Construction of Orthonormal Multivariate Wavelets

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
August 6, 2011

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Abstract

The purpose of this paper is to explain the construction of orthonormal multivariate wavelets associated with a multiresolution analysis. This paper primarily uses the work of R. A. Zalik [10], where he outlines a method of constructing orthonormal multivariate wavelets given an existing orthonormal multivariate wavelet associated with an MRA, and attempts to clarify it for a wider audience. In the last section, I use the result in constructing some orthonormal multivariate wavelets in various examples.
Acknowledgments

I’d like to thank my wife Kirsten for her encouragement throughout this process. I’d also like to thank my adviser, Dr. Zalik, for introducing me to the subject of wavelets and helping to guide my studies for the past two years, as well as for his help in my editing process. Finally, I’d like to thank Dr. Ángel San Antolín Gil for helping to focus my efforts and being available for additional help when needed.
Table of Contents

Abstract ................................................................. ii
Acknowledgments ....................................................... iii
1 Introduction ........................................................... 1
2 Main Results .......................................................... 7
3 Examples .............................................................. 15
Bibliography ............................................................. 19
Chapter 1

Introduction

In what follows, $d > 1$ will be an integer, arbitrary but fixed; $\mathbb{Z}$ will denote the set of integers and $\mathbb{R}$ the set of real numbers; boldface lowcase letters will always denote elements of $\mathbb{R}^d$; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $i$ will be reserved for the imaginary number $\sqrt{-1}$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g \rangle$, their bracket product by $[f, g]$, and the norm of $f$ by $||f||$; thus,

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(t) \overline{g(t)} dt,$$

$$[f, g](t) := \sum_{k \in \mathbb{Z}^d} f(t + k) \overline{g(t + k)},$$

and

$$||f|| := \sqrt{\langle f, f \rangle}.$$

The Fourier transform of a function $f$ will be denoted by $\hat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\hat{f}(x) := \int_{\mathbb{R}^d} e^{-2\pi \mathbf{i} \mathbf{x} \cdot \mathbf{t}} f(t) dt$$

For every $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, the operators $D^j$ and $T_k$ are defined in $L^2(\mathbb{R}^d)$ by

$$D^j f(t) := 2^{dj/2} f(2^j t)$$

and

$$T_k f(t) := f(t - k)$$
A set of functions \( \{ \psi_1, ..., \psi_m \} \subset L^2(\mathbb{R}^d) \) is called an orthonormal multivariate wavelet, if the sequence

\[
\{ D^j T_k \psi_l ; j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq l \leq m \}
\] it generates is an orthonormal basis of \( L^2(\mathbb{R}^d) \).

A multiresolution analysis (MRA) in \( L^2(\mathbb{R}^d) \) is a sequence \( \{ V_j ; j \in \mathbb{Z} \} \) of closed linear subspaces of \( L^2(\mathbb{R}^d) \) such that:

\[
V_j \subset V_{j+1} \quad \text{for every} \quad j \in \mathbb{Z} \quad (i)
\]

For every \( j \in \mathbb{Z}, f(t) \in V_j \) if and only if \( f(2t) \in V_{j+1} \) \( (ii) \)

\[
\bigcup_{j \in \mathbb{Z}} V_j \quad \text{is dense in} \quad L^2(\mathbb{R}^d). \quad (iii)
\]

There is a function \( u \) such that \( \{ T_k u ; k \in \mathbb{Z}^d \} \) is an orthonormal basis of \( V_0 \) \( (iv) \)

Let \( \mathbb{T} := [0, 1] \), and let \( \mathbb{T}^d \) denote the d-dimensional torus. A function \( f \) will be called \( \mathbb{Z}^d \)-periodic if it is defined in \( \mathbb{R}^d \), and for every \( k \in \mathbb{Z}^d \) and \( x \in \mathbb{R}^d \) we have \( f(x + k) = f(x) \).

**Claim:** It follows from the definition of MRA that there is a \( \mathbb{Z}^d \)-periodic function \( p \in L^2(\mathbb{T}^d) \) such that

\[
\widehat{u}(2x) = p(x)\widehat{u}(x) \quad \text{a.e.}
\]

**Proof.** Let \( \{ T_k u ; k \in \mathbb{Z}^d \} \) be an orthonormal basis of \( V_0 \). In particular, \( u(t) \in V_0 \). Thus, by (ii), \( u(\frac{t}{2}) \in V_{-1} \). By (i) and (iv), we can write

\[
u\left(\frac{t}{2}\right) = \sum_{k \in \mathbb{Z}^d} a_k T_k u(t)
\]
Where \( a_k = \langle u(t), T_k u(t) \rangle \). By taking the Fourier transform of both sides, we get

\[
\int_{\mathbb{R}^d} e^{-2\pi i t \cdot x} u(t/2) dt = \int_{\mathbb{R}^d} e^{-2\pi i t \cdot x} \sum_{k \in \mathbb{Z}^d} a_k u(t - k) dt.
\]

