Finitely Generated Modules Over Noncommutative Chain Rings

by

Greggory M. Scible

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
Decemeber 12, 2011

Keywords: right chain ring, right duo ring

Approved by

Ulrich Albrecht, Professor of Mathematics and Statistics
Pat Goeters, Professor of Mathematics and Statistics
Georg Hetzer, Professor of Mathematics and Statistics

Abstract

The notion of a right chain ring, a ring whose lattice of right ideals is linearly ordered by inclusion, is a generalization of a valuation ring. In this work we investigate properties of right chain rings and the structure of modules over such rings. Several results that have been established for modules over valuation rings and domains are extended to modules over non-commutative right chain rings. In this discussion the notion of a right duo ring, a ring whose right ideals are two-sided, arises naturally.

Acknowledgments

I would like to thank my advisor Dr. Ulrich Albrecht for his patient encouragement throughout the course of my Graduate studies. He has “lead from the front” as all good leaders do, and has exemplified how a professional mathematician should conduct oneself. Thanks also go to my committee members Dr. Georg Hetzer and Dr. Pat Goeters for their assistance and encouragement.

I would also like to thank my family members, especially my parents Jim and Joan for always believing in me and pushing me to explore the limits of my ability. I am deeply appreciative of their guidance and support throughout my life. Thanks also go to my sister Jennifer who has always been a source of support and whose belief in me has given me the freedom to try new things.

Finally, I could not have accomplished anything without my wife Lynette. She gave me the freedom to pursue my goals and has been a rock of support throughout our marriage.

Table of Contents

Chapter 1

Introduction

When one undertakes the study of noncommutative rings, it rapidly becomes apparent that the mathematical landscape has changed significantly from that which one encounters in the commutative setting. Things that are obvious in the commutative setting can often turn into difficult problems without commutativity. One of the best examples of this is the construction of the field of fractions of a ring R. When R is a commutative integral domain the process is relatively straightforward. The extension of this concept to noncommutative rings, however, is highly nontrivial and is indeed still a topic of current research.

In this work, we will investigate the class of noncommutative rings known as *right chain rings*, which can be described as rings having the property that, for every a, $b \in R$, either $aR \subseteq bR$ or $bR \subseteq aR$. These rings are the obvious extension of the concept of a valuation ring to the noncommutative setting. The study of valuation rings has a very extensive literature. The structure of modules over these rings has been investigated in detail in [15] and [16]. Much of this interest arises from the fact that many results and techniques arising from the study of abelian groups can be extended to the study of modules over valuation domains.

Chain rings appear in a surprising number of areas of noncommutative ring theory. In [5], references are provided showing that such rings appear as the coordinate rings of Hjelmslev planes and as building blocks for the localizations of Dedekind prime rings. Further, a domain having a distributive lattice of right ideals is characterized by the property that their localizations at maximal right ideals are right chain rings. As a final example, in 1968 Osofsky [30], extending work of Caldwell [11], established that local rings whose cyclic modules have cyclic injective hulls are right and left chain rings.

Chapter 2 contains a summary of the ring and module theory that is utilized throughout the work to follow. In Chapter 3, we begin the study of right chain rings and right duo rings. Smith and Woodward introduced the concept of a *finitely annihilated* module in [35]. In this chapter we introduce *cyclically* and *strongly cyclically* annihilated modules and establish the following result:

Theorem 1.1. *Suppose R is a ring. The following statements are equivalent:*

1) R is a right chain ring such that all cyclic right R-modules are cyclically annihilated.

2) R is a right duo, right chain ring.

3) Every finitely generated right R-module is strongly cyclically annihilated.

We utilize this result to obtain conditions on a ring R guaranteeing that R is right duo and every homomorphic image of R is strongly right bounded. In the remainder of the Chapter we investigate the ideal structure of right duo right chain rings and show that many of the results from [15] and [16] concerning ideals of valuation rings can be extended to this class of rings.

In Chapter 4 we begin the investigation of modules over right duo chain domains. A natural place to begin studying the structure of modules over any ring R is the class of finitely generated modules. In this study, RD-submodules (relatively divisible) play a distinguished role. A submodule N of a right R-module M is RD if $N \cap Mr = Nr$ for every $r \in R$. For Abelian groups, this is the usual definition of a *pure* subgroup. For modules over more general rings, we have:

Definition 1.2. *Let R be a ring and A a right R-module. A submodule B of A is said to be* ***pure*** *if every finite system of equations over B*

$$\sum_{j=1}^{m} x_j r_{ij} = b_i \quad (i = 1, \ldots, n),$$

with $r_{ij} \in R$ and unknowns $x_1, \ldots, x_m$, has a solution in B whenever it is solvable in A.

Note that B is an RD-submodule of a right R-module A if and only if, whenever $b \in B$ and the equation $xr = b$ is solvable in A, then it is solvable in B. From this we see immediately that pure submodules are RD for an arbitrary ring R. The investigation into the rings for which the converse holds is more difficult. Indeed, we have the following open question.

Open Question: If R is a right semihereditary ring, then every RD-submodule of a right R-module is pure.

Warfield has shown in [39] that an integral domain R is Prüfer if and only if the RD-property is equivalent to purity. We establish in this chapter that over right duo chain domains, purity and relative divisibility are equivalent.

In [14], Fuchs and Salce show that every finitely generated module over a valuation domain has a *pure composition series*, that is a finite chain

$$0 = M_0 < \cdots < M_{n-1} < M_n = M \tag{1.1}$$

of submodules of M such that

(i) each M_i is an RD-submodule of M.

(ii) M_{i+1}/M_i is cyclic for every i.

In Chapter 5, we show that this result extends to modules over right duo chain domains, and indeed establish a result characterizing right duo rings. The existence of essential pure submodules in finitely generated modules over this class of rings is also investigated.

Finally, in Chapter 6 we turn our attention to *duo modules*, that is, modules having the property that each submodule is fully invariant. As endomorphisms of R_R are left multiplication by elements of R, we see immediately that R_R is a duo module if and only if R is a right duo ring. These modules were introduced in [31] and the structure of finitely generated torsionfree duo modules over an integral domain is explored. We show that these results can be extended to duo Ore domains. In this setting, domains need not be commutative.

Chapter 2

Background Material

2.1 Ring Theory

In this section we will gather together the elements of Ring Theory which are used throughout the work. **Standard Assumptions:** All rings under consideration are assumed to be unital and associative. Subrings are assumed to contain the identity. By the term ideal with no additional qualifier, we mean a two-sided ideal. For an ideal I of R, we write $I \subsetneq R$ to indicate proper containment, and $I \subseteq R$ otherwise. The following material can be found in [18], [26], [41]

Special Elements of Rings

Definition 2.1. *Let R be a ring and $a \in R$. Then a is said to be*

*1) a **right zero divisor** if $ba = 0$ for some non-zero $b \in R$ and $a \neq 0$.*

*2) a **left zero divisor** if $ab = 0$ for some non-zero $b \in R$ and $a \neq 0$.*

*3) a **zero divisor** if it is both a left and a right zero divisor.*

*4) an **idempotent** if $a^2 = a$.*

*5) **nilpotent** if $a^k = 0$ for some positive integer k.*

*6) **central** if $ab = ba$ for every $b \in R$.*

Annihilators

Definition 2.2. *Let A be a non-empty subset of a ring R. Then*

*1) the **right annihilator** of A is the set $ann_r(a) = \{ b \in R \mid ab = 0 \text{ for all } a \in A \}$.*

*2) the **left annihilator** of A is the set $ann_l(A) = \{ b \in R \mid ba = 0 \text{ for all } a \in A \}$.*

*3) the **annihilator** of A is the set $ann(A) = ann_r(a) \cap ann_l(A)$.*

The right annihilator of a single element will be denoted $ann_r(a)$ instead of $ann_r(\{a\})$. The right annihilator of any nonempty subset of R is a right ideal of R. If A is a right ideal, then $ann_l(A)$ is an ideal of R. A symmetric result holds for left annihilators.

Properties of Ideals

Definition 2.3. *A left ideal I of R is*

*1) **minimal** if $I \neq 0$ and I does not properly contain any non-zero ideal of R.*

*2) **maximal** if $I \neq R$ and I is not properly contained in any proper left ideal.*

We define in a similar way minimal and maximal right ideals.

Definition 2.4. *Let P be a proper ideal of R. Then P is a*

*1) **prime** ideal if $xRy \subseteq P$ implies that either $x \in P$ or $y \in P$ for every $x, y \in R$.*

*2) **completely prime** if $xy \in P$ implies $x \in P$ or $y \in P$ for every $x, y \in R$.*

*3) **semi-prime** ideal if $xRx \subseteq P$ implies $x \in P$.*

We have the following useful characterizations of prime and semi-prime ideals.

Proposition 2.5. *[26, Prop. 10.2] Let R be a ring and P a proper ideal of R. The following statements are equivalent:*

1) P is a prime ideal.

2) For ideals I and J of R, if $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.

Proposition 2.6. *[26, Prop. 10.9] Let R be a ring and P a proper ideal of R. The following statements are equivalent:*

1) P is a semi-prime ideal.

2) For an ideal I of R, if $I^2 \subseteq P$, then $I \subseteq P$.

Properties of Rings

Definition 2.7. *A ring R is **simple** if R has no nontrivial ideals. R is **regular**(in the sense of von Neumann), if for every $a \in R$, we have $a \in aRa$ and **strongly regular** if for every $a \in R$, we have $a \in a^2R$. If 0 is a prime ideal then R is a **prime** ring, and if 0 is a semi-prime ideal then R is a **semi-prime** ring. R is a **domain** if R has no left or right zero divisors and an **integral domain** if R is a commutative domain. A **reduced** ring is a ring R which has no nontrivial nilpotent elements. R is **right duo** if every right ideal is an ideal. Finally, a **right chain ring** is a ring R such that for every $a, b \in R$, either $aR \subseteq bR$ or $bR \subseteq aR$.*

We define *left chain rings* and *left duo rings* in a similar fashion. If a ring is both a left and a right chain ring, then we simply refer to it as a *chain ring*, with a similar convention for duo rings.

The following theorems characterize regular and strongly regular rings. The proofs may be found in [41, Prop. 3.10, 3.11].

Proposition 2.8. *For a ring R, the following statements are equivalent:*

1) R is von Neumann regular.

2) Every principal left ideal is generated by an idempotent.

3) Every principal left ideal is a direct summand of R.

4) Every finitely generated left ideal is a direct summand.

5) Every right R-module is flat.

Proposition 2.9. *For a ring R, the following statements are equivalent:*

1) R is strongly regular.

2) R is regular and reduced.

3) Every principal left (right) ideal of R is generated by a central idempotent.

4) R is von Neumann regular and every left (right) ideal is an ideal.

Hence from 4), we see that strongly regular rings are duo rings.

Rings Characterized by Homological Properties

In this section we present several classes of rings characterized by homological properties.

Definition 2.10. *Let R be a ring. Then*

*1) R is a **right p.p ring** if every principal right ideal of R is projective.*

*2) R is a **right semi-hereditary** ring if every finitely generated right ideal is projective as a right R-module.*

*3) R is a **right hereditary** ring if every ideal is projective as a right R-module.*

From [18, Background], we have the following results concerning right hereditary and right semi-hereditary rings.

Theorem 2.11. *Let R be a right hereditary ring.*

1) Every submodule of a projective R-module is projective.

2) Every factor module of an injective right R-module is injective.

3) Every projective right R-module is isomorphic to a direct sum of copies of right ideals of R.

Theorem 2.12. *Let R be a right semihereditary ring.*

1) *Every finitely generated submodule of a projective right R-module is projective.*

2) *Every submodule of a flat right R-module is flat.*

3) *Every finitely generated projective right R-module is isomorphic to a direct sum of copies of right ideals of R.*

Jacobson Radical

Let R be a ring. The *Jacobson radical* of R, is defined to be the intersection of all maximal left ideals of R. It is denoted by $J(R)$, or simply J if the ring is clear from the context. In [26, Corollary 4.2], it is shown that $J(R)$ is also equal to the intersection of all maximal right ideals of R, and that $J(R)$ is a two sided ideal of R. We have the following important characterization of $J(R)$.

Proposition 2.13. *[26, pages 50–51] Let R be a ring. The following statements are equivalent:*

1) $y \in J(R)$.

2) $1 - xyz$ *is a unit for every* $x, z \in R$.

3) $My = 0$ *for every simple right R-module M.*

A ring R is *local* ring if it has a unique maximal right ideal. In [26] it is shown that this is equivalent to R having a unique maximal left ideal. Hence, in a local ring, $J(R)$ is the unique maximal ideal. In connection with the Jacobson radical we have *Nakayama's Lemma*. The proof may be found in [26, p.60]:

Theorem 2.14 (Nakayama's Lemma)**.** *Let R be a ring and I a right ideal of R. The following statements are equivalent:*

1) $I \subseteq J(R)$.

2) For any finitely generated right R-module M, $MI = M$ implies that $M = 0$.

3) For any right R-modules $N \leq M$ such that M/N is finitely generated, $N + MI = M$ implies $N = M$.

Corollary 2.15. *If M_R is finitely generated and $x_1, \ldots, x_n$ are elements of M such that the cosets $x_i + MJ$ generate M/MJ, then $x_1, \ldots, x_n$ generate M.*

Proof. Apply Nakayama's Lemma to the module $M/(x_1R + \cdots + x_nR)$. □

2.2 Skew Polynomial Rings and Power Series Rings

Let k be a ring and σ a ring endomorphism of k. Consider the set R of all polynomials in x of the form $\sum_{i=0}^{n} a_i x_i$ ($a_i \in k$). In contrast to the usual assumption that $xa = ax$ for every $a \in k$, we stipulate instead that $xa = \sigma(a)x$ for every $a \in k$. By iterating this rule we see that $x^i a = \sigma^i(a)x^i$ for all $i \geq 0$. Define addition in R as the normal addition in a polynomial ring and multiplication by

$$(\sum_i a_i x^i)(\sum_j b_j x^j) = \sum_{i,j} a_i \sigma^i(b_j) x^{i+j} = \sum_k (\sum_{i+j=k} a_i \sigma^i(b_j)) x^k.$$

It can be shown that with these definitions R forms a ring, denoted $k[x;\sigma]$, the *skew polynomial ring in x*.

We can perform a similar construction to define $k[[x;\sigma]]$, the *skew power series ring in x*, whose elements have the form $\sum_{i \geq 0} a_i x^i$. Using the rules above, it is not hard to see that $U(k[x;\sigma]) = U(k)$ and $U(k[[x;\sigma]]) = \{ a_0 + a_1 x + \cdots \mid a_0 \in U(k) \}$ (see [26, Ch. 1]).

2.3 Modules

All modules are assumed to be unital right R-modules. Similar to the convention for ideals, we write $N \subsetneq M$ for proper containment of a submodule N, and $N \subseteq M$ otherwise.

Chain Conditions

Definition 2.16. *Let R be a ring, M a right R-module, and N a submodule of M.*

*1) M is a **simple** module if M has no proper nontrivial submodules.*

*2) N is a **maximal** submodule if $N \neq M$ and N is a maximal element in the set of proper submodules of M.*

*3) N is a **fully invariant** submodule if $f(N) \subseteq N$ for every R-endomorphism of M.*

If M is a right R-module, then the set $Ann(M) = \{r \in R \mid mr = 0 \text{ for every } m \in M\}$ is the *annihilator* of M. As is the usual convention, we write $Ann(m)$ for the annihilator of a single element. We say that M is a *faithful* R-module if $Ann(M) = 0$. Equivalently, M is faithful if for every $0 \neq r \in R$ there exists $m \in M$ such that $mr \neq 0$.

The following material is taken from [19]. Let A be a set and $\mathcal{A}$ a collection of subsets of A. We say that $\mathcal{A}$ satisfies the *ascending chain condition (ACC)* if there does not exist a properly ascending infinite chain $A_1 \subset A_2 \subset \ldots$ of subsets from $\mathcal{A}$. Recall also that an element $B \in \mathcal{A}$ is a *maximal* element of $\mathcal{A}$ if there does not exist a subset in $\mathcal{A}$ that properly contains B. A right R-module M is *Noetherian* if it satisfies any of the following equivalent conditions:

Proposition 2.17. *For a module M, the following statements are equivalent:*

1) M has the ACC on submodules.

2) Every nonempty family of submodules of M has a maximal element.

3) Every submodule of M is finitely generated.

Definition 2.18. *A ring R is **right (left) Noetherian** if the module R_R $({}_RR)$ is Noetherian.*

Similarly, we say that $\mathcal{A}$ satisfies the *descending chain condition (DCC)* if there does not exist a properly descending infinite chain $A_1 \supset A_2 \supset \ldots$ of subsets from $\mathcal{A}$. Recall also that an element $B \in \mathcal{A}$ is a *minimal* element of $\mathcal{A}$ if there does not exist a subset in $\mathcal{A}$ properly contained in B. A right R-module M is *Artinian* if it satisfies any of the following equivalent conditions:

Proposition 2.19. *For a module M, the following statements are equivalent:*

1) M has the DCC on submodules.

2) Every nonempty family of submodules of M has a minimal element.

Definition 2.20. *A ring R is **right (left) Artinian** if the module R_R $({}_RR)$ is Artinian.*

The following standard results concerning modules satisfying either chain condition may be found in any standard reference on ring theory, such as [19].

Theorem 2.21. *Let N be a submodule of a module M. Then M is Noetherian (Artinian) if and only if N and M/N are both Noetherian (Artinian)*

Corollary 2.22. *Any finite direct sum of Noetherian (Artinian) modules is Noetherian (Artinian).*

Corollary 2.23. *If R is a right Noetherian (Artinian) ring, then all finitely generated right R-modules are Noetherian (Artinian). A similar statement holds for left Noetherian (Artinian) rings.*

Nonsingular Modules

The notion of the singular submodule of a module arises in the attempt to extend the concept of torsion-freeness from modules over integral domains to general non-commutative rings. The most obvious extension is to define a right R-module M to be torsion-free if and only if, for every non-zero $x \in M$ and all regular $r \in R$, one has $xr \neq 0$. Several problems arise in the use of this definition, the most notable being that the set of torsion elements $t(M)$ need not form a submodule for an arbitrary ring R.

The most useful extension of this concept has been found to be the notion of *non-singularity*, introduced by Goodearl in [18]. The singular submodule of any right R-module M over a general ring R always exists, and is equal to $t(M)$ for modules over integral domains. In this section we define the singular submodule, and list several of the more important results.

Definition 2.24. *Let A be a right R-module and B a submodule of A. We say that B is an* ***essential submodule*** *if every non-zero submodule of A has non-zero intersection with B.*

The basic properties of essential submodules may be found in [18, Ch.1] and will be used without further mention. One of the most important uses of essential submodules is the formation of the *singular submodule*.

Remark 2.25. Note that a submodule B of a right R-module A is essential if and only if, for every non-zero $a \in A$, there exists $r \in R$ such that $ar \neq 0$ and $ar \in B$. In practice, this is often used to show that a submodule is essential.

Notation 2.26. *If B is essential in A we write $B \leq_e A$.*

Definition 2.27. *For a right R-module A, we set*

$$Z(A) = \{\, x \in A \mid xI = 0 \ \textit{ for some essential right ideal } I \,\}.$$

It is shown in [18, p.30] that $Z(A)$ is a submodule of A, the *singular submodule*. Equivalently, $Z(A) = \{\, x \in A \mid Ann(x) \leq_e R \,\}$

Considering R as a right R-module, we see that $Z(R_R)$ is an ideal of R, the *right singular ideal*, denoted $Z_r(R)$. Similarly, we define the *left singular ideal* $Z_l(R)$. A module A is a *singular module* if $Z(A) = A$. On the other hand, A is a *nonsingular* module if $Z(A) = 0$. In a similar fashion, a ring R is *right nonsingular* if R_R is nonsingular. A symmetric definition applies for a *left nonsingular* ring.

The following results will be needed in the sequel.

Proposition 2.28. *[18] If $A \leq B \leq C$, then $A \leq_e C$ if and only if $A \leq_e B \leq_e C$.*

Proof. Suppose first that $A \leq_e B \leq_e C$, and $0 \neq M \leq C$. Since $B \leq_e C$, we have $M \cap B \neq 0$. Then $A \leq_e B$ implies that $A \cap (M \cap B) \neq 0$, that is $A \cap M \neq 0$. Hence $A \leq_e C$. Conversely, suppose $A \leq_e C$. Let $0 \neq M \leq B$. Then $0 \neq M \leq B \leq C$ implies that $A \cap M \neq 0$ and $A \leq_e B$. Further, if $0 \neq N \leq C$, then since A is essential in C, we have $A \cap N \neq 0$ which implies that $B \cap N \neq 0$. It follows that $B \leq_e C$. $\square$

Definition 2.29. *A **uniform module** is a non-zero module A such that every non-zero submodule of A is essential.*

Definition 2.30. *A module A has **Goldie dimension** n, written $dim(A) = n$, if it contains an essential submodule that is the direct sun of n uniform submodules.*

It is shown in [18] that $dim(A) = n$ if and only if A has an independent family of n non-zero submodules, but no independent family of more than n non-zero submodules.

