Construction of Orthonormal Multivariate Wavelets

by

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Abstract

The purpose of this paper is to explain the construction of orthonormal multivariate wavelets associated with a multiresolution analysis. This paper primarily uses the work of R. A. Zalik [10], where he outlines a method of constructing orthonormal multivariate wavelets given an existing orthonormal multivariate wavelet associated with an MRA, and attempts to clarify it for a wider audience. In the last section, I use the result in constructing some orthonormal multivariate wavelets in various examples.

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Chapter 1

Introduction

In what follows, $d > 1$ will be an integer, arbitrary but fixed; $\mathbb{Z}$ will denote the set of integers and $\mathbb{R}$ the set of real numbers; boldface lowcase letters will always denote elements of $\mathbb{R}^d$; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; i will be reserved for the imaginary number $\sqrt{-1}$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g \rangle$, their bracket product by $[f, g]$, and the norm of f by $||f||$; thus,

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})} d\mathbf{t},$$

$$[f, g](\mathbf{t}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} f(\mathbf{t} + \mathbf{k})\overline{g(\mathbf{t} + \mathbf{k})},$$

and

$$||f|| := \sqrt{\langle f, f \rangle}.$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-2\pi i \mathbf{t} \cdot \mathbf{x}} f(\mathbf{t}) d\mathbf{t}$$

For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$, the operators D^j and $T_{\mathbf{k}}$ are defined in $L^2(\mathbb{R}^d)$ by

$$D^j f(\mathbf{t}) := 2^{dj/2} f(2^j \mathbf{t})$$

and

$$T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t} - \mathbf{k})$$

A set of functions $\{\psi_1, ..., \psi_m\} \subset L^2(\mathbb{R}^d)$ is called an orthonormal multivariate wavelet, if the sequence

$$\{D^j T_{\mathbf{k}} \psi^l; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, 1 \leq l \leq m\}$$

it generates is an orthonormal basis of $L^2(\mathbb{R}^d)$.

A multiresolution analysis (MRA) in $L^2(\mathbb{R}^d)$ is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

$$V_j \subset V_{j+1} \text{ for every } j \in \mathbb{Z} \tag{i}$$

$$\text{For every } j \in \mathbb{Z}, f(\mathbf{t}) \in V_j \text{ if and only if } f(2\mathbf{t}) \in V_{j+1} \tag{ii}$$

$$\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R}^d). \tag{iii}$$

$$\text{There is a function } u \text{ such that } \{T_{\mathbf{k}} u; \mathbf{k} \in \mathbb{Z}^d\} \text{ is an orthonormal basis of } V_0. \tag{iv}$$

Let $\mathbb{T} := [0, 1]$, and let $\mathbb{T}^d$ denote the d-dimensional torus. A function f will be called $\mathbb{Z}^d$-periodic if it is defined in $\mathbb{R}^d$, and for every $\mathbf{k} \in \mathbb{Z}^d$ and $\mathbf{x} \in \mathbb{R}^d$ we have $f(\mathbf{x} + \mathbf{k}) = f(\mathbf{x})$.
Claim: It follows from the definition of MRA that there is a $\mathbb{Z}^d$-periodic function $p \in L^2(\mathbb{T}^d)$ such that

$$\widehat{u}(2\mathbf{x}) = p(\mathbf{x})\widehat{u}(\mathbf{x}) \text{ a.e.}$$

Proof. Let $\{T_{\mathbf{k}} u; \mathbf{k} \in \mathbb{Z}^d\}$ be an orthonormal basis of V_0. In particular, $u(\mathbf{t}) \in V_0$. Thus, by (ii), $u(\frac{\mathbf{t}}{2}) \in V_{-1}$. By (i) and (iv), we can write

$$u\left(\frac{\mathbf{t}}{2}\right) = \sum_{\mathbf{k} \in \mathbb{Z}^d} a_{\mathbf{k}} T_{\mathbf{k}} u(\mathbf{t})$$

Where $a_{\mathbf{k}} = \left\langle u(\frac{\mathbf{t}}{2}), T_{\mathbf{k}} u(\mathbf{t}) \right\rangle$. By taking the Fourier transform of both sides, we get

$$\int_{\mathbb{R}^d} e^{-2\pi i \mathbf{t}\cdot\mathbf{x}} u(\mathbf{t}/2) d\mathbf{t} = \int_{\mathbb{R}^d} e^{-2\pi i \mathbf{t}\cdot\mathbf{x}} \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} u(\mathbf{t}-\mathbf{k}) d\mathbf{t}.$$

By changing variables, we get

$$\int_{\mathbb{R}^d} e^{-2\pi i (2\mathbf{s})\cdot\mathbf{x}} u(\mathbf{s}) 2^d d\mathbf{s} = \int_{\mathbb{R}^d} e^{-2\pi i (\mathbf{s}+\mathbf{k})\cdot\mathbf{x}} \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} u(\mathbf{s}) d\mathbf{s}.$$

Which yields,

$$2^d \widehat{u}(2\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} \widehat{u}(\mathbf{x})$$

If we let $p(\mathbf{x}) = 2^{-d} \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}$, then the result follows. □

The function u is called a $scaling\ function$ for the MRA, and p is called the $low\ pass\ filter$ associated with u.

