \in G(M)$ and there is a positive integer n such that $nx = 0$. Note that $\|x\| \nsim M$. Indeed, if it were the case that $\|x\| \sim M$, we obtain

$$\overline{\infty} = \|0\| = \|nx\| \sim \|x\| \sim M,$$

contrary to the assumption that $M \nsim \overline{\infty}$. So, we have that $x \in G(M)$ and $\|x\| \nsim M$. Consequently, $x \in G(M^*)$. $\square$

Lemma 3.3. *If $x \in G(M)$ for some height matrix M and if n is a positive integer, then the following conditions are satisfied.*

(a) If $nx \in nG(M^*, p)$ for some prime p, then $x \in G(M^*, p)$.

(b) If $nx \in nG(M^*)$, then $x \in G(M^*)$.

Proof. To prove part (a), we have by hypothesis that $nx = ny$ for some $y \in G(M^*, p)$. Since both x and y are in $G(M)$, $x - y \in G(M)$. Moreover, $x - y \in tG$ because $n(x - y) = 0$. Therefore, by Lemma 3.2, $x - y \in G(M^*)$. Then,

$$x \in y + G(M^*) \subseteq G(M^*, p)$$

because $y \in G(M^*, p)$ and $G(M^*) \subseteq G(M^*, p)$. The proof of part (b) is similar. For again we have that $x - y \in G(M^*)$. But then

$$x \in y + G(M^*) \subseteq G(M^*)$$

since $y \in G(M^*)$. $\square$

Proposition 3.4. *If G has a decomposition basis X, then $\bigoplus_{x \in X} \langle x \rangle$ is a $*$-valuated coproduct.*

Proof. Suppose that $y \in \bigoplus_{x \in X} \langle x \rangle$ and write

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_t x_t,$$

where $x_1, x_2, \ldots, x_t$ are distinct elements of X, and $c_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, t$. We need to show that if $y \in F$, where F is one of the fully invariant subgroups of the form $G(M)$, $G(M^*)$, $G(M_p^*, p)$ or $G(M^*, p)$, then each $c_j x_j$ is in the same F. We consider, in turn, each of the four natural cases.

Case 1. $F = G(M)$. This case is clear since, by definition, $\bigoplus_{x \in X} \langle x \rangle$ is a valuated coproduct.

Case 2. $F = G(M^*)$. If $M \sim \overline{\infty}$, then $y \in G(M^*)$ implies that $y \in tG$. Then $y = 0$ since each nonzero element of $\bigoplus_{x \in X} \langle x \rangle$ has infinite order. It then follows that each $c_j x_j = 0 \in G(M^*)$. Therefore, we may assume that $M \nsim \overline{\infty}$ and write

$$y = a_1 + a_2 + \cdots + a_r$$

where for $i = 1, 2, \ldots, r$, $\|a_i\| \geq M$ and $\|a_i\| \nsim M$. Now select a positive integer k so that $k a_i \in \langle X \rangle$ for all i. Thus, for each i we have

$$k a_i = d_{i,1} x_1 + d_{i,2} x_2 + \cdots + d_{i,t} x_t + d'_{i,1} z_1 + \cdots + d'_{i,s} z_s,$$

where $x_1, x_2, \ldots, x_t$ are as above, $x_1, x_2, \ldots, x_t, z_1, \ldots, z_s$ are distinct elements of X, and all $d_{i,j}$ and $d'_{i,l}$ are in $\mathbb{Z}$ (for $j = 1, 2, \ldots, t$ and $l = 1, 2, \ldots, s$). Note that the inequalities $kM \leq \|ka_i\| \leq \|d_{i,j}x_j\|$ imply that, for all i and j, $d_{i,j}x_j \in G(kM)$ and $\|d_{i,j}x_j\| \not\sim kM$. Thus, each $d_{i,j}x_j$ is in $G((kM)^*) = kG(M^*)$. Therefore, since $\|c_jx_j\| \geq \|y\| \geq M$ and

$$kc_jx_j = \sum_{i=1}^{r} d_{i,j}x_j \in kG(M^*),$$

Lemma 3.3(b) implies that $c_jx_j \in G(M^*)$ for all j.