Proposition 2.31. *[18] Let $B \leq A$ and assume B has finite Goldie dimension. Then $dim(A) = dim(B)$ if and only if $B \leq_e A$.*

Proof. Let $B_1 \oplus \cdots \oplus B_n \leq_e B$, where the B_i are uniform and $dim(B) = n$. If $B \leq_e A$, then $B_1 \oplus \cdots \oplus B_n \leq_e A$; hence $dim(A) = n = dim(B)$. Conversely, if B is not essential in A, then there exists a non-zero submodule M of A, such that $M \cap B = 0$. Then $\{B_1, \ldots, B_n, M\}$ is an independent family of non-zero submodules of A, whence $dim(A) > n = dim(B)$. $\square$

Remark 2.32. *Note that $dim(A) = 1$ if and only if A is uniform.*

Right Quotient Rings

The material in this section presents an extension of the formation of the field of quotients of an integral domain to general rings. The following material is taken from [37].

Definition 2.33. *Let R be a ring and S a multiplicatively closed subset of R. Then a* ***right ring of fractions****, or* ***right quotient ring*** *of R with respect to S, is a ring $R[S^{-1}]$ together with a ring homomorphism $\varphi : R \longrightarrow R[S^{-1}]$ satisfying:*

i) $\varphi(s)$ is invertible for every $s \in S$,

ii) Every element of $R[S^{-1}]$ has the form $\varphi(a)\varphi(s)^{-1}$ with $s \in S$,

iii) $\varphi(a) = 0$ if and only if $as = 0$ for some $s \in S$.

In [37, Ch. II] it is shown that if $R[S^{-1}]$ exists, then it is unique up to isomorphism. The following is [37, Prop. II.1.4], and is the key result needed.

Theorem 2.34. *Let S be a multiplicatively closed subset of a ring R. $R[S^{-1}]$ exists if and only if S satisfies:*

1) If $s \in S$ and $a \in R$, there exist $t \in S$ and $b \in R$ such that $sb = at$.

2) If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.

When $R[S^{-1}]$ exists, it has the form $R[S^{-1}] = (R \times S)/\sim$, where $\sim$ is the equivalence relation defined by $(a, s) \sim (b, t)$ if and only if there exist $c, d \in R$ such that $ac = bd$ and $sc = td \in S$.

In the proof of Theorem 2.34, it is shown that the additive and multiplicative operations in $R[S^{-1}]$ are given by:

Additive structure: $(a, s) + (b, t) = (ac + bd, u)$ where $c, d \in R$ and $u = sc = td \in S$.

Multiplicative structure: $(a, s)(b, t) = (ac, tu)$, where $c \in R$ and $sc = bu$ and $u \in S$.

In practice, the equivalence class (a, s) is written a/s or as^{-1}.

Similarly, one can define the *left ring of fractions* $[S^{-1}]R$ with respect to a multiplicatively closed set S. It is shown in [37, p.51] that when $R[S^{-1}]$ and $[S^{-1}]R$ both exist, then they are naturally isomorphic. The additive and multiplicative structures of $[S^{-1}]R$ are similar to those of $R[S^{-1}]$, *mutatis mutandis*.

One of the most important examples of a multiplicatively closed set is the set S of all regular elements of a ring R. In this case, the right ring of fractions of R with respect to S is called the *classical right ring of quotients* and is denoted $Q^r(R)$, or merely Q^r if the ring R is clear from the context. In this case, we have the following important result:

Theorem 2.35. *(Ore) Let S be the set of regular elements of a ring R. Then $Q^r(R)$ exists if and only if S satisfies the right Ore condition, i.e. for $a \in R$ and $s \in S$, we have $aS \cap sR \neq \emptyset$.*

If R is a domain, then the right Ore condition reduces to $aR \cap sR \neq 0$ for all non-zero elements $a, s \in R$. Clearly this is equivalent to $I \cap J \neq 0$ for all non-zero right ideals I and J. Observe that a domain has a classical right ring of quotients if and only if $dim R_R = 1$. This condition is obvious for a right chain domain R, and therefore the right quotient ring exists for such rings and is a skew-field.

Another important example of a multiplicatively closed set is the complement, S, of a completely prime ideal P. In this case, $R[S^{-1}]$ is referred to as the *localization* of the ring R at the completely prime ideal P, and, by a standard abuse of notation, is written R_P. By Lemma 5.2 of [5], if P is a completely prime ideal of a right chain ring R, then the localization of R at P exists and $R_P = \{rs^{-1} \mid r \in R, s \in S = R \setminus P\}$.

Remark 2.36. If R is a chain domain, then R is an Ore domain, hence $Q^r(R)$ and $Q^l(R)$ both exist by Theorem 2.35. If P is a completely prime ideal of R, then, by the above the localization at P is also equal to $\{\, s^{-1}a \mid a \in R \text{ and } s \in R \setminus P \,\}$. If it is necessary to distinguish between the two, we will denote the *left* localization by ${}_PR$ and the *right* localization by R_P.

Chapter 3

Right Chain Rings and Duo Rings

3.1 Examples of Right Chain Rings

In this section we establish some of the basic properties of right chain rings. A standard reference for many of these results is [5]. All undefined terms may be found in [18, 19, 26, 32]. We include most proofs for completeness.

We first present several examples of right chain rings.

Example 3.1. Let p be a fixed prime and $k \in \mathbb{N}$. Then the ring $\mathbb{Z}_{p^k}$ is a finite chain ring.

Example 3.2. By Proposition 3.21 below, any right discrete valuation domain is a right chain domain.

The next example requires some preliminary work on ordered groups and Malcev-Neumann rings.

Definition 3.3. *A multiplicative group G with identity e is **ordered** if there exists a total order $<$ on G such that for any x, y, $z \in G$, we have*

$$x < y \text{ implies } xz < yz \text{ and } zx < zy.$$

*Given an ordered group $(G, <)$, its **positive cone** is the set*

$$P = \{\, x \in G \mid x > e \,\}.$$

P has the following easily verified properties:

P1) $P \cdot P \subseteq P$.

P2) $G \setminus \{ e \} = P \sqcup P^{-1}$.

P3) $zPz^{-1} \subseteq P$ for any $z \in G$.

Conversely, given a subset $P \subseteq G$ of a multiplicative group G satisfying $P1, P2$, and $P3$, we can define an order on G by

$$x < y \iff x^{-1}y \in P \iff yx^{-1} \in P.$$

The verification that the order given above makes G into an ordered group may be found in [26, pp. 94-95]. As an example, suppose G is an infinite cyclic group $\{ x^n \mid n \in \mathbb{Z} \}$. Let $P = \{ x^n \mid n > 0 \}$. It is easily verified that P satisfies the properties above, and hence induces an order on G where $x^n < x^m$ if and only if $m - n > 0$.

Following [26, 14.5], we detail the construction of the Malcev-Neumann ring. Modifications of this construction are used to produce chain rings having various properties. As we will be utilizing well-ordered subsets of ordered groups, the following technical lemmas are necessary. The proofs may be found in [26, Section 14.5].

Lemma 3.4. *[26, Lemma 14.16] Let $(G, <)$ be a totally ordered set. For any subset $S \subseteq G$, the following are equivalent:*

1) S is well-ordered.

2) S satisfies the DCC (any sequence $s_1 \geq s_2 \geq s_3 \geq \cdots$ in S is eventually constant).

3) Any sequence $\{ s_1, s_2, s_3, \ldots \}$ in S contains a subsequence $\{ s_{n(1)}, s_{n(2)}, s_{n(3)}, \ldots \}$, where $n(1) < n(2) < n(3) < \cdots$, such that $s_{n(1)} \leq s_{n(2)} \leq s_{n(3)} \leq \cdots$.

Lemma 3.5. *[26, Lemma 14.17] Let S and T be well-ordered subsets of a totally ordered set $(G,<)$. Then $S\cup T$ is well-ordered. If $(G,<)$ is an ordered group, then*

$$U = S\cdot T = \{\, st \mid s\in S, t\in T\,\}$$

is also well-ordered. Moreover, for any $u\in U$, there exist only finitely many ordered pairs $(s,t)\in S\times T$ such that $u = st$.

We now present the construction of the Malcev-Neumann ring. Fix a base ring R and suppose $(G,<)$ is a multiplicative ordered group. Further, fix a group homomorphism $\sigma : G \longrightarrow Aut(R)$ where the image of $g\in G$ is denoted by σ_g. As a set, the Malcev-Neumann ring $A = R((G;\sigma))$ consists of all formal sums

$$\alpha = \sum_{g\in G} \alpha_g g \quad (\alpha_g \in R)$$

such that $supp(\alpha) = \{\, g\in G \mid \alpha_g \neq 0\,\}$ is well-ordered. Addition in A is defined in the obvious fashion and multiplication by

$$\alpha\beta = (\sum_{g\in G}\alpha_g g)(\sum_{h\in G}\beta_h h) = \sum_u (\sum \alpha_g\sigma_g(\beta_h))u,$$

where the last sum is over all pairs $(g,h)\in G\times G$ such that $gh = u$. As we may restrict g and h to $supp(\alpha)$ and $supp(\beta)$, and since these supports are well-ordered, this last sum is finite by Lemma 3.5. It can be shown that with these operations, A forms a ring [26, Ch. 14].

As an example, suppose $G =< x >$ is an infinite cyclic with positive cone P as above. Then the homomorphism $\sigma : G\longrightarrow Aut(R)$ is specified by a single automorphism $\omega := \sigma_x$. The twist law in this case is $x\cdot r = \omega(r)x$. As well-ordered subsets of $\mathbb{Z}$ consist precisely of nonempty subsets that are bounded below, we see that the Malcev-Neumann ring in this case is

$$A = R((< x >, \omega)) = \{ \sum_{i=n}^{\infty} \alpha_i x^i \mid \alpha_i \in R,\, n \in \mathbb{Z} \},$$

a twisted Laurent series ring. The interest in Malcev-Neumann rings lies in part in the following nontrivial result.

Theorem 3.6 (Malcev-Neumann). *[26, Thm 14.21] Let R be a division ring and $(G, <)$ and σ as above. Then $A = R((G; \sigma))$ is a division ring.*

We consider next *valuations* of a division ring whose value group is an ordered group G.

Definition 3.7. *For a division ring D and ordered group $(G, <)$, a function*

$$\upsilon : D^* \longrightarrow G$$

is called a ***valuation*** *if the following two properties are satisfied:*

V1) $\upsilon(ab) = \upsilon(a)\upsilon(b)$ for every $a,\, b \in D^*$.

V2) $\upsilon(a + b) \geq min \{ \upsilon(a), \upsilon(b) \}$ for every $a,\, b \in D^*$ such that $a + b \neq 0$.

Given a valuation $\upsilon : D^* \longrightarrow G$, the set

$$R = \{ r \in D^* \mid \upsilon(r) \geq e \} \cup \{ 0 \}$$

is easily seen to be a subring of D^*. It is also not difficult to show that R is a duo right chain ring with unique maximal ideal $M = \{ r \in D^* \mid \upsilon(r) > e \} \cup \{ 0 \}$ (see [24, pp.216-217]).

Now let R be a division ring, $(G, <)$ an ordered group, and $\sigma : G \longrightarrow Aut(R)$ the trivial group homomorphism. As the multiplication in the Malcev-Nuemann ring A is induced by the "twist" $g \cdot r = \sigma_g(r)g$, the assumption on σ implies that $gr = rg$ for every $r \in R$ and $g \in G$. Define a map $\varphi : A^* \longrightarrow G$ by $\varphi(\alpha) = min.supp(\alpha)$.

It will be shown that φ is a valuation on the division ring A^*. Suppose $\alpha = ag + \cdots$ and $\beta = bh + \cdots$, where $a,\ b \in R^*$, $g = min.supp(\alpha)$, and $h = min.supp(\beta)$. Then $\alpha\beta = (ab)gh + \cdots$. By the properties of an ordered group, $gh = min.supp(\alpha\beta)$. Therefore $\varphi(\alpha\beta) = gh = \varphi(\alpha)\varphi(\beta)$.

For the sum, $\alpha + \beta = ag + bh + \cdots$. The smallest group element in the ordering on G appearing in the sum obviously cannot be smaller than $min\{\, g, h \,\}$. Therefore,

$$\varphi(\alpha + \beta) = min.supp(\alpha + \beta) \geq min\{\, g, h \,\} = min\{\, \varphi(\alpha), \varphi(\beta) \,\}.$$

Hence φ is a valuation. By the above, the subring $R = \{\, \alpha \in A^* \mid \varphi(\alpha) \geq e \,\} \cup \{\, 0 \,\}$ is a duo right chain domain.

3.2 Basic Properties of Right Chain Rings

In this section, some of the basic results concerning right chain rings are listed. The most complete reference for results on right chain rings is [5].

Proposition 3.8. *[5, Lemma 1.2] If R is a right chain ring, the lattice of right ideals of R is linearly ordered with respect to inclusion.*

Proof. Suppose I_1 and I_2 are right ideals of R such that $I_2 \not\subseteq I_1$. Choose $a \in I_2 \setminus I_1$ and note that if $b \in I_1$, then either $a = br$ or $b = ar$ for some $r \in R$. Clearly the first case cannot hold as then $a \in I_1$. Hence $b = ar \in I_2$. Since b was chosen arbitrarily, $I_1 \subseteq I_2$. □

Remark 3.9. Since the right ideals of a right chain ring R are linearly ordered with respect to inclusion, it is immediate that R is *local*; that is, R has a unique maximal right ideal J, which must equal the Jacobson radical of R. Throughout this work J will consistently represent the Jacobson radical of R, and $U = U(R)$ the multiplicative group of units of R. Observe that if R is a local ring, then $U(R) = R \setminus J$; see Proposition 2.13 on page 8.

Proposition 3.10. *[5, Lemma 1.2] A finitely generated right ideal I of a right chain ring R is principal.*

Proof. It suffices to establish the result for $I = aR + bR$. Since R is a right chain ring, either $aR \subseteq bR$ or $bR \subseteq aR$. Suppose the latter holds. Then clearly $I = aR$. $\square$

The following Lemma shows that a right ideal I of a right chain ring R is two-sided exactly if $UI = I$. This result is fundamental and will be used throughout the following work.

Lemma 3.11 (Test-units Lemma)**.** *[5, Lemma 1.4] Let R be a right chain ring.*

1) For every $a \in R$, $Ra \subseteq UaR$.

2) A right ideal I of R is two-sided exactly if $uI \subseteq I$ for every $u \in U$.

Proof. 1): Let $x \in R$. If x is either a unit or $xa \in aR$, then we are done. Suppose $x \in J$ and $xa \notin aR$. Since R is a right chain ring we have that $aR \subseteq xaR$ so $a = xas$ for some $s \in R$. Note that s must be in J, for otherwise s is a unit and $xa \in aR$. Then $1 + s \in U$ and $xa(1 + s) = xa + xas = xa + a = (1 + x)a$. Since $x \in J$ by assumption, $(1 + x) \in U$ and it follows that $xa = (1 + x)a(1 + s)^{-1} \in UaR$.

2): Clearly the result holds if I is a two-sided ideal. Conversely, suppose I is a right ideal such that $uI \subseteq I$ for every $u \in U$. If $a \in I$, then $Ua \subseteq I$ by assumption. Then $ra \in Ra \subseteq UaR \subseteq I$ for every $r \in R$ by (1). Therefore I is an ideal. $\square$

The following is an easy consequence of the definition of a right chain ring.

Lemma 3.12. *[5, Lemma 1.5] Let R be a right chain ring, A a right ideal and B a two-sided ideal. Then, $AB = \{ab \mid a \in A,\ b \in B\}$.*

Proof. Clearly the right hand side of the equation above is contained in AB. Conversely, if $\sum_{i=1}^{n} a_i b_i \in AB$, then since R is a right chain ring we can assume without loss of generality that $a_i R \subseteq a_1 R$ for $i = 2, \ldots, n$. Then $a_i = a_1 r_i$ and $\sum_{i=1}^{n} a_i b_i = a_1 \sum_{i=1}^{n} r_i b_i = a_1 b$ where $b = \sum_{i=1}^{n} r_i b_i$. Hence $\sum_{i=1}^{n} a_i b_i \in \{\, ab \mid a \in A,\ b \in B \,\}$ and the proof is complete. $\square$

We will now establish several results concerning prime ideals in right chain rings. Notice that completely prime ideals are always prime, and prime ideals are semiprime. In general the converses of both statements are false. See Definition 2.7 on page 6.

Proposition 3.13. *Let R be a right chain ring and P a two-sided completely prime ideal of R. If $s \notin P$, then $P = sP$.*

Proof. The inclusion $sP \subseteq P$ is immediate as P is an ideal. For the reverse inclusion, suppose $p \in P$. Since R is a right chain ring, either $sR \subseteq pR$ or $pR \subseteq sR$. The first case cannot hold, as then $s \in P$. Therefore $pR \subseteq sR$, so $p = sr$ for some $r \in R$. If $r \notin P$, then P a completely prime ideal implies $p = sr \notin P$, a contradiction. Hence, $r \in P$, and $p = sr \in sP$. □

The next result establishes the equivalence of the notions of semiprime and prime ideals in right chain rings. Further, we obtain a necessary and sufficient condition for an ideal to be completely prime.

Proposition 3.14. *[5, Lemma 1.8] Let R be a right chain ring and P a right ideal of R.*

1) P is prime if and only if it is semi-prime.

2) If P is a two-sided ideal, then P is completely prime if and only if $x^2 \in P$ implies $x \in P$ for every $x \in R$.

Proof. 1): Necessity is obvious from the definitions. For sufficiency, suppose P is semi-prime and $xRy \subseteq P$ for $x, y \in R$. Since R is a right chain ring, we may assume without loss of generality that $xR \subseteq yR$. Then $xRxR \subseteq xRyR \subseteq P$. As P is semi-prime by assumption, $x \in P$. Thus P is a prime ideal.

2): Necessity is obvious from the definition. Suppose the condition holds and $ab \in P$ for some $a, b \in R$. Since R is a right chain ring, either $a = br_1$ or $b = ar_2$ for some $r_1, r_2 \in R$. In the first case, $a^2 = aa = a(br_1) \in P$ as $ab \in P$ and P is a two-sided ideal. By the assumed condition $a \in P$. In the second case, note that $(ba)^2 = b(ab)a \in P$ as P is a two-sided ideal. Hence $ba \in P$ by our assumption. Then $b^2 = bb = bar_2 \in P$ which implies $b \in P$. Hence P is completely prime. □

Proposition 3.15. *[5, Lemma 3.2] Let R be a right Noetherian, right chain ring. Then R is right duo.*

Proof. Since R is a right Noetherian, every right ideal of R is finitely generated and thus right principal. Suppose $I = aR$ is a right ideal of R. Then by Lemma 3.11, it is enough to show that $ua \in aR$ for every unit $u \in R$. Suppose by way of contradiction that there exists a unit $u \in R$ such that $ua \notin aR$. Since R is a right chain ring we have that $aR \subsetneq uaR$, so $a = uaj$ for some $j \in J$. Then $uaR \subsetneq u^2aR \subsetneq \cdots$, which contradicts R being right Noetherian. □

3.3 Right Duo Rings

In studying the structure of modules over right chain rings we have found that additional conditions on the ring are necessary. The condition on the ring that has proven to be of the greatest importance is that the ring be *right duo*, i.e. every right ideal is two sided.

In the literature right duo rings are also referred to as *right invariant*. Observe that R is right duo if and only if for every $a \in R$ we have $Ra \subseteq aR$; similarly, R is duo if and only if $aR = Ra$ for every $a \in R$.

We now present some examples of right duo rings.

Example 3.16. As a trivial example, commutative rings are obviously duo.

Example 3.17. By Proposition 2.9 on page 7, strongly regular rings are right duo.

Example 3.18. By Proposition 3.15, right Noetherian right chain rings are right duo.

Example 3.19. In [24, pp. 214–215] it is shown that there exists a right duo ring which is not left duo. For the proof, see the next set of results.

Definition 3.20. *A domain R is a* ***right discrete valuation domain*** *if there exists a non-zero nonunit π of R such that every non-zero $a \in R$ can be written in the form $\pi^n u$ where $n \geq 0$ and $u \in U(R)$.*

Proposition 3.21. *[24, pp. 214–215] If R is a right discrete valuation domain, then R is a right duo right chain domain.*

Proof. We first show that π is neither left nor right invertible. Suppose π is right invertible. Then $\pi a = 1$ for some $a \in R$. We can write $a = \pi^n u$ for some $n \geq 0$ and unit u. Then $\pi^{n+1} u = \pi a = 1$ which implies that π is a unit, a contradiction. Next, suppose π is left invertible. Then $b\pi = 1$ for some $b \in R$. Write $b = \pi^m v$ for some $m \geq 0$ and $v \in U(R)$. Then $\pi^m v \pi = b\pi = 1$. If $m \neq 0$, this implies that π is right invertible, contradicting the above. So $m = 0$, which implies that $\pi \in U(R)$, a contradiction. Hence π is neither left nor right invertible.

Set $M = \pi R$. By the above, $M < R$. Suppose $0 \neq I$ is a proper right ideal of R and $0 \neq a \in I$. Write $a = \pi^n u$. Clearly $n \neq 0$, so $a \in \pi R = M$. Therefore M is the unique maximal right ideal of R. It follows that R is a local domain with Jacobson radical $J = M$.

Let I be a non-zero proper right ideal of R. We claim that $I = \pi^i R$ for some positive integer i. If $0 \neq a \in I$, then as above $a = \pi^n u$ for some $n > 0$. As u is a unit, this implies that $\pi^n \in I$. Let i be the least positive integer such that $\pi^i \in I$. Then $n \geq i$, so $a = \pi^n u = \pi^i \pi^{n-i} u \in \pi^i R$. Hence $I \subseteq \pi^i R$. As $\pi^i \in I$, clearly $\pi^i R \subseteq I$ and the result follows.

