# Spatial Spread and Front Propagation Dynamics of Nonlocal Monostable Equations in Periodic Habitats

by

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Abstract

This dissertation is concerned with spatial spread and front propagation dynamics of monostable equations with nonlocal dispersal in spatially periodic habitats. Such equations arise in modeling the population dynamics of many species which exhibit nonlocal internal interactions and live in spatially periodic habitats. The main results of the dissertation consist of the following four parts.

Firstly, we establish a general principal eigenvalue theory for spatially periodic nonlocal dispersal operators. Some sufficient conditions are provided for the existence of principal eigenvalue and its associated positive eigenvector for such dispersal operators. It shows that a spatially periodic nonlocal dispersal operator has a principal eigenvalue for the following three special but important cases: (i) the nonlocal dispersal is nearly local; (ii) the periodic habitat is nearly globally homogeneous or (iii) it is nearly homogeneous in a region where it is most conducive to population growth. It also provides an example which shows that in general, a spatially periodic nonlocal dispersal operator may not possess a principal eigenvalue, which reveals some essential difference between random dispersal and nonlocal dispersal. The principal eigenvalue theory established in this dissertation provides an important tool for the study of the dynamics of nonlocal monostable equations and is of also great importance in its own.

Secondly, applying the principal eigenvalue theory for nonlocal dispersal operators and comparison principle for sub- and super-solutions, we obtain one of the important features for monostable equations, that is, the existence, uniqueness, and global stability of spatially periodic positive stationary solutions to a general spatially periodic nonlocal monostable equation. In spite of the use of the principal eigenvalue theory for nonlocal dispersal operators in the proof, this feature is generic for nonlocal monostable equations in the sense it is

independent of the existence of the principal eigenvalue of the linearized nonlocal dispersal operator at the trivial solution of the monostable equation, which is of great biological importance.

Thirdly, applying the principal eigenvalue theory for nonlocal dispersal operators and comparison principle for sub- and super-solutions, we obtain another important feature for monostable equations, that is, the existence of a spatial spreading speed of a general spatially periodic nonlocal equation in any given direction, which characterizes the speed at which a species invades into the region where there is no population initially in the given direction. It is also seen that this feature is generic for nonlocal monostable equations in the same sense as above. Moreover, it is shown that spatial variation of the habitat speeds up the spatial spread of the population.

Finally, this dissertation also deals with front propagation feature for monostable equations with non-local dispersal in spatially periodic habitats. It is shown that a spatially periodic nonlocal monostable equation has in any given direction a unique stable spatially periodic traveling wave solution connecting its unique positive stationary solution and the trivial solution with all propagating speeds greater than the spreading speed in that direction for the special but important cases mentioned above, that is, (i) the nonlocal dispersal is nearly local; (ii) the periodic habitat is nearly globally homogeneous or (iii) it is nearly homogeneous in a region where it is most conducive to population growth. It remains open whether this feature is generic or not for spatially periodic nonlocal monostable equations.

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# Chapter 1

# Introduction

This dissertation is devoted to the study of spatial spread and front propagation dynamics of monostable equations with nonlocal dispersal in spatially periodic habitats. Monostable equations are widely used to model the population dynamics of many species in biology and ecology. In general, such an equation is of the following form,

$$\frac{\partial u}{\partial t} = \nu Du + u(t,x)f(x,u(t,x)), \quad x \in \Omega \subseteq \mathbb{R}^N, \tag{1.1}$$

where $u(t,x)$ represents the population density of a species at time $t$ and spatial location $x$, $D$ is a dispersal operator which measures the diffusion or redistribution of the species, $\nu > 0$ is the dispersal rate, and the term $f$ measures the growth rate of the population of the species and satisfies the so called monostablility assumptions (that is, $f(x,u) < 0$ for $u \gg 1$, $\frac{\partial f}{\partial u}(x,u) < 0$ for $u \geq 0$ and $u \equiv 0$ is linearly unstable in proper sense), the domain $\Omega \subseteq \mathbb{R}^N$ (or $\mathbb{Z}^N$) may be bounded or unbounded. Without loss of generality, $\nu$ can be chosen 1 by rescaling time $t$ and changing $f$.

Among the dispersal operators often adopted in literature are nonlocal, random or local, and discrete dispersal operators. In particular, (1.1) with $D$ being a nonlocal dispersal operator, that is,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + u(t,x)f(x,u(t,x)), \quad x \in \Omega, \tag{1.2}$$

is widely used to model the population dynamics of a species in which the movements or interactions of the organisms occur between non-adjacent spatial locations, where $k : \mathbb{R}^N \to \mathbb{R}^+$ is a nonlocal dispersal kernel function with $\int_{\mathbb{R}^N} k(x)dx = 1$. Classically, one assumes that the internal interaction of the organisms in a species is random and local (i.e. species moves randomly between the adjacent spatial locations), which leads to (1.1) with $D = \Delta$, that is, reaction-diffusion equations of the following form,

$$\frac{\partial u}{\partial t} = \Delta u + uf(x,u), \quad x \in \Omega. \tag{1.3}$$

Another dispersal strategy is nearest neighbor interaction in a patchy environment modeled by the lattice $\mathbb{Z}^N$. This leads to (1.1) with $D$ being a discrete dispersal operator, that is, the following lattice system of ordinary differential equations

$$\dot{u}_j = \sum_{k=(k_1,k_2,\cdots,k_N)\in K} (u_{j+k} - u_j) + u_j f(j,u_j), \quad j \in \Omega \tag{1.4}$$

where $K = \{(k_1, k_2, \cdots, k_N) \in \mathbb{Z}^N |\, k_1^2 + \cdots + k_N^2 = 1\}$.

Nonlocal, random, and discrete dispersal evolution equations are then of great interests in their own. They are also related to each other. For example, (1.4) can be viewed as a spatial discretization of (1.3). In order to indicate some relationship between nonlocal and random dispersal, we take $k(z) = \frac{1}{\delta^N}\tilde{k}(z/\delta)$ for some $\delta > 0$ and $\tilde{k}(\cdot) : \mathbb{R}^N \to \mathbb{R}^+$ which is smooth, symmetric, supported on $B(0,1) := \{x \in \mathbb{R}^N \mid \|x\| < 1\}$, and $\int_{\mathbb{R}^N} \tilde{k}(x)dx = 1$. Then for any smooth function $u(x)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} k(y-x)u(y)\,dy - u(x)\\
&= \int_{\mathbb{R}^N} \tilde{k}(z)\left[u(x) + \delta(\nabla u(x)\cdot z) + \frac{\delta^2}{2}\sum_{i,j=1}^N u_{x_i x_j}(x) z_i z_j + O(\delta^3)\right] dz - u(x)\\
&= \frac{\delta^2}{2N}\int_{\mathbb{R}^N} \tilde{k}(z)\|z\|^2\,dz \Delta u(x) + O(\delta^3).
\end{aligned}$$

Hence, the dispersal operator $u \mapsto \int_{\mathbb{R}^N} k(\cdot - y)u(y)dy - u(\cdot)$ "behaves" the same as the operator $u \mapsto \frac{\delta^2}{2N}\int_{\mathbb{R}^N} \tilde{k}(z)\|z\|^2\,dz\Delta u$ for $\delta \ll 1$, and $\delta$ plays basically the role of a dispersal rate.

In this dissertation, we will focus on spatially periodic nonlocal monostable equations in unbounded domains, that is, equations of the form (1.2) with $f(x, u)$ being periodic in $x$ and being of the monostable properties (see (H2) and (H3) in Charter 2 for detail). We remark that heterogeneities are present in many biological and ecological models. The periodicity of $f(x, u)$ in $x$ takes into account the periodic heterogeneities of the media of the underlying systems and monostablility assumptions reflect the natural feature for population growth models.

Common and central dynamical issues about dispersal monostable equations in unbounded domains include the understanding of spatial spread and front propagation dynamics. Here are two most fundamental dynamical problems associated to the spatial spread and front propagation dynamics of monostable equations: how fast the population spreads as time evolves? are there solutions which preserve the shape and propagate at some speed along certain direction?

The study of spatial spread and front propagation dynamics of monostable equations traces back to Fisher [19] and Kolmogorov, Petrowsky, and Piscunov [39]. In the pioneering works of Fisher [19] and Kolmogorov, Petrowsky, Piscunov [39], they studied the spatial spread and front propagation dynamics of the following special case of (1.3)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \qquad x \in \mathbb{R}. \tag{1.5}$$

Here $u$ is the frequency of one of two forms of a gene. Fisher in [19] found traveling wave solutions $u(t, x) = \phi(x - ct)$, $(\phi(-\infty) = 1, \phi(\infty) = 0)$ of all speeds $c \geq 2$ and showed that there are no such traveling wave solutions of slower speed. He conjectured that the take-over occurs at the asymptotic speed 2. This conjecture was proved in [39] by Kolmogorov, Petrowsky, and Piscunov, that is, they proved that for any nonnegative solution $u(t, x)$ of (1.5), if at time $t = 0$, $u$ is 1 near $-\infty$ and 0 near $\infty$, then $\lim_{t\to\infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$ (i.e. the population invades into the region with no initial population with speed 2).

Since then, the spatial speed and front propagation dynamics of (1.3) has been widely studied (see [1], [2], [4], [5], [6], [18], [21], [26], [31], [36], [41], [42], [44], [47], [48], [51], [52], [53], [61], [62], [63], and references therein). The spatial spreading dynamics of (1.5) has been well extended to (1.3). To be more precise, assume that $f(x,u)$ is periodic in $x$, that is $f(x+p_i\mathbf{e_i},u)=f(x,u)$ for some $p_i>0$ $(i=1,2,\cdots,N)$, $\mathbf{e_i}$ denotes the vector with a 1 in the ith coordinate and 0's elsewhere, and satisfies the following monostablility assumptions: $f\in C^1(\mathbb{R}^N\times[0,\infty),\mathbb{R})$, $\sup\limits_{x\in\mathbb{R}^N,u\geq 0}\dfrac{\partial f(x,u)}{\partial u}<0$, $f(x,u)<0$ for $x\in\mathbb{R}^N$ and $u\gg 1$, and the principal eigenvalue of

$$\begin{cases}\Delta u+a_0(x)u=\lambda u, & x\in\mathbb{R}^N\\ u(x+p_i\mathbf{e_i})=u(x), & x\in\mathbb{R}^N\end{cases}$$

is positive, where $a_0(x)=f(x,0)$. Without loss of generality, assume $\nu=1$. It has been shown that (1.3) has exactly two spatially periodic equilibrium solutions, $u=0$ and $u=u^+$, and $u=0$ is linearly unstable and $u=u^+$ is globally asymptotically stable with respect to spatially periodic perturbations (which gives a reason which the above assumptions are referred to monostability assumptions). Let $\xi\in S^{N-1}:=\{\xi\in\mathbb{R}^N|\,\|\xi\|=1\}$. It has also been shown that for every $\xi\in S^{N-1}$, there is a $c^*(\xi)\in\mathbb{R}$ such that for every $c\geq c^*(\xi)$, there is a traveling wave solution connecting $u^+$ and $u^-\equiv 0$ and propagating in the direction of $\xi$ with speed $c$, and there is no such traveling wave solution of slower speed in the direction of $\xi$. The minimal wave speed $c^*(\xi)$ is of some important spreading properties, that is,

$$\liminf_{t\to\infty}\inf_{x\cdot\xi\leq ct}(u(t,x;u_0)-u^+(x))=0\quad\forall c<c^*(\xi)$$

and

$$\limsup_{t\to\infty}\sup_{x\cdot\xi\geq ct}u(t,x;u_0)=0\quad\forall c>c^*(\xi),$$

for all nonnegative uniformly continuous bounded function $u_0$ satisfying that $u_0(x)\geq\delta_0$ for some $\delta_0>0$ and $x\in\mathbb{R}^N$ with $x\cdot\xi\ll -1$ and $u_0(x)=0$ for $x\in\mathbb{R}^N$ with $x\cdot\xi\gg 1$. Here

$u(t, x; u_0)$ denotes the solution of (1.3) with $u(0, x; u_0) = u_0(x)$ and $\inf_{x \cdot \xi \le ct}$ ( $\sup_{x\cdot\xi \ge ct}$ ) denotes the infimum (supremum) taken over all the $x \in \mathbb{R}^N$ satisfying that $x \cdot \xi \le ct$ ($x \cdot \xi \ge ct$) for given $\xi \in S^{N-1}$ and $c, t \in \mathbb{R}$. Hence $c^*(\xi)$ is also called the *spreading speed* of (1.3) in the direction of $\xi$. Moreover, it has the following variational characterization. Let $\lambda(\xi, \mu)$ be the eigenvalue of

$$\begin{cases} \Delta u - 2\mu \sum_{i=1}^N \xi_i \frac{\partial u}{\partial x_i} + (a_0(x) + \mu^2)u = \lambda u, \quad x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e_i}) = u(x), \quad x \in \mathbb{R}^N \end{cases} \tag{1.6}$$

with largest real part, where $a_0(x) = f(x, 0)$ (it is well known that $\lambda(\xi, \mu)$ is real and algebraically simple. $\lambda(\xi, \mu)$ is called the *principal eigenvalue* of (1.6) in literature). Then

$$c^*(\xi) = \inf_{\mu > 0} \frac{\lambda(\xi, \mu)}{\mu}. \tag{1.7}$$

(See [4], [5], [6], [41], [47], [48], [63] and references therein for the above mentioned properties).

Spatial spread and front propagation dynamics is also quite well studied for monostbale equations with discrete dispersal. We refer to [9], [10], [24], [32], [54], [59], [60], [62], [63], [64], [65], etc. for the study of spatial spread and front propagation dynamics dynamics of monostable equations with discrete dispersal of the form (1.4).

The objective of this dissertation is to investigate the spatial spread and front propagation of spatially periodic nonlocal monostable equations of the form (1.2).

Recently, various dynamical problems related to the spatial spread and front propagation dynamics of nonlocal dispersal equations of the form (1.2) have also been studied by many authors. See, for example, [3], [8], [11], [13], [15], [22], [23], [28], [29], [33], [34], [35], [37], [55], for the study of spectral theory for nonlocal dispersal operators and the existence, uniqueness, and stability of nontrivial positive stationary solutions. See, for example, [12], [14], [16], [40], [45], [49], [62], [63], for the study of entire solutions and the existence of spreading speeds and traveling wave solutions connecting the trivial solution $u = 0$ and a nontrivial positive stationary solution for some special cases of (1.2). In particular, if $f(x, u)$

is independent of $x$, then it is proved that (1.2) has a spreading speed $c^*(\xi)$ in every direction of $\xi \in S^{N-1}$ ($c^*(\xi)$ is indeed independent of $\xi \in S^{N-1}$ in this case) and for every $c \geq c^*(\xi)$, (1.2) has a traveling wave solution connecting $u^+$ and 0 and propagating in the direction of $\xi$ with propagating speed $c$ (see [12]).

However, most existing works on spatial spreading dynamics of monostable equations with nonlocal dispersal deal with spatially homogeneous equations (i.e. $f(x, u)$ in (1.2) is independent of $x$). There is little understanding of the spatial spread and front propagation dynamics of general nonlocal dispersal monostable equations. One major difference between (1.3) and (1.2) is that the solution operator of (1.3) in proper phase space is compact with respect to the uniform convergence on bounded subsets of $\mathbb{R}^N$ (i.e., is compact with respect to open compact topology), whereas the solution operator of (1.2) in a usual phase space does not exhibit such compactness features. It appears to be difficult to adopt many existing methods for the study of spatial spread and front propagation dynamics of random dispersal monostable equations in dealing with (1.2) in general due to the lack of compactness of the solution operator and the spatial inhomogeneity of the nonlinearity. In fact, there is even a lack of general principal eigenvalue theory for nonlocal dispersal operators and a lack of positive stationary solutions of spatially periodic nonlocal monostable equations, which are important tools/ingredients in the study of spatial spread and front propagation dynamics of monostable equations.

In this dissertation, we will then first establish a general principal eigenvalue theory for spatially periodic nonlocal dispersal operators (see chapter 4). We show that a spatially periodic nonlocal dispersal operator has a principal eigenvalue for following three special but important cases: (i) the nonlocal dispersal is nearly local; (ii) the periodic habitat is nearly globally homogeneous or (iii) it is nearly homogeneous in a region where it is most conducive to population growth. It also provides an example which shows that in general, a spatially periodic nonlocal dispersal operator may not possess a principal eigenvalue, which reveals some essential difference between random dispersal and nonlocal dispersal. The principal eigenvalue theory established in this dissertation provides an important tool for the study of the dynamics of nonlocal monostable equations and is of also great importance in its own.

Next, applying the principal eigenvalue theory for nonlocal dispersal operators and comparison principle for sub- and super-solutions, we obtain one of the important features for monostable equations, that is, the existence, uniqueness, and global stability of spatially periodic positive stationary solutions to a general spatially periodic nonlocal monostable equation (see chapter 5). In spite of the use of the principal eigenvalue theory for nonlocal dispersal operators in the proof, this feature is generic for nonlocal monostable equations in the sense it is independent of the existence of the principal eigenvalue of the linearized nonlocal dispersal operator at the trivial solution of the monostable equation, which is of great biological importance.

Furthermore, we then investigate the spatial spreading speeds of spatially periodic nonlocal monostable equations (see chapter 6). Applying the principal eigenvalue theory for nonlocal dispersal operators and comparison principle for sub- and super-solutions, we obtain another important feature for monostable equations, that is, the existence of a spatial spreading speed of a general spatially periodic nonlocal equation in any given direction, which characterizes the speed at which a species invades into the region where there is no population initially in the given direction. It is also seen that this feature is generic for nonlocal monostable equations in the same sense as above. Moreover, it is shown that spatial variation of the habitat speeds up the spatial spread of the population.

Finally, we deal with traveling wave solutions of monostable equations with non-local dispersal in spatially periodic habitats (see chapter 7). It is shown that a spatially periodic nonlocal monostable equation has in any given direction a unique stable spatially periodic traveling wave solution connecting its unique positive stationary solution and the trivial solution with all propagating speeds greater than the spreading speed in that direction for the special but important cases mentioned above, that is, (i) the nonlocal dispersal is nearly local; (ii) the periodic habitat is nearly globally homogeneous or (iii) it is nearly homogeneous in a region where it is most conducive to population growth. It remains open whether this feature is generic or not for spatially periodic nonlocal monostable equations.

The rest of the dissertation is organized as follows. In chapter 2, we state some standing notations, assumptions, definitions and the main results. In chapter 3, we develop some basic tools or fundamental theory for the use in later chapters, such as semigroup theory, comparison principle, sub- and super-solutions. We will investigate the spectral theory of nonlocal dispersal operators in chapter 4. In chapter 5, we study the existence, uniqueness and stability of stationary solutions of (1.2). In chapter 6, spatial spreading speeds of (1.2) are investigated. In chapter 7, we study the existence,uniqueness and stability of the traveling wave solutions of spatially periodic nonlocal monostable equations. The dissertation will end up with remarks, open problems, and future plan in chapter 8.

Chapter 2

Notations, Assumptions, Definitions and Main Results

In this chapter, we introduce first the standing notations, assumptions, and the definitions of principal eigenvalue of nonlocal dispersal operators, spatial spreading speeds, and traveling wave solutions of spatially periodic nonlocal monostable equations. We then state the main results of the dissertation.

## 2.1 Notations, Assumptions and Definitions

In this section, we introduce the standing notations, assumptions, and the definitions of principal eigenvalue of nonlocal dispersal operators, spatial spreading speeds, and traveling wave solutions of spatially periodic nonlocal monostable equations.

Consider (1.2) with $\Omega = \mathbb{R}^N$, that is,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + u(t,x)f(x,u(t,x)), \quad x \in \mathbb{R}^N.$$

We assume that $f(x,u)$ is periodic in $x$, that is, there are $p_i > 0$ $(i = 1, 2, \cdots, N)$ such that $f(x + p_i\mathbf{e_i}) = f(x,u)$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$. We assume that the nonlocal kernel function $k(\cdot)$ satisfies the following assumption.

**(H1)** $k(\cdot) \in C^1(\mathbb{R}^N, \mathbb{R}^+)$, $\int_{\mathbb{R}^N} k(z)dz = 1$, $k(0) > 0$ *and* $\int_{\mathbb{R}^N} k(z)e^{\mu\|z\|}dz < \infty$ *for any* $\mu > 0$.

We remark that (H1) implies that the kernel function $k(\cdot)$ is actually a smooth probability density function of some random variable X, and $k(\cdot)$ is strictly positive at the origin and the expected value of $e^{\mu|X|}$ is finite,that is, $E(e^{\mu|X|}) < \infty$. There are a lot of such examples.

For instance, the probability density functions of normal distributions and all smooth probability density functions which are positive at the origin and supported on a bounded set satisfy (H1). The following is one example which has a bounded support: $k(z) = \frac{1}{\delta_0^N}\tilde{k}(z/\delta_0)$

$$\tilde{k}(z) = \begin{cases} C\exp\left(\frac{1}{\|z\|^2-1}\right) \quad \text{for} \quad \|z\| < 1 \\ 0 \quad \text{for} \quad \|z\| \geq 1, \end{cases} \tag{2.1}$$

where $C > 0$ is chosen such that $\int_{\mathbb{R}^N} \tilde{k}(z)dz = 1$.

Let

$$X = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, u \text{ is uniformly continuous on } \mathbb{R}^N \text{ and } \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\} \tag{2.2}$$

with norm $\|u\|_X = \sup_{x\in\mathbb{R}^N} |u(x)|$, and

$$X^+ = \{u \in X \,|\, u(x) \geq 0 \quad \forall x \in \mathbb{R}^N\}. \tag{2.3}$$

Let

$$X_p = \{u \in X \,|\, u(x + p_i\mathbf{e_i}) = u(x) \ \forall x \in \mathbb{R}^N, \ i = 1, 2, \cdots, N\} \tag{2.4}$$

and

$$X_p^+ = \{u \in X_p \,|\, u(x) \geq 0 \quad \forall x \in \mathbb{R}^N\}. \tag{2.5}$$

Let $I$ be the identity map on $X_p$, and $\mathcal{K}$, $a_0(\cdot)I : X_p \to X_p$ be defined by

$$\big(\mathcal{K}u\big)(x) = \int_{\mathbb{R}^N} k(y - x)u(y)dy, \tag{2.6}$$

$$(a_0(\cdot)Iu)(x) = a_0(x)u(x), \tag{2.7}$$

where $a_0(x) = f(x, 0)$.

Throughout this dissertation, a function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is said to be *smooth* if $h(x,u)$ is $C^N$ in $x \in \mathbb{R}^N$ and $C^1$ in $u \in \mathbb{R}$. We assume that $f$ satisfies the following "monostablility" assumptions:

**(H2)** $f \in C^1(\mathbb{R}^N \times [0,\infty), \mathbb{R})$, $\sup_{x \in \mathbb{R}^N, u \geq 0} \dfrac{\partial f(x,u)}{\partial u} < 0$ *and* $f(x,u) < 0$ *for* $x \in \mathbb{R}^N$ *and* $u \gg 1$.

**(H3)** $u \equiv 0$ *is linearly unstable in* $X_p$*, that is,* $\lambda_0 := \sup\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(\mathcal{K} - I + a_0(\cdot)I)\}$ *is positive, where* $\sigma(\mathcal{K} - I + a_0(\cdot)I)$ *is the spectrum of the operator* $\mathcal{K} - I + a_0(\cdot)I$ *on* $X_p$*.*

A typical example for $f(x,u)$ is $f(x,u) = a(x) - u$ with $a(\cdot) \in X_p^+ \setminus \{0\}$. Note that (H2) and (H3) reflect the natural feature of population growth models.

Among the main techniques employed in this dissertation are the comparison principle, sub- and super-solutions, and the principal eigenvalue theory of the eigenvalue problem,

$$\big(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I\big)v = \lambda v, \quad v \in X_p, \tag{2.8}$$

where $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $a(\cdot) \in X_p$, the operator $a(\cdot)I$ has the same meaning as in (2.7) with $a_0(\cdot)$ being replaced by $a(\cdot)$, and $\mathcal{K}_{\xi,\mu} : X_p \to X_p$ is defined by

$$(\mathcal{K}_{\xi,\mu}v)(x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)v(y)dy. \tag{2.9}$$

We point out the following relation between (1.2) and (2.8): if $u(t,x) = e^{-\mu(x\cdot\xi - \frac{\lambda}{\mu}t)}\phi(x)$ with $\phi \in X_p \setminus \{0\}$ is a solution of the linearization of (1.2) at $u = 0$,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + a_0(x)u(t,x), \quad x \in \mathbb{R}^N, \tag{2.10}$$

where $a_0(x) = f(x,0)$, then $\lambda$ is an eigenvalue of (2.8) with $a(\cdot) = a_0(\cdot)$ or $\mathcal{K}_{\xi,\mu} - I + a_0(\cdot)I$ and $v = \phi(x)$ is a corresponding eigenfunction.

**Definition 2.1** (Principal eigenvalue)**.** *Let* $\sigma(K_{\xi,\mu} - I + a(\cdot)I)$ *be the spectrum of* $K_{\xi,\mu} - I + a(\cdot)I$ *on* $X_p$*.*

*(1)* $\lambda_0(\xi,\mu,a) := \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(K_{\xi,\mu} - I + a(\cdot)I)\}$ *is called the* principal spectrum point *of* $K_{\xi,\mu} - I + a(\cdot)I$.

*(2) A number* $\lambda(\xi,\mu,a) \in \mathbb{R}$ *is called* the principal eigenvalue *of* (2.8) *or* $K_{\xi,\mu} - I + a(\cdot)I$ *if it is an algebraically simple eigenvalue of* $K_{\xi,\mu} - I + a(\cdot)I$ *with an eigenfunction* $v \in X_p^+$, *and for every* $\lambda \in \sigma(K_{\xi,\mu} - I + a(\cdot)I) \setminus \{\lambda(\xi,\mu,a)\}$, $\mathrm{Re}\lambda < \lambda(\xi,\mu,a)$.

Observe that if the principal eigenvalue $\lambda(\xi,\mu,a)$ of $K_{\xi,\mu}-I+a(\cdot)I$ exists, then $\lambda(\xi,\mu,a) = \lambda_0(\xi,\mu,a)$. If $\mu = 0$, (2.8) is independent of $\xi$ and hence we put

$$\lambda_0(a) := \lambda_0(\xi,0,a) \quad \forall\, \xi \in S^{N-1}. \tag{2.11}$$

Due to the lack of compactness of the semigroup generated by $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ on $X_p$ and the inhomogeneity of $a(\cdot)$, the existence of a principal eigenvalue and eigenfunction of (2.8) cannot be obtained from standard theory (e.g. the Krein-Rutman theorem). It should be pointed out that recently the principal eigenvalue problem for nonlocal dispersal has been studied by several authors (see [35], [37], [55], etc.). However, the existing results cannot be applied directly to (2.8). We will hence develop a principal eigenvalue theory for (2.8) or $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ in chapter 4. At some places, we make the following assumption on the existence of principal eigenvalues.

**(H4)** $\lambda(\xi,\mu,a)$ *exists for all* $\xi \in S^{N-1}$ *and* $\mu \geq 0$.

In the following, $\inf\limits_{x\cdot\xi\leq r}$ $(\sup\limits_{x\cdot\xi\geq r})$ represents the infimum (supremum) taken over all the $x \in \mathbb{R}^N$ satisfying that $x\cdot\xi \leq r$ for given $\xi \in S^{N-1}$ and $r \in \mathbb{R}$. Similarly, the notations $\inf\limits_{x\cdot\xi\leq ct}$, $\inf\limits_{|x\cdot\xi|\leq ct}$, $\inf\limits_{\|x\|\leq ct}$ $(\sup\limits_{x\cdot\xi\geq ct}$, $\sup\limits_{|x\cdot\xi|\geq ct}$, $\sup\limits_{\|x\|\geq ct})$ represent the infima (suprema) taken over all the $x \in \mathbb{R}^N$ satisfying the inequalities in the notations for given $\xi \in S^{N-1}$ and $c, t \in \mathbb{R}$. For a given $\xi \in S^{N-1}$, and

$$X^+(\xi) = \{u \in X^+ \mid \liminf_{r\to-\infty} \inf_{x\cdot\xi\le r} u(x) > 0,$$
$$u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^N \quad \text{with} \quad x\cdot\xi \gg 1\}. \tag{2.12}$$

It follows from the general semigroup approach (see [27] or [50]) that (1.2) has a unique (local) solution $u(t,x;u_0)$ with $u(0,x;u_0) = u_0(x)$ for every $u_0 \in X$. Moreover, a comparison principle in the usual sense holds for solutions of (1.2), and $u(t,x;u_0)$ exists for all $t \ge 0$ if $u_0 \in X^+$ (see Proposition 3.1).

**Definition 2.2** (Spatial spreading speed)**.** *Assume that (H1) - (H3) are fulfilled and that $\xi \in S^{N-1}$. We call a number $c^*(\xi) \in \mathbb{R}$ the spatial spreading speed of* (1.2) *in the direction of $\xi$ if the following properties are satisfied:*

$$\liminf_{t\to\infty} \inf_{x\cdot\xi\le ct} u(t,x;u_0) > 0 \quad \forall c < c^*(\xi)$$

*and*

$$\limsup_{t\to\infty} \sup_{x\cdot\xi\ge ct} u(t,x;u_0) = 0 \quad \forall c > c^*(\xi)$$

*for every $u_0 \in X^+(\xi)$.*

Observe that our definition coincides with the notion of $c^*(\xi)$ in [62] provided that $f(x,u)$ is independent of $x$. The construction based definition used in [44], [62], [63] is different in the sense that our definition does not guarantee the existence of $c^*(\xi)$. In fact, we focus in this dissertation on investigating the existence and characterization of $c^*(\xi)$ for $\xi \in S^{N-1}$.

To this end, let

$$\tilde{X} = \{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable and bounded}\} \tag{2.13}$$

endowed with the norm $\|u\|_{\tilde{X}} = \sup_{x\in\mathbb{R}^N} |u(x)|$ and

$$\tilde{X}^+ = \{u \in \tilde{X} \mid u(x) \geq 0 \quad \forall x \in \mathbb{R}^N\}. \tag{2.14}$$

Observe that $X_p \subset X \subset \tilde{X}$.

To study the spatial spreading and front propagation dynamics of (1.2), we sometime need to consider the space shifted equation of (1.2)

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + u(t,x)f(x+z, u(t,x)), \quad x \in \mathbb{R}^N \tag{2.15}$$

where $z \in \mathbb{R}^N$. Let $u(t,x;u_0,z)$ be the solution of (2.15) with $u(0,x;u_0,z) = u_0(x)$ for $u_0 \in X$.

By general semigroup theory (see [27] and [50]), for any $u_0 \in \tilde{X}$ and $z \in \mathbb{R}$, (2.15) has a unique (local) solution $u(t,\cdot) \in \tilde{X}$ with $u(0,x) = u_0(x)$. Let $u(t,x;u_0,z)$ be the solution of (2.15) with $u(0,x;u_0,z) = u_0(x)$. Note that if $u_0 \in X_p$ (resp. $X$), then $u(t,\cdot;u_0,z) \in X_p$ (resp. $X$). If $u_0 \in \tilde{X}^+$, then $u(t,x;u_0,z)$ exists for all $t \geq 0$.

**Definition 2.3** (Entire solution)**.** *A measurable function $u : \mathbb{R} \times \mathbb{R}^N :\to \mathbb{R}$ is call an entire solution of* (1.2) *if $u(t,x)$ is differentiable in $t \in \mathbb{R}$ and satisfies* (1.2) *for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*

**Definition 2.4** (Traveling wave solution)**.** *Suppose that* (1.2) *has a spatially periodic positive stationary solution $u = u^+(\cdot) \in X_P^+ \setminus \{0\}$.*

*(1) An entire solution $u(t,x)$ of* (1.2) *is called a* traveling wave solution *connecting $u^+(\cdot)$ and* 0 *and propagating in the direction of $\xi$ with speed $c$ if there is a bounded measurable function $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ such that $u(t,\cdot;\Phi(\cdot,z),z)$ exists for all $t \in \mathbb{R}$,*

$$u(t,x) = u(t,x;\Phi(\cdot,0),0) = \Phi(x - ct\xi, ct\xi) \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{2.16}$$

$$u(t,x;\Phi(\cdot,z),z) = \Phi(x - ct\xi, z + ct\xi) \quad \forall t \in \mathbb{R},\ x, z \in \mathbb{R}^N, \tag{2.17}$$

$$\lim_{x\cdot\xi \to -\infty} \big(\Phi(x,z) - u^+(x+z)\big) = 0, \quad \lim_{x\cdot\xi \to \infty} \Phi(x,z) = 0 \quad \textit{uniformly in } z \in \mathbb{R}^N, \tag{2.18}$$

$$\Phi(x, z - x) = \Phi(x', z - x') \quad \forall x, x' \in \mathbb{R}^N \text{ with } x \cdot \xi = x' \cdot \xi, \tag{2.19}$$

*and*

$$\Phi(x, z + p_i \mathbf{e_i}) = \Phi(x, z) \quad \forall x, z \in \mathbb{R}^N. \tag{2.20}$$

(2) *A bounded measurable function* $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ *is said to* generate a traveling wave solution of (1.2) in the direction of $\xi$ with speed $c$ *if it satisfies* (2.17)-(2.20).

**Remark 2.1.** *Suppose that* $u(t, x) = \Phi(x - ct\xi, ct\xi)$ *is a traveling wave solution of* (1.2) *connecting* $u^+(\cdot)$ *and* 0 *and propagating in the direction of* $\xi$ *with speed* $c$. *Then* $u(t, x)$ *can be written as*

$$u(t, x) = \Psi(x \cdot \xi - ct, x) \tag{2.21}$$

*for some* $\Psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfying that* $\Psi(\eta, z + p_i \mathbf{e_i}) = \Psi(\eta, z)$, $\lim_{\eta \to -\infty} \Psi(\eta, z) = u^+(z)$, *and* $\lim_{\eta \to \infty} \Psi(\eta, z) = 0$ *uniformly in* $z \in \mathbb{R}^N$. *In fact, let* $\Psi(\eta, z) = \Phi(x, z - x)$ *for* $x \in \mathbb{R}^N$ *with* $x \cdot \xi = \eta$. *Observe that* $\Psi(\eta, z)$ *is well defined and has the above mentioned properties. In some literature, the form* (2.21) *is adopted for spatially periodic traveling wave solutions (see [41], [46], [63], and references therein).*

## 2.2 Main Results

In this section, we state the main results of the dissertation. The first two theorems are about the principal eigenvalue of (2.8).

**Theorem A.** (Sufficient conditions for the existence of principal eigenvalues)

(1) *Support that* $k(z) = \frac{1}{\delta^N} \tilde{k}(\frac{z}{\delta})$ *for some* $\delta > 0$ *and* $\tilde{k}(\cdot)$ *with* $\mathrm{supp}(\tilde{k}) = B(0, 1) := \{z \in \mathbb{R}^N \,|\, \|z\| < 1\}$. *There is* $\delta_0 > 0$ *such that for every* $0 < \delta \le \delta_0$, *the principal eigenvalue* $\lambda(\xi, \mu, a)$ *of* (2.8) *exists for all* $\xi \in S^{N-1}$ *and* $\mu \in \mathbb{R}$.

(2) *If* $a(x)$ *satisfies that* $\max_{x \in \mathbb{R}^N} a(x) - \min_{x \in \mathbb{R}^N} a(x) < \theta$ *with* $\theta = \min\{\int_{z \cdot \xi < 0} k(z) dz \,|\, \xi \in S^{N-1}\}$, *then the conclusions in (1) hold.*

*(3) If $a \in X_P \cap C^N(\mathbb{R}^N)$ and the partial derivatives of $a(x)$ up to order $N-1$ at some $x_0$ are zero, where $x_0$ is such that $a(x_0) = \max_{x \in \mathbb{R}^N} a(x)$, then the conclusions in (1) hold.*

Let $\lambda(\xi, \mu, a)$ be the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)$. Note that if $\mu = 0$, (2.8) is independent of $\xi$ and hence we put

$$\lambda(a) := \lambda(\xi, 0, a) \quad \forall\, \xi \in S^{N-1} \tag{2.22}$$

(if $\lambda(\xi, 0, a)$ exists).

Let $\bar{a} = \frac{1}{|D|} \int_D f(x,0)dx$ with $D = \prod_{i=1}^{N} [0, p_i]$ and $|D| = \prod_{i=1}^{N} p_i$ where the period vector $\mathbf{p} = (p_1, ..., p_N)$.

