Simple Techniques for Detecting Decomposability or Indecomposability of Generalized Inverse Limits

by

Scott Varagona

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Approved by

Michel Smith, Chair, Professor of Mathematics
Gary Gruenhage, Professor of Mathematics
Stewart Baldwin, Professor of Mathematics

Abstract

A continuum X is said to be decomposable if it can be written as a union of two proper subcontinua; otherwise, X is said to be indecomposable. For years, topologists have used inverse limits with continuous bonding functions to study indecomposable continua. Now that the topic of generalized inverse limits with upper semi-continuous (or "u.s.c.") bonding functions has become popular, it is natural to consider how these new kinds of inverse limits might be used to generate indecomposable (or decomposable) continua.

In this work, we build upon our past results (from "Inverse Limits with Upper Semi-Continuous Bonding Functions and Indecomposability," [13]) to obtain new and more general theorems about how to generate indecomposable (or decomposable) continua from u.s.c. inverse limits. In particular, we seek sufficient conditions for indecomposability (or decomposability) that are easily checked, just from a straightforward observation of the bonding functions of the inverse limit. Our primary focus is the case of inverse limits whose factor spaces are indexed by the positive integers, but we consider extensions to other cases as well.

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Next, I must give a very special thanks to Dr. Tom Ingram, who has constantly supported my efforts and given me wonderful feedback on my work. He has trusted me to help proofread some of his publications; moreover, he has also cited my own work in those publications. Along with Tom Ingram, I must thank Dr. Bill Mahavier, who first brought the topic of u.s.c. inverse limits into prominence. Without the work of Ingram and Mahavier to build upon, this dissertation simply could not exist!

Finally, of course I must thank my family for their perennial love and support. My academic career at Auburn has been a long journey, and my family has helped me every step of the way. Also, just as importantly, I want to thank all the amazing friends I have made as a graduate student at Auburn. You have all enriched my life in so many ways: thank you so much!

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Chapter 1

Introduction

An inverse limit space is a valuable tool for topologists who wish to study indecomposable continua. Although a non-degenerate indecomposable continuum is a complicated topological space by its very nature, it is often possible to represent such a space in a very simple way—namely, as an inverse limit space with a single continuous bonding function. On the other hand, by drawing a relatively simple bonding function f that satisfies some special properties, we can guarantee that the inverse limit space with the single bonding function f is an indecomposable continuum. In this way, we may easily generate more indecomposable continua as examples for further study.

A number of topologists have done research on the relationship between inverse limits and indecomposable continua; see Chapter 1 of [4] for highlights from the history of this topic. However, for many years, only inverse limits with continuous bonding functions had been seriously considered in the literature. Now, after the work of Mahavier [8] and both Ingram and Mahavier [3], generalized inverse limits with set-valued, upper semi-continuous (u.s.c.) bonding functions have become popular. It is therefore a natural next step to consider how these new kinds of inverse limits might be used to generate indecomposable continua.

After we gave presentations [15] and [16] addressing the issue of u.s.c. inverse limit spaces and indecomposability, other mathematicians began publishing results on this topic as well. In [5] and [4], Ingram extended some of his earlier results on indecomposability in inverse limits with continuous bonding functions to the u.s.c. case. Also, in [17], Brian Williams gave necessary and sufficient conditions for an inverse limit to have the full projection property, a property that is vital for proving that some inverse limit spaces are indecomposable continua.

Still, plenty of work remains to be done: we need to find straightforward conditions (stated in terms of the bonding functions $f_i : [0,1] \to 2^{[0,1]}$) to guarantee that the inverse limit space is an indecomposable (or decomposable) continuum. The main goal of this work is to provide many such conditions; moreover, we strive to give conditions that are simple to check in practice. Ideally, by applying the theorems given here, one can tell from a quick glance at the bonding functions whether or not an inverse limit is indecomposable.

In Chapter 2, we give basic topology definitions and state theorems that may serve as background. In Chapter 3, we define inverse limits with upper semi-continuous bonding functions and cite important theorems that will be invoked repeatedly for the rest of this work. Then, in Chapters 4 and 5, we recall our own past results (from [13]) on the main problem before delving into new material. Chapter 6 features major generalizations of our previous indecomposability theorems; using these new theorems, we may show that many more u.s.c. functions with a structure akin to the $\sin(\frac{1}{x})$ curve give us indecomposable inverse limits. In Chapter 7, we discuss a generalization of Ingram's two-pass condition; we name this new condition the "ϵ-two-pass" condition, and we consider its impact on the study of indecomposable inverse limits. Next, we look into inverse limits whose factor spaces are indexed by sets other than the positive integers: Chapter 8 addresses inverse limits indexed by large initial segments of the ordinals (i.e., "long" inverse limits), and Chapter 9 addresses inverse limits indexed by $\mathbb{Z}$ (i.e., "two-sided" inverse limits). Both of these topics have implications for the study of indecomposability as well. Finally, although we give specific examples from time to time in the theory chapters, we set aside Chapter 10 solely for additional examples. We then close with Chapter 11, a discussion of possible topics for further research.

Chapter 2

General Topology and Classic Inverse Limits Results

2.1 Background Definitions

We begin with very basic topology definitions, most of which should be covered in an introductory topology course or may be found in an introductory text, such as [7] or [10]. For a detailed discussion of ordinal numbers, see, e.g., [6].

Let X be a set and let T be a collection of subsets of X with the following properties:

1. $X \in T$;
2. $\emptyset \in T$;
3. If $\{O_i\}_{i\in\mu}$ is a collection of members of T, then $\bigcup_{i\in\mu} O_i \in T$;
4. If $\{O_i\}_{i=1}^{n}$ is a finite collection of members of T, then $\bigcap_{i=1}^{n} O_i \in T$.

Then the pair (X, T) is called a *topological space* with *topology* T. Such a topological space will often be referred to simply as X when the associated topology T is understood. The members of T are called *open sets*.

A subset K of a topological space X is *closed* if $X - K$ is open.

A topological space X is *degenerate* if it consists of only one point. Otherwise, X is *non-degenerate*.

Suppose M is a subset of a topological space X. A point $p \in X$ is a *limit point* of M if every open set containing p contains a point in M different from p.

Suppose M is a subset of a topological space X. The set of all limit points of M is denoted by M'. The *closure of* M (denoted $\overline{M}$) is $M \cup M'$.

Suppose a collection B of open sets of a space X satisfies the following property:

If $x \in X$ and O is an open set containing x, then there exists a member b of B such that $x \in b$ and $b \subseteq O$.

Then B is a *basis* for the topology on X and a member b of B is called a *basic open set of* X.

Suppose B is a collection of subsets of a set X such that

1. If $x \in X$, there exists some $b \in B$ with $x \in b$.

2. If b_1 and b_2 are members of B with $x \in b_1 \cap b_2$, then there exists some set b_3 in B with $x \in b_3 \subseteq (b_1 \cap b_2)$.

Then the collection $T = \{\bigcup R | R \subseteq B\}$ is a topology for X, and B is a basis for this topology. It is said that the topology T is *generated* by the basis B.

A topological space X is called *Hausdorff* if for every pair of distinct points $p, q \in X$, there exist disjoint open sets O_p and O_q containing p and q respectively.

A space X is called *regular* if for every closed subset H of X and point $p \in X$ not in H, there exist disjoint open sets O_H and O_p containing H and p, respectively.

A space X is called *normal* if for every pair of disjoint closed sets H and K in X, there exist disjoint open sets O_H and O_K containing H and K, respectively.

If $f : X \to Y$ is a function from X to Y, and U is a subset of X, we define $f(U) = \{f(u)|\ u \in U\}$.

Let X and Y be topological spaces, let $f : X \to Y$ be a function from X to Y, and let $x \in X$. Then f is said to be *continuous at the point* x if, whenever V is an open set in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq V$. If f is continuous at each point $x \in X$, we say f is *continuous*. A continuous function may also be called a *mapping*.

A function $f : X \to Y$ is said to be *surjective* if for each $y \in Y$, there exists some $x \in X$ with $f(x) = y$.

A function $f : X \to Y$ is said to be *1-1* if for any pair of distinct points p, q in X, $f(p) \neq f(q)$.

If $f : X \to Y$ is a function and $y \in Y$, then the *preimage of y via f* (written as $f^{-1}(y)$) is $\{x \in X |\ f(x) = y\}$. If $A \subseteq Y$, then the *preimage of A via f* (written as $f^{-1}(A)$) is $\{x \in X |\ f(x) \in A\}$.

Suppose $f : X \to Y$ is a 1-1 surjective function. Then the function $f^{-1} : Y \to X$ given by $f^{-1}(y) = x$ (where x is the unique point in X with the property that $f(x) = y$) is called the *inverse of f*.

If X and Y are topological spaces and $f : X \to Y$ is 1-1, surjective, continuous, and has a continuous inverse, then f is called a *homeomorphism* and the spaces X and Y are said to be *homeomorphic*.

If $f : X \to X$ is a function, then we denote the composition $f \circ f$ by f^2. More generally, $f^n = f \circ f^{n-1}$ for $n \geq 2$.

If $f : X \to X$ is a function and $A \subseteq X$, then we denote $f^{-1}(f^{-1}(A))$ by $f^{-2}(A)$. More generally, $f^{-n}(A) = f^{-1}(f^{-n+1}(A))$.

Let X be a topological space and let $M \subseteq X$. A collection of sets $\{O_i\}_{i\in\mu}$ in X is said to be an *open cover of* M if each O_i is open in X and $M \subseteq \bigcup_{i\in\mu} O_i$.

If $\{O_i\}_{i\in\mu}$ is a cover of X, $\gamma \subseteq \mu$, and $\{O_i\}_{i\in\gamma}$ is also a cover of X, then $\{O_i\}_{i\in\gamma}$ is called a *subcover* of the original cover $\{O_i\}_{i\in\mu}$. A subcover consisting of only finitely many members is called a *finite subcover.*

A space X is *compact* if for every open cover $\{O_i\}_{i\in\mu}$ of X, there exists a finite subcover of X. (I.e., $\{O_{i_j}\}_{j=1}^{n}$ for some positive integer n.)

A collection of subsets $\{G_i\}_{i\in\mu}$ of a space X is called a *monotonic collection* if for each pair of members G_j, G_k in the collection, either $G_j \subseteq G_k$ or $G_k \subseteq G_j$.

For each i in some arbitrary index set μ, let X_i be a topological space. Define $X = \prod_{i\in\mu} X_i$ to be the set $\{(x_i)_{i\in\mu} |\ x_i \in X_i$ for each $i\}$. Define a topology on X as follows: a basic open set containing $(x_i)_{i\in\mu}$ is given by $\prod_{i\in\mu} O_i$, where O_i is open in X_i for each i, $x_i \in O_i$ for each i, and $O_i = X_i$ for all but finitely many i.

Then X together with the topology generated by this basis may be called a *product space* (on the index set μ).

In the case of a product space on the countably infinite index set consisting of the positive integers ($\mathbb{N}$), we denote the product space by $\prod_{i=1}^{\infty} X_i$. Thus,

$\prod_{i=1}^{\infty} X_i = \{(x_1, x_2, x_3, \ldots) \mid x_i \in X_i$ for each positive integer $i\}$.

Let $X = \prod_{i \in \mu} X_i$ be a product space (with index set μ either finite or infinite). Let A be a subset of μ. Then the function $\pi_A : X \to \prod_{i \in A} X_i$ defined by $\pi_A((x_i)_{i \in \mu}) = (x_i)_{i \in A}$ is called the *projection map onto the set* A. In the special case where $A = \{j\}$ for some $j \in \mu$, we denote $\pi_{\{j\}}$ simply by π_j and we call this function the *projection map onto the* j *coordinate.*

If X, Y are topological spaces, $f : X \to Y$ is a function, and $A \subseteq X$, then f *restricted to* A (denoted by $f|_A$) is the function given by $f|_A : A \to Y$, where $f|_A(a) = f(a)$ whenever $a \in A$.

A function $f : X \to Y$ is said to be *open* if for each open subset U of X, $f(U)$ is an open subset of Y.

Suppose X is a topological space with topology T and $S \subseteq X$. Then the set S together with the topology $\hat{T} = \{S \cap O |\ O \in T\}$ is called a *subspace* of X, where $\hat{T}$ is the *subspace topology.*

Let μ be an ordinal. Then the set $\{\alpha \mid \alpha \text{ is an ordinal and } \alpha < \mu\}$ is called an *initial segment of the ordinals.* (Similarly, if $\beta < \mu$, then $\{\alpha \mid \alpha \text{ is an ordinal and } \alpha < \beta\}$ is an initial segment of μ.)

Suppose X is a topological space and $d : X \times X \to \mathbb{R}$ is a function satisfying the following properties (for all $x, y, z \in X$):

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Then the function d is said to be a *metric* on X. For a given $p \in X$ and $\epsilon > 0$, let $B(p, \epsilon) = \{x \in X \mid d(x, p) < \epsilon\}$. If the collection $\{B(p, \epsilon) \mid p \in X,\ \epsilon > 0\}$ is a basis for the space X, then X is said to be a *metric space*.

Let X be a topological space. Two subsets H and K of X are called *mutually separated* if neither set contains a point or a limit point of the other.

If X is a topological space and $M \subseteq X$, then M is *connected* if M is not the union of two mutually separated non-empty subsets of X.

A topological space X is a *continuum* if X is non-empty, compact, and connected.

A continuum that is Hausdorff (but not necessarily metric) is called a *Hausdorff continuum*.

A continuum that is metric is called a *metric continuum*.

If X is a continuum and A, a subset of X, is also a continuum, then A is called a *subcontinuum* of X. If the subcontinuum A is a proper subset of X, then A is a *proper subcontinuum* of X.