Case 3. $F = G(M_p^*, p)$. In this case we have that

$$y = a_1 + a_2 + \cdots + a_r$$

where for $i = 1, 2, \ldots, r$, $\|a_i\|_p \geq M_p$ and $|p^e a_i|_p \neq m_{p,e}$ for infinitely many $e < \omega_0$. Select a positive integer k such that $ka_i \in \langle X \rangle$ for all i. We then have

$$ka_i = d_{i,1}x_1 + d_{i,2}x_2 + \cdots + d_{i,t}x_t + d'_{i,1}z_1 + \cdots + d'_{i,s}z_s,$$

where the notation is the same as that in Case 2. For a given j, observe that

$$\sum_{i=1}^{r} d_{i,j} = kc_j. \qquad (\dagger\dagger)$$

Now temporarily fix j, and after reindexing if necessary, we may assume that $j = 1$. Thus, the proof in this case will be complete once we have shown that $c_1x_1 \in G(M_p^*, p)$.

Let p^f be the largest power of p that divides kc_1. Then condition (††) implies that p^{f+1} does not divide $d_{i,1}$ for some i. For such an i,

$$|p^{f+e}a_i|_p = |p^e kc_1 a_i|_p \leq |p^e c_1 d_{i,1} x_1|_p \leq |p^{f+e} c_1 x_1|_p$$

for all $e < \omega_0$. From this we conclude that $|p^e c_1 x_1|_p \neq m_{p,e}$ for infinitely many e. Moreover, $\|c_1 x_1\|_p \geq \|y\|_p \geq M_p$. Hence, $c_1 x_1 \in G(M_p^*, p)$.

Case 4. $F = G(M^*, p)$. In this case we have that $y = a_1 + a_2$ where $a_1 \in G(M^*)$ and $a_2 \in G(M_p^*, p) \cap G(M)$. Select a positive integer k such that $ka_i \in \langle X \rangle$ for $i = 1, 2$. We then have

$$ka_i = d_{i,1}x_1 + d_{i,2}x_2 + \cdots + d_{i,t}x_t + d'_{i,1}z_1 + \cdots + d'_{i,s}z_s,$$

where the notation is the same as that in Cases 2 and 3. For $i = 1$, Case 2 says that each $d_{1,j}x_j \in G((kM)^*)$. While for $i = 2$, Case 3 implies that each $d_{2,j}x_j \in G((kM)_p^*, p)$. Further observe that $c_j x_j \in G(M)$ for all j because $\|c_j x_j\| \geq \|y\| \geq M$. Thus, for $j = 1, 2, \ldots, t$,

$$\begin{aligned} kc_j x_j = d_{1,j}x_j + d_{2,j}x_j &\in G(kM) \cap (G((kM)^*) + G((kM)_p^*, p)) \\ &= G((kM)^*, p) = kG(M^*, p). \end{aligned}$$

Therefore, Lemma 3.3(a) shows that $c_j x_j \in G(M^*, p)$ for all j. □

Theorem 3.5. *If G has a decomposition basis X, then G is a k-group.*

Proof. We first note the fact that 0 is always a nice subgroup. Now, if S is a finite subset of G, there is a positive integer k such that $ks \in \bigoplus_{x\in X}\langle x\rangle$. Then $k\langle S\rangle \subseteq \bigoplus_{x\in X}\langle x\rangle$. So for all $s \in S$, we have that

$$ks \in \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_m\rangle$$

for some distinct $x_1, x_2, \ldots, x_m \in X$. Then, by Propositions 3.1 and 3.4,

$$k\langle S\rangle \subseteq \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots \oplus \langle x_m\rangle$$

where the coproduct is a $*$-valuated coproduct with each x_i primitive. □

Chapter 4

Torsion Free Groups with Decomposition Bases

In this chapter we show that a torsion free group has a decomposition basis if and only if it is completely decomposable (Theorem 4.3). We also show that a k-group of finite torsion free rank has a decomposition basis (Theorem 4.5). As a result, we are able to give an example of a torsion free group that is not a k-group.