We will denote the orthogonal complement of V_j in V_{j+1} by W_j. Thus, $V_{j+1} = V_j \oplus W_j$.

Let $\{\psi_1, ..., \psi_m\}$ be an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$; for $j \in \mathbb{Z}$, let P_j denote the closure of the linear span of

$$\{D^j T_{\mathbf{k}} \psi_l; \mathbf{k} \in \mathbb{Z}^d, 1 \leq l \leq m\}$$

and let $V_j := \sum_{r<j} P_r$. Note that $\psi_1, ..., \psi_m \in V_1$. We say that $\{\psi_1, ..., \psi_m\}$ is *associated* with an MRA if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\{\psi_1, ..., \psi_m\}$ is associated with M. The definition implies that $\{\psi_1, ..., \psi_m\}$ is an orthonormal multivariate wavelet associated with M if and only if $\{T_{\mathbf{k}}\psi_l; \mathbf{k} \in \mathbb{Z}^d, 1 \leq l \leq m\}$ is an orthonormal basis of W_0.

Given $\{u_1, ..., u_m\} \subset L^2(\mathbb{R}^d)$, we will adopt the following notation:

$$T(u_1, ..., u_m) := \{T_{\mathbf{k}} u_l; \mathbf{k} \in \mathbb{Z}^d, 1 \leq l \leq m\},$$

and

$$S(u_1, ..., u_m) := \overline{\text{span}}\ T(u_1, ..., u_m).$$

The following is a special case of a theorem by Guo, Lebate et al. [4, Proposition 2.1]

Theorem 1. *Assume that $T(u_1, ..., u_m)$ and $T(h_1, ..., h_n)$ are orthonormal sequences in $L^2(\mathbb{R}^d)$ such that $S(u_1, ..., u_m) = S(h_1, ..., h_m)$. Then $m = n$.*

Proof. Since $S(u_1, ..., u_m) = S(h_1, ..., h_m)$,

$$u_l(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{j=1}^{n}\langle u_l, T_{\mathbf{k}}h_j\rangle T_{\mathbf{k}}h_j(\mathbf{x}) \quad l = 1, ..., m. \quad \textit{and} \quad h_j(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{m}\langle h_j, T_{\mathbf{k}}u_l\rangle T_{\mathbf{k}}u_l(\mathbf{x})$$

This implies that

$$1 = ||u_l||^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{j=1}^{n}|\langle u_l, T_{\mathbf{k}}h_j\rangle|^2 \quad \textit{and} \quad 1 = ||h_j||^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{m}|\langle h_j, T_{\mathbf{k}}u_l\rangle|^2$$

Also note that $\langle u_l, T_{\mathbf{k}}h_j\rangle = \langle T_{-\mathbf{k}}u_l, h_j\rangle$ Thus we can show,

$$m = \sum_{l=1}^{m}||u_l||^2 = \sum_{l=1}^{m}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{j=1}^{n}|\langle u_l, T_{\mathbf{k}}h_j\rangle|^2 = \sum_{j=1}^{n}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{m}|\langle T_{-\mathbf{k}}u_l, h_j\rangle|^2 = \sum_{j=1}^{n}||h_j||^2 = n$$

□

In [7] Wilson and Weiss showed that if $\{\psi_1, ..., \psi_l\}$ is an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$ associated with a multiresolution analysis, then $m = 2^d - 1$. Hence, when combined with Theorem 1, we have that in the case of orthonormal multivariate wavelets associated with the same MRA, $m = n = 2^d - 1$.

The following theorem is from [5, p. 57], which we adapt to suit the above definition of the Fourier transform.

Theorem 2. *If ϕ is a scaling function for an MRA $\{V_j; j \in \mathbb{Z}\}$ and p is the associated low pass filter, then $h \in L^2(\mathbb{R})$ is an orthonormal wavelet associated with this MRA if and only* *if there is a measurable unimodular and $\mathbb{Z}$-periodic function $v(x)$, such that*

$$\widehat{h}(2x) = e^{2\pi ix}v(2x)\overline{p(x+1/2)}\widehat{\phi}(x) \quad a.e.$$

The main results of this paper will be generalizing the following corollary to wavelets in $L^2(\mathbb{R}^d)$.

Corollary 1. *If h is an orthonormal wavelet associated with an MRA, then ψ is an orthonormal wavelet associated with the same MRA if and only if there is a measurable unimodular and $\mathbb{Z}$-periodic function $q(x)$ such that*

$$\widehat{\psi}(x) = q(x)\widehat{h}(x) \quad a.e.$$

The following theorems will be referenced multiple times in the paper and will be included here as a reference.