A continuum X is *decomposable* if it is the union of two proper subcontinua.

If X is a continuum but X is not decomposable, then X is said to be *indecomposable*.

If X is a continuum and $p, q \in X$, then X is said to be *irreducible between* p *and* q if no proper subcontinuum of X contains both p and q.

Let X be a connected set. If $X - \{p\}$ is not connected, then p is a *cut point* of X.

A continuum with exactly 2 non-cut points is called an *arc.*

A *triod* is a union of three arcs whose intersection is exactly one point.

A *fan* is a union of infinitely many arcs, all of which have exactly one point in common.

Let X be a topological space. Suppose that A, a subset of X, is an arc with the property that whenever $O \subseteq X$ is an open set with $O \cap A \neq \emptyset$, there exists some point $p \in O$ with $p \notin A$. Then A is called a *limit arc.*

A subset A of a topological space X is said to be *nowhere dense in* X if every non-empty open subset of X contains a non-empty open set that misses A.

A subset A of a topological space X is said to be *dense in* X if every non-empty open subset of X contains a point of A.

2.2 Background Theorems

Most of the following basic theorems may be found in a standard topology text. The proofs of these theorems are omitted, but most of the proofs may be found in one or more of [7], [9], and [10]. A more general statement and proof of Theorem 2.26 may be found in [4] (Lemma 221).

Theorem 2.1. *A subset M of a topological space X is closed (in X) iff M contains all of its limit points in X.*

Theorem 2.2. *A subset M of a topological space X is closed (in X) iff $M = \overline{M}$.*

Theorem 2.3. *If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

Theorem 2.4. *A closed subset of a compact space is compact.*

Theorem 2.5. *The continuous image of a compact set is compact.*

Theorem 2.6. *(Tychonoff) If $\{X_i\}_{i\in\mu}$ is a collection of compact topological spaces, then the product space $\prod_{i\in\mu} X_i$ is compact.*

Theorem 2.7. *If $\{X_i\}_{i\in\mu}$ is a collection of continua, then the product space $\prod_{i\in\mu} X_i$ is a continuum.*

Theorem 2.8. *Let B be a basis for a topological space X. Then every open set of X is a union of members of B.*

Theorem 2.9. *The following are equivalent for a function $f : X \to Y$ from topological space X to topological space Y:*

i. f is a continuous function.

ii. If O is a (basic) open set in Y, then $f^{-1}(O)$ is open in X.

Theorem 2.10. *If $\prod_{i\in\mu} X_i$ is a product space and $A \subseteq \mu$, then the projection map π_A is continuous.*

Theorem 2.11. *If X, Y are topological spaces, $f : X \to Y$ is continuous, and $A \subseteq X$, then $f|_A$ is also continuous.*

Theorem 2.12. *If X is a compact Hausdorff space, then X is regular.*

Theorem 2.13. *If X is a compact Hausdorff space, then X is normal.*

Theorem 2.14. *If X is regular, then X is Hausdorff.*

Theorem 2.15. *If X is normal, then X is regular.*

Theorem 2.16. *The unit interval $[0, 1]$ is an arc.*

Theorem 2.17. *Suppose M is a subset of a topological space X. If M is closed and not connected, then M is the union of two disjoint closed sets H and K.*

Theorem 2.18. *The continuous image of a connected set is connected.*

Theorem 2.19. *The continuous image of a continuum is a continuum.*

Theorem 2.20. *The common part of a monotonic collection of continua is a continuum.*

Theorem 2.21. *If X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is a 1-1, surjective, continuous function, then f^{-1} is continuous (and hence, f is a homeomorphism).*

Theorem 2.22. *A topological space X is connected iff for any two distinct points $p, q \in X$, there exists a connected subset of X containing p and q.*

Theorem 2.23. *If $\{K_i\}_{i=1}^{\infty}$ is a collection of connected subsets of a space X such that $K_i \cap K_{i+1} \neq \emptyset$ for all $i \geq 1$, then $K = \bigcup_{i=1}^{\infty} K_i$ is connected.*

Theorem 2.24. *If $K \subseteq X$ is connected, then $\overline{K}$ is connected.*

Theorem 2.25. *If A is a dense subset of the topological space X, then $\overline{A} = X$.*

Theorem 2.26. *Suppose T is an arc and T is the union of two proper subcontinua H and K. If U and V are mutually exclusive connected open subsets of T, then one of U and V is a subset of one of H and K.*

Theorem 2.27. *A Hausdorff continuum X is indecomposable iff every proper subcontinuum of X is nowhere dense in X.*

2.3 Classic Inverse Limits Definitions and Theorems

As we stated in Chapter 1, the main focus of this work is inverse limits with upper semi-continuous bonding functions. However, for the sake of introduction, here we present the definition of traditional inverse limits with continuous bonding maps first. We also give basic theorems about these inverse limits to contrast with the theorems about the upper semi-continuous inverse limits given in Chapter 3. Proofs of the following theorems may be found in one or both of [4], [14].

Suppose that, for each positive integer i, X_i is a topological space and f_i is a continuous function from X_{i+1} to X_i. Let $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be the set $\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X_i \mid x_i = f_i(x_{i+1})$ for all positive integers $i\}$. Then we say $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an *inverse limit space* and a basis for the topology on $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is $\{O \cap \varprojlim\{X_i, f_i\}_{i=1}^{\infty} \mid O$ is basic open in $\prod_{i=1}^{\infty} X_i\}$. The X_i's are called the *factor spaces* of $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, and the f_i's are continuous *bonding maps*. If $f_i, f_{i+1}, \ldots, f_{j-1}$ are bonding maps, let us denote $f_i \circ f_{i+1} \circ \ldots \circ f_{j-1}$ by $f_{i,j}$.

Theorem 2.28. *Suppose $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse limit space with continuous bonding maps, $\{n_i\}_{i=1}^{\infty}$ is an increasing sequence of positive numbers, $g_i = f_{n_i, n_{i+1}}$ for each i, and $Y = \varprojlim\{X_{n_i}, g_i\}_{i=1}^{\infty}$. Then X is homeomorphic to Y.*

Theorem 2.29. *Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse limit space with continuous bonding maps. If there is a natural number N so that f_n is an onto homeomorphism for each $n \geq N$, then X is homeomorphic to X_N.*

Theorem 2.30. *Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse limit space with continuous bonding maps and suppose X_i is non-empty and compact for each i. Then X is non-empty and compact.*

Theorem 2.31. *Let $X = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse limit space with continuous bonding maps and suppose X_i is a continuum for each i. Then X is a continuum.*

We note in advance that most of these theorems are false if upper semi-continuous bonding functions are used instead of continuous bonding maps. Most importantly, as was pointed out by Ingram and Mahavier in [3], some kind of extra hypothesis must be added to Theorem 2.31 for that theorem to hold true in the upper semi-continuous case. (Contrast Theorem 2.31 with, e.g., Theorem 3.3 or Theorem 3.4.)

Chapter 3

Background on Inverse Limits with u.s.c. Bonding Functions

Suppose X and Y are compact Hausdorff spaces, and define 2^Y to be the set of all non-empty compact subsets of Y. A function $f : X \to 2^Y$ is called *upper semi-continuous* (u.s.c.) if for any $x \in X$ and open V in Y containing $f(x)$, there exists an open U in X containing x so that $f(u) \subseteq V$ for all $u \in U$. If $f : X \to 2^Y$ is u.s.c. and $f(x)$ is connected for each $x \in X$, then f is a *u.s.c. continuum-valued* function; in this case, for emphasis, we will sometimes write $f : X \to C(Y)$ instead, where $C(Y)$ is the set of all subcontinua of Y. If $f : X \to 2^Y$ is u.s.c. and $f(x) = \{y\}$ for some $x \in X$ and $y \in Y$, then although f is a set-valued function, we use the convention of writing simply $f(x) = y$. Therefore, in the case where $f : X \to 2^Y$ is u.s.c. but $f(x)$ is degenerate for all $x \in X$, we may regard f as the corresponding continuous function $f : X \to Y$.

Again, let X, Y be compact Hausdorff spaces and let $f : X \to 2^Y$ be a u.s.c. function. If $y \in Y$, then the *preimage of* y *via* f is $f^{-1}(y) = \{x \in X |\ y \in f(x)\}$. More generally, if $A \subseteq Y$, then the *preimage of* A *via* f is $f^{-1}(A) = \{x \in X |\ f(x) \cap A \neq \emptyset\}$. We say f is *surjective* if for each $y \in Y$, $f^{-1}(y)$ is non-empty. Assuming that $f : X \to 2^Y$ is a surjective u.s.c. function, the *inverse of* f, i.e., the set-valued function $f^{-1} : Y \to 2^X$, is given by $f^{-1}(y) = \{x \in X |\ y \in f(x)\}$. It will later become evident that if f is a u.s.c. surjective function, then its inverse, f^{-1}, is also a u.s.c. surjective function.

If X, Y, and Z are compact Hausdorff spaces and $f : X \to 2^Y$ and $g : Y \to 2^Z$ are u.s.c. functions, then $g \circ f : X \to 2^Z$ is the u.s.c. function given by $(g \circ f)(x) = \{z \in Z\ |\exists y \in Y$ such that $y \in f(x)$ and $z \in g(y)\}$. In the special case of a u.s.c. function $f : X \to 2^X$, we denote $f \circ f$ by f^2; moreover, for any integer $n \geq 2$, let us say $f^n = f \circ f^{n-1}$. It will also sometimes be helpful to use the following convention: whenever $f : X \to 2^X$ is a u.s.c.

function and $A \subseteq X$, let us denote the preimage of the preimage of A, i.e., $f^{-1}(f^{-1}(A))$, by $f^{-2}(A)$. More generally, for each integer $n \geq 2$, $f^{-n}(A) = f^{-1}(f^{-(n-1)}(A))$.

Given compact Hausdorff spaces X, Y and a u.s.c. function $f : X \to 2^Y$, the *graph of* f, abbreviated $G(f)$, is the set $\{(x, y) \in X \times Y \mid y \in f(x)\}$. The *inverse* of the graph is $G(f)^{-1} = \{(y, x) | (x, y) \in G(f)\}$. If $X_1, X_2, \ldots, X_n, X_{n+1}$ are compact Hausdorff spaces and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. for $1 \leq i \leq n$, then $G(f_1, f_2, \ldots, f_n) = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$.

Now suppose that, for each positive integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function. We define $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ to be the set $\{(x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X_i \mid x_i \in f_i(x_{i+1}) \text{ for all positive integers } i\}$. (For convenience, we will denote a sequence $(x_1, x_2, x_3, \ldots)$ by the boldface $\mathbf{x}$ and denote the sequence of functions $(f_1, f_2, f_3, \ldots)$ by the boldface $\mathbf{f}$. Thus, we may abbreviate $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ by $\varprojlim \mathbf{f}$.) Then we say $\varprojlim \mathbf{f}$ is an *inverse limit space with u.s.c. bonding functions*, and a basis for the topology on $\varprojlim \mathbf{f}$ is $\{O \cap \varprojlim \mathbf{f} \mid O \text{ is basic open in } \prod_{i=1}^{\infty} X_i\}$. For brevity's sake, we will sometimes call an inverse limit space with u.s.c. bonding functions simply a *u.s.c. inverse limit space*. Finally, in the special case where X is a compact Hausdorff space, $f : X \to 2^X$ is u.s.c., and $\mathbf{f} = (f, f, f, \ldots)$, we say $\varprojlim \mathbf{f}$ is the *inverse limit with the single bonding function* f. (If, in the description of a particular inverse limit, only the single bonding function $f : X \to 2^X$ is given, then it will be clear from context that $\varprojlim \mathbf{f}$ is the inverse limit with the single bonding function f.)

As stated in Chapter 2, a *continuum* is a non-empty compact connected space; a continuum that is Hausdorff (but not necessarily metric) will be called a *Hausdorff continuum*. We will usually assume that each factor space X_i is a non-degenerate Hausdorff continuum.

In [3], Ingram and Mahavier prove various theorems about inverse limits with u.s.c. bonding functions. There are four especially critical theorems (originally labeled 2.1, 3.2, 4.7, and 4.8 in [3]) that we will need as background in the next chapters, and so, we restate them here:

Theorem 3.1. *Suppose each of X and Y is a compact Hausdorff space and M is a subset of $X \times Y$ such that if x is in X then there is a point y in Y such that (x, y) is in M. Then M is closed if and only if there is an upper semi-continuous function $f : X \to 2^Y$ such that $M = G(f)$.*

Theorem 3.2. *Suppose that for each positive integer i, X_i is a non-empty compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous bonding function. Then $\varprojlim \boldsymbol{f}$ is non-empty and compact.*

Theorem 3.3. *Suppose that for each positive integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and for each x in X_{i+1}, $f_i(x)$ is connected. Then $\varprojlim \boldsymbol{f}$ is a Hausdorff continuum.*

Theorem 3.4. *Suppose that for each positive integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and for each $x \in X_i$, $f_i^{-1}(x)$ is a non-empty, connected set. Then $\varprojlim \boldsymbol{f}$ is a Hausdorff continuum.*

Chapter 4

Sufficient Conditions For Decomposability of u.s.c. Inverse Limits

Let us recall once more that a Hausdorff continuum X is *decomposable* if it is the union of two proper subcontinua; if a Hausdorff continuum X is not decomposable, X is said to be *indecomposable*. Now, suppose for each positive integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is a u.s.c. bonding function, and $\varprojlim \mathbf{f}$ is the resulting inverse limit. Our first major goal is to provide some simple means for recognizing when such a u.s.c. inverse limit is a decomposable continuum. It would be especially convenient if we could infer decomposability just from some easily-checked feature of some bonding function's graph, $G(f_i)$. In this chapter, we recall our previous results on this topic; these results and their proofs may also be found in [13], but we include them here for the sake of completeness.

