On Frobenius numbers in three variables

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Dissertation Abstract

On Frobenius numbers in three variables

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Given a set of relatively prime positive integers $\{a_1, a_2, \ldots, a_n\}$, after some point all positive integers are representable as a linear combination of the set with nonnegative coefficients. Which integer is the last one not so representable is the Frobenius problem, or the Frobenius stamp problem, and the number in question the Frobenius number of the set. While the two-variable solution is widely known, and the general solution is NP-hard, there have been several algorithmic solutions of the three-variable problem. Here we present a new bound for the Frobenius number of most relatively prime triples.

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Chapter 1

Background

Given a set of relatively prime positive numbers $\{a_1, a_2, \ldots, a_n\}$, any integer after a point is representable as a linear combination of the set using nonnegative integer coefficients. The last integer not representable as such a linear combination is called the *Frobenius number* of the set and is denoted by $g(a_1, a_2, \ldots, a_n)$. The next number, the first of the infinite sequence of representable integers, is called the *conductor* of the set and is denoted $\chi(a_1, a_2, \ldots, a_n)$. That is, $\chi(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n) + 1$.

The set of such linear combinations with nonnegative coefficients forms the *monoid* of the set, sometimes also called the *numeric semigroup* of the set. Note that such a numeric semigroup is generated by $a_1, a_2, \ldots, a_n$ and thus we will denote it by $\langle a_1, a_2, \ldots, a_n \rangle$.

1.1 Sylvester and Frobenius

Given a pair of relatively prime integers a and b, what is the last integer not representable in a and b? Two ready monetary applications have dubbed this problem the "money-changing problem" or "the stamp problem." Given two values of coins or stamps a and b, can we find a formula for computing the last number that is not representable as a linear combination of a and b? In 1884, Sylvester [13] discovered a formula in for the case of two relatively prime integers.

Theorem 1.1 *Let a, b be relatively prime integers. Then the conductor of a and b is given by $\chi(a, b) = (a-1)(b-1)$.*

Finding formulas or algorithms for computing the conductor of a given set of n relatively prime integers is often a difficult problem. Frobenius often referred to the general n-variable problem in his lectures and it has since taken to be named after him.

The three-variable case is the next logical step to attempt to extend Sylvester's result. The following result of Johnson [8] shows that it suffices to consider variables that are pairwise coprime.

Theorem 1.2 *Let a, b, c be pairwise relatively prime and d be a positive common factor of a and b. Then,*

$$g(a, b, c) = d \cdot g\left(\frac{a}{d}, \frac{b}{d}, c\right) - c(d - 1)$$

This result allows us to reduce all calculations to the pairwise coprime case. In addition, one of the following algorithms [5] uses this result.

1.2 Algorithms for computing the Frobenius number in three variables

Through the years, there have been a number of algorithms developed that compute the Frobenius number of three integers (see Johnson [8] Kennedy [9], Nijenhuis and Wilf [10], Rödseth [11], Selmer and Beyer [12], and Wilf [14]). We now highlight a select number of these.

1.2.1 Rödseth

Selmer and Beyer [12] used continuing fractions to formulate an algorithm for the Frobenius number for three pairwise coprime positive integers. Rödseth [11] almost immediately improved upon it as follows:

Step 1: Find s_0 to satisfy the congruence

$bs_0 \equiv c \pmod{a}$, where $0 \leq s_0 < a$.

Step 2: Use the Euclidean Algorithm in the form below:

$$\begin{aligned} a &= q_1 s_0 - s_1 & 0 &\leq s_1 < s_0 \\ s_0 &= q_2 s_1 - s_2 & 0 &\leq s_2 < s_1 \\ &\vdots & &\vdots \\ s_{m-1} &= q_{m+1} s_m - s_{m+1} & 0 &= s_{m+1} < s_m \end{aligned}$$

Step 3: Let $P_{-1} = 0$, $P_0 = 1$, and

$P_{i+1} = q_{i+1} P_i - P_{i-1}$ for $i = 0, 1, \ldots, m$.

Step 4: Let $a = s_{-1}$, define $\frac{s_{-1}}{P_{-1}} = \infty$, and find v, the unique integer satisfying

$$\frac{s_{v+1}}{P_{v+1}} \leq \frac{c}{b} < \frac{s_v}{P_v}.$$

Then,

$$g(a,b,c) = -a + b(s_v - 1) + c(P_{v+1} - 1) - \min\{bs_{v+1}, cP_v\}$$

Example: Let $a = 11$, $b = 30$, and $c = 117$.

Step 1: $30s_0 \equiv 117 \pmod{11}$ gives us $s_0 = 5$.

Step 2:

$$\begin{aligned} 11 &= q_1(5) - s_1 & q_1 &= 3 \quad s_1 = 4 \\ 5 &= q_2(4) - s_2 & q_2 &= 2 \quad s_2 = 3 \\ 4 &= q_3(3) - s_3 & q_3 &= 2 \quad s_3 = 2 \\ 3 &= q_4(2) - s_4 & q_4 &= 2 \quad s_4 = 1 \\ 2 &= q_5(1) - s_5 & q_5 &= 2 \quad s_5 = 0 \end{aligned}$$

Step 3: $P_{-1} = 0$, $P_0 = 1$ $P_1 = 3$, $P_2 = 5$, $P_3 = 7$, $P_4 = 9$, $P_5 = 11$.

Step 4: $\frac{s_1}{P_1} = \frac{4}{3} < \frac{117}{30} < \frac{5}{1} = \frac{s_0}{P_0}$, and so $v = 0$.

Then

$$g(11, 30, 117) = -11 + 30(5-1) + 117(3-1) - \min\{30(4),\ 117(1)\} = 226$$

1.2.2 Davison

The Rödseth [11] and Selmer-Beyer [12] algorithms were the word in the three-variable problem for about ten years. Greenberg [7] found another algorithm in 1988, which was rediscovered by Davison [5] in 1994. Davison's algorithm finds a value $f(a, b, c)$ such that $f(a, b, c) = g(a, b, c) + a + b + c$ as follows:

Step 0: Assume $a < b < c$. Find $\gcd(a, b, c)$. If $\gcd(a, b, c) \neq 1$, then $g(a, b, c)$ does not exist.

Step 1: Find $d_{12} = \gcd(a, b)$, $d_{13} = \gcd(a, c)$, and $d_{23} = \gcd(b, c)$. Set $a' = \frac{a}{d_{12}d_{13}}$, $b' = \frac{b}{d_{12}d_{23}}$, and $c' = \frac{c}{d_{13}d_{23}}$. $f(a, b, c) = d_{12}d_{13}d_{23} \cdot f(a', b', c')$. Reorder and relabel a', b', c' to $a < b < c$.

Step 2: Solve $bs \equiv c \pmod{a}$, where $0 < s < a$.

Step 3: Perform the Euclidean algorithm on (s, a).

Step 4: Let $r_{-1} = a$, $q_{-1} = 0$, $r_0 = 1$, and find k such that $\frac{r_{2k}}{q_{2k}} < \frac{c}{b} < \frac{r_{2k-2}}{q_{2k-2}}$.

Step 5: Set $\phi(t) = \frac{r_{2k-2} - tr_{2k-1}}{q_{2k-2} + tq_{2k-1}}$ and find t^* such that $\phi(t^* < \frac{c}{b} < \phi(t^* - 1)$. The bisection method is appropriate here, noting that ϕ is decreasing on $[0, a]$.

Step 6: Set $x' = r_{2k-2} - (t^* - 1)r_{2k-1}$, $y' = q_{2k-2} + (t^* - 1)q_{2k-1}$, $x'' = r_{2k-2} - t^* r_{2k-1}$, and $y'' = q_{2k-2} + t^* q_{2k-1}$. Then $f(a, b, c) = \max(bx' + cq_{2k-1}, br_{2k-1} + cy'')$.

Example: Using Davison's algorithm on $11, 30$, and 117, we find the following:

Step 1: $d_{12} = 1$, $d_{13} = 1$, and $d_{23} = 3$. Thus $a' = 11$, $b' = 10$, and $c' = 39$, and keeping in mind $d_{12}d_{13}d_{23} = 3$, rename $a = 10$, $b = 11$, and $c = 39$.

Step 2: $11s \equiv 39 \pmod{10}$ yields $s = 9$.

Step 3:

$$\begin{aligned} 10 =& q_1(9) + r_1 & \qquad q_1 = 1 \quad r_1 = 1 \\ 9 =& q_2(1) - r_2 & \qquad q_2 = 9 \quad r_2 = 0 \end{aligned}$$

Step 4: $\dfrac{b}{c} = \dfrac{39}{11} \approx 3.55$.
$\dfrac{r_0}{q_0} = 9$, and $\dfrac{r_2}{q_2} = 0$, so $k = 1$.
Step 5: $\phi(t) = \dfrac{9 - t(1)}{1 + t}$. Using the bisection method, $\phi(2) = 7/3$ and $\phi(1) = 4$, so $t^* = 2$.
Step 6: $x' = 9-(2-1)\cdot 1 = 8$, $x'' = 9-2(1) = 7$, $y' = 1+(2-1) = 2$, and $y'' = 1+2(1) = 3$.