**Theorem B.** (Influence of spatial variation) *Assume that $\lambda(\xi, \mu, a(x))$ of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ exists for any $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$. Then $\lambda(\xi, \mu, a(x)) \geq \lambda(\xi, \mu, \bar{a})$ for any $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$. $\lambda(\xi, \mu, a(x)) = \lambda(\xi, \mu, \bar{a})$ for some $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$ iff $a(x) \equiv \bar{a}$.*

We remark that the proof of Theorem A , the existence part of the principal eigenvalue, relies on techniques from the perturbation theory of Burger [7] (see [7, Proposition 2.1 and Theorem 2.2]) and on the arguments in [37, Theorem 2.6]. However, special care is required in view of the dependence of $\mathcal{K}_{\xi,\mu}$ on $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$. Note that the conclusions are independent of $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$ (i.e. for proper $\delta > 0$ and $a$, $\lambda(\xi, \mu, a)$ exists for every $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$). Theorem A (1) is proved in [37] for $\mu = 0$ with the assumption that $k(\cdot)$ is symmetric with a bounded support, that is, $k(z) = k(-z)$ supported on a ball for $z \in \mathbb{R}^N$. We extended A(1) to general kernel for all $\mu \in \mathbb{R}$. Theorem A(3) will play an important role in proving the existence of positive stationary solution and generic spreading speeds.

As it is well known, the principal eigenvalue of a random or local dispersal operator always exists. By Theorem A(1), if the nonlocal dispersal operator $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ is *nearly local* in the sense that the dispersal distance $\delta$ is sufficiently small, then we obtain a similar principal eigenvalue theory as for random dispersal operators.

Observe that $\mathcal{K}_{\xi,\mu} : X_p \to X_p$ is a compact and positive operator. If $a(x) \equiv a$ is independent of $x$, then it is not difficult to see that $\lambda(\xi, \mu, a) := a - 1 + \int_{\mathbb{R}^N} e^{-\mu z \cdot \xi} k(z) dz$ is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ for $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$. By Theorem A(2)(3), if $a(\cdot)$ has certain homogeneity features, then the nonlocal dispersal operator $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ also possesses a principal eigenvalue. More precisely, Theorem A (2) shows that if $a(x)$ is *nearly globally homogeneous* or *globally flat* in the sense that $\max_{x\in\mathbb{R}^N} a(x) - \min_{x\in\mathbb{R}^N} a(x) < \theta$, then the principal eigenvalue $\lambda(\xi, \mu, a)$ of the nonlocal dispersal operator $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ exists for $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$. Note that if $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ in (2.8) is replaced by $\nu[\mathcal{K}_{\xi,\mu} - I] + a(\cdot)I$ with a general positive dispersal rate $\nu > 0$, Theorem A (2) holds provided that $\max_{x\in\mathbb{R}^N} a(x) - \min_{x\in\mathbb{R}^N} a(x) < \nu\theta$. If $k(\cdot)$ is symmetric, then $\theta$ can be chosen 1. So A (2) holds provided that $\max_{x\in\mathbb{R}^N} a(x) - \min_{x\in\mathbb{R}^N} a(x) < \nu$, which means biologically that the variation in the habitat is less than the dispersal rate of the nonlocal dispersal operator $\mathcal{K}_{\xi,\mu} - I$. We say $a(\cdot)$ is *nearly homogeneous or flat in some region where it is most conducive to population growth in the zero-limit population* (which will be referred to as *nearly locally homogeneous* in the following) if all partial derivatives of $a(x)$ up to order $N-1$ are zero at some $x_0$ with $a(x_0) = \max_{x\in\mathbb{R}^N} a(x)$. Theorem A (3) shows that if $a(\cdot)$ is nearly locally homogenous, then for any $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$, the principal eigenvalue $\lambda(\xi, \mu, a)$ of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ exists, too. It should be pointed out that $a(x)$ is nearly globally homogeneous may not imply that it is nearly locally homogeneous.

Clearly, the "flatness" condition for $a(x)$ in Theorem A (3) is always satisfied for $N = 1$ or 2. Hence when $N = 1$ or 2, the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ exists for all $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$. In general, if $N \geq 3$ , the principal eigenvalue of (2.8) may not exist (see example in chapter 4). This reveals an essential difference between nonlocal dispersal operators and random dispersal operators.

How do spatial variations affect the principal eigenvalue (if exists)? In biological sense, Theorem B shows that spatial variation cannot reduce the principal eigenvalue of a dispersal operator with nonlocal dispersal and periodic boundary condition, and indeed it is increased except for degenerate cases.

After we established the spectral theory of the nonlocal dispersal operators, we can employ the comparison principle and construct sub- super-solutions to investigate the existence, uniqueness and stability of positive equilibrium solutions of (1.2). More precisely, we will prove the following theorem.

**Theorem C.** *(Existence, uniqueness, and stability of positive stationary solutions)*

*(1) If (H1)- (H3) hold, then* (1.2) *has exactly two stationary solutions in* $X_p^+$, $u^- \equiv 0$, *which is linearly unstable, and* $u^+(\cdot) \in X_p^+ \setminus \{0\}$, *which is globally asymptotically with respect to perturbations in* $X_p^+ \setminus \{0\}$.

*(2) If* $\bar{a} > 0$ *and (H1)-(H2) are satisfied, where* $\bar{a} := \frac{1}{p_1 p_2 \cdots p_N} \int_D f(x,0)dx$ *with* $D = [0,p_1] \times [0,p_2] \times \cdots \times [0,p_N]$. *then (H3) is satisfied and the conclusions in (1) hold.*

Let

$$u^+_{\inf} = \inf_{x \in \mathbb{R}^N} u^+(x). \tag{2.23}$$

The following four theorems are about the spatial spreading speeds of (1.2).

**Theorem D.** (Existence and symmetry of spreading speeds) *Assume (H1) - (H3).*

*(1) The spreading speed* $c^*(\xi)$ *of* (1.2) *in the direction of* $\xi \in S^{N-1}$ *exists for every* $\xi \in S^{N-1}$ *and*

$$c^*(\xi) = \inf_{\mu > 0} \frac{\lambda_0(\xi,\mu)}{\mu},$$

*where* $\lambda_0(\xi,\mu)$ *is the principal spectrum point of* (2.8) *with* $a(x) = f(x,0)$.

*(2) Assume that* $k(z) = k(-z)$ *for* $z \in \mathbb{R}^N$. $c^*(\xi) = c^*(-\xi)$ *for every* $\xi \in S^{N-1}$.

*(3) For every* $u_0 \in X^+(\xi)$ *and* $c < c^*(\xi)$,

$$\liminf_{t\to\infty} \inf_{x\cdot\xi \le ct} (u(t,x;u_0,z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*(4) For every $u_0 \in X^+(\xi)$ and $c > c^*(\xi)$,*

$$\limsup_{t\to\infty} \sup_{x\cdot\xi \geq ct} u(t,x;u_0,z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

**Theorem E.** (Spreading features of spreading speeds) *Assume (H1) - (H3).*

*(1) If $u_0 \in X^+$ satisfies that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $|x \cdot \xi| \gg 1$, then for each $c > max\{c^*(\xi), c^*(-\xi)\}$,*

$$\limsup_{t\to\infty} \sup_{|x\cdot\xi| \geq ct} u(t,x;u_0,z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*(2) Assume that $\xi \in S^{N-1}$ and $0 < c < min\{c^*(\xi), c^*(-\xi)\}$. Then for any $\sigma > 0$, and $r > 0$,*

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi| \leq ct} (u(t,x;u_0,z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N$$

*for every $u_0 \in X^+$ satisfying $u_0(x) \geq \sigma$ for all $x \in \mathbb{R}^N$ with $|x \cdot \xi| \leq r$.*

**Theorem F.** (Spreading features of spreading speeds) *Assume (H1) - (H3).*

*(1) If $u_0 \in X^+$ satisfies that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $\|x\| \gg 1$, then*

$$\limsup_{t\to\infty} \sup_{\|x\| \geq ct} u(t,x;u_0,z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*for all $c > \sup_{\xi \in S^{N-1}} c^*(\xi)$.*

*(2) Assume that $0 < c < \inf_{\xi \in S^{N-1}} \{c^*(\xi)\}$. Then for any $\sigma > 0$, there is $r > 0$ such that*

$$\liminf_{t\to\infty} \inf_{\|x\| \leq ct} (u(t,x;u_0,z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N$$

*for every $u_0 \in X^+$ satisfying $u_0(x) \geq \sigma$ for $x \in \mathbb{R}^N$ with $\|x\| \leq r$.*

To indicate the dependence of the spreading speeds on the growth rate function $f$, denote the spreading speed $c^*(\xi, f)$. If no confusion exists, we still use $c^*(\xi)$.

Let $\bar{f}(u) = \frac{1}{|D|}\int_D f(x,u)dx$, with $D = \prod_{i=1}^{N}[0, p_i]$ and $|D| = \prod_{i=1}^{N} p_i$ where the period vector $\mathbf{p} = (p_1, ..., p_N)$.

**Theorem G.** (Effect of spatial variation) Assuming that $\bar{f}(0) > 0$, $c^*(\xi, f) \geq c^*(\xi, \bar{f})$ *for any* $\xi \in S^{N-1}$. *Moreover, assuming also (H4),* $c^*(\xi, f) = c^*(\xi, \bar{f})$ *for some* $\xi \in S^{N-1}$ *iff* $f(x, 0) \equiv \bar{f}(0)$ *is independent of* $x$.

Theorems D-G extend the spreading speed theory for (1.3) to (1.2) and establish some fundamental theories for the further study of the spreading and propagating dynamics of (1.2). The next natural and important problems to address include the existence, uniqueness, and stability of traveling wave solutions of (1.2) in the direction of $\xi$ connecting $u^+$ and $u^-$ with speed $c \geq c^*(\xi)$. To explore this, we assume the existence of the principal eigenvalue of (2.8).

We now state the main results of the dissertation on traveling wave solutions. For given $\xi \in S^{N-1}$ and $c > c^*(\xi)$, let $\mu \in (0, \mu^*(\xi))$ be such that

$$c = \frac{\lambda_0(\xi, \mu, a_0)}{\mu}.$$

and $\mu^*(\xi)$ is such that

$$c^*(\xi) = \frac{\lambda_0(\xi, \mu^*(\xi), a_0)}{\mu^*(\xi)}.$$

If (H4) holds, let $\phi(\cdot) \in X_p^+$ be the positive principal eigenfunction of $\mathcal{K}_{\xi,\mu} - I + a_0(\cdot)I$ with $\|\phi(\cdot)\|_{X_p} = 1$.

**Theorem H.** (Existence of traveling wave solutions) *Assume (H1)-(H4). Then for any* $\xi \in S^{N-1}$ *and* $c > c^*(\xi)$, *there is a bounded measurable function* $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$ *such that the following hold.*

*(1)* $\Phi(\cdot, \cdot)$ *generates a traveling wave solution connecting* $u^+(\cdot)$ *and* $0$ *and propagating in the direction of* $\xi$ *with speed* $c$. *Moreover,* $\lim_{x\cdot\xi\to\infty} \frac{\Phi(x,z)}{e^{-\mu x\cdot\xi}\phi(x+z)} = 1$ *uniformly in* $z \in \mathbb{R}^N$.

*(2) Let $U(t,x;z) = u(t,x;\Phi(\cdot,z),z)(= \Phi(x-ct\xi, z+ct\xi))$. Then*

$$U_t(t,x;z) > 0 \quad \forall t \in \mathbb{R},\ x,z \in \mathbb{R}^N,$$

$$\lim_{x\cdot\xi - ct \to -\infty} U_t(t,x;z) = 0, \text{ and } \lim_{x\cdot\xi-ct\to\infty} \frac{U_t(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)} = \mu c \text{ uniformly in } z \in \mathbb{R}^N.$$

**Remark 2.2.** *Let $\Phi(x,z)$ be as in Theorem H and $\Psi(\eta,z) = \Phi(\eta\xi, z-\eta\xi)$. Then $U(t,x;z) = \Psi(x\cdot\xi - ct, z+x)$ and $\Psi(\eta,z)$ is differentiable in $\eta$ and $\Psi_\eta(\eta,z) < 0$.*

**Theorem I.** (Uniqueness and continuity of traveling wave solutions) *Assume the same conditions as in Theorem H. Let $\Phi(\cdot,\cdot)$ be as in Theorem H.*

*(1) Suppose that $\Phi_1(\cdot,\cdot)$ also generates a traveling wave solution of (1.2) in the direction of $\xi$ with speed $c$ and $\lim_{x\cdot\xi\to\infty} \frac{\Phi_1(x,z)}{\Phi(x,z)} = 1$ uniformly in $z \in \mathbb{R}$. Then $\Phi_1(x,z) \equiv \Phi(x,z)$.*

*(2) $\Phi(x,z)$ is continuous in $(x,z) \in \mathbb{R}^N$.*

**Theorem J.** (Stability of traveling wave solutions) *Assume the same conditions as in Theorem H.*

*Let $U(t,x) = U(t,x;0) = \Phi(x-ct\xi, ct\xi)$, where $\Phi(\cdot,\cdot)$ is as in Theorem H. For any $u_0 \in X^+$ satisfying that $\lim_{x\cdot\xi\to\infty} \frac{u_0(x)}{U(0,x)} = 1$ and $\liminf_{x\cdot\xi\to-\infty} u_0(x) > 0$, there holds*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| \frac{u(t,x;u_0,0)}{U(t,x)} - 1 \right| = 0.$$

We remark that by the spreading property of $c^*(\xi)$, it is not difficult to see that (1.2) has no traveling wave solutions in the direction of $\xi \in S^{N-1}$ with propagating speed smaller than $c^*(\xi)$. Theorems H-J show the existence, uniqueness, and stability of traveling wave solutions of (1.2) in any given direction with speed greater than the spreading speed in that direction for the following three special but important cases, that is, the nonlocal dispersal is nearly local; the periodic habitat is nearly globally homogeneous or it is nearly homogeneous in a region where it is most conducive to population growth in the zero-limit population. It should be pointed out that in the last case, for $N = 1, 2$, (H4) is automatically satisfied.

It remains open whether a general spatially periodic monostable equation with nonlocal dispersal in $\mathbb{R}^N$ with $N \geq 3$ has traveling wave solutions connecting the spatially periodic positive stationary solution $u^+$ and 0 and propagating with constant speeds.

## Chapter 3

## Comparison Principle and Sub- and Super-solutions

In this chapter, we establish some basic properties of solutions of equation (1.2) and some related nonlocal linear evolution equations, including the comparison principle and monotonicity of solutions with respect to initial conditions, convergence of solutions on compact sets.

### 3.1 Solutions of Evolution Equation and Semigroup Theory

For given $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $a(\cdot) \in X_p$, consider also

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)u(t,y)dy - u(t,x) + a(x)u(t,x), \quad x \in \mathbb{R}^N. \tag{3.1}$$

Let $X$ and $X_p$ be as in (2.2) and (2.4), respectively. For given $\rho \geq 0$, let

$$X(\rho) = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, \text{the function } x \mapsto e^{-\rho\|x\|}u(x) \text{ belongs to } X\} \tag{3.2}$$

equipped with the norm $\|u\|_{X(\rho)} = \sup_{x\in\mathbb{R}^N} e^{-\rho\|x\|}|u(x)|$. Note that $X(0) = X$.

It follows from the general linear semigroup theory (see [27] or [50]) that for every $u_0 \in X(\rho)$ ($\rho \geq 0$), (3.1) has a unique solution $u(t,\cdot;u_0,\xi,\mu) \in X(\rho)$ with $u(0,x;u_0,\xi,\mu) = u_0(x)$. Put

$$\Phi(t;\xi,\mu)u_0 = u(t,\cdot;u_0,\xi,\mu). \tag{3.3}$$

Note that for every $\mu \in \mathbb{R}$ and $\rho \geq 0$, there is $\omega(\mu,\rho) > 0$ such that

$$\|\Phi(t;\xi,\mu)u_0\|_{X(\rho)} \leq e^{\omega(\mu,\rho)t}\|u_0\|_{X(\rho)} \quad \forall\, t \geq 0,\ \xi \in S^{N-1},\ u_0 \in X(\rho). \tag{3.4}$$

Note also that if $u_0 \in X_p$, then $\Phi(t;\xi,\mu)u_0 \in X_p$ for $t \geq 0$.

By general nonlinear semigroup theory (see [27] or [50]), (1.2) has a unique (local) solution $u(t,x;u_0)$ with $u(0,x;u_0) = u_0(x)$ for every $u_0 \in X$. Also if $u_0 \in X_p$, then $u(t,x;u_0) \in X_p$ for $t$ in the existence interval of the solution $u(t,x;u_0)$.

Due to the spatial inhomogeneity of (1.2), it is sometime important to consider the space shifted equation (2.15) of (1.2) and the following space shifted equation of (3.1),

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)u(t,y)dy - u(t,x) + a(x+z)u(t,x), \tag{3.5}$$

where $z \in \mathbb{R}^N$. Note that if $\mu = 0$ and $a(x) = a_0(x)(:= f(x,0))$, then (3.5) reduces to the space shifted equation of the linearization equation (2.10) of (1.2) at $u = 0$,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + a_0(x+z)u(t,x), \quad x \in \mathbb{R}^N. \tag{3.6}$$

It is again a consequence of the general semigroup theory that (2.15) has a unique (local) solution $u(t,x;u_0,z)$ with $u(0,x;u_0,z) = u_0(x)$ $(z \in \mathbb{R}^N)$ for every $u_0 \in X$. Also given $u_0 \in X(\rho)$ $(\rho \geq 0)$, (3.5) has a unique solution $u(t,x;u_0,\xi,\mu,z)$ with $u(0,x;u_0,\xi,\mu,z) = u_0(x)$. We set

$$\Phi(t;\xi,\mu,z)u_0 = u(t,\cdot;u_0,\xi,\mu,z). \tag{3.7}$$

Sometimes we need study the solutions on the space $\tilde{X}$, where $\tilde{X}$ is as (2.13). For example, to get a continuous solution of (2.15), we may first investigate the existence of solution with $u_0 \in \tilde{X}$ and then prove the continuity. It is again a consequence of the general semigroup theory that (2.15) has a unique (local) solution $u(t,x;u_0,z)$ with $u(0,x;u_0,z) = u_0(x)$ $(z \in \mathbb{R}^N)$ for every $u_0 \in \tilde{X}$.

Throughout this chapter, we assume that $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$ are fixed, unless otherwise specified.

### 3.2 Sub- and Super-solutions

Let $X_p^+$ and $X^+$ be as in (2.5) and (2.3), respectively. Let

$$\mathrm{Int}(X_p^+) = \{v \in X_p | v(x) > 0, x \in \mathbb{R}^N\}. \tag{3.8}$$

For $v_1, v_2 \in X_p$, we define

$$v_1 \leq v_2 \quad (v_1 \geq v_2) \quad \text{if} \quad v_2 - v_1 \in X_p^+ \quad (v_1 - v_2 \in X_p^+),$$

and

$$v_1 \ll v_2 \quad (v_1 \gg v_2) \quad \text{if} \quad v_2 - v_1 \in \mathrm{Int}(X_p^+) \quad (v_1 - v_2 \in \mathrm{Int}(X_p^+)).$$

For $u_1, u_2 \in X$, we define

$$u_1 \leq u_2 \quad (u_1 \geq u_2) \quad \text{if} \quad u_2 - u_1 \in X^+ \quad (u_1 - u_2 \in X^+).$$

**Definition 3.1.** *A bounded Lebesgue measurable function $u(t,x)$ on $[0,T) \times \mathbb{R}^N$ is called a super-solution (or sub-solution) of* (2.15) *if for any $x \in \mathbb{R}^N$, u(t,x) is absolutely continuous on $[0,T)$(and so $\frac{\partial u}{\partial t}$ exists a.e on [0,T)) and satisfies that for each $x \in \mathbb{R}^N$,*

$$\frac{\partial u}{\partial t} \geq (or \leq) \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + f(x+z,u)u(t,x)$$

*for a.e. $t \in (0,T)$.*

Sub and super-solutions of (3.5) are defined similarly.

### 3.3 Comparison Principle and Monotonicity

**Proposition 3.1** (Comparison principle)**.**

*(1) If $u_1(t,x)$ and $u_2(t,x)$ are sub-solution and super-solution of (3.1) on $[0,T)$, respectively, $u_1(0,\cdot) \le u_2(0,\cdot)$, and $u_2(t,x) - u_1(t,x) \ge -\beta_0$ for $(t,x) \in [0,T) \times \mathbb{R}^N$ and some $\beta_0 > 0$, then*

$$u_1(t,\cdot) \le u_2(t,\cdot) \quad \text{for} \quad t \in [0,T).$$

*(2) If $u_1(t,x)$ and $u_2(t,x)$ are bounded sub- and super-solutions of (1.2) on $[0,T)$, respectively, and $u_1(0,\cdot) \le u_2(0,\cdot)$, then $u_1(t,\cdot) \le u_2(t,\cdot)$ for $t \in [0,T)$.*

*(3) For every $u_0 \in X^+$, $u(t,x;u_0)$ exists for all $t \ge 0$.*

*Proof.* (1) We prove the proposition by modifying the arguments of [35, Proposition 2.4].

First let $v(t,x) = e^{ct}\big(u_2(t,x) - u_1(t,x)\big)$. Then $v(t,x)$ satisfies

$$\frac{\partial v}{\partial t} \ge \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x) v(t,y)dy + p(x)v(t,x), \quad x \in \mathbb{R}^N \tag{3.9}$$

for $t \in (0,T)$, where $p(x) = a(x) - 1 + c$. Take $c > 0$ such that $p(x) > 0$ for all $x \in \mathbb{R}^N$. We claim that $v(t,x) \ge 0$ for $t \in [0,T)$ and $x \in \mathbb{R}^N$.

Let $p_0 = \sup_{x\in\mathbb{R}^N} p(x)$. It suffices to prove the claim for $t \in (0,T_0)$ and $x \in \mathbb{R}^N$, where $T_0 = \min\{T, \frac{1}{p_0+M}\}$ with $M = \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi} k(z)dz$. Assume that there are $\tilde{t} \in (0,T_0)$ and $\tilde{x} \in \mathbb{R}^N$ such that $v(\tilde{t},\tilde{x}) < 0$. Then there is $t^0 \in (0,T_0)$ such that

$$v_{\inf} := \inf_{(t,x)\in[0,t^0]\times\mathbb{R}^N} v(t,x) < 0.$$

Observe that there are $t_n \in (0,t^0]$ and $x^n \in \mathbb{R}^N$ such that

$$v(t_n, x^n) \to v_{\inf} \quad \text{as} \quad n \to \infty.$$

By (3.9), we have

$$\begin{aligned} v(t_n, x_n) - v(0, x_n) &\geq \int_0^{t_n} \big[ \int_{\mathbb{R}^N} e^{-\mu(y-x_n)\cdot\xi} k(y-x_n) v(t,y) dy + p(x_n) v(t, x_n) \big] dt \\ &\geq \int_0^{t_n} \big[ \int_{\mathbb{R}^N} e^{-\mu(y-x_n)\cdot\xi} k(y-x_n) v_{\inf} dy + p_0 v_{\inf} \big] dt \\ &= t_n (M + p_0) v_{\inf} \\ &\geq t^0 (M + p_0) v_{\inf} \end{aligned}$$

for $n = 1, 2, \cdots$. Note that $v(0, x_n) \geq 0$ for $n = 1, 2, \cdots$. We then have

$$v(t_n, x_n) \geq t^0 (M + p_0) v_{\inf}$$

for $n = 1, 2, \cdots$. Letting $n \to \infty$, we get

$$v_{\inf} \geq t^0 (M + p_0) v_{\inf} > v_{\inf} \quad (\text{since} \quad -\beta_0 \leq v_{\inf} < 0).$$

This is a contradiction. Hence $v(t, x) \geq 0$ for $(t, x) \in [0, T) \times \mathbb{R}^N$ and then $u_1(t, x) \leq u_2(t, x)$ for $(t, x) \in [0, T) \times \mathbb{R}^N$.

(2) Let $v(t, x) = e^{ct}(u_2(t, x) - u_1(t, x))$. Then $v(t, \cdot) \geq 0$ and $v(t, x)$ satisfies

$$\frac{\partial v}{\partial t} \geq \int_{\mathbb{R}^N} k(y-x) v(t,y) dy + p(t,x) v(t,x), \quad x \in \mathbb{R}^N$$

for $t \in (0, T)$, where

$$p(t,x) = c - 1 + f(x, u_2(t,x)) + \big[ u_1(t,x) \cdot \int_0^1 f_u(x, s u_1(t,x) + (1-s) u_2(t,x)) ds \big] v(t,x)$$

for $t \in [0, T)$, $x \in \mathbb{R}^N$. By the boundedness of $u_1(t, x)$ and $u_2(t, x)$, there is $c > 0$ such that

$$\inf_{t \in [0,T), x \in \mathbb{R}^N} p(t,x) > 0.$$

(2) then follows from the arguments in (1) with $p(x)$ and $p_0$ being replaced by $p(t,x)$ and $\sup_{t\in[0,T),x\in\mathbb{R}^N} p(t,x)$, respectively.

(3) By (H1), there is $M>0$ such that

$$u_0(x)\le M \quad \text{and} \quad f(x,M)<0 \quad \text{for} \quad x\in\mathbb{R}^N.$$

Let $u_M(t,x)\equiv M$ for $x\in\mathbb{R}^N$ and $t\in\mathbb{R}$. Then $u_M$ is a super-solution of (1.2) on $[0,\infty)$. Let $I(u_0)\subset\mathbb{R}$ be the maximal interval of existence of the solution $u(t,\cdot;u_0)$ of (1.2). Then by (2), one obtains

$$0\le u(t,x;u_0)\le M \quad \text{for} \quad x\in\mathbb{R}^N, \quad t\in I(u_0)\cap[0,\infty).$$

It then follows easily that $[0,\infty)\subset I(u_0)$ and $u(t,x;u_0)$ exists for all $t\ge 0$. □

The following remark follows by the arguments similar to those in Proposition 3.1 (1).

**Remark 3.1.** *Suppose that $u_1,u_2\in X(\rho)$ and $u_1\le u_2$. Then $\Phi(t;\xi,0,0)u_1\le\Phi(t;\xi,0,0)u_2$ for all $t>0$.*

**Proposition 3.2** (Strong monotonicity)**.** *Suppose that $u_1,u_2\in X_p$ and $u_1\le u_2$, $u_1\ne u_2$.*

*(1) $\Phi(t;\xi,\mu)u_1\ll\Phi(t;\xi,\mu)u_2$ for all $t>0$.*

*(2) $u(t,\cdot;u_1)\ll u(t,\cdot;u_2)$ for every $t>0$ at which both $u(t,\cdot;u_1)$ and $u(t,\cdot;u_2)$ exist.*

*Proof.* (1) We apply the arguments in Theorem 2.1 of [37]. First, assume that $u_0\in X_p^+\setminus\{0\}$. Then by Proposition 3.1 (1), $\Phi(t;\xi,\mu)u_0\ge 0$ for $t>0$.

We claim that $e^{\mathcal{K}_{\xi,\mu}t}u_0\gg 0$ for $t>0$. In fact, note that

$$e^{\mathcal{K}_{\xi,\mu}t}u_0=u_0+t\mathcal{K}_{\xi,\mu}u_0+\frac{t^2(\mathcal{K}_{\xi,\mu})^2u_0}{2!}+...+\frac{t^n(\mathcal{K}_{\xi,\mu})^nu_0}{n!}+...$$

Let $x_0 \in \mathbb{R}^N$ be such that $u_0(x_0) > 0$. Then there is $r > 0$ such that $u_0(x_0) > 0$ for $x \in B(x_0, r) := \{y \in \mathbb{R}^N \mid \|y - x_0\| < r\}$. This implies that

$$(\mathcal{K}_{\xi,\mu} u_0)(x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x) u_0(y) dy > 0 \quad \text{for} \quad x \in B(x_0, r+\delta).$$

By induction,

$$(\mathcal{K}_{\xi,\mu} u_0)^n(x) > 0 \quad \text{for} \quad x \in B(x_0, r + n\delta), \quad n = 1, 2, \cdots.$$

This together with the periodicity of $u_0(x)$ implies that $e^{\mathcal{K}_{\xi,\mu} t} u_0 \gg 0$ for $t > 0$.

Let $m > 1 - \min\limits_{x \in \mathbb{R}^N} a(x)$. Note that

$$\Phi(t;\xi,\mu)u_0 = u(t,\cdot;u_0) = e^{(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I + mI - mI)t} u_0 = e^{-mIt} e^{(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I + mI)t} u_0$$

and $(e^{-mIt} v)(x) = e^{-mt} v(x)$ for every $x \in \mathbb{R}^N$. Note also that

$$e^{(\mathcal{K}_{\xi,\mu} - I + a(\cdot)I + mI)t} u_0 = e^{\mathcal{K}_{\xi,\mu} t} u_0 + \int_0^t e^{\mathcal{K}_{\xi,\mu}(t-s)} (-I + a(\cdot)I + mI) u(s,\cdot;u_0) ds \quad \text{for} \quad t > 0.$$

It then follows that $\Phi(t;\xi,\mu)u_0 \gg 0$ for all $t > 0$.

Now let $u_0 = u_2 - u_1$. Then $u_0 \in X_p^+ \setminus \{0\}$. Hence $\Phi(t;\xi,\mu)u_0 \gg 0$ for $t > 0$ and then $\Phi(t;\xi,\mu)u_1 \ll \Phi(t;\xi,\mu)u_2$ for $t > 0$.

(2) Let $v(t,x) = u(t,x;u_2) - u(t,x;u_1)$ for $t \geq 0$ at which both $u(t,x;u_1)$ and $u(t,x;u_2)$ exist. Then $v(0,\cdot) = u_2 - u_1 \geq 0$ and $v(t,x)$ satisfies

$$\begin{aligned}\frac{\partial v}{\partial t} &= \int_{\mathbb{R}^N} k(y-x) v(t,y) dy - v(t,x) + f(x, u(t,x;u_2)) v(t,x) \\ &\quad + \big[u(t,x;u_1) \cdot \int_0^1 f_u(x, su(t,x;u_1) + (1-s)u(t,x;u_2)) ds\big] v(t,x), \quad x \in \mathbb{R}^N.\end{aligned}$$

(2) then follows from the arguments similar to those in (1). □

### 3.4 Convergence on Compact Sets

In this section, we investigate the convergence of solutions of (1.2) or (3.1) on compact sets.

First, we prove the following lemma.

**Lemma 3.1.** *For given $\rho_0 \geq 0$ and $\{u_n\} \in X(\rho_0)$ with $\|u_n\|_{X(\rho_0)} \leq M$ for some $M > 0$ and $n = 1, 2, \cdots$, $u_n(x) \to 0$ as $n \to \infty$ uniformly for $x$ in bounded subsets of $\mathbb{R}^N$ if and only if $u_n(x) \to 0$ in $X(\rho)$ as $n \to \infty$ for every $\rho > \rho_0$.*

*Proof.* Suppose that $u_n(x) \to 0$ as $n \to \infty$ uniformly for $x$ in bounded subsets of $\mathbb{R}^N$, that is, for any $\epsilon > 0$ and $L > 0$, there exists $N_0 \in \mathbb{N}$ such that $|u_n(x)| < \epsilon$ for all $n > N_0$ and $\|x\| \leq L$. For given $\rho > \rho_0$, pick $\hat{\rho} \in (\rho_0, \rho)$. Note that $\|u_n\|_{X(\rho_0)} \leq M$ for some $M > 0$ and $n = 1, 2, \cdots$. Then for any $\epsilon > 0$, there exists an $L > 0$ such that $|e^{-\hat{\rho}\|x\|} u_n(x)| < \epsilon$ for $\|x\| > L$ and $n = 1, 2, \cdots$. It then follows that for any $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that $|e^{-\hat{\rho}\|x\|} u_n(x)| < \epsilon$ for all $n > N_0$ and $x \in \mathbb{R}^N$. This implies that $u_n \to 0$ in $X(\rho)$ as $n \to \infty$ for every $\rho > \rho_0$.

Suppose that $u_n(x) \to 0$ in $X(\rho_0)$ as $n \to \infty$ for some $\rho_0 > 0$, that is, $\|u_n\|_{X(\rho_0)} \to 0$ as $n \to \infty$. For any $\epsilon > 0$, $L > 0$, let $\epsilon_0 = \epsilon e^{-\rho_0 L}$, then there exists a $N_0 \in \mathbb{N}$ such that $|e^{-\rho_0\|x\|} u_n(x)| < \epsilon_0$ for $x \in \mathbb{R}^N$ and $n \geq N_0$, which implies that $|u_n(x)| < e^{\rho_0(\|x\| - L)}\epsilon \leq \epsilon$, for all $n \geq N_0$ and $\|x\| \leq L$, as required. □

**Proposition 3.3** (Convergence on compact sets)**.**

*(1) If $u_n \in X$ and $u_0 \in X$ are such that $\|u_n\|_X \leq M$ for some $M > 0$ and $n = 1, 2, \cdots$, and $u_n(x) \to u_0(x)$ as $n \to \infty$ uniformly for $x$ in bounded subsets of $\mathbb{R}^N$, then $\big(\Phi(t; \xi, \mu) u_n\big)(x) \to \big(\Phi(t; \xi, \mu) u_0\big)(x)$ as $n \to \infty$ uniformly for $(t, x)$ in bounded subsets of $[0, \infty) \times \mathbb{R}^N$.*

*(2) If $u_n \in X^+$ and $u_0 \in X^+$ are such that $\|u_n\|_X \leq M$ for some $M > 0$ and $n = 1, 2, \cdots$ and $u_n(x) \to u_0(x)$ as $n \to \infty$ uniformly for $x$ in bounded subsets of $\mathbb{R}^N$,*

*then* $u(t,x;u_n)\to u(t,x;u_0)$ *as* $n\to\infty$ *uniformly for* $(t,x)$ *in bounded subsets of* $[0,\infty)\times\mathbb{R}^N$.

*Proof.* (1) First of all, by Lemma 3.1, for every given $\rho>0$, $\|u_n-u_0\|_{X(\rho)}\to 0$ as $n\to\infty$. By (3.4),

$$\|\Phi(t;\xi,\mu)u_n-\Phi(t;\xi,\mu)u_0\|_{X(\rho)}\le e^{\omega(\mu,\rho)t}\|u_n-u_0\|_{X(\rho)}$$

for $t\ge 0$ and $n=1,2,\cdots$. Then, $\|\Phi(t;\xi,\mu)u_n-\Phi(t;\xi,\mu)u_0\|_{X(\rho)}\to 0$ as $n\to\infty$ uniformly for $t$ in bounded subsets of $[0,\infty)$. This implies that

$$\big(\Phi(t;\xi,\mu)u_n\big)(x)\to\big(\Phi(t;\xi,\mu)u_0\big)(x)$$

as $n\to\infty$ uniformly for $(t,x)$ in bounded subsets of $[0,\infty)\times\mathbb{R}^N$.

(2) By Proposition 3.1 (3), $u(t,x;u_0)$ and $u(t,x;u_n)$ exist on $[0,\infty)$ for any $n\ge 1$ and there is $M^*>0$ such that $\|u(t,\cdot;u_0)\|_X\le M^*$ and $\|u(t,\cdot;u_n)\|_X\le M^*$ for $t\ge 0$ and $n=1,2,\cdots$. Let $v_n(t,x)=u(t,x;u_n)-u(t,x;u_0)$ for $x\in\mathbb{R}^N$, $t\ge 0$, and $n=1,2,\cdots$. Then

$$\frac{\partial v_n}{\partial t}=\int_{\mathbb{R}^N}k(y-x)v_n(t,y)dy-v_n(t,x)+a_n(t,x)v_n(t,x),\quad x\in\mathbb{R}^N,$$

where

$$a_n(t,x)=f(x,u(t,x;u_n))+\big[u(t,x;u_0)\cdot\int_0^1 f_u(x,su(t,x;u_0)+(1-s)u(t,x;u_n))ds\big]v_n(t,x).$$

Hence

$$v_n(t,\cdot)=e^{(\mathcal{K}-I)t}v_n(0,\cdot)+\int_0^t e^{(\mathcal{K}_{\xi,\mu}-I)(t-s)}a_n(s,\cdot)v_n(s,\cdot)ds.$$

Note that for every $\rho>0$ there are $\omega_0(\rho)\in\mathbb{R}$ and $L_0>0$ such that

$$\|e^{(\mathcal{K}-I)t}v\|_{X(\rho)}\le e^{\omega_0(\rho)t}\|v\|_{X(\rho)}\quad\forall v\in X(\rho)$$

and

$$|a_n(t,x)| \le L_0 \quad \forall t \ge 0,\ x \in \mathbb{R}^N.$$

It then follows from Gronwall's inequality that

$$\|v_n(t,\cdot)\|_{X(\rho)} \le e^{(\omega_0(\rho)+L_0)t}\|v_n(0,\cdot)\|_{X(\rho)}.$$

By the arguments in (1), we have $v_n(t,x) \to 0$ and hence $u(t,x;u_n) \to u(t,x;u_0)$ as $n \to \infty$ uniformly for $(t,x)$ in bounded subsets of $[0,\infty) \times \mathbb{R}^N$. $\square$

Chapter 4

Spectral Theory of Dispersal Operators

In this chapter, we investigate the eigenvalue problem (2.8) and prove Theorems A and B stated in the chapter 2 and some other related results which are used in the proof of the existence of spreading speeds of (1.2) in later chapters. The results of this chapter in the case that the nonlocal kernel function has compact support have already been published (see [56], [57]).