Theorem 4.1. *Suppose that for each positive integer i, X_i is a non-degenerate Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is a surjective u.s.c. function, and $f_i(x)$ is connected for each $x \in X_{i+1}$. Suppose further that, for some positive integer j, there is an open set $U \subseteq X_{j+1} \times X_j$ intersecting $G(f_j)$ so that $G(f_j) \setminus U$ is the graph of a u.s.c. function $h : X_{j+1} \to 2^{X_j}$ satisfying the following conditions:*

1) $h(x)$ is connected for all $x \in X_{j+1}$.

2) There is an open $V \subseteq X_{j+1} \times X_j$ so that $U \cap V = \emptyset$ and $G(h) \cap V \neq \emptyset$.

Then $\varprojlim \boldsymbol{f}$ is a decomposable continuum.

Proof. By Theorem 3.3, $\varprojlim \mathbf{f}$ is a continuum. To show $\varprojlim \mathbf{f}$ is decomposable, by Theorem 2.27, it suffices to find a proper subcontinuum that is not nowhere dense. Fix some positive integer j with $f_j : X_{j+1} \to 2^{X_j}$ satisfying the hypothesis, so that $G(f_j) \setminus U$ is the graph of some u.s.c. function h satisfying conditions 1 and 2. Define $\widetilde{\mathbf{f}} = (f_1, f_2, \ldots, f_{j-1}, h, f_{j+1}, \ldots)$. Then by Theorem 3.3, $\varprojlim \widetilde{\mathbf{f}}$ is a continuum. Moreover, because there is some $(p_{j+1}, p_j) \in G(f_j) \cap U$ and each f_i is surjective, there is some point $\mathbf{p} = (p_1, p_2, \ldots, p_j, p_{j+1}, \ldots)$ with $\mathbf{p} \in \varprojlim \mathbf{f} \backslash \varprojlim \widetilde{\mathbf{f}}$. So $\varprojlim \widetilde{\mathbf{f}}$ is a proper subcontinuum of $\varprojlim \mathbf{f}$.

Since there is a basic open subset $O_{j+1} \times O_j$ of V that also intersects $G(h)$, $(X_1 \times X_2 \times \ldots \times X_{j-1} \times O_j \times O_{j+1} \times X_{j+2} \times \ldots) \cap \varprojlim \mathbf{f}$ is a non-empty open subset of $\varprojlim \widetilde{\mathbf{f}}$. Thus, $\varprojlim \widetilde{\mathbf{f}}$ is not nowhere dense. $\square$

As we remarked in [13], each u.s.c. function f_i must be surjective for this theorem to succeed. There is a counterexample otherwise: Suppose that, for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is given by the graph consisting of the straight line segments from $(0,0)$ to $(1,0)$ and from $(1,0)$ to $(1, \frac{1}{2})$. (See Figure 4.1.) Then $\varprojlim \mathbf{f}$ is a single point, $(0,0,0,\ldots)$, and is therefore an indecomposable continuum. To avoid trivial counterexamples such as this, we will repeatedly assume that the bonding functions are surjective and the factor spaces are non-degenerate. See Chapter 10 for examples of bonding functions for which Theorem 4.1 actually applies. In particular, Example 10.1 shows how the conditions in Theorem 4.1 are often easy to check in practice.

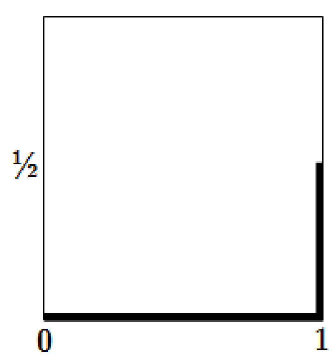

Figure 4.1: A bonding function that gives rise to a degenerate inverse limit space

Theorem 4.2. *Suppose that for each positive integer i, X_i is a non-degenerate Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is a surjective u.s.c. function, and for each $x \in X_i$, $f_i^{-1}(x)$ is connected. Suppose further that, for some positive integer j, there is an open set $U \subseteq$*

$X_{j+1} \times X_j$ *intersecting* $G(f_j)$ *so that* $G(f_j) \setminus U$ *is the graph of a u.s.c. function* $h : X_{j+1} \to 2^{X_j}$ *satisfying the following conditions:*

1) For all $x \in X_j$, $h^{-1}(x)$ *is a non-empty, connected set.*

2) There is an open $V \subseteq X_{j+1} \times X_j$ *so that* $U \cap V = \emptyset$ *and* $G(h) \cap V \neq \emptyset$.

Then $\varprojlim \boldsymbol{f}$ *is a decomposable continuum.*

Proof. By Theorem 3.4, $\varprojlim \mathbf{f}$ is a continuum. Let $\widetilde{\mathbf{f}} = (f_1, f_2, \ldots, f_{j-1}, h, f_{j+1}, \ldots)$. Again, by Theorem 3.4, $\varprojlim \widetilde{\mathbf{f}}$ is a continuum; the same argument as in Theorem 4.1 shows that $\varprojlim \widetilde{\mathbf{f}}$ is a proper subcontinuum that is not nowhere dense. □

Theorem 4.3. *Suppose that for each positive integer* i, X_i *is a non-degenerate Hausdorff continuum and* $f_i : X_{i+1} \to 2^{X_i}$ *is a surjective u.s.c. function with* $f_i(x)$ *connected for each* $x \in X_{i+1}$. *Suppose there is some point* $\boldsymbol{x} = (x_1, x_2, x_3, \ldots) \in \varprojlim \boldsymbol{f}$ *so that* $f_i^{-1}(x_i) = x_{i+1}$ *for* $i \geq 2$, *and there is an open* $U \subseteq X_2 \times X_1$ *so that* $G(f_1) \cap U$ *is a non-empty subset of* $\{x_2\} \times f_1(x_2)$. *Then* $\varprojlim \boldsymbol{f}$ *is a decomposable continuum.*

Proof. Theorem 3.3 implies that $\varprojlim \mathbf{f}$ is a continuum. Let $O_2 \times O_1$ be a basic open subset of U with $G(f_1) \cap (O_2 \times O_1)$ a non-empty subset of $\{x_2\} \times f_1(x_2)$. We note that the proper subcontinuum $f_1(x_2) \times \{x_2\} \times \{x_3\} \times \ldots$ of $\varprojlim \mathbf{f}$ contains the open set $(O_1 \times O_2 \times X_3 \times X_4 \times \ldots) \cap \varprojlim \mathbf{f}$. Thus, there exists a proper subcontinuum that is not nowhere dense. □

So far, we have seen decomposability arise in certain situations where $\varprojlim \mathbf{f}$ satisfies either Theorem 3.3 or Theorem 3.4. Now we turn to a situation where 3.3 or 3.4 no longer apply, i.e., a situation where images (or preimages) of points need not be connected. In general, if there exists some positive integer i and $x \in X_{i+1}$ with $f_i(x)$ *not* connected, then $\varprojlim \mathbf{f}$ need not be connected. However, in some special cases, $\varprojlim \mathbf{f}$ turns out to be a continuum even if $f_i(x)$ is not connected for some $x \in X_{i+1}$. One such special case is given by the following theorem, a consequence of one of Ingram's results (Theorem 3.3) in [5]: If $f : [0,1] \to 2^{[0,1]}$ is a u.s.c. bonding function that is the union of two distinct continuous functions $g, h : [0,1] \to [0,1]$, at least one of which is surjective, then $\varprojlim \mathbf{f}$ is a continuum.

Our next theorem generalizes this result and also provides another sufficient condition for decomposability.

Theorem 4.4. *Suppose for each positive integer i, X_i is a non-degenerate Hausdorff continuum and $f_i : X_{i+1} \to 2^{X_i}$ is a u.s.c. function that is the union of two u.s.c. functions $g_i, h_i : X_{i+1} \to 2^{X_i}$ satisfying the following properties:*

1) At least one of g_i, h_i is surjective.

2) $g_i(x)$ and $h_i(x)$ are connected for all $x \in X_{i+1}$.

3) $G(g_i) \cap G(h_i) \neq \emptyset$.

Then $\varprojlim \boldsymbol{f}$ is a continuum. Moreover, if there is some positive integer j so that $G(g_j) \nsubseteq G(h_j)$ and $G(h_j) \nsubseteq G(g_j)$, then $\varprojlim \boldsymbol{f}$ is a decomposable continuum.

Proof. We first show that $\varprojlim \mathbf{f}$ is a continuum. Since for each positive integer i, f_i is u.s.c. and each factor space is a continuum, it follows from Theorem 3.2 that $\varprojlim \mathbf{f}$ is compact. To show that $\varprojlim \mathbf{f}$ is connected, we will show that, for any $\mathbf{p}, \mathbf{q} \in \varprojlim \mathbf{f}$, there exists a connected subset of $\varprojlim \mathbf{f}$ that contains $\mathbf{p}$ and $\mathbf{q}$.

Let $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ and $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ be in $\varprojlim \mathbf{f}$. Then for each positive integer i, $(p_{i+1}, p_i) \in G(\alpha_i)$, where $\alpha_i \in \{g_i, h_i\}$. Similarly, for each i, $(q_{i+1}, q_i) \in G(\beta_i)$, where $\beta_i \in \{g_i, h_i\}$. Now define $\mathbf{z}_1 = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, $\mathbf{z}_2 = (\beta_1, \alpha_2, \alpha_3, \ldots)$, $\mathbf{z}_3 = (\beta_1, \beta_2, \alpha_3, \ldots)$, $\ldots$, $\mathbf{z}_i = (\beta_1, \beta_2, \ldots, \beta_{i-1}, \alpha_i, \ldots)$, etc. By property 3, $G(g_i)$ and $G(h_i)$ must intersect at some point (x_{i+1}, x_i). So, since each f_i is surjective, there exists some point $(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots)$ in $\varprojlim \mathbf{f}$ that lies in both $\varprojlim \mathbf{z}_i$ and $\varprojlim \mathbf{z}_{i+1}$. That means, for all i, $\varprojlim \mathbf{z}_i \cap \varprojlim \mathbf{z}_{i+1} \neq \emptyset$.

By Theorem 3.3, $\varprojlim \mathbf{z}_i$ is connected for each positive integer i. Since $(\varprojlim \mathbf{z}_i) \cap (\varprojlim \mathbf{z}_{i+1}) \neq \emptyset$ for each i, it follows that $K = \bigcup_{i=1}^{\infty} \varprojlim \mathbf{z}_i$ is connected. Thus, the closure of K is also connected. We already know $\mathbf{p} \in \varprojlim \mathbf{z}_1 \subseteq K$. To show that $\mathbf{q} \in \overline{K}$, we observe that, since $\varprojlim \mathbf{z}_i \subseteq K$ for each i, points of the form $(q_1, \ldots), (q_1, q_2, \ldots), \ldots, (q_1, q_2, \ldots, q_i, \ldots)$, etc., are all in K. Since $\mathbf{q}$ is a limit point of the set of these points, $\mathbf{q} \in \overline{K}$. Thus, both $\mathbf{p}$ and $\mathbf{q}$ lie in the connected set $\overline{K}$. It follows that $\varprojlim \mathbf{f}$ is connected, and hence, is a continuum.

Now, if $G(g_j) \not\subseteq G(h_j)$ and $G(h_j) \not\subseteq G(g_j)$ for some positive integer j, $\varprojlim \mathbf{f}$ can be decomposed into two proper subcontinua as follows. Let $\widetilde{\mathbf{g}} = (f_1, f_2, \ldots, f_{j-1},\ g_j, f_{j+1}, \ldots)$ and $\widetilde{\mathbf{h}} = (f_1, f_2, \ldots, f_{j-1}, h_j, f_{j+1}, \ldots)$. Then (by an argument similar to the one given above) $\varprojlim \widetilde{\mathbf{g}}$ and $\varprojlim \widetilde{\mathbf{h}}$ are both proper subcontinua of $\varprojlim \mathbf{f}$, and $(\varprojlim \widetilde{\mathbf{g}}) \cup (\varprojlim \widetilde{\mathbf{h}}) = \varprojlim \mathbf{f}$. $\square$

Corollary 4.5. *Suppose that, for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is a u.s.c. function that is the union of two distinct continuous functions $g_i, h_i : [0,1] \to [0,1]$, at least one of which is surjective. Then $\varprojlim \boldsymbol{f}$ is a decomposable continuum.*

Chapter 5

Sufficient Conditions For Indecomposability of u.s.c. Inverse Limits

Our next major goal is to give straightforward conditions on u.s.c. bonding functions $f_i : X_{i+1} \to 2^{X_i}$ that guarantee that the inverse limit space $\varprojlim \mathbf{f}$ is a non-degenerate indecomposable continuum. By way of introduction to the problem, at first we will focus only on the case where $f : [0,1] \to 2^{[0,1]}$ and $\mathbf{f} = (f, f, f, \ldots)$. That is, we will assume $\varprojlim \mathbf{f}$ is an inverse limit with a single u.s.c. bonding function $f : [0,1] \to 2^{[0,1]}$. Once again, here we recall our earlier results from [13]; we expand our results to much more general cases in the next chapter.

Let us begin by proving a lemma that gives us valuable information about the proper subcontinua of $\varprojlim \mathbf{f}$ in some special cases. An inverse limit space $\varprojlim \mathbf{f}$ with factor spaces X_i is said to have the *full projection property* if, whenever H is a proper subcontinuum of $\varprojlim \mathbf{f}$, there exists some positive integer N so that $\pi_n(H) \neq X_n$ for all $n \geq N$.