So $f(10, 11, 39) = \max(11 \cdot 8 + 39 \cdot 1, 11 \cdot 1 + 39 \cdot 3) = 128$. Therefore $f(11, 30, 117) = 3 \cdot f(10, 11, 39) = 384$, and $g(11, 30, 117) = f(11, 30, 117) - 11 - 30 - 117 = 226$.

1.2.3 Kennedy

Douglas Palmer Kennedy, in his masters' thesis [9], developed an algorithm for determining the Frobenius number of a triple $\{a, b, c\}$, where $c = ab - ax - by$, with $a < b < c$.

Step 1: For $0 \le i < a$, find α_i and β_i such that $\chi(a, b) + i = \alpha_i a + \beta_i b$.

Step 2: Let t_i range in the interval $0 \le t_i \le \frac{\chi(a,b)+i}{c}$, and find the integer k_{it_i} so that $\frac{r_{it_i} - \alpha_i + t_i(b-x)}{b} \le k_{it_i} \le \frac{\beta_i - s_{it_i} + t_i y}{a}$. r_{it_i} and s_{it_i} range as well.

Step 3: For each i and t_i, find the maximal r_{it_i} and s_{it_i} satisfying the above inequality.

Step 4: Starting at $i = 0$, find the pair (r, s) such that $ra + sb$ is maximal and exists at all i.

Then $\chi(a, b, c) = \chi(a, b) - ra - sb$.

Example: Using Kennedy's algorithm on $11, 30$, and 117, we let i range from 0 to 10. The following values were obtained:

i	α_i	β_i	t_i	k_{it_i}	bound for (r_{it_i}, s_{it_i})
0	10	6	0	0	$(10, 6)$
			1	1	$(13, 1)$
1	21	2	0	0	$(21, 2)$
2	2	9	0	0	$(2, 9)$
			1	1	$(5, 4)$
3	13	5	0	0	$(13, 5)$
			1	1	$(16, 0)$
4	24	1	0	0	$(24, 1)$
			2	1	$(0, 2)$
5	5	6	0	0	$(5, 6)$
			1	1	$(8, 1)$
6	16	4	0	0	$(16, 4)$
7	27	0	0	0	$(27, 0)$
			1	0	$(0, 6)$
			2	1	$(3, 1)$
8	8	7	0	0	$(8, 7)$
			1	1	$(11, 2)$
9	19	3	0	0	$(19, 3)$
10	0	10	0	0	$(0, 10)$
			1	1	$(3, 5)$
			2	2	$(6, 0)$

The pair (r, s) that maximizes $ra + sb$ at all i is $(3, 1)$. Thus $\chi(11, 30, 117) = 290 - 11 \cdot 3 - 30 \cdot 1 = 227$.

Since then, focus in the field has been concentrated on making algorithms faster, notably Biehoffer et al. [4]. In addition, other algorithmic work has been done using generating functions.

In spite of all these attempts, an explicit form for the Frobenius number for three variables remains elusive. In the next chapter, we provide a formula for computing $\chi(a, b, c)$ in terms of $\chi(a, b)$ for special sets of integers a, b, c.

Chapter 2

A preliminary result

In working with three variables, the obvious simplification is to refer to one variable in terms of the other two.

This result from Brauer [1] makes that simplification possible:

Theorem 2.1 *Given a relatively prime pair* (a, b) *and* c *not divisible by* a *or* b *,* c *is of exactly one of the following forms:* $c = ax + by$ *or* $c = ab - ax - by$*, with* $x, y > 0$*.*

Proof: Let u be a solution of the congruence $au \equiv c \bmod b$ so that $0 < u < b$. Then $c = au + bv$ for some $v \in \mathbf{Z}$, and $v \neq 0$. If v is positive, then c has the form $ax + by$ with $x = u$ and $y = v$. If v is negative, then $v = -y$ for $y > 0$, and so

$$\begin{aligned} c =& au + bv \\ =& ab - a(b-u) - by \\ =& ab - a(b-u) - by \end{aligned}$$

So $x = b - u$, and c is of the form $ab - ax - by$.

c is not simultaneously representable in both forms. If it were, $ab - ax - by = ax' + by'$, where $x, x', y, y' > 0$. So $ab = a(x + x') + b(y + y')$. Therefore $a|(y + y')$, which is not possible, as $(y + y') < a$.

Starting with Brauer's result we can show the following:

Lemma 2.2 *Let* a *and* b *be relatively prime and* c *be a positive integer. If* $c = ab - ax - by$ *for some integers* $x, y > 0$*,* c *is not divisible by* a *or* b*.*

Proof: First we note that since $c > 0$, $ax < ab$, so also is $by < ab$. Thus $x < b$ and $y < a$. If $a|c$, then $ab - ax - by = ar$ for some integer $r > 0$. Therefore $a(b - x - r) = by$. Since $\gcd(a, b) = 1$, this means $a|y$. However, since $a > y$, this is a contradiction. Similarly, if $b|c$, then $ab - ax - by = sb$ for some integer $s > 0$, and so $b(a - y - s) = ax$. As a and b are coprime, $b|x$, a contradiction since $b > x$. So c is not divisible by a or b.

This leads to our next conclusion:

Proposition 2.3 *Let a and b be relatively prime and c be a positive integer. Then $c \notin \langle a, b\rangle$ if and only if $c = ab - ax - by$ for some integers $x, y > 0$.*

Proof: If $c \notin \langle a, b\rangle$, c is not divisible by a or b, so $c = ab - ax - by$ for some integers $x, y > 0$. If $c = ab - ax - by$ for some integers $x, y > 0$, c is not divisible by a or b by the above, and $c \neq ax' + by'$ by Theorem 2.1. So $c \notin \langle a, b\rangle$.

The following two can be found in Kennedy:

Proposition 2.4 *Let a and b be relatively prime and c be a positive integer. Then $\chi(a, b, c) = \chi(a, b)$ if and only if $c = ax + by$ for some $x, y \geq 0$.*

Proof: We first note that $\chi(a, b, c) \leq \chi(a, b)$. Now assume $c = ax+by$ for some $x, y \geq 0$. Then $\chi(a, b) - 1 = ua + vb + wc$ for some integers $u, v, w \geq 0$. But then

$$\begin{aligned}\chi(a, b) - 1 &= ua + vb + wc \\ &= ua + vb + w(ax + by) \\ &= ra + sb\end{aligned}$$

for some $r, s \geq 0$, a contradiction. Hence $\chi(a, b, c) = \chi(a, b)$. For the converse, suppose c is not in the form $ax + by$ for any $x, y \geq 0$. Then by Proposition 2.3, $c = ab - ax - by$

for some $x, y > 0$. Furthermore $\chi(a,b) = ab - a - b + 1$, so $\chi(a,b) - 1 = ab - a - b = ab - ax - by + ax - a + by - b = c + a(x-1) + b(y-1)$. But $x - 1, y - 1 \geq 0$. So $\chi(a,b,c) \leq \chi(a,b) - 1 < \chi(a,b)$. Thus $\chi(a,b,c) \neq \chi(a,b)$.

Proposition 2.5 *Let a and b be relatively prime and c be a positive integer. Then $\chi(a,b,c) = \chi(a,b) - ra - sb$ for some $r, s \geq 0$.*

Proof: We first note that $\chi(a,b,c) - 1 \notin \langle a,b,c \rangle$ and $\langle a,b \rangle \subset \langle a,b,c \rangle$. So $\chi(a,b,c) - 1 \notin \langle a,b \rangle$. Therefore, $\chi(a,b,c) - 1 = ab - ax - by$ for some $x, y > 0$, and so

$$\begin{aligned}\chi(a,b,c) &= ab - ax - by + 1 \\ &= \chi(a,b) + a + b - ax - by \\ &= \chi(a,b) - (x-1)a - (y-1)b.\end{aligned}$$

The last result is notably only an existence proof. The real problem is finding the appropriate coefficients r and s so that $\chi(a,b,c) = \chi(a,b) - ra - sb$.

Looking at the Frobenius numbers of all triples $\{11, 30, k\}$, where n is less than 330, we noticed a pattern: for many triples, the difference in k was equal to the difference in the Frobenius number of the triples. In fact, for some triples, $\chi(11, 30, n + 11i) = \chi(11, 30, k) + 11i$, and for others, $\chi(11, 30, n + 30j) = \chi(11, 30, n) + 30j$. Further examination led to the formulation of the following:

Theorem 2.6 *Let a and b be relatively prime and c be any positive integer. Then*

1. *$c \in \langle a,b \rangle$ if and only if $\chi(a,b,c) = \chi(a,b)$*

2. *$c \notin \langle a,b \rangle$ if and only if $c = ab - ax - by$ for some $x, y > 0$. In this case,*

(a) if $ax < by$ *and* $0 < y \leq \lfloor a/2 \rfloor$*, then* $\chi(a,b,c) = \chi(a,b) - ax$

(b) if $ax > by$ *and* $0 < x \leq \lfloor b/2 \rfloor$*, then* $\chi(a,b,c) = \chi(a,b) - by$*.*

Proof: (1) follows from Proposition 2.4 while the first part of (2) follows from Proposition 2.3.

