Torsionless Modules and Minimal Generating Sets for Ideals of Integral Domains

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Vita

Wesley Robert Brown was born July 17th, 1982 in Birmingham, Alabama. In 2000, he graduated from Thompson High School in Alabaster, Alabama where he was Valedictorian of his senior class and captain of the football team. He earned a Bachelor's degree in Mathematics with a minor in Computer Science from Birmingham-Southern College in 2004. While at BSC, he served as president of Kappa Mu Epsilon Mathematics Honor Society, an editor of The Hilltop News, and was a founding member of Tri-Epsilon. He began his graduate studies at Auburn University in 2004.

Thesis Abstract

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This is a treatise of relationships between the number of elements necessary to generate the ideals of a domain and the torsionless modules of that domain. Three types of domains are identified according to natural decompositions of their torsionless modules. The descriptions of the domains follow the historical approach of Dedekind by focusing on the ideals of the domains.

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Chapter 1

Introduction

We will attempt to develop a strong relation between the number or elements necessary to generate the ideals of a domain and properties of the torsionless modules of that domain. We will consider three types of domains: principal ideal domains, the class of which is properly contained in the class of Dedekind domains, which in turn is properly contained in the class of Noetherian domains whose ideals are generated by two elements.

By definition, the ideals of a principal ideal domain are generated by a single element. For these domains, every torsionless module is a direct sum of copies of R, or free. For Dedekind domains, every ideal is generated by two elements, one of which can be arbitrarily chosen. The terminology is that ideals are $1\frac{1}{2}$-generated. For Dedekind domains, every torsionless module is projective. Finally, for the broadest class, Noetherian domains whose ideals are 2-generated, every torsionless module is a direct sum of ideals of R.

Throughout this paper, R always represents an integral domain with quotient field Q.

Chapter 2

Definitions

The following definitions are basic staples of ring and module theory but are included here due to their fundamental importance throughout this thesis.

Definition 1. *Let X be a subset of a module A over R. The intersection of all submodules of A containing X is called the* ***submodule generated by*** *X. If X is a finite set, then the submodule generated by X is said to be* ***finitely generated.***

If a module A is generated by the set $\{x_1, x_2, ..., x_n\}$ then we write $A = (x_1, x_2, ..., x_n)$.

Definition 2. *A module M over R is* ***torsion-free*** *if for all $a \in M$ and $r \in R$, $ra = 0$ implies that either $r = 0$ or $a = 0$.*

Definition 3. *A module M over R is* ***free*** *if M is isomorphic to a direct sum of copies of R.*

Definition 4. *A subset X of a torsion-free R-module M is said to be* ***linearly independent*** *if for distinct $x_1, x_2, ..., x_n \in X$ and $r_1, r_2, ..., r_n \in R$, $r_1x_1 + r_2x_2 + ... + r_nx_n = 0$ implies that $r_i = 0$ for all i.*

Definition 5. *The* ***rank*** *of a torsion-free R-module M is the cardinality of a maximal linearly independent subset of M.*

Definition 6. *A module M over R is* ***torsionless*** *if M is isomorphic to a submodule of a finite-rank free module.*

Definition 7. *A module M over R is* ***projective*** *if for every R-module homomorphism*

$$A \xrightarrow{\gamma} B \to 0$$

and any R-module homomorphism $\psi : M \to B$, then there is an R-module homomorphism $\varphi : M \longrightarrow A$ such that $\pi\varphi = \gamma$.

It is apparent that free modules are projective. However there are domains for which projective modules are not necessarily free as the example in Chapter 4 shows.

Definition 8. *Let P be a prime ideal of R. Let*

$$P \supset P_1 \supset P_2 \supset ... \supset P_n$$

be a descending chain of unique prime ideals of R such that for any other proper chain

$$P \supset P_1 \supset P_2 \supset ... \supset P_m,$$

*$m \leq n$. Then the **rank** of the prime ideal P is n.*

Definition 9. *The **Krull dimension** of R is the supremum of the ranks of the prime ideals of R.*

Definition 10. *An ideal I of R is **invertible** if $II^{-1} = R$ where $I^{-1} = \{q \in Q : qI \subseteq R\}$.*

Chapter 3

Principal Ideal Domains

We will first consider principal ideal domains

Definition 11. *A domain R is a* ***principal ideal domain****, or a PID for short, if every ideal of R can be generated by a single element.*

In the case of a PID, every torsionless R-module is free as stated below in the single result of this section.

Theorem 3.1. *Every torsionless R-module is free if and only if R is a PID.*

Proof. ($\Rightarrow$) Assume every torsionless R-module is free. Let I be an ideal of R. We will show that I is principal. Since I is an ideal, it is an R-module. R is a free R-module of rank 1 so I is a torsionless R-module and thus is free. Therefore, $I \cong \oplus_n R$ for some n. Let $a, b \in I$. Note that $b(a) + (-a)b = 0$ and thus any two elements of I are linearly dependent. However, $(1, 0, ..., 0), (0, 1, 0, ..., 0) \in \oplus_n R$ are linearly independent as long as $n > 1$. Thus, $I \cong R$ and thus I is principal.