A torsion free group G is of *rank* 1 if G is isomorphic to an additive subgroup of $\mathbb{Q}$ and has the property that if $x, y \in G$ are nonzero, then $mx = ny$ for some nonzero $m, n \in \mathbb{Z}$.

Definition 4.1. A torsion free group G is said to be *completely decomposable* if it is a direct sum of rank 1 subgroups.

Lemma 4.2. *If A is a subgroup of a group G and if p and q are relatively prime integers, then $pA \cap qA = (pq)A$.*

Proof. Clearly $(pq)A \subseteq pA \cap qA$. For the reverse inclusion, suppose that $x \in pA \cap qA$. Then, $x = pa_1 = qa_2$ where $a_1, a_2 \in A$. Since $(p, q) = 1$, $rp + sq = 1$ for some $r, s \in \mathbb{Z}$, which implies that

$$a_1 = rpa_1 + sqa_1 = rqa_2 + sqa_1 = q(ra_2 + sa_1) \in qA.$$

But then,

$$x = pa_1 \in p(qA) = (pq)A. \qquad \square$$

If N is a subgroup of a torsion free group G, define $N_* = \{x \in G : nx \in N$ for some nonzero integer $n\}$. Observe that N_* is a pure subgroup of G and is the smallest pure subgroup of G that contains N.

Theorem 4.3. *A torsion free group G has a decomposition basis X if and only if G is completely decomposable.*

Proof. Suppose that G is a torsion free abelian group and that X is a decomposition basis for G. Observe that each $\langle x\rangle_*$ with $x \in X$ has rank 1. For, suppose $y, z \in \langle x\rangle_*$. Then $my \in \langle x\rangle$ and $nx \in \langle x\rangle$ for some nonzero integers m, n. So $my = lx$ and $nz = rx$ for some nonzero integers l, r. But then

$$(rm)y = (rl)x = (ln)z.$$

Next we claim that the sum $\sum_{x\in X}\langle x\rangle_*$ is direct. Indeed, if for some $x_1 \in X$ and $y \in G$ we have that

$$y \in \langle x_1\rangle_* \cap \sum_{x\in X\setminus\{x_1\}} \langle x\rangle_*,$$

then there are a finite number of distinct elements $x_2, x_3, \ldots, x_k \in X\setminus\{x_1\}$ such that $y \in \sum_{i=2}^{k}\langle x_i\rangle_*$. Thus,

$$y = a_1 = \sum_{i=2}^{k} a_i$$

where each $a_i \in \langle x_i \rangle_*$. Now select a positive integer n such that $na_i \in \langle x_i \rangle$ for all i. Then, since $\bigoplus_{x \in X} \langle x \rangle$ is a direct sum,

$$ny \in \langle x_1 \rangle \cap \sum_{i=2}^{k} \langle x_i \rangle = 0.$$

Since G is torsion free and $n \neq 0$, $y = 0$. We conclude that $\sum_{x \in X} \langle x \rangle_* = \bigoplus_{x \in X} \langle x \rangle_*$.