If $c \notin \langle a,b \rangle$, then $c = ab - ax - by$ by Brauer [1]. Conversely, if $c = ab - ax - by$ and $a|c$, then $a|y$ since a and b are relatively prime, a contradiction since $y < a$. Similarly, c is not divisible by b. So $c \notin \langle a,b \rangle$ again by Brauer [1].

We now assume the first part of (2) and argue that if $ax < by$ and $0 < y \leq \lfloor a/2 \rfloor$ then $\chi(a,b,c) = \chi(a,b) - ax$.

Suppose $\chi(a,b,c) < \chi(a,b) - ax$. Then $\chi(a,b) - ax - 1 = ra + sb + t(ab - ax - by)$ for some $r, s, t \geq 0$ where $c = ab - ax - by$. Note that $t > 0$ since otherwise $\chi(a,b) - 1 = (r+x)a + sb$, a contradiction.

Suppose t is even. Then $\chi(a,b) - 1 = (r + x + \frac{t}{2}(b - 2x))a + (s + \frac{t}{2}(a - 2y))b$. But $y \leq \frac{a}{2}$ and so $x \leq \frac{b}{2}$ since $ax < by$. Thus $\chi(a,b) - 1 \in \langle a,b \rangle$, a contradiction.

Now suppose t is odd. Then

$$\begin{aligned}
\chi(a,b) - 1 &= (r + x + \frac{t-1}{2}(b-2x))a + (s + \frac{t-1}{2}(a-2y))b + \frac{1}{2}(b-2x)a + \frac{1}{2}(a-2y)b \\
&= (r + x + \frac{t-1}{2}(b-2x))a + (s + \frac{t-1}{2}(a-2y))b + c
\end{aligned}$$

Define u and v as follows:

$$\begin{aligned}
u &= r + x + \frac{t-1}{2}(b-2x) \\
v &= s + \frac{t-1}{2}(a-2y)
\end{aligned}$$

So

$$
\begin{aligned}
c &= ab - a - b - ua - vb \\
&= ab - (u+1)a - (v+1)b
\end{aligned}
$$

Therefore $u + 1 = x$ since $c = ab - ax - by$ uniquely. That is, $r + \frac{t-1}{2}(b - 2x) + 1 = 0$, a contradiction since $t > 0$ and $2x \leq b$. Thus $\chi(a, b, c) \geq \chi(a, b) - ax$.

Suppose now that $\chi(a, b, c) > \chi(a, b) - ax$. By Kennedy [9], $\chi(a, b, c) = \chi(a, b) - ra - sb$ for some $r, s \geq 0$. So $\chi(a, b) - ra - sb > \chi(a, b) - ax$, and so $ra + sb < ax$. Thus $r < x$ and since $ax < by$, $s < y$.

We know that $\chi(a, b) - ra - sb - 1$ is not in $\langle a, b, c\rangle$. But

$$
\begin{aligned}
\chi(a, b) - ra - sb - 1 &= ab - a - b - ra - sb \\
&= ab - ax - by + ax + by - a - b - ra - sb \\
&= c + (x - r - 1)a + (y - s - 1)b.
\end{aligned}
$$

But $x - r - 1 \geq 0$, and $y - s - 1 \geq 0$, so $c + (x - r - 1)a + (y - s - 1)b \in \langle a, b, c\rangle$, a contradiction. Thus $\chi(a, b, c) \leq \chi(a, b) - ax$, and so $\chi(a, b, c) = \chi(a, b) - ax$.

Now we assume the second part of (2) and argue if $ax > by$ and $0 < x \leq \lfloor b/2 \rfloor$, then $\chi(a, b, c) = \chi(a, b) - by$

Suppose $\chi(a, b, c) < \chi(a, b) - by$. Then $\chi(a, b) - by - 1 = ra + sb + tc$ for some $r, s, t \geq 0$. Again, note that $t > 0$, because otherwise $\chi(a, b) - 1 \in \langle a, b\rangle$, a contradiction.

Suppose t is even. Then $\chi(a, b) - 1 = (r + \dfrac{t}{2}(b - 2x))a + (s + y + \dfrac{t}{2}(a - 2y))b$. Since $x \leq b/2$ and $by < ax$, then also $y \leq a/2$, and $\chi(a, b) - 1 \in \langle a, b\rangle$, a contradiction.

Now suppose t is odd. Then

$$\begin{aligned}\chi(a,b)-1 &= (r+\frac{t-1}{2}(b-2x))a+(s+y+\frac{t-1}{2}(a-2y))b+\frac{1}{2}(b-2x)a+\frac{1}{2}(a-2y)b \\ &= (r+\frac{t-1}{2}(b-2x))a+(s+y+\frac{t-1}{2}(a-2y))b+c\end{aligned}$$

Define u and v as follows:

$$\begin{aligned}u &= r+\frac{t-1}{2}(b-2x) \\ v &= s+y+\frac{t-1}{2}(a-2y)\end{aligned}$$

So $\chi(a,b)-1=ab-a-b=ua+vb+c$, and so $c=ab-a-b-ua-vb=ab-(u+1)a-(v+1)b$. But $c=ab-ax-by$ uniquely by Brauer [1]. So $v+1=y$, or equivalently,$s+\frac{t-1}{2}(a-2y)+1=0$. As $s\geq 0$, $t>0$ and $a>2y$, this is a contradiction. Thus $\chi(a,b,c)\geq\chi(a,b)-by$.

Suppose now that $\chi(a,b,c)>\chi(a,b)-by$. By 2.5, we know that $\chi(a,b,c)=\chi(a,b)-ra-sb$ for some $r,s\geq 0$, and so $\chi(a,b)-ra-sb>\chi(a,b)-by$. So $ra+sb<by$, so $s<y$, and since $by<ax$, $r<x$. We know that $\chi(a,b)-ra-sb-1$ is not in $\langle a,b,c\rangle$. But

$$\begin{aligned}\chi(a,b)-ra-sb-1 &= ab-a-b-ra-sb \\ &= ab-ax-by+ax+by-a-b-ra-sb \\ &= c+(x-r-1)a+(y-s-1)b.\end{aligned}$$

Since $x-r-1\geq 0$ and $y-s-1\geq 0$, $\chi(a,b,c)-1=c+(x-r-1)a+(y-s-1)b\in\langle a,b,c\rangle$, a contradiction. Thus $\chi(a,b,c)\leq\chi(a,b)-by$, and so $\chi(a,b,c)=\chi(a,b)-by$.

Example: Let $a = 11$, $b = 30$ and $c = 152$. Then the solution to $30q = 152 (\text{mod } 11)$ is $q = 8$. But then $bq > c$ and so $c \notin \langle 11, 30 \rangle$. Now $ab - c = 330 - 152 = 178$, and solving $30y = 178 (\text{mod } 11)$ gives $y = 3$ and so $x = \dfrac{178 - 30(3)}{11} = 8$. Thus, $by = 90 > 88 = ax$ and $y = 3 < \lfloor \dfrac{11}{2} \rfloor$. Hence, $\chi(11, 30, 152) = \chi(a, b) - ax = 29 \cdot 10 - 11 \cdot 8 = 202$ by the theorem above.

The condition $0 < y \leq \lfloor \dfrac{a}{2} \rfloor$ is necessary. For let $c = 117$ and a, b be as in the above. Then $117 = 330 - 213 = 330 - 11 \cdot 3 - 30 \cdot 6$ and so $x = 3$ and $y = 6$. We note that $ax < by$ and $y = 6 > \lfloor \dfrac{a}{2} \rfloor$. And $\chi(11, 30) - ax = 290 - 33 = 257 \neq 227 = \chi(11, 30, 117)$. In fact, $\chi(11, 30, 117) = \chi(11, 30) - 11 \cdot 3 - 30 \cdot 1$.