By changing variables, we get

\[
\int_{\mathbb{R}^d} e^{-2\pi i (2s) \cdot x} u(s) 2^d ds = \int_{\mathbb{R}^d} e^{-2\pi i (s+k) \cdot x} \sum_{k \in \mathbb{Z}^d} a_k u(s) ds.
\]

Which yields,

\[
2^d \hat{u}(2x) = \sum_{k \in \mathbb{Z}^d} a_k e^{-2\pi i k \cdot x} \hat{u}(x)
\]

If we let \( p(x) = 2^{-d} \sum_{k \in \mathbb{Z}^d} a_k e^{-2\pi i k \cdot x} \), then the result follows.

The function \( u \) is called a scaling function for the MRA, and \( p \) is called the low pass filter associated with \( u \).

We will denote the orthogonal complement of \( V_j \) in \( V_{j+1} \) by \( W_j \). Thus, \( V_{j+1} = V_j \oplus W_j \).

Let \( \{\psi_1, ..., \psi_m\} \) be an orthonormal multivariate wavelet in \( L^2(\mathbb{R}^d) \); for \( j \in \mathbb{Z} \), let \( P_j \) denote the closure of the linear span of

\[
\{ D^j T_k \psi_l; k \in \mathbb{Z}^d, 1 \leq l \leq m \}
\]

and let \( V_j := \sum_{r < j} P_r \). Note that \( \psi_1, ..., \psi_m \in V_1 \). We say that \( \{\psi_1, ..., \psi_m\} \) is associated with an MRA if \( M := \{V_j; j \in \mathbb{Z}\} \) is a multiresolution analysis. If this is the case, we also say that \( \{\psi_1, ..., \psi_m\} \) is associated with \( M \). The definition implies that \( \{\psi_1, ..., \psi_m\} \) is an orthonormal multivariate wavelet associated with \( M \) if and only if \( \{T_k \psi_l; k \in \mathbb{Z}^d, 1 \leq l \leq m\} \) is an orthonormal basis of \( W_0 \).

Given \( \{u_1, ..., u_m\} \subset L^2(\mathbb{R}^d) \), we will adopt the following notation:

\[
T(u_1, ..., u_m) := \{T_k u_l; k \in \mathbb{Z}^d, 1 \leq l \leq m\}.
\]
and
\[ S(u_1, \ldots, u_m) := \text{span} \ T(u_1, \ldots, u_m). \]

The following is a special case of a theorem by Guo, Lebate et al. [4, Proposition 2.1]

**Theorem 1.** Assume that \( T(u_1, \ldots, u_m) \) and \( T(h_1, \ldots, h_n) \) are orthonormal sequences in \( L^2(\mathbb{R}^d) \) such that \( S(u_1, \ldots, u_m) = S(h_1, \ldots, h_m) \). Then \( m = n \).

**Proof.** Since \( S(u_1, \ldots, u_m) = S(h_1, \ldots, h_m) \),

\[
  u_l(x) = \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^n \langle u_l, T_k h_j \rangle T_k h_j(x) \quad l = 1, \ldots, m \quad \text{and} \quad h_j(x) = \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^m \langle h_j, T_k u_l \rangle T_k u_l(x)
\]

This implies that

\[
  1 = ||u_l||^2 = \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^n |\langle u_l, T_k h_j \rangle|^2 \quad \text{and} \quad 1 = ||h_j||^2 = \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^m |\langle h_j, T_k u_l \rangle|^2
\]

Also note that \( \langle u_l, T_k h_j \rangle = \langle T_{-k} u_l, h_j \rangle \) Thus we can show,

\[
  m = \sum_{l=1}^m ||u_l||^2 = \sum_{l=1}^m \sum_{k \in \mathbb{Z}^d} \sum_{j=1}^n |\langle u_l, T_k h_j \rangle|^2 = \sum_{j=1}^n \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^m |\langle T_{-k} u_l, h_j \rangle|^2 = \sum_{j=1}^n ||h_j||^2 = n
\]

\[ \Box \]

In [7] Wilson and Weiss showed that if \( \{\psi_1, \ldots, \psi_l\} \) is an orthonormal multivariate wavelet in \( L^2(\mathbb{R}^d) \) associated with a multiresolution analysis, then \( m = 2^d - 1 \). Hence, when combined with Theorem 1, we have that in the case of orthonormal multivariate wavelets associated with the same MRA, \( m = n = 2^d - 1 \).