($\Leftarrow$) Assume R is a PID. Let K be a torsionless R-module. Then,

$$0 \neq K \cong K^{'} \leq F \cong \oplus_n R,$$

where F is some free R-module. Without loss of generality, we may identify K with $K^{'}$. Now, let $\pi : F \longrightarrow R$ be a projection map such that $\pi(K) = I \neq 0$, for some ideal I of R. Since I is principal, we can write $I = aR$. Let $x \in K$ such that $\pi(x) = a$. Now, define $\varphi : I \longrightarrow K$ by $\varphi(ar) = rx$. Since $\pi\varphi = 1_I$, $K \cong Im\varphi \oplus K_0$, where $K_0 = Ker(\pi|_K)$. Note that $Im\varphi = xR \cong R$. Also,

$$Ker\pi = R \oplus R \oplus ... \oplus 0 \oplus ... \oplus R,$$

where 0 corresponds to the copy of R being projected. Since $K_0 \leq Ker\pi$, induction applies to show that K_0, and thus K, is a direct sum of copies of R and is thus free. □

We can now state the following corollary, which creates a symmetry with the first result of the next section.

Corollary 3.1. *Every torsionless R-module is free if and only if every ideal is free.*

Since free modules are projective, the proof of this corollary follows in the exact manner of the first proof of the next section.

Chapter 4

Dedekind Domains

We will begin this section by characterizing domains whose torsionless modules are projective. We will then relate this to the class of Dedekind domains defined below.

Theorem 4.1. *Every torsionless R-module is projective if and only if every ideal is projective.*

Proof. ($\Rightarrow$) Assume every torsionless R-module is projective. Let I be an ideal of R. Then I is an R-module. Also, since R is a free R-module of rank 1, I is torsionless. Thus, I is projective.

($\Leftarrow$) Assume every ideal of R is projective. Let K be a torsionless R-module. Then $0 \neq K \cong K^{'} \leq F \cong \oplus_n R$. As in the proof of Theorem 3.1, we will identify K with $K^{'}$. Since F is free, it is projective. Let $\pi : F \to R$ be a projection map such that $\pi(K) = I \neq 0$. I is an ideal of R and is thus projective. Consider the sequence:

$$0 \to Ker(\pi|_K) \to K \xrightarrow{\pi} I \to 0.$$

Since I is projective, we know there exists $\varphi : I \to K$ such that $\pi\varphi = 1_I$. So, $K \cong Im\varphi \oplus Ker(\pi|_K)$. $Im\varphi$ is isomorphic to an ideal of R and is thus projective. As in the proof of Theorem 3.1, inductively $Ker(\pi|_K)$ is a direct sum of ideals and so is a direct sum of projective modules and is thus projective. Thus, K is a direct sum of projective modules and is projective.

$\square$

Definition 12. *A domain R is called a **Dedekind domain** if every proper ideal of R is the product of a finite number of prime ideals of R.*

Every principal ideal domain is a Dedekind domain. In fact, every nonzero element $a \in R$ can be factored as $a = p_1 p_2 ... p_m$ with each p_j a prime element of R. Thus, $(a) = (p_1)(p_2)...(p_n)$ There are, however, Dedekind domains that are not principal. An example is

$$\mathbf{Z}[\sqrt{10}] = \{a + b\sqrt{10} : a, b \in \mathbf{Z}\}$$

as discussed in [1].

We now relate the result from the theorem proved above to Dedekind domains.

Theorem 4.2. *The following are equivalent for a domain R:*

(a) R is a Dedekind Domain.

(b) Every proper ideal of R is uniquely the product of a finite number of prime ideals.

(c) Every nonzero (fractional) ideal in R is invertible.

(d) Every (fractional) ideal of R is projective.

(e) R is Noetherian, integrally closed, and every nonzero prime ideal is maximal.

(f) R is Noetherian and for every nonzero prime ideal P of R the localization R_P is a principal ideal domain.

The proof of this theorem is rather sprawling. We will give a sketch here. A complete proof can be found in [1]. A related proof can be found in [4]. The equivalence of (c) and (d) will be discussed again in Lemma 5.7.

Sketch of Proof. $(a) \Rightarrow (b)$ We begin with I, a proper ideal of R such that $I = P_1P_2...P_m = Q_1Q_2...Q_n$ for primes P_i and Q_j. Since $I \subset P_i, Q_j$, any P_i contains some Q_j. Once it is shown that every nonzero prime ideal of a Dedekind domain is invertible and maximal, the proof follows by cancellation and induction.

$(b) \Rightarrow (c)$ This follows from the fact that a finite product of ideals $I_1I_2 \cdots I_n$ of a domain is invertible if and only if each I_J is invertible and the previously stated fact that every nonzero prime ideal of a Dedekind domain is invertible.

$(c) \Rightarrow (d)$ If I is a nonzero ideal of R, then $II^{-1} = R$ implies that there are $a_1, a_2, ..., a_n \in I$ and $b_1, b_2, ..., b_n \in I^{-1}$ such that $a_1b_1 + a_2b_2 + ... + a_nb_n = 1$. The epimorphism $\phi : \oplus_n R \to I$ given by $\phi(r_1, r_2, ..., r_n) = r_1a_1 + r_2a_2 + ... + r_na_n$ is split by $f : I \to \oplus_n R$ where $f(a) = (ab_1, ab_2, ..., ab_n)$. Thus, I is projective.