Since each $\langle x \rangle_*$ with $x \in X$ is torsion free of rank 1, this part of the proof will be complete once we have shown that $G = \bigoplus_{x \in X} \langle x \rangle_*$. For a given $y \in G$, the fact that $G/\langle X \rangle$ is torsion implies there is a positive integer n, distinct $x_1, x_2, \ldots, x_k \in X$ and $c_1, c_2, \ldots, c_k \in \mathbb{Z}$ such that

$$ny = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.$$

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ be the prime factorization of n. Since $\bigoplus_{x \in X} \langle x \rangle$ is a valuated coproduct,

$$e_j \leq |p_j^{e_j} y|_{p_j} = |ny|_{p_j} \leq |c_i x_i|_{p_j}$$

for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, t$. We then have that

$$c_i x_i \in p_j^{e_j} G \cap \langle x_i \rangle \subseteq p_j^{e_j} G \cap \langle x_i \rangle_* = p_j^{e_j} \langle x_i \rangle_*.$$

Therefore, for each i,

$$c_i x_i \in \bigcap_{j=1}^{t} p_j^{e_j} \langle x_i \rangle_*$$

so that $c_i x_i \in n\langle x_i \rangle_*$ by repeated applications of Lemma 4.2. Hence,

$$ny = na_1 + na_2 + \cdots + na_k = n(a_1 + a_2 + \cdots + a_k)$$

with $a_i \in \langle x_i \rangle_*$ for $i = 1, 2, \ldots, k$. Since $n \neq 0$ and G is torsion free, it follows that

$$y = a_1 + a_2 + \cdots + a_k \in \bigoplus_{x \in X} \langle x \rangle_*$$

and we conclude that $G = \bigoplus_{x \in X} \langle x \rangle_*$.

Conversely, suppose that G is completely decomposable. Say $G = \bigoplus_{i \in I} A_i$ where each A_i has rank 1. In each A_i, select a nonzero element x_i. Now set $X = \{x_i\}_{i \in I}$. We claim that X is a decomposition basis for G. To see that $G/\langle X \rangle$ is torsion, suppose that $g \in G$. Then there is a finite subset $\{i(1), i(2), \ldots, i(n)\} \subseteq I$ with

$$g = a_{i(1)} + a_{i(2)} + \cdots + a_{i(n)}$$

and $a_{i(j)} \in A_{i(j)}$ for $j = 1, 2, \ldots, n$. For each j, there are nonzero integers k_j, l_j with $k_j a_{i(j)} = l_j x_{i(j)}$ which implies that $k_j a_{i(j)} \in \langle X \rangle$. Now, let $k = lcm\{k_1, k_2, \ldots, k_n\}$. Then k has the property that $kg \in \langle X \rangle$ and hence $G/\langle X \rangle$ is torsion. Finally, since $\bigoplus_{i \in I} A_i$ is a valuated coproduct in G, $\bigoplus_{i \in I} \langle x_i \rangle$ is valuated. □

Definition 4.4. The *torsion free rank* of a group G is the cardinality of a maximal $\mathbb{Z}$-independent subset of G consisting only of elements of infinite order.

Theorem 4.5. *If G is a k-group of finite torsion free rank, then G has a decomposition basis.*

Proof. Let $\{a_1, a_2, \ldots, a_k\}$ be a maximal $\mathbb{Z}$-independent subset of G consisting of elements of infinite order. Since G is a k-group, there are primitive elements $x_1, x_2, \ldots, x_n \in G$ with $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ a $*$-valuated coproduct such that there is some nonzero integer m with $ma_i \in N$ for $i = 1, 2, \ldots, k$. Observe that if g is any element of G, there is some positive integer l with $lg \in \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_k \rangle$. Hence, G/N is torsion. We conclude that $\{x_1, x_2, \ldots, x_n\}$ is a decomposition basis for G. □

One consequence of Theorem 4.3 and the last result is that any torsion free group of finite rank cannot be a k-group unless it is completely decomposable. For an example, let p_1, p_2, p_3 be distinct prime numbers and let

$$G = \frac{\mathbb{Z}[1/p_1] \oplus \mathbb{Z}[1/p_2] \oplus \mathbb{Z}[1/p_3]}{\langle (1,1,1) \rangle_*}.$$

It is known that G is a torsion free group of rank 2 that is not completely decomposable. For example, see [1]. Hence, G is not a k-group.