The following theorem is from [5, p. 57], which we adapt to suit the above definition of the Fourier transform.

**Theorem 2.** If \( \phi \) is a scaling function for an MRA \( \{V_j; j \in \mathbb{Z}\} \) and \( p \) is the associated low pass filter, then \( h \in L^2(\mathbb{R}) \) is an orthonormal wavelet associated with this MRA if and only
if there is a measurable unimodular and $\mathbb{Z}$-periodic function $v(x)$, such that

$$\hat{h}(2x) = e^{2\pi i x} v(2x) \overline{p(x + 1/2)} \widehat{\phi}(x) \quad \text{a.e.}$$

The main results of this paper will be generalizing the following corollary to wavelets in $L^2(\mathbb{R}^d)$.

**Corollary 1.** If $h$ is an orthonormal wavelet associated with an MRA, then $\psi$ is an orthonormal wavelet associated with the same MRA if and only if there is a measurable unimodular and $\mathbb{Z}$-periodic function $q(x)$ such that

$$\hat{\psi}(x) = q(x) \hat{h}(x) \quad \text{a.e.}$$

The following theorems will be referenced multiple times in the paper and will be included here as a reference.

**Theorem 3.** (Parseval’s Identity) Let $f \in L^2(\mathbb{T}^d)$, and $c_k := \hat{f}(k)$ be the Fourier coefficients of $f$. Then

$$\sum_{k \in \mathbb{Z}^d} |c_k|^2 = ||f||^2_{L^2(\mathbb{T}^d)}$$

**Theorem 4.** (Plancherel’s Theorem) Let $f, g \in L^2(\mathbb{R}^d)$. Then,

$$\int_{\mathbb{R}^d} |f(t)|^2 dt = \int_{\mathbb{R}^d} |\hat{f}(x)|^2 dx$$

**Corollary 2.**

$$\int_{\mathbb{R}^d} f(t) \overline{g(t)} dt = \int_{\mathbb{R}^d} \hat{f}(x) \overline{\hat{g}(x)} dx$$

**Theorem 5.** (Fubini’s Theorem) Let $X, Y$ be measure spaces. If

$$\int_{X \times Y} |f(x, y)| d(x, y) < \infty.$$
Then
\[\int_X \int_Y |f(x, y)| dy dx = \int_Y \int_X |f(x, y)| dx dy = \int_{X \times Y} |f(x, y)| d(x, y)\]

Corollary 3. If
\[\sum_n \int_A |f(n, x)| dx < \infty,\]

then,
\[\sum_n \int_A f(n, x) dx = \int_A \sum_n f(n, x) dx\]

Theorem 6. (Gram-Schmidt Orthogonalization) Let \(\{u_1, ..., u_n\} \in S\) linearly independent, where \(S\) is an inner product space. Then we can find a set \(\{\tilde{u}_1, ..., \tilde{u}_n\} \in S\) of orthonormal vectors that span the same space.
Chapter 2
Main Results

**Lemma 1.** (a) $T(u_1...u_m)$ is an orthogonal sequence in $L^2(\mathbb{R}^d)$ if and only if

$$\hat{[\hat{u}_l, \hat{u}_j]}(x) = 0 \quad \text{a.e.,} \quad l, j = 1, ..., m \quad l \neq j$$

(b) $T(u_1...u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ if and only if

$$\hat{[\hat{u}_l, \hat{u}_j]}(x) = \delta_{l,j} \quad \text{a.e.,} \quad l, j = 1, ..., m$$

*Proof.* It suffices to prove (b).

Let $a, b \in \mathbb{Z}^d$ and $k = b - a$. Then

$$\langle T_a u_l, T_b u_j \rangle = \langle u_l, T_k u_j \rangle$$

$$= \int_{\mathbb{R}^d} u_l(t) \overline{u_j(t-k)} dt$$

$$= \int_{\mathbb{R}^d} \hat{u}_l(x) \overline{\hat{u}_j(x)} e^{2\pi i k \cdot x} dx \quad \text{(Plancherel’s Theorem)}$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \hat{u}_l(y+n) \overline{\hat{u}_j(y+n)} e^{2\pi i k \cdot (y+n)} dy \quad \text{("periodize" the integral)}$$

$$= \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} \hat{u}_l(y+n) \overline{\hat{u}_j(y+n)} e^{2\pi i k \cdot y} dy \quad \text{(Fubini’s Theorem)}$$

$$= \int_{\mathbb{T}^d} \hat{[\hat{u}_l, \hat{u}_j]}(y) e^{2\pi i k \cdot y} dy \quad (1)$$