Lemma 5.1. *Let $f : [0,1] \to 2^{[0,1]}$ be a u.s.c. function with the property that $\varprojlim \boldsymbol{f}$ is a continuum. Suppose that, for some $A \subsetneq [0,1]$, $f|_{[0,1]\setminus A}$ is a function, $f([0,1] \setminus A) = [0,1]$, and $P = \{(p_1, p_2, \ldots) \in \varprojlim \boldsymbol{f} \mid p_i \notin A \text{ for all } i\}$ is a dense subset of $\varprojlim \boldsymbol{f}$. Then $\varprojlim \boldsymbol{f}$ has the full projection property.*

Proof. Assume by way of contradiction that there is some proper subcontinuum H of $\varprojlim \mathbf{f}$ so that, for each positive integer n, there exists some $m \geq n$ such that $\pi_m(H) = [0,1]$. For any such m, we know that $[0,1] = f([0,1] \setminus A) = f(\pi_m(H) \setminus A) \subseteq \pi_{m-1}(H)$; from this it follows that $\pi_i(H) = [0,1]$ for all $i \leq m$. Thus, since infinitely many positive integers m with $\pi_m(H) = [0,1]$ exist, we have that $\pi_n(H) = [0,1]$ for each positive integer n. We will now show that $P \subseteq H$.

Let $\mathbf{p} = (p_1, p_2, p_3, \ldots) \in P$. Then since $\pi_1(H) = [0,1]$, there exists some point in H of form $(p_1, ?, ?, ?, \ldots)$. Since $\pi_2(H) = [0,1]$, there exists some point in H of form $(?, p_2, ?, ?, \ldots)$. However, $p_2 \notin A$ and $f|_{[0,1]\setminus A}$ is a function, so $f(p_2)$ is unique; therefore, $f(p_2) = p_1$. That means some point of form $(p_1, p_2, ?, ?, ?, \ldots)$ lies in H. A similar argument shows that some point of form $(p_1, p_2, \ldots, p_{i-1}, p_i, ?, ?, \ldots)$ lies in H for all i; since $\mathbf{p}$ is a limit point of the set of all such points, and H is closed, $\mathbf{p} \in H$. Thus, $P \subseteq H$. But then $\overline{P} \subseteq H$; because P is dense, $\overline{P} = \varprojlim \mathbf{f}$, so $\varprojlim \mathbf{f} \subseteq H$. Therefore, $\varprojlim \mathbf{f} = H$, contradicting the assumption that H is a proper subcontinuum. $\square$

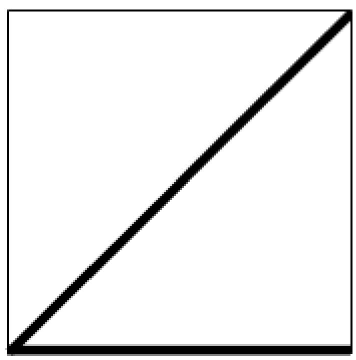

Figure 5.1: A bonding function that gives rise to an inverse limit without the full projection property

A remark about the full projection property is in order. Any inverse limit $\varprojlim \mathbf{f}$ with a continuous surjective bonding function $f : [0,1] \to [0,1]$ has the full projection property automatically. However, if f is u.s.c., then in general $\varprojlim \mathbf{f}$ need not have the full projection property. Example 131 in [4] illustrates this point: For each positive integer i, let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be the graph consisting of the straight line segments joining $(0,0)$ to $(1,0)$ and $(0,0)$ to $(1,1)$. (See Figure 5.1.) Then $H = \{(x,x,x,\ldots) \mid x \in [0,1]\}$ is a proper subcontinuum of $\varprojlim \mathbf{f}$, but $\pi_i(H) = [0,1]$ for all i. Thus, some kind of additional hypotheses (like those in Lemma 5.1) are required for the full projection property to hold.

With this lemma in hand, we may prove the following theorem. We note in advance that the example motivating this theorem is the inverse limit generated using the u.s.c. function

$f : [0,1] \to 2^{[0,1]}$ whose graph is topologically equivalent to a $\sin(\frac{1}{x})$ curve (see Figure 5.2). In this case, the set A is simply $\{0\}$.

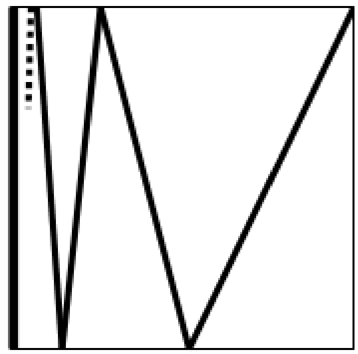

Figure 5.2: A u.s.c. function whose graph is topologically equivalent to a $\sin(\frac{1}{x})$ curve

Theorem 5.2. *Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. and there is some non-empty closed nowhere dense set $A \subseteq [0,1]$ with the property that:*

1) $f(a) = [0,1]$ for all $a \in A$.

2) $f|_{[0,1]\setminus A}$ is an open continuous function.

3) For each $a \in A$, $y \in [0,1]$ and $\epsilon > 0$:

i. If $\exists b \in [0,1]$ with $b > a$, then there exists some $x_1 \in [0,1] \setminus A$ such that $x_1 \in (a, a+\epsilon)$ and $f(x_1) = y$.

ii. If $\exists b \in [0,1]$ with $b < a$, then there exists some $x_2 \in [0,1] \setminus A$ such that $x_2 \in (a-\epsilon, a)$ and $f(x_2) = y$.

Then $\varprojlim \boldsymbol{f}$ is an indecomposable continuum.

Proof. $\varprojlim \mathbf{f}$ is a continuum since $f(x)$ is connected for each $x \in [0,1]$. It remains to show that $\varprojlim \mathbf{f}$ is indecomposable.

First, let $P = \{(p_1, p_2, p_3, \ldots) \in \varprojlim \mathbf{f} \mid p_i \notin A \ \forall i\}$. We will show that P is dense in $\varprojlim \mathbf{f}$. Thus, we need to show that, for each positive integer n, if $O_1, O_2, \ldots, O_n \subseteq [0,1]$ are arbitrary opens sets such that $O = (O_1 \times O_2 \times \cdots \times O_n \times [0,1] \times \ldots) \cap \varprojlim \mathbf{f}$ is non-empty, then O contains some point in P.

Proof by induction on n:

If $n = 1$, then $O = (O_1 \times [0,1] \times [0,1] \times \cdots) \cap \varprojlim \mathbf{f}$ and, since A is nowhere dense, O_1 contains a point p_1 not in A. By condition 3, there exists $p_2 \notin A$ such that $f(p_2) = p_1$, there exists $p_3 \notin A$ such that $f(p_3) = p_2$, etc. It follows that $\mathbf{p} = (p_1, p_2, p_3, \ldots) \in O$, and $\mathbf{p} \in P$ also.

Now we assume the claim is true for n; we need to show it is true for $n+1$. So, suppose $O = (O_1 \times O_2 \times \cdots \times O_n \times O_{n+1} \times [0,1] \times \cdots) \cap \varprojlim \mathbf{f}$ is non-empty. We need to show that O contains a point in P.

We will begin by showing that there is some point $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in O$ with $x_{n+1} \notin A$. There is at least some $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in O$, since O is non-empty; if $x_{n+1} \notin A$, we are done. So, suppose $x_{n+1} \in A$. Then because $x_{n+1} \in O_{n+1}$, which is open, there exists some $\epsilon > 0$ such that $((x_{n+1} - \epsilon, x_{n+1} + \epsilon) \cap [0,1]) \subseteq O_{n+1}$. By condition 3, there exists some $z \in ((x_{n+1} - \epsilon, x_{n+1} + \epsilon) \cap [0,1])$ with $z \notin A$ and $f(z) = x_n$. That means $(x_1, x_2, \ldots, x_n, z, \ldots) \in O$.

In any case, there exists some point $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in O$ with $x_{n+1} \notin A$. Now let $\widehat{O_{n+1}} = O_{n+1} \setminus A$. Since A is closed and $x_{n+1} \in \widehat{O_{n+1}}$, it follows that $\widehat{O_{n+1}}$ is open and non-empty. Moreover, by condition 2, $f(\widehat{O_{n+1}})$ is open. Since $f(\widehat{O_{n+1}})$ contains x_n, which lies in O_n, we have that $O_n \cap f(\widehat{O_{n+1}})$ is open and non-empty. It follows that $W = (O_1 \times O_2 \times \cdots \times O_{n-1} \times (O_n \cap f(\widehat{O_{n+1}})) \times [0,1] \times \cdots) \cap \varprojlim \mathbf{f}$ contains $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$ and is therefore a basic open set that satisfies the inductive hypothesis. So, W contains a point $(p_1, p_2, \ldots, p_{n-1}, p_n, ?, ?, \ldots) \in P$. Since $p_n \in f(\widehat{O_{n+1}})$, there exists some $p_{n+1} \in O_{n+1} \setminus A$ such that $f(p_{n+1}) = p_n$. But (by condition 3) there exists $p_{n+2} \in [0,1] \setminus A$ such that $f(p_{n+2}) = p_{n+1}$, there exists $p_{n+3} \in [0,1] \setminus A$ such that $f(p_{n+3}) = p_{n+2}$, etc. So, we have shown that $(p_1, p_2, \ldots, p_n, p_{n+1}, p_{n+2}, \ldots)$, a point in P, lies in O. This means P is dense in $\varprojlim \mathbf{f}$.

By condition 2, $f|_{[0,1]\setminus A}$ is a function. By condition 3, $f([0,1] \setminus A) = [0,1]$. Thus, the hypothesis of Lemma 5.1 is satisfied; this means that $\varprojlim \mathbf{f}$ has the full projection property.

Finally, suppose by way of contradiction that $\varprojlim \mathbf{f}$ is a union of two proper subcontinua H and K. Because $\varprojlim \mathbf{f}$ has the full projection property, there exists some positive integer N such that $\pi_n(H) \neq [0,1]$ and $\pi_n(K) \neq [0,1]$ for all $n \geq N$. Since A is non-empty, there exists some $a \in A$ lying in either $\pi_{N+1}(H)$ or $\pi_{N+1}(K)$; without loss of generality, assume $a \in \pi_{N+1}(H)$. Since $\pi_{N+1}(K) \neq [0,1]$, $\pi_{N+1}(H)$ must be an arc; in particular, $\pi_{N+1}(H)$ must contain either $[a-\epsilon, a]$ or $[a, a+\epsilon]$ for some $\epsilon > 0$. In either case, by condition 3, $f(\pi_{N+1}(H) \setminus A) = [0,1]$, which forces $\pi_N(H) = [0,1]$. (Contradiction.)

So $\varprojlim \mathbf{f}$ is indecomposable and the proof is complete. $\square$

We improve on this result in various ways in the next chapter. For now, let us turn to a much different condition that also guarantees indecomposability. First, however, we must define the "itinerary space" of an inverse limit. Suppose that, for each positive integer i, $\mathcal{P}_i$ is a partition of $[0,1]$. That is, $\mathcal{P}_i$ is a collection of subsets $P_1^i, P_2^i, \ldots, P_{n_i}^i$ of $[0,1]$ so that $[0,1] = \bigcup_{k=1}^{n_i} P_k^i$ and $P_j^i \cap P_l^i = \emptyset$ if $j \neq l$. Suppose for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is u.s.c. and $\varprojlim \mathbf{f}$ is the associated inverse limit space. If $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \varprojlim \mathbf{f}$, and for $i \geq 2$, α_i is the unique member of $\mathcal{P}_i$ containing x_i, then the *itinerary representation of* $\mathbf{x}$ is $\phi(\mathbf{x}) = (x_1, \alpha_2, \alpha_3, \ldots)$. If for each positive integer i, $\mathcal{P}_i$ is assumed to have some topology $\mathcal{T}_{\mathcal{P}_i}$, then $\mathcal{I} = [0,1] \times \prod_{i=2}^{\infty} \mathcal{P}_i$ is a product space with the standard product topology. Thus, $\phi : \varprojlim \mathbf{f} \to \mathcal{I}$ is given by $\phi((x_1, x_2, x_3, \ldots)) = (x_1, \alpha_2, \alpha_3, \ldots)$. That is, ϕ maps $\mathbf{x} \in \varprojlim \mathbf{f}$ to its unique itinerary representation $\phi(\mathbf{x})$. Then we call $\phi(\varprojlim \mathbf{f})$ the *itinerary space* of $\varprojlim \mathbf{f}$.

Our approach toward itinerary spaces here is inspired by that of Stewart Baldwin in [1]. We intend to form the partitions $\mathcal{P}_i$, each with respective topology $\mathcal{T}_{\mathcal{P}_i}$, so that ϕ turns out to be a homeomorphism between $\varprojlim \mathbf{f}$ and the itinerary space of $\varprojlim \mathbf{f}$. Of course, our choice of partition depends heavily on the nature of the bonding functions (f_i), so we now introduce the kind of bonding function that interests us here. Recall that if g is a u.s.c. bonding function, then the inverse of the graph $G(g)$ is given by $G(g)^{-1} = \{(y,x) | (x,y) \in G(g)\}$. Let the graph of $f : [0,1] \to 2^{[0,1]}$ be given by $G(g)^{-1} \cup G(h)^{-1}$, where, for some fixed $a \in (0,1)$,

1) $g : [0,1] \to [0,1]$ is a non-decreasing continuous function with $g(0) = 0$, $g(1) = a$, and $g((0,1)) = (0,a)$.

2) $h : [0,1] \to [0,1]$ is a non-increasing continuous function with $h(0) = 1$, $h(1) = a$, and $h((0,1)) = (a,1)$.