$(d) \Rightarrow (c)$ If I is a nonzero ideal of R, then there is a split epimorphism $\phi : \oplus_n R \to I \to 0$. So, there exists $b_i \in I^{-1}$ for $i = 1, 2, ...n$ such that for $a_i = \phi(e_i)$, where e_i is the standard basis vector in $\oplus_n R$, $a_1b_1 + a_2b_2 + ... + a_nb_n = 1$. Thus $II^{-1} = R$ and I is invertible.

$(c) \Rightarrow (e)$ Since every invertible fractional ideal of R is a finitely generated R-module, and since a commutative ring is Noetherian if and only if every ideal of R is finitely generated, we can conclude that R is Noetherian.

If $u \in K$, the quotient field of R, then Ru is a finitely generated R-module and is thus a fractional ideal of R. So, $R[u]$ is invertible. Since $R[u]R[u] = R[u]$,

$$R[u] = RR[u] = (R[u]^{-1}R[u])R[u] = R[u]^{-1}R[u] = R$$

which implies that $u \in R$ and thus R is integrally closed.

Next, we will show that any nonzero prime ideal is maximal. Let P be a nonzero prime ideal of R. There is some maximal ideal M containing P. M is invertible and thus $M^{-1}P$

is a fractional ideal of R and $M^{-1}P \subset M^{-1}M = R$ which means $M^{-1}P$ is an ideal of R. Also, since $M(M^{-1}P) = RP = P$ and P is prime either $M \subset P$ or $M^{-1}P \subset P$. In the latter case,

$$R \subset M^{-1} = M^{-1}R = M^{-1}PP^{-1} \subset PP^{-1} \subset R$$

which means that $M^{-1} = R$. But then $R = MM^{-1} = MR = M$, a contradiction since M is maximal. So, it must be that $M \subset P$. Therefore $M = P$ and P is maximal.

$(e) \Rightarrow (f)$ This implication is established using several results in [1]. First, we know that R_P is an integrally closed integral domain. Also, every ideal of R_P is of the form I_P where I is an ideal of R and furthermore every nonzero prime ideal of R_P is of the form I_P where I is a nonzero prime ideal of R that is contained in P. By assumption, every nonzero prime ideal of R is maximal and thus P_P is the unique nonzero prime ideal of R_P. This implies that R_P is a principal ideal domain.

$(f) \Rightarrow (a)$ First, we will show that any ideal I of R is invertible. II^{-1} is an ideal of R and thus if $II^{-1} \neq R$ then there is some maximal ideal M containing I. M is prime so by assumption, I_M is principal. Let r/q be the element that generates I_M, where $r \in I$ and $q \notin M$. R is Noetherian and thus I is finitely generated. Let $\{b_1, b_2, ..., b_n\}$ be a generating set for I. For each i, $b_i/1_R = (r_i/q_i)(r/q) \in I_M$ where each $r_i \in I$ and each $q_i \notin M$. Thus $q_iqb_i = r_ir \in I$. Let $t = qq_1q_2...q_n \notin M$. Then for all i, we have $(t/r)b_i = tb_i/r \in R$ and thus $(t/r) \in I^{-1}$. Finally, we have that $t = (t/r)r \in II^{-1} \subset M$. But $t \notin M$ and so it must be that $II^{-1} = R$. This shows that I is invertible.

Then for each I, we can choose a maximal ideal $M \neq R$ that contains I. The invertibility of M and I implies that I must be properly contained in the ideal IM^{-1} since otherwise we have

$$R = RR = (II^{-1})(MM^{-1}) = I^{-1}(IM^{-1})M = I^{-1}IM = RM = M,$$

a contradiction since M is maximal.

For any proper ideal J of R, we can now set up a chain of ascending ideals containing J that must terminate since R is Noetherian. But, the ideals are of the form discussed in the last paragraph and thus the only way the chain can terminate is for the terminal link to be R itself. This allows us to express J as a product of maximal and thus prime ideals. This shows that R is Dedekind.

□

Stated here, Nakayama's lemma is a well known result of ring theory.

Nakayama's Lemma 4.3. *Let S be a domain. If $J(S)$ is the intersection of all maximal ideals of S, K is a finitely-generated S-module, and L is a submodule of K, then*

$$K = L + J(S)K$$

implies that $K = L$.

Since our main theme is to relate torsionless modules to the magnitude of maximal generating sets of ideals, we now introduce the following concept.

Definition 13. *A nonzero ideal I of a domain R is said to be* $1\frac{1}{2}$**-generated** *if for any $a \in I$, $a \neq 0$, there exists $b \in I$ such that I is generated by $\{a,b\}$.*

Theorem 4.4. *A domain R is Dedekind if and only if every nonzero ideal I of R is one-and-a-half generated.*

Proof. (⇐) We know that R being Dedekind is equivalent to the statement R is Noetherian and for all maximal ideals M of R, the localization R_P is a PID. Assume that every ideal I of R can be generated by {a,b} as above. Thus, R is clearly Noetherian.