By the above, the ideals of R form a chain

$$0 \subsetneq \cdots \subsetneq \pi^{i+1} R \subsetneq \pi^i R \subsetneq \cdots \subsetneq \pi R \subsetneq R$$

where the inclusions are strict since π is not right invertible. It follows that R is a right chain ring. Further, R is right noetherian and hence right duo. $\square$

Using the proof of Proposition 3.21, it is can be shown that the function $v : R^* \longrightarrow \mathbb{Z}$ given by $v(a) = n$ if $a = \pi^n u$, is a valuation. Hence, a right discrete valuation ring is a ring whose value group is $\mathbb{Z} \cup \{\infty\}$. The next proposition shows that the concept of a duo ring is not right-left symmetric.

Proposition 3.22. *[24, pp. 214–215] There exists a right duo right chain domain that is not left duo.*

Proof. We will exhibit the existence of a noncommutative discrete valuation domain. Let k be a field and $\sigma \in End_k(k)$, where $\sigma(k) \subsetneq k$. Let $R = k[[x;\sigma]]$, the ring of skew power series of the form $\sum_{i\geq 0} x^i a_i$ $(a_i \in k)$ with multiplication given by $ax = x\sigma(a)$.

Set $\pi = x$ and suppose $\alpha \in R$. If a_n is the least non-zero coefficient of α, then $\alpha = x^n a_n + x^{n+1}a_{n+1} + \cdots$, so $\alpha = x^n(a_n + xa_{n+1} + \cdots)$. Then $a_n + xa^n + \cdots \in U(R)$ as $a_n \neq 0$. It follows that every element of R can be written in the form $\pi^n u$ for some $u \in U(R)$. We conclude that R is a right discrete valuation ring, and is therefore a right duo right chain ring.

Note that since k is a field, R is a domain. If R is left duo, then $xR \subseteq Rx$. Let $a \in k \setminus \sigma(k)$. Then $xa \in xR \subseteq Rx$, so $xa = bx = x\sigma(b)$ for some $b \in R$. Then $x(a - \sigma(b)) = 0$ and it follows that $a = \sigma(b)$, contradicting the choice of a. Hence R is not left duo. $\square$

The following example, presented in [6, Example 6.16], shows the existence of a right duo right chain ring that is not a domain. The ring is also neither a left chain ring nor a left duo ring.

Example 3.23. Let k be a division ring and $\alpha : k \to k$ a monomorphism such that $\alpha(k) \subsetneq k$. Define a multiplication on $R = k \times k$ by

$$(f, g)(f', g') = (ff', \alpha(f)g' + gf'),$$

with componentwise addition. It is routine to verify that with this operation R is a ring with $1_R = (1, 0)$ and $0_R = (0, 0)$. A direct calculation shows that the Jacobson radical of

R is $J = \{ (0, g) \in R \mid g \in k \}$. As $J = (0, g')R$ for any $0 \neq g' \in k$, the only possible non-zero right ideals are J and R. Hence R is a right duo right chain ring. R is not a domain as $(0, 1)^2 = 0_R$. To see that R is not a left chain ring, choose $h, h' \in k \setminus \alpha(k)$. If $R(0, h) \subseteq R(0, h')$ then $(0, h) = (f, g)(0, h) = (0, \alpha(f)h') = (0, \alpha(fh'))$. Thus, $h \in \alpha(k)$, a contradiction. By symmetry, $R(0, h') \nsubseteq R(0, h)$. A similar argument shows that R is also not left duo.

In [38], Thierrin discussed several properties of *duo* rings. Several of his results can be extended to the more general case of right duo rings. The proof of the first result requires modifications for the right duo case; the other results carry through verbatim for right duo rings and the proofs are due to Thierrin.

Proposition 3.24. *Every idempotent e of a right duo ring R is central.*

Proof. In [26] it is shown that an idempotent $e \in R$ is central if and only if $eRf = 0 = fRe$, where $f = 1 - e$ is the complementary idempotent orthogonal to e. Suppose R is right duo. Then $Re \subseteq eR$ and $Rf \subseteq fR$. These relations imply that $fRe \subseteq feR = 0$ and $eRf \subseteq efR = 0$. So we have that $eRf = 0 = fRe$ and by the above, e is central. □

Proposition 3.25. *[38] Every prime right ideal in a right duo ring is completely prime.*

Proof. Suppose R is right duo, P a prime right ideal, and $xy \in P$. Set $T = \{t \in R \mid xt \in P\}$. Then T is a right ideal. As R is right duo, T is a two-sided ideal. Then $y \in T$ implies $Ry \subseteq T$. Hence $xRy \subseteq P$. Since P is prime, $x \in P$ or $y \in P$, and we conclude that P is completely prime. □

Proposition 3.26. *[38] The set of nilpotent elements of a right duo ring R form an ideal N, which is equal to the intersection of the completely prime ideals of R.*

Proof. Let I be the intersection of the completely prime ideals P_i of R and N the set of nilpotent elements. If $a^n = 0$, then $a^n \in P_i$ for every i. Since P_i is completely prime, it follows that $a \in P_i$ and thus $N \subseteq I$. By Proposition 3.25, I is the intersection of the prime ideals of R. By a result of [29], I is a nil ideal. Then each element of I is nilpotent, and therefore $I \subseteq N$ and the proof is complete. $\square$

Let $N(R)$ denote the set of all nilpotent elements of a ring R, and $P(R)$ the prime radical of R. Then R is 2-*primal* if $P(R) = N(R)$. This class of rings was introduced by Birkenmeier, Heatherly and Lee [8] in the context of nearrings. The previous result establishes immediately that:

Corollary 3.27. *Right duo rings are* 2*-primal.*

Obviously every right semi-hereditary ring is right p.p. If R is a right chain ring, then Proposition 3.10 implies that the converse holds. The next result establishes the equivalence of several conditions on a right duo right chain ring.

Theorem 3.28. *Let R be a right duo right chain ring. Then the following conditions are equivalent.*

1) R is semiprime.

2) R is a domain.

3) R is right p.p.

4) R is right semi-hereditary.

5) R is right nonsingular.

Proof. 1) implies 2): Suppose R is semiprime. Since R is a right chain ring, Proposition 3.14 implies that R is a prime ring, so 0 is a prime ideal. Since R is right duo, 0 is completely prime. Hence R is a domain.

2) implies 3): Let aR be a non-zero principal right ideal of R. Since R has no zero divisors, the map $R \to aR$ given by $r \mapsto ar$ is an isomorphism. Hence $R \simeq aR$ and therefore aR is projective.

3) implies 4): Since R is a right chain ring, finitely generated right ideals are principal. Hence by 3), finitely generated right ideals are projective.

4) implies 5): As R is a right chain ring, it suffices to establish the result for principal right ideals. Suppose $0 \neq a \in R$. We have an exact sequence

$$0 \longrightarrow ann_r(a) \longrightarrow R \xrightarrow{f} aR \longrightarrow 0,$$

where $f(x) = xa$.

As aR is projective by assumption, the sequence splits. Hence, $ann_r(a)$ is a direct summand of R and cannot be essential. Thus, $Z_r(R) = 0$, and R is right nonsingular.

5) implies 1): Suppose I is a right ideal of R and $I^2 = 0$. Since R is a right chain ring, the lattice of right ideals is linearly ordered with respect to inclusion. Hence every right ideal of R is essential. Then $I^2 = 0$ and R right nonsingular imply that $I = 0$. Thus, R is semiprime by Proposition 2.6. □

Proposition 2.9 yields that R is strongly regular if and only if R is a von Neumann regular duo ring. Using this result, we can show that von Neumann regular right chain rings are duo rings.

Proposition 3.29. *Let R be a von Neumann regular right chain ring. Then R is strongly regular.*

Proof. Suppose $0 \neq a \in R$. Since R is von Neumann regular there exists $b \in R$ such that $a = aba$. Since R is a right chain ring, either $b \in aR$ or $a \in bR$. If the first case holds then $b = ar$ for some $r \in R$. Then $a = aba = aara = a^2ra \in a^2R$.

If $b \notin aR$, then $a = bj$ for some $j \in J$. Then $a = aba = bjba$. From this we see that $(1 - bjb)a = 0$. By Proposition 2.13, $1 - bjb$ is a unit of R. Therefore $a = 0$, a contradiction. Hence the first case holds and R is strongly regular. □

Corollary 3.30. *A von Neumann regular right chain ring is a duo ring.*

Proof. By Proposition 3.29, von Neumann regular right chain rings are strongly regular. The result follows by Proposition 2.9. □

The following extension of [26, Exercise 10.19] shows that right duo rings behave in many ways like commutative rings. Consider the following conditions on a ring R:

1) Every ideal of R is semiprime.

2) Every ideal of R is idempotent.

3) R is von Neumann regular.

Then for any ring R we have 3) $\Rightarrow$ 2) $\Rightarrow$ 1). The problem as stated is that 1) $\Rightarrow$ 3) if R is a commutative ring. This result also holds for right duo rings.

Lemma 3.31. *Let R be a right duo ring. Then R/I is reduced for every semiprime ideal I of R.*

Proof. Suppose $x^2 \in I$ for some $x \in R$. Then $Rx \subseteq xR$ implies $xRx \subseteq x^2R \subseteq I$. As I is semiprime, $x \in I$. □

Proposition 3.32. *The following are equivalent for a right duo ring R:*

1) Every ideal of R is semiprime.

2) Every ideal of R is idempotent.

3) R is strongly regular.

Proof. For completeness we present the proof in its entirety. 3) implies 2): Since R is strongly regular, it is von Neumann regular. Suppose I is an ideal of R and $a \in I$. Then $a \in aRa \subseteq I^2$, so that $I \subseteq I^2$.

2) implies 1): trivial

1) implies 3). Suppose $a \in R$. Then a^2R is a semiprime ideal. Hence R/a^2R is reduced by Lemma 3.31. As $a + a^2R$ is a nilpotent element of R/a^2R, we have that $a \in a^2R$. Thus R is strongly regular. □

The following observation is remarkably useful, particularly in the study of finitely generated modules over right duo right chain rings. This property of right duo rings allows us to extend many results established for valuation rings to right chain rings.

Proposition 3.33. *Let R be a right duo ring and M a right R-module. Then for every $a \in M$ we have $Ann(a) = Ann(aR)$.*

Proof. Suppose $a \in M$. Then $Ann(a)$ is a two-sided ideal of R so that $R \cdot Ann(a) \subseteq Ann(a)$. Then $(aR)Ann(a) \subseteq a \cdot Ann(a) = 0$ which says precisely that $Ann(a) \subseteq Ann(aR)$. The other inclusion is obvious and the result follows. □

The following consequence of Proposition 3.33 will be used extensively in the study of finitely generated modules over right duo right chain rings.

Proposition 3.34. *Let R be a right duo ring and suppose $M = x_1R + \cdots + x_nR$ is a finitely generated right R-module. Then $Ann_R M = \bigcap_{i=1}^n Ann_R(x_i)$.*

Proof. Clearly $Ann_R M \subseteq \bigcap_{i=1}^n Ann_R(x_i)$. Suppose $t \in \bigcap_{i=1}^n Ann_R(x_i)$ and $m \in M$. Then $m = \sum_{i=1}^n x_i r_i$, for some $r_i \in R$. Observe that $x_i t = 0$ yields $x_i r_i t = 0$. It follows that $mt = \sum_{i=1}^n x_i r_i t = \sum_{i=1}^n x_i t s_i = 0$. Thus, $t \in Ann_R M$.

□

A partial converse to 3.33 also holds; namely, if R is a right chain ring with the property that every finitely generated right R-module satisfies the conclusion of 3.33, then R is right duo. To establish this we introduce the concept of a *cyclically annihilated* module.

Chapter 4

Modules Over Right Duo Right Chain Rings

4.1 Finitely Annihilated Modules

In this section we consider *finitely annihilated* modules, a notion attributed to P. Gabriel [17] by Smith and Woodward in [35]. More specifically, we consider modules that are *cyclically annihilated* over right duo right chain rings. This leads to a natural generalization of results obtained by various authors, and a characterization of this class of rings.

Definition 4.1. *Let R be a ring. A module M_R is* ***finitely annihilated*** *if there exist elements $m_1, \ldots, m_k$ in M such that $Ann_R(M) = Ann_R(m_1, \ldots, m_k)$. If we have that $Ann_R(M) = Ann_R(m)$ for some $m \in M$, then M is* ***cyclically annihilated****. A finitely generated right R-module M is* ***strongly cyclically annihilated*** *if the equality above holds for one of the generators of M, i.e., if $M = x_1R + \cdots + x_nR$, then $Ann_R(M) = Ann_R(x_i)$ for some i.*

Definition 4.2. *A ring R is* ***strongly right bounded*** *if every non-zero right ideal of R contains a non-zero two-sided ideal.*

Remark 4.3. Note that right duo rings are obviously strongly right bounded.

Theorem 4.4. *The following statements are equivalent for a ring R:*

1) R is a right chain ring such that all cyclic right R-modules are cyclically annihilated.

2) R is a right duo right chain ring.

3) Every finitely generated right R-module is strongly cyclically annihilated.

Proof. 1) implies 2): Suppose R is a right chain ring such that every cyclic right R-module is cyclically annihilated. Let I be an ideal of R. We claim that if M is a cyclic right R/I-module, then $M_{R/I}$ is cyclically annihilated. To see this, note that M is a cyclic right R module via the operation $m \cdot r = m(r + I)$. As M_R is cyclically annihilated by hypothesis, there exists $a \in M_R$ such that $M_R \cdot Ann_R(a) = 0$. An easy calculation shows that $Ann_{R/I}(a) = Ann_R(a)/I$. Hence $M_{R/I} \cdot Ann_{R/I}(a) = 0$ and the claim follows. Therefore, R/I is a right chain ring such that every cyclic right R/I-module is cyclically annihilated.

We now show that R is strongly right bounded. For $0 \neq a \in R$, the cyclic right R-module R/aR cyclically annihilated. Hence, $Ann_R(R/aR) = Ann_R(x+aR)$ for some $x \in R$. Thus $J = Ann_R(x + aR) = \{\, r \in R \mid xr \in aR \,\}$ is a two-sided ideal of R which is clearly contained in aR. Since R is a right chain ring, aR is essential in R. Thus, R/aR is a singular R-module and $Ann(y + aR)$ is essential in R for all $y \in R$. In particular, $J = Ann(x + aR)$ is essential, and thus non-zero. So R is a strongly right bounded ring, and this allows us to establish the existence of a largest non-zero two-sided ideal contained in aR as follows: set $I_0 = \sum\{\, 0 \neq K \trianglelefteq R \mid K \subseteq aR \,\}$, and note that the sum is nonempty as it contains J. Clearly I_0 is the largest non-zero two-sided ideal contained in aR.

Suppose $I_0 \neq aR$. By what has been shown, R/I_0 is a strongly right bounded right chain ring. By the argument above, R/I_0 is cyclically annihilated. Hence, the right ideal $(a + I_0)R/I_0$ contains a non-zero two-sided ideal I_1/I_0, where I_1 is an ideal of R such that $I_0 \subsetneq I_1 \subseteq (a + I_0)R$. As $I_0 \subseteq aR$, we have that $I_0 \subsetneq I_1 \subseteq aR$. Since I_0 is the largest two-sided ideal of R contained in aR, we obtain a contradiction. Thus, $I_0 = aR$ and aR is a two-sided ideal of R.

2) implies 3): Suppose that $M = x_1R + \cdots + x_nR$. Since R is a right chain ring we may order the right ideals $\{\, Ann_R(x_i) \mid i = 1, \ldots, n \,\}$ so that $Ann_R(x_1) \subseteq \cdots \subseteq Ann_R(x_n)$. It follows from Proposition 3.34 that $Ann_R M = \bigcap_{i=1}^{n} Ann_R(x_i) = Ann_R(x_1)$. Thus M is strongly cyclically annihilated.

3) implies 1): As strongly cyclically annihilated modules are obviously cyclically annihilated, it remains to show that R is a right chain ring. We will show first that R is right duo. Consider the module $M = R/aR$ for some $a \in R$. By 3), $Ann_R(M) = Ann_R(1 + aR) = aR$. Thus aR is a two-sided ideal of R.

For $a,\ b \in R$, consider the module $M = R/aR \oplus R/bR$, which is generated by the elements $\{\,(1 + aR, 0), (0, 1 + bR)\,\}$. By 3) we may assume without loss of generality that

$$Ann_R(M) = Ann_R(1 + aR) = aR.$$

On the other hand, if $x \in Ann_R(M)$, then $(0, 0) = (1 + aR, 1 + bR)x = (x + aR, x + bR)$, so that $x \in aR \cap bR$. Thus, $aR = Ann_R(M) \subseteq aR \cap bR \subseteq bR$, and R is a right chain ring. $\square$

Given the last result, we can establish the following corollary to Proposition 3.33.

Corollary 4.5. *Let R be a right chain ring. Suppose every finitely generated module M_R has the property that $Ann_R(a)$ is an ideal for every $a \in M$. Then R is right duo.*

Proof. Suppose $M = a_1R + \cdots + a_nR$ for $a_1, \ldots, a_n$ in M. By hypothesis, $Ann(a_i)$ is an ideal for every i. Hence $Ann(M) = \bigcap_i Ann(a_i)$. Since R is a right chain ring we may order the annihilators (reindexing if necessary) as $Ann(a_1) \subseteq \cdots \subseteq Ann(a_n)$. Then $Ann(M) = Ann(a_1)$, and M is strongly cyclically annihilated. By Theorem 4.4, R is right duo. $\square$

We now consider a condition on a ring similar to the considerations above.

Definition 4.6. *A ring R is a **right $*$-ring** if every finitely generated right R-module M is cyclically annihilated.*

If R is a right duo right chain ring, then by Theorem 4.4, every finitely generated right R-module is strongly cyclically annihilated. As strongly cyclically annihilated modules are obviously cyclically annihilated, we have

Corollary 4.7. *Let R be a right duo, right chain ring. Then R is a right $*$-ring.*

However, right $*$-rings need not be right chain rings as shown by:

Proposition 4.8. *If R and S are right $*$-rings, then $R \times S$ is a right $*$-ring.*

Proof. Let K be a finitely generated $T = R \times S$-module. Then there exist finitely generated modules M_R and N_S such that $K = M \bigoplus N$ and $MS = 0$ and $NR = 0$. By hypothesis there exist $x \in M$ and $y \in N$ such that $Ann_R(x) = Ann_R(M)$ and $Ann_S(y) = Ann_S(N)$. Clearly $(x, y) \in M \oplus N = K$ yields $Ann_T(K) \subseteq Ann_T((x, y))$. On the other hand, suppose $(r, s) \in T = R \times S$ satisfies $(x, y)(r, s) = (0, 0)$. Then, $xr = ys = 0$, and it follows that $r \in Ann_R(x) = Ann_R(M)$, and $s \in Ann_S(y) = Ann_S(N)$. Thus, we see that (r, s) is an element of $Ann_T(M \oplus N) = Ann_T(K)$ as desired. □

In [36], the following characterization of right Artinian rings is established:

Theorem 4.9. *The following are equivalent for a ring R.*

1) R is right Artinian.

2) Every right R-module is finitely annihilated.

3) Every countably generated right R-module is finitely annihilated.

4) R satisfies the descending chain condition on two-sided ideals and every cyclic right R-module is finitely annihilated.

If the ring in question is taken to be a right chain ring, the above theorem can be specialized as follows. The proof generally follows that of [36], using the results above to account for the nature of the ring in question.

Theorem 4.10. *Let R be a right chain ring. The following are equivalent:*

1) R is right Artinian.

2) Every right R-module is cyclically annihilated.

3) Every countably generated right R-module is cyclically annihilated.

4) R satisfies the descending chain condition on two-sided ideals and every cyclic right R-module is cyclically annihilated.

Proof. 1) implies 2): Let M_R be a right R-module and choose $a \in M$ such that $Ann(a)$ is minimal. Since R is right Artinian, it is right Noetherian. Then by Proposition 3.15, R is right duo. Thus, $Ann(m)$ is a two-sided ideal of R for every $m \in M$. Then $Ann(a) \subseteq \bigcap_{m\in M} Ann(m) = Ann(M)$. It follows that $Ann(M) = Ann(a)$ and M is cyclically annihilated.

2) implies 3): Obvious.

3) implies 4): Suppose $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a descending chain of two-sided ideals of R. Consider the countably generated right R-module $M = \bigoplus_{n<\omega} R/I_n$. By hypothesis there exists $x \in M$ such that $Ann(M) = Ann(x)$. There also exists a positive integer k such that $xR \subseteq R/I_1 \oplus \cdots \oplus R/I_k$. The hypotheses also guarantee that R is right duo so that $Ann(x) = Ann(xR)$. Then

$$I_k = Ann((R/I_1) \oplus \cdots \oplus (R/I_k)) \subseteq Ann(xR) = Ann(x) = Ann(M) = \bigcap_{n<\omega} I_n.$$

Hence, the chain terminates and the claim follows by Theorem 4.4.

4) implies 1): Let J be the Jacobson radical of R. By a result of Bessenrodt, Brungs, and Törner [5, Prop. 3.16], it is enough to show that J is nilpotent. Since R has the d.c.c. for ideals, we have that $J^n = J^{n+1}$ for some integer n. Set $B = J^n$ and suppose that $B \neq 0$. Then $B^2 = (J^n)(J^n) = J^{2n} = J^n = B$. Set $A = B \cap Ann(B)$. Then $A \subsetneq B$ as $A = B$ implies $B^2 = B = 0$. Since R is a right chain ring and every cyclic right R-module is cyclically annihilated, Theorem 4.4 implies that R is a right duo ring. Hence, by a result of Birkenmeier and Tucci [9, Prop. 6], every homomorphic image of R is strongly right

bounded. Thus, B/A contains a non-zero two-sided ideal C/A. By the d.c.c. on ideals, we may assume that C is the minimal two-sided ideal such that $A \subsetneq C \subseteq B$.