Then we say f is a *steeple* with turning point a. Note that the graph of any steeple f is closed, so that a steeple is automatically u.s.c. by Theorem 3.1. See Examples 10.5 and 10.6 in Chapter 10 for samples of bonding functions that are (and are *not*) steeples.

Now suppose that, for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is a steeple with turning point a_{i+1}. Then for each $i \geq 2$, we define the *steeple partition with center* a_i to be $\mathcal{P}_i = \{L_i, C_i, R_i\}$, where $L_i = [0, a_i), C_i = \{a_i\}$, and $R_i = (a_i, 1]$. For a given i, the topology $\mathcal{T}_{\mathcal{P}_i}$ on $\mathcal{P}_i$ will be $\{\mathcal{P}_i, \emptyset, \{L_i\}, \{R_i\}, \{L_i, R_i\}\}$. Let $\mathcal{I} = [0,1] \times \prod_{i=2}^{\infty} \mathcal{P}_i$, so that $\mathcal{I}$ is a product space with the usual product topology. Direct inspection reveals that the itinerary space of $\varprojlim \mathbf{f}$ is the subspace $\widehat{\mathcal{I}}$ of $\mathcal{I}$ consisting of all points of the following forms:

1) $(x_1, \alpha_2, \alpha_3, \alpha_4, \ldots)$, where $x_1 \in (0,1)$ and $\alpha_i \in \{L_i, R_i\}$ for each $i \geq 2$.

2) $(0, L_2, L_3, L_4, \ldots)$

3) $(0, L_2, L_3, \ldots, L_k, R_{k+1}, C_{k+2}, \alpha_{k+3}, \alpha_{k+4}, \ldots)$, where $\alpha_i \in \{L_i, R_i\}$ for each $i \geq k+3$.

4) $(0, R_2, C_3, \alpha_4, \alpha_5, \ldots)$, where $\alpha_i \in \{L_i, R_i\}$ for each $i \geq 4$.

5) $(1, C_2, \alpha_3, \alpha_4, \ldots)$, where $\alpha_i \in \{L_i, R_i\}$ for $i \geq 3$.

Because the only open set in $\mathcal{P}_i$ that contains C_i is $\mathcal{P}_i$ itself, $\mathcal{P}_i$ is not Hausdorff, and neither is $\mathcal{I}$. However, as we will see in the following lemma (and as was first pointed out in [1]), $\widehat{\mathcal{I}}$ turns out to be a Hausdorff subspace of $\mathcal{I}$.

Lemma 5.3. *Suppose for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is a steeple with turning point a_{i+1}. Then $\varprojlim \boldsymbol{f}$ is a Hausdorff continuum. Moreover, if for $i \geq 2$, $\mathcal{P}_i$ is the steeple partition with center a_i, then $\phi : \varprojlim \boldsymbol{f} \to \mathcal{I}$ is a homeomorphism between $\varprojlim \boldsymbol{f}$ and $\widehat{\mathcal{I}}$.*

Proof. For each positive integer i, the graph of each f_i satisfies the hypothesis of Theorem 3.1, so each f_i is u.s.c.; indeed, by construction, $f_i(x)$ is connected for each positive integer i and each $x \in [0,1]$. Thus, by Theorem 3.3, $\varprojlim \mathbf{f}$ is a Hausdorff continuum.

We have already observed that, since each f_i is a steeple, $\phi(\varprojlim \mathbf{f}) = \widehat{\mathcal{I}}$; it remains to verify that ϕ is a homeomorphism between $\varprojlim \mathbf{f}$ and $\widehat{\mathcal{I}}$. Of course ϕ maps onto its range. If $\mathbf{x}, \mathbf{y} \in \varprojlim \mathbf{f}$ and $\mathbf{x} \neq \mathbf{y}$, then either $x_1 \neq y_1$ or, for some $i \geq 2$, x_i and y_i lie in different members of $\mathcal{P}_i$, so that the itinerary representations of $\mathbf{x}$ and $\mathbf{y}$ must differ. So ϕ is 1-1. To show that ϕ is continuous, let $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \varprojlim \mathbf{f}$ so that $\phi(\mathbf{x}) = (x_1, \alpha_2, \alpha_3, \ldots)$, and let $O = O_1 \times O_2 \times \ldots \times O_n \times \mathcal{P}_{n+1} \times \ldots$ be a basic open set in $\mathcal{I}$ containing $\phi(\mathbf{x})$. For convenience, shrink O to a smaller basic open set that also contains $\phi(\mathbf{x})$, i.e., $\widetilde{O} = O_1 \times \{\beta_2\} \times \{\beta_3\} \times \ldots \times \{\beta_n\} \times \mathcal{P}_{n+1} \times \ldots$, where $\beta_i = \alpha_i$ if $\alpha_i = L_i$ or R_i, and $\beta_i = \mathcal{P}_i$ if $\alpha_i = C_i$. Then let $U = O_1 \times \gamma_2 \times \gamma_3 \times \ldots \times \gamma_n \times [0,1] \times \ldots$, where $\gamma_i = \beta_i$ if $\beta_i = L_i$ or R_i and $\gamma_i = [0,1]$ if $\beta_i = \mathcal{P}_i$. U contains $\mathbf{x}$, and $\phi(U) \subseteq \widetilde{O} \subseteq O$, so ϕ is continuous.

$\varprojlim \mathbf{f}$ is compact. Therefore, to show that ϕ^{-1} is continuous, it will suffice to show that $\widehat{\mathcal{I}}$ is Hausdorff. Suppose that $\mathbf{y} = (y_1, y_2, \ldots)$ and $\mathbf{z} = (z_1, z_2, \ldots)$ with $\mathbf{y}, \mathbf{z} \in \widehat{\mathcal{I}}$ and $\mathbf{y} \neq \mathbf{z}$. If $y_1 \neq z_1$, then there exist disjoint open sets $O_{y_1}, O_{z_1} \subseteq [0,1]$ containing y_1 and z_1, respectively, so that $O_{y_1} \times \mathcal{P}_2 \times \ldots$ and $O_{z_1} \times \mathcal{P}_2 \times \ldots$ are disjoint open sets in $\mathcal{I}$ containing $\mathbf{y}$ and $\mathbf{z}$. So, suppose $y_1 = z_1$. There are three subcases:

1) If $y_1 = z_1$ are not 0 or 1, then the remaining coordinates are all L_i's and R_i's. Since $\mathbf{y} \neq \mathbf{z}$, there must be some coordinate $k \geq 2$ for which one of y_k, z_k is L_k and the other is R_k. Thus, if $U_k = \{L_k\}$, $V_k = \{R_k\}$, and $U_i = V_i = \mathcal{P}_i$ for all $i \neq k$, then $[0,1] \times \prod_{i=2}^{\infty} U_i$ and $[0,1] \times \prod_{i=2}^{\infty} V_i$ are disjoint open sets, one containing $\mathbf{y}$ and the other containing $\mathbf{z}$.

2) If $y_1 = z_1 = 1$, then $y_2 = z_2 = C_2$ and the remaining coordinates of $\mathbf{y}$ and $\mathbf{z}$ are only L_i's and R_i's. Again, since $\mathbf{y} \neq \mathbf{z}$, there must be some coordinate $k \geq 3$ for which one of y_k, z_k is L_k and the other is R_k. Thus, if $U_k = \{L_k\}$, $V_k = \{R_k\}$, and $U_i = V_i = \mathcal{P}_i$ for all $i \neq k$, then $[0,1] \times \prod_{i=2}^{\infty} U_i$ and $[0,1] \times \prod_{i=2}^{\infty} V_i$ are disjoint open sets, one containing $\mathbf{y}$ and the other containing $\mathbf{z}$.

3) If $y_1 = z_1 = 0$, then there are two subcases:

A) $\mathbf{y}$ and $\mathbf{z}$ both have their first R in the kth coordinate. So,

$\mathbf{y} = (0, L_2, L_3, \ldots, L_{k-1}, R_k, C_{k+1}, y_{k+2}, y_{k+3}, \ldots)$, and

$\mathbf{z} = (0, L_2, L_3, \ldots, L_{k-1}, R_k, C_{k+1}, z_{k+2}, z_{k+3}, \ldots)$. Since $\mathbf{y} \neq \mathbf{z}$, for some $j \geq k+2$, one of y_j, z_j is L_j and the other is R_j. Then two disjoint open sets containing $\mathbf{y}$ and $\mathbf{z}$ respectively can be found in a manner similar to that of case 1 and 2.

B) $\mathbf{y}$ has its first R in the kth coordinate but $\mathbf{z}$ either has no R's at all or its first R lies in the jth coordinate, where (without loss of generality) $k > j$. If $\mathbf{z}$ has no R's at all, then $\mathbf{y}$ and $\mathbf{z}$ can be separated by open sets like those in case 1 and 2; if $\mathbf{z}$ has its first R in the jth coordinate, then $\mathbf{y}$'s jth coordinate is L_j, and thus, it is again easy to separate $\mathbf{y}$ and $\mathbf{z}$ with open sets.

All possible cases have been addressed, so $\phi(\varprojlim \mathbf{f}) = \widehat{\mathcal{I}}$ is Hausdorff. Hence, ϕ^{-1} is continuous, and ϕ is a homeomorphism. $\square$

Theorem 5.4. *Suppose for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is a steeple with turning point a_i. Then $\varprojlim \boldsymbol{f}$ is homeomorphic to the bucket-handle continuum.*

Proof. Let $g : [0,1] \to [0,1]$ be the standard tent map, i.e., the function whose graph consists of straight line segments from $(0,0)$ to $(\frac{1}{2}, 1)$, and from $(\frac{1}{2}, 1)$ to $(1,0)$. Then g is a steeple, so $\varprojlim \mathbf{g}$, the bucket-handle continuum, is homeomorphic to $\widehat{\mathcal{I}}$. Since $\varprojlim \mathbf{f}$ is also homeomorphic to $\widehat{\mathcal{I}}$, $\varprojlim \mathbf{f} \cong \varprojlim \mathbf{g}$. $\square$

Thus, because the bucket-handle is indecomposable, so is any inverse limit space on unit intervals with bonding functions that are steeples. It is possible to give an alternate argument for indecomposability here that does not involve itineraries; we do so, and generalize the "steeple" construction a bit further, in Chapter 7. For a more detailed discussion of itineraries, see, e.g., [1]. The author is indebted to Michel Smith and Tom Ingram for their help in refining our description of "steeple" bonding functions.

Chapter 6

Further Results on Indecomposability

Most of the results in the last two chapters have already been proven in the previous work by the author, [13], but we have included them again here for the sake of completeness. Now we seek new and more general results on detecting indecomposability in u.s.c. inverse limits. The main results in this chapter are generalizations of the important Theorem 5.2. However, we will begin with some indecomposability results that are more miscellaneous, but still interesting and useful.

Theorem 6.1. *Suppose that, for each positive integer i, $f_i : [0,1] \to 2^{[0,1]}$ is u.s.c., surjective, and continuum-valued. If $\varprojlim \boldsymbol{f}$ has the full projection property and for each i, either $f_i(0) = [0,1]$ or $f_i(1) = [0,1]$, then $\varprojlim \boldsymbol{f}$ is an indecomposable continuum.*

Proof. Assume by way of contradiction that $\varprojlim \mathbf{f}$ is a union of two proper subcontinua, H and K. By the full projection property, there must exist some large enough integer N so that for all $n \geq N$, $\pi_n(H) \neq [0,1]$ and $\pi_n(K) \neq [0,1]$. Let us consider the case when $f_N(0) = [0,1]$. One of $\pi_{N+1}(H)$ or $\pi_{N+1}(K)$, but not both, must contain 0; without loss of generality, assume $\pi_{N+1}(H)$ contains 0 but $\pi_{N+1}(K)$ does not. Thus, the set $R = \{(x_1, x_2, x_3, \ldots) \in \varprojlim \mathbf{f} \mid x_{N+1} = 0\}$ must be a subset of H. However, since $f_N(0) = [0,1]$, it follows that $\pi_N(R) = [0,1]$, so that $\pi_N(H) = [0,1]$ also. But it was already stated that $\pi_N(H) \neq [0,1]$, so we have a contradiction. The case when $f_N(1) = [0,1]$ is similar. $\square$

Theorem 6.2. *Suppose for each positive integer i, X_i is a Hausdorff continuum that is irreducible between two of its points (a_i and b_i), and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c., surjective, and continuum-valued. If $\varprojlim \boldsymbol{f}$ has the full projection property and for all i, either $f_i(a_{i+1}) = X_i$ or $f_i(b_{i+1}) = X_i$, then $\varprojlim \boldsymbol{f}$ is an indecomposable continuum.*

Proof. Assume by way of contradiction that $\varprojlim \mathbf{f}$ is a union of two proper subcontinua, H and K. By the full projection property, there must exist some large enough integer N so that for all $n \geq N$, $\pi_n(H) \neq X_n$ and $\pi_n(K) \neq X_n$. Now let us consider the case where $f_N(a_{N+1}) = X_N$. Since X_{N+1} is irreducible from a_{N+1} to b_{N+1}, and $X_{N+1} = \pi_{N+1}(H) \cup \pi_{N+1}(K)$, it follows that one of $\pi_{N+1}(H)$ or $\pi_{N+1}(K)$, but not both, must contain a_{N+1}. Without loss of generality, assume $\pi_{N+1}(H)$ contains a_{N+1} but $\pi_{N+1}(K)$ does not. Thus, the set $R = \{(x_1, x_2, x_3, \ldots) \in \varprojlim \mathbf{f} \mid x_{N+1} = a_{N+1}\}$ must be a subset of H. However, since $f_N(a_{N+1}) = X_N$, it follows that $\pi_N(R) = X_N$, so that $\pi_N(H) = X_N$ also. But it was already stated that $\pi_N(H) \neq X_N$, so we have a contradiction. The case when $f_N(b_{N+1}) = X_N$ is similar. $\square$