Now, let P be a maximal ideal of R. Let J be an ideal of R_P. So, $J = R_P I$ for some ideal I of R. Choose $a \in PI$ and write $I = Ra + Rb$. Then,

$$J = R_P(Ra + Rb) = R_P a + R_P b,$$

and $a \in PI \subseteq PJ$. Thus

$$R_P b + PJ = J,$$

so by Nakayama's Lemma, $J = R_P b$. Thus, J is principal and R_P is a PID. By Theorem 4.2, R is Dedekind.

(⇒) Let I be a nonzero ideal of R. Let $a \in I$ be nonzero. R is Dedekind so the ideal Ra is a product of maximal ideals, say $Ra = P_1 P_2 ... P_m$ for maximal ideals $P_1, P_2, ... P_m$. Then, $aR \subseteq P$ implies $P_i \subseteq P$ for some i and therefore these are the only maximal ideals that contain a. By Theorem 4.2, if R is Dedekind, then R is Noetherian and for every maximal ideal P, R_P is a PID. So, R_{P_i} is a PID for $i = 1, 2, ...m$ and so the ideal IR_{P_i} is a principal ideal of R_{P_i}. Thus, for each i there is some $b_i \in I$ such that $IR_{P_i} = R_{P_i} b_i$. We claim that there is a $b \in I$ such that $IR_{P_i} = R_{P_i} b$ for all i. Order $P_1, P_2, ... P_m$ such that $P_1, P_2, ... P_k$ are distinct and for $j > k$, P_j is equal to one of $P_1, P_2, ... P_k$. Now, $\prod_{j=1, j \neq i}^{k} P_j$ is not contained in P_i for any i so

$$P_i + \prod_{j=1, j \neq i}^{k} P_j = R.$$

Then clearly,

$$P_iI + (\prod_{j=1,j\neq i}^{k} P_j)I = I$$

and for each $i = 1, 2, ...m$, we can write $b_i = c_i + d_i$ where $c_i \in P_i$ and $d_i \in (\prod_{j=1,j\neq i}^{k} P_j)I$. Note that b_i is congruent to d_i modulo P_iI for all i. Let $b = d_1 + d_2 + ... + d_m$. Then for any i, d_j is congruent to zero modulo P_iI if $j \neq i$. So, $b \equiv b_i \pmod{P_i}$ for all i. Then

$$IR_{P_i} = R_{P_i}b_i = R_{P_i}b_i + P_iIR_{P_i}.$$

But since $b \equiv b_i \pmod{P_i}$, this is equal to $R_{P_i}b + P_iIR_{P_i}$. Moreover, as in the reverse implication, the only maximal ideal of R_{P_i} is $P_iR_{P_i}$ and since

$$IR_{P_i} = R_{P_i}b + P_iIR_{P_i} = R_{P_i}b + (IR_{P_i})(P_iR_{P_i}),$$

we can apply Nakayama's Lemma and arrive at the equality: $IR_{P_i} = R_{P_i}b$ for all i.

Let $J = Ra + Rb$. Note that $J \subseteq I$. Also note that $IR_P = I_P = J_P = JR_P$ for any maximal ideal P of R. We can see this by first considering the case where P is not one of our $P_1, P_2, ...P_m$. In this case,

$$R_P = (Ra)_P \subseteq J_P \subseteq I_P \subseteq R_P.$$

If P is one of our $P_1, P_2, ...P_m$ then

$$IR_P = R_Pb \subseteq JR_P \subseteq IR_P.$$

To complete the proof, we will show that $I = J$. Since $J \subseteq I$, we can consider $I^{-1}J \subseteq R$. Now, note that

$$R_P(I^{-1}J) = (R_P I^{-1})(R_P J) = (R_P I^{-1})(R_P I) = R_P(I^{-1}I) = R_P$$

for any maximal ideal P. Assume $I^{-1}J \neq R$. $I^{-1}J$ is an ideal and so it is contained in some maximal ideal P. But then

$$R_P = R_P(I^{-1}J) \subseteq R_P P \subset R_P,$$

while $R_P P \neq R_P$ since P is a proper ideal, a contradiction. Thus, $I^{-1}J = R$ and finally $I = J$.

□

So, we can follow the logical chain laid out above and summarize the ideas of this section in the following theorem that follows the form of the other results in this paper. As we have seen, these equivalent conditions are met precisely when R is a Dedekind domain.

Theorem 4.5. *Every torsionless R-module is projective (hence a direct sum of rank-1 projective ideals) if and only if every ideal of R is one-and-a-half generated.*

Chapter 5

Noetherian Domains

In this section, we will work within the class of Noetherian domains (domains that satisfy the Ascending Chain Condition for ideals).

Proposition 5.1. *A domain R is **Noetherian** if and only if every prime ideal of R is finitely generated.*

We will rely upon the following standard in commutative ring theory, as well as some well-known results whose proofs are beyond the scope of this thesis.

Krull Principal Ideal Theorem. *Let R be a Noetherian domain. If P is a minimal prime over some nonzero principal ideal aR, then P is a maximal ideal.*

Krull Intersection Theorem. *If I is an ideal in a Noetherian domain R, then $\cap_n I^n = 0$.*

The proof of the Krull Principal Ideal Theorem. is contained in [3]. We now state a necessary definition.

Definition 14. *An R-module M is **reflexive** if $Hom(Hom(M, R), R) \cong M$. A domain R is **reflexive** if every torsionless R-module is reflexive.*

The following is the main result of this section.