Remark 2.7 Let $T = \mathbb{N} - < a, b >$ and $R = \{c \in T : 0 < x \leq \lfloor \frac{b}{2} \rfloor \text{ and } 0 < y \leq \lfloor \frac{a}{2} \rfloor\}$. Then $|R| \geq \frac{(a-1)}{2} \cdot \frac{(b-1)}{2} = \frac{1}{4}(a-1)(b-1)$. But $|T| = \frac{1}{2}\chi(a, b)$ by Sylvester 1.1. So $|R| \geq \frac{1}{2}|T|$. That is, the formula above calculates $\chi(a, b, c)$ for at least half of the elements in T. Moreover, if $c \in T - R$ and $c < a$ or $c < b$, then $\chi(a, b, c)$ may be calculated using the theorem above. Consequently the theorem calculates $\chi(a, b, c)$ for a much larger number of elements c in T. For example, in the example above, let $c = 6$. Then $6 \in T - R$ and clearly $30 \in < 11, 6 >$. So $\chi(30, 11, 6) = \chi(11, 6) = 50$ by the theorem. Similarly, let $c = 20$. Then $20 \in T$ and $20 = 330 - 30(3) - 11(20)$ and so $x = 3$ and $y = 20$. But $y = 20 > \lfloor \frac{a}{2} \rfloor = 15$ and so $20 \in T - R$. If we now let $a = 11, b = 20$ and $c = 30$, then $\gcd(11, 20) = 1$ and $30 \notin < 11, 20 >$. Furthermore, $30 = 220 - (11)(10) - (20)(4)$ and so $x = 10$ and $y = 4$. Thus $ax = 110 > 80 = by$ and $x = 10 \leq \lfloor \frac{20}{2} \rfloor$. Hence $\chi(30, 11, 20) = \chi(11, 20) - (20)(4) = 190 - 80 = 110$ by the theorem.

Chapter 3

Main Results

Lemma 3.1 *If* $n = \lceil \frac{y}{a-y} \rceil$ *and* $n(ab - by) < ab - ax$*, then* $y \leq \frac{an}{n+1}$ *and* $x \leq \frac{b}{n+1}$.

Proof: By the definition of n, $n \geq \frac{y}{a-y}$. Further,

$$
\begin{aligned}
n(a - y) &\geq y \\
an - ny &\geq y \\
an &\geq (n + 1)y \\
\frac{an}{n+1} &\geq y
\end{aligned}
$$

Because of this, and because $n(ab - by) < ab - ax$,

$$
\begin{aligned}
ax &< ab - nab + nby \\
&\leq ab - nab + nb\frac{an}{n+1} \\
ax &\leq \frac{ab}{n+1} \\
x &\leq \frac{b}{n+1}
\end{aligned}
$$

Lemma 3.2 *If* $m = \lceil \frac{x}{b-x} \rceil$ *and* $m(ab - ax) < ab - by$*, then* $x \leq \frac{bm}{m+1}$ *and* $y \leq \frac{a}{m+1}$.

Proof: By the definition of m, $m \geq \frac{x}{b-x}$. Further,

$$m(b-x) \geq x$$
$$mb - mx \geq x$$
$$mb \geq (m+1)x$$
$$\frac{mb}{m+1} \geq x$$

Because of this, and because $m(ab-ax) < ab - by$,

$$\begin{aligned} by &< ab - mab + max \\ &\leq ab - mab + mb\frac{bm}{m+1} \\ by &\leq \frac{ab}{m+1} \\ y &\leq \frac{a}{m+1} \end{aligned}$$

3.1 First Result

Theorem 3.3 *Let a, b, c be relatively prime and let $c = ab - ax - by$. If $n = \lceil \frac{y}{a-y} \rceil$ and $m = \lceil \frac{x}{b-x} \rceil$, then*

$$\chi(a,b,c) \geq \begin{cases} \chi(a,b) - nax, & if \quad n(ab-by) < ab - ax; \\ \chi(a,b) - mby, & if \quad m(ab-ax) < ab - by. \end{cases} \tag{3.1}$$

We will prove this in parts:

3.1.1 Part 1: $n(ab-by) < ab-ax$

We assume the condition in the first part, and argue that $\chi(a,b,c) = \chi(a,b) - nax$.

First we suppose not, and look for a contradiction. If given the above condition, $\chi(a,b,c) < \chi(a,b) - nax$, then $\chi(a,b)-nax-1$ is in the monoid $\langle a,b,c\rangle$. So $\chi(a,b)-nax-1 = ra+sb+tc$ for nonnegative r,s,t.

This can be rewritten $\chi(a,b)-1 = ra+nax+sb+tc$. Since we know for certain that the left-hand side of the last is not in the monoid $\langle a,b\rangle$, we know that t is positive.

Now there are multiple possibilities, based on the relationship between t and n.

If $n+1 > t$, then $\chi(a,b)-1 = ra+nax-tax+sb+tab-tby$. This can be rewritten as $\chi(a,b)-1 = a(r+x(n-t))+b(s+t(a-y))$. Since $(n-t)$ is nonnegative, as is $(a-y)$, this is a contradiction. So $t \geq n+1$.

So now we suppose that $n+1$ divides t. This divisibility allows us to break up the term tc as follows: $\left(\frac{t}{n+1}\right)ab + \left(\frac{nt}{n+1}\right)ab - tax - tby$.

This means that $\chi(a,b)-1 = ra+nax+\left(\frac{t}{n+1}\right)ab - tax + sb + \left(\frac{nt}{n+1}\right)ab - tby$. Grouping like terms gives us

$$\chi(a,b)-1 = a\left(r+nx+\left(\frac{t}{n+1}\right)(b-(n+1)x)\right) + b\left(s+\left(\frac{t}{n+1}\right)(na-(n+1)y)\right)$$

Since $y < \frac{na}{n+1}$ and $x < \frac{b}{n+1}$, by 3.1 this is a contradiction. So $(n+1)$ does not divide t.

Since $(n+1)$ does not divide t, Euler says that $t = (n+1)q+p$ for some $0 < p \leq n$. With this form, we know that $(n+1)$ does divide $(t-p)$.

We then take the equation we had before, and regroup the terms thus: $\chi(a,b)-1 = ra+nax+sb+(t-p)(ab-ax-by)+p(ab-ax-by)$.

And then dividing by $(n+1)$,

$\chi(a,b)-1 = ra+nax+(\frac{t-p}{n+1})(ab-(n+1)ax)+sb+(\frac{t-p}{n+1})(nab-(n+1)by)+p(ab-ax-by)$

If we let $u := r + nx + (\frac{t-p}{n+1})(b-(n+1)x)$, and $v := s + (\frac{t-p}{n+1})(na-(n+1)y)$, then $\chi(a,b)-1 = ua+vb+pc$. By the same lemmas as above, $u \geq 0$ and $v \geq 0$.

Suppose a or b divides p. Then $\chi(a,b)-1 \in \langle a,b \rangle$, a contradiction.

So, since $\chi(a,b) = ab-a-b+1$, $ab-a-b = ua+vb+pc$, or $pc = ab-(u+1)-(v+1)$. And as neither a or b divides p, this is unique by Brauer.

However, $pc = pab - pax - pby$.

Rearranging this, $pc = ab - pax - b(py-(p-1)a)$.

We know that $y > \frac{(n-1)a}{n}$ from the definition of n. As $n \geq p$, $y > \frac{p-1}{p}a$. The coefficient on b is therefore nonnegative. This is also unique, therefore there must be equality:

$ab - pax - b(py-(p-1)a) = ab-(u+1)a-(v+1)b$, and so $u+1 = px$.

$r + nx + (\frac{t-p}{n+1})(b-(n+1)x) + 1 = px$. Rewrite to read as follows:

$r+(n-p)x+(\frac{t-p}{n+1})(b-(n+1)x)+1 = 0$. As the three terms added to 1 are nonnegative, this is a contradiction.

No t exists such that $\chi(a,b)-nax-1 = ra+sb+tc$, so $\chi(a,b,c) \geq \chi(a,b)-nax$

3.1.2 Part 2: $m(ab-ax) < ab-by$

This follows similarly to Part 1. We assume $m(ab-ax) < ab-by$ and argue $\chi(a,b,c) = \chi(a,b)-mby$.

First we assume not. Suppose $\chi(a,b,c) < \chi(a,b)-mby$. Then $\chi(a,b)-mby+1 = ra+sb+tc$ for some $r,s,t \geq 0$. Note that $t > 0$ since otherwise $\chi(a,b)-1 = ra+sb+mby$, a contradiction, as $\chi(a,b)-1 \notin \langle a,b \rangle$.

The rest follows based on the relationship between m and t. Suppose $m+1>t$.

$$\chi(a,b)-1 = ra+sb+t(ab-ax-by)+mby$$
$$= ra+sb+ta(b-y)+by(m-t)$$

As $b>y$ and $m\geq t$, this implies $\chi(a,b)-1\in\langle a,b\rangle$, a contradiction. So $t\geq m+1$.

Suppose $(m+1)|t$. Then $\chi(a,b)-1 = ra+sb+\dfrac{t}{m+1}(ab)+\dfrac{mt}{m+1}(ab)-tax-tby$. Rearranging, the following is evident:

$$\chi(a,b)-1 = a(r+\tfrac{t}{m+1}(mb-x(m+1)))+b(s+my+\tfrac{t}{m+1}(a-y(m+1)))$$

As $m\geq \dfrac{x}{b-x}$, and $y\leq \dfrac{a}{m+1}$ by 3.2 $\chi(a,b)-1\in\langle a,b\rangle$, a contradiction. So $m+1$ does not divide t.