Theorem 6.3. *Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. continuum-valued with $f(0) = [0,1]$, and $\varprojlim \boldsymbol{f}$ is an indecomposable continuum. Then $\{0\} \times [0,1]$ is a limit arc of $G(f)$.*

Proof. Suppose by way of contradiction that $\{0\} \times [0,1]$ is not a limit arc of $G(f)$. Then there is some point $(0, y)$ that is not a limit point of $G(f) \setminus (\{0\} \times [0,1])$. That means there exist some small $\epsilon_1, \epsilon_2 > 0$ so that the open set of form $[0, \epsilon_1) \times (y - \epsilon_2, y + \epsilon_2)$ contains no element of $G(f) \setminus (\{0\} \times [0,1])$. If $[0, \epsilon_1) \times (y - \epsilon_2, 1]$ contains no element of $G(f) \setminus (\{0\} \times [0,1])$, then $[0, \epsilon_1) \times (y - \epsilon_2, 1]$ may be used as the open set U in Theorem 4.1, contradicting that $\varprojlim \mathbf{f}$ is indecomposable. A contradiction is similarly reached if $[0, \epsilon_1) \times [0, y + \epsilon_2)$ contains no element of $G(f) \setminus (\{0\} \times [0,1])$. Thus, $[0, \epsilon_1) \times [y + \epsilon_2, 1]$ and $[0, \epsilon_1) \times [0, y - \epsilon_2]$ must respectively contain points (a_1, b_1) and (a_2, b_2) from $G(f) \setminus (\{0\} \times [0,1])$. Since $G(f)$ is connected, $G(f)$ must contain either a point (a_1, b_3) in $[0, \epsilon_1) \times [0, y - \epsilon_2]$ or a point (a_2, b_4) in $[0, \epsilon_1) \times [y + \epsilon_2, 1]$. In either case, we have a contradiction because f has been shown not to be continuum-valued. $\square$

Theorem 6.4. *Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. continuum-valued with $f(1) = [0,1]$, and $\varprojlim \boldsymbol{f}$ is an indecomposable continuum. Then $\{1\} \times [0,1]$ is a limit arc of $G(f)$.*

Proof. The argument is almost identical to that of the previous theorem. $\square$

The fact that 0 and 1 are endpoints of $[0,1]$ was used strongly in the proof of the previous two theorems. If instead $f(a) = [0,1]$ for some $a \in (0,1)$, then there is a counterexample. Let the graph of $f : [0,1] \to 2^{[0,1]}$ consist of the straight line segment between the points $(0,0)$ and $(\frac{1}{2}, 1)$, the straight line segment between the points $(\frac{1}{2}, 1)$ and $(\frac{1}{2}, 0)$, and the straight line segment between the points $(\frac{1}{2}, 0)$ and $(1,1)$. (This is Example 209 in [4].) Then f is u.s.c. and continuum-valued, $f(\frac{1}{2}) = [0,1]$, and (as proved by Ingram) $\varprojlim \mathbf{f}$ is indecomposable. However, $\{\frac{1}{2}\} \times [0,1]$ is not a limit arc of $G(f) \setminus (\{\frac{1}{2}\} \times [0,1])$.

Theorem 6.5. *(Michel Smith) Suppose that for each positive integer i, X_i is an indecomposable continuum and $f_i : X_{i+1} \to 2^{X_i}$ is a surjective u.s.c. bonding function. If $\varprojlim \boldsymbol{f}$ is a continuum with the full projection property, then $\varprojlim \boldsymbol{f}$ is indecomposable.*

Proof. Assume the hypothesis and suppose by way of contradiction that $\varprojlim \mathbf{f}$ is decomposable. Then $\varprojlim \mathbf{f}$ is a union of two proper subcontinua H and K. Since $\varprojlim \mathbf{f}$ has the full projection property, there exists some positive integer N so that, for all $n \geq N$, $\pi_n(H)$ and $\pi_n(K)$ are proper subcontinua of X_n. However, since each bonding function f_i is surjective, it must be the case that $\pi_N(\varprojlim \mathbf{f}) = X_N$. From this it follows that $\pi_N(H) \cup \pi_N(K) = X_N$, contradicting that X_N is indecomposable. □

Now, we seek to generalize Theorem 5.2. The next few theorems and lemmas may be thought of as a warm-up before the most significant and useful result in this chapter, Theorem 6.14.

Lemma 6.6. *For each positive integer i, let X_i be a Hausdorff continuum. Suppose that $f_i : X_{i+1} \to 2^{X_i}$ is a u.s.c. function for each i and $\varprojlim \boldsymbol{f}$ is a continuum. Suppose further that there is a sequence of sets $A_2, A_3, \ldots$ so that, for each $i \geq 2$, $A_i \subsetneq X_i$, $f_{i-1}|_{X_i \setminus A_i}$ is a function, and $f_{i-1}(X_i \setminus A_i) = X_{i-1}$. Finally, suppose that $P = \{(p_1, p_2, \ldots) \in \varprojlim \boldsymbol{f} \,|\forall i \geq 2,\ p_i \notin A_i\}$ is a dense subset of $\varprojlim \boldsymbol{f}$. Then $\varprojlim \boldsymbol{f}$ has the full projection property.*

Proof. Assume by way of contradiction that there is some proper subcontinuum H of $\varprojlim \mathbf{f}$ so that, for each positive integer n, there exists some $m \geq n$ such that $\pi_m(H) = X_m$. For any such $m \geq 2$, we know that $X_{m-1} = f_{m-1}(X_m \setminus A_m) = f_{m-1}(\pi_m(H) \setminus A_m) \subseteq \pi_{m-1}(H)$; from this it follows that $\pi_i(H) = X_i$ for each $i \leq m$. Thus, since infinitely many positive integers m with $\pi_m(H) = X_m$ exist, we have that $\pi_n(H) = X_n$ for each positive integer n. We will now show that $P \subseteq H$.

Let $\mathbf{p} = (p_1, p_2, p_3, \ldots) \in P$. Then since $\pi_1(H) = X_1$, there exists some point in H of form $(p_1, ?, ?, ?, \ldots)$. Since $\pi_2(H) = X_2$, there exists some point in H of form $(?, p_2, ?, ?, \ldots)$. However, $p_2 \notin A_2$ and $f_1|_{X_2 \setminus A_2}$ is a function, so $f_1(p_2)$ is unique; therefore, $f_1(p_2) = p_1$. That means some point of form $(p_1, p_2, ?, ?, ?, \ldots)$ lies in H. A similar argument shows that some point of form $(p_1, p_2, \ldots, p_{i-1}, p_i, ?, ?, \ldots)$ lies in H for all i; since $\mathbf{p}$ is a limit point of the set of all such points, and H is closed, $\mathbf{p} \in H$. Thus, $P \subseteq H$. But then $\overline{P} \subseteq H$; because P is dense, $\overline{P} = \varprojlim \mathbf{f}$, so $\varprojlim \mathbf{f} \subseteq H$. Therefore, $\varprojlim \mathbf{f} = H$, contradicting the assumption that H is a proper subcontinuum. □

Theorem 6.7. *Suppose for each positive integer i, X_i is a Hausdorff continuum and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. Suppose that, for each $i \geq 2$, there is some non-empty closed nowhere dense set $A_i \subseteq X_i$ with the property that:*

1) $f_{i-1}(a) = X_{i-1}$ for all $a \in A_i$.

2) $f|_{X_i \setminus A_i}$ is an open continuous function.

3) For each $a \in A_i$, $y \in X_{i-1}$ and open $U_a \subseteq X_i$ containing a, there exists some $x \in X_i \setminus A_i$ with $x \in U_a$ and $f_{i-1}(x) = y$.

4) For each $a \in A_i$, if H is a non-degenerate subcontinuum of X_i containing a, then H must contain a subset $\widetilde{H}$ of $X_i \setminus A_i$ for which $\overline{f(\widetilde{H})} = X_i$.

Then $\varprojlim \boldsymbol{f}$ is an indecomposable continuum.

Proof. $\varprojlim \mathbf{f}$ is a continuum since f_i is continuum-valued for each positive integer i. It remains to show that $\varprojlim \mathbf{f}$ is indecomposable.

First, let $P = \{(p_1, p_2, p_3, \ldots) \in \varprojlim \mathbf{f} \mid p_i \notin A_i \ \forall i \geq 2\}$. We will show that P is dense in $\varprojlim \mathbf{f}$. Thus, we need to show that, for each positive integer n, if $O_1, O_2, \ldots, O_n$ are arbitrary opens subsets of $X_1, X_2, \ldots, X_n$ respectively so that $O = (O_1 \times O_2 \times \cdots \times O_n \times X_{n+1} \times \ldots) \cap \varprojlim \mathbf{f}$ is non-empty, then O contains some point in P.

Proof by induction on n:

If $n = 1$, then $O = (O_1 \times X_2 \times X_3 \times \cdots) \cap \varprojlim \mathbf{f}$. O_1 is open and non-empty, so O_1 contains a point p_1. By condition 3, there exists $p_2 \in X_2 \setminus A_2$ such that $f_1(p_2) = p_1$, there exists $p_3 \in X_3 \setminus A_3$ such that $f_2(p_3) = p_2$, etc. It follows that $\mathbf{p} = (p_1, p_2, p_3, \ldots) \in O$, and $\mathbf{p} \in P$ also.

Now we assume the claim is true for n; we need to show it is true for $n+1$. So, suppose $O = (O_1 \times O_2 \times \cdots \times O_n \times O_{n+1} \times X_{n+2} \times \cdots) \cap \varprojlim \mathbf{f}$ is non-empty. We need to show that O contains a point in P.

We will begin by showing that there is some point $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in O$ with $x_{n+1} \notin A_{n+1}$. There is at least some $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in O$, since O is non-empty; if $x_{n+1} \notin A_{n+1}$, we are done. So, suppose $x_{n+1} \in A_{n+1}$. Then because $x_{n+1} \in O_{n+1}$, which is open, by condition 3, there exists some $z \in O_{n+1}$ with $z \notin A_{n+1}$ and $f_n(z) = x_n$. That means $(x_1, x_2, \ldots, x_n, z, \ldots) \in O$.

In any case, there exists some point $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in O$ with $x_{n+1} \notin A_{n+1}$. Now let $\widehat{O_{n+1}} = O_{n+1} \setminus A_{n+1}$. Since A_{n+1} is closed and $x_{n+1} \in \widehat{O_{n+1}}$, it follows that $\widehat{O_{n+1}}$ is open and non-empty. Moreover, by condition 2, $f_n(\widehat{O_{n+1}})$ is open. Since $f_n(\widehat{O_{n+1}})$ contains x_n, which lies in O_n, we have that $O_n \cap f_n(\widehat{O_{n+1}})$ is open and non-empty. It follows that $W = (O_1 \times O_2 \times \cdots \times O_{n-1} \times (O_n \cap f_n(\widehat{O_{n+1}})) \times X_{n+1} \times \cdots) \cap \varprojlim \mathbf{f}$ contains $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$ and is therefore a basic open set that satisfies the inductive hypothesis. So, W contains a point $(p_1, p_2, \ldots, p_{n-1}, p_n, ?, ?, \ldots) \in P$. Since $p_n \in f_n(\widehat{O_{n+1}})$, there exists some $p_{n+1} \in O_{n+1} \setminus A_{n+1}$ such that $f_n(p_{n+1}) = p_n$. But (by condition 3) there exists $p_{n+2} \in X_{n+2} \setminus A_{n+2}$ such that $f_{n+1}(p_{n+2}) = p_{n+1}$, there exists $p_{n+3} \in X_{n+3} \setminus A_{n+3}$ such that $f_{n+2}(p_{n+3}) = p_{n+2}$, etc. So, we have shown that $(p_1, p_2, \ldots, p_n, p_{n+1}, p_{n+2}, \ldots)$, a point in P, lies in O. This means P is dense in $\varprojlim \mathbf{f}$.

By condition 2, for each $i \geq 2$, $f_{i-1}|_{X_i \setminus A_i}$ is a function. By condition 3, for each $i \geq 2$, $f_{i-1}(X_i \setminus A_i) = X_{i-1}$. Thus, the hypothesis of Lemma 6.6 is satisfied; this means that $\varprojlim \mathbf{f}$ has the full projection property.

Finally, suppose by way of contradiction that $\varprojlim \mathbf{f}$ is a union of two proper subcontinua H and K. Because $\varprojlim \mathbf{f}$ has the full projection property, there exists some positive integer N such that $\pi_n(H) \neq X_n$ and $\pi_n(K) \neq X_n$ for all $n \geq N$. Since A_{N+1} is non-empty, there exists some $a \in A_{N+1}$ lying in either $\pi_{N+1}(H)$ or $\pi_{N+1}(K)$; without loss of generality, assume $a \in \pi_{N+1}(H)$. Since $\pi_{N+1}(H) \neq X_{N+1}$ and $\pi_{N+1}(K) \neq X_{N+1}$, it follows that $\pi_{N+1}(H)$ must be a non-degenerate subcontinuum of X_{N+1} containing a. By condition 4, $\pi_{N+1}(H)$ must contain a subset $\widetilde{H}$ of $X_{N+1} \setminus A_{N+1}$ for which $\overline{f_N(\widetilde{H})} = X_N$. But $f_N(\widetilde{H}) \subseteq \pi_N(H)$; since $\pi_N(H)$ is closed, it follows that $\overline{f_N(\widetilde{H})} = X_N \subseteq \pi_N(H)$. Thus, $X_N = \pi_N(H)$, which gives us a contradiction.