Throughout this chapter, $X_p$ is as in (2.4), $a : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function, $a \in X_p$, and

$$a_{\max} = \max_{x \in \mathbb{R}^N} a(x), \quad a_{\min} = \min_{x \in \mathbb{R}^N} a(x).$$

$a(\cdot)I : X_p \to X_p$ has the same meaning as in (2.7) with $a_0(\cdot)$ being replaced by $a(\cdot)$ and $\mathcal{K}_{\xi,\mu} : X_p \to X_p$ is understood as in (2.9), $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$. We first introduce in 4.1 some important operators related to $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ or (2.8) and explore some basic properties of the eigenvalue problems associated with these operators. We then prove Theorems A in 4.2 and derive in 4.3 from Theorems A some results on the spectral radius of some operator related to $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$.

## 4.1 Evolution Operators and Eigenvalue Problems

In this section, we introduce some evolution operators related to the operator $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$, explore the basic properties of the eigenvalue problems associated to these operators, and discuss the relations between the eigenvalues of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ and its related operators. If no confusion occurs, we may write the principal eigenvalue $\lambda(\xi, \mu, a)$ of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ (if exists) as $\lambda(\xi, \mu)$.

First of all, we introduce a compact operator associated to $\mathcal{K}_{\xi,\mu}$ based on the perturbation idea in [7]. This operator plays an important role in the proofs of Theorems A in the next section. For given $\alpha > -1 + a_{\max}$, let $U_{\xi,\mu,\alpha} : X_p \to X_p$ be defined as follows

$$(U_{\xi,\mu,\alpha}u)(x) = \int_{\mathbb{R}^N} \frac{e^{-\mu(y-x)\cdot\xi}k(y-x)u(y)}{\alpha + 1 - a(y)} dy. \tag{4.1}$$

Observe that $U_{\xi,\mu,\alpha}$ is a compact and positive operator on $X_p$. Let $r(U_{\xi,\mu,\alpha})$ be the spectral radius of $U_{\xi,\mu,\alpha}$.

**Proposition 4.1.** *(1) $\alpha > -1 + a_{\max}$ is an eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ or* (2.8) *iff 1 is an eigenvalue of the eigenvalue problem*

$$U_{\xi,\mu,\alpha}v = \lambda v.$$

*(2) For $\alpha > -1 + a_{\max}$, 1 is an eigenvalue of $U_{\xi,\mu,\alpha}$ with a positive eigenfunction iff $r(U_{\xi,\mu,\alpha}) = 1$.*

*(3) If there is $\alpha > -1 + a_{\max}$ with $r(U_{\xi,\mu,\alpha}) > 1$, then there is $\alpha_0 > \alpha$ such that $r(U_{\xi,\mu,\alpha_0}) = 1$.*

*(4) If $\alpha > -1 + a_{\max}$ is an eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ or* (2.8) *with a positive eigenfunction, then it is the principal eigenvalue of* (2.8)*.*

*Proof.* (1) and (2) follow from Proposition 2.1 of [7].

(3) and (4) follow from Theorem 2.2 of [7]. □

By Proposition 4.1, the spectral radius of $U_{\xi,\mu,\alpha}$ provides a useful tool for the investigation of those eigenvalues of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ which are greater than $-1 + a_{\max}$. The following proposition shows that if $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ possesses a principal eigenvalue, then it must be greater than $-1 + a_{\max}$.

**Proposition 4.2.** *If $\lambda(\xi,\mu)$ is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$, then $\lambda(\xi,\mu) > -1 + a_{\max}$.*

*Proof.* Since $\lambda(\xi,\mu)$ is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$, there is an eigenfunction $\psi \in X_p^+ \setminus \{0\}$ such that

$$\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)\psi(y)dy - \psi(x) + a(x)\psi(x) = \lambda(\xi,\mu)\psi(x), \quad x \in \mathbb{R}^N. \tag{4.2}$$

Note that $u(t,x) = e^{\lambda(\xi,\mu)t}\psi(x)$ is a solution of (3.1). By Proposition 3.2, $\psi \in \text{Int}(X_p^+)$.

Let $x_0 \in \mathbb{R}^N$ be such that $a(x_0) = a_{\max}$. By $\psi \in \text{Int}(X_p^+)$,

$$\int_{\mathbb{R}^N} e^{-\mu(y-x_0)\cdot\xi} k(y-x_0)\psi(y)dy > 0.$$

This together with (4.2) implies that

$$\lambda(\xi,\mu)\psi(x_0) > -\psi(x_0) + a(x_0)\psi(x_0).$$

Hence $\lambda(\xi,\mu) > -1 + a_{\max}$. □

Next, consider the evolution equation (3.1) associated with the operator $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$. Let $\Phi(t;\xi,\mu)$ be the solution operator of (3.1) given in (3.3) and $\Phi^p(t;\xi,\mu) : X_p \to X_p$ be defined by

$$\Phi^p(t;\xi,\mu) = \Phi(t;\xi,\mu)|_{X_p} \tag{4.3}$$

for $t \geq 0$, $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$. Let $r(\Phi^p(1;\xi,\mu))$ and $\sigma(\Phi^p(1;\xi,\mu))$ be the spectral radius and the spectrum of $\Phi^p(1;\xi,\mu)$, respectively. The following lemma states the relationship between the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ and the spectral radius of $\Phi^p(1;\xi,\mu)$ and follows easily (see [27, Theorems 1.5.2 and 1.5.3]).

**Lemma 4.1.** *The principal eigenvalue $\lambda(\xi,\mu)$ of (2.8) exists if and only if $r(\Phi^p(1;\xi,\mu))$ is an algebraically simple eigenvalue of $\Phi^p(1;\xi,\mu)$ with an eigenfunction in $X_p^+$ and for every*

$\tilde{\lambda} \in \sigma\big(\Phi^p(1;\xi,\mu)\big) \setminus \{r(\Phi^p(1;\xi,\mu))\}$, $|\tilde{\lambda}| < r(\Phi^p(1;\xi,\mu))$. *Moreover, if* $\lambda(\xi,\mu)$ *exists, then* $\lambda(\xi,\mu) = \ln r(\Phi^p(1;\xi,\mu))$ .

Therefore, the spectral radius of $\Phi^p(1;\xi,\mu)$ plays an important role in the investigation of the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ or (2.8). We next establish some further observations for $r(\Phi^p(1;\xi,\mu))$.

Note that $\Phi(t;\xi,0)$ is independent of $\xi \in S^{N-1}$. We put

$$\tilde{\Phi}(t) = \Phi(t;\xi,0) \tag{4.4}$$

for $\xi \in S^{N-1}$.

For given $u_0 \in X$ and $\mu \in \mathbb{R}$, letting $u_0^{\xi,\mu}(x) = e^{-\mu x\cdot\xi}u_0(x)$, then $u_0^{\xi,\mu} \in X(|\mu|)$. The following lemma follows directly from the uniqueness of solutions of (3.1).

**Lemma 4.2.** *For given* $u_0 \in X$, $\xi \in S^{N-1}$, *and* $\mu \in \mathbb{R}$, $\Phi(t;\xi,\mu)u_0 = e^{\mu x\cdot\xi}\tilde{\Phi}(t)u_0^{\xi,\mu}$.

Observe that for each $x \in \mathbb{R}^N$, there is a measure $m(x;y,dy)$ such that

$$(\tilde{\Phi}(1)u_0)(x) = \int_{\mathbb{R}^N} u_0(y)m(x;y,dy). \tag{4.5}$$

Moreover, by $(\tilde{\Phi}(1)u_0(\cdot - p_ie_i))(x) = (\tilde{\Phi}(1)u_0(\cdot))(x - p_ie_i)$ for $x \in \mathbb{R}^N$ and $i = 1,2,\cdots,N$,

$$\int_{\mathbb{R}^N} u_0(y)m(x-p_ie_i;y,dy) = \int_{\mathbb{R}^N} u_0(y-p_ie_i)m(x;y,dy) = \int_{\mathbb{R}^N} u_0(y)m(x;y+p_ie_i,dy)$$

and hence

$$m(x-p_ie_i;y,dy) = m(x;y+p_ie_i,dy) \tag{4.6}$$

for $i = 1,2,\cdots,N$.

By Lemma 4.2, we have

$$(\Phi(1;\xi,\mu)u_0)(x) = \int_{\mathbb{R}^N} e^{\mu(x-y)\cdot\xi}u_0(y)m(x;y,dy), \quad u_0 \in X.$$

**Proposition 4.3.** *For every $u \in \mathrm{Int}(X_p^+)$,*

$$\inf_{x\in\mathbb{R}^N} \frac{\int_{\mathbb{R}^N} e^{\mu(x-y)\cdot\xi}u(y)m(x;y,dy)}{u(x)} \le r(\Phi^p(1;\xi,\mu)) \le \sup_{x\in\mathbb{R}^N} \frac{\int_{\mathbb{R}^N} e^{\mu(x-y)\cdot\xi}u(y)m(x;y,dy)}{u(x)}.$$

*Proof.* By [20, Theorems 3.6 and 4.3], the spectral radius of the nonnegative operator $\Phi^p(1;\xi,\mu)$ is bounded by the lower and upper Collatz-Wielandt numbers of $u$ for every $u \in \mathrm{Int}(X_p^+)$, which are defined by $\sup\{\lambda \ge 0 : \lambda u \le \Phi^p(1;\xi,\mu)u\}$ and $\inf\{\lambda \ge 0 : \lambda u \ge \Phi^p(1;\xi,\mu)u\}$, respectively. The inequality then follows. □

In proving the existence of spreading speeds of (1.2) in chapter 6, properly truncated operators of $\Phi(1;\xi,\mu)$ are used. We therefore introduce them next.

Let $\zeta : \mathbb{R} \to [0,1]$ be a smooth function satisfying that

$$\zeta(s) = \begin{cases} 1 & \text{for} \quad |s| \le 1 \\ 0 & \text{for} \quad |s| \ge 2. \end{cases} \tag{4.7}$$

For a given $B > 0$, define $\Phi_B(1;\xi,\mu) : X \to X$ by

$$(\Phi_B(1;\xi,\mu)u_0)(x) = \int_{\mathbb{R}^N} e^{\mu(x-y)\cdot\xi}u_0(y)\zeta(\|y-x\|/B)m(x;y,dy). \tag{4.8}$$

Define $\Phi_B^p(1;\xi,\mu) : X_p \to X_p$ by

$$\Phi_B^p(1;\xi,\mu) = \Phi_B(1;\xi,\mu)|_{X_p}. \tag{4.9}$$

Similarly, let $r(\Phi_B^p(1;\xi,\mu))$ and $\sigma(\Phi_B^p(1;\xi,\mu))$ be the spectral radius and the spectrum of $\Phi_B^p(1;\xi,\mu)$, respectively.

**Lemma 4.3.**

$$\|\Phi_B^p(1;\xi,\mu) - \Phi^p(1;\xi,\mu)\|_{X_p} \to 0 \quad \text{as} \quad B \to \infty$$

*uniformly for $\mu$ in bounded sets and $\xi \in S^{N-1}$.*

*Proof.* It suffices to prove that

$$\int_{\|y-x\|\geq B} e^{\mu\|y-x\|} m(x; y, dy) \to 0 \quad \text{as} \quad B \to \infty$$

uniformly for $\mu$ in bounded sets and for $x \in \mathbb{R}^N$.

For given $\mu_0 > 0$ and $n \in \mathbb{N}$, let $u_n \in X(\mu_0 + 1)$ be such that

$$u_n(x) = \begin{cases} e^{\mu_0\|x\|} \quad \text{for} \quad \|x\| \geq n \\ 0 \quad \text{for} \quad \|x\| \leq n-1 \end{cases}$$

and

$$0 \leq u_n(x) \leq e^{\mu_0 n} \quad \text{for} \quad \|x\| \leq n.$$

Then $\|u_n\|_{X(\mu_0+1)} \to 0$ as $n \to \infty$. Therefore, $\|\tilde{\Phi}(1)u_n\|_{X(\mu_0+1)} \to 0$ as $n \to \infty$. This together with Lemma 3.1 implies that

$$\int_{\mathbb{R}^N} u_n(y) m(x; y, dy) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for $x$ in bounded subsets of $\mathbb{R}^N$ and then

$$\int_{\|y\|\geq n} e^{\mu_0\|y\|} m(x; y, dy) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for $x$ in bounded subsets of $\mathbb{R}^N$. The later implies that

$$\int_{\|y-x\|\geq n} e^{\mu\|y-x\|} m(x; y, dy) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for $|\mu| \le \mu_0$ and $x$ in bounded subset of $\mathbb{R}^N$. By (4.6), for every $1 \le i \le N$,

$$\begin{aligned}\int_{\|y-(x+p_ie_i)\|\ge n} e^{\mu\|y-(x+p_ie_i)\|} m(x+p_ie_i; y, dy) &= \int_{\|y-x\|\ge n} e^{\mu\|y-x\|} m(x+p_ie_i; y+p_ie_i, dy)\\ &= \int_{\|y-x\|\ge n} e^{\mu\|y-x\|} m(x; y, dy).\end{aligned}$$

We then have

$$\int_{\|y-x\|\ge n} e^{\mu\|y-x\|} m(x; y, dy) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for $|\mu| \le \mu_0$ and $x \in \mathbb{R}^N$. The lemma now follows. □

## 4.2 Existence of the Principal Eigenvalue

In this section, we prove Theorems A. Throughout this section, $U_{\xi,\mu,a}$ is understood as in (4.1), and $r(U_{\xi,\mu,a})$ denotes the spectral radius of $U_{\xi,\mu,a}$. We may simply write $U_\alpha$ for $U_{\xi,\mu,a}$ if no confusion can occur.

*Proof of Theorem A.* (1) We prove the existence of a $\delta_0 > 0$ and the existence of a principal eigenvalue $\lambda(\xi, \mu, a)$ for all $0 < \delta < \delta_0$, $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$. By Proposition 4.1, it suffices to prove the existence of $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$, $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$, there exists an $\alpha > -1 + a_{\max}$ such that $r(U_\alpha) > 1$.

Let

$$M_0 = \inf_{\xi \in S^{N-1}} \left( \frac{2(a_{\max} - a_{\min} + 1)}{\int_{z\cdot\xi>0} (z\cdot\xi)^2 k(z) dz} \right)^{1/2}.$$

We first prove the existence of an $\alpha > -1 + a_{\max}$ such that $r(U_\alpha) > 1$ for every $\xi \in S^{N-1}$, $\delta > 0$, and $\mu \in \mathbb{R}$ with $|\mu| > \frac{M_0}{\delta}$.

In fact, for $v(x) \equiv 1$ and every $0 < \epsilon < 1$ and $\mu > 0$, we have

$$
\begin{aligned}
(U_{-1+a_{\max}+\epsilon}v)(x) &= \int_{\mathbb{R}^N} \frac{e^{-\mu(y-x)\cdot\xi}k(y-x)}{a_{\max}+\epsilon-a(y)}dy \\
&\geq \int_{\mathbb{R}^N} \frac{e^{-\mu(y-x)\cdot\xi}k(y-x)}{a_{\max}-a_{\min}+\epsilon}dy \\
&\geq \int_{z\cdot\xi<0} \frac{e^{-\mu z\cdot\xi}k(z)}{a_{\max}-a_{\min}+\epsilon}dz \\
&\geq \frac{1}{a_{\max}-a_{\min}+\epsilon}\big(1+\frac{\mu^2\delta^2}{2!}\int_{z\cdot\xi<0}(z\cdot\xi)^2k(z)dz \\
&\qquad +\frac{\mu^4\delta^4}{4!}\int_{z\cdot\xi<0}(z\cdot\xi)^4k(z)dz+\cdots\big) \\
&\geq \frac{\mu^2\delta^2}{M_0^2}. \qquad (4.10)
\end{aligned}
$$

Similarly, we have $(U_{-1+a_{\max}+\epsilon}v)(x) \geq \frac{\mu^2\delta^2}{M_0^2}$ for $\mu < 0$. Hence if $|\mu|\delta > M_0$, then for $0 < \epsilon \ll 1$, there is $\gamma > 1$ such that

$$(U_{-1+a_{\max}+\epsilon}v)(x) > \gamma v(x) \quad \forall x \in \mathbb{R}^N.$$

This implies that $r(U_{-1+a_{\max}+\epsilon}) > 1$.

We then only need to prove that there is a $\delta_0 > 0$ and an $\alpha > -1+a_{\max}$ with $r(U_\alpha) > 1$ for all $0 < \delta < \delta_0$, $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$ with $\mu\delta \leq M_0$. We prove this by applying arguments similar to those in [37, Theorem 2.6].

Let $D = [0,p_1]\times[0,p_2]\times\cdots\times[0,p_N]$. Assume that $x_0 \in D$ is such that $a(x_0) = a_{\max}$. Without loss of generality, we may assume that $x_0 \in \text{Int}(D)$.

Then for every $0 < \epsilon < 1$, there is some $\eta > 0$ such that $a(x_0) - a(x) < \epsilon$ for $x \in B(\eta, x_0) \subset D$, where $B(\eta,x_0) = \{x \in \mathbb{R}^N \,|\, \|x-x_0\| < \eta\}$. Let $v(\cdot) \in X_p$ be such that

$v(x) = \psi(\|x - x_0\|)$ for $x \in D$, where

$$\psi(r) = \begin{cases} \cos(\frac{\pi r}{2\eta}) & \text{if } 0 \le r \le \eta \\ \\ 0 & \text{if } r > \eta \end{cases}$$

Let $0 < \delta < \frac{\eta}{2}$ and $0 < \epsilon_1 < 1$. Also let $D_1 = B(\frac{\eta}{2}, x_0)$ , $D_2 = B(\eta, x_0) \setminus D_1$. For $x \in D_2$, let $\tilde{D}(\delta, x) = B(\delta, x) \cap B(\|x - x_0\|, x_0)$.

Observe that for $x \in B(\frac{\eta}{2}, x_0)$, $v(x) \ge \frac{\sqrt{2}}{2}$. For $x \in D_2$ and $y \in \tilde{D}(\delta, x)$, $v(y) \ge v(x)$. For $x \in D \setminus B(\eta, x_0)$, $v(x) = 0$. Observe also that there are $C > 0$ (independent of $\epsilon$) and $\delta_1 > 0$ such that

$$\inf_{x \in D_1} \int_{B(\eta/2, x_0)} e^{-\mu(y-x)\cdot\xi} k(y - x) dy \ge C, \quad \inf_{x \in D_2} \int_{\tilde{D}(\delta, x)} e^{-\mu(y-x)\cdot\xi} k(y - x) dy \ge C$$

for $0 < \delta < \delta_1$, $\xi \in S^{N-1}$, and $0 \le |\mu|\delta \le M_0$.

Clearly, for each $\gamma > 1$,

$$(U_{a_{\max}-\epsilon_1} v)(x) \ge \gamma v(x) \quad \text{for} \quad x \in D \setminus B(\eta, x_0). \tag{4.11}$$

If $x \in D_1$, we have

$$\begin{aligned} (U_{a_{\max}-\epsilon_1} v)(x) &\ge \int_D \frac{e^{-\mu(y-x)\cdot\xi} k(y - x) v(y)}{1 - a(y) + a_{\max} - \epsilon_1} dy \\ &\ge \frac{1}{1 - \epsilon_1 + \epsilon} \int_{B(\eta, x_0)} e^{-\mu(y-x)\cdot\xi} k(y - x) v(y) dy \\ &\ge \frac{\sqrt{2}}{2(1 - \epsilon_1 + \epsilon)} \int_{B(\eta/2, x_0)} e^{-\mu(y-x)\cdot\xi} k(y - x) dy \\ &\ge \frac{\sqrt{2} C}{2(1 - \epsilon_1 + \epsilon)} \\ &\ge \frac{\sqrt{2} C}{2(1 - \epsilon_1 + \epsilon)} v(x). \end{aligned} \tag{4.12}$$

If $x \in D_2$, we have

$$
\begin{aligned}
(U_{a_{\max}-\epsilon_1} v)(x) &\geq \int_D \frac{e^{-\mu(y-x)\cdot\xi} k(y-x) v(y)}{1-a(y)+a_{\max}-\epsilon_1} dy \\
&\geq \frac{1}{1-\epsilon_1+\epsilon} \int_D e^{-\mu(y-x)\cdot\xi} k(y-x) v(y) dy \\
&\geq \frac{v(x)}{1-\epsilon_1+\epsilon} \int_{\tilde{D}(\delta,x)} e^{-\mu(y-x)\cdot\xi} k(y-x) dy \\
&\geq \frac{C v(x)}{1-\epsilon_1+\epsilon}. \qquad (4.13)
\end{aligned}
$$

Let $M = \frac{\sqrt{2}C}{2(1-\epsilon_1+\epsilon)}$. By (4.11)-(4.13) and the periodicity of $v$, we obtain

$$
(U_{a_{\max}-\epsilon_1} v)(x) \geq M v(x) \quad \text{for all} \quad x \in \mathbb{R}^N.
$$

Choose $\epsilon$ and $\epsilon_1$ such that $0 < \epsilon < \frac{\sqrt{2}}{2} C$ and $1 > \epsilon_1 > 1 + \epsilon - \frac{\sqrt{2}C}{2}$. Let $\delta_0 = \min\{\delta_1, \frac{\eta}{2}\}$. Then $M > 1$ and $r(U_{a_{\max}-\epsilon_1}) \geq M > 1$. Thus (1) is proved.

(2)

By the arguments in (4.10), we have for $v(x) \equiv 1$ and every $0 < \epsilon < 1$ that

$$
\begin{aligned}
(U_{-1+a_{\max}+\epsilon} v)(x) &= \int_{\mathbb{R}^N} \frac{e^{-\mu(y-x)\cdot\xi} k(y-x)}{a_{\max}+\epsilon-a(y)} dy \\
&\geq \frac{\theta}{a_{\max}-a_{\min}+\epsilon}
\end{aligned}
$$

for $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$. Hence if $a_{\max} - a_{\min} < \theta$, then for $0 < \epsilon \ll 1$, there is $\gamma > 1$ such that

$$
(U_{-1+a_{\max}+\epsilon} v)(x) > \gamma v(x).
$$

This implies that $r(U_{-1+a_{\max}+\epsilon}) > 1$. It then follows from Proposition 4.1 that the principal eigenvalue $\lambda(\xi, \mu, a)$ of (2.8) exists for $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$.

(3)

Let $x_0 \in D$ be such that $a(x_0) = a_{\max}$. Also, without loss of generality, we may assume that $x_0 \in \mathrm{Int}(D)$. Since the partial derivatives of $a(x)$ up to order $N-1$ at $x_0$ are zero, there is $M > 0$ such that

$$a(x_0) - a(y) \le M\|x_0 - y\|^N \quad \text{for} \quad y \in \mathbb{R}^N. \tag{4.14}$$

Fix $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$. Let $\sigma > 0$ be such that $\sigma < \frac{\delta}{2}$ and $B(2\sigma, x_0) \subset D$. Let $v^* \in X_p^+$ be such that $v^*(x) = 1$ if $x \in B(\sigma, x_0)$ and $v^*(x) = 0$ if $x \in D \setminus B(2\sigma, x_0)$.

Clearly, for every $x \in D \setminus B(2\sigma, x_0)$ and $\gamma > 1$,

$$(U_{-1+a_{\max}+\epsilon} v^*)(x) > \gamma v^*(x) = 0. \tag{4.15}$$

For $x \in B(2\sigma, x_0)$, there is $\tilde{M} > 0$ such that

$$e^{-\mu(y-x)\cdot\xi} k(y-x) \ge \tilde{M}$$

for $y \in B(\sigma, x_0)$. It then follows that for $x \in B(2\sigma, x_0)$

$$(U_{-1+a_{\max}+\epsilon} v^*)(x) \ge \int_{B(\sigma,x_0)} \frac{e^{-\mu(y-x)\cdot\xi} k(y-x)}{M\|x_0 - y\|^N + \epsilon} dy \ge \int_{B(\sigma,x_0)} \frac{\tilde{M}}{M\|x_0 - y\|^N + \epsilon} dy. \tag{4.16}$$

Note that $\int_{B(\sigma,x_0)} \frac{\tilde{M}}{M\|x_0-y\|^N} dy = \infty$. This together with the periodicity of $v^*(x)$ implies that for $0 < \epsilon \ll 1$, there is $\gamma > 1$ such that

$$(U_{-1+a_{\max}+\epsilon} v^*)(x) > \gamma v^*(x) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{4.17}$$

Hence $r(U_{-1+a_{\max}+\epsilon}) > 1$. It then follows from Proposition 4.1 that the principal eigenvalue $\lambda(\xi, \mu, a)$ of (2.8) exists for $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$. □

Next we prove a proposition about the comparison of principal eigenvalue on the $a(\cdot)$ of (2.8), which will also be used in the later chapters.

**Proposition 4.4.** *Assume that $a_1(x) \le \tilde{a}_1(x)$. If for given $\mu \in \mathbb{R}$, and $\xi \in S^{N-1}$, both, $\lambda(\xi, \mu, a_1)$ and $\lambda(\xi, \mu, \tilde{a}_1)$, exist, then*

$$\lambda(\xi, \mu, a_1) \le \lambda(\xi, \mu, \tilde{a}_1).$$

*Proof.* Consider the following two evolution equations,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)u(t,y)dy - u(t,x) + a_1(x)u(t,x) \tag{4.18}$$

and

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)u(t,y)dy - u(t,x) + \tilde{a}_1(x)u(t,x). \tag{4.19}$$

For given $u^0 \in X_p$, let $u(t,\cdot;u^0)$ and $\tilde{u}(t,\cdot;u^0)$ be the solutions of (4.18) and (4.19) with $u(0,\cdot;u^0) = u^0$ and $\tilde{u}(0,\cdot;u^0) = u^0$, respectively. Put

$$\Psi(t)u^0 = u(t,\cdot;u^0), \quad \tilde{\Psi}(t)u^0 = \tilde{u}(t,\cdot;u^0).$$

Let $r(\Psi(1))$ and $r(\tilde{\Psi}(1))$ be the spectral radius of $\Psi(1)$ and $\tilde{\Psi}(1)$, respectively. By [56, Lemma 3.1],

$$\lambda(\xi, \mu, a_1) = \ln r(\Psi(1))$$

and

$$\lambda(\xi, \mu, \tilde{a}_1) = \ln r(\tilde{\Psi}(1)).$$

By the fact that $a_1 \le \tilde{a}_1$ and Proposition 3.1, we have that for any $u^0 \ge 0$, $\Psi(t,\cdot;u^0) \le \tilde{\Psi}(t,\cdot;u^0)$ for any $t \ge 0$. It then follows that $r(\Psi(1)) \le r(\tilde{\Psi}(1))$ and then $\lambda(\xi,\mu,a_1) \le \lambda(\xi,\mu,\tilde{a}_1)$. □

**Theorem 4.1.** *(1) For each $\xi \in S^{N-1}$, $\lambda(\xi, \mu, a)$ is convex in $\mu$;*

*(2) There are $m > 0$ and $\mu_0 > 0$ such that $\lambda(\xi, \mu, a) \geq m\mu^2$ for all $\mu \geq \mu_0$ and $\xi \in S^{N-1}$;*

*(3) If $\lambda(\xi, 0, a) > 0$, then for every $\xi \in S^{N-1}$, there is a $\mu^*(\xi) \in (0, \infty)$ such that*

$$\frac{\lambda(\xi, \mu^*(\xi), a)}{\mu^*(\xi)} = \inf_{\mu>0} \frac{\lambda(\xi, \mu, a)}{\mu} \tag{4.20}$$

*and*

$$\frac{\lambda(\xi, \mu, a)}{\mu} > \frac{\lambda(\xi, \mu^*(\xi), a)}{\mu^*(\xi)} \quad \text{for} \quad 0 < \mu < \mu^*(\xi). \tag{4.21}$$

*Proof.* (1) Fix $\xi \in S^{N-1}$. By Lemma 4.1, $\hat{\lambda}(\mu_i) := r(\Phi^p(1; \xi, \mu_i))$ is an eigenvalue of $\Phi^p(1; \xi, \mu_i)$ with a positive eigenfunction $\psi_i$ $(i = 1, 2)$. Hence

$$\hat{\lambda}(\mu_i) = \frac{(\Phi^p(1; \xi, \mu_i)\psi_i)(x)}{\psi_i(x)} = \frac{\int_{\mathbb{R}^N} e^{\mu_i(x-y)\cdot\xi}\psi_i(y)m(x; y, dy)}{\psi_i(x)} \quad \forall x \in \mathbb{R}^N$$

for $i = 1, 2$. For given $0 \leq t \leq 1$, let $\psi_3 = \psi_1^t \psi_2^{1-t}$. By Hölder's inequality,

$$\begin{aligned}
[\hat{\lambda}(\mu_1)]^t[\hat{\lambda}(\mu_2)]^{1-t} &= [\frac{\int_{\mathbb{R}^N} e^{\mu_1(x-y)\cdot\xi}\psi_1(y)m(x; y, dy)}{\psi_1(x)}]^t[\frac{\int_{\mathbb{R}^N} e^{\mu_2(x-y)\cdot\xi}\psi_2(y)m(x; y, dy)}{\psi_2(x)}]^{1-t} \\
&\geq \int_{\mathbb{R}^N} [\frac{e^{\mu_1(x-y)\cdot\xi}\psi_1(y)}{\psi_1(x)}]^t[\frac{e^{\mu_2(x-y)\cdot\xi}\psi_2(y)}{\psi_2(x)}]^{1-t} m(x; y, dy) \\
&= \frac{\int_{\mathbb{R}^N} e^{(t\mu_1+(1-t)\mu_2)(x-y)\cdot\xi}\psi_3(y)m(x; y, dy)}{\psi_3(x)} \quad \forall x \in \mathbb{R}^N.
\end{aligned}$$

Applying Proposition 4.3, we get

$$[\hat{\lambda}(\mu_1)]^t[\hat{\lambda}(\mu_2)]^{1-t} \geq \sup_{x\in\mathbb{R}^N} \frac{\int_{\mathbb{R}^N} e^{(t\mu_1+(1-t)\mu_2)(x-y)\cdot\xi}\psi_3(y)m(x; y, dy)}{\psi_3(x)} \geq r(\Phi^p(1; \xi, t\mu_1 + (1-t)\mu_2).$$

Thus,

$$\ln[\hat{\lambda}(\mu_1)]^t[\hat{\lambda}(\mu_2)]^{1-t} \geq \ln(r(\Phi^p(1; \xi, t\mu_1 + (1-t)\mu_2)).$$

By Lemma 4.1 again,

$$t\lambda(\xi,\mu_1,a)+(1-t)\lambda(\xi,\mu_2,a)\geq\lambda(\xi,t\mu_1+(1-t)\mu_2,a),$$

that is, $\lambda(\xi,\mu,a)$ is convex in $\mu$.

(2) Note that by Proposition 4.4 $\lambda(\xi,\mu,a)\geq\lambda(\xi,\mu,a_{\min})$, and

$$\lambda(\xi,\mu,a_{\min})=\int_{\mathbb{R}^N}e^{-\mu y\cdot\xi}k(y)dy-1+a_{\min}$$

with 1 as an eigenfunction. Let $m_n(\xi)=\int_{y\cdot\xi<0}\frac{(-y\cdot\xi)^n}{n!}k(y)dy$. Then, for $\mu>0$

$$\begin{aligned}\int_{\mathbb{R}^N}e^{-\mu y\cdot\xi}k(y)dy-1+a_{\min}&=\sum_{n=0}^{\infty}\int_{\mathbb{R}^N}\frac{(-\mu y\cdot\xi)^n}{n!}k(y)dy-1+a_{\min}\\&\geq m_2(\xi)\mu^2+\sum_{n=2}^{\infty}m_{2n}(\xi)\mu^{2n}+a_{\min}\end{aligned}$$

Let $m:=\inf_{\xi\in S^{N-1}}m_2(\xi)(>0)$ and $\mu_0>0$ be such that $\sum_{n=2}^{\infty}m_{2n}(\xi)\mu^{2n}>|a_{\min}|$ for $\mu\geq\mu_0$. Then $m$ and $\mu_0$ have the required property.

(3) By (2), $\frac{\lambda(\xi,\mu,a)}{\mu}\to\infty$ as $\mu\to\infty$. By $\lambda(\xi,0,a)>0$, $\frac{\lambda(\xi,\mu,a)}{\mu}\to\infty$ as $\mu\to 0+$. This implies that there is $\mu^*(\xi)>0$ such that (4.20) and (4.21) hold. □

## 4.3 Spectral Radius of the Truncated Evolution Operator

In this section, we derive from Theorems A and B some important properties of the spectral radius of the truncated operator $\Phi_B^p(1;\xi,\mu)$ of $\Phi^p(1;\xi,\mu)$ discussed in 3.1.

For a fixed $\xi\in S^{N-1}$, let $r_B(\xi,\mu)=r(\Phi_B^p(1;\xi,\mu))$ and $\lambda_B(\xi,\mu)=\ln r(\Phi_B^p(1;\xi,\mu))$. Denote by $\lambda_B'(\xi,\mu)$ the partial derivative of $\lambda_B(\xi,\mu)$ with respect to $\mu$. We establish the following theorem for $\lambda_B(\xi,\mu)$, which is analogous to Theorem A for $\lambda(\xi,\mu)$.

**Theorem 4.2.** *Let $\xi \in S^{N-1}$ be given. Assume that (2.8) has the principal eigenvalue $\lambda(\xi,\mu)$ for $\mu \in \mathbb{R}$, that $\lambda(\xi,0) > 0$, and that $\frac{\lambda(\xi,\mu^*(\xi))}{\mu^*(\xi)} < \frac{\lambda(\xi,\mu^*(\xi)+k_0)}{\mu^*(\xi)+k_0}$ for some $k_0 > 0$, where $\mu^*(\xi)$ is as in Theorem A. Then we have:*

*(1) There is $B_0 > 0$ such that for each $B \geq B_0$ and $|\mu| \leq \mu^*(\xi) + k_0$, $r(\Phi_B^p(1;\xi,\mu))$ is an algebraically simple eigenvalue of $\Phi_B^p(1;\xi,\mu)$ with a positive eigenfunction. Moreover, $\lambda_B(\xi,0) > 0$ and $\frac{\lambda_B(\xi,\mu^*(\xi))}{\mu^*(\xi)} < \frac{\lambda_B(\xi,\mu^*(\xi)+k_0)}{\mu^*(\xi)+k_0}$.*

*(2) For each $B \geq B_0$, $\ln r(\Phi_B^p(1;\xi,\mu))$ (i.e. $\lambda_B(\xi,\mu)$) is convex in $\mu$ for $|\mu| \leq \mu^*(\xi) + k_0$.*

*(3) For a given $B \geq B_0$, define*

$$\mu_B^*(\xi) := \inf\{\tilde{\mu}_B^*(\xi) \mid \frac{\lambda_B(\xi,\tilde{\mu}_B^*(\xi))}{\tilde{\mu}_B^*(\xi)} = \inf_{0<\mu\leq\mu^*(\xi)+k_0} \frac{\lambda_B(\xi,\mu)}{\mu}\}.$$

*Then (i) $\mu_B^*(\xi) > 0$ and $\lambda_B'(\xi,\mu) < \frac{\lambda_B(\xi,\mu)}{\mu}$ for $0 < \mu < \mu_B^*(\xi)$.*

*(ii) For every $\epsilon > 0$, there exists some $\mu_\epsilon > 0$ such that for $\mu_\epsilon < \mu < \mu_B^*(\xi)$,*

$$-\lambda_B'(\xi,\mu) < -\frac{\lambda_B(\xi,\mu_B^*(\xi))}{\mu_B^*(\xi)} + \epsilon.$$

*(iii) For every $\epsilon > 0$, there is $B_1 \geq B_0$ such that if $B$ also satisfies $B \geq B_1$, then*

$$|\frac{\lambda(\xi,\mu^*(\xi))}{\mu^*(\xi)} - \frac{\lambda_B(\xi,\mu_B^*(\xi))}{\mu_B^*(\xi)}| < \epsilon.$$

*Proof.* (1) It follows from Lemma 4.3 and the perturbation theory of the spectrum of bounded linear operators (see [38]).

(2) It follows from arguments similar to those in Theorem 4.1.

(3) Fixing $\xi \in S^{N-1}$, we set $\lambda_B(\mu) = \lambda_B(\xi,\mu)$ and $r_B(\mu) = r_B(\xi,\mu)$ for simplifying notations. By (1), $0 < \mu_B^*(\xi) < \mu^*(\xi) + k_0$.

For $0 < \mu \leq \mu^*(\xi) + k_0$, let

$$\Psi(\mu) = \frac{\lambda_B(\mu)}{\mu} \quad \text{and} \quad \psi(\mu) = (\mu\Psi(\mu))^{'} \equiv \lambda_B^{'}(\mu) = \frac{r_B^{'}(\mu)}{r_B(\mu)}.$$

By the convexity of $\lambda_B(\mu)$ in $\mu \in (-\mu^*(\xi) - k_0, \mu^*(\xi) + k_0)$, $\psi^{'} \geq 0$ for $0 < \mu < \mu^*(\xi) + k_0$. Note that

$$\Psi^{'}(\mu) = \frac{1}{\mu}[\psi(\mu) - \Psi(\mu)], \tag{4.22}$$

and

$$(\mu^2\Psi^{'})^{'} = \mu\psi^{'}(\mu) \geq 0 \tag{4.23}$$

for $0 < \mu < \mu^*(\xi) + k_0$.