Theorem 5.2. *The following are equivalent for a Noetherian domain R:*

(a) Every ideal of R is 2-generated.

(b) Every fractional overring of R is reflexive.

(c) Every nonzero ideal of R is projective over its ring of endomorphisms.

(d) Every torsionless module is isomorphic to a direct sum of ideals of R.

Note that the equivalence of items (a) and (d) is the primary result of this chapter. Before we prove this result, we will establish a necessary foundation through a series of lemmas.

Lemma 5.3. *If R is Noetherian and M and A torsionless modules, then for any prime ideal P of R,*

$$Hom(M, A)_P = Hom(M_P, A_P).$$

Proof. Since $Hom(M_P, A_P) = \{\phi \in Hom(QM, QA) \mid \phi(M_P) \subseteq A_P\}$, then clearly $Hom(M, A)_P \subseteq Hom(M_P, A_P)$.

Let $\phi \in Hom(M_P, A_P)$. M is torsionless and thus finitely-generated. Let $x_1, x_1, \ldots, x_n$ be a finite generating set for M; then there exists an $r \in R \setminus P$ such that $r\phi(x_j) \in A$ for all j. But then

$$\phi = (1/r) \cdot r\phi \in Hom(M, A)_P.$$

□

Lemma 5.4. *A Noetherian domain R is reflexive if and only if R has Krull dimension 1 and M^{-1} can be generated by two elements for each maximal ideal M of R.*

We will give a sketch of this proof. For more detail see [2].

Sketch of Proof. ($\Rightarrow$) If R is reflexive, then every ideal of R is reflexive. Let P be a nonzero minimal prime ideal of R. By results in [2], there exist elements $a, b \in R$ such that $P = \{r \in R | ra \in Rb\}$. Let $q = a/b$ and $x = q + R$. Let M be a maximal ideal of R containing P. As in the proof of Lemma 5.10 below, $Rx \cap (M^{-1}/R) \neq 0$ and thus there must be some $r \in R$ such that $rq \in M^{-1} \setminus R$. Therefore, r is not an element of P, but

$Mr \subset \{s \in R : sa \in Rb\} = P$ implying that $M = P$. Thus, every nonzero prime ideal is maximal and R has Krull dimension 1.

Let M be a maximal ideal of R. First, we claim that $M^{-1}/R \cong R/M$. This is because there is a one-to-one correspondence between the set of ideals between M and R and the set of ideals between M^{-1} and R. Thus, M^{-1}/R is simple and the result follows. Our next claim is that this implies that M^{-1} is generated by two elements. This is clear since if we take some $u \in M^{-1} \backslash R$, then M^{-1} can be generated by 1 and u.

($\Leftarrow$) Once it is shown that the R-module Q/R is injective, the proof proceeds as follows. Let P be a rank 1 prime ideal of R. It can be shown that P must then be reflexive and thus $P^{-1}/R \neq 0$. It is sufficient to show that P is maximal (this is done by showing that R/P is a field) and thus Q/R is a universal injective R-module, meaning that it is injective and contains every simple R-module. It can then be shown that this is equivalent to R being reflexive.

□

The following lemma is a a weaker version of one direction of our main result of this section. It requires the domain to be local.

Lemma 5.5. *If R is a local domain such that every finitely-generated, torsion-free module is a direct sum of modules of rank 1, then every finitely-generated, torsion-free R-module of rank 1 is 2-generated.*

Sketch of Proof. Let I be a nonzero, finitely-generated R module. It will be sufficient to show that I is 2-generated. Let $a_1, a_2, ..., a_n$ be a minimal generating set for I. Assume that n is at least 2. Let $x = (a_1, a_2, ..., a_n) \in \oplus_n R = F$. Let B be the submodule of F generated by x. B is torsion-free of rank 1 and F/B is finitely-generated of rank $n-1$. Then by the hypothesis, $F/B = C_1 \oplus C_2 \oplus ... \oplus C_{n-1}$ where each C_i is torsion-free of rank 1.

It can be shown that there is a decomposition $F = F_1 \oplus F_2 \oplus ... \oplus F_{n-1}$ such that if $B_i = F_i \cap B$ then $B = B_1 \oplus B_2 \oplus ... \oplus B_{n-1}$ and $C_i \cong F_i/B_i$. Since B is torsion-free of rank 1, it is indecomposable. So, we can assume it is contained in one of the summands of F, say $B \subseteq F_1$. From here we can use the fact that the coordinates of x relative to any basis for F are a generating set for I and the minimality of such a generating set to conclude that B can not be properly contained in any proper summand of F and thus $F = F_1$. So $n - 1 = 1$ and $n = 2$.

□

Lemma 5.6. *If every nonzero ideal of a ring R is contained in only a finite number of maximal ideals of R, then any ideal I of R can be generated by by $max\{2, k\}$ elements, where k is the supremum over the maximal ideals M of R of the number of elements necessary to generate I_M as an R_M-module.*

More thorough treatments of the previous three lemmas appear in [2].

Lemma 5.7. *Let I be a finitely generated ideal of a domain R. Then the following are equivalent:*

(a) I is projective.

(b) I is invertible.