So then $t=(m+1)q+p$ for some positive q and $0<p\leq m$. Therefore, $(m+1)|(t-p)$. Still, $\chi(a,b)-1 = ra+sb+mby+t(ab-ax-by)$. Using the above divisibility,

$$\chi(a,b)-1 = ra+sb+(t-p)c+pc$$
$$= a(r+\frac{t-p}{m+1}(mb-(m+1)x))+b(s+my+\frac{t-p}{m+1}(a-(m+1)y))+pc$$

For conciseness' sake, define

$$u = r+\frac{t-p}{m+1}(mb-(m+1)x)$$
$$v = s+my+\frac{t-p}{m+1}(a-(m+1)y)$$

So $\chi(a,b)-1 = ua+vb+pc$. If a or b divides p, then $\chi(a,b)-1\in\langle a,b\rangle$, a contradiction. So $pc = ab-a-b-ua-vb$, and by Brauer [1], this is unique. But also $pc = pab-pax-pby$, which can be written $pc = ab - a(px - (p-1)b) - pby$. Since $m(ab - ax) < ab - by$, $x > \dfrac{b(m-1)}{m} > \dfrac{b(p-1)}{p}$, this is also unique, and so the two are equal, and so $v + 1 = py$. So $s + (m-p)y + \frac{t-p}{m+1}(a - (m+1)y) + 1 = 0$. As $y \leq \frac{a}{m+1}$, this is a contradiction. So $\chi(a, b, c) \geq \chi(a, b) - mby$.

3.2 Second Result

Theorem 3.4 *Let a, b, c be relatively prime and let $c = ab - ax - by$. If $n = \lceil \dfrac{y}{a-y} \rceil$, $m = \lceil \dfrac{x}{b-x} \rceil$, then*

$$\chi(a,b,c) \geq \begin{cases} \chi(a,b) - (n-1)ax - (a - n(a-y))b, & \textit{if } ab - ax < n(ab - by) \textit{ and } x \leq \frac{b}{n+1}; \\ \chi(a,b) - (m-1)by - (b - m(b-x)a, & \textit{if } ab - by < m(ab - ax) \textit{ and } y \leq \frac{a}{m+1}. \end{cases} \tag{3.2}$$

3.2.1 Part 1: $ab - ax < n(ab - by)$

First we suppose not, and look for a contradiction. If given the above conditions, $\chi(a, b, c) < \chi(a, b) - (n-1)ax - (a - n(a-y))b$, then $\chi(a, b) - (n-1)ax - (a - n(a-y))b = ra + sb + tc$ for some $r, s \geq 0$, and $t \geq 1$.

Thus $\chi(a, b) - 1 = ra + (n-1)ax + sb + (a - n(a-y))b + t(ab - ax - by)$.

If $t < n$, then the above can be rewritten thus: $\chi(a, b) - 1 = a(r + (n-1)x - tx) + b(s + (a - n(a-y)) + t(a-y))$, a contradiction. So $t \geq n$

The expression can be expanded thus: $\chi(a, b) - 1 = ra + nax - ax + sb + ab - nab + nby + tc$, or $\chi(a, b) - 1 = ra + ab - ax + sb + (t-n)c$. If $t = n$, then $\chi(a, b) - 1 = a(r + (b-x)) + sb$, a contradiction. So $t \neq n$, and so $t > n$, or $t \geq n + 1$.

Suppose $(n+1)|t$. Then $\chi(a,b)-1 = ra+(n-1)ax+\left(\frac{t}{n+1}\right)(b-(n+1)x)a+sb+(a-n(a-y))b+\left(\frac{t}{n+1}\right)(na-(n+1)y)b$, a contradiction. So $n+1$ does not divide t.

As $t \geq (n+1)$ and $n+1$ does not divide t, $t=(n+1)q+p$ for some $p \leq n$, and so $(n+1)|(t-p)$.

Therefore $\chi(a,b)-1 = ra+(n-1)ax+sb+(a-n(a-y))b+(t-p)(ab-ax-by)+pc$. As $(n+1)|(t-p)$, this can be rewritten as $\chi(a,b)-1 = ra+(n-1)ax+sb+(a-n(a-y))b+\left(\frac{t-p}{n+1}\right)a(b-(n+1)x)+\left(\frac{t-p}{n+1}\right)b(na-(n+1)y)+pc$.

For conciseness, define u and v as follows:

$$u = \left[r+(n-1)x+\left(\frac{t-p}{n+1}\right)(b-(n+1)x)\right]$$
$$v = \left[s+(a-n(a-y))+\left(\frac{t-p}{n+1}\right)(na-(n+1)y)\right]$$

This way $\chi(a,b)-1 = ab-a-b = ua+vb+pc$. If $a|p$ or $b|p$, this is a contradiction. Otherwise, $pc = ab-(u+1)a-(v+1)b$, which is unique by Brauer.

But $pc = p(ab-ax-by) = ab-pax-b(py-(p-1)a)$, which is also unique by Brauer. So $px = u+1$ and $py-(p-1)a = v+1$. Therefore $\left[r+(n-1)x+\left(\frac{t-p}{n+1}\right)(b-(n+1)x)\right]+1-px = 0$, or, $\left[r+(n-1-p)x+\left(\frac{t-p}{n+1}\right)(b-(n+1)x)\right]+1=0$. If $p \leq (n-1)$, this is a contradiction.

If $p=n$, look at the b coefficients:

$$py-(p-1)a = \left[s+(a-n(a-y))+\left(\frac{t-p}{n+1}\right)(na-(n+1)y)\right]+1$$
$$\left[s+(a-n(a-y))+\left(\frac{t-p}{n+1}\right)(na-(n+1)y)\right]+(p-1)a-py+1=0$$

, or, since $p = n$,

$$s + \left(\frac{t-p}{n+1}\right)(na - (n+1)y) + 1 = 0.$$

As s and $\left(\frac{t-p}{n-1}\right)(na-(n+1)y)$ are both nonnegative, this is a contradiction, and $\chi(a,b,c) \geq \chi(a,b) - (n-1)ax - (a - n(a-y))b$.

3.2.2 Part 2: $ab - by < m(ab - ax)$

Suppose that $\chi(a,b,c) < \chi(a,b) - (m-1)by - (b - m(b-x))a$. Then $\chi(a,b) - (m-1)by - (b - m(b-x))a - 1 = ra + sb + tc$ for some $r, s \geq 0$ and $t \geq 1$, as in the previous cases. Rearranging, we get

$$\chi(a,b) - 1 = [r + b - m(b-x)]a + [s + (m-1)y]b + t(ab - ax - by) \tag{3.3}$$

If $t < m$, the above can be rewritten thus: $\chi(a,b) - 1 = a[r + (b - m(b-x)) + t(b-x)] + b[s + (m-1)y - ty]$, and the coefficients on a and b are both nonnegative, a contradiction. So $t \geq m$.

The above inequality 3.3 can be expanded like this:

$$\begin{aligned}\chi(a,b) - 1 &= ra + ab - mab + max + sb + mby - by + tc \\ &= ra + sb + (a-x)b + (t-m)c\end{aligned}$$

If $m = t$, this is a contradiction. So $t > m$, or equivalently, $t \geq m + 1$.

Suppose $(m+1)|t$. Then inequality 3.3 can be written thus:

$$\chi(a,b) - 1 = [r + (b - m(b-x)) + \tfrac{t}{m+1}(mb - (m+1)x)]a + [s + (m-1)y + \tfrac{t}{m+1}(a - (m+1)y)]b$$

As each coefficient is a nonnegative integer, this is a contradiction. So $(m+1)$ does not divide t.

So $t = (m+1)q + p$ for some positive q and $0 < p \leq m$. Therefore $(m+1)|(t-p)$. And so $\chi(a,b) - 1 = ra + (b - m(b-x))a + sb + (m-1)by + (t-p)(ab - ax - by) + pc$
$= a[r + (b - m(b-x)) + \frac{t-p}{m+1}(mb - (m+1)x)] + b[s + (m-1)y + \frac{t-p}{m+1}(a - (m+1)y)] + pc.$

Assign u and v the following values:

$$u = r + (b - m(b-x)) + \frac{t-p}{m+1}(mb - (m+1)x)$$
$$v = s + (m-1)y + \frac{t-p}{m+1}(a - (m+1)y)$$

And so $\chi(a,b) - 1 = ua + vb + pc$. If $a|p$ or $b|p$, this is a contradiction. By Theorem 1.1,

$$pc = ab - a - b - ua - vb = ab - (u+1)a - (v+1)b,$$

and by Theorem 2.1, this is unique.