Theorem 3. *(Parseval's Identity) Let $f \in L^2(\mathbb{T}^d)$, and $c_{\mathbf{k}} := \widehat{f}(\mathbf{k})$ be the Fourier coefficients of f. Then*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{k}}|^2 = ||f||^2_{L^2(\mathbb{T}^d)}$$

Theorem 4. *(Plancherel's Theorem) Let $f, g \in L^2(\mathbb{R}^d)$. Then,*

$$\int_{\mathbb{R}^d} |f(\mathbf{t})|^2 d\mathbf{t} = \int_{\mathbb{R}^d} |\widehat{f}(\mathbf{x})|^2 d\mathbf{x}$$

Corollary 2.

$$\int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}d\mathbf{t} = \int_{\mathbb{R}^d} \widehat{f}(\mathbf{x})\overline{\widehat{g}(\mathbf{x})}d\mathbf{x}$$

Theorem 5. *(Fubini's Theorem) Let X, Y be measure spaces. If*

$$\int_{X\times Y} |f(x,y)|d(x,y) < \infty.$$

Then

$$\int_X \int_Y |f(x,y)| dy dx = \int_Y \int_X |f(x,y)| dx dy = \int_{X\times Y} |f(x,y)| d(x,y)$$

Corollary 3. *If*

$$\sum_n \int_A |f(n,\mathbf{x})| d\mathbf{x} < \infty,$$

then,

$$\sum_n \int_A f(n,\mathbf{x}) d\mathbf{x} = \int_A \sum_n f(n,\mathbf{x}) d\mathbf{x}$$

Theorem 6. *(Gram-Schmidt Orthogonalization) Let* $\{\mathbf{u}_1, ..., \mathbf{u}_n\} \in S$ *linearly independent, where* S *is an inner product space. Then we can find a set* $\{\tilde{\mathbf{u}}_1, ..., \tilde{\mathbf{u}}_n\} \in S$ *of orthonormal vectors that span the same space.*

Chapter 2

Main Results

Lemma 1. *(a) $T(u_1...u_m)$ is an orthogonal sequence in $L^2(\mathbb{R}^d)$ if and only if*

$$[\widehat{u}_l, \widehat{u}_j]\,(\mathbf{x}) = 0 \;\text{ a.e.}, \;\; l, j = 1, ..., m \;\; l \neq j$$

(b) $T(u_1...u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ if and only if

$$[\widehat{u}_l, \widehat{u}_j]\,(\mathbf{x}) = \delta_{l,j} \;\text{ a.e.}, \;\; l, j = 1, ..., m$$

Proof. It suffices to prove (b).

Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ and $\mathbf{k} = \mathbf{b} - \mathbf{a}$. Then

$$\begin{aligned}
\langle T_{\mathbf{a}} u_l, T_{\mathbf{b}} u_j \rangle &= \langle u_l, T_{\mathbf{k}} u_j \rangle \\
&= \int_{\mathbb{R}^d} u_l(\mathbf{t})\overline{u_j(\mathbf{t}-\mathbf{k})}d\mathbf{t} \\
&= \int_{\mathbb{R}^d} \widehat{u}_l(\mathbf{x})\overline{\widehat{u}_j(\mathbf{x})}e^{2\pi i \mathbf{k}\cdot\mathbf{x}}d\mathbf{x} \;\; \text{(Plancherel's Theorem)} \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{T^d} \widehat{u}_l(\mathbf{y}+\mathbf{n})\overline{\widehat{u}_j(\mathbf{y}+\mathbf{n})}e^{2\pi i \mathbf{k}\cdot(\mathbf{y}+\mathbf{n})}d\mathbf{y} \;\; \text{(“periodize” the integral)} \\
&= \int_{\mathbb{T}^d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{u}_l(\mathbf{y}+\mathbf{n})\overline{\widehat{u}_j(\mathbf{y}+\mathbf{n})}e^{2\pi i \mathbf{k}\cdot\mathbf{y}}d\mathbf{y} \;\; \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{T}^d} [\widehat{u}_l, \widehat{u}_j]\,(\mathbf{y})e^{2\pi i \mathbf{k}\cdot\mathbf{y}}d\mathbf{y} \qquad (1)
\end{aligned}$$

Thus $\langle T_{\mathbf{a}} u_l, T_{\mathbf{b}} u_j \rangle$ are the Fourier coefficients of $[\widehat{u}_l, \widehat{u}_j]$, and Parseval's Identity implies that

$$||[\widehat{u}_l, \widehat{u}_j]||^2_{L^2(\mathbb{T}^d)} = \sum_{\mathbf{k}\in\mathbb{Z}^d} |\langle u_l, T_{\mathbf{k}} u_j \rangle|^2 \qquad (2)$$

Assume $T(u_1, ..., u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$. Then for $l \neq j$, we have that the right hand side of (2) is equal to 0, which implies that $[\widehat{u_l}, \widehat{u_j}](\mathbf{x}) = 0$ a.e.. When $l = j$, we have that the $\langle u_l, T_\mathbf{k} u_l\rangle$ are the Fourier coefficients of the function 1, and by the uniqueness of Fourier coefficients (since $[\widehat{u_l}, \widehat{u_l}](\mathbf{x})$ is $\mathbb{Z}^d$-periodic and in $L^2(\mathbb{T}^d)$), we have that $[\widehat{u_l}, \widehat{u_l}](\mathbf{x}) = 1$ a.e., and thus $[\widehat{u_l}, \widehat{u_j}](\mathbf{x}) = \delta_{l,j}$ a.e..