(c) I_P is principal for all maximal ideals P of R.

Proof. $(a \Rightarrow b)$ Assume I is projective. Then for any sequence:

$$R \oplus R \oplus ... \oplus R \xrightarrow{\phi} I \rightarrow 0,$$

there is $f : I \to R \oplus R \oplus ... \oplus R$ that splits ϕ. Define ϕ by

$$\phi(r_1, r_2, ..., r_n) = r_1 a_1 + r_2 a_2, ... + r_n a_n,$$

where $\{a_1, a_2, ..., a_n\}$ is a finite generating set for I, and let $f : I \to R \oplus R \oplus ... \oplus R$ be such that $f(x) = (f_1 x, f_2 x, ..., f_n x)$. Then $f_j \in I^{-1}$ for all j. By definition, $\phi f(x) = x$. But then

$$\phi(f_1 x, f_2 x, ..., f_n x) = (f_1 a_1 x + f_2 a_2 x + ... + f_n a_n x) = x(f_1 a_1 + f_2 a_2 + ... + f_n a_n) = x.$$

This implies that $f_1 a_1 + f_2 a_2 + ... + f_n a_n = 1$ and since each $f_j \in I^{-1}$ and each $a_j \in I$, this means that $f_1 a_1 + f_2 a_2 + ... + f_n a_n \in I^{-1} I$ and so $1 \in I^{-1} I$. Therefore, $I^{-1} I = R$ and I is invertible.

($b \Rightarrow c$) Assume $II^{-1} = R$. Then $I_P I_P^{-1} = R_P$ for any maximal ideal P of R. Thus $1 \in I_P I_P^{-1}$. So, assume $a_1 b_1 + ... + a_n b_n = 1$. Thus, it must be that for some i, $a_i b_i = u$,a unit. Therefore $a_i b_i u^{-1} = (u^{-1} b_i) a_i = 1$. Let $c \in I_P$. $c = c(u^{-1} b_i) a_i = (cu^{-1} b_i) a$. This shows that $I_P = R_P a$ and thus I_P is principal.

($c \Rightarrow b$) Assume I_P is principal for all maximal P. Furthermore, assume that I is not invertible. Then $II^{-1} \neq R$. II^{-1} is an ideal of R so it must be contained in some maximal P. Then $I_P (I^{-1})_P \subseteq P_P$. It follows from Lemma 5.3 that $(I_P)^{-1} = (I^{-1})_P$ where $(I_P)^{-1}$ is the inverse of I_P over R_P. So, I_P is not invertible over R_P. But this is a contradiction since over the local ring R_P, principal and invertible are equivalent. So, it must be that I is invertible over R.

($b \Rightarrow a$) Assume $II^{-1} = R$. So, there must be some $a_1 b_1 + a_2 b_2 + ... + a_n b_n \in II^{-1}$ that is equal to 1. Thus I can be generated by $\{a_1, a_2, ..., a_n\}$ since for any $c \in I$, $c = ((cb_1) a_1 + ... + (cb_n) a_n)$. Since this set generates I, we only need to consider a map

$\phi : R \oplus R \oplus ... \oplus R \to I$ such that $\phi(r_1, r_2, ..., r_n) = r_1 a_1 + r_2 a_2 + ... + r_3 a_3$. Then if we consider $f : I \to R \oplus R \oplus ... \oplus R$ such that $f(x) = (b_1 x, b_2 x, ..., b_n x)$, then

$$\phi f(x) = \phi(b_1 x, b_2 x, ..., b_n x) = x(b_1 a_1 + b_2 a_2 + ... + b_n a_n) = x.$$

Thus, I is projective and the proof is complete. □

Lemma 5.8. *If R is Noetherian and satisfies (a), (b), (c), or (d), then R has Krull dimension* 1.

Proof. (a) Assume every ideal of R is 2-generated. Without loss of generality, we can assume that R is a local domain with maximal ideal M. We will then assume that this maximal ideal has rank greater than 1.

Let x and y be the elements that generate M. Consider R/Rx, which we will call $\bar{R}$. Note that this is a local ring with maximal ideal $\bar{M} = M/Mx$. Note that this ideal is generated by $y + Rx$ and is thus principal. By the Krull Intersection Theorem, we have that $\bigcap_n M^n = 0$ and so $\bigcap_n \bar{M}^n = 0$ since $\bar{M}^n = (M^n + Rx)/Rx$. We claim that either $\bar{R}$ is an integral domain or $\bar{M}^n = 0$ for some n.

Assume $\bar{M}^n \neq 0$ for all n. Let $r, s \in R$ and assume $(r + Rx)(s + Rx) = rs + Rx = 0$ but r and s are not elements of Rx. Since $\bigcap_n \bar{M}^n = 0$, there are m and n such that $r + Rx \in \bar{M}^n \setminus \bar{M}^{n+1}$ and $s + Rx \in \bar{M}^m \setminus \bar{M}^{m+1}$. Now, $\bar{M}^n = \{ay^n + Rx : a \in R\}$ and $\bar{M}^m = \{by^m + Rx : b \in R\}$. So, we can write $r + Rx = ay^n + Rx$ where $a \in R \setminus M$ and $s + Rx = by^n + Rx$ where $b \in R \setminus M$. Then $(r + Rx)(s + Rx) = aby^{n+m} + Rx$. Since ab is not in M, it is a unit and there exists $c \in R \setminus M$ such that $cab = 1_R$. So,

$$c(r + Rx)(s + Rx) = y^{n+m} + Rx = 0.$$

Therefore, $y^{n+m} \in Rx$ and thus

$$\bar{M}^{n+m} = \{dy^{n+m} + Rx : d \in R\} = 0,$$

a contradiction. Thus, if $\bar{M}^n \neq 0$ for all n, then $\bar{R}$ is an integral domain.