Let $x \in C \setminus A$ and note that $xB \neq 0$. As $x \in C$, a two-sided ideal of R, we have $xB \subseteq C$. Since R is right duo, xB is also a two-sided ideal. Further, if $xB \subseteq A$, then $xB \subseteq B \cap Ann(B) \subseteq Ann(B)$. Then $0 = (xB)B = xB^2 = xB$, a contradiction. So $xB \nsubseteq A$ and by the choice of C we have $xB = C$. Then there exists $b \in B$ such that $xb = x$ so that $x(1-b) = 0$. Hence, $b \in B \subseteq J$, and therefore $1-b$ is a unit. Thus, $x = 0$, a contradiction. Hence $B = 0$ and consequently R is right Artinian. $\square$

4.2 Ideal Structure of Right Chain Rings

Valuation domains play a key role in many areas of commutative ring theory, especially in algebraic geometry. The interplay between valuation domains and valuations on a field are well known. A similar thing occurs when one considers chain domains. Notice that a chain domain R is an Ore domain and hence the quotient ring Q of R exists and is a division ring. Hence, constructing chain domains gives rise to division rings. The converse question, i.e., finding a right chain domain in a given division ring is also of interest. We will show below that, for chain domains, the situation is similar to that of valuation domains. Recall that by a *chain domain*, with no qualifier, we mean a domain that is a right and left chain ring. A similar convention holds for a *chain ring*, an *Ore domain*, etc.

Proposition 4.11. *Suppose R is a domain. The following statements are equivalent:*

1) R is a chain ring.

2) R is an Ore domain with the property that for every non-zero $q \in Q$, either $q \in R$ or $q^{-1} \in R$, where Q is the quotient ring of R.

Proof. 1) implies 2): Since R is a chain domain, ${}_RR_R$ is a uniform right and left R-module and hence an Ore domain. Suppose $0 \neq q = ab^{-1} \in Q$ and $q \notin R$. Then $a \notin Rb$. Since R is a left chain ring by assumption this implies that $Rb \subsetneq Ra$ so we may write $b = ta$ for some $t \in R$. Then $q^{-1} = ba^{-1} = t \in R$ as claimed.

2) implies 1): As R is a right Ore domain the right quotient ring Q^r exists. Suppose a and b are non-zero elements of R such that $Ra \nsubseteq Rb$. Then $ab^{-1} \notin R$. By hypothesis $ba^{-1} = (ab^{-1})^{-1} \in R$. Hence $b \in Ra$ and therefore $Rb \subseteq Ra$ so that R is a left chain ring. As R is also a left Ore domain Q^l also exists and $Q^l = Q^r = Q$. By a symmetric argument with $Q^r = Q$, R is also a right chain ring. □

The following, which appears without proof in [6, Lemma 6.1], now follows easily from Corollary 4.11.

Corollary 4.12. *Suppose R is a subring of a division ring Q. The following statements are equivalent:*

1) R is a chain domain with skew field of quotients Q.

2) For every non-zero $q \in Q$, either $q \in R$ or $q^{-1} \in R$.

As noted in [6], Corollary 4.12 immediately implies that every overring of a chain domain R in Q is also a chain domain. In fact Brungs and Törner show in [10] that each such overring of R is a localization of R at a prime ideal. Unfortunately these properties do not extend to right chain domains, in that there exists a right duo right chain domain R with quotient ring Q, and an overring S of R in Q, such that S is neither a right nor a left chain ring. See [6, Example 7.1].

We now turn our attention to a class of modules that play a key role in the investigation of modules over valuation domains.

Definition 4.13. *A right R module U is* ***uniserial*** *if its submodules are linearly ordered with respect to inclusion. Equivalently given $a, b \in U$, either $aR \leq bR$ or $bR \leq aR$.*

Obviously submodules and quotients of uniserial modules are uniserial, uniserial modules are uniform and therefore indecomposable, and R is a right chain ring if and only if R_R is a uniserial right R-module.

Proposition 4.14. *Let R be a chain domain and Q the right quotient ring of R. Then*

1) Q is a uniserial R module, and

2) every proper R-submodule of Q is isomorphic to a right ideal of R.

Proof. 1): Let U and V be R-submodules of Q and suppose $U \not\subseteq V$. Choose $a \in U \setminus V$ and let $b \in V$. Clearly $b = 0$ implies $b \in U$, so assume $b \neq 0$. Then we claim that $a^{-1}b \in R$. Suppose to the contrary that $a^{-1}b \notin R$. Then by Proposition 4.11, $b^{-1}a = (a^{-1}b)^{-1} \in R$. Thus, $a = b(b^{-1}a) \in V$, a contradiction. Hence $a^{-1}b \in R$, and therefore $b \in aR \subseteq U$. It follows that $V \subseteq U$ and Q is a uniserial R-module.

2): Suppose M is a proper R-submodule of Q. If $M \subseteq R$, then M is a right ideal of R and the claim is established. Suppose R is properly contained in M. Since M is proper there exists a regular element $t \in R$ such that $t^{-1} \notin M$. Since Q is uniserial by 1), either $t^{-1}R \leq M$ or $M \leq t^{-1}R$. Clearly the first case cannot hold as this forces $t^{-1} \in M$. So we have that $M < t^{-1}R$. Then $tM < R$ and t regular implies that left multiplication by t is a monomorphism. It follows that $M \simeq tM$. □

Definition 4.15. *Let R be a right Ore domain with right quotient ring Q, and suppose K is an R-submodule of Q. If I is a subset of Q, then the **residual** of I and K is defined as the set $(K : I)_r = \{\, q \in Q \mid Iq \leq K \,\}$.*

Remark 4.16. *If R a right duo ring, then $(K : xR)_r = (K : x)_r$ for every $x \in R$.*

Lemma 4.17. *Let R be a right Ore domain with right quotient ring Q, and suppose I_R and K_R are R-submodules of Q. Then $(K : I)_r$ is an R-submodule of Q_R.*

Proof. Suppose $x, y \in (K : I)_r$ and $a \in I$. Then $ax = k_1$ and $ay = k_2$ for some $k_1, k_2 \in K$. Therefore $a(x - y) = ax - ay = k_1 - k_2 \in K$, so $x - y \in (K : I)_r$. If $r \in R$ then $a(xr) = (ax)r = k_1 r \in K$ as K is an R-submodule. Hence $xr \in (K : I)_r$. It follows that $(K : I)_r$ is an R-submodule of Q_R. □

Proposition 4.18. *Let I and K be proper non-zero right ideals of a right duo, right chain ring R. For $0 \neq x \in R$, we have $I = (K : x)_r$ if and only if $K = xI$.*

Proof. If $0 \neq K = xI$ and $a \in I$, then $xa \in K$, so $a \in (K : x)_r$. Thus, $I \subseteq (K : x)_r$. For the reverse inclusion, suppose $a \in (K : x)_r$. Then $xa \in K = xI$ so $xa = xb$ for some $b \in I$. From this we see $a - b \in ann_r(x)$. Since $xI \neq 0$, $I \nsubseteq ann_r(x)$. As R is a right chain ring, it follows that $ann_r(x) \subsetneq I$. Therefore $a - b \in I$. Thus, $b \in I$ implies that $a \in I$. Hence, $(K : x)_r \subseteq I$.

Conversely, suppose $I = (K : x)_r$. If $a \in I$, then $a \in (K : x)_r$ and $xa \in K$. Hence $xI \subseteq K$. Suppose by way of contradiction that $xI \subsetneq K$, and choose $a \in K \setminus xI$. Since R is a right chain ring, we consider two cases: $a = xb'$ and $x = ab$, for some $b, b' \in R$. In the first case, since $a \in K$ and $a = xb'$, we have that $b' \in (K : x)_r = I$. Then $a = xb' \in xI$, contrary to the choice of a. Suppose the second case, $x = ab$, holds. Since K is a right ideal of R and $a \in K$, we have $R \leq (K : a)_r$. We obtain a chain of inclusions $I < R \leq (K : a)_r$, and claim that $(K : a)_r \leq (K : x)_r$ as well. To see this, suppose $y \in (K : a)_r$. Then $ay = k$ for some $k \in K$. Since $x = ab$, it follows that $xy = aby$. Since R is right duo, $by \in Ry \subseteq yR$, so $by = yb'$ for some $b' \in R$. Then $xy = aby = ayb' = kb' \in K$. It follows that $y \in (K : x)_r$, as claimed. Combining these results we have a chain of inclusions $I < R \leq (K : a)_r \leq (K : x)_r$. This is a contradiction as $I = (K : x)_r$ by assumption. It follows that $xI = K$ and the proof is complete. $\square$

Corollary 4.19. *Suppose I and K are proper non-zero right ideals of a right duo right chain ring R. Then for every $r \in R$, $rI = rK \neq 0$ implies $I = K$.*

Proof. By Proposition 4.18, $I = (rI : r)$ for every proper non-zero right ideal I. Thus, $I = (rI : r) = (rK : r) = K$. $\square$

Let I be a non-zero ideal of a right chain ring R. Following Fuchs-Salce [15], we set $U_l(I) = \{ r \in R \mid rI = I \}$ and define $I^{\#}$ as the set $R \setminus U_l(I)$. Then $I^{\#} = \{ r \in R \mid rI < I \}$. We will show that, under certain conditions, $I^{\#}$ is a completely prime ideal containing I.

Lemma 4.20. *Let R be a right duo right chain domain and I a non-zero proper ideal of R. Then,*

1) $I^{\#}$ is a completely prime ideal containing I.

2) If I is a principal left ideal of R, then $I^{\#} = J$.

3) If I is a completely prime ideal of R, then $I^{\#} = I$.

Proof. 1): We will show that $I^{\#}$ is a right ideal of R. Suppose $r, s \in I^{\#}$. Since R is a right chain ring, we may assume without loss of generality that $s = rt$ for some $t \in R$. Then $r - s = r - rt = r(1-t)$. Suppose by way of contradiction that $r - s \notin I^{\#}$. Then $r - s \in U_l(I)$ so $(r - s)I = I$. Since $r \in I^{\#}$, we may choose an element $a \in I \setminus rI$. Then there exists $x \in I$ such that $(r - s)x = a$. So $a = (r - s)x = r(1 - t)x \in rI$, contradicting the choice of a.

Suppose now that $r \in I^{\#}$ and $x \in R$. If $rx \notin I^{\#}$, then $rx \in U_l(I)$ and $(rx)I = I$. Since $r \in I^{\#}$, there exists $a \in I \setminus rI$. Then there exists $y \in I$ such that $a = (rx)y \in rI$, a contradiction. Thus $rx \in I^{\#}$ and $I^{\#}$ is a right ideal.

The proof that $I^{\#}$ is completely prime follows a similar line of argument. Suppose $rs \in I^{\#}$ and $r \notin I^{\#}$. Then choosing $a \in I \setminus rsI$, there exists an element $b \in I$ such that $a = rb$ since $rI = I$. If $s \notin I^{\#}$, then there exists $c \in I$ such that $sc = b$. Then $a = rb = rsc \in rsI$, a contradiction. Hence, $s \in I^{\#}$ and it follows that $I^{\#}$ is a completely prime right ideal.

Finally, if $a \in I \setminus I^{\#}$, then there exists an element $b \in I$ such that $ab = a$. Then $a(1 - b) = 0$. As R is a domain and $a \neq 0$, we have that $b = 1$, a contradiction as I is a proper ideal. Hence, $I \subseteq I^{\#}$.

2): Suppose $0 \neq I = Ra$. It is always the case that $I^{\#} \subseteq J$, since $I^{\#}$ is a proper right ideal. For the reverse inclusion, if $j \in J$ and $j \notin I^{\#}$, then $jI = I$. So $a = jsa$ for some $s \in R$. Hence, $(1 - js)a = 0$ and $1 - js \in U(R)$. Thus, $a = 0$, a contradiction. It follows that $J \subseteq I^{\#}$.

3): Suppose $a \in R \setminus I$. Since R is a right chain ring, this implies that $I \subsetneq aR$. If $b \in I$, then $b = ac$ for some $c \in R$. Since I is completely prime and $a \notin I$, we must have $c \in I$. Then $I \subseteq aI \subseteq I$, so $I = aI$ and therefore $a \in U_l(I) = R \setminus I^{\#}$. Hence, $R \setminus I \subseteq R \setminus I^{\#}$, which implies $I^{\#} \subseteq I$. As $I \subseteq I^{\#}$ by 1), it follows that $I = I^{\#}$. □

Definition 4.21. *Let I be an ideal of a ring R. We say that I is a* ***regular*** *ideal of R if I contains a regular element.*

By [15, Lemma II.4.4], if I is a non-zero ideal of a valuation domain R, then $I^{\#}$ is a $R_{I^{\#}}$-module in a natural way, where $R_{I^{\#}}$ is the localization of R at the completely prime ideal $I^{\#}$ (see Theorem 2.34). In order to extend this result to the non-commutative setting, we will use the left ring of fractions. In what follows we will use the *left* localization at $I^{\#}$. Then ${}_{I^{\#}}R = \{\, s^{-1}a \mid s \in U_l(I) \text{ and } a \in R \,\}$ (see Remark 2.36). We also extend [15, Proposition I.4.6] to right duo chain domains.

Theorem 4.22. *Let R is a right chain ring and I a non-zero regular ideal of R. If L is the left localization of R at the completely prime ideal $I^{\#}$, then*

1) If R is a right duo ring, the set of zero divisors of R is contained in $I^{\#}$.

2) If R is a right duo chain domain, then I_R is a left L-module.

3) If R is a right duo chain domain, then $End_R(I_R) \simeq L$.

Proof. 1): Suppose $b \in R$ is a right zero divisor. Then there exists a non-zero x in R such that $xb = 0$. If $b \notin I^{\#}$, then $bI = I$. Let $a \in I$ be a regular element, and choose $y \in I$ such that $by = a$. Then $xa = x(by) = (xb)y = 0$, which is a contradiction as a is regular. Therefore $I^{\#}$ contains all right zero divisors.

Similarly, if $b \in R$ is a left zero divisor, then $bx = 0$ for some non-zero $x \in R$. If $b \notin I^{\#}$, then $bI = I$. Choosing $a \in I$ regular, there exists $y \in I$ such that $by = a$. As R is right duo there exists $y' \in R$ such that $yx = xy'$ for some $y' \in R$. Then $ax = (by)x = (bx)y' = 0$, again contradicting the fact that a is regular. Hence $I^{\#}$ contains all left divisors of zero as well.

2): Since R is a chain domain, R is right and left Ore. As $I^{\#}$ is a two-sided completely prime ideal of R, the left and right localizations at $I^{\#}$ exist. Recall that the left localization of R at the completely prime ideal $I^{\#}$ is $L = \{ s^{-1}a \mid s \in U_l(I) \text{ and } a \in R \}$. If $s \in U_l(I)$ is non-zero, then s is a regular element of R by 1). Let $\lambda_s : I_R \longrightarrow I_R$ be left multiplication by s. Since s is in $U_l(I)$ we have that $sI = I$, so λ_s is an epimorphism of I_R. As s is regular, we have in addition that λ_s is injective. Hence $\lambda_s \in Aut_R(I_R)$ and it follows that $s^{-1}I \subseteq I$. Therefore we may define $(s^{-1}a) \cdot x = s^{-1}(ax)$ for every $s^{-1}a \in L$ and $x \in I$. It will be shown that I_R is a left L-module.

We first show that the multiplication given above is well-defined. Suppose we have elements $s^{-1}a$, $t^{-1}b \in L$ such that $s^{-1}a \sim t^{-1}b$. By Theorem 2.34, there exist c, $d \in R$ such that $ca = db$ and $cs = dt \in U_l(I)$. Then $s^{-1}c^{-1} = t^{-1}d^{-1}$, so $s^{-1} = t^{-1}d^{-1}c$. If $x \in I$, then $s^{-1}ax = t^{-1}d^{-1}cax = t^{-1}d^{-1}dbx = t^{-1}bx$. It follows that the operation is well defined. The verification of the module axioms is mechanical using Theorem 2.34 and the remarks following.

3): Note that since R is a right and left chain domain, it is right and left Ore. Hence the right and left quotient rings Q^r and Q^l exist. We first identify $End_R(I_R)$ with a subring of Q^l. Let $q \in Q^l$ and $\lambda_q : Q^l \longrightarrow Q^l$ be left multiplication by q. If $\varphi \in End_R(I_R)$, then since Q_R is an injective right R-module, there exists an R-homomorphism $\psi : Q_R \longrightarrow Q_R$ such that $\psi|_I = \varphi$. As $Hom_R(Q_R, Q_R) = Hom_Q(Q_R, Q_R)$, there exists $q' \in Q$ such that $\psi = \lambda_{q'}$. Then for $i \in I$, $\varphi(i) = \lambda_{q'}(i) = q'i$. Define $\sigma : End_R(I_R) \longrightarrow Q$ by $\sigma(\varphi) = q'$. It is easily verified that σ is a ring monomorphism and hence $End_R(I_R)$ can be viewed as a subring of Q^l.

With this identification, we show that $End_R(I_R) = L$. Suppose $0 \neq s^{-1}a \in L$. If $x \in I$, then $s^{-1}ax \in I$ and we conclude that $ax = sy$ for some $y \in I$. Computing inside Q^l we obtain that $s^{-1}ax = y \in I$. Hence, left multiplication by $s^{-1}a$ is an endomorphism of I_R.

For the reverse inclusion, suppose $\varphi \in End_R(I_R)$. By the identification above, $\varphi = \lambda_q$ for some $q \in Q^l$. If $qI = I$, then $q \in L$ and we are done, so suppose $qI < I$. As $q \in Q^l$, $q = s^{-1}a$ for some regular s. We consider two cases: $aR \subseteq sR$ and $sR \subseteq aR$. In the first case, $a = st$ for some $t \in R$. Then $s^{-1}a = s^{-1}st = t \in R \subseteq L$. In the second case, $s = at$ for some $t \in R$. Then inside Q^l we have that $s^{-1} = t^{-1}a^{-1}$. Hence $q = s^{-1}a = t^{-1}a^{-1}a = t^{-1}$ so that $q = t^{-1}$. Then $qI \subseteq I$ implies that $t^{-1}I \subseteq I$ which implies that $I \subseteq tI \subseteq I$ so $tI = I$. Thus, $t \in U_l(I)$ and therefore $q \in L$. $\square$

4.3 Modules over Right Chain rings

Let n be a positive integer. If A is an abelian group, then the set

$$A[n] = \{\, a \in A \mid na = 0 \,\}$$

forms a subgroup of A consisting of elements whose order divides n. If R is a commutative ring and M an R-module, then for any $r \in R$, the set $M[r] = \{\, m \in M \mid mr = 0 \,\}$ is easily seen to be a submodule of M. For general rings, $M[r]$ need not form a submodule of M. If R is a right duo ring, however, $M[r]$ is a submodule of M for every $r \in R$.

Lemma 4.23. *Let R be a ring. Then $M[r]$ is a submodule of M for every right R-module M and every $r \in R$ if and only if R is right duo.*

Proof. Suppose R is right duo and choose $m, n \in M[r]$. Then $mr = 0$ and $nr = 0$. Hence $(m - n)r = mr - nr = 0$, so $M[r]$ is an additive subgroup of M. To show that $M[r]$ is a submodule of M, suppose $m \in M[r]$ and $s \in R$. Since R is right duo, $sr \in Rr \subseteq rR$, so we can write $sr = rt$ for some $t \in R$. Then $(ms)r = m(sr) = m(rt) = (mr)t = 0$. Therefore, $ms \in M[r]$ and it follows that $M[r]$ is a submodule of M.

Conversely, suppose the condition holds and $a \in R$. Consider the right R-module $M = R/aR$. Then $M[a] = \{\, r + aR \in R/aR \mid ra \in aR \,\}$. In particular, $1 + aR \in M[a]$. As

$M[a]$ is a submodule of M, $R + aR = (1 + aR)R \in M[a]$. Thus, $Ra \subseteq aR$ and therefore R is right duo. $\square$

We can also form the set $M[I] = \{ m \in M \mid I \leq Ann(m) \}$, where I is a right ideal of R. Once again, if R is commutative, $M[I]$ is a submodule of M. We have:

Lemma 4.24. *Let R be right duo and M a right R-module. Then for every right ideal I of R, $M[I]$ is a submodule of M.*

Proof. Clearly $M[I]$ is an additive subgroup of M. To see that $M[I]$ is a submodule of M, suppose $m \in M[I]$ and $r \in R$. Since R is right duo, I is an ideal of R. Then for every $a \in I$, $ra \in I$. As $I \leq Ann(m)$, it follows that $(mr)a = m(ra) = 0$. Hence $mr \in M[I]$ and $M[I]$ is a submodule of M. $\square$

Consider the set $M[I^+] = \{ m \in M \mid I < Ann(m) \}$, where I is a right ideal of R. Once again, if R is a valuation ring, it is not difficult to see that $M[I^+]$ is a submodule of M. This can be extended to right chain rings.

Lemma 4.25. *Let R be a right duo right chain ring and M a right R-module. Then for every right ideal I of R, $M[I^+]$ is a submodule of M.*

Proof. Suppose $m, n \in M[I^+]$. Then $I < Ann(m)$ and $I < Ann(n)$, so we may choose $r \in Ann(m) \setminus I$ and $s \in Ann(n) \setminus I$. Since R is a right chain ring, this implies that I is properly contained in the right ideals rR and sR. Once again, by the right chain ring property, we may assume that $rR \subseteq sR$. So we have $I \subset rR \subseteq sR$. Hence, $r = st$ for some $t \in R$. Then $(m - n)r = mr - nr = mr - nst = 0$. Therefore $r \in Ann(m - n)$, and as $r \notin I$, we see that $m - n \in M[I^+]$. So $M[I^+]$ is an additive subgroup of M.