By (4.22) and $\Psi^{'}(\mu_B^*(\xi)) = 0$, we have

$$\lambda_B^{'}(\mu_B^*(\xi)) = \psi(\mu_B^*(\xi)) = \Psi(\lambda_B^*(\xi)) = \frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)}.$$

By the definition of $\mu_B^*(\xi)$,

$$\frac{\lambda_B(\mu)}{\mu} > \frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)} \quad \text{for} \quad \mu \in (0, \mu_B^*(\xi)).$$

By $\lambda_B^{''}(\mu) \equiv \psi^{'}(\mu) \geq 0$ for $\mu \in (0, \mu^*(\xi) + k_0)$,

$$\lambda_B^{'}(\mu) \leq \lambda_B^{'}(\mu_B^*(\xi)) \quad \text{for} \quad \mu \in (0, \mu_B^*(\xi)).$$

It then follows that

$$\lambda_B^{'}(\mu) < \frac{\lambda_B(\mu)}{\mu} \quad \text{for} \quad \mu \in (0, \mu_B^*(\xi)).$$

(i) is thus proved.

By the continuity of $\lambda_B^{'}(\mu)$, for every $\epsilon > 0$, there is $\mu_\epsilon > 0$ such that

$$\lambda_B^{'}(\mu_B^*(\xi)) - \lambda_B^{'}(\mu) < \epsilon \quad \text{for} \quad \mu \in (\mu_\epsilon, \mu_B^*(\xi)).$$

This together with $\lambda_B^{'}(\mu_B^*(\xi)) = \frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)}$ implies that

$$-\lambda_B^{'}(\mu) < -\frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)} + \epsilon \quad \text{for} \quad \mu \in (\mu_\epsilon, \mu_B^*(\xi)).$$

Hence (ii) holds.

Note that for every $0 < \epsilon \ll 1$, there are $0 < \tilde{\mu}_\epsilon^- < \mu^*(\xi) < \tilde{\mu}_\epsilon^+ < \mu^*(\xi) + k_0$ such that

$$\frac{\lambda(\xi, \tilde{\mu}_\epsilon^-)}{\tilde{\mu}_\epsilon^-} = \frac{\lambda(\xi, \tilde{\mu}_\epsilon^+)}{\tilde{\mu}_\epsilon^+} = \frac{\lambda(\xi, \mu^*(\xi))}{\mu^*(\xi)} + \frac{\epsilon}{2} \geq \frac{\lambda(\xi, \mu)}{\mu} \geq \frac{\lambda(\xi, \mu^*(\xi))}{\mu^*(\xi)} \quad \text{for} \quad \mu \in [\tilde{\mu}_\epsilon^-, \tilde{\mu}_\epsilon^+].$$

Note also that there is $B_1 \geq B_0$ such that if $B \geq B_1$, then

$$\frac{\lambda(\xi, \mu)}{\mu} - \frac{\lambda_B(\mu)}{\mu} < \frac{\epsilon}{4} \quad \text{for} \quad \mu \in [\tilde{\mu}_\epsilon^-, \tilde{\mu}_\epsilon^+]$$

holds. This implies that

$$\frac{\lambda_B(\mu^*(\xi))}{\mu^*(\xi)} < \min\{\frac{\lambda_B(\tilde{\mu}_\epsilon^-)}{\tilde{\mu}_\epsilon^-}, \frac{\lambda_B(\tilde{\mu}_\epsilon^+)}{\tilde{\mu}_\epsilon^+}\}.$$

By (4.23) and $\Psi^{'}(\mu_B^*(\xi)) = 0$,

$$\Psi^{'}(\mu) \leq 0 \quad \text{for} \quad \mu \in (0, \mu_B^*(\xi)) \quad \text{and} \quad \Psi^{'}(\mu) \geq 0 \quad \text{for} \quad \mu \in (\mu_B^*(\xi), \mu^*(\xi) + k_0).$$

We thus must have

$$\mu_B^*(\xi) \in [\tilde{\mu}_\epsilon^-, \tilde{\mu}_\epsilon^+]$$

and then

$$|\frac{\lambda(\xi,\mu^*(\xi))}{\mu^*(\xi)} - \frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)}| \le |\frac{\lambda(\xi,\mu^*(\xi))}{\mu^*(\xi)} - \frac{\lambda(\mu_B^*(\xi))}{\mu_B^*(\xi)}| + |\frac{\lambda(\mu_B^*(\xi))}{\mu_B^*(\xi)} - \frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)}| < \epsilon,$$

i.e. (iii) holds. □

## 4.4 Remarks

It is sometime important to study the space shifted equation (3.5) of (3.1). Let $\Phi(t;\xi,\mu,z)$ be the solution operator of (3.5) given in (3.7) and

$$\Phi^p(t;\xi,\mu,z) = \Phi(t;\xi,\mu,z)|_{X_p}. \tag{4.24}$$

Note that $\Phi(t;\xi,0,z)$ is independent of $\xi \in S^{N-1}$. Put

$$\tilde{\Phi}(t;z) = \Phi(t;\xi,0,z) \tag{4.25}$$

for $\xi \in S^{N-1}$.

The following remarks are easy to derive.

**Remark 4.1.** *For every $z \in \mathbb{R}^N$, $(\Phi(t;\xi,\mu,z)u_0)(x) = (\Phi(t;\xi,\mu)u_0(\cdot - z))(x+z)$ and $(\tilde{\Phi}(t;z)u_0)(x) = (\tilde{\Phi}(t)u_0(\cdot - z))(x+z)$.*

**Remark 4.2.** *For every $z \in \mathbb{R}^N$,*

$$(\tilde{\Phi}(1;z)u_0)(x) = \int_{\mathbb{R}^N} u_0(y-z)m(x+z;y,dy) \tag{4.26}$$

*and*

$$(\Phi(1;\xi,\mu,z)u_0)(x) = \int_{\mathbb{R}^N} e^{\mu(x+z-y)\cdot\xi} u_0(y-z)m(x+z;y,dy). \tag{4.27}$$

**Remark 4.3.** *If the principal eigenvalue $\lambda(\xi,\mu)$ of (2.8) exists and $\phi(x;\xi,\mu)$ is a corresponding eigenfunction, then for every $z\in\mathbb{R}^N$, $\lambda=\lambda(\xi,\mu)$ is an eigenvalue and $\phi(x;\xi,\mu,z):=\phi(x+z;\xi,\mu)$ is a corresponding eigenfunction of the following space shifted eigenvalue problem of (2.8),*

$$\begin{cases}\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)v(y)dy - v(x) + a(x+z)v(x) = \lambda v, \quad x\in\mathbb{R}^N \\ v(x+p_ie_i)=v(x), \quad i=1,2,\cdots,N, \quad x\in\mathbb{R}^N.\end{cases} \tag{4.28}$$

Let $\zeta:\mathbb{R}\to[0,1]$ be a smooth function satisfying (4.7). For a given $B>0$, define $\Phi_B(1;\xi,\mu,z):X\to X$ by

$$(\Phi_B(1;\xi,\mu,z)u_0)(x) = \int_{\mathbb{R}^N} e^{\mu(x+z-y)\cdot\xi}u_0(y-z)\zeta(\|y-x-z\|/B)m(x+z;y,dy) \tag{4.29}$$

Let

$$\Phi_B^p(1;\xi,\mu,z) = \Phi_B(1;\xi,\mu,z)|_{X_p}. \tag{4.30}$$

**Remark 4.4.** It follows from the arguments of Lemma 4.3 that

$$\|\Phi_B^p(1;\xi,\mu,z) - \Phi^p(1;\xi,\mu,z)\|_{X_p}\to 0 \quad \text{as} \quad B\to\infty$$

*uniformly for $\mu$ in bounded sets and $z\in[0,p_1]\times[0,p_2]\times\cdots\times[0,p_N]$.*

**Remark 4.5.** *The spectral radius $r(\Phi_B^p(1;\xi,\mu,z))$ of $\Phi_B^p(1;\xi,\mu,z)$ is independent of $z\in\mathbb{R}^N$. If $r(\Phi_B^p(1;\xi,\mu))$ is an eigenvalue and $\phi_B(x;\xi,\mu)$ is a corresponding eigenfunction of $\Phi_B^p(1;\xi,\mu)$, then $r(\Phi_B^p(1;\ \xi,\mu,z))(=r(\Phi_B^p(1;\xi,\mu)))$ is an eigenvalue of $\Phi_B^p(1;\xi,\mu,z)$ with the eigenfunction $\phi_B(x+z;\xi,\mu)$.*

### 4.5 An Example

In this section, we give an example which shows that the principal eigenvalue of (2.8) may not exist in case that $N \geq 3$.

**Example.** Let $q(x)$ be a smooth $p$-periodic function defined as follows

$$q(x) = \begin{cases} e^{\frac{\|x\|^2}{\|x\|^2-\sigma^2}} & \text{for} \quad \|x\| < \sigma \\ 0 \quad \text{for} \quad \sigma \leq \|x\| \leq 1/2, \end{cases}$$

where $p = (1, 1, \cdots, 1)$ (i.e. $q(x + e_i) = q(x)$ for $i = 1, 2, \cdots, N$) and $0 < \sigma < 1/2$. Note that $q_{\max} = 1$, $q(x)$ decreases as $\|x\|$ increases and $q(x) \leq e^{-\frac{\|x\|^2}{\sigma^2}}$ for $\|x\| \leq 1/2$. Let $k(z) = \frac{1}{\delta^N}\tilde{k}(\frac{z}{\delta})$, where $\tilde{k}(\cdot)$ be as in (2.1). Then for given $M > 1$, $\mathcal{K}_{\xi,\mu} - I + Q^M$ or (2.8) with $\mu = 0$ and $a(x) = Mq(x)$ has no principal eigenvalue for $0 < \sigma \ll 1$ and $\delta \gg 1$, where $Q^M = Mq(\cdot)I$.

In fact, let $U_\alpha^M = U_{\xi,0,\alpha}$, where $U_{\xi,0,\alpha}$ is as in (4.1) with $\mu = 0$ and $a(\cdot) = Mq(\cdot)$. If $\lambda^*$ is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + Q^M$, then by Propositions 4.1 and 4.2, $\lambda^* > -1 + M$ and $r(U_{\lambda^*}^M) = 1$. Observe that for every $\epsilon > 0$ and $u(x) \equiv 1$,

$$\begin{aligned}(U_{-1+M+\epsilon}^M 1)(x) &= \int_{\mathbb{R}^N} \frac{k(y-x)}{\epsilon + M - Mq(y)} dy \\ &\leq \frac{1}{\epsilon + M}\int_{\mathbb{R}^N} k(y-x)dy + C\frac{N^*(\delta)}{\delta^N}\int_{\|y\|\leq\sigma} \frac{1}{\epsilon + M\big(1 - e^{-\frac{\|y\|^2}{\sigma^2}}\big)} dy \\ &\leq \frac{1}{\epsilon + M} + C\frac{N^*(\delta)\sigma^N}{\delta^N}\int_{\|y\|\leq 1} \frac{1}{\epsilon + M\big(1 - e^{-\|y\|^2}\big)} dy,\end{aligned}$$

where $N^*(\delta)$ is the total number of disjoint unit hypercubes in $\mathbb{R}^N$ whose vertices have integer coordinates and lie inside the ball $B(x, \delta + \sqrt{N}) = \{y \in \mathbb{R}^N | \|y - x\| \leq \delta + \sqrt{N}\}$. We then have $N^*(\delta) = O(\delta^N)$ as $\delta \to \infty$ and hence

$$\frac{N^*(\delta)}{\delta^N} = O(1) \quad \text{as} \quad \delta \to \infty.$$

Note that when $N \geq 3$, there is $\tilde{M}$ such that

$$\int_{\|y\| \leq 1} \frac{1}{\epsilon + M\big(1 - e^{-\|y\|^2}\big)} dy \leq \frac{\tilde{M}}{M}$$

for all $\epsilon > 0$ and $M > 0$. This implies that for every $\epsilon > 0$ and $M > 0$,

$$(U^M_{-1+M+\epsilon} 1)(x) \leq \frac{1}{\epsilon + M} + C \frac{N^*(\delta)\sigma^N}{\delta^N} \frac{\tilde{M}}{M}.$$

Therefore when $N \geq 3$, there is $0 < \sigma_0 < 1$ such that

$$U^M_{-1+M+\epsilon} 1 \leq 1 - \sigma_0$$

for $0 < \sigma < \sigma_0$, $\delta \gg 1$ and any $\epsilon > 0$. It then follows that

$$r(U^M_{-1+M+\epsilon}) \leq 1 - \sigma_0$$

and hence $r(U^M_{\lambda^*}) \leq 1 - \sigma_0$, a contradiction. Therefore, for the given $M > 1$ and $0 < \sigma \ll 1$, $\mathcal{K}_{\xi,\mu} - I + Q^M$ has no principal eigenvalue for $\delta \gg 1$.

We remark that the principal eigenvalue $\lambda^*$ of $\mathcal{K}_{\xi,\mu} - I + Q^M$ (if exists) depends on the parameters $\delta$, $M$, $\sigma$, and $N$. To see the dependence of $\lambda^*$ (if exists) on $M$, fix $N \geq 3$, $\delta > 0$, and $0 < \sigma < 1/2$ such that

$$C\tilde{M} \frac{N^*(\delta)\sigma^N}{\delta^N} < 1.$$

Let $\lambda^*(M) = \lambda^*$ (if $\lambda^*$ exists) and $\phi^M$ be the corresponding positive eigenfunction with $\|\phi^M(\cdot)\|_{X_p} = 1$. By Theorem B (1), if $0 < M < 1$, then $\lambda^*(M)$ exists. By the above arguments, $\lambda^*(M)$ does not exists for $M \gg 1$. We claim that there is $M^* > 1$ such that $\lambda^*(M)$ exists for $0 < M < M^*$ and $\lambda^*(M)$ does not exists for $M \geq M^*$. Moreover,

$$\lim_{M \to M^*-} \lambda^*(M) = -1 + M^* \tag{4.31}$$

and

$$\lim_{M\to M^*-}\phi^M(x)=0 \quad \forall \quad x\in\mathbb{R}^N\setminus\{\sum_{i=1}^N k_ie_i\,|\,k_i\in\mathbb{Z}\}. \tag{4.32}$$

In fact, note that $U_\alpha^M$ is well defined for $\alpha=-1+M$ (since $N\geq 3$). It follows directly that $r(U_{-1+M}^M)$ decreases as $M$ increases and there is $M^*>1$ such that

$$r(U_{-1+M}^M)\begin{cases} >1 & \text{for} \quad 0<M<M^* \\ =1 & \text{for} \quad M=M^* \\ <1 & \text{for} \quad M>M^*. \end{cases}$$

We then have that for $0<M<M^*$, the principal eigenvalue $\lambda^*(M)$ of $\mathcal{K}_{\xi,\mu}-I+Q^M$ exists, $\lambda^*(M)>-1+M$, and the principal eigenvalue of $\mathcal{K}_{\xi,\mu}-I+Q^M$ does not exists for $M\geq M^*$. Moreover, it is clear that $\lim_{M\to M^*-} r(U_{-1+M}^M)=1$ and hence (4.31) holds. Note that for every $0<M<M^*$,

$$\int_{\mathbb{R}^N}k(y-x)\phi^M(y)dy=(1+\lambda^*(M)-Mq(x))\phi^M(x) \quad \forall x\in\mathbb{R}^N \tag{4.33}$$

and hence

$$0\leq\int_{\mathbb{R}^N}k(y)\phi^M(y)dy\leq(1+\lambda^*(M)-Mq(0))\phi^M(0)\leq 1+\lambda^*(M)-M.$$

This implies that

$$\lim_{M\to M^*-}\int_{\mathbb{R}^N}k(y)\phi^M(y)dy=0$$

and then

$$\lim_{M\to M^*-}\phi^M(x)=0 \quad \text{for} \quad a.e.\ x \quad \text{with} \quad \|x\|\leq\delta. \tag{4.34}$$

If $x_0 \in \mathbb{R}^N$ is such that $\lim_{M\to M^*-} \phi^M(x_0) = 0$, then by (4.33),

$$\lim_{M\to M^*-} \int_{\mathbb{R}^N} k(y-x_0)\phi^M(y)dy = \lim_{M\to M^*-} (1+\lambda^*(M) - Mq(x_0))\phi^M(x_0) = 0$$

and hence

$$\lim_{M\to M^*-} \phi^M(x) = 0 \quad \text{for} \quad a.e.\ x \quad \text{with} \quad \|x - x_0\| \le \delta. \tag{4.35}$$

By (4.34), (4.35) and induction, we have

$$\lim_{M\to M^*-} \phi^M(x) = 0 \quad \text{for} \quad a.e.\ x \in \mathbb{R}^N. \tag{4.36}$$

By (4.33) and (4.36), for every $x \in \mathbb{R}^N \setminus \{\sum_{i=1}^N k_i e_i \,|\, k_i \in \mathbb{Z}\}$,

$$\phi^M(x) = \frac{1}{1+\lambda^*(M) - Mq(x)} \int_{\mathbb{R}^N} k(y-x)\phi^M(y)dy \to 0,$$

and $M \to M^*-$, that is, (4.32) holds.

### 4.6 Effects of Spatial Variation on Principal Eigenvalue

To explore the effects of spatial variations on principal eigenvalue, recall that $\lambda(\xi, \mu, a)$(if exists) is the principal eigenvalue of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$.

Let $\bar{a} = \frac{1}{|D|}\int_D f(x,0)dx$ with $D = \prod_{i=1}^N [0, p_i]$ and $|D| = \prod_{i=1}^N p_i$ where the period vector $\mathbf{p} = (p_1, ..., p_N)$.

Assume that the principal eigenvalues of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ exist. Let $\lambda(\xi, \mu, \bar{a})$ be the principal eigenvalues of $\mathcal{K}_{\xi,\mu} - I + \bar{a}I$ and we have

$$\lambda(\xi, \mu, \bar{a}) = \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi} k(z)dz - 1 + \bar{a}. \tag{4.1}$$

In this section, we prove Theorem B. Theorem B reveals the important fact that spatial variations cannot reduce the principal eigenvalues of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$, and they are indeed increased except for degenerate cases.

The proof of Theorem B employs the Jensen's inequality (see [43, Theorem 2.2])

$$F(\frac{1}{|D|}\int_D g(x)dx) \leq \frac{1}{|D|}\int_D F(g(x))dx \tag{4.2}$$

for any continuous function $g : D \to (c, d)$ and strictly convex function $F : (c, d) \to \mathbb{R}$ with equality occurring, iff $g(x)$ is a constant function.

*Proof of Theorem B.* Suppose that $\phi(x)$ is a strictly positive principal eigenvector of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$. First we divide both sides of (2.8) by $\phi(x)$ and obtain

$$\frac{\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\phi(y)dy - \phi(x) + a(x)\phi(x)}{\phi(x)} = \lambda(\xi,\mu,a), \quad x \in \mathbb{R}^N.$$

Integrating with respect to x over $D$ yields

$$\int_D [\frac{\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\phi(y)dy - \phi(x) + a(x)\phi(x)}{\phi(x)}]dx = \int_D \lambda(\xi,\mu,a)dx$$

or

$$\lambda(\xi,\mu,a) = \frac{1}{|D|}\int_D\int_{\mathbb{R}^N} \frac{e^{-\mu(y-x)\cdot\xi}k(y-x)\phi(y)}{\phi(x)}dydx - 1 + \frac{1}{|D|}\int_D a(x)dx.$$

Since $\lambda(\xi,\mu,\bar{a}) = \int_{\mathbb{R}^N} e^{-\mu y\cdot\xi}k(y)dy - 1 + \bar{a}$, $\lambda(\xi,\mu,a) \geq \lambda(\xi,\mu,\bar{a})$ follows from

$$\frac{1}{|D|}\int_D\int_{\mathbb{R}^N} \frac{e^{-\mu(y-x)\cdot\xi}k(y-x)\phi(y)}{\phi(x)}dydx \geq \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}k(z)dz$$

or

$$\frac{1}{|D|}\int_D\int_{\mathbb{R}^N} \frac{e^{-\mu z\cdot\xi}k(z)\phi(x+z)}{\phi(x)}dzdx \geq \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}k(z)dz$$

or

$$\frac{1}{|D|}\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}k(z)\int_D \frac{\phi(x+z)}{\phi(x)}dxdz \geq \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}k(z)dz. \tag{4.3}$$

Moreover, $\lambda(\xi,\mu,a)=\lambda(\xi,\mu,\bar{a})$ iff

$$\frac{1}{|D|}\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}k(z)\int_D \frac{\phi(x+z)}{\phi(x)}dxdz = \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}k(z)dz. \tag{4.4}$$

To prove (4.3), it suffices to prove that

$$\frac{1}{|D|}\int_D \frac{\phi(x+z)}{\phi(x)}dx \geq 1 \quad \forall z\in\mathbb{R}^N. \tag{4.5}$$

Note that $F(x)=-\ln x$ is a strictly convex function on $(0,\infty)$. By (4.2),

$$-\frac{1}{|D|}\int_D \ln[\frac{\phi(x+z)}{\phi(x)}]dx \geq -\ln[\frac{1}{|D|}\int_D \frac{\phi(x+z)}{\phi(x)}dx]. \tag{4.6}$$

By the periodicity of $\phi(x)$, we have $\int_D \ln\phi(x+z)dx=\int_D \ln\phi(x)dx$ for any $z\in\mathbb{R}^N$. Hence, (4.6) implies that

$$\begin{aligned}\ln[\frac{1}{|D|}\int_D \frac{\phi(x+z)}{\phi(x)}dx] &\geq \frac{1}{|D|}\int_D \ln[\frac{\phi(x+z)}{\phi(x)}]dx\\ &= \frac{1}{|D|}\int_D \ln[\phi(x+z)]dx - \frac{1}{|D|}\int_D \ln[\phi(x)]dx\\ &=0\end{aligned}$$

Therefore, $\frac{1}{|D|}\int_D \frac{\phi(x+z)}{\phi(x)}dx \geq 1$ and thus $\lambda(\xi,\mu,a)\geq\lambda(\xi,\mu,\bar{a})$. Moreover, by (4.4), $\lambda(\xi,\mu,a)=\lambda(\xi,\mu,\bar{a})$ iff

$$\frac{1}{|D|}\int_D \frac{\phi(x+z)}{\phi(x)}dx = 1 \quad \forall z\in\mathbb{R}^N. \tag{4.7}$$

By (4.2) again, the equality occurs in (4.6) iff $\frac{\phi(x+z)}{\phi(x)}\equiv 1$ for any $z\in\mathbb{R}^N$, which is equivalent to $\phi(x)\equiv$ constant since z is arbitrary. This implies that $\lambda(\xi,\mu,a)=\lambda(\xi,\mu,\bar{a})$ iff $a(x)\equiv\bar{a}$. The proof is thus completed. □

Chapter 5

# Positive Stationary Solutions of Spatially Periodic Nonlocal Monostable Equations

In this chapter, we study the existence, uniqueness and stability of positive equilibrium solutions of (1.2), and prove Theorem C. The results of this chapter in the case that the nonlocal kernel function has compact support will be published in the Proceedings of the American Mathematical Society (see [57]).

For convenience, we introduce the following assumption:

**(H5)** $a(\cdot) \in C^N(\mathbb{R}^N) \cap X_p$ *and the partial derivatives of* $a(x)$ *up to order* $N-1$ *at some* $x_0$ *are zero, where* $x_0$ *is such that* $a(x_0) = \max_{x \in \mathbb{R}^N} a(x)$.

Suppose that $u = u^*$ is an equilibrium solution of (1.2) in $X_p^+ \setminus \{0\}$. $u = u^*$ is said to be *globally asymptotically stable in* $X_p^+ \setminus \{0\}$ if for any $u_0 \in X_p^+ \setminus \{0\}$, $u(t, \cdot; u_0) \to u^*$ in $X_p$ as $t \to \infty$. We first prove two lemmas, which will also be used to prove some theorems in next chapter. Throughout this section, $X_p$ is as in (2.4), $a \in X_p$, and $a_{\max} = \max_{x \in \mathbb{R}^N} a(x)$, $a_{\min} = \min_{x \in \mathbb{R}^N} a(x)$. $a(\cdot)I : X_p \to X_p$ has the same meaning as in (2.7) with $a_0(\cdot)$ being replaced by $a(\cdot)$ and $K_{\xi,\mu} : X_p \to X_p$ is understood as in (2.9), $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$.

**Lemma 5.1.** *Suppose that* $\{a_n\}, \{a^n\} \subset X_p$ *satisfy that*

$$a_n(\cdot) \le a(\cdot) \le a^n(\cdot) \quad \text{for} \quad n \ge 1 \quad \text{and} \quad \|a_n - a^n\|_{X_p} \to 0 \quad \text{as} \quad n \to \infty.$$

*Then for any* $\xi \in S^{N-1}$ *and* $\mu \in \mathbb{R}$,

$$\lambda_0(\xi, \mu, a_n) \le \lambda_0(\xi, \mu, a) \le \lambda_0(\xi, \mu, a^n) \quad \text{for} \quad n \ge 1 \tag{5.1}$$

*and*

$$\lambda_0(\xi, \mu, a_n) - \lambda_0(\xi, \mu, a^n) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.2}$$

*Proof.* By Propositions 3.1 and 3.2,

$$r(\Phi(1; \xi, \mu, a_n)) \le r(\Phi(1; \xi, \mu, a)) \le r(\Phi(1; \xi, \mu, a^n)) \quad \forall n \ge 1, \ \xi \in S^{N-1}, \ \mu \in \mathbb{R}.$$

This together with the spectral theorem for bounded linear operators (see [50]) implies (5.1). By (5.1), for any $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $\epsilon > 0$,

$$\lambda_0(\xi, \mu, a - \epsilon) \le \lambda_0(\xi, \mu, a_n) \le \lambda_0(\xi, \mu, a) \le \lambda_0(\xi, \mu, a^n) \le \lambda_0(\xi, \mu, a + \epsilon) \quad \forall n \gg 1. \tag{5.3}$$

This together with $\lambda_0(\xi, \mu, a \pm \epsilon) = \lambda_0(\xi, \mu, a) \pm \epsilon$ implies (5.2). □

**Lemma 5.2.** *Given $a \in X_p$, $\lambda_0(\xi, \mu, a) \ge \lambda_0(\xi, \mu, \bar{a})$ for any $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$.*

*Proof.* Take $a_n \in C^N(\mathbb{R}^N) \cap X_p$ such that $a_n$ satisfies (H5) and

$$a_n(\cdot) \le a(\cdot) \quad \text{for} \quad n \ge 1 \quad \text{and} \quad \|a_n - a\|_{X_p} \to 0 \quad \text{as} \quad n \to \infty.$$

By Theorem A, $\lambda(\xi, \mu, a_n)$ exists and $\lambda(\xi, \mu, a_n) = \lambda_0(\xi, \mu, a_n)$ for $n \ge 1$. By Theorem B, $\lambda_0(\xi, \mu, a_n) \ge \lambda_0(\xi, \mu, \bar{a}_n)$ for $n \ge 1$. The lemma follows by letting $n \to \infty$ and applying Lemma 5.1. □

To prove the theorem, we will apply Proposition 3.1 and so first we provide a sub-solution and a super-solution of (2.15).

**Proposition 5.1.** *Assume (H4) and let $\phi$ be the positive principal eigenfunction of $\mathcal{K} - I + a(\cdot)I$ with $\|\phi\|_{X_p} = 1$. Then for any $z \in \mathbb{R}^N$ and $0 < b \ll 1$, $\underline{\hat{v}}(t, x; z, b) := b\phi(x + z)$ is a sub-solution of* (2.15).

*Proof.* Fix $z \in \mathbb{R}^N$. Observe that

$$\int_{\mathbb{R}^N} k(y-x)\phi(y+z)dy - \phi_0(x+z) + f(x+z,0)\phi(x+z) = \lambda_0\phi(x+z) \quad \forall x \in \mathbb{R}^N.$$

Observe also that $\max_{x\in\mathbb{R}^N} \lambda_0\phi(x+z) > 0$ and then

$$\lambda_0 b\phi(x+z) \geq (f(x+z,0) - f(x+z, b\phi(x+z)))b\phi(x+z) \quad \forall 0 < b \ll 1.$$

It then follows that

$$\int_{\mathbb{R}^N} k(y-x)b\phi(y+z)dy - b\phi(x+z) + f(x+z, b\phi(x+z))b\phi(x+z) \geq 0 \quad \forall x \in \mathbb{R}^N,\ 0 < b \ll 1.$$

Hence $\underline{\hat{v}}(t, x; z, b)$ is a sub-solution of (2.15) for $0 < b \ll 1$. □

Since the positive principal eigenvalue of $\mathcal{K} - I + a_0(\cdot)I$ may not exist, we will construct a new sub-solution by applying Lemma 5.1.

**Proposition 5.2.** *There exists an $\epsilon$ and $0 < b \ll 1$ such that for any $z \in \mathbb{R}^N$, $\underline{v}(t, x; z, b) := b\phi_n(x+z)$ is a sub-solution of (2.15), where $\phi_n$ be the positive principal eigenfunction of $\mathcal{K} - I + a_n(\cdot)I$ with $\|\phi_n\|_{X_p} = 1$, with $\|a_n - a_0\| < \epsilon(> 0)$ and $a_n$ is such that (H5) holds.*

*Proof.* This follows by arguments similar to those in Proposition 5.1. □

**Proposition 5.3.** *For $d \gg 1$, $z \in \mathbb{R}^N$, $\bar{v}(t, x; z) \equiv d$ is a super-solution of (2.15).*

*Proof.* By direct calculation, we have

$$\begin{aligned}
&\frac{\partial \bar{v}}{\partial t} - [\int_{\mathbb{R}^N} k(y-x)\bar{v}(t,y;z)dy - \bar{v}(t,x;z) + f(x+z,\bar{v})\bar{v}(t,x;z)] \\
\geq &- d[\int_{\mathbb{R}^N} k(y-x)dy - 1 + f(x+z, d)] \\
\geq &0.
\end{aligned}$$

The proposition thus follows. □

**Lemma 5.3.** *Assume (H1) and (H2). (1.2) has at most one positive stationary solution $u^+(\cdot)$ in $X_p^+$. If there is a positive stationary solution $u^+(\cdot) \in X_p^+$, it is globally asymptotically stable with respect to perturbations in $X_p^+$.*

*Proof.* It follows from the arguments in [37, Lemma 3.3]. □

*Proof of Theorem C.* (1) It follows easily that $u(t, \cdot; \bar{v})$ is monotonically decreasing as $t$ increasing, where $\bar{v}$ is as in Proposition 5.3. Let $u^+(x) = \lim_{t\to\infty} u(t, x; \bar{v})$. Then $u^+(x)$ is upper semicontinuous and satisfied that $\int_{\mathbb{R}^N} k(y-x)u^+(y)dy - u^+(x) + f(x+z, u^+(x))u^+(x) = 0$. Then $u^+(x)[1 - f(x+z, u^+(x))] = \int_{\mathbb{R}^N} k(y-x)u^+(y)dy > 0$, which implies that $f(x+z, u^+(x)) < 1$.

Let $g(x) = \int_{\mathbb{R}^N} k(y-x)u^+(y)dy$ and $y = u^+(x)$. Let $F(x, z, y) = g(x) - y + f(x+z, y)y$ and then $F(x, z, y) = 0$. Since $\frac{\partial F(x,z,y)}{\partial y} = -1 + f(x+z, y) + f_u(x+z, y)y < 0$, by Implicit Function Theorem, $u^+(x)$ is continuous.

Similarly, let $u^-(x) = \lim_{t\to\infty} u(t, x; \underline{v})$ and then is also a positive stationary solution of (2.15). By Lemma 5.3, $u^-(x) = u^+(x)$

For any $u_0 \in X_p^+ \setminus \{0\}$, for $t_0 > 0$, there exist $\bar{v}$ and $\underline{v}$ such that $\bar{v} > u(t_0, x; u_0) > \underline{v}$, where $\underline{v}$ is as in Proposition 5.2. By Proposition 3.1, $u(t, x; \bar{v}) > u(t + t_0, x; u_0) > u(t, x; \underline{v})$.

Then, $u = u^+$ is a globally asymptotically stable stationary solutions with respect to the perturbations in $X_p^+ \setminus \{0\}$. (1) then follows.

(2) By Lemma 5.2, we have $\lambda_0(a) \geq \lambda_0(\bar{a}) = \bar{a}$¿0 and then (H3) is satisfied. Thus the conclusions in (1) hold. □

**Remark 5.1.** *Assume (H1), (H2), and (H3). Then*

$$\lim_{t\to\infty}(u(t, x; \alpha^+, z) - u^+(x+z)) = 0$$

*holds uniformly in $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$ for every $\alpha^+ > 0$. Here $\alpha^+$ in $u(t, x; \alpha^+, z)$ stands for the constant function with value $\alpha^+$.*

Chapter 6

Spreading Speeds of Spatially Periodic Nonlocal Monostable Equations

In this chapter, we investigate the spatial spreading speeds of (1.2) and prove Theorems D-G. To do so, we first introduce a so-called spreading speed interval $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ of (1.2) in the direction of $\xi \in S^{N-1}$ and establish basic properties. We will prove the existence of spreading speed of (1.2) in the direction of $\xi \in S^{N-1}$ by showing that $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is a singleton and $c^*_{\inf}(\xi)(= c^*_{\sup}(\xi))$ is the spreading speed of (1.2) in the direction of $\xi$. The results of this chapter in the case that the nonlocal kernel function has compact support have been published (see [29], [56], [57]).

## 6.1 Spreading Speed Intervals

Throughout this section, $X_p$ is as in (2.4), $X$ is as in (2.2), and $X^+(\xi)$ is as in (2.12) ($\xi \in S^{N-1}$). We assume (H1) - (H3). and so, (1.2) has a unique positive stable periodic equilibrium solution $u^+(x)$ in $X_p$. Let $u^+_{\inf}$ be as in (2.23). For simplifying notations set

$$\liminf_{x\cdot\xi\to-\infty} u_0(x) = \lim_{r\to-\infty} \inf_{x\cdot\xi\le r} u_0(x), \quad \limsup_{x\cdot\xi\to\infty} u_0(x) = \lim_{r\to\infty} \sup_{x\cdot\xi\ge r} u_0(x)$$

for given $u_0 \in X$ and $\xi \in S^{N-1}$. For given $u(t,\cdot) \in X$, $\xi \in S^{N-1}$, and $c \in \mathbb{R}$, put

$$\liminf_{x\cdot\xi\le ct, t\to\infty} u(t,x) = \liminf_{t\to\infty} \inf_{x\cdot\xi\le ct} u(t,x), \quad \limsup_{x\cdot\xi\ge ct, t\to\infty} u(t,x) = \limsup_{t\to\infty} \sup_{x\cdot\xi\ge ct} u(t,x),$$

$$\liminf_{|x\cdot\xi|\le ct, t\to\infty} u(t,x) = \liminf_{t\to\infty} \inf_{|x\cdot\xi|\le ct} u(t,x), \quad \limsup_{|x\cdot\xi|\ge ct, t\to\infty} u(t,x) = \limsup_{t\to\infty} \sup_{|x\cdot\xi|\ge ct} u(t,x),$$

and

$$\liminf_{\|x\|\le ct, t\to\infty} u(t,x) = \liminf_{t\to\infty} \inf_{\|x\|\le ct} u(t,x), \qquad \limsup_{\|x\|\ge ct, t\to\infty} u(t,x) = \limsup_{t\to\infty} \sup_{\|x\|\ge ct} u(t,x).$$

**Definition 6.1.** *For a given vector $\xi \in S^{N-1}$, let*

$$C^*_{\inf}(\xi) = \Big\{ \ c \mid \forall\ u_0 \in X^+(\xi),\ \liminf_{x\cdot\xi\le ct, t\to\infty} (u(t,x;u_0) - u^+(x)) = 0 \Big\}$$

*and*

$$C^*_{\sup}(\xi) = \Big\{ \ c \mid \forall\ u_0 \in X^+(\xi),\ \limsup_{x\cdot\xi\ge ct, t\to\infty} u(t,x;u_0) = 0 \Big\}.$$

*Define*

$$c^*_{\inf}(\xi) = \sup\ \{\ c \mid c \in C^*_{\inf}(\xi)\}, \quad c^*_{\sup}(\xi) = \inf\ \{\ c \mid c \in C^*_{\sup}(\xi)\}.$$

*We call $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ the* spreading speed interval *of* (1.2) *in the direction of $\xi$.*

Observe that if $c_1 \in C^*_{\inf}(\xi)$ and $c_2 \in C^*_{\sup}(\xi)$, then $c_1 < c_2$. Hence $c^*_{\inf}(\xi) \le c^*_{\sup}(\xi)$ for all $\xi \in S^{N-1}$.

To establish basic properties of the spreading speed intervals of (1.2), we first construct some useful sub- and super-solutions of (1.2) and its space shifted equation (2.15). Recall that $u(t,x;u_0,z)$ denotes the solution of (2.15) with $u(0,x;u_0,z) = u_0(x)$ for $u_0 \in X$ and $z \in \mathbb{R}^N$.