But $pc = p(ab - ax - by) = ab - a(px - (p-1)b) - pby$. As $x \leq \frac{(m-1)b}{m} \leq \frac{(p-1)b}{p}$, the coefficients on a and b are positive integers, and so this also is unique. So the two can be equated, $v + 1 = py$, or

$$s + (m-1)y + \frac{t-p}{m+1}(a - (m+1)y) + 1 = py$$
$$s + (m-1)y - py + \frac{t-p}{m+1}(a - (m+1)y) + 1 = 0$$
$$s + (m-1-p)y + \frac{t-p}{m+1}(a - (m+1)y) + 1 = 0$$

If $p < m$, each term added to 1 is nonnegative, so the above is a contradiction. If $p = m$, look instead at $u + 1 = px - (p-1)b$.

$$\begin{aligned} r + (b - m(b-x)) + \frac{t-p}{m+1}(mb - (m+1)x) + 1 &= mx - (m-1)b \\ r + (b - m(b-x)) - mx + (m-1)b + \frac{t-p}{m+1}(mb - (m+1)x) + 1 &= 0 \\ r + \frac{t-p}{m+1}(mb - (m+1)x) + 1 &= 0 \end{aligned}$$

As each of the above terms is nonnegative, and one is positive, this also is a contradiction. Therefore there is no t so that $\chi(a,b) - (m-1)by - (b - m(b-x))a - 1 = ra + sb + tc$, and $\chi(a,b) - (m-1)by - (b - m(b-x))a \leq \chi(a,b,c)$.

One should note that when $n = m = 1$, the above results reduce to the preliminary result in Chapter 2.

Chapter 4

Future Work

The obvious place to start with future work is the attempt to show the previous results hold with equality. The examples in Appendices A and B, as well as several others not included here, support this conjecture.

Previous attempts have involved trying to find some number of c to add to the conjectured conductor that will result in the same sort of contradiction that served us in this case in our preliminary result.

Ideas for further attempts involve reevaluating the formulation of the inequality, or perhaps working strictly with x and y, and not m and n.

The following result of Brauer and Seelbinder [3] suggests that the previous results may have some corollary in more than three dimensions. First, define T as follows:
Let $d_1 = a_1, d_2 = \gcd(a_1, a_2), \ldots, d_k = \gcd(a_1, a_2, \ldots a_k)$. Then

$$\begin{aligned} T &= T(a_1, a_2, \ldots, a_k) \\ &= \frac{a_2 d_1}{d_2} + \frac{a_3 d_2}{d_3} + \ldots \frac{a_k d_{k-1}}{d_k} \end{aligned}$$

Theorem 4.1 *Let $a_1, a_2, \ldots, a_k$ be relatively prime positive integers. Then every integer m is representable in at least one of the forms*

1. $m = a_1x_1 + a_2x_2 + \ldots + a_kx_k \qquad (x_\kappa > 0 for \kappa = 1, 2, \ldots, k)$

2. $m = T - a_1x_1 - a_2x_2 - \ldots - a_kx_k \qquad (x_\kappa \geq 0 for \kappa = 1, 2, \ldots, k)$

In the pairwise relatively prime case with four variables, $T = ab + c$.

In particular, let $k = 4$. Then one can show that $\chi(a,b,c,d) = \chi(a,b) - ra - sb - tc$ for some $r, s, t \geq 0$. One then can develop an algorithm as in Kennedy [9] to find r, s, t for given pairwise nonnegative integers a, b, c, d.

Note that if $d = ax + by + cz$ for some $x, y, z \geq 0$, then the monoid generated by $\langle a, b, c, d\rangle$ is the same as the one generated by $\langle a, b, c\rangle$ and so $\chi(a,b,c,d) = \chi(a,b,c)$ and so $r = s = t = 0$.

Now suppose d is not in $\langle a, b, c\rangle$, then

$$\begin{aligned} d &= T - ax - by - cz \\ &= ab + c - ax - by - cz \\ &= ab - ax - by - c(z-1) \end{aligned}$$

If $i \geq 0$, then $\chi(a,b) + i = \alpha_i a + \beta_i b$ for some nonnegative integers α_i, β_i. So

$$\begin{aligned} \chi(a,b) - ra - sb - tc + i &= \alpha_i a + \beta_i b - ra - sb - tc - w_i + w_i d \\ &= \alpha_i a + \beta_i b - ra - sb - tc - w_i(ab - ax - by - c(z-1)) + w_i d \end{aligned}$$

$$= [\alpha_i - r + w_i(x-b) + k_i b]a + [\beta_i - s + w_i y - k_i a]b + [-t + w_i(z-1)]c + w_i d$$

where k_i, w_i are nonnegative integers.

So, at each i, w_i, we would find $r_{iw_i}, s_{iw_i}, t_{iw_i}$ so that there is an integer k_{iw_i} such that

$$\frac{r - \alpha_i + w_i(b-x)}{b} \leq k_{iw_i} \leq \frac{\beta_i - s_{iw_i} + w_i y}{a}$$

and $t_{iw_i} \leq w_i(z-1)$.

Then we proceed as in Kennedy [9] to find values for r, s, and t. This approach has been demonstrated to work in some examples.

Note that the case $z = 0$ needs to be considered separately.

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Appendices

Appendix A

$11, 30, k$ data set

Remembering that $\chi(a, b, c) = \chi(a, b) - ra - sb$, we break down the following data as follows:

x	k $g(11, 30, k)$ r, s
	y

Note that $g(11, 30) = 289$.

10	19 179 10,0	8 69 20,0									
9	49 234 5,0	38 179 10,0	27 127 12,1	16 83 16,1	5 39 20,1						
8	79 256 3,0	68 223 6,0	57 190 9,0	46 157 12,0	35 124 15,0	24 97 12,2	13 75 14,2	2 9 20,2			
7	109 267 2,0	98 245 4,0	87 223 6,0	76 201 8,0	65 179 10,0	54 157 12,0	43 135 14,0	32 113 16,0	21 100 9,3	10 89 10,3	
6	139 267 2,0	128 245 4,0	117 226 3,1	106 215 4,1	95 204 5,1	84 193 6,1	73 182 7,1	62 171 8,1	51 160 9,1	40 149 10,1	29 108 11,2
5	169 278 1,0	158 267 2,0	147 256 3,0	136 245 4,0	125 234 5,0	114 223 6,0	103 212 7,0	92 201 8,0	81 190 9,0	70 179 10,0	59 168 11,0
4	199 278 1,0	188 267 2,0	177 256 3,0	166 245 4,0	155 234 5,0	144 223 6,0	133 212 7,0	122 201 8,0	111 190 9,0	100 179 10,0	89 169 0,4
3	229 278 1,0	218 267 2,0	207 256 3,0	196 245 4,0	185 234 5,0	174 223 6,0	163 212 7,0	152 201 8,0	141 199 0,3	130 199 0,3	119 199 0,3
2	259 278 1,0	248 267 2,0	237 256 3,0	226 245 4,0	215 234 5,0	204 229 0,2	193 229 0,2	182 229 0,2	171 229 0,2	160 229 0,2	149 229 0,2
1	289 278 1,0	278 267 2,0	267 259 0,1	256 259 0,1	245 259 0,1	234 259 0,1	223 259 0,1	212 259 0,1	201 259 0,1	190 259 0,1	179 259 0,1
	1	2	3	4	5	6	7	8	9	10	11

6	18 97 12,2	7 45 13,3								
5	48 157 12,0	37 146 13,0	26 139 0,5	15 139 0,5	4 29 11,5					
4	78 169 0,4	67 169 0,4	56 169 0,4	45 169 0,4	34 125 4,4	23 95 4,5	12 73 6,5	1 -1		
3	108 199 0,3	97 199 0,3	86 199 0,3	75 199 0,3	64 177 2,3	53 155 4,3	42 133 6,3	31 111 8,3	20 109 0,6	9 46 3,7
2	138 229 0,2	127 229 0,2	116 229 0,2	105 229 0,2	94 207 2,2	83 185 4,2	72 169 0,4	61 169 0,4	50 169 0,4	39 136 3,4
1	168 259 0,1	157 259 0,1	146 259 0,1	135 259 0,1	124 237 2,1	113 229 0,2	102 229 0,2	91 229 0,2	80 229 0,2	69 199 0,3
	12	13	14	14	15	16	17	18	19	20

2	28 109 0,6	17 87 2,6	6 49 0,8			
1	58 199 0,3	47 177 2,3	36 169 0,4	25 139 0,5	14 137 2,6	3 19 0,9
	21	22	23	24	25	26

Appendix B

28, 57, k data set

Remembering that $\chi(a,b,c) = \chi(a,b) - ra - sb$, we break down the following data as follows:

x	k $g(28,57,k)$ r,s
	y

Note that $g(28,57) = 1511$.