Conversely, assume $[\widehat{u_l}, \widehat{u_j}] = \delta_{l,j}$ a.e.. Then when $l \neq j$, (1) implies that $\langle T_\mathbf{a} u_l, T_\mathbf{b} u_j\rangle = \int_{\mathbb{T}^d} 0 \; d\mathbf{y} = 0$. When $l = j$, (1) implies that $\langle T_\mathbf{a} u_l, T_\mathbf{b} u_l\rangle = \int_{\mathbb{T}^d} e^{2\pi i \mathbf{k}\cdot\mathbf{y}} d\mathbf{y} = \delta_{\mathbf{a},\mathbf{b}}$. Thus $T(u_1, ..., u_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$. □

Lemma 2. *If $T(h_1, ..., h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and $S(u_1, ..., u_m) \subset S(h_1 ... h_m)$, then there are $\mathbb{Z}^d$-periodic functions $p_{l,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$, uniquely defined a.e., such that*

$$\widehat{u_l}(\mathbf{x}) = \sum_{r=1}^{m} p_{l,r}(\mathbf{x})\widehat{h_r}(\mathbf{x}) \quad \text{a.e.,} \quad l = 1, ..., m \tag{3}$$

Proof. Since $T(h_1 ... h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ we can write the orthogonal projection of u_l onto $S(h_j)$, which we will denote by $u_{l,r}$, as

$$u_{l,r}(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{l,r,\mathbf{k}} h_r(\mathbf{t} - \mathbf{k}).$$

Where $a_{l,r,\mathbf{k}} = \langle u_l, T_\mathbf{k} h_r\rangle$, and since $S(u_1 ... u_m) \subset S(h_1 ... h_m)$, we can write

$$u_l(\mathbf{t}) = \sum_{r=1}^{m} u_{l,r}(\mathbf{t}) = \sum_{r=1}^{m} \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{l,r,\mathbf{k}} h_r(\mathbf{t} - \mathbf{k}).$$

If we take the Fourier Transform of both sides, we get

$$\begin{aligned}\widehat{u_l}(\mathbf{x}) &= \sum_{r=1}^{m} \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{l,r,\mathbf{k}} \widehat{h_r}(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} \\ &= \sum_{r=1}^{m} \widehat{h_r}(\mathbf{x}) \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{l,r,\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\end{aligned}$$

If we let $p_{l,r}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{l,r,\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}$, then the result follows. □

Lemma 3. *Assume that $T(h_1, ..., h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and that $S(u_1, ..., u_m) \subset S(h_1, ..., h_m)$, and assume there are $\mathbb{Z}^d$-periodic functions $p_{l,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ such that (3) is satisfied. Then $T(u_1, ..., u_m)$ is an orthonormal sequence if and only if*

$$\sum_{r=1}^{m} p_{l,r}(\mathbf{x})\overline{p_{j,r}(\mathbf{x})} = \delta_{l,j} \quad \text{a.e.,} \quad l, j = 1, ..., m \tag{4}$$

Proof. Let $u_{l,r}$ denote the orthogonal projection of u_l onto $S(h_r)$. Then

$$\widehat{u_{l,r}}(\mathbf{x}) = p_{l,r}(\mathbf{x})\widehat{h_r}(\mathbf{x}) \;\; \text{a.e.} \;\; l, r = 1, ..., m.$$

Note that $\widehat{u_l}(\mathbf{x}) = \sum_{r=1}^{m} \widehat{u_{l,r}}(\mathbf{x})$ and that since $T(h_1, ..., h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, $u_{l,r}$ is orthogonal to $u_{j,s}$ for any $r \neq s$.

Hence,

$$\begin{aligned}
\langle u_l, T_{\mathbf{k}} u_j \rangle &= \left\langle \sum_{r=1}^{m} u_{l,r}(\mathbf{t}), \sum_{s=1}^{m} T_{\mathbf{k}} u_{j,s}(\mathbf{t}) \right\rangle \\
&= \int_{\mathbb{R}^d} \sum_{r=1}^{m} u_{l,r}(\mathbf{t})\overline{u_{j,r}(\mathbf{t}-\mathbf{k})} d\mathbf{t} \quad \text{(by orthogonality)} \\
&= \int_{\mathbb{R}^d} \sum_{r=1}^{m} \widehat{u_{l,r}}(\mathbf{x})\overline{\widehat{u_{j,r}}(\mathbf{x})} e^{2\pi i \mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \quad \text{(Plancherel's Theorem)} \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{\mathbb{T}^d} \sum_{r=1}^{m} \widehat{u_{l,r}}(\mathbf{y}+\mathbf{n})\overline{\widehat{u_{j,r}}(\mathbf{y}+\mathbf{n})} e^{2\pi i \mathbf{k}\cdot(\mathbf{y}+\mathbf{n})} d\mathbf{y} \\
&= \int_{\mathbb{T}^d} \left(\sum_{r=1}^{m} [\widehat{u_{l,r}}, \widehat{u_{j,r}}](\mathbf{y}) \right) e^{2\pi i \mathbf{k}\cdot\mathbf{y}} d\mathbf{y} \quad \text{(Fubini's Theorem)}
\end{aligned}$$