Chapter 5

Torsion Free Separable Groups

In this chapter we show that a torsion free separable group is a k-group (that does not necessarily have a decomposition basis).

Definition 5.1. A torsion free group G is called *separable* if every finite subset of G is contained in a completely decomposable direct summand of G.

Lemma 5.2. *If $G = A \oplus B$, then for every prime p and ordinal α, $p^\alpha G = p^\alpha A \oplus p^\alpha B$.*

Proof. Clearly $p^\alpha A \oplus p^\alpha B \subseteq p^\alpha G$. So it suffices to prove the reverse inclusion. We proceed by transfinite induction on α. If $\alpha = 1$, then $x \in pG$ gives that $x = py$ for some $y \in G$. Now write $y = a + b$ where $a \in A$ and $b \in B$. Then

$$x = py = p(a + b) = pa + pb \in pA \oplus pB \subseteq pG.$$

Therefore, $pG = pA \oplus pB$. We finish the proof by considering two cases.

Case 1. $\alpha = \beta + 1$ for some β. By induction, $p^\beta G = p^\beta A \oplus p^\beta B$. The base case then provides that $p(p^\beta G) = p(p^\beta A) \oplus p(p^\beta B)$. That is, $p^\alpha G = p^\alpha A \oplus p^\alpha B$.

Case 2. α is a limit ordinal. Then $p^\beta G = p^\beta A \oplus p^\beta B$ for all $\beta < \alpha$. Now if $x \in p^\beta G$ for each $\beta < \alpha$ (that is, if $x \in \bigcap_{\beta<\alpha} p^\beta G = p^\alpha G$), then $x = a_\beta + b_\beta$ where $a_\beta \in p^\beta A$ and $b_\beta \in p^\beta B$. Also, $x \in A \oplus B$ and so $x = a + b$ for some $a \in A$ and $b \in B$. Then for all β,

$a + b = a_\beta + b_\beta$ implies that

$$a - a_\beta = b_\beta - b \in A \cap B = 0.$$

Therefore, $a = a_\beta \in p^\beta A$ and $b = b_\beta \in p^\beta B$ for all $\beta < \alpha$. Hence, $a \in p^\alpha A$ and $b \in p^\alpha B$ results in $x \in p^\alpha A \oplus p^\alpha B$. □

Corollary 5.3. *If $G = A \oplus B$, then $A \oplus B$ is a valuated coproduct.*

Proof. If $x = a + b$ where $a \in A$, $b \in B$ and $|x|_p = \alpha$ for some prime p and ordinal α, then

$$x \in p^\alpha G = p^\alpha A \oplus p^\alpha B$$

by Lemma 5.2. Writing $x = a_1 + b_1$ with $a_1 \in p^\alpha A$, $b_1 \in p^\alpha B$ we have that $|a_1|_p \geq \alpha$ and $|b_1|_p \geq \alpha$. Now if both $|a_1|_p > \alpha$ and $|b_1|_p > \alpha$, then $\alpha < |(a_1 + b_1)|_p = |x|_p$, a contradiction. We conclude that $|a_1|_p = \alpha$ or $|b_1|_p = \alpha$. Therefore,

$$|x|_p = min\{|a_1|_p, |b_1|_p\} = |a_1|_p \wedge |b_1|_p.$$ □

Observe that Corollary 5.3 says that if $G = A \oplus B$, then $G(M) = A(M) \oplus B(M)$ for every height matrix M.

Proposition 5.4. *If $G = A \oplus B$, then $A \oplus B$ is a $*$-valuated coproduct.*

Proof. Suppose $x \in F$ where F is one of the fully invariant subgroups $G(M)$, $G(M^*)$, $G(M_p^*, p)$ or $G(M^*, p)$. We need to show that $x \in (A \cap F) \oplus (B \cap F)$. We consider, in turn, each of the four natural cases.

Case 1. $x \in G(M)$. Corollary 5.3 provides that the coproduct is valuated.