27	29 755 27,0	1 -1									
26	86 1147 13,0	58 783 26, 0	30 419 39,0	2 55 52,0							
25	143 1259 9,0	115 1007 18,0	87 782 24,1	59 558 32,1	31 334 40,1	3 53 48,2					
24	200 1343 6,0	172 1175 12,0	144 1007 18,0	116 839 24,0	88 671 30,0	60 503 36,0	32 335 42,0	4 167 48,0			
23	257 1371 5,0	229 1231 10,0	201 1091 15,0	173 951 20,0	145 811 25,0	117 671 30,0	89 556 28,3	61 444 32,3	33 332 36,3	5 79 43,4	
22	314 1399 4,0	286 1287 8,0	258 1175 12,0	230 1063 16,0	202 951 20,0	174 839 24,0	146 727 28,0	118 615 32,0	90 527 27,4	62 443 30,4	34 359 33,4
21	371 1427 3,0	343 1343 6,0	315 1259 9,0	287 1175 12,0	259 1091 15,0	231 1007 18,0	203 923 21,0	175 839 24,0	147 755 27,0	119 671 30,0	91 587 33,0
20	428 1427 3,0	400 1343 6,0	372 1259 9,0	344 1175 12,0	316 1091 15,0	288 1007 18,0	260 923 21,0	232 839 24,0	204 779 18,4	176 723 20,4	148 667 22,4
19	485 1427 3,0	457 1343 6,0	429 1286 6,1	401 1230 8,1	373 1174 10,1	345 1118 12,1	317 1062 14,1	289 1006 16,1	261 950 18,1	233 894 20,1	205 838 22,1
18	542 1455 2,0	514 1399 4,0	486 1343 6,0	458 1287 8,0	430 1231 10,0	402 1175 12,0	374 1119 14,0	346 1063 16,0	318 1007 18,0	290 951 20,0	262 895 22,0
17	599 1455 2,0	571 1399 4,0	543 1343 6,0	515 1287 8,0	487 1231 10,0	459 1175 12,0	431 1119 14,0	403 1063 16,0	375 1007 18,0	347 951 20,0	319 895 22,0
16	656 1455 2,0	628 1399 4,0	600 1343 6,0	572 1287 8,0	544 1231 10,0	516 1175 12,0	488 1119 14,0	460 1063 16,0	432 1031 9,4	404 1003 10,4	376 975 11,4
15	713 1455 2,0	685 1399 4,0	657 1343 6,0	629 1287 8,0	601 1257 5,2	573 1229 6,2	545 1201 7,2	517 1173 8,2	489 1145 9,2	461 1117 10,2	433 1089 11,2
14	770 1483 1,0	742 1455 2,0	714 1427 3,0	686 1399 4,0	658 1371 5,0	630 1343 6,0	602 1315 7,0	574 1287 8,0	546 1259 9,0	518 1231 10,0	490 1203 11,0
	1	2	3	4	5	6	7	8	9	10	11

22	6 107 42,4										
21	63 503 36,0	35 419 39,0	7 335 42,0								
20	120 611 24,4	92 555 26,4	64 499 28,4	36 359 33,4	8 191 39,4						
19	177 782 24,1	149 726 26,1	121 670 28,1	93 557 30,2	65 501 32,2	37 388 34,3	9 161 36,6				
18	234 839 24,0	206 783 26,0	178 727 28,0	150 671 30,0	122 615 32,0	94 579 17,8	66 551 18,8	38 523 19,8	10 159 32,8		
17	291 839 24,0	263 805 13,6	235 777 14,6	207 749 15,6	179 721 16,6	151 693 17,6	123 665 18,6	95 637 19,6	67 468 23,7	39 356 27,7	11 159 32,8
16	348 947 12,4	320 919 13,4	292 891 14,4	264 863 15,4	236 835 16,4	208 807 17,4	180 779 18,4	152 751 19,4	124 639 23,4	96 527 27,4	68 439 22,8
15	405 1061 12,2	377 1033 13,2	349 1005 14,2	321 977 15,2	293 949 16,2	265 921 17,2	237 893 18,2	209 865 19,2	181 753 23,3	153 695 21,4	125 667 22,4
14	462 1175 12,0	434 1147 13,0	406 1119 14,0	378 1091 15,0	350 1063 16,0	322 1035 17,0	294 1007 18,0	266 979 19,0	238 951 20,0	210 923 21,0	182 895 22,0
	12	13	14	15	16	17	18	19	20	21	22

16	40 383 24,8	12 215 27,10				
15	97 611 24,4	69 497 24,6	41 357 29,6	13 185 27,10		
14	154 867 23,0	126 839 24,0	98 811 25,0	70 783 26,0	42 755 27,0	14 727 28,0
	23	24	25	26	27	28

13	827 1483 1,0	799 1455 2,0	771 1427 3,0	743 1399 4,0	715 1371 5,0	687 1343 6,0	659 1315 7,0	631 1287 8,0	603 1259 9,0	575 1231 10,0	547 1203 11,0
12	884 1483 1,0	856 1455 2,0	828 1427 3,0	800 1399 4,0	772 1371 5,0	744 1343 6,0	716 1315 7,0	688 1287 8,0	660 1259 9,0	632 1231 10,0	604 1203 11,0
11	941 1483 1,0	913 1455 2,0	885 1427 3,0	857 1399 4,0	829 1371 5,0	801 1343 6,0	773 1315 7,0	745 1287 8,0	717 1259 9,0	689 1231 10,0	661 1203 11,0
10	998 1483 1,0	970 1455 2,0	942 1427 3,0	914 1399 4,0	886 1371 5,0	858 1343 6,0	830 1315 7,0	802 1287 8,0	774 1259 9,0	746 1231 10,0	718 1203 11,0
9	1055 1483 1,0	1027 1455 2,0	999 1427 3,0	971 1399 4,0	943 1371 5,0	915 1343 6,0	887 1315 7,0	859 1287 8,0	831 1259 9,0	803 1231 10,0	775 1203 11,0
8	1112 1483 1,0	1084 1455 2,0	1056 1427 3,0	1028 1399 4,0	1000 1371 5,0	972 1343 6,0	944 1315 7,0	916 1287 8,0	888 1259 9,0	860 1231 10,0	832 1203 11,0
7	1169 1483 1,0	1141 1455 2,0	1113 1427 3,0	1085 1399 4,0	1057 1371 5,0	1029 1343 6,0	1001 1315 7,0	973 1287 8,0	945 1259 9,0	917 1231 10,0	889 1203 11,0
6	1226 1483 1,0	1198 1455 2,0	1170 1427 3,0	1142 1399 4,0	1114 1371 5,0	1086 1343 6,0	1058 1315 7,0	1030 1287 8,0	1002 1259 9,0	974 1231 10,0	946 1203 11,0
5	1283 1483 1,0	1255 1455 2,0	1227 1427 3,0	1199 1399 4,0	1171 1371 5,0	1143 1343 6,0	1115 1315 7,0	1087 1287 8,0	1059 1259 9,0	1031 1231 10,0	1003 1226 0,5
4	1340 1483 1,0	1312 1455 2,0	1284 1427 3,0	1256 1399 4,0	1228 1371 5,0	1200 1343 6,0	1172 1315 7,0	1144 1287 8,0	1116 1283 0,4	1088 1283 0,4	1060 1283 0,4
3	1397 1483 1,0	1369 1455 2,0	1341 1427 3,0	1313 1399 4,0	1285 1371 5,0	1257 1343 6,0	1229 1340 0,3	1201 1340 0,3	1173 1340 0,3	1145 1340 0,3	1117 1340 0,3
2	1454 1483 1,0	1426 1455 2,0	1398 1427 3,0	1370 1399 4,0	1342 1397 0,2	1314 1397 0,2	1286 1397 0,2	1258 1397 0,2	1230 1397 0,2	1202 1397 0,2	1174 1397 0,2
1	1511 1483 1,0	1483 1455 2,0	1455 1454 0,1	1427 1454 0,1	1399 1454 0,1	1371 1454 0,1	1343 1454 0,1	1315 1454 0,1	1287 1454 0,1	1259 1454 0,1	1231 1454 0,1
	1	2	3	4	5	6	7	8	9	10	11

13	519	491	463	435	407	379	351	323	295	267	239
	1175	1147	1119	1091	1063	1035	1007	979	951	923	895
	12,0	13,0	14,0	15,0	16,0	17,0	18,0	19,0	20,0	21,0	22,0
12	576	548	520	492	464	436	408	380	352	324	296
	1175	1147	1119	1091	1063	1035	1007	979	951	923	895
	12,0	13,0	14,0	15,0	16,0	17,0	18,0	19,0	20,0	21,0	22,0
11	633	605	577	549	521	493	465	437	409	381	353
	1175	1147	1119	1091	1063	1035	1007	979	951	923	895
	12,0	13,0	14,0	15,0	16,0	17,0	18,0	19,0	20,0	21,0	22,0
10	690	662	634	606	578	550	522	494	466	438	410
	1175	1147	1119	1091	1063	1035	1007	979	951	941	941
	12,0	13,0	14,0	15,0	16,0	17,0	18,0	19,0	20,0	0,10	0,10
9	747	719	691	663	635	607	579	551	523	495	467
	1175	1147	1119	1091	1063	1035	1007	998	998	998	998
	12,0	13,0	14,0	15,0	16,0	17,0	18,0	0,9	0,9	0,9	0,9
8	804	776	748	720	692	664	636	608	580	552	524
	1175	1147	1119	1091	1063	1055	1055	1055	1055	1055	1055
	12,0	13,0	14,0	15,0	16,0	0,8	0,8	0,8	0,8	0,8	0,8
7	861	833	805	777	749	721	693	665	637	609	581
	1175	1147	1119	1112	1112	1112	1112	1112	1112	1112	1112
	12,0	13,0	14,0	0,7	0,7	0,7	0,7	0,7	0,7	0,7	0,7
6	918	890	862	834	806	778	750	722	694	666	638
	1175	1169	1169	1169	1169	1169	1169	1169	1169	1169	1169
	12,0	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6	0,6
5	975	947	919	891	863	835	807	779	751	723	695
	1226	1226	1226	1226	1226	1226	1226	1226	1226	1226	1226
	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5	0,5
4	1032	1004	976	948	920	892	864	836	808	780	752
	1283	1283	1283	1283	1283	1283	1283	1283	1283	1283	1283
	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4	0,4
3	1089	1061	1033	1005	977	949	921	893	865	837	809
	1340	1340	1340	1340	1340	1340	1340	1340	1340	1340	1340
	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3	0,3
2	1146	1118	1090	1062	1034	1006	978	950	922	894	866
	1397	1397	1397	1397	1397	1397	1397	1397	1397	1397	1397
	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2	0,2
1	1203	1175	1147	1119	1091	1063	1035	1007	979	951	923
	1454	1454	1454	1454	1454	1454	1454	1454	1454	1454	1454
	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1	0,1
	12	13	14	15	16	17	18	19	20	21	22