To see that $M[I^+]$ is an R-submodule of M, suppose $m \in M[I^+]$ and $r \in R$. Then $I < Ann(m)$, so we may choose $s \in Ann(m) \setminus I$. As R is right duo, $Ann(m)$ is an ideal of R. Hence $rs \in Ann(m)$. Then $(mr)s = m(rs) = 0$. As $s \notin I$, it follows that $mr \in M[I^+]$. Hence $M[I^+]$ is a submodule of M. $\square$

Given the results above, we can easily establish the following.

Proposition 4.26. *Let R be a right duo right chain ring and M a right R-module. Then for every right ideal I of R:*

1) $M[I]$ and $M[I^+]$ are fully invariant submodules of M.

2) If $f \in Hom_R(M,N)$, then f carries $M[I]$ into $N[I]$ and $M[I^+]$ into $N[I^+]$.

Proof. 1): Suppose $f \in End_R(M)$. If $a \in f(M[I])$, then $a = f(m)$ where $m \in M[I]$. So if $x \in I$, then $mx = 0$. Hence $ax = f(m)x = f(mx) = 0$, and therefore $I \leq Ann(a)$. It follows that $a \in M[I]$ whence $M[I]$ is fully invariant. The second statement is proven in a similar fashion.

2): If $f(m) \in f(M[I])$, then $m \in M[I]$. Hence, $I \leq Ann(m)$. So for every $x \in I$, $mx = 0$ implies that $f(m)x = f(mx) = 0$. Therefore $I \leq Ann(f(m))$ and $f(m) \in N[I]$. The proof of the second statement is similar. □

Definition 4.27. *Let R be a ring and M a right R-module. The* ***socle*** *of M, written $soc(M)$, is the sum of all simple submodules of M.*

Proposition 4.28. *Let R be a right duo right chain ring and M a right R-module. Then $soc(M) = M[J]$.*

Proof. If K is a simple submodule of M, then $K \simeq R/M$, where M is a maximal ideal of R. As R is local, we have that $K \leq M$ is simple if and only if $K \simeq R/J$. So the socle of M is isomorphic to a sum of copies of R/J. If $a + J \in R/J$, then $J \leq Ann_R(a + J)$, i.e. $a + J \in M[J]$. Hence any finite sum of such elements, that is, any element of $soc(M)$, is in $M[J]$.

For the reverse inclusion, suppose $0 \neq a \in M[J]$. Then $J \leq Ann(a)$. As J is maximal, this implies that $Ann(a) = J$. Hence $aR \simeq R/J$, and therefore $a \in soc(M)$. □

Chapter 5

Finitely Generated Modules

In this chapter we begin the investigation of finitely generated modules over right chain rings. This work presents a natural extension to the noncommutative setting of results by Kaplansky, Warfield, Fuchs, Salce, and Zanardo concerning finitely generated modules over valuation domains.

5.1 RD-submodules

The notion of a pure subgroup of an Abelian group may be extended to modules over arbitrary rings in several ways. One of the most useful extensions is the concept of an *RD*-submodule of a right R-module. This section introduces some of the basic properties of these submodules. In addition, we provide a characterization of *RD*-submodules of modules over right duo chain rings.

Definition 5.1. *Let R be a ring and M a right-R module. A submodule N of M is an **RD-submodule** if for every $r \in R$, we have $Nr = N \cap Mr$. As the inclusion $Nr \subseteq N \cap Mr$ always holds, the RD property amounts to showing that $N \cap Mr \subseteq Nr$ for every $r \in R$.*

Notation 5.2. *We will denote the fact that N is an RD-submodule of M by $N \leq_{RD} M$.*

Example 5.3. If G is an abelian group, then a subgroup H of G is **pure** if for $h \in H$, $h = ng$ for an integer n and $g \in G$ implies $h = nh_1$ for some $h_1 \in H$. It is easily seen that the *RD*-property for $R = \mathbb{Z}$ is equivalent to purity.

The following establishes some of the basic properties of *RD*-submodules.

Proposition 5.4. *Let R be a ring and M a right R-module. If L and N are submodules of M, then*

1) If $M = N \bigoplus K$, then $N \leq_{RD} M$.

2) If $L \leq_{RD} N$ and $N \leq_{RD} M$, then $L \leq_{RD} M$.

3) If $L \leq N \leq M$ and $N \leq_{RD} M$, then $N/L \leq_{RD} M/L$.

4) If $L \leq N \leq M$ and $L \leq_{RD} M$, then $N/L \leq_{RD} M/L$ implies $N \leq_{RD} M$.

Proof. 1): Let $r \in R$ and suppose $mr \in N$ for $m \in M$. Then $m = n + k$ for some $n \in N$, $k \in K$. Hence $mr = (n+k)r = nr + kr$. So $mr - nr \in N \cap K$ as $mr \in N$ by hypothesis. Since the sum is direct, we have that $mr - nr = 0$. Therefore $mr = nr$, and $N \leq_{RD} M$.

2): Let $r \in R$ and suppose $mr \in L$ for some $m \in M$. Then $L \leq N$ implies that $mr \in N \cap Mr = Nr$ as $N \leq_{RD} M$. Since $mr \in L$ also, we have that $mr \in L \cap Nr = Lr$ by the assumption that L is an RD-submodule of N. It follows that $L \leq_{RD} M$.

3): Suppose $mr + L \in N/L$ for some $m \in M, r \in R$. Then $mr \in N \bigcap Mr = Nr$ as $N \leq_{RD} M$ by hypothesis. Hence $mr + L \in (N/L)r$ and we have that N/L is an RD submodule of M/L.

4): Suppose $mr \in N$ for some $m \in M, r \in R$. Then we have that $mr + L = (m+L)r \in N/L \bigcap (M/L)r = (N/L)r$ as $N/L \leq_{RD} M/L$. Therefore $mr \in Nr$ and $N \leq_{RD} M$. □

Remark 5.5. *Note that 4) says that preimages of RD-submodules are RD.*

Theorem 5.6. *Let R be a right duo chain ring. Then the following are equivalent for a right R module M and every $a \in M$:*

1) aR is an RD-submodule of M.

2) If $0 \neq ar = zs$ for r, $s \in R$ and $z \in M$, then $Rr \subseteq Rs$.

3) $Ann(x) \subseteq Ann(a)$ for every $x \in a + MJ$.

Proof. 1) implies 2): Assume that aR is an RD-submodule of M. Suppose r, $s \in R$ and $z \in M$ are such that $0 \neq ar = zs$. Since aR is RD, we can write $ar = ar's$ for some $r' \in R$. Assume by way of contradiction that $Rs \subsetneq Rr$. Then $s = jr$ for some $j \in R$ which cannot be a unit of R. Thus, $j \in J$ and $ar = ar's = ar'jr$ so that $a(1 - r'j)r = 0$. Note that $u = 1 - r'j \in U(R)$ as $j \in J$. So $ur \in Ann(a) = Ann(aR)$ as R is right duo. In particular, $ar = au^{-1}(ur) = 0$, a contradiction.

2) implies 3): Suppose that $(a + y)r = 0$ but $ar \neq 0$ for some $y \in MJ$. Then since R is a chain ring, we can write $y = mj$ for some $m \in M$ and $j \in J$. Then $0 \neq ar = (-m)(jr)$. By 2), $Rr \subseteq R(jr)$ which implies that $(1 - r'j)r = 0$ for some $r' \in R$. Then $r = 0$ as $1 - r'j \in U(R)$, a contradiction.

3) implies 1): Suppose $0 \neq ar = ms$ for some r, $s \in R$ and $m \in M$. If $Rr \subseteq Rs$ then $r = ts$ for some $t \in R$. So $ms = ar = ats \in aRs$ and we are done. So suppose $Rs \subsetneq Rr$. Then $s = jr$ for some $j \in J$. We have $0 \neq ar = ms = mjr$ and therefore $(a - mj)r = 0$. Then, $ar = 0$ by 3), a contradiction. □

5.2 *RD*-Composition Series

In this section we characterize right duo rings using the existence of an RD-composition series for finitely generated modules over right chain rings. This result naturally extends a result for commutative rings by Fuchs and Salce in [14].

Definition 5.7. *A ring R has the **right unique kernel property** provided that, whenever I and K are right ideals of R such that $R/I \cong R/K$, then $I = K$.*

Lemma 5.8. *A right chain ring R has the right unique kernel property if and only if R is a right duo ring.*

Proof. Suppose that R is right duo, and consider an isomorphism $\phi : R/I \to R/K$ for right ideals I and K of R. Observe that I and K are two-sided since R is duo. Select $r \in R$ such that $\phi(1+I) = r+K$. For every $a \in K$, we have $\phi(a+I) = \phi(1+I)a = ra+K = 0$. Since ϕ is one-to-one, $a \in I$, and thus $K \subseteq I$. By symmetry, $I = K$.

Conversely, suppose that R has the right unique kernel property, and let I be any right ideal of R. Consider a unit $u \in R$, and define $\phi : R/I \to R/uI$ by $\phi(r+I) = ur+uI$. Then,

$r - s \in I$ if and only if $ur - us \in uI$, and ϕ is a well-defined monomorphism. Moreover, if $uv = 1$, then $\phi(v + I) = 1 + uI$, and ϕ is onto. Hence, $I = uI$ by the right unique kernel property. Since R is a right chain ring, this implies that I is two-sided. $\square$

Definition 5.9. *A series of submodules* $0 = M_0 < \cdots < M_{n-1} < M_n = M$ *of a module* M *satisfying*

1) each M_i *is an RD-submodule of* M *and*

2) M_{i+1}/M_i *is cyclic for every* i*,*

is an ***RD-composition series*** *of* M*. The sequence* $A_i = Ann(M_i/M_{i-1})$ *of right ideals of* R *is the* ***annihilator sequence*** *of the RD-composition series. The annihilator sequence is* ***nondecreasing*** *if* $A_i \leq A_{i+1}$ *for every* $i = 1, 2, \ldots, n-1$*.*

Theorem 5.10. *The following are equivalent for a chain ring* R*:*

1) R *is right duo.*

2) Every finitely generated right R*-module* M *admits a finite chain*

$$0 = M_0 < \cdots < M_{n-1} < M_n = M$$

of RD-submodules such that $M_{i+1}/M_i \simeq R/I_i$ *for right ideals* I_i *of* R *with the property that* $I_1 \subseteq \cdots \subseteq I_n$*. Moreover, if* $0 = M'_0 < \cdots < M'_n = M$ *is another chain of RD-submodules of* M *such that* $M'_{i+1}/M'_i \simeq R/I'_i$ *for right ideals* $I'_1 \subseteq \cdots \subseteq I'_n$*, then* $I'_i = I_i$ *for every* i*.*

Proof. Suppose R is right duo. Let $m + MJ \in M/MJ$ and $r + J \in R/J$. Define an operation $(m + MJ)(r + J) = mr + MJ$. If $r - r' \in J$, then $mr - mr' = m(r - r') \in MJ$ and therefore $mr + MJ = mr' + MJ$. Thus, M/MJ is a right R/J-module. Since M/MJ is finitely generated, it is a finite dimensional vector space over the division ring R/J, and hence has a finite basis, say $\{x_i + MJ \,|\, i = 1, 2, \ldots, n\}$. As $MJ + (\sum_i x_i R) = M$ and M is finitely generated, the x_i generate M by Nakayama's Lemma.

As R is right duo, $Ann_R(M) = \bigcap_{i=1}^n Ann_R(x_i)$. Since R is a chain ring, we can order the annihilators of the x_i (reindexing if necessary) as $Ann_R(x_1) \subseteq \cdots \subseteq Ann_R(x_n)$. So there exists at least one k such that $Ann_R(x_k) = Ann_R(M)$. Let $1 \leq j \leq n$ be chosen largest such that $Ann_R(M) = Ann_R(x_1) = \cdots = Ann_R(x_j)$. We claim that there exists an $i \in \{1, \ldots, j\}$ such that $Ann_R(x) \subseteq Ann_R(x_i)$ for every $x \in x_i + MJ$. Suppose by way of contradiction that for every $k \leq j$ there exists $x'_k \in x_k + MJ$ such that $Ann_R(x_k) \subsetneq Ann_R(x'_k)$.

Then M is generated by the set $\{x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_n\}$ by another application of Nakayama's Lemma. Ordering the annihilators as an increasing chain as above we see that $Ann_R(M) = Ann_R(x_l)$ for at least one l. By the choice of j we must have $l \leq j$. Hence, $Ann_R(x_l) = Ann_R(x'_l)$ which contradicts our assumption. Hence the claim is established. Without loss of generality, we may assume the element so obtained is x_1. Then by Theorem 5.6, $x_1 R$ is an RD-submodule of M.

Set $M_1 = x_1 R$ and induct on the number n of generators. Clearly the result holds for $n = 1$. By the induction hypothesis, M/M_1 has an RD-composition series

$$0 = L_0/M_1 \leq L_1/M_1 \leq \ldots L_{n-1}/M_1 = M/M_1$$

where $M_1 \leq L_1 \leq L_2 \leq \ldots \leq L_{n-1} = M$ are submodules of M. Then

$$(L_{i+1}/M_1)/(L_i/M_1) \simeq L_{i+1}/L_i$$

is cyclic. As preimages of RD-submodules are RD by Proposition 5.4, we see that the L_i form an RD-composition series of M.

To see that the annihilator condition is satisfied, note

$$Ann(L_i/L_{i-1}) = Ann((L_i/M_i)/(L_{i-1}/M_i)) \subseteq Ann((L_{i+1}/M_i)/(L_i/M_i)) = Ann(L_{i+1}/L_i).$$

As M_i/M_{i-1} is cyclic, A_i satisfies $R/A_i \simeq M_i/M_{i-1}$ as R is a right duo ring. Set $I_i = A_i$. To establish the uniqueness claim, suppose $0 = M'_0 < \cdots < M'_n = M$ is another RD-composition series of M, with $I'_i = Ann(M'_i/M'_{i-1})$. Then $M_i/M_{i-1} \simeq R/I_i$ and $M'_i/M'_{i-1} \simeq R/I'_i$, where the right ideals I_i and I'_i are unique. Observe that $I_1 \subseteq \cdots \subseteq I_n$ and $I'_1 \subseteq \cdots \subseteq I'_n$.

Suppose that $I_1 \neq I'_1$. By Lemma 5.8, $R/I_1 \not\simeq R/I'_1$, and hence $M_1/M_0 \not\simeq M'_1/M'_0$. But then $M_1/M_0 \simeq M'_j/M'_{j-1}$ for some $j > 1$. Hence, $I_1 = I'_j$. In the same way, we can find $k > 1$ such that $I'_1 = I_k$. Then $I_1 \leq I_k = I'_1 \leq I'_j = I_1$, which implies that $I_1 = I'_1$, a contradiction. Thus, $I_1 = I'_1$. An induction on the length of the composition series establishes the claim.

For the converse, consider a right ideal I of R, and choose a unit u in R. Then $R/I = (u^{-1} + I)R$ so that $R/I \simeq R/uI$. By uniqueness, $I = uI$. As R is a chain ring, this implies that I is two-sided and thus, R is right duo. □

Let M be a finitely generated right R-module, where R is a right duo chain ring. Then, M has a composition series

$$\{0\} = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n = M$$

such that U_i/U_{i-1} is cyclic, say $U_i/U_{i-1} \cong R/I_i$ for some right ideal I_i. The last theorem shows that the factors U_i/U_{i-1} determine the "annihilators" $I_1, \ldots, I_n$ uniquely only if R is a right duo ring. Therefore, we will assume that R is right duo whenever we consider composition series for a finitely generated module.

5.3 Equivalence of Relative Divisibility and Purity

Purity in the sense of Cohn is an additional extension of the notion of a pure subgroup of an Abelian group to modules over arbitrary rings. Warfield has shown in [39] that if R is an integral domain, then R is Prüfer if and only if relative divisibility is equivalent to purity. This result naturally extends to modules over right duo chain domains. In establishing this equivalence, we utilize the existence of an RD-composition series established in the previous section.

Definition 5.11. *Let R be a ring and A a right R-module. A submodule B of A is said to be* ***pure*** *if every finite system of equations over B*

$$\sum_{j=1}^{m} x_j r_{ij} = b_i \quad (i = 1, \ldots, n),$$

with $r_{ij} \in R$ and unknowns $x_1, \ldots, x_m$, has a solution in B whenever it is solvable in A.

Note that $B \leq_{RD} A$ is equivalent to the condition that whenever $b \in B$ and the equation $xr = b$ is solvable in A, then it is solvable in B. From this we see immediately that pure submodules are RD for an arbitrary ring R. The investigation into the rings for which the converse holds is more difficult. Indeed, we have the following open question.

Open Question: If R is a right semihereditary ring, then every RD-submodule of a right R-module is pure.

This is a natural question to ask, given the result of Warfield noted above. We need several preliminary result in order to establish the equivalence for modules over right duo chain domains.

Definition 5.12. *An exact sequence $\mathcal{E}\colon\ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is* ***RD-exact*** *if the map $A \longrightarrow B$ embeds A in B as an RD-submodule. A right R-module M is* ***cyclically presented*** *if it is of the form R/xR for some $x \in R$.*

We have the following Lemma from [39]:

Lemma 5.13. *[39] For an exact sequence $\mathcal{E} :\ 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ of right R-modules, the following are equivalent:*

1) $\mathcal{E}$ is RD-exact.

2) The induced map $\beta_* : Hom_R(R/rR, B) \longrightarrow Hom_R(R/rR, C)$ is surjective for every $r \in R$.

3) Then induced map $1 \otimes \alpha : R/rR \otimes_R A \longrightarrow R/rR \otimes_R B$ is injective for every $r \in R$.

Note that 2) says that cyclically presented modules have the projective property relative to RD-exact sequences. The following proof follows that of Warfield [39] verbatim.

Theorem 5.14. *A finitely presented module M_R over a right duo chain domain is a direct sum of cyclically presented modules.*

Proof. Since M_R is finitely presented, by Theorem 5.10 there exists an RD-composition series of M

$$0 = M_0 < \cdots < M_{n-1} < M_n = M.$$

As M_{n-1} is finitely generated and M finitely presented, the quotient M/M_{n-1} is again finitely presented. Hence the quotient M/M_{n-1} is finitely presented and cyclic, so that we can find a finitely generated right ideal I_n such that $M/M_{n-1} \simeq R/I_n$. Since R is a right chain domain, I_n is cyclic. Thus, M/M_{n-1} is cyclically presented. By Lemma 5.13 M/M_{n-1} has the projective property relative to RD-exact sequences. Hence, $0 \longrightarrow M_{n-1} \longrightarrow M \longrightarrow M/M_{n-1} \longrightarrow 0$ splits and $M \simeq M_{n-1} \oplus M/M_{n-1}$. As M_{n-1} is again finitely presented, the result follows by induction. $\square$

We have the following characterization for pure exact sequences. The proof may be found in [25, Theorem 4.89].

Theorem 5.15. *For any short exact sequence $\mathcal{E}: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules, the following are equivalent:*

1) $\mathcal{E}$ is pure-exact.

2) $\mathcal{E} \oplus_R C'$ is exact for any finitely presented left R-module C'.

3) A is a pure submodule of B.

4) Given a commutative diagram (of right R-modules)

$$\begin{array}{ccccc} & & R^n & \xrightarrow{\sigma} & R^m \\ & & \downarrow f & & \downarrow g \\ 0 & \longrightarrow & A & \longrightarrow & B \end{array}$$

there exists $\theta \in Hom_R(R^m, A)$ such that $\theta\sigma = f$.

5) Any finitely presented right R-module M has the projective property relative to $\mathcal{E}$.

The following proof is due to Warfield [39].

Theorem 5.16. *Over right duo chain domains, purity and relative divisibility are equivalent.*

Proof. We will show that an RD-exact sequence $\mathcal{E}: \ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is pure exact. Let M be a finitely presented right R-module. By Theorem 5.14, M is a direct sum of cyclically presented modules. By Lemma 5.13 each of these have the projective property relative to $\mathcal{E}$, hence M has this same property. By Theorem 5.15, $\mathcal{E}$ is pure-exact. □

5.4 A Jordan-Hölder Type Theorem

Given the existence of an RD-composition series, a natural question to ask is if any two such series are isomorphic. We show that this is indeed the case for finitely generated modules over right duo right chain rings. The results below follow the work of Fuchs and Salce [16], and Salce and Zanardo [33] closely.

By modifying the proof of Salce and Zanardo in [33, Lemma 1.2], we are able to extend the following result to modules over right duo chain rings.

Proposition 5.17. *Let R be a right duo, right chain ring and M a right R-module. If N is an RD-submodule of M such that $M/N = (a+N)R$ is cyclic and $Ann(M/N) \leq Ann(N)$, then $M = aR \oplus N$.*

Proof. Clearly $M = aR + N$. Suppose $ar \in N$. Since N is an RD-submodule, we have that $ar \in N \cap Mr = Nr$ and we can write $ar = nr$ for some $n \in N$. Then $(a-n)r = 0$ so

$r \in Ann(a-n) = Ann((a-n)R)$ by Lemma 3.33. If we can show that $r \in Ann(M/N)$, then by the assumed condition on annihilators, $Nr = 0$ and therefore $ar = 0$ and the sum is direct.