Thus, we have that $\langle u_l, T_{\mathbf{k}} u_j \rangle$ are Fourier coefficients of $\sum_{r=1}^{m} [\widehat{u_{l,r}}, \widehat{u_{j,r}}](\mathbf{x})$. But these are the same Fourier coefficients as $[\widehat{u_l}, \widehat{u_j}](\mathbf{x})$, found in our proof of Lemma 1. Hence, by the uniqueness of Fourier coefficients, $[\widehat{u_l}, \widehat{u_j}](\mathbf{x}) = \sum_{r=1}^{m} [\widehat{u_{l,r}}, \widehat{u_{j,r}}](\mathbf{x})$ a.e. and thus, by Lemma 1,

$T(u_1, ..., u_m)$ is an orthonormal sequence if and only if

$$\sum_{r=1}^{m} [\widehat{u_{l,r}}, \widehat{u_{j,r}}](\mathbf{x}) = [\widehat{u_l}, \widehat{u_j}](\mathbf{x}) = \delta_{l,j} \quad a.e. \quad l, j = 1, ..., m$$

But,

$$\begin{aligned}
[\widehat{u_{l,r}}, \widehat{u_{j,r}}](\mathbf{x}) &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{u_{l,r}}(\mathbf{x}+\mathbf{k})\overline{\widehat{u_{j,r}}(\mathbf{x}+\mathbf{k})} \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} p_{l,r}(\mathbf{x}+\mathbf{k})\widehat{h}_r(\mathbf{x}+\mathbf{k})\overline{p_{j,r}(\mathbf{x}+\mathbf{k})\widehat{h}_r(\mathbf{x}+\mathbf{k})} \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} p_{l,r}(\mathbf{x})\overline{p_{j,r}(\mathbf{x})}\widehat{h}_r(\mathbf{x}+\mathbf{k})\overline{\widehat{h}_r(\mathbf{x}+\mathbf{k})} \quad \text{(since } p \text{ is } \mathbb{Z}^d\text{-periodic)} \\
&= p_{l,r}(\mathbf{x})\overline{p_{j,r}(\mathbf{x})}\left[\widehat{h}_r, \widehat{h}_r\right](\mathbf{x}) \\
&= p_{l,r}(\mathbf{x})\overline{p_{j,r}(\mathbf{x})} \quad \text{(by Lemma 1)}
\end{aligned}$$

Hence, $T(u_1, ..., u_m)$ is an orthonormal sequence if and only if

$$\begin{aligned}
\delta_{l,j} &= [\widehat{u_l}, \widehat{u_j}](\mathbf{x}) \quad \text{a.e.} \quad l, j = 1, ..., m \quad \text{(Lemma 1)} \\
&= \sum_{r=1}^{m} [\widehat{u_{l,r}}, \widehat{u_{j,r}}](\mathbf{x}) \\
&= \sum_{r=1}^{m} p_{l,r}(\mathbf{x})\overline{p_{j,r}(\mathbf{x})}
\end{aligned}$$

□

Lemma 4. *Assume that $T(u_1, ..., u_m)$ and $T(h_1, ..., h_m)$ are orthonormal sequences in $L^2(\mathbb{R}^d)$. Then $S(u_1, ..., u_m) = S(h_1, ..., h_m)$ if and only if there are $\mathbb{Z}^d$-periodic functions $p_{l,r}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ that satisfy (3) and the matrix*

$$P(\mathbf{x}) := \Big(p_{l,r}(\mathbf{x})\Big)_{l,r=1}^{m} \tag{5}$$

is nonsingular almost everywhere.