So $\varprojlim \mathbf{f}$ is indecomposable and the proof is complete. □

Next, we present a sequence of lemmas that will lead to the very powerful Theorem 6.14.

Lemma 6.8. *Suppose $f : X \to Y$ is a continuous function and there is some closed nowhere dense subset A of X such that $f|_{X \setminus A}$ is open. Then if B is a nowhere dense subset of Y, $f^{-1}(B)$ is a nowhere dense subset of X.*

Proof. Let B be nowhere dense in Y. If $f^{-1}(B)$ is empty, we are done. So, suppose $f^{-1}(B)$ is non-empty. Assume by way of contradiction that there exists a non-empty open $U \subseteq X$ such that every non-empty open subset of U meets $f^{-1}(B)$. Since A is closed and nowhere dense, $U \setminus A$ is a non-empty open subset of U. Hence, $f(U \setminus A)$ is open in Y, and therefore $f(U \setminus A)$ contains an open set V that misses B. That means the (non-empty) open set $f^{-1}(V)$ misses $f^{-1}(B)$. But then $f^{-1}(V) \cap (U \setminus A)$ is a non-empty open subset of U that misses $f^{-1}(B)$. (Contradiction.) □

Lemma 6.9. *If $A \subseteq X$ is nowhere dense in X and X is an open subset of Y, then A is nowhere dense in Y.*

Proof. Let O be open in Y. If O misses X, then O misses A. If O meets X, then $O \cap X$ is open in Y, so that $O \cap X$ contains an open set U that misses A. Since U is open in X and X is open in Y, U is open in Y. $\square$

Lemma 6.10. *If $A, B \subseteq X$ are both nowhere dense in X, then $A \cup B$ is nowhere dense in X.*

Proof. Let A, B be nowhere dense in X and assume $O \subseteq X$ is open. Then O contains an open set U that misses A. In turn, U contains an open set V that misses B. Hence, O contains the open set V which misses $A \cup B$. $\square$

Lemma 6.11. *If $A \subseteq X$ is nowhere dense in X, then $\overline{A}$ is nowhere dense in X.*

Proof. Suppose not, i.e., $A \subseteq X$ is nowhere dense but $\overline{A}$ is not. Then there exists an open set O in X such that every open $U \subseteq O$ contains a point in $\overline{A}$. Now since A is nowhere dense, O contains an open set U that misses A. But U must contain a point in $\overline{A}$, where of course $\overline{A} = A \cup A'$. Therefore, U must contain a point in A'. That means, by the definition of limit point, U must contain a point in A. (Contradiction.) $\square$

Lemma 6.12. *Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. and there is some non-empty closed nowhere dense set $A \subseteq [0,1]$ with the property that:*

1) $f(a) = [0,1]$ for all $a \in A$.

2) $f|_{[0,1]\setminus A}$ is a continuous function, and for some $B \subseteq [0,1] \setminus A$ that is closed and nowhere dense in $[0,1] \setminus A$, $f|_{[0,1]\setminus(A \cup B)}$ is open.

3) For each $a \in A$, $y \in (0,1)$ and interval U_a of form (c,a) or (a,c) in $[0,1]$, there exists some $x \in U_a \setminus A$ with $f(x) = y$.

Then if $O = (O_1 \times O_2 \times \cdots \times O_n \times [0,1] \times \cdots) \cap \varprojlim \boldsymbol{f}$ is a non-empty basic open set in $\varprojlim \boldsymbol{f}$, $\pi_n(O)$ is a disjoint union $V \cup Z$, where V is a non-empty open set and Z is a set *so that if $z \in Z$, then every non-empty subset of $[0,1]$ of form (d,z) or (z,d) contains some point of V.*

Proof. We prove the lemma by induction on n. If $n = 1$, then since the bonding function f is surjective, $\pi_1(O)$ is O_1 itself. O_1 may be written as $V \cup Z$, where $V = O_1$ and $Z = \emptyset$. So assume the lemma is true for n; we must show it is true for $n+1$. Let $O = (O_1 \times O_2 \times \cdots \times O_n \times O_{n+1} \times [0,1] \times \cdots) \cap \varprojlim \mathbf{f}$ be basic open and non-empty. The inductive hypothesis applies to the set $\widetilde{O} = (O_1 \times O_2 \times \cdots \times O_n \times [0,1] \times \cdots) \cap \varprojlim \mathbf{f}$; so, $\pi_n(\widetilde{O})$ is a disjoint union $V \cup Z$ where V is open and non-empty, and if $z \in Z$, then every non-empty subset of $[0,1]$ of form (d,z) or (z,d) contains some point in V. We must show that the analogous statement holds true for $\pi_{n+1}(O)$.

First, we note that $\pi_{n+1}(O)$ is the intersection of O_{n+1} with $f^{-1}(\pi_n(\widetilde{O}))$. By the inductive assumption, $\pi_n(\widetilde{O}) = V \cup Z$, as already described. So now, we consider $f^{-1}(\pi_n(\widetilde{O}))$. Since f is a continuous function on $[0,1] \setminus A$ and, by condition 3, $(0,1) \subseteq f([0,1] \setminus A)$, we have that the preimage of V via $f|_{[0,1]\setminus A}$ is a non-empty open set U in $[0,1] \setminus A$. (Moreover, since $[0,1] \setminus A$ is open in $[0,1]$, U is open in $[0,1]$.) Since $f(a) = [0,1]$ for each $a \in A$, we have that the preimage of V via $f|_A$ is A itself. Thus, $f^{-1}(V) = U \cup A$. The preimage of the set Z, $f^{-1}(Z)$, may be written as a disjoint union $A \cup W$ for some $W \subseteq [0,1]$.

Now if $a \in A$, we claim that every open interval in $[0,1]$ of form (d,a) or (a,d) contains a point of U. For, if we have (d,a), we may pick any $y \in V \setminus \{0,1\}$, and then, by condition 3, there exists some $x \in (d,a) \setminus A$ with $f(x) = y$. That means $f|_{[0,1]\setminus A}^{-1}(V)$ contains x, and $x \in U$. Since $x \in (d,a)$, we are done. A similar argument applies in the case of (a,d).

Next, suppose $w \in W$; we intend to show that every open set in $[0,1]$ of form (d,w) and (w,d) contains a point in U. Since $w \in W$, we know $w \notin A$; that implies that (since A is closed and $[0,1]$ is regular) there exists some $\delta > 0$ so that $(w-\delta, w+\delta)$ misses A. Therefore, on $(w-\delta, w+\delta)$, f is a continuous function. Let N be a large enough integer so that $(w - \frac{1}{N}, w) \subseteq (w-\delta, w) \cap (d,w)$. We will show that $(w - \frac{1}{N}, w)$ contains a point of U. First, we note that $f((w - \frac{1}{N}, w])$ cannot be identically $f(w)$; for, if it were, then that would contradict the assumption that $f|_{[0,1]\setminus(A\cup B)}$ is an open map (since B was assumed to be nowhere dense). So, for some $x \in (w - \frac{1}{N}, w)$, either $f(x) > f(w)$ or $f(x) < f(w)$. Either way, by the Intermediate Value Theorem, since f is a continuous map on $[x, w]$, on that interval f must achieve every value between $f(x)$ and $f(w)$. But $f(w)$ is in Z. So, if $f(x) > f(w)$, the open set $(f(w), f(x))$ contains points in V; if $f(x) < f(w)$, the open set $(f(x), f(w))$ contains points in V. Either way, $U = f^{-1}(V) \setminus A$ meets (x, w), and that shows that every open set of form (d, w) contains a point in U. A similar argument shows that every open set of form (w, d) contains a point in U.

Thus, we have shown that $f^{-1}(\pi_n(\widetilde{O}))$ consists exactly of $U \cup f^{-1}(Z)$, where U is a non-empty open set and each point t in $f^{-1}(Z)$ has the property that every non-empty subset of $[0,1]$ of form (d, t) or (t, d) contains some point in U. We note that, since V and Z were disjoint, $f^{-1}(V)$ and $f^{-1}(Z)$ have only the points of A in common; thus, $U = f^{-1}(V) \setminus A$ and $f^{-1}(Z)$ are disjoint sets. Now, we consider $f^{-1}(\pi_n(\widetilde{O})) \cap O_{n+1}$, which equals $(U \cap O_{n+1}) \cup (f^{-1}(Z) \cap O_{n+1})$. Note that $U \cap O_{n+1}$ and $f^{-1}(Z) \cap O_{n+1}$ are disjoint sets, and $U \cap O_{n+1}$ is open. Now, suppose $t \in f^{-1}(Z) \cap O_{n+1}$, and let (d, t) be some open interval in $[0, 1]$. Since $t \in O_{n+1}$ and O_{n+1} is open, we know that (for some small $\epsilon > 0$) O_{n+1} must contain an open interval $(t - \epsilon, t + \epsilon)$. Hence, O_{n+1} must contain $(d, t) \cap (t - \epsilon, t + \epsilon)$, which equals $(\max\{d, t - \epsilon\}, t)$. But $(\max\{d, t - \epsilon\}, t)$ must contain points in U, so that (d, t) contains points in $U \cap O_{n+1}$. A similar argument may be given for an open interval (t, d). Finally, the open set $U \cap O_{n+1}$ is non-empty; for, if $U \cap O_{n+1} = \emptyset$, then because O was non-empty (and thus, $f^{-1}(\pi_n(\widetilde{O})) \cap O_{n+1}$ was non-empty), $f^{-1}(Z) \cap O_{n+1}$ must be non-empty. However, by the above argument, since O_{n+1} contains points in $f^{-1}(Z)$, it follows that O_{n+1} must contain points of U. Hence, $U \cap O_{n+1}$ is non-empty, which is a contradiction; we conclude that $U \cap O_{n+1}$ was non-empty in the first place.

We have therefore demonstrated that $f^{-1}(\pi_n(\widetilde{O})) \cap O_{n+1}$ is a union of two disjoint sets satisfying the condition given in the inductive hypothesis. However, $\pi_{n+1}(O)$ equals $f^{-1}(\pi_n(\widetilde{O})) \cap O_{n+1}$, so the proof is complete. $\square$

Lemma 6.13. *Let $f : [0,1] \to 2^{[0,1]}$ be a u.s.c. function with the property that $\varprojlim \boldsymbol{f}$ is a continuum. Suppose that, for some $A \subsetneq [0,1]$, $f|_{[0,1]\setminus A}$ is a function, $(0,1) \subseteq f([0,1] \setminus A)$, and $P = \{(p_1, p_2, \ldots) \in \varprojlim \boldsymbol{f} \mid p_i \notin A \cup \{0,1\} \text{ for all } i\}$ is a dense subset of $\varprojlim \boldsymbol{f}$. Then $\varprojlim \boldsymbol{f}$ has the full projection property.*

Proof. Assume by way of contradiction that there is some proper subcontinuum H of $\varprojlim \mathbf{f}$ so that, for each positive integer n, there exists some $m \geq n$ such that $\pi_m(H) = [0,1]$. For any such m, we know that $(0,1) \subseteq f([0,1] \setminus A) = f(\pi_m(H) \setminus A) \subseteq \pi_{m-1}(H)$; since $\pi_{m-1}(H)$ is closed and contains $(0,1)$, it follows that $\pi_{m-1}(H) = [0,1]$. Similarly, $\pi_i(H) = [0,1]$ for all $i \leq m$. Thus, since infinitely many positive integers m with $\pi_m(H) = [0,1]$ exist, we have that $\pi_n(H) = [0,1]$ for each positive integer n. We will now show that $P \subseteq H$.

Let $\mathbf{p} = (p_1, p_2, p_3, \ldots) \in P$. Then since $\pi_1(H) = [0,1]$, there exists some point in H of form $(p_1, ?, ?, ?, \ldots)$. Since $\pi_2(H) = [0,1]$, there exists some point in H of form $(?, p_2, ?, ?, \ldots)$. However, $p_2 \notin A$ and $f|_{[0,1]\setminus A}$ is a function, so $f(p_2)$ is unique; therefore, $f(p_2) = p_1$. That means some point of form $(p_1, p_2, ?, ?, ?, \ldots)$ lies in H. A similar argument shows that some point of form $(p_1, p_2, \ldots, p_{i-1}, p_i, ?, ?, \ldots)$ lies in H for all i; since $\mathbf{p}$ is a limit point of the set of all such points, and H is closed, $\mathbf{p} \in H$. Thus, $P \subseteq H$. But then $\overline{P} \subseteq H$; because P is dense, $\overline{P} = \varprojlim \mathbf{f}$, so $\varprojlim \mathbf{f} \subseteq H$. Therefore, $\varprojlim \mathbf{f} = H$, contradicting the assumption that H is a proper subcontinuum. □

We are now ready to prove the main theorem of this chapter.

Theorem 6.14. *Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. and there is some non-empty closed nowhere dense set $A \subseteq [0,1]$ with the property that:*

1) $f(a) = [0,1]$ for all $a \in A$.

2) $f|_{[0,1]\setminus A}$ is a continuous function, and for some $B \subseteq [0,1] \setminus A$ that is closed and nowhere dense in $[0,1] \setminus A$, $f|_{[0,1]\setminus(A\cup B)}$ is open.

3) For each $a \in A$, $y \in (0,1)$ and interval U_a of form (c,a) or (a,c) in $[0,1]$, there exists some $x \in U_a \setminus A$ with $f(x) = y$.

Then $\varprojlim \boldsymbol{f}$ *is an indecomposable continuum.*

Proof. Since $f(x)$ is connected for each $x \in [0,1]$, $\varprojlim \mathbf{f}$ is a continuum. Now we must show it is indecomposable.