Let $m + N \in M/N$. Then $m + N = as + N$ for some $s \in R$. Since $r \in Ann((a-n)R)$, we have, in particular, that $[(a-n)s]r = (as - ns)r = asr - nsr = 0$, i.e. $asr \in N$. It follows that $(m+N)r = (as+N)r = asr + N = N$, so $r \in Ann(M/N)$. $\square$

The next result follows from Proposition 5.17.

Corollary 5.18. *Let R be a right duo chain ring and M a right R-module. Suppose*

$$\mathcal{R}: \quad 0 = M_0 < \cdots < M_{n-1} < M_n = M$$

is a RD-composition series of M of length n, with nondecreasing annihilator sequence $A_1 \leq \cdots \leq A_n$. If $M_k/M_{k-1} = (x_k + M_{k-1})R$ and $A_1 = \cdots = A_k$ for some $k < n$, then $M_k = \bigoplus_{i=1}^{k} x_i R$. In particular, if the annihilator sequence of $\mathcal{R}$ is constantly equal to $Ann(M)$, then M is the direct sum of cyclic submodules.

Proof. The proof is by induction on k. The result is trivial for $k = 1$ as $M_1 = x_1 R$ is cyclic. Suppose $k = 2$. We will show that M_2 is the direct sum of cyclic modules using Proposition 5.17. By assumption, we have that $A_1 = A_2$. Recall that $A_1 = Ann(M_1) = Ann(x_1 R) = Ann(M)$. So we have that $Ann(M) = A_1 = A_2$. Since M_1 is an RD-submodule of M, we obviously have that M_1 is RD in M_2. Further, $M_2/M_1 = (x_2 + M_1)R$ is cyclic. By hypothesis, $A_2 = Ann(M_2/M_1) = A_1 = Ann(M_1)$. Hence the hypotheses of Proposition 5.17 are satisfied, and therefore $M_2 = M_1 \bigoplus x_2 R$. As $M_1 = x_1 R$ is cyclic, the result follows by induction. $\square$

The following result extends [33, Lemma 1.3].

Lemma 5.19. *Let R be a right duo chain ring. Then every RD-composition series of a finitely generated right R-module M is isomorphic to one whose annihilator sequence is non-decreasing.*

Proof. Suppose we have an RD-composition series

$$\mathcal{R}: \quad 0 = M_0 < \cdots < M_{n-1} < M_n = M$$

of M with annihilator sequence $\{ A_i \mid i = 1, \ldots, n-1 \}$. Since $\mathcal{R}$ is an RD-composition series of M, M_i/M_{i-1} is cyclic, say $M_i/M_{i-1} = (x_i + M_{i-1})R$. Suppose there exists an $i \in \{1, 2, \ldots n-1\}$ such that $A_{i+1} < A_i$. The idea of the proof is to replace the submodule M_i by a submodule M_i' so that $M_i/M_{i-1} \simeq M_i'/M_{i-1}$ and $A_i' = Ann(M_i'/M_{i-1}) \leq A_{i+1}$. Consider the module $K = M_{i+1}/M_{i-1}$ and its submodule $N = M_i/M_{i-1}$. It will be shown that these modules satisfy the hypotheses of Proposition 5.17.

By Proposition 5.4 N is an RD-submodule of K. Further, $K/N \cong M_{i+1}/M_i$ so K/N is cyclic. With respect to this isomorphism, we see that $K = (x_{i+1} + M_{i-1})R + N$. As

$$Ann(K/N) = A_{i+1} < A_i = Ann(N),$$

the hypotheses of Proposition 5.17 are met. Hence $K = N \bigoplus (x_{i+1} + M_{i-1})R$.

Set $M_i' = x_{i+1}R + M_{i-1}$. Note that $M_{i-1} \leq M_i' \leq M_{i+1}$. We claim that M_i'/M_{i-1} is isomorphic to M_{i+1}/M_i. Define $\phi : M_i'/M_{i-1} \longrightarrow M_{i+1}/M_i$ by $\phi(x_i r + M_{i-1}) = x_{i+1}r + M_i$. To see that ϕ is well defined, suppose $x_{i+1}r + M_{i-1} = x_{i+1}s + M_{i-1}$. Then $x_{i+1}(r-s) \in M_{i-1} \subseteq M_i$. ϕ is clearly an epimorphism. Suppose $x_{i+1}r \in M_i$. Then $r \in A_{i+1} \subseteq A_i$. Hence, $x_{i+1}r + M_{i-1} \in (M_i/M_{i-1}) \cap (M_i'/M_{i-1}) = 0$ as $K = N \bigoplus (x_{i+1} + M_{i-1})R$. Hence, ϕ is a monomorphism, and therefore an isomorphism.

It follows that M_i'/M_{i-1} is cyclic. We also have that M_{i+1}/M_i' is cyclic, as

$$M_{i+1}/M_i' \simeq (M_{i+1}/M_{i-1})/(M_i'/M_{i-1}) = K/(M_i'/M_{i-1}) \simeq N = M_i/M_{i-1}.$$

Since M_{i-1} is an RD-submodule of M and $M_i/M_{i-1} \leq_{RD} M/M_{i-1}$, Proposition 5.4 implies that M_i' is an RD-submodule of M.

It remains to be shown that $Ann(M_i'/M_{i-1}) \leq Ann(M_{i+1}/M_i')$. Suppose we have that $r \in Ann(M_i'/M_{i-1})$. Then $M_i'r \subseteq M_{i-1}$. Therefore $x_{i+1}Rr \subseteq M_{i-1}$. Recalling that $M_{i+1}/M_i' = x_{i+1}R + M_i'$, we see that $(M_{i+1}/M_i')r = x_{i+1}Rr + M_i' \subseteq M_i'$. Hence $r \in Ann(M_{i+1}/M_i')$.

In a finite number of steps we thus obtain a pure composition series with non-decreasing annihilator sequence isomorphic to $\mathcal{P}$. $\square$

Definition 5.20. *Let R be a ring and M a finitely generated R-module. If $\mathcal{R}$ is an RD-composition series of M, say $0 = M_0 < M_1 < \ldots < M_n = M$, then we call n the* ***length*** *of $\mathcal{R}$, written $\ell(M)$. The minimal number of generators of M is denoted* ***gen(M)****.*

The next three results are due to Salce and Zanardo presented in [33]. The proofs carry over without change for right duo, right chain rings.

Lemma 5.21. *Let R be a right duo chain ring. Then the length of an RD-composition series $\mathcal{R}$ of a finitely generated right R-module M is equal to $gen(M)$.*

Proof. By Nakayama's Lemma, a set of elements of M generate M if and only if their cosets modulo MJ generate M/MJ. Hence $gen(M) = dim_{R/J}\, M/MJ$. If $M_i/M_{i-1} = (x_i + M_{i-1})R$, $i = 1, \ldots, n$, then the elements x_i generate M; hence $gen(M) \leq n$. Since M/MJ is a finite dimensional vector space over the division ring R/J, the independence of the elements x_i modulo MJ establishes the converse.

Assume by way of contradiction that $\sum_{i=1}^{n} x_i r_i \in MJ$ where $r_i \in R$ for every i and at least one of the r_i is a unit. Let k be the least index such that $r_k \notin J$. Then we have that $\sum_{i=1}^{k} x_i r_i \in M_k \cap MJ = M_k J$. It follows that $\sum_{i=1}^{k} x_k r_k = (\sum_{i=1}^{k} x_i s_i)j$ for a suitable $j \in J$. Hence, $x_k(r_k - s_k j) \in M_{k-1}$. Then $r_k \notin J$ implies that $r_k - s_k j \notin J$, and therefore is a unit. It follows that $x_k \in M_{k-1}$, a contradiction. $\square$

We now examine submodules of M of the form Mr.

Lemma 5.22. *[33, Lemma 1.5] Let M be a finitely generated right R-module over a duo chain ring R. Suppose M has an RD-composition series of length n with non-decreasing annihilator sequence $\{ A_i \mid i = 1, \dots, n \}$. Let $r \in R$. Then*

1) If $r \notin \cup_{i=1}^{n} A_i$, then $\ell(Mr) = \ell(M)$.

2) If $r \in A_{k+1} \setminus A_k$ for $0 \leq k \leq n-1$, then $Mr = M_k r$ and $\ell(Mr) = k$.

Proof. 1): Note first that since R is duo, rR is an ideal of R and $M_k r$ is a submodule of M for every $k = 1, \dots, n$. Consider the series

$$0 < M_1 r < \cdots < M_n r = Mr.$$

It will be shown that this is a pure composition series of Mr.

Suppose $m(rt) \in M_j rR$ for some $t \in R$ and $m \in M$. Then $m(rt) \in M_j$ since $M_j r \subseteq M_j$. Further, $m(rt) \in M(rt)$ since $m \in M$. Hence $m(rt) \in M_j \cap M(rt)$. By the purity of M_j in M, $m(rt) \in M_j \cap M(rt) = M_j(rt) = (M_j r)t \subseteq M_j(rt)$. Therefore $M_j r$ is pure in Mr.

By assumption, M_j/M_{j-1} is cyclic, say $M_j/M_{j-1} = (x_j + M_{j-1})R$. We claim that $M_j r/M_{j-1} r = (x_j r + M_{j-1} r)R$. Clearly $(x_j r + M_{j-1} r)R \subseteq M_j r/M_{j-1} r$. For the reverse inclusion, suppose $m_j(rs) + M_{j-1} r \in M_j r/M_{j-1} r$, where $m_j \in M_j$. Then we have that $m_j + M_{j-1} r \in M_j/M_{j-1}$. Since $M_j/M_{j-1} = (x_j + M_{j-1})R$, there exists $t \in R$ such that $m_j + M_{j-1} r = x_j t + M_{j-1}$. Hence, $m_j(rs) + M_{j-1} r = x_j trs + M_{j-1} r$. Since R is right duo, $Rr \subseteq rR$; so there exists $t' \in R$ such that $tr = rt'$. Putting the equations above together, we have that $m_j(rs) + M_{j-1} r = x_j trs + M_{j-1} r = x_j rt's + M_{j-1} r$, which is an element of $(x_j r + M_{j-1} r)R$.

We conclude that $M_j r/M_{j-1} r$ is cyclic. Therefore, the sequence above is a pure composition series for Mr and $\ell(Mr) = \ell(M)$.

2): Fix k, $0 \leq k \leq n-1$. Suppose j satisfies $k \leq j \leq n-1$. We claim that for such a j, $M_{j+1} r = M_j r$. Since M_j is pure in M, it follows easily that M_j is pure in M_{j+1}. Since $M_j \leq M_{j+1}$ for every $j = 1, \dots, n$, we also have $M_j r \subseteq M_{j+1} r$. For the reverse inclusion, note that $r \in A_{j+1} \setminus A_j$ implies that $M_{j+1}r \subseteq M_j$. So if $m_{j+1}r \in M_{j+1}r$, then $m_{j+1}r \in M_j$. By the purity of M_j in M_{j+1}, we have that $m_{j+1}r \in M_j r$. We conclude that $M_{j+1}r = M_j r$ for $k \leq j \leq n-1$. So for $k \leq j \leq n-1$, we have $M_k r = \cdots = M_n r = Mr$. By (1) and Lemma 5.22, it follows that $\ell(Mr) = \ell(M) = k$. □

The next result establishes the isomorphism of any two RD-composition series of a finitely generated right R module over a duo chain ring.

Theorem 5.23. *Let M be a finitely generated right R-module over a duo chain ring R. Then any two RD-composition series of M are isomorphic.*

Proof. By Lemma 5.21, any RD-composition series of M has length equal to $gen(M)$. Hence we may assume the series have the same length. We may also assume, by Lemma 5.19, that the annihilator sequences A_i and B_i are nondecreasing and have the same length.

In the proof of Theorem 5.10, it was shown that the initial non-zero submodule M_1 in any RD-composition series of M was chosen as $M_1 = x_1R$, where $Ann(x_1) = Ann(M)$. Since R is right duo, $Ann(M) = Ann(x_1) = Ann(x_1R) = Ann(M_1)$. Therefore, $A_1 = Ann(M) = B_1$. Assume $A_k = B_k$ for some $1 < k < n$. Suppose $r \in A_{k+1} \setminus A_k$. By Lemma 5.22, $\ell(MrR) = k$. Since $r \notin A_k$, and $A_k = B_k$ by the induction hypothesis, we have that $r \notin B_k$. But then r must be in B_{k+1} and $A_{k+1} \subseteq B_{k+1}$. By a symmetric argument, $B_{k+1} \subseteq A_{k+1}$, whence $A_{k+1} = B_{k+1}$ and the result follows by induction. □

5.5 Essential Pure Submodules

The goal of the next set of results is to establish, in any finitely generated right R-module M, the existence of an essential pure submodule of M that is the direct sum of cyclic modules and has the same Goldie dimension as M. We require several technical lemmas, following [33], with modifications for the noncommutative case.

Proposition 5.24. *Let R be a right duo chain ring and M a finitely generated right R-module. Then for every $0 \neq x \in M$, there exists $r \in R$ such that $x \in Mr \setminus MJr$.*

Proof. Let $0 \neq x \in M$, and suppose $0 = M_0 < M_1 < \ldots < M_n = M$ is an RD-composition series of M. Then, there exists an index $i > 0$ such that $x \in M_i \setminus M_{i-1}$. The proof is by induction on i. If $i = 1$, then $x \in M_1 = x_1R$, a cyclic module. Hence there exists an $r \in R$ such that $x = x_1r$, whence $x \in Mr$. Suppose by way of contradiction that $x \in MJr$. Then $x = mj'r$ for some $j' \in J$ and then $x \in M_1 \cap Mj'r = M_1j'r$ by the purity of M_1 in M. As $M_1 = x_1R$ we can write $x = x_1sj'r$. As J is an ideal we may set $j = sj' \in J$ so that $x_1r = x = x_1jr$. Since x_1R is an RD-submodule of M, 5.6.3 implies that $Ann_R(y) \subseteq Ann_R(x_1)$ for every $y \in x_1+MJ$. As $x_1-x_1j \in x_1+MJ$ and $r \in Ann_R(x_1-x_1j)$, we have that $r \in Ann_R(x_1)$. But then $x = x_1r = 0$, a contradiction.

Suppose now that $i > 1$. Set $\bar{x} = x + M_{i-1}$ and $\overline{M} = M/M_{i-1}$. Then $\bar{x}$ is a non-zero element of M_i/M_{i-1}. Since M_i/M_{i-1} is cyclic with generator $x_i + M_{i-1}$, we have that $\bar{x} = \bar{x}_ir$ for some $r \in R$. Hence $\bar{x} \in \overline{M}r$. We claim that $x \notin \overline{M}Jr$. If $x \in \overline{M}Jr$, then $\bar{x} = \bar{m}jr$ for some $j \in J$, and we can write $\bar{x} = \bar{x}_ir = \bar{x}_ijr$. Arguing as above, noting that the purity of M_i/M_{i-1} in M/M_{i-1} implies that $\bar{x}_iR$ is pure in $\overline{M}$, we obtain that $r \in Ann(\bar{x}_i)$. Therefore $x = x_ir \in M_{i-1}$, a contradiction. So we have that $x \in \overline{M}r \setminus \overline{M}Jr$.

If $x \in Mr$, then the proof is complete as $x \in MJr$ implies $\bar{x} \in \overline{M}Jr$, contradicting the above. Suppose that $x \notin Mr$. We can write $x = mr + y$ for suitable $m \in M$ and $y \in M_{i-1}$. Then by the induction hypothesis, there exists an $s \in R$ such that $y \in Ms \setminus MJs$. Note that $y \notin Mr$ as $x \notin Mr$ by hypothesis, and $x = mr + y$. Since R is a chain ring, we have that $Rr \subseteq Rs$ or $Rs \subseteq Rr$. As $Rs \subseteq Rr$ implies that $y \in Mr$, it must be the case that $Rr \subseteq Rs$. Therefore $r = ts$ for some $t \in R$. Hence $x = mr + y = m(ts) + y = (mt)s + y \in Ms \setminus MJs$ and the proof is complete. □

The following Corollary is an immediate consequence of Proposition 5.24:

Corollary 5.25. *If $0 \neq x \in M$, then there exists $r \in R$ and $y \in M \backslash MJ$ such that $x = yr$.*

In the work to follow, we find it necessary to consider conditions on a ring R which guarantee that $aJ = Ja$ for every $a \in R$, where J is the Jacobson radical of R. This is not always the case, as witnessed by the following:

Example 5.26. Let R be the 2×2 lower triangular matrix ring with entries from a field k. By [18, Corollary 4.9], R is a right and left Artinian hereditary ring. Since R is Artinian, the Jacobson radical is equal to the nilradical N of R. Let e_{ij} be the standard matrix units. Then $N = re_{21}$, where $r \in k$. If $a = e_{11}$, then a calculation shows that $aJ = 0$. On the other hand, again by an easy calculation, $Ja = J$. Hence, $aJ \neq aJ$.

In the class of rings under consideration, however, we are able to obtain a positive result.

Proposition 5.27. *Suppose R is a chain domain. Then $aJ = Ja$ for every $a \in R$ if and only if R is a duo ring.*

Proof. Suppose R is a duo ring, that is, $aR = Ra$ for every $a \in R$. Without loss of generality, we may assume $a \neq 0$. By symmetry, it is enough to show that $Ja \subseteq aJ$. If this is not the case, observe that $xar = xr'a \in Ja$ for every $x \in J$ and $r \in R$, since R is duo. Hence, Ja and aJ are right ideals of R. Since $Ja \nsubseteq aJ$ and R is a chain domain, we obtain $aJ \subsetneq Ja$. Thus, Ja/aJ is a non-zero submodule of aR/aJ, for $Ja \subseteq Ra = aR$.

Consider $\phi : R/J \to aR/aJ$ given by $\phi(r + J) = ar + aJ$. If $r_1 - r_2 \in J$, then $ar_1 - ar_2 \in aJ$. Thus, ϕ is a well-defined epimorphism. If $\phi(r + J) = 0$, then $ar \in aJ$ which implies $ar = ax$ for some $x \in J$. Thus, $a(r - x) = 0$. If $r \notin J$, then $rs = 1$ for some $s \in R$. Then $0 = a(r - x) = a(rs - xs) = a(1 - xs)$, which is a contradiction as $1 - xs$ is a unit of R. Hence, ϕ is an isomorphism.

Since R is a right chain ring, J is a maximal ideal. Then R/J is a simple R-module. Hence, aR/aJ is simple which implies $Ja/aJ = aR/aJ$. We conclude $Ja = aR = Ra$. Thus, there exists $y \in J$ such that $ya = a$, which implies $(y - 1)a = 0$. But then $a = 0$, as $y - 1$ is a unit. Thus, $Ja \subseteq aJ$.

Conversely, suppose $aJ = Ja$ for every $a \in R$. Again, without loss of generality, we may assume $a \neq 0$. It is enough to show that $Ra \subseteq aR$. Suppose $x \in R$ and consider xa. If $x \in J$, then $xa \in Ja = aJ \subseteq aR$. Hence, $Ra \subseteq aR$ as claimed. Thus, we may assume that x is a unit of R. Since R is a chain domain, either $xaR \subseteq aR$, or $aR \subseteq xaR$. In the first case, $xa \in aR$ and the result holds. Suppose the second case holds. Then $a = xaj$ for some $j \in J$. Choose $j' \in J$ such that $aj = j'a$. Then $a = xaj = xj'a$ which implies $(1 - xj')a = 0$. As $1 - xj'$ is a unit of R, this implies $a = 0$, a contradiction. □

The following technical result is crucial in establishing the main result. The proof is based on ideas found in [33].

Theorem 5.28. *Let R be a duo chain ring, and consider a finitely generated right R-module M. Suppose $0 = M_0 < M_1 < \ldots < M_n = M$ is an RD-composition series of M with nondecreasing annihilator sequence A_i, $i = 1, \ldots, n$. If M_{n-1} is not essential in M, then M has a non-zero cyclic summand.*

Proof. Let $M = x_1R + \cdots + x_nR$ and suppose $0 \neq y \in M$ is such that $yR \cap M_{n-1} = 0$. Then

$$y = \sum_{i=1}^{n} x_i a_i \, (a_i \in R). \tag{5.1}$$

It will be shown that there is no loss of generality in assuming $y \in M \backslash MJ$. Suppose $y \in MJ$; then by Corollary 5.25, there exists an $r \in R$ and $x \in M \setminus MJ$ such that $y = xr$. Moreover, $xR \cong R/I$ for some right ideal I of R. As R is a chain ring, R/I is uniserial and therefore $dim(xR) = 1$. Hence yR is an essential submodule of xR and $xR \cap M_{n-1} = 0$.