Case 2. $x \in G(M^*)$. If $M \sim \overline{\infty}$, then $G(M^*) = tG(M)$. So since $G(M) = A(M) \oplus B(M)$ we have that $tG(M) = tA(M) \oplus tB(M)$. More precisely,

$$G(M^*) = A(M^*) \oplus B(M^*) \subseteq (A \cap G(M^*)) \oplus (B \cap G(M^*)).$$

If $M \nsim \overline{\infty}$, then $x = x_1 + x_2 + \cdots + x_n$ where $\|x_i\| \geq M$ and $\|x_i\| \nsim M$. Also, for each i, $x_i = a_i + b_i$ where $a_i \in A$ and $b_i \in B$. We claim that $\|a_i\| \nsim M$ for all i. Indeed, if $\|a_i\| \sim M$, there are positive integers k, l such that $M \leq k\|a_i\|$ and $\|a_i\| \leq lM$. But then $\|x_i\| \leq \|a_i\| \leq lM$ and $\|x_i\| \geq M$. That is, $\|x_i\| \sim M$, a contradiction. Therefore, $\|a_i\| \nsim M$, and by symmetry, $\|b_i\| \nsim M$. We now obtain

$$\begin{aligned} x = \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i &\in A(M^*) \oplus B(M^*) \\ &\subseteq (A \cap G(M^*)) \oplus (B \cap G(M^*)), \end{aligned}$$

as desired.

Case 3. $x \in G(M_p^*, p)$. If $x \in G(M_p^*, p)$, then $x = x_1 + x_2 + \cdots + x_n$ where each x_j has the property that $\|x_j\|_p \geq M_p$ but $|p^i x_j|_p \neq m_{p,i}$ for infinitely many i. Now write $x_j = a_j + b_j$ where $a_j \in A$ and $b_j \in B$. Then

$$\|a_j\|_p \wedge \|b_j\|_p = \|x_j\|_p \geq M_p$$

gives that both $\|a_j\|_p \geq M_p$ and $\|b_j\|_p \geq M_p$. Hence, for all $i < \omega_0$,

$$|p^i a_j|_p \wedge |p^i b_j|_p = |p^i x_j|_p$$

gives that both $|p^i a_j|_p \neq m_{p,i}$ and $|p^i b_j|_p \neq m_{p,i}$ for infinitely many i. Therefore,

$$x \in A(M_p^*, p) \oplus B(M_p^*, p) \subseteq (A \cap G(M_p^*, p)) \oplus (B \cap G(M_p^*, p)).$$

Case 4. $x \in G(M^*, p)$. In this case,

$$\begin{aligned} G(M^*, p) &= G(M) \cap (G(M_p^*, p) + G(M^*)) \\ &= (A(M) \oplus B(M)) \cap [(A(M_p^*, p) \oplus B(M_p^*, p)) + (A(M^*) \oplus B(M^*))] \\ &= (A(M) \oplus B(M)) \cap [(A(M_p^*, p) + A(M^*)) \oplus (B(M_p^*, p) + B(M^*))] \\ &\subseteq (A(M) \cap (A(M_p^*, p) + A(M^*))) \oplus (B(M) \cap (B(M_p^*, p) + B(M^*))) \\ &= A(M^*, p) \oplus B(M^*, p) \\ &\subseteq (A \cap G(M^*, p)) \oplus (B \cap G(M^*, p)). \end{aligned}$$

□

Corollary 5.5. *Let $G = A \oplus B$ with A torsion-free of rank 1. If $0 \neq a \in A$, then $\langle a \rangle \oplus B$ is $*$-valuated and a is primitive in G.*

Proof. Observe that $\{a\}$ is a decomposition basis for A. Then by Proposition 3.2, a is primitive in A. So if $na \in G(M^*, p)$, it must be that either $M \nsim \|a\|^A = \|a\|^G$ or