Proof. First, assume there are $\mathbb{Z}^d$-periodic functions $p_{l,j}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ that satisfy (1) and the matrix (5) is nonsingular almost everywhere. Let

$$U(\mathbf{x}) := \begin{pmatrix} \widehat{u_1}(\mathbf{x}) \\ \vdots \\ \widehat{u_m}(\mathbf{x}) \end{pmatrix} \quad \text{and} \quad H(\mathbf{x}) := \begin{pmatrix} \widehat{h_1}(\mathbf{x}) \\ \vdots \\ \widehat{h_m}(\mathbf{x}) \end{pmatrix}.$$

Then

$$U(\mathbf{x}) = P(\mathbf{x})H(\mathbf{x}) \quad \text{a.e.}$$

If $P(\mathbf{x})$ is nonsingular almost everywhere, setting

$$Q(\mathbf{x}) := \begin{cases} [P(\mathbf{x})]^{-1} & \text{if } P(\mathbf{x}) \text{ is nonsingular} \\ 0 & \text{if } P(\mathbf{x}) \text{ is singular} \end{cases},$$

yields that $Q(\mathbf{x})$ is $\mathbb{Z}^d$-periodic and

$$H(\mathbf{x}) = Q(\mathbf{x})U(\mathbf{x}) \quad \text{a.e.}$$

If we let

$$Q(\mathbf{x}) := \Big(q_{l,r}(\mathbf{x})\Big)_{l,r=1}^{m},$$

then

$$\widehat{h_l}(\mathbf{x}) = \sum_{r=1}^{m} q_{l,r}(\mathbf{x})\widehat{u_r}(\mathbf{x}).$$

We then have

$$
\begin{aligned}
1 = ||\widehat{h_l}||^2 = ||\sum_{r=1}^{m} q_{l,r}\widehat{u_r}||^2 &= \left\langle \sum_{r=1}^{m} q_{l,r}(\mathbf{x})\widehat{u_r}(\mathbf{x}), \sum_{s=1}^{m} q_{l,s}(\mathbf{x})\widehat{u_s}(\mathbf{x}) \right\rangle \\
&= \int_{\mathbb{R}^d} \sum_{r=1}^{m}\sum_{s=1}^{m} q_{l,r}(\mathbf{x})\widehat{u_r}(\mathbf{x})\overline{q_{l,s}(\mathbf{x})\widehat{u_s}(\mathbf{x})}d\mathbf{x} \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d}\int_{\mathbb{T}^d} \sum_{r=1}^{m}\sum_{s=1}^{m} q_{l,r}(\mathbf{y}+\mathbf{k})\widehat{u_r}(\mathbf{y}+\mathbf{k})\overline{q_{l,s}(\mathbf{y}+\mathbf{k})\widehat{u_s}(\mathbf{y}+\mathbf{k})}d\mathbf{y} \\
&= \int_{\mathbb{T}^d} \sum_{r=1}^{m}\sum_{s=1}^{m} q_{l,r}(\mathbf{y})\overline{q_{l,s}(\mathbf{y})}\sum_{k\in\mathbb{Z}^d}\widehat{u_r}(\mathbf{y}+\mathbf{k})\overline{\widehat{u_s}(\mathbf{y}+\mathbf{k})}d\mathbf{y} \quad \text{(since } q \text{ is } \mathbb{Z}^d \text{ periodic)} \\
&= \int_{\mathbb{T}^d} \sum_{r=1}^{m}\sum_{s=1}^{m} q_{l,r}(\mathbf{y})\overline{q_{l,s}(\mathbf{y})}[\widehat{u_r},\widehat{u_s}](\mathbf{y})d\mathbf{y} \\
&= \int_{\mathbb{T}^d} \sum_{r=1}^{m} |q_{l,r}(\mathbf{y})|^2 d\mathbf{y} \quad \text{(by Lemma 1)} \\
&\geq \int_{\mathbb{T}^d} |q_{l,n}(\mathbf{y})|^2 d\mathbf{y} \quad \text{for any } n \in [1,...,m] \\
&= ||q_{l,n}||^2_{L^2(\mathbb{T}^d)}
\end{aligned}
$$

Therefore $q_{l,n} \in L^2(\mathbb{T}^d)$ for $l, n = 1, ..., m$ and thus $S(u_1, ..., u_m) = S(h_1, ..., h_m)$.

Conversely, assume that $S(u_1, ..., u_m) = S(h_1, ..., h_m)$. Then (3) implies that there are $\mathbb{Z}^d$-periodic matrices

$$
P(\mathbf{x}) = \Big(p_{l,r}(\mathbf{x})\Big)_{l,r=1}^{m} \quad \text{and} \quad Q(\mathbf{x}) = \Big(q_{l,r}(\mathbf{x})\Big)_{l,r=1}^{m}
$$

such that

$$
\begin{gathered}
p_{l,r}, q_{l,r} \in L^2(\mathbb{T}^d), \quad l, r = 1, ..., m \\
U(\mathbf{x}) = P(\mathbf{x})H(\mathbf{x}) \quad \text{a.e.} \\
H(\mathbf{x}) = Q(\mathbf{x})U(\mathbf{x}) \quad \text{a.e.}
\end{gathered}
$$

Thus

$$
U(\mathbf{x}) = P(\mathbf{x})Q(\mathbf{x})U(\mathbf{x}) \quad \text{a.e.}
$$

Which implies that

$$P(\mathbf{x})Q(\mathbf{x}) = I \quad \text{a.e.}$$

and thus $P(\mathbf{x})$ is nonsingular almost everywhere. □

Theorem 7. *Assume that $T(h_1, ..., h_m)$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, and let $\{u_1, ..., u_n\}$ be a set of functions defined on $\mathbb{R}^d$. Then $T(u_1, ..., u_n)$ is an orthonormal sequence and*

$$S(h_1, ..., h_m) = S(u_1, ..., u_n)$$

if and only if $m = n$, there are $\mathbb{Z}^d$-periodic functions $p_{l,r}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ such that (3) is satisfied and the matrix (5) is orthogonal almost everywhere.