Thus $\langle T_a u_l, T_b u_j \rangle$ are the Fourier coefficients of $[\hat{u}_l, \hat{u}_j]$, and Parseval’s Identity implies that

$$||[\hat{u}_l, \hat{u}_j]||_{L^2(\mathbb{T}^d)}^2 = \sum_{k \in \mathbb{Z}^d} |\langle u_l, T_k u_j \rangle|^2 \quad (2)$$
Assume $T(u_1, ..., u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$. Then for $l \neq j$, we have that the right hand side of (2) is equal to 0, which implies that $[\hat{u}_l, \hat{u}_j](x) = 0$ a.e.. When $l = j$, we have that the $\langle u_l, T_k u_j \rangle$ are the Fourier coefficients of the function 1, and by the uniqueness of Fourier coefficients (since $[\hat{u}_l, \hat{u}_j](x)$ is $\mathbb{Z}^d$-periodic and in $L^2(\mathbb{T}^d)$), we have that $[\hat{u}_l, \hat{u}_l](x) = 1$ a.e., and thus $[\hat{u}_l, \hat{u}_j](x) = \delta_{l,j}$ a.e..

Conversely, assume $[\hat{u}_l, \hat{u}_j] = \delta_{l,j}$ a.e.. Then when $l \neq j$, (1) implies that $\langle T_a u_l, T_b u_j \rangle = \int_{\mathbb{T}^d} 0 \ dy = 0$. When $l = j$, (1) implies that $\langle T_a u_l, T_b u_l \rangle = \int_{\mathbb{T}^d} e^{2\pi i k \cdot y} dy = \delta_{a,b}$. Thus $T(u_1, ..., u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$.

**Lemma 2.** If $T(h_1, ..., h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and $S(u_1, ..., u_m) \subset S(h_1...h_m)$, then there are $\mathbb{Z}^d$-periodic functions $p_{l,j}(x) \in L^2(\mathbb{T}^d)$, uniquely defined a.e., such that

$$\hat{u}_l(x) = \sum_{r=1}^{m} p_{l,r}(x) \hat{h}_r(x) \ a.e., \ l = 1, ..., m \tag{3}$$

**Proof.** Since $T(h_1...h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ we can write the orthogonal projection of $u_l$ onto $S(h_j)$, which we will denote by $u_{l,r}$, as

$$u_{l,r}(t) = \sum_{k \in \mathbb{Z}^d} a_{l,r,k} h_r(t - k).$$

Where $a_{l,r,k} = \langle u_l, T_k h_r \rangle$, and since $S(u_1...u_m) \subset S(h_1...h_m)$, we can write

$$u_l(t) = \sum_{r=1}^{m} u_{l,r}(t) = \sum_{r=1}^{m} \sum_{k \in \mathbb{Z}^d} a_{l,r,k} h_r(t - k).$$

If we take the Fourier Transform of both sides, we get

$$\hat{u}_l(x) = \sum_{r=1}^{m} \sum_{k \in \mathbb{Z}^d} a_{l,r,k} \hat{h}_r(x) e^{-2\pi i k \cdot x}$$

$$= \sum_{r=1}^{m} \hat{h}_r(x) \sum_{k \in \mathbb{Z}^d} a_{l,r,k} e^{-2\pi i k \cdot x}$$
If we let $p_{l,r}(x) = \sum_{k \in \mathbb{Z}^d} a_{l,r,k} e^{-2\pi i k \cdot x}$, then the result follows.

**Lemma 3.** Assume that $T(h_1, \ldots, h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and that $S(u_1, \ldots, u_m) \subset S(h_1, \ldots, h_m)$, and assume there are $\mathbb{Z}^d$-periodic functions $p_{l,j}(x) \in L^2(\mathbb{T}^d)$ such that (3) is satisfied. Then $T(u_1, \ldots, u_m)$ is an orthonormal sequence if and only if

$$\sum_{r=1}^m p_{l,r}(x)\overline{p_{j,r}(x)} = \delta_{l,j} \text{ a.e., } l, j = 1, \ldots, m \quad (4)$$

**Proof.** Let $u_{l,r}$ denote the orthogonal projection of $u_l$ onto $S(h_r)$. Then

$$\hat{u}_{l,r}(x) = p_{l,r}(x)\hat{h}_r(x) \text{ a.e. } l, r = 1, \ldots, m.$$ 

Note that $\hat{u}_l(x) = \sum_{r=1}^m \hat{u}_{l,r}(x)$ and that since $T(h_1, \ldots, h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, $u_{l,r}$ is orthogonal to $u_{j,s}$ for any $r \neq s$.