Let $\eta(s)$ be the function defined by

$$\eta(s) = \frac{1}{2}(1 + \tanh\frac{s}{2}), \quad s \in \mathbb{R}. \tag{6.1}$$

Observe that

$$\eta^{'}(s) = \eta(s)(1-\eta(s)), \quad s \in \mathbb{R} \tag{6.2}$$

and

$$\eta''(s) = \eta(s)(1-\eta(s))(1-2\eta(s)), \quad s \in \mathbb{R}. \tag{6.3}$$

Without loss of generality, we may assume that $f(x,u) = 0$ for $u \ll 0$. For otherwise, let $\tilde{\zeta}(\cdot) \in C^\infty(\mathbb{R})$ be such that $\tilde{\zeta}(u) = 1$ for $u \geq 0$ and $\tilde{\zeta}(u) = 0$ for $u \ll 0$. We replace $f(x,u)$ by $f(x,u)\tilde{\zeta}(u)$. Hence we may also assume that there is $u^- < 0$ such that for any $u_0 \in X$ with $u^- \leq u_0 \leq 0$ and $z \in \mathbb{R}^N$,

$$u^- \leq u(t,\cdot;u_0,z) \leq 0 \quad \text{for} \quad t \geq 0. \tag{6.4}$$

**Proposition 6.1.** *Assume (H1) - (H3). Let* $\alpha_\pm$ ($u^- \leq \alpha_- \leq 0 < \alpha_+ \leq 2u^+_{\inf}$) *be given constants. There is* $C_0 > 0$ *such that for every* $C \geq C_0$*, every* $\xi \in S^{N-1}$ *and every* $z \in \mathbb{R}^N$*, the following properties hold:*

1) *letting* $v^\pm(t,x;z) = u(t,x;\alpha_\pm,z)\eta(x\cdot\xi + Ct) + u(t,x;\alpha_\mp,z)(1-\eta(x\cdot\xi+Ct))$, $v^+$ *and* $v^-$ *are super- and sub-solutions of* (2.15) *on* $[0,\infty)$*, respectively;*

2) *letting* $w^\pm(t,x;z) = u(t,x;\alpha_\mp,z)\eta(x\cdot\xi - Ct) + u(t,x;\alpha_\pm,z)(1-\eta(x\cdot\xi-Ct))$, $w^+$ *and* $w^-$ *are super- and sub-solutions of* (2.15) *on* $[0,\infty)$*, respectively.*

*Proof.* We prove that $v^+(t,x;z)$ with $z = 0$ is a super-solution of (1.2). Other statements can be proved similarly. We write $v^+(t,x)$ for $v^+(t,x;0)$.

First, by Taylor expansion,

$$
\begin{aligned}
&f(x,u(t,x;\alpha_+))\eta(x\cdot\xi+Ct)+f(x,u(t,x;\alpha_-))(1-\eta(x\cdot\xi+Ct))\\
&\quad -f\big(x,u(t,x;\alpha_+)\eta(x\cdot\xi+Ct)+u(t,x;\alpha_-)(1-\eta(x\cdot\xi+Ct)\big)\\
&=f(x,u(t,x;\alpha^+)-u(t,x;\alpha^-)+u(t,x;\alpha^-))\eta(x\cdot\xi+Ct)\\
&\quad +f(x,u(t,x;\alpha_-))(1-\eta(x\cdot\xi+Ct))\\
&\quad -f(x,(u(t,x;\alpha^+)-u(t,x;\alpha^-))\eta(x\cdot\xi+Ct)+u(t,x;\alpha^-))\\
&=\Big(f_u(x,\tilde{u}^*(t,x)+u(t,x;\alpha^-))-f_u(x,\tilde{u}^*(t,x)\eta(x\cdot\xi+Ct)+u(t,x;\alpha^-))\Big)\cdot\\
&\qquad (u(t,x;\alpha^+)-u(t,x;\alpha^-))\eta(x\cdot\xi+Ct)\\
&=f_{uu}(x,u^{**}(t,x))\big(u^*(t,x)-u(t,x;\alpha_-)\big)\big(u(t,x;\alpha_+)-u(t,x;\alpha_-)\big)\eta^{'}(x\cdot\xi+Ct)
\end{aligned}
$$

where $u^*(t,x)=\tilde{u}^*(t,x)+u(t,x;\alpha^-)$ and $u^{**}(t,x)$ and $u^*(t,x)$ are between $u(t,x;\alpha_-)$ and $u(t,x;\alpha_+)$. Then a direct computation yields

$$
\begin{aligned}
v_t^+(t,x)-[\int_{\mathbb{R}^N}&k(y-x)v^+(t,y)dy-v^+(t,x)]-f(x,v^+(t,x))\\
&=\eta^{'}(x\cdot\xi+Ct)\Big\{C\big(u(t,x;\alpha_+)-u(t,x;\alpha_-)\big)\\
&\quad -\int_{\mathbb{R}^N}k(y-x)(u(t,y;\alpha^+)-u(t,y;\alpha^-))\frac{\eta(y\cdot\xi+Ct)-\eta(x\cdot\xi+Ct)}{\eta^{'}(x\cdot\xi+Ct)}dy\\
&\quad -f_{uu}(x,u^{**}(t,x))\big(u^*(t,x)-u(t,x;\alpha_-)\big)(u(t,x;\alpha_+)-u(t,x;\alpha_-))\Big\}.
\end{aligned}
$$

Note that there are $M_0$ and $M_1>0$ such that

$$u(t,x;\alpha_+)-u(t,x;\alpha_-)\geq M_0\quad\text{for all}\quad t\geq 0,\quad x\in\mathbb{R}^N,$$

$$|\frac{\eta(y\cdot\xi+Ct)-\eta(x\cdot\xi+Ct)}{\eta^{'}(x\cdot\xi+Ct)}|\leq M_1\quad\text{for all}\quad t\geq 0,\quad x,y\in\mathbb{R}^N,\quad \|y-x\|\leq\delta.$$

It then follows that there is $C_0 > 0$ such that for every $C \geq C_0$, $v^+(t,x)$ is a super-solution of (1.2). □

**Proposition 6.2.** *Assume (H1) - (H3). For every $\xi \in S^{N-1}$, the following properties hold:*

*(1) if there is $u_0^* \in X^+(\xi)$ such that*

$$\liminf_{x\cdot\xi \leq ct, t\to\infty} \big(u(t,x;u_0^*,z) - u^+(x+z)\big) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N,$$

*then $c \leq c^*_{\inf}(\xi)$;*

*(2) if $c < c^*_{\inf}(\xi)$, then for each $u_0 \in X^+(\xi)$,*

$$\liminf_{x\cdot\xi \leq ct, t\to\infty} (u(t,x;u_0,z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*Proof.* It can be proved by arguments similar to those in [30, Lemma 3.4]. □

**Proposition 6.3.** *Assume (H1) - (H3). For every $\xi \in S^{N-1}$, the following properties hold:*

*(1) if there is $u_0^* \in X^+(\xi)$ such that*

$$\limsup_{x\cdot\xi \geq ct, t\to\infty} u(t,x;u_0^*,z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N,$$

*then $c \geq c^*_{\sup}(\xi)$;*

*(2) if $c > c^*_{\sup}(\xi)$, then for every $u_0 \in X^+(\xi)$,*

$$\limsup_{x\cdot\xi \geq ct, t\to\infty} u(t,x;u_0,z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*Proof.* It can be proved by arguments similar to those in [30, Lemma 3.5]. □

**Corollary 6.1.** *Assume (H1) - (H3). $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is a finite interval for all $\xi \in S^{N-1}$.*

*Proof.* Fix $\xi \in S^{N-1}$.

Let $\alpha_- = 0 < \alpha_+ \le u^+_{\inf}$ be given constants. There is $u_0^* \in X^+(\xi)$ such that

$$w^+(0,x;z) = \alpha_-\eta(x\cdot\xi) + \alpha_+(1-\eta(x\cdot\xi)) \ge u_0^*(x), \quad x \in \mathbb{R}^N$$

for all $z \in \mathbb{R}^N$. Then by Propositions 3.1 and 6.1,

$$\begin{aligned} w^+(t,x;z) &= u(t,x;\alpha_-,z)\eta(x\cdot\xi - C_0t) + u(t,x;\alpha_+,z)(1-\eta(x\cdot\xi - C_0t)) \\ &\ge u(t,x;z,u_0^*) \end{aligned}$$

for $t \ge 0$, and $x, z \in \mathbb{R}^N$. This implies that for $C > C_0$,

$$\limsup_{x\cdot\xi\ge Ct, t\to\infty} u(t,x;z,u_0^*) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

Therefore by Proposition 6.3, $c^*_{\sup}(\xi) \le C_0$.

Now let $u^+_{\inf} > \alpha_+ > 0 > \alpha_- \ge u^-$ be a given constant, where $u^-$ satisfies (6.4). There is $u_0^{**} \in X^+(\xi)$ such that

$$v^-(0,x;z) = \alpha_-\eta(x\cdot\xi) + \alpha_+(1-\eta(x\cdot\xi)) \le u_0^{**}(x)$$

for $x, z \in \mathbb{R}^N$. Then by Propositions 3.1 and 6.1 again,

$$\begin{aligned} v^-(t,x;z) &= u(t,x;\alpha_-,z)\eta(x\cdot\xi + C_0t) + u(t,x;\alpha_+,z)(1-\eta(x\cdot\xi + C_0t)) \\ &\le u(t,,x;u_0^{**},z) \end{aligned}$$

for $t \ge 0$, and $x, z \in \mathbb{R}^N$. This implies that for $C < -C_0$,

$$\liminf_{x\cdot\xi\le Ct, t\to\infty} (u(t,x;u_0,z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

Therefore by Proposition 6.2, $c^*_{\inf}(\xi) \geq -C_0$.

Hence $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ is a finite interval. $\square$

Let

$$\tilde{X}^+(\xi) = \{u \in X^+ \mid \liminf_{x\cdot\xi\to-\infty} u_0(x) > 0, \limsup_{x\cdot\xi\to\infty} u_0(x) = 0\}. \tag{6.5}$$

**Proposition 6.4.** *Assume (H1) - (H3).*

*(1) Let $\xi \in S^{N-1}$, $u_0 \in \tilde{X}^+(\xi)$, and $c \in \mathbb{R}$ be given. If there are $\delta_0$ and $T_0 > 0$ such that*

$$\liminf_{x\cdot\xi\leq cnT_0, n\to\infty} u(nT_0, x; u_0, z) \geq \delta_0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N, \tag{6.6}$$

*then for every $c^{'} < c$,*

$$\liminf_{x\cdot\xi\leq c^{'}t, t\to\infty} (u(t, x; u_0, z) - u^+(x + z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*(2) Let $c \in \mathbb{R}$ and $u_0 \in X$ with $u_0 \geq 0$ be given. If there are $\delta_0$ and $T_0 > 0$ such that*

$$\liminf_{|x\cdot\xi|\leq cnT_0, n\to\infty} u(nT, x; u_0, z) \geq \delta_0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N, \tag{6.7}$$

*then for every $c^{'} < c$,*

$$\liminf_{|x\cdot\xi|\leq c^{'}t, t\to\infty} (u(t, x; u_0, z) - u^+(x + z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*(3) Let $c \in \mathbb{R}$ and $u_0 \in X$ with $u_0 \geq 0$ be given. If there are $\delta_0$ and $T_0 > 0$ such that*

$$\liminf_{\|x\|\leq cnT_0, n\to\infty} u(nT, x; u_0, z) \geq \delta_0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N, \tag{6.8}$$

*then for every $c^{'} < c$,*

$$\liminf_{\|x\| \le c^{'}t, t\to\infty} (u(t,x;u_0,z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

*Proof.* (1) First, for given $c^{'} < c$, there is $n_0 \in \mathbb{N}$ such that

$$u(nT_0, x+y; u_0, z) \ge \frac{\delta_0}{2} \quad \text{for} \quad z \in \mathbb{R}^N,\ x\cdot\xi \le (c-c^{'})nT_0,\ y\cdot\xi \le c^{'}nT_0,\ n \ge n_0. \tag{6.9}$$

Let $\tilde{u}_0(x) \equiv \frac{\delta_0}{2}$. For each $\epsilon > 0$, there exists $n_1 \ge n_0$ such that

$$u(t,x;\tilde{u}_0,z) \ge u^+(x+z) - \epsilon \quad \text{for} \quad t \ge n_1T_0,\ x,z \in \mathbb{R}^N. \tag{6.10}$$

For a given $B > 1$, let $\tilde{u}_B(\cdot) \in X$ be such that $0 \le \tilde{u}_B(x) \le \frac{\delta_0}{2}$ for $x \in \mathbb{R}^N$, $\tilde{u}_B(x) = \frac{\delta_0}{2}$ for $x \cdot \xi \le B-1$, and $\tilde{u}_0(x) = 0$ for $x\cdot\xi \ge B$. By Proposition 3.3, Remark 5.1 and (6.10), there is $\tilde{B}_0 > 1$ such that for each $B \ge \tilde{B}_0$,

$$u(t,0;\tilde{u}_B,z) \ge u^+(z) - 2\epsilon \quad \text{for} \quad n_1T_0 \le t \le (n_1+1)T_0, \quad z \in \mathbb{R}^N. \tag{6.11}$$

Note that $(c-c^{'})nT_0 \to \infty$ as $n \to \infty$. Hence there is $n_2 \ge n_1$ such that

$$(c-c^{'})nT_0 \ge \tilde{B}_0 + c^{'}(n_1+1)T_0 \quad \text{for} \quad n \ge n_2.$$

This together with (6.9) implies that

$$u(nT_0, \cdot + x + c^{'}nT_0\xi + c^{'}\tau\xi; u_0, z) \ge \tilde{u}_{\tilde{B}_0}(\cdot)$$

for all $x \in \mathbb{R}^N$ with $x \cdot \xi \leq 0$, all $\tau$ with $n_1 T_0 \leq \tau \leq (n_1+1)T_0$, and all $n \geq n_2$. Given $n \geq n_2$ and $(n+n_1)T_0 \leq t < (n+n_1+1)T_0$, let $\tau = t - nT_0$. Then $n_1 T_0 \leq \tau < (n_1+1)T_0$ and

$$\begin{aligned} u(t, x + c^{'} t\xi; u_0, z) &= u(\tau, x + c^{'} t\xi; u(nT_0, \cdot; u_0, z), z) \\ &= u(\tau, 0; u(nT_0, \cdot + x + c^{'} nT_0 \xi + c^{'} \tau\xi; u_0, z), z + x + c^{'} t\xi) \\ &\geq u^+(x + z + c^{'} t\xi) - 2\epsilon \end{aligned}$$

for all $x \in \mathbb{R}^N$ with $x \cdot \xi \leq 0$. It then follows that

$$u(t, x; u_0, z) \geq u^+(x+z) - 2\epsilon \quad \text{for} \quad z \in \mathbb{R}^N,\ x \cdot \xi \leq c^{'} t,\ t \geq (n_1 + n_2)T_0.$$

(1) is thus proved.

(2) It can be proved by arguments similar to those in (1).

(3) It can also be proved by arguments similar to those in (1). For the reader's convenience, we provide a proof in the following.

First, for a given $c^{'} < c$, there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$u(nT_0, x + y; u_0, z) \geq \frac{\delta_0}{2} \quad \text{for} \quad z \in \mathbb{R}^N,\ \|x\| \leq (c - c^{'})nT_0,\ \|y\| \leq c^{'} nT_0. \tag{6.12}$$

Let $\tilde{u}_0(x) \equiv \frac{\delta_0}{2}$. Then, for each $\epsilon > 0$, there is $n_1 \geq n_0$ such that

$$u(t, x; \tilde{u}_0, z) \geq u^+(x+z) - \epsilon \quad \text{for} \quad t \geq n_1 T_0,\ \ x, z \in \mathbb{R}^N. \tag{6.13}$$

For a given $B > 1$, let $\tilde{u}_B(\cdot) \in X$ be such that $0 \leq \tilde{u}_B(x) \leq \frac{\delta_0}{2}$ for $x \in \mathbb{R}^N$, $\tilde{u}_B(x) = \frac{\delta_0}{2}$ for $\|x\| \leq B-1$, and $\tilde{u}_0(x) = 0$ for $\|x\| \geq B$. By Proposition 3.3, Remark 5.1, and (6.12), there exists $\tilde{B}_0 > 1$ such that

$$u(t, 0; \tilde{u}_B, z) \geq u^+(z) - 2\epsilon \quad \text{for} \quad n_1 T_0 \leq t \leq (n_1+1)T_0, \quad z \in \mathbb{R}^N \tag{6.14}$$

for all $B \geq \tilde{B}_0$.

Note that $(c - c^{'})nT_0 \to \infty$ as $n \to \infty$. Hence there is $n_2 \geq n_1$ such that

$$(c - c^{'})nT_0 \geq \tilde{B}_0 + c^{'}(n_1 + 1)T_0 \quad \text{for} \quad n \geq n_2.$$

This together with (6.12) implies that

$$u(nT_0, \cdot + x; u_0, z) \geq \tilde{u}_{\tilde{B}_0}(\cdot)$$

for each $n \geq n_2$ and each $x \in \mathbb{R}^N$ with $\|x\| \leq c^{'}nT_0 + c^{'}(n_1 + 1)T_0$. For given $n \geq n_2$ and $(n + n_1)T_0 \leq t < (n + n_1 + 1)T_0$, let $\tau = t - nT_0$. Then $n_1T_0 \leq \tau < (n_1 + 1)T_0$ and

$$\begin{aligned} u(t, x; u_0, z) &= u(\tau, x; u(nT_0, \cdot; u_0, z), z) \\ &= u(\tau, 0; u(nT_0, \cdot + x; u_0, z), z + x) \\ &\geq u^+(x + z) - 2\epsilon \end{aligned}$$

for all $x \in \mathbb{R}^N$ with $\|x\| \leq c^{'}t(\leq c^{'}(n + n_1 + 1)T_0)$. This implies that

$$u(t, x; u_0, z) \geq u^+(x + z) - 2\epsilon$$

for $t \geq (n_1 + n_2)T_0$ and $\|x\| \leq c^{'}t$. (3) is thus proved. □

## 6.2 Spreading Speeds under the Assumption of the Existence of a Principal Eigenvalue

In this section, we investigate the spreading speeds of (1.2) and prove Theorems D, E and F stated in the chapter 2 under the assumptions (H1)-(H4).

Recall that $u(t, x; u_0)$ denotes the solution of (1.2) with $u(0, \cdot; u_0) = u_0 \in X$ and $u(t, x; u_0, z)$ denotes the solution of (2.15) with $u(0, \cdot; u_0, z) = u_0 \in X$. Note that $u(t, x; u_0, 0) =$

$u(t,x;u_0)$. In the following, $\Phi(t;\xi,\mu,z)$ and $\Phi_B(1;\xi,\mu,z)$ denote the solution operators of (3.5) with $a(x) = a_0(x)(= f(x,0))$ given in (3.7) and the truncated operator of $\Phi(1;\xi,\mu,z)$ given in (4.29), respectively.

*Proof of Theorem D.* (1) Fix $\xi \in S^{N-1}$. Put $\lambda(\mu) = \lambda(\xi,\mu)$. By Theorem 4.1, there is $\mu^* = \mu^*(\xi) \in (0,\infty)$ such that

$$\inf_{\mu>0} \frac{\lambda(\mu)}{\mu} = \frac{\lambda(\mu^*)}{\mu^*}.$$

It is easy to see that $c^*(\xi)$ exists and $c^*(\xi) = \frac{\lambda(\mu^*)}{\mu^*}$ if and only if $c^*_{\inf}(\xi) = c^*_{\sup}(\xi) = \frac{\lambda(\mu^*)}{\mu^*}$.

We first prove that $c^*_{\sup}(\xi) \leq \frac{\lambda(\mu^*)}{\mu^*}$.

Since $f(x,u) = f(x,0) + f_u(x,\eta)u$ for some $0 \leq \eta \leq u$, we have, by assumption (H2), $f(x,u) \leq f(x,0)$ for $u \geq 0$. If $u_0 \in X^+$ , then

$$u(t,x;u_0) \leq (\Phi(t;\xi,0,0)u_0)(x) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{6.15}$$

Suppose that $\phi(\mu,x) \in X_p^+$ is a principal eigenvector of (2.8) with $a(x) = a_0(x)(= f(x,0)$, that is, $(\mathcal{K}_{\xi,\mu} - I + a_0(\cdot)I)\phi(\mu,x) = \lambda(\mu)\phi(\mu,x)$ with $\mu > 0$. It can easily be verified that $(\Phi(t;\xi,0,0)\tilde{u}_0)(x) = Me^{-\mu(x\cdot\xi-\tilde{c}t)}\phi(\mu,x)$ with $\tilde{u}_0 = Me^{-\mu x\cdot\xi}\phi(\mu,x)$ for $\tilde{c} = \frac{\lambda(\mu)}{\mu}$ and $M > 0$. For any $u_0 \in X^+(\xi)$, choose $M > 0$ large enough such that $\tilde{u}_0 \geq u_0$. Then by Proposition 3.1 and Remark 3.1 we have $u(t,x;u_0) \leq (\Phi(t;\xi,0,0)u_0)(x) \leq (\Phi(t;\xi,0,0)\tilde{u}_0)(x) = Me^{-\mu(x\cdot\xi-\tilde{c}t)}\phi(\mu,x)$. Hence

$$\limsup_{x\cdot\xi\geq ct,t\to\infty} u(t,x;u_0) = 0 \quad \text{for} \quad \text{every} \quad c > \tilde{c}.$$

This implies that $c^*_{\sup}(\xi) \leq \frac{\lambda(\mu)}{\mu}$ for any $\mu > 0$ and then

$$c^*_{\sup}(\xi) \leq \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}. \tag{6.16}$$

We then prove that $c^*_{\inf}(\xi) \geq \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$. We will do so by modifying the arguments in [44] and [62].

First of all, for every $\epsilon_0 > 0$, there is $b_0 > 0$ such that

$$f(x,u) \geq f(x,0) - \epsilon_0 \quad \text{for} \quad 0 \leq u \leq b_0, \quad x \in \mathbb{R}^N. \tag{6.17}$$

Choose $B \gg 1$ such that Theorem 4.2 holds. Observe that if $u_0 \in X^+$ is so small that $0 \leq u(t,x;u_0,z) \leq b_0$ for $t \in [0,1]$, $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$, then

$$u(1,x;u_0,z) \geq e^{-\epsilon_0}(\Phi(1;\xi,0,z)u_0)(x) \geq e^{-\epsilon_0}(\Phi_B(1;\xi,0,z)u_0)(x) \tag{6.18}$$

for $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$.

Let $r_B(\mu)$ be the spectral radius of $\Phi_B(1;\xi,\mu,0)$ and $\lambda_B(\mu) = \ln r_B(\mu)$. By Theorem 4.2 (1), $r_B(\mu)$ is an eigenvalue of $\Phi_B(1;\xi,\mu,0)$ with a positive eigenfunction $\phi(\mu,x)$ for $|\mu| \leq \mu^*(\xi) + k_0$.

By Theorem 4.2 (3), for each $\epsilon_1 > 0$, there exists a $B > 0$ such that

$$-\frac{\lambda_B(\mu^*_B(\xi))}{\mu^*_B(\xi)} \leq -\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} + \epsilon_1, \tag{6.19}$$

where $\mu^*_B(\xi)$ is as in Theorem 4.2 (3). Moreover, there is $\mu_{\epsilon_1}$ such that

$$-\lambda_B'(\mu) < -\frac{\lambda_B(\mu^*_B(\xi))}{\mu^*_B(\xi)} + \epsilon_1 \tag{6.20}$$

for $\mu_{\epsilon_1} < \mu < \mu^*_B(\xi)$. In the following, we fix $\mu \in (\mu_{\epsilon_1}, \mu^*_B(\xi))$. By Theorem 4.2 (3) again, we can choose $\epsilon_0 > 0$ so small that

$$\lambda_B(\mu) - \mu r_B'(\mu)/r_B(\mu) - \epsilon_0 > 0. \tag{6.21}$$

Let

$$\kappa(\mu, z) = \frac{\phi_\mu(\mu, z)}{\phi(\mu, z)}.$$

For given $\gamma > 0$ and $z \in \mathbb{R}^N$, define

$$\tau(\gamma, z) = \frac{1}{\gamma} tan^{-1} \frac{\int_{\mathbb{R}^N} \phi(\mu, y) e^{-\mu(y-z)\cdot\xi} \zeta(\|y-z\|/B) \sin\gamma(-(y-z)\cdot\xi + \kappa(\mu, y)) m(z; y, dy)}{\int_{\mathbb{R}^N} \phi(\mu, y) e^{-\mu(y-z)\cdot\xi} \zeta(\|y-z\|/B) \cos\gamma(-(y-z)\cdot\xi + \kappa(\mu, y)) m(z; y, dy)}.$$

It is not difficult to prove that

$$\lim_{\gamma\to 0} \tau(\gamma, z) = \lambda_B^{'}(\mu) + \kappa(\mu, z) \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

Choose $\gamma > 0$ so small that $\gamma(B + |\tau(z)| + |\kappa(\mu, z)|) < \pi$ for all $z \in \mathbb{R}^N$ and

$$\kappa(\mu, z) - \tau(\gamma, z) < -\lambda_B^{'}(\mu) + \epsilon_1 \tag{6.22}$$

for $z \in [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N]$.

For given $\epsilon_2 > 0$ and $\gamma > 0$, define

$$v(s, x) = \begin{cases} \epsilon_2 \phi(\mu, x) e^{-\mu s} \sin\gamma(s - \kappa(\mu, x)), & 0 \le s - \kappa(\mu, x) \le \frac{\pi}{\gamma} \\ 0, & \text{otherwise.} \end{cases} \tag{6.23}$$

Let

$$v^*(x; s, z) = v(x\cdot\xi + s - \kappa(\mu, z) + \tau(\gamma, z), x + z).$$

Choose $\epsilon_2 > 0$ so small that

$$0 \le u(t, x; v^*(\cdot; s, z), z) \le b_0 \quad \text{for} \quad t \in [0, 1], \quad x, z \in \mathbb{R}^N.$$

Let

$$\eta(\gamma, \mu, z) = -\kappa(\mu, z) + \tau(\gamma, z).$$

Then for $0 \le s - \kappa(\mu, z) \le \frac{\pi}{\gamma}$, we have

$$
\begin{aligned}
& u(1,0;v^*(\cdot;s,z),z) \\
& \ge e^{-\epsilon_0}\Phi_B(1;\xi,0,z)v^*(\cdot;s,z) \\
& \ge \epsilon_2 e^{-\epsilon_0}\int_{\mathbb{R}^N}\Big[\phi(\mu,y)e^{-\mu[(y-z)\cdot\xi+s+\eta(\gamma,\mu,z)]}\cdot\sin\gamma[(y-z)\xi+s+\eta(\gamma,\mu,z)-\kappa(\mu,y)] \\
& \qquad \cdot\zeta(\|y-z\|/B)\Big]m(z;y,dy) \\
& = e^{-\epsilon_0}v(s,z)e^{-\mu\eta(\gamma,\mu,z)}\frac{\sec\gamma\tau(\gamma,z)}{\phi(\mu,z)}\int_{\mathbb{R}^N}\Big[\phi(\mu,y)e^{-\mu(y-z)\cdot\xi}\cdot\cos\gamma(-(y-z)\cdot\xi+\kappa(\mu,y)) \\
& \qquad \cdot\zeta(\|y-z\|/B)\Big]m(z;y,dy).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& \lim_{\gamma\to 0} e^{-\epsilon_0}e^{-\mu\eta(\gamma,\mu,z)}\frac{\sec\gamma\tau(\gamma,z)}{\phi(\mu,z)}\int_{\mathbb{R}^N}\Big[\phi(\mu,y)e^{-\mu(y-z)\cdot\xi}\cdot\cos\gamma(-(y-z)\cdot\xi+\kappa(\mu,y)) \\
& \qquad \cdot\zeta(\|y-z\|/B)\Big]m(z;y,dy) \\
& = e^{-\epsilon_0}e^{-\mu r_B^{'}(\mu)/r_B(\mu)}r_B(\mu) \\
& = e^{\lambda_B(\mu)-\mu r_B^{'}(\mu)/r_B(\mu)-\epsilon_0} \\
& > 1 \quad (\text{by} \quad (6.21)).
\end{aligned}
$$

It then follows that for $0 \le s - \kappa(\mu, z) \le \frac{\pi}{\gamma}$,

$$
u(1,0;v^*(\cdot;s,z),z) \ge v(s,z) = v^*((\kappa(\mu,z)-\tau(\gamma,z))\xi;s,(-k(\mu,z)+\tau(\gamma,z))\xi+z).
$$

Clearly, the above equality holds for all $s \in \mathbb{R}$.

Let $\bar{s}(x)$ be such that $v(\bar{s}(x),x) = \max_{s\in\mathbb{R}} v(s,x)$. Let

$$
\bar{v}(s,x) = \begin{cases} v(\bar{s}(x),x), & s \le \bar{s}(x) - \frac{\pi}{\gamma} \\ v(s+\frac{\pi}{\gamma},x), & s \ge \bar{s}(x) - \frac{\pi}{\gamma}. \end{cases}
$$

Set

$$\bar{v}^*(x; s, z) = \bar{v}(x \cdot \xi + s - \kappa(\mu, z) + \tau(\gamma, z), x + z).$$

We then have

$$u(1, 0; \bar{v}^*(\cdot; s, z), z) \geq \bar{v}(s, z) = \bar{v}^*((\kappa(\mu, z) - \tau(\gamma, z))\xi; s, (-\kappa(\mu, z) + \tau(\gamma, z))\xi + z)$$

for $s \in \mathbb{R}$ and $z \in \mathbb{R}^N$.

Let

$$v_0(x; z) = \bar{v}(x \cdot \xi, x + z).$$

Note that $\bar{v}(s, x)$ is non-increasing in $s$. Hence we have

$$\begin{aligned}
u(1, x; v_0(\cdot; z), z) &= u(1, 0; v_0(\cdot + x; z), x + z) \\
&= u(1, 0; \bar{v}^*(\cdot; x \cdot \xi + \kappa(\mu, x + z) - \tau(\gamma, x + z), x + z), x + z) \\
&\geq \bar{v}(x \cdot \xi + \kappa(\mu, x + z) - \tau(\gamma, x + z), x + z) \\
&\geq \bar{v}(x \cdot \xi - \lambda_B^{'}(\mu) + \epsilon_1, x + z) \quad (\text{by} \quad (6.22)) \\
&\geq \bar{v}\Big(x \cdot \xi - \frac{\lambda_B(\mu_B^*(\xi))}{\mu_B^*(\xi)} + 2\epsilon_1, x + z\Big) \quad (\text{by} \quad (6.20)) \\
&\geq \bar{v}\Big(x \cdot \xi - \frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} + 3\epsilon_1, x + z\Big) \quad (\text{by} \quad (6.19)) \\
&= v_0\Big(x - [\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} - 3\epsilon_1)]\xi, [\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} - 3\epsilon_1]\xi + z\Big)
\end{aligned}$$

for $z \in [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N]$. Let $\tilde{c}^*(\xi) = \frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} - 3\epsilon_1$. Then

$$u(1, x; v_0(\cdot, z), z) \geq v_0(x - \tilde{c}^*(\xi)\xi, \tilde{c}^*(\xi)\xi + z)$$

for all $z \in \mathbb{R}^N$. We also have

$$\begin{aligned} u(2, x; v_0(\cdot, z), z) &\geq u(1, x; v_0(\cdot - \tilde{c}^*(\xi)\xi, \tilde{c}^*(\xi)\xi + z), z) \\ &= u(1, x - \tilde{c}^*(\xi)\xi; v_0(\cdot, \tilde{c}^*(\xi)\xi + z), \tilde{c}^*(\xi)\xi + z) \\ &\geq v_0(x - 2\tilde{c}^*(\xi)\xi, 2\tilde{c}^*(\xi) + z) \end{aligned}$$

for all $z \in \mathbb{R}^N$. By induction, we have

$$u(n, x; v_0(\cdot, z), z) \geq v_0(x - n\tilde{c}^*(\xi)\xi, n\tilde{c}^*(\xi) + z)$$

for $n \geq 1$ and $z \in \mathbb{R}^N$. This together with Proposition 6.4 implies that

$$c^*(\xi) \geq \tilde{c}^*(\xi) = \frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} - 3\epsilon_1.$$

Since $\epsilon_1$ is arbitrary, we must have

$$c^*_{\inf}(\xi) \geq \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}. \tag{6.24}$$

By (6.16) and (6.24), we have $c^*(\xi)$ exists and $c^*(\xi) = \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}$.

(2) Let $D_{\mathbf{i}} = [i_1p_1, (i_1+1)p_1] \times [i_2p_2, (i_2+1)p_2] \times \ldots \times [i_Np_N, (i_N+1)p_N] (\mathbf{i} = (i_1, i_2, \cdots, i_N) \in \mathbb{Z}^N)$. Let $\lambda_1 = \lambda(\xi, \mu)$ and $\lambda_2 = \lambda(-\xi, \mu)$ be the principal eigenvalues of $\mathcal{K}_{\xi,\mu} - I + a(\cdot)I$ and $K_{-\xi,\mu} - I + a(\cdot)I$ with eigenfunctions $\psi_1, \psi_2 \in \text{Int}(X_p^+)$, respectively. It suffices to prove that $\lambda_1 = \lambda_2$. Observe that

$$\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)\psi_1(y)dy - \psi_1(x) + a(x)\psi_1(x) = \lambda_1\psi_1(x), \quad x \in \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} e^{\mu(y-x)\cdot\xi} k(y-x)\psi_2(y)dy - \psi_2(x) + a(x)\psi_2(x) = \lambda_2\psi_2(x), \quad x \in \mathbb{R}^N.$$

Multiplying the first equality by $\psi_2(x)$ and the second one by $\psi_1(x)$ and then integrating both equations over $D_{\mathbf{0}}$, we get

$$\int_{D_{\mathbf{0}}}[\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\psi_1(y)dy\psi_2(x)-\psi_1(x)\psi_2(x)+a(x)\psi_1(x)\psi_2(x)]dx = \lambda_1\int_{D_{\mathbf{0}}}\psi_1(x)\psi_2(x)dx$$

and

$$\int_{D_{\mathbf{0}}}[\int_{\mathbb{R}^N} e^{\mu(y-x)\cdot\xi}k(y-x)\psi_2(y)dy\psi_1(x)-\psi_2(x)\psi_1(x)+a(x)\psi_2(x)\psi_1(x)]dx = \lambda_2\int_{D_{\mathbf{0}}}\psi_2(x)\psi_1(x)dx.$$

Therefore, in order to derive $\lambda_1 = \lambda_2$, we only need to prove

$$\int_{D_{\mathbf{0}}}\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\psi_1(y)\psi_2(x)dydx = \int_{D_{\mathbf{0}}}\int_{\mathbb{R}^N} e^{\mu(y-x)\cdot\xi}k(y-x)\psi_2(y)\psi_1(x)dydx.$$

To this end, it suffices to prove that for each $\mathbf{i} = (i_1, i_2, \cdots, i_N) \in \mathbb{Z}^N$, one has

$$\int_{D_{\mathbf{0}}}\int_{D_{\mathbf{i}}} e^{-\mu(y-x)\cdot\xi}k(y-x)\psi_1(y)\psi_2(x)dydx = \int_{D_{\mathbf{0}}}\int_{D_{-\mathbf{i}}} e^{\mu(y-x)\cdot\xi}k(y-x)\psi_2(y)\psi_1(x)dydx.$$

For given $\mathbf{i} = (i_1, i_2, \cdots, i_N) \in \mathbb{Z}^N$, let $z_l = y_l - i_l p_l$ and $w_l = x_l - i_l p_l$, for $l = 1, 2, ..., N$. We have

$$\begin{aligned}
&\int_{D_{\mathbf{0}}}\int_{D_{\mathbf{i}}} e^{-\mu(y-x)\cdot\xi}k(y-x)\psi_1(y)\psi_2(x)dydx\\
&=\int_{D_{\mathbf{i}}}\int_{D_{\mathbf{0}}} e^{-\mu(y-x)\cdot\xi}k(y-x)\psi_1(y)\psi_2(x)dxdy\\
&=\int_{D_{\mathbf{0}}}\int_{D_{-\mathbf{i}}} e^{-\mu(z-w)\cdot\xi}k(z-w)\psi_1(z_1+i_1p_1, ..., z_N+i_Np_N)\psi_2(w_1+i_1p_1, ..., w_N+i_Np_N)dwdz\\
&=\int_{D_{\mathbf{0}}}\int_{D_{-\mathbf{i}}} e^{\mu(w-z)\cdot\xi}k(z-w)\psi_2(w)\psi_1(z)dwdz.
\end{aligned}$$

This proves (2).

(3) It follows from (1) and Proposition 6.2 (2).