Proof. Assume that $T(h_1, ..., h_m)$ and $T(u_1, ..., u_n)$ are orthonormal and such that $S(h_1, ..., h_m) = S(u_1, ..., u_n)$. Then $m = n$ by Theorem 1. Lemma 2 implies that (3) is satisfied. Since (3) is satisfied and $T(h_1, ..., h_m)$ is orthonormal, Lemma 3 implies that (4) is satisfied. If we define $P_l(\mathbf{x})$ as the l-th row of $P(\mathbf{x})$, we see that the left hand side of (4) is equivalent to $P_l(\mathbf{x}) \cdot P_j(\mathbf{x})$, which tells us that (5) is orthogonal.

Now assume $m = n$, there are $\mathbb{Z}^d$-periodic functions $p_{l,r}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ such that (3) is satisfied and (5) is orthogonal a.e.. Since (3) is satisfied,

$$S(u_1, ..., u_m) \subset S(h_1, ..., h_m).$$

Since (5) is orthogonal a.e., (4) is satisfied. We can then use Lemma 3 to show that $T(u_1, ..., u_m)$ is an orthonormal sequence. Since (5) is orthogonal a.e., it is also nonsingular a.e., and we can then use Lemma 4 to conclude that

$$S(h_1, ..., h_m) = S(u_1, ..., u_m).$$

□

As we remarked above, if $\{\phi_1, ..., \phi_m\}$ is an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$ associated with an MRA, then $m = 2^d - 1$. Thus, an immediate consequence of Theorem 7 is

Theorem 8. *Assume that $\{\phi_1, ..., \phi_m\}$ is an orthonormal multivariate wavelet in $L^2(\mathbb{R}^d)$ associated with an MRA, and let $\{\psi_1, ..., \psi_n\}$ be a set of functions defined in $L^2(\mathbb{R}^d)$. Then $\{\psi_1, ..., \psi_n\}$ is an orthonormal multivariate wavelet associated with the same MRA as $\{\phi_1, ..., \phi_m\}$, if and only if $m = n = 2^d - 1$, and there are $\mathbb{Z}^d$-periodic functions $p_{l,r}(\mathbf{x}) \in L^2(\mathbb{T}^d)$ such that*

$$\widehat{\psi}_l(\mathbf{x}) = \sum_{r=1}^{m} p_{l,r}(\mathbf{x})\widehat{\phi}_r(\mathbf{x}) \quad \text{a.e.}, \quad l = 1, ..., m$$

and the matrix (5) is orthogonal a.e.

Chapter 3

Examples

We'll start with some basic constructions of the matrix $P(\mathbf{x})$, and then move to some more complicated formulations. The following definitions will hold for all of the examples section.

$$\phi(x) = \chi_{[0,1)}(x)$$

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \\ 0 & \text{elsewhere} \end{cases}$$

Note that $\psi(x)$ is known as the Haar Wavelet. For more information on its construction and properties, see [5, 6, 7]

Let

$$\psi_1(x, y) = \phi(x)\psi(y)$$

$$\psi_2(x, y) = \psi(x)\phi(y)$$

$$\psi_3(x, y) = \psi(x)\psi(y)$$

The construction of $\{\psi_1, \psi_1, \psi_3\}$ is outlined in [6, p.82], and is shown to be orthonormal wavelet in $L^2(\mathbb{R}^2)$ generated by the scaling function $\phi(x, y) = \phi(x)\phi(y)$. For information regarding the construction of multivariate wavelets from a scaling function, see [9, Theorem

9].

Since each ψ_j is separable, the Fourier transforms are easily found and equal to,

$$\widehat{\psi}_1(u,v) = \widehat{\phi}(u)\widehat{\psi}(v)$$
$$\widehat{\psi}_2(u,v) = \widehat{\psi}(u)\widehat{\phi}(v)$$
$$\widehat{\psi}_3(u,v) = \widehat{\psi}(u)\widehat{\psi}(v)$$

Example 1. *Let*

$$P(u,v) = \begin{pmatrix} cos(2\pi u) & -sin(2\pi u) & 0 \\ sin(2\pi u) & cos(2\pi u) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a basic rotation matrix, which has the property of being $\mathbb{Z}^2$-periodic and orthogonal, hence fulfilling the conditions for Theorem 8. If we apply this to our equation, we get,

$$\begin{pmatrix} \widehat{\tilde{\psi}}_1(u,v) \\ \widehat{\tilde{\psi}}_2(u,v) \\ \widehat{\tilde{\psi}}_3(u,v) \end{pmatrix} = \begin{pmatrix} cos(2\pi u) & -sin(2\pi u) & 0 \\ sin(2\pi u) & cos(2\pi u) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\psi}_1(u,v) \\ \widehat{\psi}_2(u,v) \\ \widehat{\psi}_3(u,v) \end{pmatrix}$$