Hence,

$$\langle u_l, T_k u_j \rangle = \left\langle \sum_{r=1}^m u_{l,r}(t), \sum_{s=1}^m T_k u_{j,s}(t) \right\rangle$$

$$= \int_{\mathbb{R}^d} \sum_{r=1}^m u_{l,r}(t)\overline{u_{j,r}(t-k)} dt \text{ (by orthogonality)}$$

$$= \int_{\mathbb{R}^d} \sum_{r=1}^m \hat{u}_{l,r}(x)\overline{\hat{u}_{j,r}(x)} e^{2\pi i k \cdot x} dx \text{ (Plancherel’s Theorem)}$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \hat{u}_{l,r}(y+n)\overline{\hat{u}_{j,r}(y+n)} e^{2\pi i k \cdot (y+n)} dy$$

$$= \int_{\mathbb{T}^d} \left( \sum_{r=1}^m [\hat{u}_{l,r}, \hat{u}_{j,r}](y) \right) e^{2\pi i k \cdot y} dy \text{ (Fubini’s Theorem)}$$

Thus, we have that $\langle u_l, T_k u_j \rangle$ are Fourier coefficients of $\sum_{r=1}^m [\hat{u}_{l,r}, \hat{u}_{j,r}](x)$. But these are the same Fourier coefficients as $[\hat{u}_l, \hat{u}_j](x)$, found in our proof of Lemma 1. Hence, by the uniqueness of Fourier coefficients, $[\hat{u}_l, \hat{u}_j](x) = \sum_{r=1}^m [\hat{u}_{l,r}, \hat{u}_{j,r}](x) \text{ a.e.}$ and thus, by Lemma 1,
\( T(u_1, \ldots, u_m) \) is an orthonormal sequence if and only if

\[
\sum_{r=1}^{m} [\hat{u}_{l,r}, \hat{u}_{j,r}](x) = [\hat{u}_l, \hat{u}_j](x) = \delta_{l,j} \quad \text{a.e.} \quad l, j = 1, \ldots, m
\]

But,

\[
[\hat{u}_{l,r}, \hat{u}_{j,r}](x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_{l,r}(x + k) \hat{u}_{j,r}(x + k)
\]

\[
= \sum_{k \in \mathbb{Z}^d} p_{l,r}(x + k) \hat{h}_r(x + k) p_{j,r}(x + k) \hat{h}_r(x + k)
\]

\[
= \sum_{k \in \mathbb{Z}^d} p_{l,r}(x) p_{j,r}(x) \hat{h}_r(x + k) \hat{h}_r(x + k) \quad \text{(since } p \text{ is } \mathbb{Z}^d\text{-periodic)}
\]

\[
= p_{l,r}(x) p_{j,r}(x) \quad \text{(by Lemma 1)}
\]

Hence, \( T(u_1, \ldots, u_m) \) is an orthonormal sequence if and only if

\[
\delta_{l,j} = [\hat{u}_l, \hat{u}_j](x) \quad \text{a.e.} \quad l, j = 1, \ldots, m \quad \text{(Lemma 1)}
\]

\[
= \sum_{r=1}^{m} [\hat{u}_{l,r}, \hat{u}_{j,r}](x)
\]

\[
= \sum_{r=1}^{m} p_{l,r}(x) p_{j,r}(x)
\]

\[\square\]

**Lemma 4.** Assume that \( T(u_1, \ldots, u_m) \) and \( T(h_1, \ldots, h_m) \) are orthonormal sequences in \( L^2(\mathbb{R}^d) \).

Then \( S(u_1, \ldots, u_m) = S(h_1, \ldots, h_m) \) if and only if there are \( \mathbb{Z}^d\)-periodic functions \( p_{l,r}(x) \in L^2(\mathbb{T}^d) \) that satisfy (3) and the matrix

\[
P(x) := \left( p_{l,r}(x) \right)_{l,r=1}^{m}
\]

is nonsingular almost everywhere.
Proof. First, assume there are $\mathbb{Z}^d$-periodic functions $p_{i,j}(x) \in L^2(\mathbb{T}^d)$ that satisfy (1) and the matrix (5) is nonsingular almost everywhere. Let

$$U(x) := \begin{pmatrix} \hat{u}_1(x) \\ \vdots \\ \hat{u}_m(x) \end{pmatrix} \quad \text{and} \quad H(x) := \begin{pmatrix} \hat{h}_1(x) \\ \vdots \\ \hat{h}_m(x) \end{pmatrix}.$$  