So assume that $y \in M \backslash MJ$ and let j be the largest index such that $a_j \notin J$. Then a_j is a unit and we may assume without loss of generality that $a_j = 1$. If $j = n$ then $M = yR \oplus M_{n-1}$ and the proof is complete. Assume $j < n$ and set $N = \sum_{i \neq j} x_i R$. Clearly $M = N + yR$, so we need only show that $N \cap yR = 0$. Assume that $N \cap yR \neq 0$. Then we have a relation

$$0 \neq yr = \sum_{i \neq j} x_i b_i, (b_i \in R). \tag{5.2}$$

We claim that if $j < h \leq n$ then there exist a relation

$$x_h r_h = \sum_{i=1}^{h-1} x_i a_{h,i} \quad \text{with} \tag{5.3}$$

$$r_h \in Jr, \quad a_{h,j} \in Rr \setminus Jr \tag{5.4}$$

The proof is by induction on h and we show first that the claim holds for $h = n$. From (5.1) and (5.2) we obtain $x_n(b_n - a_n r) = \sum_{i=1}^{n-1} x_i(a_i r - b_i)$ where $b_j = 0$. Suppose $b_n \in A_n = Ann(M/M_{n-1})$. Then $0 \neq yr = \sum_{i \neq j} x_i b_i \in yR \cap M_{n-1}$, a contradiction. Similarly, $a_n r \notin A_n$. Since R is a chain ring, either $rR \subseteq b_n R$ or $b_n R \subseteq rR$. If the first case holds, then $r = b_n t$ for some $t \in R$. Recall that $a_n \in J$ by our assumption on j. Hence, $a_n b_n \in J b_n = b_n J$ by Proposition 5.27. Thus, $a_n b_n = b_n a'_n$ for some $a'_n \in J$. Then $b_n - a_n r = b_n - a_n b_n t = b_n - b_n a'_n t = b_n(1 - a'_n t)$. As $a'_n \in J$, $(1 - a'_n t)$ is a unit of R. Hence, $x_n b_n (1 - a'_n t) \in M_{n-1}$, which implies $x_n b_n \in M_{n-1}$. We conclude that $b_n \in A_n$, a contradiction. Thus, $b_n R \subsetneq rR$, and $b_n = rs$ for some non-unit $s \in R$. So $b_n \in rJ = Jr$, and thus $a_n \in J$ implies that $b_n - a_n r \in Jr$.

Set $r_n = b_n - a_n r$ and note that (5.3) and the first part of (5.4) hold for $h = n$. To establish the final claim in (4), recall that $b_j = 0$ and that $a_j \notin J$. So we have that $a_j r \in Rr \backslash Jr$. Setting $a_{n,j} = a_j r$, we see that the final claim in (5.4) holds and the result is established for $h = n$.

Assume by induction that the result holds for $h > j + 1$. So we have a relation

$$x_h r_h = \sum_{i=1}^{h-1} x_i a_{h,i} \tag{5.5}$$

such that $r_h \in Jr$ and $a_{h,j} \in Rr \backslash Jr$.

Since $r_h \in Jr$, we have that $x_h r_h \in MJr \cap M_{h-1} = M_{h-1}Jr$ by the purity of M_{h-1}. Therefore we can write $x_h r_h = \sum_{i=1}^{h-1} x_i c_i qr$ where $q \in J, c_i \in R$. Subtracting, we see that $x_{h-1}(a_{h,h-1} - c_{h-1}qr) = \sum_{i=1}^{h-2} x_i(c_i qr - a_{hi})$. By the same argument as above, using the fact that $r \notin A_{h-1}$, we will show that $a_{h,h-1} - c_{h-1}qr \in Jr$.

Observe first that $c_{h-1}qr \in Jr$ as $q \in J$. Therefore our claim reduces to showing that $a_{h,h-1} \in Jr$. If $Ra_{h,h-1} \subsetneq Rr$, then $a_{h,h-1} = jr$ for some $j \in J$ and the claim follows. Suppose $Rr \subseteq Ra_{h,h-1}$. Then $r = sa_{h,h-1}$ for some $s \in R$. Utilizing Proposition 5.27 in an identical fashion as above, we obtain that $x_h a_{h,h-1} \in M_{h-1}$. Hence, $a_{h,h-1} \in A_{h-1}$. Thus, $r = sa_{h,h-1} \in A_{h-1}$, a contradiction. So we have that $a_{h,h-1} - c_{h-1}qr \in Jr$. Moreover, $a_{h,j} \in Rr \setminus Jr$ implies that $c_j qr - a_{h,j} \in Rr \setminus Jr$. Setting $r_{h-1} = a_{h,h-1} - c_{h-1}qr$ and $a_{h-1,j} = c_j qr - a_{h,j}$ for $i \leq h-2$, the claim is established for $h-1$ and the result follows by induction.

Now set $h = j+1$. By the result above, we have a relation $x_{j+1}r_{j+1} = \sum_{i=1}^{j} x_i a_{j+1,j}$ where $r_{j+1} \in Jr$ and $a_{j+1,j} \in Rr \setminus Jr$. Repeating the argument above, using the fact that $r_{j+1} \in Jr$ and the relative divisibility of M_j, we can write $x_j(a_{j+1,j} - d_j qr) \in M_{j-1}$ where $d_j \in R$ and $q \in J$. Then $r \notin A_{j-1}$ implies that $a_{j+1,j} - d_j qr \in Jr$. Thus, $q \in J$ implies that $d_j qr \in Jr$. Hence, $a_{j+1,j} \in Jr$, a contradiction. It follows that $N \cap yR = 0$. □

Now that these technical results are established, the proof of the following is an easy consequence of Lemma 5.28. The proofs of the following results are due to Fuchs and Salce [16].

Theorem 5.29. *Over a right duo, chain ring R, every finitely generated right R-module M contains an essential pure submodule which is the direct sum of cyclic modules such that $dim(B) = dim(M)$.*

Proof. We induct on $n = gen(M)$. If $n = 1$, then the result is trivial, so assume $n > 1$. By induction, M_{n-1} has an essential pure submodule B' that is the direct sum of $dim(M_{n-1})$ non-zero cyclic submodules. If M_{n-1} is essential in M, then B' is essential in M. Set $B = B'$

and note that B has finite Goldie dimension. Then by Proposition 2.31, $B \leq_e M$ implies that $dim(B) = dim(M)$ and the result follows. If M_{n-1} is not essential in M, then by Lemma 5.28, there exists $0 \neq y \in M$ and a submodule N of M such that $M = yR \oplus N$. Note that $gen(N) = n - 1$, so by the induction hypothesis, N contains an essential pure submodule B'', which is the direct sum of $dim(N)$ non-zero cyclic submodules. Set $B = yR \oplus B''$. Then B is an essential pure submodule of M that is the direct sun of non-zero cyclic submodules and the proof is complete. □

Recall that the minimal number of elements needed to generate M is denoted $gen(M)$ and we have shown that $gen(M) = dim_{R/J} M/MJ$. Also, the number of factors in a pure composition series of M is written $\ell(M)$ and $\ell(M) = gen(M)$ for a finitely generated right R-module M over a right duo, chain ring. As a result of Theorem 5.29 and the observations above, we can obtain an upper estimate on the Goldie dimension of a finitely generated right R-module.

Corollary 5.30. *For a finitely generated right R-module over a right duo chain ring R, the following holds:*

$$dim(M) \leq gen(M).$$

Proof. Let B be an essential pure submodule which is the direct sum of non-zero cyclic submodules. Since B is pure, we have that $BJ = B \cap MJ$. Then $(B + MJ)/MJ$ is a submodule of M/MJ, and $B/BJ = B/B \cap MJ \cong (B + MJ)/MJ$. Since M/MJ is a finite dimensional vector space over the division ring R/J, $dim_{R/J} B/BJ \leq dim_{R/J} M/MJ$. By the observations above, we have

$$dim(M) = dim(B) = dim_{R/J} B/BJ \leq dim_{R/J} M/MJ = gen(M). \qquad \square$$

The final result of this section is a criteria for a finitely generated right R-module over a right duo, chain ring to be a direct sum of cyclic modules.

Corollary 5.31. *A finitely generated right R-module over a right duo, chain ring R is the direct sum of cyclic modules if and only if $gen(M) = dim(M)$.*

Proof. If M is the direct sum of cyclic modules, the clearly $gen(M) = dim(M)$. Conversely, assume $gen(M) = dim(M)$. By Theorem 5.29, M contains an essential pure submodule B that is the sum of non-zero cyclic submodules and $dim(B) = dim(M) = gen(M)$. Suppose B is a proper submodule of M and $dim(B) = gen(M)$. Then

$$B/BJ \simeq B/(B \cap MJ) = (B + MJ)/MJ \subsetneq M/MJ.$$

Then $B + MJ \subsetneq M$, a contradiction. Hence, $gen(M) > gen(B) = dim(B) = gen(M)$, an obvious contradiction. Thus $M = B$ and the result follows. $\square$

Chapter 6

Duo Modules

6.1 Quasiprojective Modules

Quasiprojective modules are a generalization of the familiar concept of projective modules. One of the first studies of quasiprojective modules was undertaken by Wu and Jans in [42] and was motivated by work on the dual concept of quasiinjective modules.

Definition 6.1. *Let M be a right R-module. A right R-module U is* ***M-projective*** *if every diagram*

$$\begin{tikzcd} & U \arrow[d, "f"] & \\ M \arrow[r, "\pi"'] & M/T \arrow[r] & 0 \end{tikzcd}$$

can be embedded in a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "f"] \arrow[dl, dashed, "f'"'] & \\ M \arrow[r, "\pi"'] & M/T \arrow[r] & 0 \end{tikzcd}$$

where π is the canonical map. If M is M-projective, then M is said to be a ***quasiprojective*** *module.*

In [42] the following Proposition is established.

Proposition 6.2. *[42, Proposition 2.1] Let M be a right R-module. If*

$$0 \longrightarrow T \longrightarrow P \longrightarrow M \longrightarrow 0$$

is exact with P projective and T fully invariant in P, then M is quasiprojective.

As a corollary, we obtain the following.

Corollary 6.3. *Let R be a right duo ring. Then every cyclic right R-module is quasiprojective.*

Proof. Suppose aR is a cyclic right R-module and $\varphi : R \longrightarrow aR$ is left multiplication by a. Then we have an exact sequence $0 \longrightarrow ann_r(a) \longrightarrow R \xrightarrow{\varphi} aR \longrightarrow 0$. As R is right duo, $ann_r(a)$ is a fully invariant submodule. Since R is projective, the result follows by Proposition 6.2. □

The following basic properties of M-projective modules are established in Anderson-Fuller [2, Proposition 16.2].

Theorem 6.4. *[2] Let U be a right R-module.*

1) *If $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of right R-modules, then U is M-projective if and only if U is projective relative to both K and N.*

2) *If U is projective relative to each of $M_1, \ldots, M_n$, then U is $\oplus_{i=1}^{n} M_i$-projective. Moreover, if U is finitely generated and M_α-projective ($\alpha \in A$), then U is projective relative to $\oplus_A M_\alpha$.*

Herrmann has investigated quasiprojective modules in [21], however a complete classification of such modules remains open (see [16, Problem 21]). There seems to have been little work done concerning quasiprojective modules over more general rings. As a start, we offer the following which extends a result presented in [21]. The proof follows that of Herrmann closely.

Theorem 6.5. *Let R be a chain domain and I a proper right ideal of R. If Q is the quotient ring of R, then I is quasiprojective if and only if it is Q-projective.*

Proof. Suppose I is quaisprojective and $f : I \longrightarrow Q/J$, where $J \subsetneq Q$. We claim that f is not an epimorphism. Suppose to the contrary that $f(I) = Q/J$. Then there exists an ideal

L such that $I/L \simeq Q/J$, which is a divisible R-module. Hence, $(I/L)r = I/L$ for every non-zero $r \in R$. Hence, $I = Ir + L$ for every non-zero $r \in R$. Choose $0 \neq c \in I$. Then $c = c_1c + l$ for some $c_1 \in I$ and $l \in L$. This implies $c - c_1c \in L$ and therefore $(1 - c_1)c \in L$. As $(1 - c_1)$ is a unit of R, it follows that $c \in L$. Hence, $I = L$ and therefore $Q/J = 0$, a contradiction.

Thus $im(f) = V/J$, where V is a proper R-submodule of Q. Then V is isomorphic to a right ideal K of R, so without loss of generality we set $V = K$ and $im(f) = K/J$. There is an embedding $R \longrightarrow I$ whose restriction embeds K in I. As I is quasiprojective it is K-projective by Theorem 6.4.

Then we have the following commutative diagram.

$$\begin{tikzcd}
 & I \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\
K \arrow[r] \arrow[d, "incl."'] & K/J \arrow[d, "incl."] \\
Q \arrow[r, "\pi"'] & Q/J
\end{tikzcd}$$

where π is the canonical map. Then $f = \pi g$ and I is Q-projective. As $I < Q$, the converse follows from Theorem 6.4. $\square$

6.2 Duo Modules and Strongly Right Bounded Modules

Definition 6.6. *Let R be a ring and M a right R-module. We say that M is a **duo** module if every submodule N of M is fully invariant.*

Example 6.7. The following are some examples of duo modules.

1. *Since elements of the endomorphism ring of R_R are precisely left multiplication by elements of R, the module R_R is a duo module if and only if R is a right duo ring.*

2. *For a ring R, a right R-module M is called a* ***multiplication module*** *if for every submodule N of M, there exists an ideal I of R such that $N = MI$. If f is an endomorphism of such a module, then*

$$f(N) = f(MI) = f(M)I \subseteq MI = N.$$

   Thus, multiplication modules are duo modules.

3. *By [12], Noetherian uniserial modules are duo.*

As a consequence of (3) above, we obtain the well known result that right Noetherian right chain rings are right duo rings. A simple observation is given by the following Lemma.

Lemma 6.8. *[31] A right R-module M is a duo module if and only if for every $m \in M$ and $f \in End_R(M)$, we have that $f(m) \in mR$.*

A ring is **strongly right bounded** if every non-zero right ideal contains a non-zero two-sided ideal. Right duo rings are obvious examples of strongly right bounded rings. The following extends this concept to right R-modules.

Definition 6.9. *A right R-module is* ***strongly right bounded*** *if every non-zero submodule of M contains a non-zero submodule that is fully invariant in M.*

Thus, R_R is strongly right bounded if and only if R is a strongly right bounded ring. In [13] it is shown that a strongly right bounded ring has the property that every non-zero right ideal is an essential extension of a two-sided ideal. The following shows that this result extends to strongly right bounded modules as well.

Lemma 6.10. *Let M be a strongly right bounded, right R-module. Then every non-zero submodule is an essential extension of a fully invariant submodule.*

Proof. Let $0 \neq N$ be a submodule of M and

$$T = \sum \{ L \leq M \mid L \text{ is fully invariant and } L \leq N \}.$$

Then T is fully invariant in M and $T \leq N$. Suppose K is a non-zero submodule of N and $T \cap K = 0$. Since M is strongly right bounded, there exists a non-zero fully invariant submodule K' contained in K. But then $0 \neq K' \subseteq T \cap K$, a contradiction. Thus N is an essential extension of T. $\square$

The following Lemma contains some well known properties of fully invariant submodules. The proof is included for completeness. We will write $N \trianglelefteq M$ to denote that N is fully invariant in M.

Lemma 6.11. *Let M be a right R-module. Then*

1) *The sum and intersection of any family of fully invariant submodules of M is again fully invariant.*

2) *If $X \leq Y$ are submodules of M such that $Y \trianglelefteq M$ and $X \trianglelefteq Y$, then $X \trianglelefteq M$.*

3) *If $M = \oplus_{i \in I} M_i$ and $N \trianglelefteq M$, then $N = \oplus_{i \in I}(M_i \cap N)$.*

4) *Suppose $N \lneq K$ are submodules of M. If N is a fully invariant submodule of M and K/N is fully invariant in M/N, then K is fully invariant in M.*

Proof. 1): This is clear.

2): Let $f \in End_R(M)$. Then $f(Y) \subseteq Y$ as Y is fully invariant. Thus, $f|_Y$ is an endomorphism of Y so that $f|_Y(X) \subseteq X$. As $X \subseteq Y$ we have $f(X) = f|_Y(X) \subseteq X$.

3): For each $j \in I$ let $p_j : M \longrightarrow M_j$ and $i_j : M_j \longrightarrow M$ be the canonical projections and injections respectively. Then $i_j p_j$ is an endomorphism of M. Thus, $i_j p_j(N) \subseteq N$ for every $j \in I$. Then $N \subseteq \oplus_j \, i_j p_j(N) \subseteq \oplus_j \, (M_j \cap N) \subseteq N$, whence $N = \oplus_j \, (M_j \cap N)$.

4): Let $f \in End_R(M)$. Define $f^* : M/N \to M/N$ by $f^*(m + N) = f(m) + N$. As N is a fully invariant submodule of M, f^* is a well-defined endomorphism of M/N. Then $f(K) + N = f^*(K/N) \subseteq K/N$ as K/N is fully invariant. Thus, $f(K) \subseteq K$ and K is fully invariant in M. $\square$

Clearly any homomorphic image of a right duo ring is right duo. However, there exist strongly right bounded rings that have homomorphic images that are not strongly right bounded [9]. In [9] it is also shown that a ring R is right duo if and only if every homomorphic image of R is strongly right bounded. The following result extends this to quasiprojective duo modules.

Theorem 6.12. *Let M be a quasiprojective right R-module. The following are equivalent:*

1) [31] M is a duo module.

2) Every homomorphic image of M is strongly right bounded.

Proof. 1) implies 2): Suppose M is quasi-projective and duo. As duo modules are obviously strongly right bounded, it is enough to show that every homomorphic image of M is duo. Suppose N and K are submodules of M such that $N \subsetneq K$, and $f \in End_R(M/N)$. Since M is quasi-projective, there exists $g \in End_R(M)$ such that $g(m) + N = f(m + N)$ for every $m \in M$. As M is a duo module, we have that $g(K) \subseteq K$. Then it follows that $f(K/N) = g(K) + N \subseteq K/N$. Hence, M/N is a duo module.

2) implies 1): Suppose there exists $0 \neq N \leq M$ such that N is not fully invariant. By hypothesis, M is strongly right bounded, and therefore N properly contains a non-zero fully invariant submodule K. Consider the following sum:

$$T = \sum \{ L \leq M \mid L \text{ is fully invariant and } L \lneq N \}.$$

Then T is a fully invariant submodule of M, and by assumption $T \lneq N$. Thus N/T is a non-zero submodule of M/T. Since M/T is strongly right bounded, there exists a non-zero fully invariant submodule $K/T \leq N/T$. Then $T \subsetneq K \subseteq N$. By Lemma 6.11, K is a fully invariant submodule of M, and by the choice of T we must have that $K = N$, a contradiction. $\square$

We obtain as a corollary the result of Birkenmeier and Tucci.

Corollary 6.13. *Let R be a ring. Then R is right duo if and only if every homomorphic image of R is strongly right bounded.*

6.3 Duo Modules over Domains

In [31] the following Theorem is established.

Theorem 6.14. *[31, Th. 3.3] Suppose R is an integral domain. The following are equivalent for a torsion-free uniform module U.*

i) U is duo.

ii) U contains a non-zero cyclic fully invariant submodule.

iii) $\mathcal{O}_l(U) = R$, where $\mathcal{O}_l(U) = \{q \in Q \mid qU \subseteq U\}$.

The following example shows that this Theorem cannot be extended to general rings.

Example 6.15. Let $R = M_2(\mathbb{Z})$ be the ring of 2×2 matrices with integer entries and $U = e_1 R$, where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then U is a torsion-free, uniform right R-module. A calculation shows that $\mathcal{O}_l(U) = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} \neq R$.

We now classify the right and left Ore domains for which this equivalence remains valid.

Theorem 6.16. *Let R be a right and left Ore domain. The following are equivalent:*

1) R is a right and left duo ring.

2) The equivalence of i), ii), and iii) in Theorem 6.14 is valid for all torsion-free uniform right R-modules U.

Proof. 1) implies 2): Suppose U is a torsion-free uniform right R-module. We will show that i), ii), and iii) above are equivalent.

i) implies ii): Trivial.

ii) implies iii): We will first show that U is a left R-submodule of Q. Let $r \in R$ and $u \in U$. Then $u = ab^{-1}$ for some $a, b \in R$. So $ru = rab^{-1} = ar_1b^{-1}$ since $Ra \subseteq aR$. As R is left duo, we also have that $bR \subseteq Rb$ so that $br_1 = r_2b$, or equivalently, $r_1b^{-1} = b^{-1}r_2$. Thus, $ru = rab^{-1} = ar_1b^{-1} = ab^{-1}r_2 = ur_2 \in U$. It follows that $R \subseteq \mathcal{O}_l(U)$.

Conversely, suppose $q \in \mathcal{O}_l(U)$, that is, $qU \subseteq U$. By hypothesis there exists $x \in U$ such that xR is non-zero and fully invariant. We claim that $xR \subseteq Rx$. To see this, write $x = ab^{-1}$ for $a, b \in R$ and suppose $r \in R$. Then

$$xr = ab^{-1}r = ar_1b^{-1} = r_2ab^{-1} = r_2x \in Rx.$$

As left multiplication by q is an R-endomorphism of U we have $qx = xr = sx$ for $r, s \in R$. As U is torsion-free, it follows that $q = s \in R$.

iii) implies i): Suppose N is a non-zero submodule of U and $\varphi \in End_R(U_R)$. Since U is uniform, φ is left multiplication by some $q \in Q$. Hence, $q \in \mathcal{O}_l(U) = R$, so φ is left multiplication by some $r \in R$. Then for every $x \in N$ we have $\varphi(x) = rx \in Rx = xR \subseteq N$ which implies that N is fully invariant.

2) implies 1): Since R is a right Ore domain, R_R is uniform. By hypothesis R_R is a duo module, i.e., R is a right duo ring. If $0 \neq c \in R$, then $cR \simeq R$ and thus cR is a right duo ring as well. Then $(cR)_R$ is a duo module so that $\mathcal{O}_l(cR) = R$. An easy calculation shows that $\mathcal{O}_l(cR) = cRc^{-1}$. Thus, $R = \mathcal{O}_l(cR) = cRc^{-1}$ for every non-zero $c \in R$. It follows that $cR = Rc$ for every $c \in R$, whence R is a right and left duo ring. □

We now consider finitely generated torsionfree duo modules. Our result extends [31, Thm 3.7]. We need a result of Levy in the proof.