(4) It follows from (1) and Proposition 6.3 (2). □

*Proof of Theorem E.* (1) For given $u_0$ in (1), there are $u_0^{\pm} \in X^+(\pm\xi)$ such that

$$u_0(\cdot) \leq u_0^{\pm}(\cdot).$$

By Proposition 6.3, for every $c > c^*(\xi)$,

$$\limsup_{x\cdot\xi \leq ct, t\to\infty} u(t,x;u_0^+,z) = 0, \qquad \limsup_{x\cdot(-\xi)\leq ct, t\to\infty} u(t,x;u_0^-,z) = 0$$

uniformly in $z \in \mathbb{R}^N$. By Proposition 3.1 and Proposition 3.2,

$$u(t,x;u_0,z) \leq u(t,x;u_0^{\pm},z) \quad \text{for} \quad t \geq 0,\ x, z \in \mathbb{R}^N.$$

It then follows that

$$\limsup_{|x\cdot\xi|\geq ct, t\to\infty} u(t,x;u_0,z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

(2) First, we claim that for each $\sigma > 0$, there is $r_\sigma > 0$ such that

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi|\leq ct} (u(t,x;u_0) - u^+(x)) = 0 \tag{6.25}$$

for every $u_0 \in X^+$ satisfying $u_0(x) \geq \sigma$ for all $x \in \mathbb{R}^N$ with $|x\cdot\xi| \leq r_\sigma$. By Proposition 3.1 and Proposition 3.2, we only need to consider $\sigma$ satisfying $0 < \sigma < u^+_{\inf}$.

Given $\xi \in S^{N-1}$, assume $0 < c < c^*(\xi)$. For $0 < \sigma < u^+_{\inf}$, let $\tilde{u}^\sigma(\cdot) \in C(\mathbb{R},\mathbb{R})$ be such that $\tilde{u}^\sigma(r) \geq 0$ for $r \in \mathbb{R}$ and

$$\tilde{u}^\sigma(r) = \begin{cases} \sigma, & r \leq 0 \\ 0, & r \geq 1. \end{cases}$$

Let

$$u^{\sigma,\pm\xi}(x) = \tilde{u}^\sigma(x\cdot(\pm\xi)).$$

By the definition of $c^*(\pm\xi)$,

$$\liminf_{x\cdot(\pm\xi)\le ct, t\to\infty}(u(t,x;u^{\sigma,\pm\xi},z)-u^+(x+z))=0 \quad \text{uniformly in} \quad z\in\mathbb{R}^N.$$

Take any $0<\tilde{c}<c$. For given $B>0$, let $\tilde{u}^\sigma_B\in C(\mathbb{R},\mathbb{R})$ be such that $\tilde{u}^\sigma_B(r)\ge 0$ and

$$\tilde{u}^\sigma_B(r)=\begin{cases}\tilde{u}^\sigma(r), & -B\le r\\ 0, & r\le -B-1.\end{cases}$$

Let

$$u^{\sigma,\pm\xi}_B(x)=\tilde{u}^\sigma_B(x\cdot(\pm\xi)).$$

Then

$$u(t,x;u^{\sigma,\pm\xi}_B,z)\to u(t,x;u^{\sigma,\pm\xi},z)$$

as $B\to\infty$ in open compact topology. This implies that there are $T>\frac{1}{c-\tilde{c}}$ and $B_0>0$ such that given $B\ge B_0$,

$$u(T,x;u^{\sigma,\pm\xi}_B,z)\ge\sigma$$

for $0\le x\cdot(\pm\xi)\le cT$, $\|x\|\le 2cT$, and $z\in\mathbb{R}^N$. Note that for each $x\in\mathbb{R}^N$ with $0\le x\cdot(\pm\xi)\le cT$, there is a vector $q$ such that $q\cdot\xi=0$ and $\|(x-q)\|\le 2cT$. It then follows that

$$u(T,x;u^{\sigma,\pm\xi}_B,z)=u(T,x-q;u^{\sigma,\pm\xi}_B,z+q)\ge\sigma$$

for $0\le x\cdot(\pm\xi)\le cT$, and $z\in\mathbb{R}^N$.

Let $r_\sigma>0$ be such that $r_\sigma>B_0+1$. Assume that $u_0\ge 0$ satisfies $u_0(x)\ge\sigma$ for $|x\cdot\xi|\le r_\sigma$. Then

$$u_0(\cdot\pm r\xi)\ge u^{\sigma,\pm\xi}_B(\cdot)\quad\text{for all } r \quad \text{with} \quad 0\le\pm r\le r_\sigma-1.$$

It then follows from the above arguments that

$$u(T, x; u_0, z) \geq \sigma \quad \text{for} \quad -r_\sigma - cT + 1 \leq x \cdot \xi \leq r_\sigma + cT - 1$$

for all $z \in \mathbb{R}^N$. This together with $T > \frac{1}{c-\tilde{c}}$ implies that

$$u(T, x; u_0, z) \geq \sigma \quad \text{for} \quad |x \cdot \xi| \leq r_\sigma + \tilde{c}T.$$

By induction, we have

$$u(nT, x; u_0, z) \geq \sigma \quad \text{for} \quad |x \cdot \xi| \leq r_\sigma + \tilde{c}nT,\ n = 1, 2, \cdots.$$

Then by Proposition 6.4, one obtains for each $0 < c^{'} < \tilde{c}$ that

$$\liminf_{|x \cdot \xi| \leq c^{'}t, t \to \infty} (u(t, x; u_0, z) - u^+(x + z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

By the arbitrariness of $c^{'}$ and $\tilde{c}$ with $0 < c^{'} < \tilde{c} < c(< c^*(\xi))$, we have

$$\liminf_{|x \cdot \xi| \leq ct, t \to \infty} (u(t, x; u_0, z) - u^+(x + z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

Next we claim that (2) can be proved by arguments similar to those in [41, Corollary 2.16]. In fact, let $\sigma > 0$ and $r > 0$ be given. Suppose that $u_0 \in X^+$ satisfies $u_0(x) \geq \sigma$ for all $x \in \mathbb{R}^N$ with $|x \cdot \xi| \leq r$. Note that there is $m > 0$ such that

$$-1 + f(x, u(t, x; u_0)) \geq -m \quad \forall x \in \mathbb{R}^N,\ t \geq 0.$$

Then

$$u_t(t, x; u_0) \geq \int_{\mathbb{R}^N} k(y - x) u(t, y; u_0) dy - m u(t, x; u_0)$$

and hence

$$(e^{mt}u(t,x;u_0))_t \geq \int_{\mathbb{R}^N} k(y-x)e^{mt}u(t,y;u_0)dy.$$

This together with Proposition 3.1 implies that

$$e^{mt}u(t,\cdot;u_0) \geq e^{Kt}u_0$$

where $e^{Kt} = I + Kt + \frac{K^2t^2}{2!} + \cdots$ and $Ku$ is defined as in (2.6) with $u \in X_p$ being replaced by $u \in X$. It is then not difficult to see that there is $\rho \in (0,1)$ such that

$$\rho\sigma < \inf_{x\in\mathbb{R}^N} u^+(x) \quad \text{and} \quad u(1,x;u_0) \geq \rho\sigma \quad \text{for} \quad |x\cdot\xi| \leq r_{\rho\sigma}.$$

Let $v_0(x) = \frac{1}{\rho}u(1,x;u_0)$. Then by (6.25),

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi|\leq ct} (u(t,x;v_0) - u^+(x)) = 0. \tag{6.26}$$

By (H2) and Proposition 3.1, we have

$$u(t+1,x;u_0) \equiv u(t,x;\rho v_0) \geq \rho u(t,x;v_0). \tag{6.27}$$

By (6.26) and (6.27), there is $T > 0$ such that

$$u(T,x;u_0) \geq \rho\sigma \quad \text{for} \quad |x\cdot\xi| \leq r_{\rho\sigma}. \tag{6.28}$$

By (6.25) and (6.28),

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi|\leq ct} (u(t+T,x;u_0) - u^+(x)) = 0. \tag{6.29}$$

(2) then follows from the arbitrariness of $c$ with $0 < c < \min\{c^*(\xi), c^*(-\xi)\}$.

□

*Proof of Theorem F.* (1) Fix $c > \sup_{\xi \in S^{N-1}} c^*(\xi)$.

First, let $u_0$ be as in (1). For every given $\xi \in S^{N-1}$, there is $\tilde{u}_0(\cdot;\xi) \in X^+(\xi)$ such that $u_0(\cdot) \le \tilde{u}_0(\cdot;\xi)$. Then by Proposition 3.1 and Proposition 3.2,

$$0 \le u(t, x; u_0, z) \le u(t, x; \tilde{u}_0(\cdot;\xi), z)$$

for $t > 0$, $x \in \mathbb{R}^N$, and $z \in \mathbb{R}^N$. It then follows from Proposition 6.3 that

$$0 \le \limsup_{x\cdot\xi \ge ct, t\to\infty} u(t, x; u_0, z) \le \limsup_{x\cdot\xi \ge ct, t\to\infty} u(t, x; \tilde{u}_0(\cdot;\xi), z) = 0$$

uniformly in $z \in \mathbb{R}^N$.

Take any $c^{'} > c$. Consider all $x \in \mathbb{R}^N$ with $\|x\| = c^{'}$. By the compactness of $\partial B(0, c^{'}) = \{x \in \mathbb{R}^N | \, \|x\| = c^{'}\}$, there are $\xi_1, \xi_2, \cdots, \xi_K \in S^{N-1}$ such that for every $x \in \partial B(0, c^{'})$, there is $k$ $(1 \le k \le K)$ such that $x \cdot \xi_k \ge c$. Hence for every $x \in \mathbb{R}^N$ with $\|x\| \ge c^{'}t$, there is $1 \le k \le K$ such that $x \cdot \xi_k = \frac{\|x\|}{c^{'}}\left(\frac{c^{'}}{\|x\|}x\right) \cdot \xi_k \ge \frac{\|x\|}{c^{'}} c \ge ct$. By the above arguments,

$$0 \le \limsup_{x\cdot\xi_k \ge ct, t\to\infty} u(t, x; u_0, z) \le \limsup_{x\cdot\xi_k \ge ct, t\to\infty} u(t, x; \tilde{u}_0(\cdot;\xi_k), z) = 0$$

uniformly for $z \in \mathbb{R}^N$, $k = 1, 2, \cdots K$. This implies that

$$\limsup_{\|x\| \ge c^{'}t, t\to\infty} u(t, x; u_0, z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

Since $c^{'} > c$ and $c > \sup_{\xi \in S^{N-1}} c^*(\xi)$ are arbitrary, we have that for $c > \sup_{\xi \in S^{N-1}} c^*_{\sup}(\xi)$,

$$\limsup_{\|x\| \ge ct, t\to\infty} u(t, x; u_0, z) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

(2) First of all, for given $x_0 \in \mathbb{R}^N$ and $r > 0$, let $B(x_0, r) = \{x \in \mathbb{R}^N \,|\, \|x - x_0\| < r\}$. Let $0 < \sigma < u^+_{\inf}$ and $v_0(s)$ be a smooth function satisfying that $v_0^{'}(s) \le 0$ for $s \in \mathbb{R}$, $v_0(s) = \sigma$

for $s \le -1$, and $v_0(s) = 0$ for $s \ge 0$. Let

$$\tilde{v}_0(s) = \begin{cases} v_0(s-2) & \text{for} \quad s \ge 0 \\ v_0(-s-2) & \text{for} \quad s \le 0. \end{cases}$$

For a given $B > 0$, let

$$u_0^B(x) = \begin{cases} \tilde{v}_0(\frac{\|x\|}{B}) & \text{for} \quad \|x\| \le B \\ \tilde{v}_0(1 + \|x\| - B) & \text{for} \quad \|x\| > B. \end{cases}$$

Fix $0 < c < \inf_{\xi \in S^{N-1}} c^*(\xi)$ and take any $c_1, c_2, c_3, c_4$ with $0 < c_4 < c_3 < c_2 < c_1 < c$. It then suffices to prove for $B \gg 1$ that

$$\liminf_{\|x\| \le c_4 t, t \to \infty} (u(t, x; u_0^B, z) - u^+(x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

To this end, first, for a given $\xi \in S^{N-1}$, let

$$u_0^\xi(x) = v_0(x \cdot \xi).$$

We claim that there is $T^* > \max\{\frac{1}{c-c_1}, \frac{1}{c_1-c_2}, \frac{1}{c_2-c_3}, \frac{1}{c_3-c_4}\}$ such that for every $\xi \in S^{N-1}$,

$$u(t, x; u_0^\xi(\cdot), z) > \sigma \quad \text{for} \quad t \ge T^*, \; x \cdot \xi \le c_3 t, \; z \in \mathbb{R}^N. \tag{6.30}$$

In fact, for every $\xi \in S^{N-1}$, by Proposition 6.2, there is $T(\xi) > 0$ such that

$$u(t, x; u_0^\xi(\cdot), z) > \sigma \quad \text{for} \quad t \ge T(\xi), \; x \cdot \xi \le ct, \; z \in \mathbb{R}^N.$$

In particular,

$$u(T(\xi), x; u_0^\xi(\cdot), z) > \sigma \quad \text{for} \quad x \cdot \xi \le cT(\xi), \; z \in \mathbb{R}^N.$$

Let $0 < \delta_0 < (c - c_1)T(\xi)$. Then

$$u(T(\xi), x; u_0^{\xi}(\cdot), z) > \sigma \quad \text{for} \quad x \in \mathrm{cl}\big(B(c_1 T(\xi)\xi, \delta_0)\big),\ z \in \mathbb{R}^N.$$

Note that for given $\rho > 0$, $\xi_n \in S^{N-1}$, and $z_n \in \mathbb{R}^N$ with $\xi_n \to \xi$ and $z_n \to z$,

$$\|u_0^{\xi_n}(z - z_n + \cdot) - u_0^{\xi}(\cdot)\|_{X(\rho)} \to 0$$

as $n \to \infty$. Observe also that

$$\begin{aligned} u(T(\xi), x; u_0^{\xi_n}, z_n) &= u(T(\xi), x; u_0^{\xi_n}, z + z_n - z) \\ &= u(T(\xi), x + z_n - z; u_0^{\xi_n}(z - z_n + \cdot), z). \end{aligned}$$

Then by Proposition 3.3,

$$u(T(\xi), x; u_0^{\xi_n}, z_n) \to u(T(\xi), x; u_0^{\xi}, z)$$

as $n \to \infty$ uniformly for $x$ in compact sets. This implies that there is $\delta_\xi > 0$ such that for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$,

$$c_1 T(\xi)\bar{\xi} \in B(c_1 T(\xi)\xi, \delta_0)$$

and

$$u(T(\xi), x; u_0^{\bar{\xi}}, z) > \sigma$$

for $x \in \mathrm{cl}\big(B(c_1 T(\xi)\xi, \delta_0)\big)$ and $z \in \mathbb{R}^N$. Hence for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$,

$$u(T(\xi), c_1 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}, z) > \sigma \quad \text{for} \quad z \in \mathbb{R}^N. \tag{6.31}$$

Observe that

$$u(T(\xi), \eta + c_1 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}, z) = u(T(\xi), c_1 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}(\cdot + \eta), z + \eta)$$
$$= u(T(\xi), c_1 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}(\cdot), z + \eta)$$

for all $\eta \in \mathbb{R}^N$ with $\eta \cdot \bar{\xi} = 0$. Then by (6.31), it follows for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$ that

$$U(T(\xi), x; u_0^{\bar{\xi}}, z) > \sigma \quad \text{for} \quad x \cdot \bar{\xi} = c_1 T(\xi),\ z \in \mathbb{R}^N. \tag{6.32}$$

Observe also that for each $x \in \mathbb{R}^N$ with $x \cdot \bar{\xi} \leq c_1 T(\xi)$, there is $x^{'} \in \mathbb{R}^N$ such that $x^{'} \cdot \bar{\xi} \geq 0$, $(x + x^{'}) \cdot \bar{\xi} = c_1 T(\xi)$, and

$$u_0^{\bar{\xi}}(\cdot - x^{'}) \geq u_0^{\bar{\xi}}(\cdot).$$

Then by (6.32), one has for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$ that

$$u(T(\xi), x; u_0^{\bar{\xi}}, z) = u(T(\xi), x + x^{'}; u_0^{\bar{\xi}}(\cdot - x^{'}), z - x^{'}) > \sigma \quad \text{for} \quad x \cdot \bar{\xi} \leq c_1 T(\xi). \tag{6.33}$$

Therefore, for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$,

$$u(T(\xi), x + c_2 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}, z) > \sigma \quad \text{for} \quad x \cdot \bar{\xi} \leq (c_1 - c_2) T(\xi).$$

This implies that

$$U(T(\xi), \cdot + c_2 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}, z) \geq u_0^{\bar{\xi}}(\cdot)$$

for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$. It then follows by induction, from Proposition 3.1 and Proposition 3.2 that

$$u(nT(\xi), \cdot + nc_2 T(\xi)\bar{\xi}; u_0^{\bar{\xi}}, z) \geq u_0^{\bar{\xi}}(\cdot)$$

for $n = 1, 2, \cdots$ and $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$.

By the arguments of Proposition 6.4 (1), there is $T^*(\xi)$ such that for $\bar{\xi} \in B(\xi, \delta_\xi) \cap S^{N-1}$,

$$u(t, x; u_0^{\bar{\xi}}, z) > \sigma \quad \text{for} \quad t \geq T^*(\xi),\ x \cdot \bar{\xi} \leq c_3 t,\ z \in \mathbb{R}^N.$$

Then by the compactness of $S^{N-1}$, there is $T^*$ such that (6.30) holds for every $\xi \in S^{N-1}$. This proves the claim.

Now given $\xi \in S^{N-1}$ and $B > c_3 T^*$, let

$$u_0^{B,\xi}(x) = u_0^B(x + (B+1)\xi),$$

$$\bar{u}_0^{B,\xi}(x) = u_0^B(x - c_3 T^* \xi),$$

and

$$\tilde{u}_0^{B,\xi}(x) = \min\{\bar{u}_0^{B,\xi}(x), u_0^{B,\xi}(x)\}.$$

Then for every $\rho > 0$,

$$\|\tilde{u}_0^{B,\xi}(x) - u_0^{\xi}(x)\|_{X(\rho)} \to 0$$

as $B \to \infty$. Hence

$$u(T^*, x; \tilde{u}_0^{B,\xi}, z) \to u(T^*, x; u_0^{\xi}, z)$$

as $B \to \infty$ uniformly for $x$ in bounded sets and $z \in \mathbb{R}^N$. By (6.30) and arguments similar to those in (6.31), there is $B^+(\xi)$ and $\tilde{\delta}_\xi^+ > 0$ such that for $B \geq B^+(\xi) > c_3 T^*$ and $\tilde{\xi} \in B(\xi, \tilde{\delta}_\xi^+) \cap S^{N-1}$,

$$u(T^*, c_3 \tilde{\xi} T^*; \tilde{u}_0^{B,\tilde{\xi}}, z) > \sigma$$

for $z \in \mathbb{R}^N$.

Observe that for every $\beta \in [-c_3 T^*, B+1]$ and $\tilde{\xi} \in B(\xi, \tilde{\delta}_\xi^+) \cap S^{N-1}$,

$$u_0^B(\cdot + \beta\tilde{\xi}) \geq \tilde{u}_0^{B,\tilde{\xi}}(\cdot)$$

and

$$u(T^*, c_3\tilde{\xi}T^* + \beta\tilde{\xi}; u_0^B, z) = u(t, c_3\tilde{\xi}T^*; u_0^B(\cdot + \beta\tilde{\xi}), z + \beta\tilde{\xi}).$$

It then follows that

$$u(T^*, \tilde{\beta}\xi; u_0^B, z) > \sigma \quad \text{for} \quad 0 \le \tilde{\beta} \le B + 1 + c_3T^*.$$

Similarly, there is $B^-(\xi) > c_3T^*$ and $\tilde{\delta}_\xi^-$ such that for $B > B^-(\xi)$ and $\tilde{\xi} \in B(\xi, \tilde{\delta}_\xi^-) \cap S^{N-1}$,

$$u(T^*, \tilde{\beta}\xi; u_0^B, z) > \sigma \quad \text{for} \quad -B - 1 - c_3T^* \le \tilde{\beta} \le 0.$$

Let $B(\xi) = \max\{B^+(\xi), B^-(\xi)\}$ and $\tilde{\delta}_\xi = \min\{\tilde{\delta}_\xi^-, \tilde{\delta}_\xi^+\}$. Then we have that for every $\tilde{\xi} \in B(\xi, \tilde{\delta}_\xi) \cap S^{N-1}$,

$$u(T^*, \beta\tilde{\xi}; u_0^B, z) > \sigma \quad \text{for} \quad -B - 1 - c_3T^* \le \beta \le B + 1 + c_3T^*.$$

By the compactness of $S^{N-1}$, there is $B^* > c_3T^*$ such that

$$u(T^*, x; u_0^B, z) > \sigma \quad \text{for} \quad \|x\| \le B + 1 + c_3T^*,\ z \in \mathbb{R}^N.$$

Hence for $B \ge B^*$,

$$u(T^*, \cdot; u_0^B, z) \ge u_0^{B+c_3T^*}(\cdot) \quad \text{for} \quad z \in \mathbb{R}^N.$$

By induction, Proposition 3.1 and Proposition 3.2, one obtains for $B \ge B^*$ that

$$u(nT^*, \cdot; u_0^B, z) \ge u_0^{B+c_3nT^*}(\cdot) \quad \text{for} \quad z \in \mathbb{R}^N.$$

This together with Proposition 6.4 (3) implies that for every $B \ge B^*$,

$$\liminf_{\|x\| \le c_4t, t\to\infty} (u(t, x; u_0^B, z) - u^+(x + z)) = 0$$

uniformly in $z \in \mathbb{R}^N$.

Finally, we can prove that $r$ can be chosen to be independent of $\sigma$ by arguments similar to those in Theorem E(2). This completes the proof. $\square$

## 6.3 Spreading Speeds in the General Case

In this section, we investigate the existence and characterization of the spreading speeds of (1.2) without the assumption (H4).

**Lemma 6.1.** *Assume (H1) - (H3). For every $\xi \in S^{N-1}$, there is $\mu^*(\xi) \in (0,\infty)$ such that*

$$\frac{\lambda_0(\xi, \mu^*(\xi), a_0)}{\mu^*(\xi)} = \inf_{\mu>0} \frac{\lambda_0(\xi, \mu, a_0)}{\mu}.$$

*Proof.* First, it is not difficult to see that $\lambda_0(\xi,\mu,a_0)$ is continuous in $\mu$. By (H2), $\lambda_0(\xi,0,a_0) > 0$ and hence $\lim_{\mu\to 0+} \frac{\lambda_0(\xi,\mu,a_0)}{\mu} = \infty$. By Theorem 4.1 and Theorem B, $\lim_{\mu\to\infty} \frac{\lambda_0(\xi,\mu,a_0)}{\mu} = \infty$. The lemma then follows. $\square$

*Proof of Theorem D.* (1) First, we prove that $c^*_{\sup}(\xi) \le \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}$.

Let $a^n(\cdot) \in C^N(\mathbb{R}^N) \cap X_p$ be such that $a^n$ satisfies (H5),

$$a^n \ge a_0 \quad \text{for} \quad n \ge 1 \quad \text{and} \quad \|a^n - a\|_{X_p} \to 0 \quad \text{as} \quad n \to \infty.$$

Then by Lemma 5.1,

$$\lambda_0(\xi,\mu,a^n) \to \lambda_0(\xi,\mu,a_0) \quad \text{as} \quad n \to \infty.$$

Let $\phi^n$ be the positive eigenfunction of $K_{\xi,\mu} - I + a^n(\cdot)I$ corresponding to $\lambda(\xi,\mu,a^n) = \lambda_0(\xi,\mu,a^n)$ with $\|\phi^n\|_{X_p} = 1$. Note that

$$uf(x,u) \le uf(x,0) \le a^n(x)u \quad \text{for} \quad x \in \mathbb{R}^N,\ u \ge 0$$

and

$$\big(\Phi(t,\xi,0,a^n)u_{\xi,\mu}\big)(x)=e^{-\mu(x\cdot\xi-\frac{\lambda_0(\xi,\mu,a^n)}{\mu}t)}\phi^n(x),$$

where $u_{\xi,\mu}(x)=e^{-\mu x\cdot\xi}\phi^n(x)$. Hence by Proposition 3.1 and Proposition 3.2, for any $\mu>0$,

$$u(t,x;u_{\xi,\mu})\le e^{-\mu(x\cdot\xi-\frac{\lambda_0(\xi,\mu,a^n)}{\mu}t)}\phi^n(x)\quad\text{for}\quad t\ge 0.$$

This implies that

$$c^*_{\sup}(\xi)\le\frac{\lambda_0(\xi,\mu,a^n)}{\mu}\quad\forall\ \mu>0,\ n\ge 1$$

and then

$$c^*_{\sup}(\xi)\le\frac{\lambda_0(\xi,\mu,a_0)}{\mu}\quad\forall\mu>0.$$

Therefore,

$$c^*_{\sup}(\xi)\le\inf_{\mu>0}\frac{\lambda_0(\xi,\mu,a_0)}{\mu}.\tag{6.34}$$

Next, we prove $c^*_{\inf}(\xi)\ge\inf_{\mu>0}\frac{\lambda_0(\xi,\mu,a_0)}{\mu}$. For any $\epsilon>0$, there is $\delta_0>0$ such that such that

$$f(x,u)\ge f(x,0)-\epsilon\quad\text{for}\quad x\in\mathbb{R}^N,\ 0<u<\delta_0.$$

Let $a_n(\cdot)\in C^N(\mathbb{R}^N)\cap X_p$ be such that $a_n$ satisfies (H5),

$$f(\cdot,0)-2\epsilon\le a_n(\cdot)\le f(\cdot,0)-\epsilon\quad\forall n\ge 1.$$

Let

$$c^*_n(\xi)=\inf_{\mu>0}\frac{\lambda(\xi,\mu,a_n)}{\mu}.$$

Applying the arguments in Theorem D, there is $u_0(\cdot;z)\in X^+(\xi)$ such that

$$\liminf_{x\cdot\xi\to-\infty}\inf_{z\in\mathbb{R}^N}u_0(x;z)>0$$

and

$$u(1,x;u_0(\cdot;z),z) \geq u_0(x-(c_n^*(\xi)-\epsilon)\xi;(c_n^*(\xi)-\epsilon)\xi+z) \quad \forall z \in \mathbb{R}^N.$$

This implies that

$$u(m,x;u_0(\cdot;z),z) \geq u_0(x-m(c_n^*(\xi)-\epsilon)\xi;m(c_n^*(\xi)-\epsilon)\xi+z) \quad \forall m \geq 1,\ z \in \mathbb{R}^N.$$

Then by Proposition 6.4,

$$c_{\inf}^*(\xi) \geq c_n^*(\xi)-\epsilon.$$

By Lemma 5.1,

$$c_{\inf}^*(\xi) \geq \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)-2\epsilon}{\mu} - \epsilon.$$

Letting $\epsilon \to 0$, by Lemma 6.1, we have

$$c_{\inf}^*(\xi) \geq \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}. \tag{6.35}$$

By (6.34) and (6.35),

$$c_{\sup}^*(\xi) = c_{\inf}^*(\xi) = \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}.$$

Hence $c^*(\xi)$ exists and

$$c^*(\xi) = \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}.$$

(2) (3) (4) They can be proved by arguments similar to those in last section. □

*Proof of Theorem E.* (1) Fix $c > \max\{c^*(\xi), c^*(-\xi)\}$. As in the proof of Theorem D (1), let $a^n(\cdot) \in C^N(\mathbb{R}^N) \cap X_p$ be such that $a^n$ satisfies (H5),

$$a^n \geq a_0 \quad \text{for} \quad n \geq 1 \quad \text{and} \quad \|a^n - a\|_{X_p} \to 0 \quad \text{as} \quad n \to \infty.$$

Choose $\mu > 0$ and $n \gg 1$ such that

$$\frac{\lambda_0(\xi, \mu, a^n)}{\mu} < c.$$

Choose $M > 1$ such that

$$u_0(x) \leq M e^{-\mu x\cdot\xi} \phi^n(x),$$

where $\phi^n(x)$ is the positive eigenfunction of $K_{\xi,\mu} - I + a^n(\cdot)I$ corresponding to $\lambda(\xi, \mu, a^n) = \lambda_0(\xi, \mu, a^n)$ with $\|\phi^n\|_{X_p} = 1$. By arguments similar to those in Theorem E (1),

$$u(t, x; u_0) \leq e^{-\mu(x\cdot\xi - \frac{\lambda_0(\xi,\mu,a^n)}{\mu}t)} \phi^n(x) \quad \text{for} \quad t \geq 0.$$

This implies that

$$\limsup_{t\to\infty} \sup_{x\cdot\xi\geq ct} u(t, x; u_0) = 0. \tag{6.36}$$

Similarly, it can be proved that

$$\limsup_{t\to\infty} \sup_{x\cdot\xi\leq -ct} u(t, x; u_0) = 0. \tag{6.37}$$

(1) thus follows from (6.36) and (6.37).

(2) It follows from arguments in last section. □

*Proof of Theorem F.* (1) It can be proved by arguments similar to those in last section.

(2) It can be proved by arguments similar to those in last section. □

### 6.4 Effects of Spatial Variations on Spreading Speeds

In this section, we will investigate the effects of spatial variations on spreading speeds and prove the Theorem G. Let

$$\bar{f}(u) = \frac{1}{|D|} \int_D f(x, u) dx.$$

We assume that $\bar{f}$ satisfy $\bar{f}(0) > 0$ .

Then $\bar{f}$ is also monostable type functions. Let $c^*(\xi, \bar{f})$ be the spreading speeds of the following averaged equations of (6.38),

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y - x)u(t, y)dy - u(t, x) + u\bar{f}(u). \tag{6.38}$$

*Proof of Theorem G.* First, let $a_0(x) = f(x, 0)$, and by Theorem D, for any $\xi \in S^{N-1}$, there is $\mu^*(\xi) > 0$ such that

$$c^*(\xi, f) = \frac{\lambda_0(\mu^*(\xi), \xi, a_0(x))}{\mu^*(\xi)}.$$

Then by Theorem B, Lemma 5.2 and Theorem D,

$$c^*(\xi, f) = \frac{\lambda_0(\mu^*(\xi), \xi, a_0(x))}{\mu^*(\xi)} \geq \frac{\lambda(\mu^*(\xi), \xi, \bar{a})}{\mu^*(\xi)} \geq c^*(\xi, \bar{f}).$$

Now assuming (H4), for some $\xi \in S^{N-1}$, $c^*(\xi, f) = c^*(\xi, \bar{f})$, then we must have

$$\frac{\lambda(\mu^*(\xi), \xi, a_0(x))}{\mu^*(\xi)} = \frac{\lambda(\mu^*(\xi), \xi, \bar{a})}{\mu^*(\xi)}.$$

By Theorem B again, we must have $a_0(x) \equiv \bar{a}$.

□

Theorem G shows that it is a generic scenario that spatial variation increases the spreading speed.

Chapter 7

Traveling Wave Solutions of Spatially Periodic Nonlocal Monostable Equations

In this chapter, we explore the traveling wave solutions of (1.2) and prove Theorems H-J. To this end, we first construct some sub- and super-solutions to be used in the proofs of the main results. We then study the existence, uniqueness, and stability of traveling wave solutions of (1.2). The results of this chapter have been submitted for publication (see [58]).

Throughout this chapter, we assume that (H1)-(H4). Biologically, we are only interested in nonnegative solutions of (1.2). Hence, without loss of generality, we make the following technical assumption throughout this chapter, $f(x,u) = f(x,0)$ *for* $u \le 0$.

## 7.1 Sub- and Super-solutions

Let $a_0(x) = f(x,0)$.

For given $\xi \in S^{N-1}$, let $\mu^*(\xi)$ be such that

$$c^*(\xi) = \frac{\lambda_0(\xi, \mu^*(\xi), a_0)}{\mu^*(\xi)}.$$

Fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let $0 < \mu < \mu_1 < \min\{2\mu, \mu^*(\xi)\}$ be such that $c = \frac{\lambda_0(\xi,\mu,a_0)}{\mu}$ and $\frac{\lambda_0(\xi,\mu,a_0)}{\mu} > \frac{\lambda_0(\xi,\mu_1,a_0)}{\mu_1} > c^*(\xi)$. Let $\phi(\cdot)$ and $\phi_1(\cdot)$ be positive eigenfunctions of $\mathcal{K}_{\xi,\mu} - I + a_0(\cdot)I$ associated to $\lambda_0(\xi,\mu,a_0)$ and $\lambda_0(\xi,\mu_1,a_0)$ with $\|\phi(\cdot)\|_{X_p} = 1$ and $\|\phi_1(\cdot)\|_{X_p} = 1$, respectively. If no confusion occurs, we may write $\lambda_0(\xi,\mu,a_0)$ as $\lambda(\mu)$.

For given $d_1 > 0$, let

$$\underline{v}^1(t,x;z,T,d_1) = e^{-\mu(x\cdot\xi + cT - ct)}\phi(x+z) - d_1 e^{-\mu_1(x\cdot\xi + cT - ct)}\phi_1(x+z). \tag{7.1}$$

We may write $\underline{v}^1(t,x;z,T)$ for $\underline{v}^1(t,x;z,T,d_1)$ for fixed $d_1 > 0$ or if no confusion occurs.

**Proposition 7.1.** *For any $z \in \mathbb{R}^N$ and $T > 0$, $\underline{v}^1(t, x; z, T)$ is a sub-solution of* (2.15) *provided that $d_1$ is sufficiently large.*

*Proof.* First of all, let $\varphi = e^{-\mu(x\cdot\xi + cT - ct)}\phi(x + z)$ and $\varphi_1 = d_1 e^{-\mu_1(x\cdot\xi + cT - ct)}\phi_1(x + z)$. Let $M = \max_{x\in\mathbb{R}^N} \phi(x)(> 0)$. Let $L > 0$ be such that $-f_u(x + z, u) \leq L$ for $0 \leq u \leq M$. Let $d_0$ be defined by

$$d_0 = \max\{\frac{\max_{x\in\mathbb{R}^N} \phi(x)}{\min_{x\in\mathbb{R}^N} \phi_1(x)}, \frac{L \max_{x\in\mathbb{R}^N} \phi^2(x)}{(\mu_1 c - \lambda(\mu_1)) \min_{x\in\mathbb{R}^N} \phi_1(x)}\}$$

Fix $z \in \mathbb{R}^N$ and $T > 0$. We prove that $\underline{v}^1(t, x; z, T)$ is a sub-solution of (2.15) for $d_1 \geq d_0$, that is, for any $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,

$$\frac{\partial \underline{v}^1}{\partial t} - [\int_{\mathbb{R}^N} k(y - x)\underline{v}^1(t, y; z, T)dy - \underline{v}^1(t, x; z, T) + f(x + z, \underline{v}^1(t, x; z, T))\underline{v}^1(t, x; z, T)] \leq 0. \tag{7.2}$$

First, for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) \leq 0$, $f(x + z, \underline{v}^1(t, x; z, T)) = f(x + z, 0)$. Hence

$$\begin{aligned}&\frac{\partial \underline{v}^1}{\partial t} - [\int_{\mathbb{R}^N} k(y - x)\underline{v}^1(t, y; z, T)dy - \underline{v}^1(t, x; z, T) + f(x + z, \underline{v}^1(t, x; z, T))\underline{v}^1(t, x; z, T)]\\ &= -(\mu_1 c - \lambda(\mu_1))\varphi_1 \leq 0.\end{aligned}$$

Therefore (7.2) holds for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) \leq 0$.