Then

$$U(x) = P(x)H(x) \quad \text{a.e.}\)$$

If $P(x)$ is nonsingular almost everywhere, setting

$$Q(x) := \begin{cases} [P(x)]^{-1} & \text{if } P(x) \text{ is nonsingular} \\ 0 & \text{if } P(x) \text{ is singular} \end{cases},$$

yields that $Q(x)$ is $\mathbb{Z}^d$-periodic and

$$H(x) = Q(x)U(x) \quad \text{a.e.}.$$  

If we let

$$Q(x) := \left( q_{l,r}(x) \right)_{l,r=1}^m,$$

then

$$\hat{h}_l(x) = \sum_{r=1}^m q_{l,r}(x)\hat{u}_r(x).$$
We then have

\begin{equation}
1 = \|\hat{h}_l\|^2 = \|\sum_{r=1}^{m} q_{l,r} \hat{u}_r\|^2 = \left\langle \sum_{r=1}^{m} q_{l,r}(x) \hat{u}_r(x), \sum_{s=1}^{m} q_{l,s}(x) \hat{u}_s(x) \right\rangle \\
= \int_{\mathbb{R}^d} \sum_{r=1}^{m} \sum_{s=1}^{m} q_{l,r}(x) \hat{u}_r(x) \overline{q_{l,s}(x) \hat{u}_s(x)} \, dx \\
= \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \sum_{r=1}^{m} \sum_{s=1}^{m} q_{l,r}(y+k) \hat{u}_r(y+k) \overline{q_{l,s}(y+k) \hat{u}_s(y+k)} \, dy \\
= \int_{\mathbb{T}^d} \sum_{r=1}^{m} \sum_{s=1}^{m} q_{l,r}(y) \overline{q_{l,s}(y)} \sum_{k \in \mathbb{Z}^d} \hat{u}_r(y+k) \overline{\hat{u}_s(y+k)} \, dy \\
= \int_{\mathbb{T}^d} |q_{l,r}(y)|^2 \, dy \quad \text{(by Lemma 1)} \\
\geq \int_{\mathbb{T}^d} |q_{l,n}(y)|^2 \, dy \quad \text{for any } n \in [1,\ldots,m] \\
= \|q_{l,n}\|^2_{L^2(\mathbb{T}^d)}
\end{equation}

Therefore \(q_{l,n} \in L^2(\mathbb{T}^d)\) for \(l, n = 1, \ldots, m\) and thus \(S(u_1, \ldots, u_m) = S(h_1, \ldots, h_m)\).

Conversely, assume that \(S(u_1, \ldots, u_m) = S(h_1, \ldots, h_m)\). Then (3) implies that there are \(\mathbb{Z}^d\)-periodic matrices

\begin{equation}
P(x) = \left( p_{l,r}(x) \right)_{l,r=1}^{m} \quad \text{and} \quad Q(x) = \left( q_{l,r}(x) \right)_{l,r=1}^{m}
\end{equation}

such that

\begin{align*}
p_{l,r}, q_{l,r} &\in L^2(\mathbb{T}^d), \quad l, r = 1, \ldots, m \\
U(x) &= P(x)H(x) \quad \text{a.e.} \\
H(x) &= Q(x)U(x) \quad \text{a.e.}
\end{align*}

Thus

\begin{equation}
U(x) = P(x)Q(x)U(x) \quad \text{a.e.}
\end{equation}
Which implies that
\[ P(x)Q(x) = I \quad \text{a.e.} \]
and thus \( P(x) \) is nonsingular almost everywhere.

**Theorem 7.** Assume that \( T(h_1, ..., h_m) \) is an orthonormal sequence in \( L^2(\mathbb{R}^d) \), and let \( \{u_1, ..., u_n\} \) be a set of functions defined on \( \mathbb{R}^d \). Then \( T(u_1, ..., u_n) \) is an orthonormal sequence and
\[
S(h_1, ..., h_m) = S(u_1, ..., u_n)
\]
if and only if \( m = n \), there are \( \mathbb{Z}^d \)-periodic functions \( p_{l,r}(x) \in L^2(\mathbb{T}^d) \) such that (3) is satisfied and the matrix (5) is orthogonal almost everywhere.