Theorem 6.17. *[27, Theorem 6.1] Let R be a right semihereditary ring having a semisimple two-sided quotient ring. Then every finitely generated right R-module is the direct sum of its torsion submodule and a finite set of right ideals of R.*

Given this result we can establish the following.

Theorem 6.18. *Let R be a duo chain domain. The following are equivalent for a finitely generated torsionfree right R-module M.*

1) M is duo.

2) M contains a non-zero cyclic fully invariant submodule.

3) M is a uniform module and $\mathcal{O}_l(M) = R$.

Proof. 1) implies 2): trivial

2) implies 3): Note first that R is right semihereditary by Theorem 3.28. Further, R is right and left Goldie and thus has right and left quotient rings that coincide. Choose $0 \neq m \in M$ such that mR is fully invariant in M. Then there exists a submodule L of M such that $mR \oplus L \leq_e M$ so that $M/(mR \oplus L)$ is a finitely generated torsion right R-module. Thus there exists a regular element $a \in R$ such that $Ma \subseteq mR \oplus L$.

As M is torsionfree, it follows from Theorem 6.17 that M is a finitely generated projective right R-module. By [28, Lemma 5.2], we have that $MM^* = End(M)$. Define $f : mR \oplus L \longrightarrow R$ by $f(mr + x) = r$, and suppose $u \in L$. Then $f \in M^*$ so that $uf \in MM^* = End(M)$. It follows that $ua = uf(ma) \in mR \cap L = 0$ as mR is fully invariant. Thus, $ua = 0$ and M torsionfree imply that $u = 0$. Then $L = 0$ so that mR is an essential submodule of M. As $mR \simeq R$, a uniform module, it follows that M is uniform and $\mathcal{O}_l(M) = R$ by Theorem 6.16.

3) implies 1): Follows from Theorem 6.16. □

6.4 Maximal RD-submodules

This section is motivated by results obtained by Goldsmith and Zanardo [20] for torsionfree modules over valuation domains. We show that several results may be extended to a certain class of noncommutative rings. The proofs are motivated by their results.

Throughout the following section R is a semiprime right and left Goldie ring and M a right R-module.

We first note that the set of regular elements $\mathcal{C}_R(0)$ of R is right Ore; hence the set

$$t(M) = \{ a \in M \mid ax = 0 \text{ for some } x \in \mathcal{C}_R(0) \}$$

is a submodule of M. Recall that a right R-module is *torsion-free* if $t(M) = 0$ and *torsion* if $t(M) = M$. The following result is well known, but we could not locate a proof so one is provided.

Proposition 6.19. *Let R be a semiprime right Goldie ring and M a torsion-free right R-module. Then a submodule N of M is RD if and only if M/N is torsion-free.*

Proof. Suppose first that N is an RD-submodule of M and let $m + N \in t(M/N)$. Then there is a regular element $r \in R$ such that $mr \in N$. Since N is RD, $mr \in N \cap Mr = Nr$, so $mr = nr$ for some $n \in N$. Then $(m - n)r = 0$ so that $m - n \in t(M) = 0$. Thus, $m \in N$ and $t(M/N) = 0$.

Conversely, suppose $t(M/N) = 0$ and $mr \in N$ for some $m \in M$ and $r \in R$. Then $(m + N)r = 0$ in M/N. If $r \in \mathcal{C}_R(0)$, then $m + N \in t(M/N) = 0$ so that $m \in N$. In this case, it follows that N is an RD-submodule of M. If $r \notin \mathcal{C}_R(0)$, then rR is not an essential right ideal of R. So there exists a non-zero right ideal K of R such that $rR \cap K = 0$. Then the sum $K + rK + r^2K + \cdots$ is direct, contradicting $dim_R(R_R) < \infty$. □

Suppose M_R is torsion-free and N is a submodule of M. The set

$$\{ x \in M \mid xr \in N \text{ for some } r \in \mathcal{C}_R(0) \}$$

is the *RD-hull* of N, denoted N_*.

Lemma 6.20. *If M_R is torsion-free and N a submodule of M, then*

1) N_* is an RD-submodule of M.

2) $N \subseteq N_*$, and if K is an RD-submodule of M containing N, then $N_* \subseteq K$.

Proof. 1): We will show first that N_* is a submodule of M. Suppose x_1, $x_2 \in N_*$. Then there exist r_1, $r_2 \in \mathcal{C}_R(0)$ such that $x_i r_i = 0$ for $i = 1, 2$. Since $\mathcal{C}_R(0)$ is right Ore, there exists $y \in x_1 R \cap x_2 R \cap \mathcal{C}_R(0)$. Then $(x_1 - x_2)y \in N$ and $x_1 - x_2 \in N_*$. If $r \in R$, then $rz = r_1 s$ for some $z \in \mathcal{C}_R(0)$ and $s \in R$. So $x_1 rz = x_1 r_1 s \in N$, and z regular implies that $x_1 r \in N_*$.

Suppose that $m + N_* \in t(M/N_*)$. Then there exists a regular element $r \in R$ such that $mr \in N_*$. By the definition of N_*, this implies that there is a regular $s \in R$ such that $mrs \in N$. As R is semiprime right Goldie and r and s are regular, we have that rs is regular. Thus, $m \in N_*$ and it follows that $t(M/N_*) = 0$. By the preceding Proposition, N_* is an RD-submodule of M.

2): Clearly $N \subseteq N_*$. Suppose K is an RD-submodule of M containing N and let $x \in N_*$. Then $xr \in N \subseteq K$ for some regular r, so that $xr \in K \cap Mr = Kr$. But then there exists $k \in K$ such that $(x - k)r = 0$, and therefore $x - k \in t(M)$. As M is torsion-free, it follows that $x \in K$. Thus, $N_* \subseteq K$. □

Proposition 6.21. *Let R be semiprime right Goldie and M a torsionfree right R-module. Then a submodule N of M is a maximal RD-submodule if and only if $dim_R(M/N) = 1$.*

Proof. Suppose first that $dim(M/N) = 1$ so that $M/N \simeq J_R \subseteq Q^r_R$. Then J is a uniform right R-module so every proper non-zero submodule K of J is essential. Let $a \in J$. The right ideal $I = \{ r \in R \mid ar \in K \}$ is an essential right ideal of R. Since R is semiprime right Goldie, there exists a regular element $y \in I$. Then $ay \in K$ and it follows that J/K is a non-zero torsion module. Thus no proper non-zero submodule of J is RD. It follows that N is a maximal RD-submodule of M.

Conversely, suppose N is a maximal RD-submodule. Then M/N has no proper non-zero RD-submodules. If $dim_R(M/N) \neq 1$, then there exist non-zero submodules U_1/N and U_2/N such that $U_1/N \cap U_2/N = 0$. Choose $0 \neq \bar{a} \in U_1/N$. Then the RD-hull of the cyclic submodule generated by $\bar{a}$ is a proper non-zero RD-submodule of M/H, which is a contradiction.

$\square$

Proposition 6.22. *Every right R-module M contains a maximal RD-submodule.*

Proof. Let E be an injective hull of M. Since R is semiprime right Goldie, $E = \bigoplus_{i \in I} S_i$, where each S_i is a simple right Q^r-module. Note that each S_i has Goldie dimension 1. Fix $j \in I$ and let $E' = \bigoplus_{i \neq j} S_i$. If $N = M \cap E'$, then

$$M/N = M/(M \cap E') \simeq M + E'/E' \subseteq E/E' \simeq S_j.$$

Thus, $dim(M/N) = 1$ and N is a maximal RD-submodule of M by Proposition 6.21. $\square$

We recall the following definition from [20]:

Definition 6.23. *Let R be a ring. A cardinal τ is the **level of coherency** of R if τ is the smallest cardinal such that, for every short exact sequence*

$$0 \longrightarrow N \longrightarrow X \longrightarrow J \longrightarrow 0,$$

where J is a right ideal of R and X is finitely generated, we have $gen\, N \leq \tau$. Suppose R has a two-sided quotient ring Q. Then we define $\delta_R = sup\,\{\, gen\, J\,\}$ where J ranges over the R-submodules of Q.

The next three results establish several cardinality arguments that are needed in the proof of the main result.

Proposition 6.24. *Let R be a semiprime Goldie ring and τ the level of coherency of R. If N is an RD-submodule of a finitely generated torsionfree R-module X, then $gen\, N \leq \tau$.*

Proof. Assume, without loss of generality, that $N \neq X$. The proof is by induction on $dim_R(X)$. If X has Goldie dimension 1, then $dim_R(X/N) \leq 1$, so X/N is isomorphic to a finitely generated right ideal of R. As τ is the level of coherency of R and X is finitely generated, it follows that $gen\, N \leq \tau$.

Now assume $gen\, N > 1$. Choose a maximal RD-submodule H of X containing N. Then $dim_R(X/H) = 1$ and therefore $gen\, H = \kappa \leq \tau$. If κ is finite then H is finitely generated, N is RD in H, and $dim\, H < dim\, X$. Therefore by the induction hypothesis, $dim\, N \leq \tau$.

If κ is an infinite cardinal, choose a set of generators $\{\, x_\alpha \,\}_{\alpha<\kappa}$ of H indexed by κ. For every finite subset F of κ, let $N_F = N \cap (\sum_{\alpha\in F} x_\alpha R)$. Then N_F is relatively divisible in $\sum_{\alpha\in F} x_\alpha R$ and

$$dim(\sum_{\alpha\in F} x_\alpha R) \leq dim\, H < dim\, X.$$

By the induction hypothesis $gen\, N_F \leq \tau$. Then $N = \sum_F N_F$ implies

$$gen\, N \leq \sum_F gen\, N_F \leq \kappa\tau = \tau$$

and the proof is complete. □

Theorem 6.25. *Let R be a semiprime Goldie ring and τ the level of coherency of R. If N is an RD-submodule of the torsionfree right R-module M, then*

$$gen\, N \leq \tau(gen\, M).$$

Proof. Let $gen\, M = \kappa$. Choose a set of generators $\{\, x_\alpha \,\}_{\alpha<\kappa}$ of M indexed by κ. For every finite subset F of κ, let $N_F = N \cap (\sum_{\alpha\in F} x_\alpha R)$. Since N_F is an RD-submodule of $\sum_{\alpha\in F} x_\alpha R$, by Proposition 6.24 $gen\, N_F \leq \tau$. Then $N = \sum_F N_F$ and it follows that $gen\, N \leq \sum_F gen\, N_F \leq \tau\kappa$, as desired. □

Corollary 6.26. *Suppose R is a semiprime Goldie ring, M a torsionfree right R-module with $gen\,(M) \leq \delta_R$, and N is an RD-submodule of M. Then $gen\,(N) \leq \delta_R$ as well.*

Proof. Let τ be the level of coherency of R and consider an exact sequence

$$0 \longrightarrow N \longrightarrow X \longrightarrow J \longrightarrow O$$

where J is a right ideal of R and X is finitely generated. As $gen\,(M) \leq \delta_R$ by assumption, 6.25 implies that $gen\,(N) \leq \tau \cdot gen\,(M) \leq \tau \cdot \delta_R$. We complete the proof by showing that $\tau \leq \delta_R$. As τ is the smallest cardinal such that $gen\,(N) \leq \tau$ we need only show that $gen\,(N) \leq \delta_R$.

Since X is a finitely generated nonsingular module over R, [18, Proposition 2.12] implies that X can be embedded into a finite direct sum of copies of Q. As N is isomorphic to a submodule of X, we may assume without loss of generality that there is an embedding $0 \longrightarrow N \longrightarrow Q^n$ for some natural number n. The proof is by induction on n. If $n = 1$ then N is isomorphic to an R-submodule of Q and the result is clear. Suppose we have an embedding $0 \longrightarrow N \longrightarrow Q^{n+1}$. Set $N' = N \cap Q^n \subseteq Q^n$. The the induction hypothesis implies that $gen\,(N') \leq \delta_R$. We also have $N/N' = N/(Q^n \cap N) \simeq (N + Q^n)/Q^n \subseteq Q$. Hence $gen\,(N/N') \leq \delta_R$ and it follows that $gen\,(N) \leq \delta_R$ $\square$

Given these preliminary results, we can now establish the main theorem.

Theorem 6.27. *Let R be a semiprime Goldie ring with classical quotient ring Q such that $R \neq Q$. If M is a torsionfree right R-module such that*

1) Every maximal RD-submodule of M is a direct sum of rank 1 projective modules.

2) $gen\,M > \delta_R$.

Then M is a direct sum of rank 1 projective modules.

Proof. Let H be a maximal RD-submodule of M. Then $dim(M/H) = 1$ which implies that M/H is isomorphic to an R-submodule of Q. Thus $gen\,M/H \leq \delta_R$. Let $\pi : M \to M/H$ be the canonical epimorphism. For each generator x_i of M/H, choose $y_i \in M$ such that

$\pi(y_i) = x_i$ and let A be the submodule of M generated by the y_i. Clearly $gen\,A \leq \delta_R$ as well.

We claim that $M = A + H$. To see this, suppose $x \in M$. Then we have $x + H = (\sum_i y_i r_i) + H$ so that $x - (\sum_i y_i r_i) \subseteq H$. But $\sum_i y_i r_i \in A$ and thus $x \in A + H$. An obvious cardinality argument shows that $gen\,H > \delta_R$.

Now, $A/(A \cap H) \simeq (A + H)/H = M/H$ so that $dim(A/A \cap H) = 1$ and $A \cap H$ is an RD-submodule of A. Since $gen\,A \leq \delta_R$, 6.26 implies that $gen\,(A \cap H) \leq \delta_R$. Write $H = \bigoplus_{i \in I} P_i$ where each P_i is a rank 1 projective module. Decompose H as $H = H_0 \bigoplus H_1$, where H_0 and H_1 are rank 1 projective, $A \cap H \subseteq H_0$, and $gen\,H_1 > \delta_R$. Let $N = A + H_0$. Then $M = A + H = A + H_0 + H_1 = N + H_1$. We claim that the sum $N + H_1$ is direct. Suppose $a + h_0 = h_1$, where $a \in A$, $h_0 \in H_0$ and $h_1 \in H_1$. Then $a = h_1 - h_0 \in A \cap H \subseteq H_0$, say $h_1 - h_0 = h_0'$. Thus, $h_1 = h_0' + h_0 \in H_1 \cap H_0$ whence $h_1 = 0$.

So $M = N \bigoplus H_1$. Now decompose H_1 as $H_1 = H_2 \bigoplus P_{i_0}$, where P_{i_0} is rank 1 projective. Then $M = N \bigoplus H_1 = N \bigoplus H_2 \bigoplus P_{i_0}$, and $M/(N \bigoplus H_2) \simeq P_{i_0}$. Thus, $N \bigoplus H_2$ is a maximal RD-submodule of M and hence a direct sum of rank 1 projective modules. It follows that M is the direct sum of rank 1 projectives.

□

Bibliography

[1] Albrecht, Ulrich ; Dauns, John ; Fuchs, Laszlo. *Torsion-freeness and non-singularity over right p.p.-rings*; J. Algebra 285 (2005), no. 1, 98–119.

[2] Anderson, Frank W. ; Fuller, Kent R. *Rings and categories of modules*; Graduate Texts in Mathematics, Vol. 13. Springer-Verlag, New York-Heidelberg, 1974.

[3] Beachy, John A. *On quasi-Artinian rings* J. London Math. Soc. (2) 3 1971 449–452.

[4] Beachy, John A. ; Blair, William D. *Finitely annihilated modules and orders in Artinian rings*; Comm. Algebra 6 (1978), no. 1, 1–34.

[5] Bessenrodt, Christine, Brungs, H.H., and Törner, G; *Right Chain Rings, Pt 1.*; Schriftenreihe des Fachbereiches Mathematik, Universitt Duisburg 74 (1985).

[6] Bessenrodt, Christine, Brungs, H.H., and Törner, G; *Right Chain Rings, Pt 2a.*; Schriftenreihe des Fachbereiches Mathematik, Universitt Duisburg 74 (1986).

[7] Bessenrodt, Christine, Brungs, H.H., and Törner, G; *Right Chain Rings, Pt 2b.*; Schriftenreihe des Fachbereiches Mathematik, Universitt Duisburg 74 (1986).

[8] Birkenmeier, Gary F. ; Heatherly, Henry E. ; Lee, Enoch K.; *Completely prime ideals and associated radicals*; Ring theory (Granville, OH, 1992), 102–129, World Sci. Publ., River Edge, NJ, 1993.

[9] Birkenmeier, Gary F. ; Tucci, Ralph P; *Homomorphic images and the singular ideal of a strongly right bounded ring* Comm. Algebra 16 (1988), no. 6, 1099–1112.

[10] Brungs, H. H. ; Törner, G.; *Chain rings and prime ideals*; Arch. Math. (Basel) 27 (1976), no. 3, 253–260.

[11] Caldwell, William H.; *Hypercyclic rings*; Pacific J. Math. 24 1968 29–44.

[12] Clark, John ; Lomp, Christian ; Vanaja, Narayanaswami ; Wisbauer, Robert; *Lifting modules. Supplements and projectivity in module theory* Frontiers in Mathematics. Birkhuser Verlag, Basel, 2006.

[13] Faith, Carl ; Page, Stanley; *FPF ring theory. Faithful modules and generators of mod-RR* London Mathematical Society Lecture Note Series, 88. Cambridge University Press, Cambridge, 1984.

[14] Fuchs, L. ; Salce, L.; *Uniserial modules over valuation rings*; J. Algebra 85 (1983), no. 1, 14–31.

[15] Fuchs, L., and Salce, L.; *Modules over valuation domains*; Lecture Notes in Pure and Applied Mathematics, 97. Marcel Dekker, Inc., New York, 1985.

[16] Fuchs, L., and Salce, L.; *Modules over non-Noetherian domains*; Mathematical Surveys and Monographs, 84. American Mathematical Society, Providence, RI, 2001.

[17] Gabriel, Pierre; *Des catgories abliennes*; (French) Bull. Soc. Math. France 90 1962 323–448.

[18] Goodearl, K. R.; *Ring theory*; Pure and Applied Mathematics, No. 33. Marcel Dekker, Inc., New York-Basel, 1976.

[19] Goodearl, K. R., and Warfield, R. B., Jr; *An introduction to noncommutative Noetherian rings*; London Mathematical Society Student Texts, 16. Cambridge University Press, Cambridge, 1989.

[20] Goldsmith, B.; Zanardo, P.; *On maximal relatively divisible submodules*; to appear.

[21] Herrmann, Peter; *Self-projective modules over valuation rings.* Arch. Math. (Basel) 43 (1984), no. 4, 332–339.

[22] Hodges, Timothy J.; Osterburg, James; *A rank two indecomposable projective module over a Noetherian domain of Krull dimension one*; Bull. London Math. Soc. 19 (1987), no. 2, 139–144.

[23] Johnson, R. E. ; Wong, E. T. *Quasi-injective modules and irreducible rings*; J. London Math. Soc. 36 1961 260–268.

[24] Lam, T. Y.; *Exercises in classical ring theory*; Problem Books in Mathematics. Springer-Verlag, New York, 1995.

[25] Lam, T. Y.; *Lectures on modules and rings* ; Graduate Texts in Mathematics, 189. Springer-Verlag, New York, 1999.

[26] Lam, T. Y.; *A first course in noncommutative rings*; Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 2001.

[27] Levy, Lawrence; *Torsion-free and divisible modules over non-integral-domains*; Canad. J. Math. 15 1963 132–151.

[28] McConnell, J. C. ; Robson, J. C.; *Noncommutative Noetherian rings*; Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI, 2001.

[29] McCoy, Neal H.; *Prime ideals in general rings*; Amer. J. Math. 71, (1949). 823–833.

[30] Osofsky, B. L.; *Noncommutative rings whose cyclic modules have cyclic injective hulls*; Pacific J. Math. 25 1968 331–340.

[31] Özcan, A. . ; Harmanci, A. ; Smith, P. F.; *Duo modules* Glasg. Math. J. 48 (2006), no. 3, 533–545.

[32] Rotman, Joseph J.; *An introduction to homological algebra*; Pure and Applied Mathematics, 85. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.

[33] Salce, Luigi, and Zanardo, Paolo; *Finitely generated modules over valuation rings* Comm. Algebra 12 (1984), no. 15-16, 1795–1812.

[34] Shores, Thomas S. ; Lewis, William J.; *Serial modules and endomorphism rings*; Duke Math. J. 41 (1974), 889–909.

[35] Smith, Patrick F. ; Woodward, Alex R.; *On finitely annihilated modules and Artinian rings* Mediterr. J. Math. 3 (2006), no. 2, 301–311.

[36] Smith, P. F. ; Woodward, A. R.; *Artinian rings and the H-condition* Bull. London Math. Soc. 38 (2006), no. 4, 571–574.

[37] Stenstrom, Bo; *Rings of quotients*; Die Grundlehren der Mathematischen Wissenschaften, Band 217. Springer-Verlag, New York-Heidelberg, 1975.

[38] Thierrin, G.; *On duo rings*; Canad. Math. Bull. 3 1960 167–172.

[39] Warfield, R. B., Jr.; *Purity and algebraic compactness for modules*; Pacific J. Math. 28 1969 699–719.

[40] Warfield, R. B., Jr.; *Decomposability of finitely presented modules*; Proc. Amer. Math. Soc. 25 1970 167–172.

[41] Wisbauer, Robert; *Foundations of module and ring theory*; Algebra, Logic and Applications, 3. Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[42] Wu, L. E. T. ; Jans, J. P.; *On quasi projectives*; Illinois J. Math. 11 1967 439–448.