Next, consider $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ with $\underline{v}^1(t, x; z, T) > 0$. By $d_1 \geq d_0$, we must have $x \cdot \xi + cT - ct \geq 0$. Then $\underline{v}^1(t, x; z, T) \leq e^{-\mu(x\cdot\xi+cT-ct)}\phi(x + z) \leq \phi(x + z) \leq M$. Note that for $0 < y < M$,

$$
\begin{aligned}
-(\mu_1 c-\lambda(\mu_1)) - f_u(x+z,y)\frac{(\varphi)^2}{\varphi_1} &\le -(\mu_1 c-\lambda(\mu_1)) + L\frac{(\varphi)^2}{\varphi_1}\\
&= -(\mu_1 c-\lambda(\mu_1)) + \frac{L\phi^2(x+z)}{d_1\phi_1(x+z)}e^{(\mu_1-2\mu)(x\cdot\xi+cT-ct)}\\
&\le -(\mu_1 c-\lambda(\mu_1)) + \frac{L\max\limits_{y\in\mathbb{R}^N}\phi^2(y)}{d_1\max\limits_{y\in\mathbb{R}^N}\phi_1(y)}\\
&\le 0.
\end{aligned}
$$

Therefore, for $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ with $\underline{v}^1(t,x;z,T)>0$,

$$
\begin{aligned}
&\frac{\partial \underline{v}^1}{\partial t} - [\int_{\mathbb{R}^N} k(y-x)\underline{v}^1(t,y;z,T)dy - \underline{v}^1(t,x;z,T) + f(x+z,\underline{v}^1)\underline{v}^1(t,x;z,T)]\\
=&\mu c\varphi - \mu_1 c\varphi_1 - [\int_{\mathbb{R}^N} k(y-x)\underline{v}^1(t,y;z,T)dy - \underline{v}^1(t,x;z,T) + f(x+z,\underline{v}^1)\underline{v}^1(t,x;z,T)]\\
=&(\mu c-\lambda(\mu))\varphi - (\mu_1 c-\lambda(\mu_1))\varphi_1 + f(x+z,0)\underline{v}^1(t,x;z,T) - f(x+z,\underline{v}^1)\underline{v}^1(t,x;z,T)\\
=&-(\mu_1 c-\lambda(\mu_1))\varphi_1 - f_u(x+z,y)(\varphi-\varphi_1)^2 \qquad \big(\text{for some } y\in(0,M)\big)\\
\le&-(\mu_1 c-\lambda(\mu_1))\varphi_1 - f_u(x+z,y)(\varphi)^2\\
=&[-(\mu_1 c-\lambda(\mu_1)) - f_u(x+z,y)\frac{(\varphi)^2}{\varphi_1}]\varphi_1\\
\le&0.
\end{aligned}
$$

Hence (7.2) also holds for $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ with $\underline{v}^1(t,x;z,T)>0$. The proposition then follows. □

Let $\lambda(0)$ be the principal eigenvalue and $\phi_0$ be the positive principal eigenfunction of $\mathcal{K}-I+a_0(\cdot)I$ with $\|\phi_0\|_{X_p}=1$. Observe that there exists sufficiently large $M>0$ such that $\underline{v}^1(t,x_0;z,T)\ge \frac{1}{2}e^{-\mu(x\cdot\xi+cT-ct)}\phi(x+z)$ for $x\cdot\xi+cT-ct>M$. Thus we have $\underline{v}^1(t,x_0;z,T)\ge \frac{1}{2}e^{-\mu(x\cdot\xi+cT-ct)}\min\limits_{x\in\mathbb{R}^N}\{\phi(x)\}$ for $x\cdot\xi+cT-ct>M$. For any $\delta_0>0$, let $M_1$ be such that $M_1-M>\delta_0$ and $\hat{b}=\frac{1}{2}e^{-\mu M_1}\min\limits_{x}\phi(x)$. Then we have $\underline{v}^1(t,x_0;z,T)\ge \frac{1}{2}e^{-\mu(x\cdot\xi+cT-ct)}\min\limits_{x\in\mathbb{R}^N}\{\phi(x)\}\ge\hat{b}$ for any $M<x\cdot\xi+cT-ct<M_1$. Let $0<b\ll 1$, such that

$b \max_{x\in\mathbb{R}^N} \phi_0(x) < \hat{b}$ and

$$-f_2(x+z,\eta)(b\phi_0(x+z)) < \frac{1}{2}\lambda(0), \tag{7.3}$$

where $f_2(\cdot,\cdot)$ is the partial derivative of $f(x,u)$ with respect to the second coordinate and $\eta$ is such that $f(x+z,0) - f(x+z,b\phi_0(x+z)) = -f_2(x+z,\eta)(b\phi_0(x+z))$. Therefore, $\underline{v}^1(t,x;z,T) > b\phi_0(x+z)$ for any $M < x\cdot\xi + cT - ct < M_1$.

**Proposition 7.2.** *Let $\delta_0$ be such that $\int_{\|z\|>\delta_0} k(z)dz \le \hat{\sigma} := \frac{1}{2}\lambda(0)\min_{x\in\mathbb{R}^N}\{\phi_0(x)\}/\max_{x\in\mathbb{R}^N}\{\phi_0(x)\}$. Let $0 < b \ll 1$ and $M_1 > M > 0$ be such that $\underline{v}^1(t,x_0;z,T) > b\phi_0(x+z)$ for any $M < x\cdot\xi + cT - ct < M_1$ with $M_1 - M > \delta_0$ and (7.3) holds. Let*

$$\underline{u}(t,x;z,T,d_1,b) = \begin{cases} \max\{b\phi_0(x+z), \underline{v}^1(t,x;z,T,d_1)\} & \text{for} \quad x\cdot\xi + cT - ct < M \\ \underline{v}^1(t,x;z,T,d_1) & \text{for} \quad x\cdot\xi + cT - ct \ge M. \end{cases}$$

*Then $\underline{u}(t,x;z,T,d_1,b)$ is a sub-solution of (2.15).*

*Proof.* First, it is not difficult to see that for any $x,z\in\mathbb{R}^N$, there are at most two $t$s such that $b\phi_0(x+z) = \underline{v}^1(t,x;z,T)$. Hence for any fixed $x,z\in\mathbb{R}^N$, $\underline{u}(t,x;z,T)(:= \underline{u}(t,x;z,t,d_1,b))$ is absolutely continuous in $t$ and is differentiable in $t$ for a.e. $t$. Moreover, we claim that for any $t$ at which $\underline{u}(t,x;z,T)$ is differentiable, there holds

$$\frac{\partial \underline{u}(t,x;z,T)}{\partial t} \le \int_{\mathbb{R}^N} k(y-x)\underline{u}(t,y;z,T)dy - \underline{u}(t,x;z,T) + \underline{u}(t,x;z,T)f(x+z,\underline{u}(t,x;z,T)).$$

By observation, $\underline{u}(t,x;z,T,d_1,b) \ge \underline{v}^1(t,x;z,T,d_1)$ for all $x\in\mathbb{R}^N$. If $\underline{u}(t,x;z,T,d_1,b) = \underline{v}^1(t,x;z,T,d_1)$, it is easy to verify that,

$$\frac{\partial \underline{u}(t,x;z,T)}{\partial t} \le \int_{\mathbb{R}^N} k(y-x)\underline{u}(t,y;z,T)dy - \underline{u}(t,x;z,T) + \underline{u}(t,x;z,T)f(x+z,\underline{u}(t,x;z,T)). \tag{7.4}$$

Otherwise, $\underline{u}(t,x;z,T,d_1,b) = b\phi_0(x+z)$ for $x\in D_0$ , where $D_0 := \{x|\underline{u}(t,x;z,T,d_1,b) = b\phi_0(x+z), \quad x\cdot\xi + cT - ct \le M\}$. Note that $b\phi_0(x+z) \le \underline{u}(t,x;z,T,d_1,b)$ for $x\in D_1 :=$

$\{x|x \cdot \xi + cT - ct \leq M_1\}$. Thus,

$$\int_{D_1} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy \leq 0. \tag{7.5}$$

Let $D_2 := \{x|x \cdot \xi + cT - ct > M_1\}$. Note that $D_0 \subset \{x|x \cdot \xi + cT - ct \leq M\}$. If $x \in D_0$ and $y \in D_2$, then $\|y-x\| \geq (y-x)\cdot\xi \geq M_1 - M \geq \delta_0$, which implies $\{y|y \in D_2, x \in D_0\} \subset \{y|\|y-x\| > \delta_0\}$. Thus,

$$\begin{aligned}
&\int_{D_2} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy \\
\leq &\int_{\|y-x\|>\delta_0} k(y-x)b\phi_0(y+z)dy \\
\leq &\int_{\|z\|>\delta_0} k(z)\max_{x\in\mathbb{R}^N}\{b\phi_0(x)\}dy \\
\leq &\hat{\sigma}\max_{x\in\mathbb{R}^N}\{b\phi_0(x)\} \\
= &\frac{1}{2}\lambda(0)b\min_{x\in\mathbb{R}^N}\{\phi_0(x)\}
\end{aligned}$$

Thus, for $x \in D_0$,

$$\int_{D_2} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy \leq \frac{1}{2}\lambda(0)b\phi_0(y+z). \tag{7.6}$$

If $x \in D_0$, we have

$$
\begin{aligned}
&\frac{\partial \underline{u}(t,x;z,T,d_1,b)}{\partial t} - [\int_{\mathbb{R}^N} k(y-x)\underline{u}(t,y;z,T,d_1,b)dy - \underline{u}(t,x;z,T,d_1,b) \\
&\quad + f(x+z,\underline{u}(t,x;z,T,d_1,b))\underline{u}(t,x;z,T,d_1,b)] \\
=& -[\int_{\mathbb{R}^N} k(y-x)\underline{u}(t,y;z,T,d_1,b)dy - b\phi_0(x+z) + f(x+z,b\phi_0(x+z))b\phi_0(x+z)] \\
=& -[\int_{\mathbb{R}^N} k(y-x)b\phi_0(y+z)dy - b\phi_0(x+z) + f(x+z,0)b\phi_0(x+z)] \\
&\quad + \int_{\mathbb{R}^N} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy \\
&\quad + [f(x+z,0)b\phi_0(x+z) - f(x+z,b\phi_0(x+z))b\phi_0(x+z)] \\
=& -\lambda(0)b\phi_0(x+z) + \int_{\mathbb{R}^N} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy - f_2(x+z,\eta)(b\phi_0(x+z))^2 \\
=& -\lambda(0)b\phi_0(x+z) + \int_{D_1} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy \\
&\quad + \int_{D_2} k(y-x)[b\phi_0(y+z) - \underline{u}(t,y;z,T,d_1,b)]dy - f_2(x+z,\eta)(b\phi_0(x+z))^2.
\end{aligned}
$$

Together with the inequalities (7.3), (7.5) and (7.6), we have for $x \in D_0$,

$$
\frac{\partial \underline{u}(t,x;z,T)}{\partial t} \le \int_{\mathbb{R}^N} k(y-x)\underline{u}(t,y;z,T)dy - \underline{u}(t,x;z,T) + \underline{u}(t,x;z,T)f(x+z,\underline{u}(t,x;z,T)). \tag{7.7}
$$

By (7.4) and (7.7), we proved the claim.

Therefore, $\underline{u}(t,x;z,T)$ is a sub-solution of (2.15). □

For given $d_2 \ge 0$, let

$$
\bar{v}(t,x;z,T,d_2) = e^{-\mu(x\cdot\xi+cT-ct)}\phi(x+z) + d_2 e^{-\mu_1(x\cdot\xi+cT-ct)}\phi_1(x+z)
$$

and

$$
\bar{u}(t,x;z,T,d_2) = \min\{\bar{v}(t,x;z,T,d_2), u^+(x+z)\}.
$$

We may write $\bar{v}(t,x;z,T)$ and $\bar{u}(t,x;z,T)$ for $\bar{v}(t,x;z,T,d_2)$ and $\bar{u}(t,x;z,T,d_2)$, respectively, if no confusion occurs.

**Proposition 7.3.** *For any $d_2 \ge 0$, $z \in \mathbb{R}^N$, and $T > 0$, $\bar{u}(t,x;z,T)$ is a super-solution of (2.15).*

*Proof.* It suffices to prove that $\bar{v}(t, x; z, T)$ is a super-solution.

Let $\varphi_2 = d_2 e^{-\mu_1(x\cdot\xi + cT - ct)}\phi_1(x+z)$. By direct calculation, we have

$$
\begin{aligned}
&\frac{\partial \bar{v}}{\partial t} - [\int_{\mathbb{R}^N} k(y-x)\bar{v}(t,y;z,T)dy - \bar{v}(t,x;z,T) + f(x+z,\bar{v})\bar{v}(t,x;z,T)] \\
\geq &\frac{\partial \bar{v}}{\partial t} - [\int_{\mathbb{R}^N} k(y-x)\bar{v}(t,y;z,T)dy - \bar{v}(t,x;z,T) + f(x+z,0)\bar{v}(t,x;z,T)] \\
= &(\mu_1 c - \lambda(\mu_1))\varphi_2 \\
\geq &0.
\end{aligned}
$$

The proposition thus follows. □

In the rest of this section, we fix $d_1^* \gg 1$, $d_2^* \geq 0$, and $0 < b^* \ll 1$. Let

$$
u^-_{0,z,T}(x) = \underline{\mathrm{u}}(0, x; z, T, d_1^*, b^*) \quad \text{and} \quad u^+_{0,z,T}(x) = \bar{u}(0, x; z, T, d_2^*). \tag{7.8}
$$

Then by Proposition 7.2,

$$
\begin{aligned}
u(t, x; u^-_{0,z,T}, z) &\geq \underline{\mathrm{u}}(t, x; z, T) \\
&= \underline{\mathrm{u}}(0, x; z, T - t) \\
&= u^-_{0,z,T-t}(x).
\end{aligned}
$$

Similarly,

$$
u(t, x; u^+_{0,z,T}, z) \leq u^+_{0,z,T-t}(x).
$$

**Proposition 7.4.** *For any given $z \in \mathbb{R}^N$, the following hold:*

*(1) For any $t_2 > t_1 > 0$,*

$$
u(t_2 + t, x; u^-_{0,z,t_2}, z) \geq u(t_1 + t, x; u^-_{0,z,t_1}, z) \quad \forall t > -t_1, \; x \in \mathbb{R}^N;
$$

*(2)* $u(t_2 + t, x; u^+_{0,z,t_2}, z) \leq u(t_1 + t, x; u^+_{0,z,t_1}, z) \quad \forall t > -t_1, \; x \in \mathbb{R}^N.$

*Proof.* (1) For given $z \in \mathbb{R}^N$ and $t_2 > t_1 > 0$, by Proposition 7.2,

$$
\begin{aligned}
u(t_2 - t_1, x; u^-_{0,z,t_2}, z) &\geq \underline{u}(t_2 - t_1, x; z, t_2) \\
&= u^-_{0,z,t_2-(t_2-t_1)}(x) \\
&= u^-_{0,z,t_1}(x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
u(t_2 + t, x; u^-_{0,z,t_2}, z) &= u(t_1 + t, x; u(t_2 - t_1, \cdot; u^-_{0,z,t_2}, z), z) \\
&\geq u(t_1 + t, x; u^-_{0,z,t_1}, z).
\end{aligned}
$$

(1) is thus proved.

(2) It follows by arguments similar to those in (1) and Proposition 7.3. □

### 7.2 Existence and Uniqueness of Traveling Wave Solutions

In this section, we investigate the existence of traveling wave solutions of (1.2) and prove Theorem H.

Let $u^{\pm}_{0,z,T}$ be as in (7.8). Let

$$
\Phi^{\pm}(x, z) = \lim_{\tau \to \infty} u(\tau, x; u^{\pm}_{0,z,\tau}, z) \tag{7.9}
$$

and

$$
U^{\pm}(t, x; z) = \lim_{\tau \to \infty} u(t + \tau, x; u^{\pm}_{0,z,\tau}, z). \tag{7.10}
$$

By Proposition 7.4, the limits in the above exist for all $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. Moreover, it is easy to see that $\Phi^-(x, z)$ is lower semi-continuous in $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\Phi^+(x, z)$ is upper semi-continuous.

We will show that $u = U^+(t, x; 0)$ and $u = U^-(t, x; 0)$ are traveling wave solutions of (1.2) in the direction of $\xi$ with speed $c$ generated by $\Phi^+(\cdot, \cdot)$ and $\Phi^-(\cdot, \cdot)$, respectively, and that $\Phi(\cdot, \cdot) := \Phi^+(\cdot, \cdot)$ satisfies Theorem H(1)-(2).

To this end, we first prove some lemmas.

**Lemma 7.1.** *For each* $z \in \mathbb{R}^N$, $u(t,x) = U^{\pm}(t,x;z)$ *are entire solutions of* (2.15).

*Proof.* We prove the case that $u(t,x) = U^{+}(t,x;z)$. The other case can be proved similarly.

Fix $z \in \mathbb{R}^N$. Observe that for any $x \in \mathbb{R}^N$,

$$u(t+\tau, x; u^{+}_{0,z,\tau}, z) = u(\tau, x; u^{+}_{0,z,\tau}, z) + \int_0^t \int_{\mathbb{R}^N} k(y-x) u(s+\tau, y; u^{+}_{0,z,\tau}, z) dy ds$$
$$+ \int_0^t \big[ -u(s+\tau, x; u^{+}_{0,z,\tau}, z) + u(s+\tau, x; u^{+}_{0,z,\tau}, z) f(x+z, u(s+\tau, x; u^{+}_{0,z,\tau}, z))\big] ds$$

Letting $\tau \to \infty$, we have

$$u(t,x) = u(0,x) + \int_0^t \big[ \int_{\mathbb{R}^N} k(y-x) u(s,y) dy - u(s,x) + u(s,x) f(x+z, u(s,x))\big] ds.$$

This implies that $u(t,x)$ is differentiable in $t$ and satisfies (2.15) for all $t \in \mathbb{R}$. $\square$

Observe that

$$U^{\pm}(t,x;z) = u(t,x;\Phi^{\pm}(\cdot,z),z) \ \forall t \in \mathbb{R},\ x,z \in \mathbb{R}^N.$$

**Lemma 7.2.** $u(t,x;\Phi^{\pm}(\cdot,z),z) = \Phi^{\pm}(x - ct\xi, z + ct\xi)$, $\lim_{x\cdot\xi \to -\infty} (\Phi^{\pm}(x,z) - u^{+}(x+z)) = 0$ *and* $\lim_{x\cdot\xi\to\infty} \dfrac{\Phi^{\pm}(x,z)}{e^{-\mu x\cdot\xi}\phi(x+z)} = 1$ *uniformly in* $z \in \mathbb{R}^N$.

*Proof.* We prove the lemma for $\Phi^{+}(\cdot,\cdot)$. It can be proved similarly for $\Phi^{-}(\cdot,\cdot)$.

First of all, we have

$$\begin{aligned} u(t,x;\Phi^{+}(\cdot,z),z) &= \lim_{\tau\to\infty} u(t,x;u(\tau,x;u^{+}_{0,z,\tau},z),z) \\ &= \lim_{\tau\to\infty} u(t+\tau,x;u^{+}_{0,z,\tau},z) \\ &= \lim_{\tau\to\infty} u(t+\tau, x - ct\xi; u^{+}_{0,z+ct\xi,t+\tau}, z + ct\xi) \\ &= \Phi^{+}(x - ct\xi, z + ct\xi). \end{aligned}$$

Note that

$$
\begin{aligned}
e^{-\mu(x\cdot\xi-ct)}\phi(x+z)-d_1e^{-\mu_1(x\cdot\xi-ct)}\phi_1(x+z) &\le \underline{u}(t+T,x;z,T)\\
&\le u(t,x;\Phi^+(\cdot,z),z)\\
&\le \bar{u}(t+T,x;z,T)\\
&= e^{-\mu(x\cdot\xi-ct)}\phi(x+z)+d_2e^{-\mu_1(x\cdot\xi-ct)}\phi_1(x+z)
\end{aligned}
$$

for $t\in\mathbb{R}$ and $x,z\in\mathbb{R}^N$. Thus $\lim_{x\cdot\xi-ct\to\infty}\frac{\Phi^+(x-ct\xi,z+ct\xi)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)}=1$, which is equivalent to $\lim_{x\cdot\xi\to\infty}\frac{\Phi^+(x,z)}{e^{-\mu x\cdot\xi}\phi(x+z)}=1$, uniformly in $z\in\mathbb{R}^N$.

We now prove that $\lim_{x\cdot\xi\to-\infty}\big(\Phi^+(x,z)-u^+(x+z)\big)=0$ uniformly in $z\in\mathbb{R}^N$. Observe that there is $M>0$ such that

$$
U^+(t,x,z)\ge U^-(t,x,z)\ge b^*\phi_0(x+z)\quad \text{for}\quad x\cdot\xi-ct\le M,\ z\in\mathbb{R}^N.
$$

By Proposition 3.3, for any $\epsilon>0$, there are $T>0$ and $\eta^*\in\mathbb{R}$ such that

$$
|U^+(T,x,z)-u^+(x+z)|<\epsilon\quad\text{for}\quad x\cdot\xi\le\eta^*,\ z\in\mathbb{R}^N.
$$

This implies that

$$
|\Phi^+(x,z)-u^+(x+z)|\le\epsilon\quad\text{for}\quad x\cdot\xi\le\eta^*+cT,\ z\in\mathbb{R}^N
$$

and hence $\lim_{x\cdot\xi\to-\infty}\big(\Phi^+(x,z)-u^+(x+z)\big)=0$ uniformly in $z\in\mathbb{R}^N$. □

**Corollary 7.1.** *Both $\Phi^+(\cdot,\cdot)$ and $\Phi^-(\cdot,\cdot)$ generate traveling wave solutions of* (1.2) *in the direction of $\xi$ with speed $c$.*

*Proof.* First of all, by Lemmas 7.1 and 7.2, both $\Phi^+(\cdot,\cdot)$ and $\Phi^-(\cdot,\cdot)$ satisfy (2.17) and (2.18).

Next, for any $x,x'\in\mathbb{R}^N$ with $x\cdot\xi=x'\cdot\xi$, $z\in\mathbb{R}^N$, and $\tau\in\mathbb{R}$, we have

$$
\begin{aligned}
u(\tau,x';u^{\pm}_{0,z-x',\tau}(\cdot),z-x') &= u(\tau,x;u^{\pm}_{0,z-x',\tau}(\cdot+x'-x),z-x'+(x'-x))\\
&= u(\tau,x;u^{\pm}_{0,z-x,\tau}(\cdot),z-x).
\end{aligned}
$$

This implies that $\Phi^{\pm}(\cdot,\cdot)$ satisfies (2.19).

Observe now that $u^{\pm}_{0,z+p_i\mathbf{e_i},\tau} = u^{\pm}_{0,z,\tau}$ for any $\tau \in \mathbb{R}$ and $z \in \mathbb{R}^N$. It then follows that $\Phi^{\pm}(x, z + p_i\mathbf{e_i}) = \Phi^{\pm}(x, z)$ and hence $\Phi^{\pm}(\cdot,\cdot)$ satisfies (2.20).

Therefore, both $\Phi^{+}(\cdot,\cdot)$ and $\Phi^{-}(\cdot,\cdot)$ generate traveling wave solutions of (1.2) in the direction of $\xi$ with speed $c$. □

**Lemma 7.3.** $\lim\limits_{x\cdot\xi - ct\to-\infty} U^{\pm}_t(t,x;z) = 0$ *uniformly in* $z \in \mathbb{R}^N$.

*Proof.* Note that

$$U^{\pm}_t(t,x;z) = \int_{\mathbb{R}^N} k(y-x)U^{\pm}(t,y;z)dy - U^{\pm}(t,x;z) + U^{\pm}(t,x;z)f(x+z,U^{\pm}(t,x;z))$$

and thus

$$\begin{aligned}\lim_{x\cdot\xi-ct\to-\infty} U^{\pm}_t(t,x;z) = &\lim_{x\cdot\xi-ct\to-\infty}\Big[U^{\pm}_t(t,x;z) - \int_{\mathbb{R}^N} k(y)u^{+}(y+x+z)dy + u^{+}(x+z)\\ &- u^{+}(x+z)f(x+z,u^{+}(x+z))\Big]\\ = &\lim_{x\cdot\xi-ct\to-\infty}\Big[\int_{\mathbb{R}^N} k(y)\big(U^{\pm}(t,x+y;z) - u^{+}(x+y+z)\big)dy\\ &- \big(U^{\pm}(t,x;z) - u^{+}(x+z)\big)\\ &+ \big(U^{\pm}(t,x;z)f(x+z,U^{\pm}(t,x;z)) - u^{\pm}(x+z)f(x+z,u^{+}(x+z))\big)\Big]\end{aligned}$$

It suffices to prove that $\lim\limits_{x\cdot\xi-ct\to-\infty}\int_{\mathbb{R}^N} k(y)\big(U^{\pm}(t,x+y;z)-u^{+}(x+y+z)\big)dy \to 0$ uniformly in $z \in \mathbb{R}^N$. For any $\epsilon > 0$. Since $U^{\pm}(t,x+y;z)-u^{+}(x+y+z)$ is bounded and $k(\cdot)$ satisfies (H1), then there exists a $\hat{\delta} > 0$ such that $\int_{\|y\|>\hat{\delta}} k(y)\big(U^{\pm}(t,x+y;z)-u^{+}(x+y+z)\big)dy < \frac{\epsilon}{2}$ for $z \in \mathbb{R}^N$. Since $\lim\limits_{x\cdot\xi-ct\to-\infty}[U^{\pm}(t,x;z) - u^{+}(x+z)] = 0$ uniformly in $z \in \mathbb{R}^N$, there exists an $L > 0$ such that $U^{\pm}(t,x;z) - u^{+}(x+z) < \frac{\epsilon}{2}$ for $x\cdot\xi - ct < -L$ and $z \in \mathbb{R}^N$. Thus, $\int_{\|y\|\le\hat{\delta}} k(y)\big(U^{\pm}(t,x+y;z) - u^{+}(x+y+z)\big)dy < \frac{\epsilon}{2}$ for $x\cdot\xi - ct < -L-\hat{\delta}$ and $z \in \mathbb{R}^N$. Therefore, $\int_{\mathbb{R}^N} k(y)\big(U^{\pm}(t,x+y;z)-u^{+}(x+y+z)\big)dy < \epsilon$ for $x\cdot\xi-ct < -L-\hat{\delta}$ and $z \in \mathbb{R}^N$. This completes the proof. □

**Lemma 7.4.** $\lim_{x\cdot\xi - ct\to\infty} \dfrac{U_t^{\pm}(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)} = \mu c$ *uniformly in* $z\in\mathbb{R}^N$.

*Proof.* We prove the lemma for $U^+(t,x;z)$. It can be proved similarly for $U^-(t,x;z)$.

First, let $U(t,x;z) = U^+(t,x;z)$. By Lemma 7.2, for any $\epsilon > 0$, there is $M > 0$ such that for any $x, z \in \mathbb{R}^N$ and $t \in \mathbb{R}$ with $x\cdot\xi - ct \geq M$,

$$\Big|\frac{U(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}} - \phi(x+z)\Big| < \epsilon \tag{7.11}$$

and

$$|f(x+z, U(t,x;z)) - f(x+z,0)| < \epsilon. \tag{7.12}$$

Observe that

$$\mu c\phi(x+z) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)\phi(y+z)dy - \phi(x+z) + a_0(x+z)\phi(x+z) \tag{7.13}$$

for all $x, z \in \mathbb{R}^N$, where $a_0(x+z) = f(x+z,0)$, and

$$U_t(t,x;z) = \int_{\mathbb{R}^N} k(y-x)U(t,y;z)dy - U(t,x;z) + U(t,x;z)f(x+z,U(t,x;z)) \tag{7.14}$$

for all $t\in\mathbb{R}$ and $x,z\in\mathbb{R}^N$. By (7.11)-(7.14), we have

$$\begin{aligned}
\Big|\frac{U_t(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)} - \mu c\Big| &= \frac{1}{\phi(x+z)}\Big|\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)\Big(\frac{U(t,y;z)}{e^{-\mu(y\cdot\xi-ct)}} - \phi(y+z)\Big)dy \\
&\quad - \Big(\frac{U(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}} - \phi(x+z)\Big) \\
&\quad + \Big(\frac{U(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}} - \phi(x+z)\Big)f(x+z,U(t,x;z)) \\
&\quad + \phi(x+z)\big(f(x+z,U(t,x;z)) - f(x+z,0)\big)\Big| \\
&\leq \epsilon\Big[\int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}k(y-x)dy \\
&\quad + 1 + |f(x+z,U(t,x;z))| + \phi(x+z)\Big]
\end{aligned}$$

for all $x, z \in \mathbb{R}^N$ and $t \in \mathbb{R}$ with $x \cdot \xi - ct \geq M + \delta_0$, where $\delta_0$ is the nonlocal dispersal distance in (1.2). It then follows that

$$\lim_{x\cdot\xi - ct \to \infty} \frac{U_t^{\pm}(t, x; z)}{e^{-\mu(x\cdot\xi - ct)}\phi(x+z)} = \mu c$$

uniformly in $z \in \mathbb{R}^N$. □

*Proof of Theorem H.* Let $\Phi(x, z) = \Phi^+(x, z)$ and $U(t, x; z) = U^+(t, x; z)$. Note that $U(t, x; z) = u(t, x; \Phi(\cdot, z), z))$. We show that $\Phi(\cdot, \cdot)$ and $U(\cdot, \cdot; \cdot)$ satisfy Theorem H(1) and (2), respectively.

(1) It follows from Corollary 7.1 and Lemma 7.2.

(2) By Lemmas 7.3 and 7.4, we only need to prove that $U_t(t, x; z) > 0$ for all $(t, x, z) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

For any $t_1 < t_2$, we have

$$u^+_{0,z,t_1}(x) \geq u^+_{0,z,t_2}(x) \quad \forall x, z \in \mathbb{R}^N.$$

Hence

$$\begin{aligned}
u(t_1, x; \Phi^+(\cdot, z), z) &= u(t_2 + t_1 - t_2, x; \Phi^+(\cdot, z), z) \\
&= \lim_{n\to\infty} u(t_2, x; u(n + t_1 - t_2, \cdot; u^+_{0,z,n}, z), z) \\
&\leq \lim_{n\to\infty} u(t_2, x; u(n + t_1 - t_2, \cdot; u^+_{0,z,n+t_1-t_2}, z), z) \\
&= u(t_2, x; \Phi^+(\cdot, z), z).
\end{aligned}$$

Therefore, $U(t, x; z) = u(t, x; \Phi^+(\cdot, z), z)$ is nondecreasing as $t$ increases.

Let $v(t, x; z) = u_t(t, x; \Phi^+(\cdot, z), z)$. Then $v(t, x; z) \geq 0$. By Lemma 7.4, for any $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$, the set $\{x \in \mathbb{R}^N \,|\, v(t, x; z) > 0\}$ has positive Lebesgue measure. Note that $v(t, x; z)$ satisfies

$$v_t(t, x; z) = \int_{\mathbb{R}} k(y - x)v(t, y; z)dy - v(t, x; z) + a(t, x; z)v(t, x; z) \tag{7.15}$$

where $a(t,x;z) = f(x+z, u(t,x;\Phi^+(\cdot,z),z)) + u(t,x;\Phi^+(\cdot,z),z) f_u(x+z, u(t,x;\Phi^+(\cdot,z),z))$. Then by Proposition 3.1, we have

$$v(t,x;z) > 0 \quad \forall t \in \mathbb{R},\ x,z \in \mathbb{R}^N.$$

This implies that $U_t(t,x;z) > 0$ for all $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. □

Next, we investigate the uniqueness and continuity of traveling wave solutions of (1.2) and prove Theorem I by the "squeezing" techniques developed in [9] and [25].

Throughout this section, we fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let $\mu^*$ be such that

$$c^*(\xi) = \frac{\lambda_0(\xi,\mu^*,a_0)}{\mu^*} < \frac{\lambda_0(\tilde{\xi},\mu,a_0)}{\tilde{\mu}} \quad \forall \tilde{\mu} \in (0,\mu^*).$$

We fix $c > c^*(\xi)$ and $\mu \in (0,\mu^*)$ with $\frac{\lambda_0(\xi,\mu,a_0)}{\mu} = c$ and assume that $U^{\pm}(t,x;z)$ and $\Phi^{\pm}(x,z)$ are as in section 7.2. We put $\Phi(x,z) = \Phi^+(x,z)$ and $U(t,x;z) = U^+(t,x;z)$. Let $U_1(t,x;z) = u(t,x;\Phi_1(\cdot,z),z) (\equiv \Phi_1(x - ct\xi, z + ct\xi))$.

We first prove some lemmas, some of which will also be used in next section. By Lemmas 7.2 and 7.4, there is $M_0 > 0$ such that

$$0 < \sup_{x\cdot\xi - ct \ge M_0, z \in \mathbb{R}^N} \frac{U(t,x;z)}{U_t(t,x;z)} < \infty. \tag{7.16}$$

Observe that there is $\sigma_0 > 0$ such that

$$U(t,x;z) \ge \sigma_0 \quad \text{for} \quad x\cdot\xi - ct \le M_0. \tag{7.17}$$

Let

$$\eta_0 = \inf_{0 < u \le 2u^+_{\sup}} (-f_u(x,u))\sigma_0, \tag{7.18}$$

where $u^+_{\sup} = \sup_{x \in \mathbb{R}^N} u^+(x)$. Throughout the rest of this section, $M_0$, $\sigma_0$, $\eta_0$ are fixed and satisfy (7.16)-(7.18).

**Lemma 7.5.** *Let $\epsilon_0 \in (0,1)$ and $\eta \in (0,(1-\epsilon_0)\eta_0)$. There is $l > 0$ such that for each $\epsilon \in (0,\epsilon_0)$,*

$$H^{\pm}(t,x;z) = (1 \pm \epsilon e^{-\eta t})U(t \mp l\epsilon e^{-\eta t}, x; z), \forall t \geq 0\ x, z \in \mathbb{R}^N$$

*are super-/sub-solution of* (2.15).

*Proof.* First we prove that $H^+(t,x;z)$ is a super-solution of (2.15). Let $h = \epsilon e^{-\eta t}$ and $\tau = t - l\epsilon e^{-\eta t}$. Then

$$H^+(t,x;z) = (1+h)U(\tau,x;z), \forall t \geq 0,\ x, z \in \mathbb{R}^N.$$

By direct calculation, we have

$$\begin{aligned}
&\frac{\partial H^+(t,x;z)}{\partial t} - [\int_{\mathbb{R}^N} k(y-x)H^+(t,y;z)dy - H^+(t,x;z) + H^+(t,x;z)f(x+z, H^+(t,x;z))] \\
&= -\eta h U(\tau,x;z) + (1+l\eta h)[(\mathcal{K}-I)H^+ + f(x+z,U)H^+] - [(\mathcal{K}-I)H^+ + f(x+z,H)H^+] \\
&= -\eta h U(\tau,x;z) + l\eta h[(\mathcal{K}-I)H^+ + f(x+z,U)H^+] + [f(x+z,U) - f(x+z,H)]H^+ \\
&= -\eta h U(\tau,x;z) + l\eta h(1+h)U_t(\tau,x;z) + [f(x+z,U) - f(x+z,H^+)](1+h)U(\tau,x;z) \\
&= h\eta U(\tau,x;z)[-1 + l(1+h)\frac{U_t(\tau,x+z)}{U(\tau,x+z)} - f_u(x+z,u^*(\tau,x;z))(1+h)U(\tau,x;z)/\eta],
\end{aligned}$$

where $u^*(\tau,x;z)$ is some number between $U(\tau,x;z)$ and $H^+(t,x;z)$. We only need to prove that

$$-1 + l(1+h)\frac{U_\tau(\tau,x;z)}{U(\tau,x;z)} - f_u(x+z,U^*(\tau,x;z))(1+h)U(\tau,x)/\eta \geq 0 \qquad (7.19)$$

for all $t \geq 0$ and $x, z \in \mathbb{R}^N$.

If $t \geq 0$ and $x \in \mathbb{R}^N$ are such that $x \cdot \xi - c\tau \leq M_0$, by (7.17), (7.18), and the fact that $U_t(\tau,x,;z) > 0$, (7.19) holds.

If $t \geq 0$ and $x \in \mathbb{R}^N$ are such that $x \cdot \xi - c\tau \geq M_0$, and $l \geq \sup_{x\cdot\xi - c\tau \geq M_0} \frac{U(\tau,x;z)}{U_t(\tau,x;z)}$, then (7.19) also holds.