Assume $\bar{M}^n = 0$ for some positive n. But then $M \subseteq Rx$ and by the Krull Principal Ideal Theorem, the rank of M is no greater than 1. So, $\bar{R}$ is an integral domain. Thus Rx is a prime ideal of R since if $(r + Rx)(s + Rx) = 0$ then either $(r + Rx)$ or $(s + Rx)$ must be zero and thus either r or s is in Rx.

Since the rank of M is greater than one, M is not principal and thus x and y are distinct elements and they are linearly independent modulo M^2. But, M^2 can be generated by two elements and it must be that those two elements are among x^2, y^2, and xy. So, one of these must be generated by the other two. If x^2 is generated by y^2 and xy, then $M^2 \subset Rx$ and by the Krull Principal Ideal Theorem, the rank of M is at most 1. Similarly, y^2 cannot be generated by the other two. So, it must be that $xy = ax^2 + by^2 \in Rx$ for some $a, b \in M$. Now, $by^2 \in Rx$ and Rx is a prime ideal. Thus, either $b \in Rx$ or $y^2 \in Rx$. But Ry is not contained in Rx and so y^2 must not be in Rx. So, $b \in Rx$. Let $b = cx$ with $c \in R$. Then $xy = ax^2 + cxy^2$ and $y = ax + cy^2$. But this means $y \in M^2$, a contradiction. So, the rank of M must be 1 and thus R has Krull dimension 1.

(b) By Lemma 5.4, R has Krull dimension 1.

(c) Let P be a nonzero prime ideal of R. By Lemma 5.3, given an ideal I of R, $End(I)_P = End(I_P)$ and so (c) holds for the domain R_P. It is enough to show that no prime ideal properly contains another prime ideal and therefore, without loss of generality, we may assume that R is local with maximal ideal P. Note that P is invertible over its ring S of endomorphisms. Let P' be a maximal ideal of S. Then P' contains P and there is no other prime ideal P'' containing P because S is integral over R. Thus, P' is a minimal prime of S containing the principal ideal P and so by the Krull Principal Ideal Theorem, P' must also be a minimal prime. But S is integral over R, and so there can be no proper prime ideals of R contained in P since if there were there would have to be a proper prime ideal of S contained in P'. Thus, R has Krull dimension 1.

(d) As in part (a), assume that R is local with maximal ideal M. Then by Lemma 5.5, R can be generated by two elements. We can then retrace our steps from (a).

□

Lemma 5.9. *If R is Noetherian of Krull dimension 1 and M is a reflexive R-module, then for $I = trace_R(M) = Hom(M,R)M$, one has*

(i) I^{-1} is a ring, and

(ii) M is an I^{-1}-module.

Proof. Let M be any module and let $a \in I^{-1} = Hom(I,R)$. If $f : M \to R$, then $f(M) \subseteq I$ and so $af(M) \subseteq aI \subseteq R$ implying that af is again in $Hom(M,R)$. When $M = I^{-1}$, $aI^{-1} \subseteq I^{-1}$ and so $I^{-1}I^{-1} \subseteq I^{-1}$. Furthermore, $I \subseteq R$ implies $R \subseteq I^{-1}$. Hence I^{-1} is a ring and $Hom(M,R)$ is an I^{-1}-module. This latter fact implies that $M \cong Hom(Hom(M,R),R)$ is also an I^{-1}-module. □

Lemma 5.10. *If R is a reflexive, local domain with maximal ideal P, then every R-submodule, L, of Q containing R properly contains P^{-1}.*

Proof. Let $x \in L \setminus R$. Let $L_0 = Rx + R$. Consider L_0^{-1}, an ideal of R. If $L_0^{-1} = R$, then $L_0 = (L_0^{-1})^{-1} = R$, a contradiction since $x \in L \setminus R$. Thus, $L_0^{-1} \subseteq P$. This implies that $P^{-1} \subseteq (L_0^{-1})^{-1} = L_0 \subseteq L$. □

Lemma 5.11. *If M is a torsionless module of rank ≥ 2 over a Noetherian domain R of Krull dimension 1 such that $trace_R(M) = R$, then $M \cong R \oplus M'$ for some module M'.*

The proof of the above lemma appears in [5]. We can now begin the proof of the main result of this section.

Proof. Note that by Lemma 5.8, under any of the conditions, R is Noetherian of Krull dimension 1 so we will assume this as a matter of course.