Recall that $cos(2\pi u) = \frac{1}{2}(e^{2\pi iu} + e^{-2\pi iu})$ and $sin(2\pi u) = \frac{1}{2i}(e^{2\pi iu} - e^{-2\pi iu})$. Hence,

$$\begin{pmatrix} \tilde{\psi}_1(x,y) \\ \tilde{\psi}_2(x,y) \\ \tilde{\psi}_3(x,y) \end{pmatrix} = \begin{pmatrix} \left[\frac{1}{2}\Big(\phi(x+1) + \phi(x-1)\Big) - \frac{1}{2i}\Big(\phi(x+1) - \phi(x-1)\Big)\right]\psi(y) \\ \left[\frac{1}{2i}\Big(\psi(x+1) - \psi(x-1)\Big) + \frac{1}{2}\Big(\psi(x+1) + \psi(x-1)\Big)\right]\phi(y) \\ \psi(x)\psi(y) \end{pmatrix}$$

Note that in the above example, the new wavelets are linear combinations of shifted versions of our original $\phi(\cdot)$ and $\psi(\cdot)$ in each variable. Thus, it is clear that the new wavelets have bounded support since $\phi(\cdot)$ and $\psi(\cdot)$ have bounded support. This will hold true for all of the future examples by the same reasoning.

Example 2. *For a more general case, we can look back to our introduction of* $p_{l,r}(\mathbf{x})$*, where we defined it as* $\sum_{\mathbf{k}\in\mathbb{Z}^d} a_{l,r,\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}$*. Hence, we can choose each* $p_{l,r}(u,v)$ *to be linear combinations of two-dimensional* $\mathbb{Z}^d$*-periodic complex exponentials. We will maintain the orthogonality condition by only using the main diagonal and single terms, rather than linear combinations.*

Let

$$P(u,v) = \begin{pmatrix} e^{4\pi iu}e^{12\pi iv} & 0 & 0 \\ 0 & -e^{2\pi iu}e^{-6\pi iv} & 0 \\ 0 & 0 & e^{-2\pi iu} \end{pmatrix}$$

Once again, finding the inverse Fourier transform is simple since the wavelet and each $p_{l,r}(u,v)$ are separable. Thus,

$$\begin{pmatrix} \tilde{\psi}_1(x,y) \\ \tilde{\psi}_2(x,y) \\ \tilde{\psi}_3(x,y) \end{pmatrix} = \begin{pmatrix} \phi(x+2)\psi(y+6) \\ -\psi(x+1)\phi(y-3) \\ \psi(x-1)\psi(y) \end{pmatrix}$$

Example 3. *For the most general case, we begin with three rows which are linearly independent, and use the Gram-Schmidt Process to orthogonalize the rows, and thus create an orthogonal matrix.*

Let

$$\tilde{P}(u,v) = \begin{pmatrix} 3e^{4\pi iu} & 0 & 4 \\ e^{4\pi iu} & 4e^{-2\pi iu}e^{-2\pi iv} & 0 \\ 0 & 0 & 5e^{10\pi iv} \end{pmatrix}$$

Note that all these rows are linearly independent. If we define our inner product

$$\langle P_l(\mathbf{x}), P_j(\mathbf{x})\rangle := \sum_{r=1}^{3} \int_{\mathbb{T}^2} p_{l,r}(u,v)\overline{p_{j,r}(u,v)}dudv,$$

then by using Gram-Schmidt orthogonalization on our example $\tilde{P}(u,v)$, we get that

$$P(u,v) = \begin{pmatrix} \frac{3}{5}e^{4\pi iu} & 0 & \frac{4}{5} \\ \frac{4}{25\sqrt{26}}e^{4\pi iu} & \frac{1}{\sqrt{26}}e^{-2\pi i(u+v)} & -\frac{3}{25\sqrt{26}} \\ 0 & 0 & e^{10\pi iv} \end{pmatrix}$$

Which yields our new wavelet,

$$\begin{pmatrix} \tilde{\psi}_1(x,y) \\ \tilde{\psi}_2(x,y) \\ \tilde{\psi}_3(x,y) \end{pmatrix} = \begin{pmatrix} \frac{3}{5}\phi(x+2)\psi(y) + \frac{4}{5}\psi(x)\psi(y) \\ \frac{4}{25\sqrt{26}}\phi(x+2)\psi(y) + \frac{1}{\sqrt{26}}\psi(x-1)\phi(y-1) - \frac{3}{25\sqrt{26}}\psi(x)\psi(y) \\ \psi(x)\psi(y+5) \end{pmatrix}$$

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