**Proof.** Assume that \( T(h_1, ..., h_m) \) and \( T(u_1, ..., u_n) \) are orthonormal and such that \( S(h_1, ..., h_m) = S(u_1, ..., u_n) \). Then \( m = n \) by Theorem 1. Lemma 2 implies that (3) is satisfied. Since (3) is satisfied and \( T(h_1, ..., h_m) \) is orthonormal, Lemma 3 implies that (4) is satisfied. If we define \( P_l(x) \) as the \( l \)-th row of \( P(x) \), we see that the left hand side of (4) is equivalent to \( P_l(x) \cdot P_j(x) \), which tells us that (5) is orthogonal.

Now assume \( m = n \), there are \( \mathbb{Z}^d \)-periodic functions \( p_{l,r}(x) \in L^2(\mathbb{T}^d) \) such that (3) is satisfied and (5) is orthogonal a.e.. Since (3) is satisfied,
\[
S(u_1, ..., u_m) \subset S(h_1, ..., h_m).
\]
Since (5) is orthogonal a.e., (4) is satisfied. We can then use Lemma 3 to show that \( T(u_1, ..., u_m) \) is an orthonormal sequence. Since (5) is orthogonal a.e., it is also nonsingular a.e., and we can then use Lemma 4 to conclude that
\[
S(h_1, ..., h_m) = S(u_1, ..., u_m).
\]
As we remarked above, if \( \{ \phi_1, ..., \phi_m \} \) is an orthonormal multivariate wavelet in \( L^2(\mathbb{R}^d) \) associated with an MRA, then \( m = 2^d - 1 \). Thus, an immediate consequence of Theorem 7 is

**Theorem 8.** Assume that \( \{ \phi_1, ..., \phi_m \} \) is an orthonormal multivariate wavelet in \( L^2(\mathbb{R}^d) \) associated with an MRA, and let \( \{ \psi_1, ..., \psi_n \} \) be a set of functions defined in \( L^2(\mathbb{R}^d) \). Then \( \{ \psi_1, ..., \psi_n \} \) is an orthonormal multivariate wavelet associated with the same MRA as \( \{ \phi_1, ..., \phi_m \} \), if and only if \( m = n = 2^d - 1 \), and there are \( \mathbb{Z}^d \)-periodic functions \( p_{l,r}(x) \in L^2(\mathbb{T}^d) \) such that

\[
\hat{\psi}_l(x) = \sum_{r=1}^{m} p_{l,r}(x) \hat{\phi}_r(x) \text{ a.e., } l = 1, ..., m
\]

and the matrix (5) is orthogonal a.e.
Chapter 3
Examples

We’ll start with some basic constructions of the matrix $P(x)$, and then move to some more complicated formulations. The following definitions will hold for all of the examples section.

$$
\phi(x) = \chi_{[0,1]}(x)
$$

$$
\psi(x) = \begin{cases} 
1 & \text{if } x \in [0, \frac{1}{2}) \\
-1 & \text{if } x \in [\frac{1}{2}, 1) \\
0 & \text{elsewhere}
\end{cases}
$$

Note that $\psi(x)$ is known as the Haar Wavelet. For more information on its construction and properties, see [5, 6, 7]

Let

$$
\psi_1(x,y) = \phi(x)\psi(y)
$$

$$
\psi_2(x,y) = \psi(x)\phi(y)
$$

$$
\psi_3(x,y) = \psi(x)\psi(y)
$$

The construction of $\{\psi_1, \psi_1, \psi_3\}$ is outlined in [6, p.82], and is shown to be orthonormal wavelet in $L^2(\mathbb{R}^2)$ generated by the scaling function $\phi(x,y) = \phi(x)\phi(y)$. For information regarding the construction of multivariate wavelets from a scaling function, see [9, Theorem]
Since each \( \psi_j \) is separable, the Fourier transforms are easily found and equal to,

\[
\hat{\psi}_1(u, v) = \hat{\phi}(u)\hat{\psi}(v) \\
\hat{\psi}_2(u, v) = \hat{\psi}(u)\hat{\phi}(v) \\
\hat{\psi}_3(u, v) = \hat{\psi}(u)\hat{\psi}(v)
\]

**Example 1.** Let

\[
P(u, v) = \begin{pmatrix}
cos(2\pi u) & -sin(2\pi u) & 0 \\
sin(2\pi u) & cos(2\pi u) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

This is a basic rotation matrix, which has the property of being \( \mathbb{Z}^2 \)-periodic and orthogonal, hence fulfilling the conditions for Theorem 8. If we apply this to our equation, we get,

\[
\begin{pmatrix}
\hat{\psi}_1(u, v) \\
\hat{\psi}_2(u, v) \\
\hat{\psi}_3(u, v)
\end{pmatrix} = \begin{pmatrix}
cos(2\pi u) & -sin(2\pi u) & 0 \\
sin(2\pi u) & cos(2\pi u) & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\hat{\psi}_1(u, v) \\
\hat{\psi}_2(u, v) \\
\hat{\psi}_3(u, v)
\end{pmatrix}
\]