$(a) \Rightarrow (b)$ If S is a fractional overring of R, and J is an ideal of S, then $rJ \subseteq R$ for some $0 \neq r \in R$, and so $rJ = Ra + Rb$ for some $a, b \in rJ$. Then $rJ = Sa + Sb$ from which it follows that $J = S(a/r) + S(b/r)$ is 2-generated over S. Therefore, it suffices to show that a domain whose ideals are 2-generated is reflexive, and it is enough to argue that R is reflexive. But, by Lemma 5.4, R being reflexive is equivalent to R having Krull dimension 1 and M^{-1} being 2-generated. We know that R has Krull dimension 1 and also that M^{-1} is isomorphic to an ideal of R and is thus 2-generated. Therefore, R is reflexive as desired.

$(b) \Rightarrow (c)$ An ideal I of R is projective over its endomorphism ring S if and only if $Hom(I, S)I = S$, which in turn holds if and only if $S_P = Hom(I, S)_P I_P = Hom(I_P, S_P)I_P$ for every prime ideal P of R. We know by Lemma 5.3 that the second equality holds. Therefore, we may assume that R is local with maximal ideal P.

$P^{-1}P$ is an ideal of R, so either $P^{-1}P = R$ or $P^{-1}P = P$ since P is the maximal ideal of R. If $P^{-1}P = R$, then P is invertible. Let $I \subseteq P$ be an ideal of R. It can be seen that either I is principal or $I \subseteq \bigcap P^n = 0$. This shows that, in this case, R is a PID. By Lemma 5.7, any ideal I is then projective. So, assume that $P^{-1}P \subseteq P$. But then,

$$P^{-1}P^{-1}P \subseteq P^{-1}P \subseteq P,$$

and thus, $P^{-1}P^{-1} \subseteq P^{-1}$. So P^{-1} is an overring of R. Moreover, if S is any fractional overring of R, then $S^{-1} \subseteq R$ implies that $S^{-1} \subseteq P$. Thus, $R_1 = P^{-1} \subseteq (S^{-1})^{-1} = S$ since R is reflexive.

Let I be an ideal of R. If $I^{-1}I \subseteq P$, then $R_1I^{-1}I \subseteq R_1P = P$ and thus $R_1I^{-1} \subseteq I^{-1}$. Consequently, $I = Hom(I^{-1}, R)$ is also an R_1-module. Thus, either I is principal (as when $P^{-1}P = R$) or I is a module over R_1.

We will argue that either R_1 is a pid, or, R_1 is a reflexive local domain whose maximal ideal P_1 properly contains P. Since P has endomorphism ring R_1 and R_1 is reflexive, P is invertible over R_1 by the argument in the last paragraph. Write $P = R_1a$. Recall that

$$0 \to R/P \to R_1/P \to R_1/R \to 0$$

exact, implies that R_1/P has length 2 as an R (or R/P) module. If P_1, P_2 are distinct maximal ideals of R_1, then

$$R_1/P_1P_2 \cong R_1/P_1 \oplus R_1/P_2,$$

by the Chinese Remainder Theorem. Since $P \subseteq P_1P_2$, we must have $P = P_1P_2$. Thus P_1 and P_2 are invertible since $P_1^{-1} = P_2P^{-1}$ and $P_2^{-1} = P_1P^{-1}$ Thus, they are also principal, and R is a pid. Otherwise, R_1 is local with maximal ideal P_1. If $P = P_1$, then R_1 is a pid.

Let S be an overring of R and let I be an ideal with endomorphism ring S. If $S = R$, then I is principal by the same argument used earlier in this proof. Otherwise, $R_1 \subseteq S$ by Lemma 5.10. In this case, either $S = R_1$, or $R_2 = P_1^{-1} \subseteq S$ where P_1 is the maximal ideal of R_1. In the first case, again, I is principal, and in the second case, either $S = R_2$ or S contains P_2^{-1} where P_2 is the maximal ideal of R_2. We claim that this ascending chain of ideals must terminate at some point and thus leave us with the conclusion that I is principal and thus invertible over S and finally projective over S. To see that this chain terminates, recall that R is Noetherian and note that S is finitely generated as an R-module. Thus S is a Noetherian R-module and thus satisfies the ascending chain condition on ideals and thus our ascending chain must terminate, leaving us with the conclusion that I is projective over S.

$(c) \Rightarrow (b)$ This proof, while not excessively long, is beyond the scope of this paper. It can be found in [6].

$(b) \Rightarrow (d)$ Let $I = trace_R(M)$ and let S be the endomorphism ring of I. Since $(b) \Rightarrow (c)$, M is a reflexive R-module and so by Lemma 5.9, I^{-1} is a ring and M is an I^{-1}-module. But I is also an I^{-1}-module and so $I^{-1}I \subseteq I$ which implies $I^{-1} \subseteq S$. Clearly $S \subseteq I^{-1}$ and so $S = I^{-1}$. Thus, M is an S-module. Because I is projective over S, it follows that $trace_S(M) = S$. Hence by Lemma 5.11, $M = M' \oplus S$, and induction applies to M' to show that M is a direct sum of fractional ideals of R.

$(d) \Rightarrow (a)$ For any maximal ideal M of R, every ideal of the local ring R_M can be generated by two elements by Lemma 5.5. We know that R has Krull dimension 1 and thus any nonzero prime ideal of R is contained in only one maximal ideal of R and any element of R is contained in only a finite number of maximal ideals of R. So, we can apply Lemma 5.6 and conclude that every ideal of R can be generated by two elements.

□

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