By the similar arguments above, we can prove that $H^-(t,x;z)$ is a sub-solution of (2.15). This completes the proof. □

**Lemma 7.6.** *Let $\epsilon_0 \in (0,1)$ and $\eta \in (0,(1-\epsilon_0)\eta_0)$ be given and $l$ be as in Lemma 7.5. For any given $0 < \epsilon_1 \le \epsilon_0$, there exists constant $M_1(\epsilon_1) > 0$ such that for all $\epsilon \in (0, \epsilon_1]$*

$$(1-\epsilon)U(t+3l\epsilon, x; z) \le U(t, x; z) \le (1+\epsilon)U(t-3l\epsilon, x; z) \ \forall t \in \mathbb{R},\ x, z \in \mathbb{R}^N,\ x-ct \le -M_1(\epsilon_1).$$

*Proof.* Let $h(s) = (1+s)U(t-3ls, x; z)$. Then, $h'(s) = U(t-3ls, x; z) - 3lU_t(t-3ls, x; z)$. By Lemma 7.3, there exists a $M(\epsilon_1) > 0$ such that $h'(s) > 0$ for $s \in [-\epsilon_1, \epsilon_1]$, $x - ct \le -M_1(\epsilon_1)$, and $z \in \mathbb{R}^N$. Hence, the lemma follows. □

**Lemma 7.7.** *For any $\epsilon > 0$, there exists a constant $C(\epsilon) \ge 1$ such that*

$$U_1(t-2\epsilon, x; z) \le U(t, x; z) \le U_1(t+2\epsilon, x; z) \quad \forall t \in \mathbb{R},\ x, z \in \mathbb{R}^N,\ x\cdot\xi - ct \ge C(\epsilon).$$

*Proof.* It follows from the fact that

$$\begin{aligned}\lim_{x\cdot\xi-ct\to\infty} \frac{U_1(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)} &= \lim_{x\cdot\xi-ct\to\infty} \frac{U_1(t,x;z)}{U(t,x;z)} \frac{U(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)} \\ &= \lim_{x\cdot\xi-ct\to\infty} \frac{U(t,x;z)}{e^{-\mu(x\cdot\xi-ct)}\phi(x+z)} \\ &= 1\end{aligned}$$

uniformly in $z \in \mathbb{R}^N$. □

**Lemma 7.8.** *Let $\epsilon_0 \in (0,1)$ and $\eta_0$, $l$ be as in Lemma 7.5. For any given $\epsilon \in (0, \epsilon_0)$, there is $\tau > 0$ such that*

$$(1-\epsilon e^{-\eta t})U(t-\tau+l\epsilon e^{-\eta t}, x) \le U_1(t, x; z) \le (1+\epsilon e^{-\eta t})U(t+\tau-l\epsilon e^{-\eta t}, x; z)$$

*for all $x, z \in \mathbb{R}^N$ and $t \ge 0$.*

*Proof.* First by Theorem C(1) and 3.1,

$$0 < U(t, x; z) < u^+(x+z) \quad \text{and} \quad 0 < U_1(t, x; z) < u^+(x+z) \quad \forall t \in \mathbb{R},\ x, z \in \mathbb{R}^N.$$

Then by Lemma 7.7, there exists a constant $C(1)$ such that

$$U_1(t,x;z) \geq U(t-2,x;z) \quad \forall t \in \mathbb{R}\ x,z \in \mathbb{R}^N,\ x\cdot\xi - ct \geq C(1).$$

By (2.18), there is $t_1 \geq 2$ such that

$$U_1(t,x;z) \geq (1-\epsilon)U(t-t_1,x;z) \quad \forall\ t \in \mathbb{R},\ x,z \in \mathbb{R}^N,\ x\cdot\xi - ct < C(1).$$

Thus

$$U_1(0,x;z) \geq (1-\epsilon)U(-t_1,x;z) = (1-\epsilon)U(-(t_1+l\epsilon)+l\epsilon,x;z) \quad \forall\, x,z \in \mathbb{R}^N.$$

It then follows Lemma 7.5 that

$$U_1(t,x;z) \geq (1-\epsilon e^{-\eta t})U(t-(t_1+l\epsilon)+l\epsilon e^{-\eta t},x;z) \quad \forall t \geq 0, x,z \in \mathbb{R}^N.$$

Similarly, it can be proved that there is $t_2 \geq 2$ such that

$$U_1(t,x;z) \leq (1+\epsilon e^{-\eta t})U(t+t_2+l\epsilon - l\epsilon e^{-\eta t},x;z) \quad \forall t \geq 0,\ x,z \in \mathbb{R}^N.$$

The lemma then follows with $\tau = \max\{t_1+l\epsilon, t_2+l\epsilon\}$. □

**Lemma 7.9.** *Let $\tau > 0, t_1 > 0$, and $M \in \mathbb{R}$ be given. Suppose that $W^{\pm}(t,x;t_1,z)$ are the solution of* (2.15) *with initial*

$$W^{\pm}(0,x;t_1,z) = U(t_1 \pm \tau,x;z)\varsigma(x-ct_1-M) + U(t_1 \pm 2\tau,x;z)(1-\varsigma(x-ct_1-M)),$$

*where $\varsigma(s)=0$ for $s \leq 0$ and $\varsigma(s)=1$ for $s>0$. Then*

$$W^{+}(1,x;t_1,z) \leq (1+\epsilon)U(t_1+1+2\tau-3l\epsilon,x;z)$$

*and*

$$W^{-}(1,x;t_1,z) \geq (1-\epsilon)U(t_1+1-2\tau+3l\epsilon,x;z)$$

*for all $x,z \in \mathbb{R}^N$ with $x-c(1+t_1) \leq M$ provided that $0<\epsilon \ll 1$.*

*Proof.* We give a proof for $W^-(1, x; t_1, z)$. The case of $W^+$ can be proved similarly. Note that

$$W^-(0, x; t_1, z) \geq U(t_1 - 2\tau, x; z) \quad \forall x, z \in \mathbb{R}^N.$$

It then follows that

$$W^-(1, x; t_1, z) > U(1 + t_1 - 2\tau, x; z) \quad \forall x, z \in \mathbb{R}^N.$$

Take an $\epsilon_1 \in (0, \epsilon_0]$. By Lemma 7.6, for any $\epsilon \in (0, \epsilon_1]$,

$$W^-(1, x; t_1, z) > (1 - \epsilon)U(1 + t_1 - 2\tau + 3l\epsilon, x; z) \quad \forall x \cdot \xi - c(t_1 + 1) \leq -M(\epsilon_1),\ z \in \mathbb{R}^N.$$

We claim that for $0 < \epsilon \ll 1$,

$$W^-(1, x; t_1, z) > (1 - \epsilon)U(1 + t_1 - 2\tau + 3l\epsilon, x; z) \quad \forall x \cdot \xi - c(t_1 + 1) \in [-M(\epsilon_1), M],\ z \in \mathbb{R}^N.$$

In fact, let $W(t, x; z) = W^-(t, x; t_1, z) - U(t + t_1 - 2\tau, x; z)$ and

$$\begin{aligned} h = \inf_{t \in [0,1], x, z \in \mathbb{R}^N} \{ & [W^-(t, x; t_1, z) f(x + z, u(t, x; u_{0,z}, z)) \\ & - U(t + t_1 - 2\tau, x; z) f(x + z, U(t + t_1 - 2\tau, x; z))] \\ & \cdot \frac{1}{W^-(t, x; t_1, z) - U(t + t_1 - 2\tau, x; z)} \}. \end{aligned}$$

Then

$$W(0, x; z) = \begin{cases} U(t_1 - \tau, x; z) - U(t_1 - 2\tau, x; z) & \text{for} \quad x \cdot \xi - ct_1 > M \\ 0 \quad \text{for} \quad x \cdot \xi - ct_1 \leq M \end{cases}$$

and

$$W_t(t, x; z) \geq \int_{\mathbb{R}^N} k(y - x) W(t, y; z) dy - W(t, x; z) + hW(t, x; z) \quad \forall t \in [0, 1],\ x, z \in \mathbb{R}^N.$$

It then follows that

$$W(1, \cdot; z) \geq e^{-1+h}(W(0, \cdot; z) + \mathcal{K}W(0, \cdot; z) + \frac{\mathcal{K}^2}{2!}W(0, \cdot; z) + \cdots),$$

where $\mathcal{K}W(0,\cdot;z)$ is defined as in (2.6) with $u$ being replaced by $W(0,\cdot;z)$. By Lemma 7.2, there are $\tilde{\sigma}>0$ and $\tilde{M}>0$ such that

$$U(t_1-\tau,x;z)-U(t_1-2\tau,x;z)\geq\tilde{\sigma}\quad\forall\, x,z\in\mathbb{R}^N\ \text{ with }\ \tilde{M}\leq x\cdot\xi-ct_1\leq\tilde{M}+1.\quad(7.20)$$

This implies that

$$W(1,x;z)\geq U(1+t_1-2\tau+3l\epsilon,x;z)-U(1+t_1-2\tau,x;z)\qquad(7.21)$$

for $x\cdot\xi-c(t_1+1)\in[-M(\epsilon_1),M]$ and $z\in\mathbb{R}^N$ provided that $0<\epsilon\ll 1$. By (7.20) and (7.21), we have

$$\begin{aligned}W^-(1,x;t_1,z)&=W(1,x;z)+U(1+t_1-2\tau,x;z)\\&\geq U(1+t_1-2\tau+3l\epsilon,x;z)\\&\geq(1-\epsilon)U(1+t_1-2\tau+3l\epsilon,x;z)\end{aligned}$$

for $x\cdot\xi-c(1+t_1)\leq M$ and $z\in\mathbb{R}^N$ provided that $0<\epsilon\ll 1$. □

*Proof of Theorem I.* (1) Let

$$A^+=\{\tau\geq 0\,|\,\limsup_{t\to\infty}\sup_{x,z\in\mathbb{R}^N}\frac{U_1(t,x;z)}{U(t+2\tau,x;z)}\leq 1\}$$

and

$$A^-=\{\tau\geq 0\,|\,\liminf_{t\to\infty}\inf_{x,z\in\mathbb{R}^N}\frac{U_1(t,x;z)}{U(t-2\tau,x;z)}\geq 1\}.$$

By Lemma 7.8, $A^\pm\neq\emptyset$. Let

$$\tau^+=\inf\{\tau\,|\,\tau\in A^+\},\quad \tau^-=\inf\{\tau\,|\,\tau\in A^-\}.$$

We first claim that $\tau^\pm\in A^\pm$. In fact, let $\tau_n\in A^+$ be such that $\tau_n\to\tau^+$. Then for any $0<\epsilon<1$, there are $t_n\to\infty$ such that

$$\frac{U_1(t,x;z)}{U(t+2\tau_n,x;z)}\leq 1+\epsilon\quad\forall x,z\in\mathbb{R}^N,\ t\geq t_n$$

and

$$\frac{U(t+2\tau^+,x;z)-U(t+2\tau_n,x;z)}{U(t+2\tau_n,x;z)} > -\epsilon \quad \forall n \gg 1,\ t\in\mathbb{R},\ x,z\in\mathbb{R}^N.$$

Observe that

$$\frac{U_1(t,x;z)}{U(t+2\tau^+,x;z)} = \frac{U_1(t,x;z)}{U(t+2\tau_n,x;z)}\frac{U(t+2\tau_n,x;z)}{U(t+2\tau^+,x;z)}$$

and

$$\begin{aligned}\frac{U(t+2\tau_n,x;z)}{U(t+2\tau^+,x;z)} &= \frac{1}{1+\frac{U(t+2\tau^+,x;z)-U(t+2\tau_n,x;z)}{U(t+2\tau_n,x;z)}}\\ &\le \frac{1}{1-\epsilon}\\ &\le 1+\epsilon \quad \forall n\gg 1.\end{aligned}$$

Fix $n \gg 1$. Then

$$\sup_{x,z\in\mathbb{R}^N} \frac{U_1(t,x;z)}{U(t+2\tau^+,x;z)} \le (1+\epsilon)^2 \quad \forall t \ge t_n.$$

This implies that $\tau^+ \in A^+$. Similarly, we have $\tau^- \in A^-$.

Next we claim that $\tau^\pm = 0$. Assume that $\tau^- > 0$. Note that

$$\liminf_{t\to\infty} \inf_{x,z\in\mathbb{R}^N} \frac{U_1(t,x;z)}{U(t-2\tau^-,x;z)} \ge 1.$$

Hence for any $\bar\epsilon > 0$, there is $t_0 > 0$ such that

$$\frac{U_1(t_0,x;z)}{U(t_0-2\tau^-,x;z)} \ge 1-\bar\epsilon \quad \forall x,z\in\mathbb{R}^N.$$

This implies that

$$U_1(t_0,x;z) \ge (1-\bar\epsilon)U(t_0-2\tau^-,x;z) \ge U^+(t_0-2\tau^-,x;z)-\hat\epsilon$$

where $\hat\epsilon = \bar\epsilon \max_{t,x,z} U^+(t,x,z)$. By Lemma 7.7, for $x\cdot\xi - ct_0 \ge M := C(\tau^-/2)$,

$$U_1(t_0,x;z) \ge U(t_0-\tau^-,x;z).$$

This implies that

$$U_1(t_0,x;z) \geq U(t_0-2\tau^-,x;z)(1-\zeta(x\cdot\xi-ct_0-M))+U(t_0-\tau^-,x;z)\zeta(x\cdot\xi-ct_0-M)-\hat{\epsilon}.$$

Note that there is $K>0$ such that $U_1(t,x;z)+\hat{\epsilon}e^{Kt}$ is a super-solution of (2.15) for $t\in[0,1]$ provided that $0<\hat{\epsilon}\ll 1$. Then by Lemma 7.9, for $0<\bar{\epsilon}\ll 1$ and $0<\epsilon\ll 1$,

$$U_1(t_0+1,x;z)+\hat{\epsilon}e^K \geq (1-\epsilon)U(t_0+1-2\tau^-+3l\epsilon,x;z) \quad \forall x\cdot\xi-c(t_0+1)\leq M,\ z\in\mathbb{R}^N,$$

where $l$ is as in Lemma 7.5. Hence for $0<\bar{\epsilon}\ll\epsilon\ll 1$,

$$U_1(t_0+1,x;z)\geq(1-2\epsilon)U(t_0+1-2z^-+3l\epsilon,x;z) \quad \forall x\cdot\xi-c(t_0+1)\leq M,\ z\in\mathbb{R}^N.$$

By Lemma 7.7 again, for $x\cdot\xi-c(t_0+1)\geq M$ , $z\in\mathbb{R}^N$, and $0<\epsilon\ll 1$,

$$\begin{aligned} U_1(t_0+1,x;z) &> U(t_0+1-\tau^-,x;z) \\ &\geq (1-2\epsilon)U(t_0+1-\tau^-,x;z) \\ &\geq (1-2\epsilon)U(t_0+1-2\tau^-+3l\epsilon,x;z). \end{aligned}$$

Therefore for $0<\epsilon\ll 1$,

$$U_1(t_0+1,x;z)\geq(1-2\epsilon)U(t_0+1-2\tau^-+3l\epsilon,x;z) \quad \forall x,z\in\mathbb{R}^N.$$

By Lemma 7.5,

$$U_1(t_0+t+1,x;z)\geq(1-2\epsilon e^{-\tau t})U(t_0+1+t-2\tau^-+2l\epsilon e^{-\eta t}+l\epsilon,x;z) \quad \forall t\geq 0,\ x,z\in\mathbb{R}^N.$$

It then follows that

$$\tau^- - \frac{l\epsilon}{2}\in A^-.$$

this is a contradiction. Therefore $\tau^-=0$. Similarly, we have $\tau^+=0$.

We now prove that $\Phi_1(x,z)=\Phi(x,z)$. Recall that $U_1(t,x;z)=\Phi_1(x-ct\xi,z+ct\xi)$ and $U(t,x;z)=\Phi(x-ct\xi,z+ct\xi)$. Hence

$$\begin{aligned}\inf_{x,z\in\mathbb{R}^N}\frac{U_1(t,x;z)}{U(t,x;z)}&=\inf_{x,z\in\mathbb{R}^N}\frac{\Phi_1(x-ct\xi,z+ct\xi)}{\Phi(x-ct\xi,z+ct\xi)}\\&=\inf_{x,z\in\mathbb{R}^N}\frac{\Phi_1(x,z)}{\Phi(x,z)}\end{aligned}$$

and

$$\begin{aligned}\sup_{x,z\in\mathbb{R}^N}\frac{U_1(t,x;z)}{U(t,x;z)}&=\sup_{x,z\in\mathbb{R}^N}\frac{\Phi_1(x-ct\xi,z+ct\xi)}{\Phi(x-ct\xi,z+ct\xi)}\\&=\sup_{x,z\in\mathbb{R}^N}\frac{\Phi_1(x,z)}{\Phi(x,z)}\end{aligned}$$

This together with $\tau^{\pm}=0$ implies that

$$\inf_{x,z\in\mathbb{R}^N}\frac{\Phi_1(x,z)}{\Phi(x,z)}=\sup_{x,z\in\mathbb{R}^N}\frac{\Phi_1(x,z)}{\Phi(x,z)}=1.$$

We then must have $\Phi_1(x,z)\equiv\Phi(x,z)$.

(2) Let $\Phi_1(x,z)=\Phi^-(x,z)(=U^-(0,x;z))$. By (1), $\Phi^-(x,z)=\Phi(x,z)$. Recall that $\Phi^-(x,z)$ is lower semi-continuous and $\Phi^+(x,z)$ is upper semi-continuous. We then must have that $\Phi(x,z)$ is continuous in $(x,z)\in\mathbb{R}^N\times\mathbb{R}^N$. □

**Corollary 7.2.** *Let $\Phi(x,z)$ be as in Theorem H. Then*

$$\lim_{\tau\to\infty}u(\tau,x;\underline{u}(0,\cdot;z,\tau,d_1,b),z)=\lim_{\tau\to\infty}u(\tau,x;\bar{u}(0,\cdot;z,\tau,d_2),z)=\Phi(x,z)$$

*for all $d_1\gg 1$, $d_2>0$, $0<b\ll 1$, and $x,z\in\mathbb{R}^N$.*

*Proof.* By the arguments of Theorem I (1), for any $d_1\gg 1$ and $0<b\ll 1$,

$$\lim_{\tau\to\infty}u(\tau,x;\underline{u}(0,\cdot;z,\tau,d_1,b),z)=\Phi^+(x,z)(=\Phi(x,z))\quad\forall x,z\in\mathbb{R}^N,$$

and for any $d_2 \gg 1$,

$$\lim_{\tau\to\infty} u(\tau, x; \bar{u}(0,\cdot; z, \tau, d_2), z) = \Phi^-(x,z)(= \Phi(x,z)) \quad \forall x, z \in \mathbb{R}^N.$$

The corollary then follows. □

### 7.3 Stability of Traveling Wave Solutions

In this section, we investigate the stability of traveling wave solutions of (1.2) and prove Theorem J. Throughout this section, we fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let $\mu^*$ be such that

$$c^*(\xi) = \frac{\lambda_0(\mu^*, \xi, a_0)}{\mu^*} < \frac{\lambda_0(\tilde{\xi}, \mu, a_0)}{\tilde{\mu}} \quad \forall \tilde{\mu} \in (0, \mu^*).$$

We fix $c > c^*(\xi)$ and $\mu \in (0, \mu^*)$ with $\frac{\lambda_0(\xi,\mu,a_0)}{\mu} = c$. Let $U(t,x;z) = U^+(t,x;z)$, where $U^+(t,x;z)$ is as in section 7.2. Put $u(t,x) = u(t,x;u_0,0)$, where $u_0$ is as in Theorem I, and put $U(t,x) = U^+(t,x;0)$. First we prove some lemmas, which are analogues of Lemmas 7.7-7.9.

**Lemma 7.10.** *For any $\epsilon > 0$, there exists a constant $C_0(\epsilon) \geq 1$ such that*

$$u(t-2\epsilon, x) \leq U(t,x) \leq u(t+2\epsilon, x) \quad \forall x\cdot\xi - ct \geq C_0(\epsilon),\ t \geq 2\epsilon.$$

*Proof.* This is an analogue of Lemma 7.7 and can be proved by properly modifying the arguments in Lemma 7.7. For clarity, we provide a proof in the following.

First we prove that there exists a constant $C_1(\epsilon) \geq 1$ such that $U(t,x) \leq u(t+2\epsilon, x)$ for all $x\cdot\xi - ct \geq C_1(\epsilon)$. Note that for given $\epsilon > 0$, there exists a $L > 0$ such that

$$e^{-\mu(x\cdot\xi + c\epsilon)}\phi(x) < u_0(x) < e^{-\mu(x\cdot\xi - c\epsilon)}\phi(x) \quad \forall x\cdot\xi > L.$$

Choose $d_1$ large enough such that $\underline{v}^1$ (see (7.1)) is a sub-solution of (1.2) and $e^{-\mu(x\cdot\xi+c\epsilon)}\phi(x) - d_1 e^{-\mu_1(x\cdot\xi+c\epsilon)}\phi_1(x) \leq 0$ for all $x \in \mathbb{R}^N$ with $x\cdot\xi \leq L$. Then by Proposition 3.1,

$$u(t,x) \geq e^{-\mu(x\cdot\xi - c(t-\epsilon))}\phi(x) - d_1 e^{-\mu_1(x\cdot\xi - c(t-\epsilon))}\phi_1(x) \quad \forall x \in \mathbb{R}^N,\ t \geq 0.$$

Thus,

$$u(t+\epsilon,x) \geq \underline{v}^1(t,x) = e^{-\mu(x\cdot\xi - ct)}\phi(x) - d_1 e^{-\mu_1(x\cdot\xi - ct)}\phi_1(x) \quad \forall x \in \mathbb{R}^N,\ t \geq 0.$$

Observe that there exists a constant $C_1(\epsilon) \geq 1$ such that

$$\underline{v}^1(t+\epsilon,x) \geq U(t,x) \quad \forall x - ct \geq C_1(\epsilon).$$

Therefore,

$$u(t+2\epsilon,x) \geq \underline{v}^1(t+\epsilon,x) \geq U(t,x) \quad \forall x - ct \geq C_1(\epsilon), t \geq 0.$$

Next we prove that there exists a constant $C_2(\epsilon) \geq 1$ such that $U(t,x) \geq u(t-2\epsilon,x)$ for all $x\cdot\xi - ct \geq C_2(\epsilon)$. Note that there are $d_2 > 0$ and $L > 0$ such that

$$u_0(x) \leq \min\{e^{-\mu x\cdot\xi}\phi(x) + d_2 e^{-\mu_1 x\cdot\xi}\phi_1(x), Lu^+(x)\} \quad \forall x \in \mathbb{R}^N.$$

By Proposition 3.1,

$$u(t,x) \leq \min\{e^{-\mu(x\cdot\xi - ct)}\phi(x) + d_2 e^{-\mu_1(x\cdot\xi - ct)}\phi_1(x), u(t,x;Lu^+(\cdot),0)\} \quad \forall x \in \mathbb{R}^N, t \geq 0.$$

On the other hand, we have

$$\lim_{x\cdot\xi - ct \to \infty} \frac{U(t,x)}{e^{-\mu(x\cdot\xi - c(t-2\epsilon))}\phi(x)} = e^{2\mu c\epsilon} > 1$$

Therefore, there exists $C_2(\epsilon) \geq 1$ such that

$$e^{-\mu(x - c(t-2\epsilon))}\phi(x) < U(t,x) \quad \forall x\cdot\xi - ct \geq C_2(\epsilon)$$

Thus it follows that

$$u(t-2\epsilon,x) \leq U(t,x) \quad \forall x\cdot\xi - ct \geq C_2(\epsilon),\ t \geq 2\epsilon.$$

The lemma then follows by choosing $C_0(\epsilon) = \max\{C_1(\epsilon), C_2(\epsilon)\}$. □

**Lemma 7.11.** *Let $\epsilon_0$, $\eta$, and $l$ be as in Lemma 7.5. For given $\epsilon \in (0, \epsilon_0)$, there are $t_\pm > 0$ and $\tau_\pm > 0$ such that*

$$(1-\epsilon e^{-\eta(t-t_-)})U(t-\tau_- + l\epsilon e^{-\eta(t-t_-)}, x) \le u(t,x) \le (1+\epsilon e^{-\eta(t-t_+)})U(t+\tau_+ - l\epsilon e^{-\eta(t-t_+)}, x)$$

*for all $x \in \mathbb{R}^N$ and $t \ge \max\{t_-, t_+\}$.*

*Proof.* This is an analogue of Lemma 7.8 and can be proved by properly modifying the arguments in Lemma 7.8. For clarity, we also provide a proof in the following.

By Lemma 7.10, there exists a constant $C_0(1)$ such that

$$u(t,x) \ge U(t-2, x) \quad \forall x\cdot\xi - ct \ge C_0(1),\ t \ge 2.$$

Observe that there are $t_- > 2$ and $\tau_- > 2 + l\epsilon$ such that

$$u(t_-, x) \ge (1-\epsilon)U(t_- - (\tau_- - l\epsilon), x) \quad \forall x\cdot\xi - ct_- \le C_0(1).$$

Thus,

$$u(t_-, x) \ge (1-\epsilon)U(t_- - \tau_- + l\epsilon, x) \quad \forall x \in \mathbb{R}^N.$$

By Lemma 7.5,

$$u(t,x) \ge (1-\epsilon e^{-\eta(t-t_-)})U(t-\tau_- + l\epsilon e^{-\eta(t-t_-)}, x) \quad \forall t \ge t_-,\ x \in \mathbb{R}^N.$$

Similarly, by Lemma 7.10, there exists a constant $C_0(1)$ such that

$$u(t,x) \le U(t+2, x) \quad \forall x - ct \ge C_0(1),\ t \ge 2.$$

Observe that there are $t_+ > 0$ and $\tau_+ > 2 + l\epsilon$ such that

$$u(t_+, x) \le (1+\epsilon)U(t_+ + \tau_+ - l\epsilon, x) \quad \forall x\cdot\xi - ct_0 \le C_0(1)$$

By Lemma 7.5 again,

$$u(t,x) \le (1+\epsilon e^{-\eta(t-t_+)})U(t+\tau_+ + -l\epsilon e^{-\eta(t-t_+)}, x) \quad \forall t \ge t_+,\ x \in \mathbb{R}^N.$$

The lemma then follows. □

**Lemma 7.12.** *Let $\tau > 0$, $t_1 > 0$, and $M \in \mathbb{R}$ be given. Suppose that $w^{\pm}(\cdot, x; t_1)$ are the solution of* (1.2) *for $t \ge 0$ with the initial conditions*

$$w^{\pm}(0,x;t_1) = U(t_1 \pm \tau, x)\varsigma(x - ct_1 - M) + U(t_1 \pm 2\tau, x)(1 - \varsigma(x - ct_1 - M)) \quad \forall x \in \mathbb{R}^N,$$

*where $\varsigma(s) = 0$ for $s \le 0$ and $\varsigma(s) = 1$ for $s > 0$. Then*

$$w^{+}(1,x;t_1) \le (1+\epsilon)U(t_1 + 1 + 2\tau - 3l\epsilon)$$
$$w^{-}(1,x;t_1) \ge (1-\epsilon)U(t_1 + 1 - 2\tau + 3l\epsilon),$$

*for all $x \cdot \xi - ct_1 \le M + c$ and $0 < \epsilon \ll 1$.*

*Proof.* This is an analogue of Lemma 7.9 and can be proved by properly modifying the arguments in Lemma 7.9. Again, for clarity, we provide a proof in the following.

First we consider $w^-$. Note that

$$w^{-}(0,x;t_1) = U(t_1 - 2\tau, x) \quad \forall x \cdot \xi - ct_1 \le M,$$

and

$$w^{-}(0,x;t_1) = U(t_1 - \tau, x) > U(t_1 - 2\tau, x) \quad \forall x \cdot \xi - ct_1 > M.$$

By Proposition 3.1,

$$w^{-}(1,x;t_1) > U(t_1 + 1 - 2\tau, x) \quad \forall x \in \mathbb{R}^N.$$

By Lemma 7.6, for $0 < \epsilon \le \epsilon_1 < \epsilon_0$,

$$U(t_1 + 1 - 2\tau, x) \ge (1-\epsilon)U(t_1 + 1 - 2\tau + 3l\epsilon, x) \quad \forall\, x \in \mathbb{R}^N,\ x \cdot \xi - c(t_1+1) \le -M_1(\epsilon_1).$$

By arguments similar to those in Lemma 7.9, we can prove that

$$w^-(1,x;t_1) > (1-\epsilon)U(t_1+1-2\tau+3l\epsilon,x) \quad \forall\, x\in\mathbb{R}^N,\ x\cdot\xi - c(t_1+1)\in[-M_1(\epsilon_1),M]$$

provided that $0<\epsilon\ll 1$. We then have

$$\begin{aligned} w^-(1,x;t_1) &= w(1,x)+U(1+t_1-2\tau,x)\\ &\geq U(1+t_1-2\tau+3l\epsilon,x)\\ &\geq (1-\epsilon)U(1+t_1-2\tau+3l\epsilon,x)\end{aligned}$$

for $x\in\mathbb{R}^N$ with $x\cdot\xi-c(1+t_1)\leq M$ provided that $0<\epsilon\ll 1$. The statement for $w^-$ then follows.

Similarly, we can prove the case of $w^+$. Hence, the lemma follows. □

*Proof of Theorem J.* First of all, let

$$A_0^+ := \{\tau\geq 0|\ \limsup_{t\to\infty}\sup_{x\in\mathbb{R}^N}\frac{u(t,x)}{U(t+2\tau,x)}\leq 1\}$$

and

$$A_0^- := \{\tau\geq 0|\ \liminf_{t\to\infty}\inf_{x\in\mathbb{R}^N}\frac{u(t,x)}{U(t-2\tau,x)}\geq 1\}.$$

Define

$$\tau_0^+ := \inf\{\tau|\tau\in A_0^+\},\quad \tau_0^- := \inf\{\tau|\tau\in A_0^-\}.$$

By Lemma 7.11, $A_0^\pm\neq\emptyset$. Hence $\tau_0^\pm$ are well defined.

By the similar arguments as in the proof of $\tau^\pm\in A^\pm$ in Theorem I, we have that $\tau_0^\pm\in A_0^\pm$.

It then suffices to prove that $\tau_0^\pm=0$. This can be proved again by the similar arguments as in the proof of $\tau^\pm=0$ in Theorem I. For clarity, we provide a proof for the case of $\tau_0^+$. We prove $\tau_0^+=0$ by contradiction. Suppose for the contrary that $\tau_0^+>0$. Then by the definition of $\tau_0^+$, for any given $\hat\epsilon>0$, there exists $t_0>0$ such that

$$u(t_0,x) \le U(t_0+2\tau_0^+,x)+\hat{\epsilon},\ \forall x\in\mathbb{R}^N.$$

Let $w^+(t,x;t_0)$ be the solution of (1.2) for $t\ge 0$ with initial condition given by

$$w^+(0,x;t_0)=U(t_0+\tau_0^+,x)\varsigma(x\cdot\xi-ct_0-M)+(1-\varsigma(x\cdot\xi-ct_0-M))U(t_0+2\tau_0^+,x),$$

where $M=C_0(\frac{\tau_0^+}{2})+c\tau_0^+$.

Then, $w^+(0,x;t_0)=U(t_0+2\tau_0^+,x)$ for $x\cdot\xi-ct_0\le M$, which implies

$$u(t_0,x)\le w^+(0,x;t_0)+\hat{\epsilon},\quad \forall x\cdot\xi-ct_0\le M.$$

On the other hand, by Lemma 7.10,

$$u(t_0,x)\le U(t_0+\tau_0^+,x)\quad \forall x\cdot\xi-c(t_0+\tau_0^+)\ge C_0(\frac{\tau_0^+}{2}).$$

Hence

$$u(t_0,x)\le U(t_0+\tau_0^+,x),\quad \forall x\cdot\xi-ct_0\ge M.$$

Therefore,

$$u(t_0,x)\le w^+(0,x;t_0)+\hat{\epsilon},\quad \forall x\in\mathbb{R}^N.$$

Note that there is $K>0$ such that $w^+(t,x;t_0)+\hat{\epsilon}e^{Kt}$ is a super-solution of (1.2) for $t\in[0,1]$ provided that $0<\hat{\epsilon}\ll 1$. By Proposition 3.1,

$$u(t_0+1,x)\le w^+(1,x;t_0)+\hat{\epsilon}e^K,\quad \forall x\in\mathbb{R}^N.$$

Thus, by Lemma 7.12,

$$u(t_0+1,x)\le (1+\epsilon)U(t_0+1+2\tau_0^+-3l\epsilon,x)+\hat{\epsilon}e^K,\quad \forall x\cdot\xi-ct_0\le M+c$$

provided that $0<\hat{\epsilon}\ll 1$ and $0<\epsilon\ll 1$. Choose $\hat{\epsilon}$ to be sufficiently small, we have

$$u(t_0+1,x)\le (1+2\epsilon)U(t_0+1+2\tau_0^+-3l\epsilon,x),\quad \forall x\cdot\xi-ct_0\le M+c.$$

On the other hand, by Lemma 7.10 again,

$$u(t_0+1,x) \le U(t_0+1+\tau_0^+,x) \quad \forall x\cdot\xi - c(t_0+1+\tau_0^+) \ge C_0(\frac{\tau_0^+}{2}).$$

This implies that for $0<\epsilon\ll 1$,

$$u(t_0+1,x) < (1+2\epsilon)U(t_0+1+2\tau_0^+ - 3l\epsilon, x) \quad \forall x\cdot\xi - ct_0 \ge c + c\tau_0^+ + C_0(\frac{\tau_0^+}{2}) = c+M.$$

Hence, for $0<\epsilon\ll 1$,

$$u(t_0+1,x) \le (1+2\epsilon)U(t_0+1+2\tau_0^+ - 3l\epsilon, x), \quad \forall x\in\mathbb{R}^N.$$

By Proposition 3.1 and Lemma 7.5,

$$u(t+t_0+1,x) \le (1+2\epsilon e^{-\eta t})U(t+t_0+1+2\tau_0^+ - 2l\epsilon - l\epsilon e^{-\eta t}), \quad \forall t>0, x\in\mathbb{R}^N.$$

Letting $t\to\infty$, we have $\tau_0^+ - l\epsilon \in A^+$ for $0<\epsilon\ll 1$, which contradicts the definition of $\tau_0^+$. Hence we must have $\tau_0^+ = 0$. □

# Chapter 8

## Concluding Remarks, Open Problems, and Future Plan

In this dissertation, we studied the spatial spread and front propagation dynamics of monostable equations with nonlocal dispersal in spatially periodic habitats. We first established a general principal eigenvalue theory for spatially periodic nonlocal dispersal operators. More precisely, we investigated the following eigenproblem,

$$\int_{\mathbb{R}^N} k(y-x)v(y)dy - v(x) + a(x)v(x) = \lambda v, \quad v \in X_p,$$

where $a(x) \in X_p$, and provided some sufficient conditions for the existence of principal eigenvalue and its associated positive eigenvector. The principal eigenvalue theory established in this dissertation provides an important tool for the study of nonlinear evolution equations with nonlocal dispersal and is also of great interest in its own.

Applying the principal eigenvalue theory for nonlocal dispersal operators and comparison principle for sub- and super-solutions, we obtained the existence, uniqueness, and global stability of spatially periodic positive stationary solutions to a general spatially periodic nonlocal monostable equation. It should be pointed out that in [13], the authors also provided some sufficient conditions for the existence of a principal eigenfunction of some nonlocal operators on some bounded or unbounded domain. Similar statements to Theorem C(1) are proved in [13] for time independent nonlocal KPP equations. We learned the work [13] while the paper [57] was almost finished. The proof of Theorem C(1) in this dissertation or [57] is different from that in [13].

Applying the principal eigenvalue theory for nonlocal dispersal operators and comparison principle for sub- and super-solutions, we proved the existence of a spatial spreading speed of a general spatially periodic nonlocal equation in any given direction, which characterizes the speed at which a species invades into the region where there is no population initially in the given direction. Moreover, it is shown that spatial variation of the habitat speeds up the spatial spread of the population.

We remark that though we used the principal eigenvalue theory for nonlocal dispersal operators in their proofs, the existence, uniqueness, and stability of spatially periodic positive stationary solutions and the existence of spreading speeds are generic features for nonlocal monostable equations in the sense that they are independent of the existence of the principal eigenvalue of the linearized nonlocal dispersal operator at the trivial solution of the monostable equation, which is of great biological importance.

Assuming the existence of the principal eigenvalues of certain nonlocal dispersal operators related to the linearized nonlocal dispersal operator at the trivial solution of the monostable equation, we showed that a spatially periodic nonlocal monostable equation has in any given direction a unique stable spatially periodic traveling wave solution connecting its unique positive stationary solution and the trivial solution with all propagating speeds greater than the spreading speed in that direction. It should be pointed out that in [17], J. Coville, J. Dávila and S. Martínez proved the the existence of the traveling wave solutions for nonlocal dispersal with KPP nonlinearity for speed $c \geq c^*(\xi)$. But they did not investigate the uniqueness and stability of the traveling wave solutions. We learned the work [17], while I completed almost all the work of this dissertation with my adviser Dr W. Shen and submitted the joint work [58]. We did not include the case with the speed $c = c^*(\xi)$. But in our work, we further investigated the uniqueness and stability of the traveling wave solutions. Since we did independently, the methods in [17] and [58]are also different. We remark that in [17], the kernel is symmetric with bounded support and in [58], the kernel is also supported on a bounded ball. In this dissertation, we extended the kernel to a more general case.

Along the line of my dissertation, there are several important open problems. We discuss the following three problems.

**Open problem 1.** *In [17], assuming the existence of the principal eigenvalues of certain nonlocal dispersal operators related to the linearized nonlocal dispersal operator at the trivial solution of* (1.2)*, the authors proved that* (1.2) *has a traveling wave solution in the given direction of* $\xi \in S^{N-1}$ *with speed* $c = c^*(\xi)$. *It is an open question whether the traveling wave solution in a given direction of* $\xi \in S^{N-1}$ *with speed* $c = c^*(\xi)$ *is unique and stable.*

Among the main techniques in proving the existence of traveling wave solutions are comparison principle and sub- and super-solutions. Recall that on the construction of sub- or super- solutions, the positive principal eigenfunctions play important roles. We proved the "monstable" feature of our equation and the existence of spreading speed no matter (H4) is satisfied or not. However, we only proved the case under the assumption (H4) for traveling wave solutions and in [17], the authors also assumed (H4). Then the following is an open question.

**Open problem 2.** *It also remains open whether a general spatially periodic monostable equation with nonlocal dispersal in* $\mathbb{R}^N$ *with* $N \geq 3$ *has traveling wave solutions connecting the spatially periodic positive stationary solution* $u^+$ *and* 0 *and propagating with constant speeds.*

If we add the temporal variable $t$ to the growth rate function $f$, the following problem is of great biological interest and is very challenging mathematically.

**Open problem 3.** *How about the spatial spread and front propagation dynamics of the nonlocal monostable equations involving both space and temporal variations, which is modeled by the following equation,*

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + u(t,x)f(t,x,u(t,x)), \quad x \in \Omega? \tag{8.1}$$

As for my future research plan, here are some of the problems I attempt to study in the near future.

• I would like to continue my study on spatial spread and front propagation dynamics of monostable stable equations with nonlocal dispersal, in particular, I plan to investigate the open problems mentioned above.

• I would like to investigate the front propagation dynamics of other types of evolution equations with nonlocal dispersal arising in applied problems, including nonlocal evolution equations with combustion type and bistable type nonlinearities.

• I also would like to extend my study of evolution equations with deterministic inhomogeneity to equations with random inhomogeneity.

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