$m_{p,i} \neq |p^i na|_p^A = |p^i na|_p^G$ for infinitely many i. Thus, a is primitive in G. Finally, note that since $A \oplus B$ is $*$-valuated, $\langle a \rangle \oplus B$ must be as well. $\square$

Theorem 5.6. *If G is a torsion free separable group, then G is a k-group.*

Proof. Suppose $S = \{x_1, x_2, \ldots, x_n\}$ is a finite subset of G. Then $S \subseteq C$ where $G = C \oplus B$ for some B and completely decomposable C of finite rank. Write $C = A_1 \oplus A_2 \oplus \cdots \oplus A_m$ where each A_i is torsion free of rank 1. For each i, select a nonzero $a_i \in A_i$. Then there is a positive integer k such that $kx_i \in \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_m \rangle$. Observe that repeated applications of Corollary 5.5 then gives that each a_i is primitive and that the coproduct is $*$-valuated. $\square$

Example 5.7. We claim that $G = \prod_{\aleph_0} \mathbb{Z}$ is a k-group that does not have a decomposition basis. We note that G is indeed a k-group since by Theorem 139 of [4], G is separable, and by Theorem 5.6, torsion free separable groups are k-groups. Now, if G had a decomposition basis, it would be a direct sum of rank 1 groups by Theorem 4.3. Then Proposition 96.2 of [3] (due to Mishina [15]) provides that each rank 1 summand of G is isomorphic to $\mathbb{Z}$. This would mean that $G = \prod_{\aleph_0} \mathbb{Z}$ is free, a contradiction in light of Corollary 52 of [4] which states that $\prod_{\alpha} \mathbb{Z}$ is not free for any cardinal $\alpha \geq \aleph_0$. Hence, G does not have a decomposition basis.

We conclude by noting that Example 3.1 of [6] provides an example of a torsion free k-group that is not separable.

Bibliography

[1] D. Arnold, *Finite rank torsion free abelian groups and rings*, Lecture Notes in Math., vol. 931, Springer-Verlag, New York, 1982.

[2] L. Fuchs, *Infinite abelian groups*, vol. I, Academic Press, New York, 1970.

[3] L. Fuchs, *Infinite abelian groups*, vol. II, Academic Press, New York, 1973.

[4] P. Griffith, *Infinite abelian group theory*, Univ. of Chicago Press, Chicago, Ill., 1970.

[5] P. Hill, *On the classification of abelian groups*, photocopied manuscript, 1967.

[6] P. Hill and C. Megibben, *Torsion free groups*, Trans. Amer. Math. Soc. **295** (1986), 735-751.

[7] P. Hill and C. Megibben, *Knice subgroups of mixed groups*, Abelian Group Theory, Gordon-Breach, New York, 1987, pp. 89-109.

[8] P. Hill and C. Megibben, *Mixed groups*, Trans. Amer. Math. Soc. **334** (1992), 121-142.

[9] R. Hunter and F. Richman, *Global Warfield groups*, Trans. Amer. Math. Soc. **266** (1981), 555-572.

[10] R. Hunter, F. Richman, and E. Walker, *Warfield modules*, Lecture Notes in Math., vol. 616, Springer-Verlag, New York, 1977, pp. 87-123.

[11] R. Hunter, F. Richman, and E. Walker, *Existence theorems for Warfield groups*, Trans. Amer. Math. Soc. **235** (1978), 345-362.

[12] C. Megibben and W. Ullery, *Isotype Warfield subgroups of global Warfield groups*, Rocky Mountain J. Math. **32** (2002), 1523-1542.

[13] C. Megibben and W. Ullery, *The sequentially pure projective dimension of global groups with decomposition bases*, J. Pure and Applied Algebra **187** (2004), 183-205.

[14] C. Megibben and W. Ullery, *On global abelian k-groups*, Houston J. Math. **31** (2005), 675-692.

[15] A. Mishina, *On the direct summands of complete direct sums of torsion free abelian groups of rank 1*, Sibirsk. Mat. Ž. **3** (1962), 242-249.