Recall that \( \cos(2\pi u) = \frac{1}{2}(e^{2\pi i u} + e^{-2\pi i u}) \) and \( \sin(2\pi u) = \frac{1}{2i}(e^{2\pi i u} - e^{-2\pi i u}) \). Hence,

\[
\begin{pmatrix}
\hat{\psi}_1(x, y) \\
\hat{\psi}_2(x, y) \\
\hat{\psi}_3(x, y)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}\left(\phi(x + 1) + \phi(x - 1)\right) - \frac{1}{2i}\left(\phi(x + 1) - \phi(x - 1)\right) \\
\frac{1}{2i}\left(\psi(x + 1) - \psi(x - 1)\right) + \frac{1}{2}\left(\psi(x + 1) + \psi(x - 1)\right) \\
\psi(x)\psi(y)
\end{pmatrix}
\]

Note that in the above example, the new wavelets are linear combinations of shifted versions of our original \( \phi(\cdot) \) and \( \psi(\cdot) \) in each variable. Thus, it is clear that the new wavelets have bounded support since \( \phi(\cdot) \) and \( \psi(\cdot) \) have bounded support. This will hold true for all of the future examples by the same reasoning.
Example 2. For a more general case, we can look back to our introduction of $p_{l,r}(x)$, where we defined it as $\sum_{k \in \mathbb{Z}^d} a_{l,r,k} e^{2\pi i k \cdot x}$. Hence, we can choose each $p_{l,r}(u,v)$ to be linear combinations of two-dimensional $\mathbb{Z}^d$-periodic complex exponentials. We will maintain the orthogonality condition by only using the main diagonal and single terms, rather than linear combinations.

Let

$$P(u, v) = \begin{pmatrix} e^{4\pi i u} e^{12\pi i v} & 0 & 0 \\ 0 & -e^{2\pi i u} e^{-6\pi i v} & 0 \\ 0 & 0 & e^{-2\pi i u} \end{pmatrix}$$

Once again, finding the inverse Fourier transform is simple since the wavelet and each $p_{l,r}(u,v)$ are separable. Thus,

$$\begin{pmatrix} \tilde{\psi}_1(x, y) \\ \tilde{\psi}_2(x, y) \\ \tilde{\psi}_3(x, y) \end{pmatrix} = \begin{pmatrix} \phi(x + 2) \psi(y + 6) \\ -\psi(x + 1) \phi(y - 3) \\ \psi(x - 1) \psi(y) \end{pmatrix}$$

Example 3. For the most general case, we begin with three rows which are linearly independent, and use the Gram-Schmidt Process to orthogonalize the rows, and thus create an orthogonal matrix.

Let

$$\tilde{P}(u, v) = \begin{pmatrix} 3e^{4\pi i u} & 0 & 4 \\ e^{4\pi i u} & 4e^{-2\pi i u} e^{-2\pi i v} & 0 \\ 0 & 0 & 5e^{10\pi i v} \end{pmatrix}$$

Note that all these rows are linearly independent. If we define our inner product

$$\langle P_l(x), P_j(x) \rangle := \sum_{r=1}^3 \int_{\mathbb{T}^2} p_{l,r}(u,v) p_{j,r}(u,v) dudv,$$
then by using Gram-Schmidt orthogonalization on our example $\tilde{P}(u, v)$, we get that

$$P(u, v) = \begin{pmatrix} \frac{3}{5}e^{4\pi i u} & 0 & \frac{4}{5} \\ \frac{4}{25\sqrt{26}} e^{4\pi i u} & \frac{1}{\sqrt{26}} e^{-2\pi i (u+v)} & -\frac{3}{25\sqrt{26}} \\ 0 & 0 & e^{10\pi i v} \end{pmatrix}$$

Which yields our new wavelet,

$$\begin{pmatrix} \tilde{\psi}_1(x, y) \\ \tilde{\psi}_2(x, y) \\ \tilde{\psi}_3(x, y) \end{pmatrix} = \begin{pmatrix} \frac{3}{5}\phi(x+2)\psi(y) + \frac{4}{5}\psi(x)\psi(y) \\ \frac{4}{25\sqrt{26}}\phi(x+2)\psi(y) + \frac{1}{\sqrt{26}}\psi(x-1)\phi(y-1) - \frac{3}{25\sqrt{26}}\psi(x)\psi(y) \\ \psi(x)\psi(y+5) \end{pmatrix}$$
Bibliography