13	211 867 23,0	183 839 24,0	155 811 25,0	127 783 26,0	99 770 0,13	71 770 0,13	43 574 7,13	15 209 18,14			
12	268 867 23,0	240 839 24,0	212 827 0,12	184 827 0,12	156 827 0,12	128 827 0,12	100 743 3,12	72 575 9,12	44 407 15,12	16 239 21,12	
11	325 884 0,11	297 884 0,11	269 884 0,11	241 884 0,11	213 884 0,11	185 884 0,11	157 828 2,11	129 716 6,11	101 604 10,11	73 492 14,11	45 380 18,11
10	382 941 0,10	354 941 0,10	326 941 0,10	298 941 0,10	270 941 0,10	242 941 0,10	214 885 2,10	186 773 6,10	158 687 5,12	130 631 7,12	102 575 9,12
9	439 998 0,9	411 998 0,9	383 998 0,9	355 998 0,9	327 998 0,9	299 998 0,9	271 970 1,9	243 914 3,9	215 858 5,9	187 802 7,9	159 746 9,9
8	496 1055 0,8	468 1055 0,8	440 1055 0,8	412 1055 0,8	384 1055 0,8	356 1055 0,8	328 1027 1,8	300 971 3,8	272 915 5,8	244 859 7,8	216 803 9,8
7	553 1112 0,7	525 1112 0,7	497 1112 0,7	469 1112 0,7	441 1112 0,7	413 1112 0,7	385 1084 1,7	357 1028 3,7	329 972 5,7	301 916 7,7	273 860 9,7
6	610 1169 0,6	582 1169 0,6	554 1169 0,6	526 1169 0,6	498 1169 0,6	470 1169 0,6	442 1141 1,6	414 1085 3,6	386 1029 5,6	358 973 7,6	330 917 9,6
5	667 1226 0,5	639 1226 0,5	611 1226 0,5	583 1226 0,5	555 1226 0,5	527 1226 0,5	499 1198 1,5	471 1142 3,5	443 1086 5,5	415 1030 7,5	387 974 9,5
4	724 1283 0,4	696 1283 0,4	668 1283 0,4	640 1283 0,4	612 1283 0,4	584 1283 0,4	556 1255 1,4	528 1199 3,4	500 1143 5,4	472 1087 7,4	444 1055 0,8
3	781 1340 0,3	753 1340 0,3	725 1340 0,3	697 1340 0,3	669 1340 0,3	641 1340 0,3	613 1312 1,3	585 1256 3,3	557 1200 5,3	529 1169 0,6	501 1169 0,6
2	838 1397 0,2	810 1397 0,2	782 1397 0,2	754 1397 0,2	726 1397 0,2	698 1397 0,2	670 1369 1,2	642 1313 3,2	614 1283 0,4	586 1283 0,4	558 1283 0,4
1	895 1454 0,1	867 1454 0,1	839 1454 0,1	811 1454 0,1	783 1454 0,1	755 1454 0,1	727 1426 1,1	699 1397 0,2	671 1397 0,2	643 1397 0,2	615 1397 0,2
	23	24	25	26	27	28	29	30	31	32	33

11	17 291 11,16										
10	74 519 11,12	46 351 17,12	18 209 18,14								
9	131 690 11,9	103 634 13,9	75 578 15,9	47 522 17,9	19 485 0,18						
8	188 747 11,8	160 691 13,8	132 635 15,8	104 599 0,16	76 599 0,16	48 431 6,16	20 263 12,16				
7	245 804 11,7	217 748 13,7	189 713 0,14	161 713 0,14	133 713 0,14	105 629 3,14	77 545 6,14	49 461 9,14	21 377 12,14		
6	302 861 11,6	274 827 0,12	246 827 0,12	218 827 0,12	190 827 0,12	162 743 3,12	134 659 6,12	106 575 9,12	78 491 12,12	50 429 2,18	22 231 5,20
5	359 941 0,10	331 941 0,10	303 941 0,10	275 941 0,10	247 941 0,10	219 857 3,10	191 773 6,10	163 689 9,10	135 656 0,15	107 628 1,15	79 516 5,15
4	416 1055 0,8	388 1055 0,8	360 1055 0,8	332 1055 0,8	304 1055 0,8	276 971 3,8	248 887 6,8	220 827 0,12	192 827 0,12	164 799 1,12	136 687 5,12
3	473 1169 0,6	445 1169 0,6	417 1169 0,6	389 1169 0,6	361 1169 0,6	333 1085 3,6	305 1001 6,6	277 998 0,9	249 998 0,9	221 970 1,9	193 858 5,9
2	530 1283 0,4	502 1283 0,4	474 1283 0,4	446 1283 0,4	418 1283 0,4	390 1199 3,4	362 1169 0,6	334 1169 0,6	306 1169 0,6	278 1141 1,6	250 1055 0,8
1	587 1397 0,2	559 1397 0,2	531 1397 0,2	503 1397 0,2	475 1397 0,2	447 1340 0,3	419 1340 0,3	391 1340 0,3	363 1340 0,3	335 1312 1,3	307 1283 0,4
	34	35	36	37	38	39	40	41	42	43	44

5	51 404 9,15	23 259 4,20								
4	108 599 0,16	80 543 2,16	52 403 7,16	24 287 3,20						
3	165 827 0,12	137 771 2,12	109 656 0,15	81 572 3,15	53 403 1,18	25 286 1,21				
2	222 1055 0,8	194 999 2,8	166 941 0,10	138 857 3,10	110 799 1,12	82 685 1,14	54 515 3,16	26 315 2,10		
1	279 1283 0,4	251 1227 2,4	223 1226 0,5	195 1169 0,6	167 1141 1,6	139 1084 1,7	111 998 0,9	83 885 2,10	55 742 1,13	27 485 0,18
	45	46	47	48	49	50	51	52	53	54

Appendix C

Notes on the style-file project

These style-files for use with LaTeX are maintained by Darrel Hankerson[1] and Ed Slaminka[2].

In 1990, department heads and other representatives met with Dean Doorenbos and Judy Bush-Crofton (then responsible for manuscript approval). This meeting was prompted by a memorandum[3] from members of the mathematics departments concerning the *Thesis and Dissertation Guide* and the approval process. There was wide agreement among the participants (including Dean Doorenbos) to support the basic recommendations outlined in the memorandum. The revised *Guide* reflected some (but not all) of the agreements of the meeting.

Ms Bush-Crofton was supportive of the plan to obtain "official approval" of these style files.[4] Unfortunately, Ms Bush-Crofton left the Graduate School before the process was completed. In 1994, we find that we are returning to some of the same problems which were resolved at the 1990 meeting.

In Summer 1994, I sent several memoranda to Ms. Ilga Trend of the Graduate School, reminding her of the agreements made at the 1990 meeting. Professors A. Scottedward Hodel and Stan Reeves provided additional support. In short, it is essential that the Graduate School honor its commitments of the 1990 meeting. It should be emphasized that Dean Doorenbos is to thank for the success of that meeting.

Maintaining these LaTeX files has been more work than I expected. If the Graduate School rejects your manuscript based on items controlled by the style-files, please contact Darrel Hankerson or Ed Slaminka so that we can coordinate an appropriate response.

Finally, there have been several requests for additions to the package (mostly formatting changes for figures, etc.). While such changes are not really part of the thesis-style package, it could be beneficial to collect these options and distribute with the package (making it easier on the next student). I'm especially interested in changes needed by various departments.

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[3] Originally, the memorandum was presented to Professor Larry Wit. A copy is available on request.

[4] Followup memoranda gave a definition of "official approval." Copies will be sent on request.