We will show that the set $P = \{(p_1, p_2, \ldots) \in \varprojlim \mathbf{f} \mid p_i \notin A \cup \{0,1\} \text{ for all } i\}$ is dense in $\varprojlim \mathbf{f}$. To that end, let $O = (O_1 \times O_2 \times \cdots \times O_n \times [0,1] \times \cdots) \cap \varprojlim \mathbf{f}$ be a non-empty basic open subset of $\varprojlim \mathbf{f}$; we will show that O contains a point in P. We begin by noting that, by Lemma 6.10, $A \cup \{0,1\} = A_1$ is nowhere dense in $[0,1]$. Now $f^{-1}(A_1)$ is the union of the sets $f|_{[0,1]\setminus A}^{-1}(A_1)$ and A; since (by Lemma 6.8) $f|_{[0,1]\setminus A}^{-1}(A_1)$ is nowhere dense in $[0,1] \setminus A$, we conclude from Lemma 6.9 that $f|_{[0,1]\setminus A}^{-1}(A_1)$ is also nowhere dense in $[0,1]$. We already know A is nowhere dense, so we have shown that $f^{-1}(A_1)$ is a union of two nowhere dense sets and is therefore also nowhere dense. Let us call $f^{-1}(A_1) \cup \{0,1\} = A_2$. Again, by Lemma 6.10, A_2 is nowhere dense. Next, by similar reasoning, $f^{-1}(A_2) \cup \{0,1\} = A_3$ is nowhere dense. Continuing this way, we find that $A_n = f^{-1}(A_{n-1}) \cup \{0,1\}$ is nowhere dense. We note that A_n contains all the points of $A, f^{-1}(A), f^{-2}(A), \ldots, f^{-(n-1)}(A)$, as well as $\{0,1\}, f^{-1}(\{0,1\}), f^{-2}(\{0,1\}), \ldots, f^{-(n-1)}(\{0,1\})$.

Now, we note that (by Lemma 6.12) the projection of O onto the nth factor space, $\pi_n(O)$, contains a non-empty open set. Since A_n was nowhere dense, $\pi_n(O) \setminus A_n$ is non-empty. Thus, there exists a point $x_n \in \pi_n(O) \setminus A_n$ such that $x_n \notin A \cup \{0,1\}, f(x_n) = x_{n-1} \notin A \cup \{0,1\}, f^2(x_n) = x_{n-2} \notin A \cup \{0,1\}$, and so forth, so that $x_i \notin A \cup \{0,1\}$ for each positive integer $i \leq n$. We may also use condition 3 to select some element x_{n+1} of $f^{-1}(x_n) \setminus (A \cup \{0,1\})$, and then select some element x_{n+2} of $f^{-1}(x_{n+1}) \setminus (A \cup \{0,1\})$, and so forth. The sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$ is therefore an element of P. Since $x_i \notin A$ for each positive integer i, f acts as a function on each x_i; so, because $x_n \in \pi_n(O)$, $\mathbf{x} \in O$. Thus, we have shown that $P = \{(p_1, p_2, \ldots) \in \varprojlim \mathbf{f} \mid p_i \notin A \cup \{0,1\} \text{ for all } i\}$ is dense in $\varprojlim \mathbf{f}$. From this, we conclude using Lemma 6.13 that $\varprojlim \mathbf{f}$ has the full projection property.

Finally, suppose by way of contradiction that $\varprojlim \mathbf{f}$ is the union of two proper subcontinua H and K. Since $\varprojlim \mathbf{f}$ has the full projection property, there exists some large enough integer N so that if $n \geq N$, $\pi_n(H)$ and $\pi_n(K)$ are proper subcontinua of $[0, 1]$. Since A is non-empty, there is some $a \in A$ and either $a \in \pi_{N+1}(H)$ or $a \in \pi_{N+1}(K)$. Without loss of generality, assume $a \in \pi_{N+1}(H)$. Then for some small $\epsilon > 0$, either $(a - \epsilon, a) \subseteq \pi_{N+1}(H)$ or $(a, a + \epsilon) \subseteq \pi_{N+1}(H)$. In either case, by condition 3, for any $y \in (0, 1)$ there exists some x in $(a - \epsilon, a) \setminus A$ or in $(a, a + \epsilon) \setminus A$ with $f(x) = y$. That implies that $\pi_N(H)$ must contain $(0, 1)$. But then, since $\pi_N(H)$ is closed, $\pi_N(H) = [0, 1]$. This is a contradiction, so the proof is complete. $\square$

With this grand theorem, we may detect indecomposability easily in a great many more cases: See Example 10.8 in Chapter 10.

Chapter 7

A Generalization of the Two-Pass Condition

The two-pass condition, as described by Ingram in [5] and later in [4], is important to the question of when indecomposability arises in inverse limits with u.s.c. bonding functions. Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. Then f satisfies the *two-pass condition* if there are mutually exclusive connected open subsets U and V of $[0,1]$ so that $f|_U$ and $f|_V$ are mappings and $\overline{f(U)} = \overline{f(V)} = [0,1]$. A consequence of Ingram's Theorem 4.3 from [5] is the following:

Theorem 7.1. (Ingram) *Suppose $f : [0,1] \to 2^{[0,1]}$ is a u.s.c. function satisfying the two-pass condition. Then if $\varprojlim \boldsymbol{f}$ is a continuum that has the full projection property, $\varprojlim \boldsymbol{f}$ is an indecomposable continuum.*

Our goal in this section is to further explore the relationship between the two-pass condition, the full projection property, and indecomposability. In particular, we will introduce a new generalization of the two-pass condition that applies to a wider variety of u.s.c. graphs. The idea that such a generalization was possible arose in a discussion with Michel Smith, when he noted that a certain u.s.c. graph comes within ϵ of satisfying the two-pass condition, for any choice of $\epsilon > 0$. As it turns out, having a function f that "almost" satisfies the two-pass condition is enough to prove an indecomposability theorem analogous to Ingram's.

Suppose $f : [0,1] \to 2^{[0,1]}$ is u.s.c. Then f satisfies the *ϵ-two-pass condition* if $\forall \epsilon > 0$ there exist mutually exclusive connected open sets $U, V \subseteq [0,1]$ so that, for some $\{a,b\} \subseteq U$ and $\{c,d\} \subseteq V$, $f|_{\{a,b\}}$ and $f|_{\{c,d\}}$ are mappings, $f(a)$ and $f(c)$ lie within ϵ of 0, and $f(b)$ and $f(d)$ lie within ϵ of 1.

Theorem 7.2. *Suppose the u.s.c. function $f : [0,1] \to 2^{[0,1]}$ satisfies the ϵ-two-pass condition and $\varprojlim \boldsymbol{f}$ is a continuum with the full projection property. Then $\varprojlim \boldsymbol{f}$ is indecomposable.*

Proof. Suppose $\varprojlim \mathbf{f} = H \cup K$, a union of two proper subcontinua. By the full projection property, there is a positive integer N such that $\pi_n(H) \neq [0,1]$ and $\pi_n(K) \neq [0,1]$ for all $n \geq N$.

We consider the sets $\pi_N(H)$ and $\pi_N(K)$. Because these sets are both proper subcontinua of $[0,1]$ whose union is $[0,1]$, one of them must contain 0 and the other must contain 1. Without loss of generality, suppose $\pi_N(H)$ contains 0 and $\pi_N(K)$ contains 1, so that $\pi_N(H) = [0,h]$ for some $0 < h < 1$ and $\pi_N(K) = [k,1]$ for some $0 < k < 1$. Let $\epsilon = \min\{1-h, k\}$, and now consider $\pi_{N+1}(H)$ and $\pi_{N+1}(K)$. Since f satisfies the ϵ-two-pass condition, there exist mutually exclusive open subsets U and V of $[0,1]$ with some $\{a,b\} \subseteq U$ and $\{c,d\} \subseteq V$ such that $f|_{\{a,b\}}$ and $f|_{\{c,d\}}$ are mappings, $f(a)$ and $f(c)$ lie within ϵ of 0, and $f(b)$ and $f(d)$ lie within ϵ of 1.

Since $\pi_{N+1}(H) \cup \pi_{N+1}(K) = [0,1]$, by Theorem 2.26, one of U and V is a subset of one of $\pi_{N+1}(H)$ and $\pi_{N+1}(K)$. We now examine the case in which $U \subseteq \pi_{N+1}(H)$. Because $\{a,b\} \subseteq U$, we have $\{a,b\} \subseteq \pi_{N+1}(H)$. Thus, because $f|_{\{a,b\}}$ is a mapping, it follows that $f(a) \in \pi_N(H)$ and $f(b) \in \pi_N(H)$. But $f(a)$ and $f(b)$ lie within ϵ of 0 and 1, respectively, contradicting the way ϵ was chosen. The remaining cases may be handled similarly. $\square$

The following examples may shed more light on the relationship between the full projection property, the two-pass condition, the ϵ-two-pass condition, and indecomposability. Let the graph of f_1 be given by the straight line segments from $(0,0)$ to $(\frac{1}{2},1)$, from $(\frac{1}{2},1)$ to $(\frac{1}{2},0)$, and from $(\frac{1}{2},0)$ to $(1,1)$. Let the graph of f_2 be given by the straight line segments from $(0,0)$ to $(\frac{1}{2},1)$, from $(\frac{1}{2},1)$ to $(\frac{1}{2},\frac{1}{2})$, and from $(\frac{1}{2},\frac{1}{2})$ to $(1,1)$. (These are the graphs from Examples 3.4 and 3.5 in Ingram's paper, [5]; see Figures 7.1 and 7.2 below.) Let the graph of f_3 be the same as the graph of f_j from Example 10.2 in Chapter 10. (See Figure 7.3.) Let the graph of f_4 be the same as the graph topologically equivalent to the closure of a $\sin(\frac{1}{x})$ curve, as seen in Chapter 5. (See Figure 7.4.) Finally, let the graph of f_5 be a somewhat distorted version of the graph of f_4, as shown in Figure 7.5.

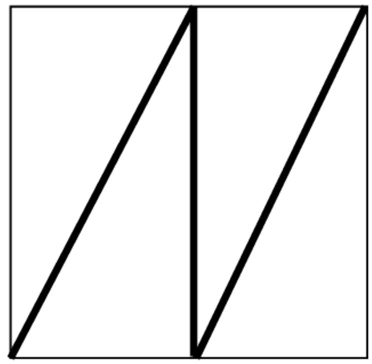

Figure 7.1: f_1

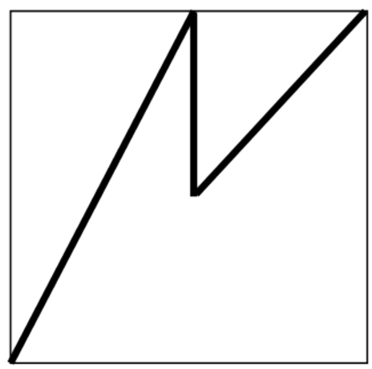

Figure 7.2: f_2

f_1 satisfies the two-pass condition and $\varprojlim \mathbf{f}_1$ has the full projection property (as shown directly by Ingram in [5] and [4]), so $\varprojlim \mathbf{f}_1$ is indecomposable. f_2 does not satisfy the two-pass condition and $\varprojlim \mathbf{f}_2$ does not have the full projection property; moreover, $\varprojlim \mathbf{f}_2$ is not indecomposable. f_3 satisfies the two-pass condition but by Theorem 4.1, $\varprojlim \mathbf{f}_3$ is decomposable. That means, by Theorem 7.1, f_3 does not have the full projection property. f_4 satisfies the two-pass condition and $\varprojlim \mathbf{f}_4$ has the full projection property (as shown in Theorem 5.2), so that $\varprojlim \mathbf{f}_4$ is indecomposable. Finally, f_5 does not satisfy the two-pass condition; however, it does satisfy the ϵ-two-pass condition. Since it may be shown directly that $\varprojlim \mathbf{f}_5$ has the full projection property, it follows that $\varprojlim \mathbf{f}_5$ is indecomposable. (Although, it must be admitted that this conclusion could have been reached other ways, e.g., by applying Theorem 6.14.)

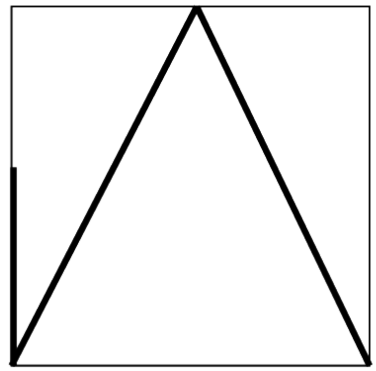

Figure 7.3: f_3

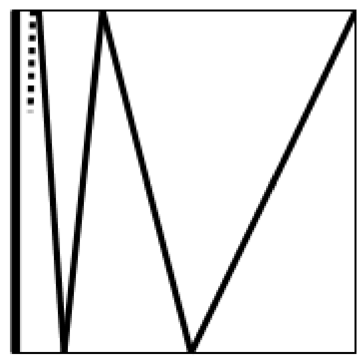

Figure 7.4: f_4

Let f_6 be the "steeple function" given in Chapter 10, Example 10.5. (See Figure 7.6.) In Chapter 5, we used itineraries to prove that the inverse limit with this single bonding function is indecomposable. Still, it would be helpful to have an alternate proof that does not resort to itineraries. We note that f_6 does not satisfy the two-pass condition; however, it does satisfy the ϵ-two-pass condition. Thus, if we can prove that the corresponding inverse limit has the full projection property, then by Theorem 7.2, $\varprojlim \mathbf{f}_6$ is an indecomposable continuum.

The next major theorem (Theorem 7.4) implies that any inverse limit with bonding functions that are steeples has the full projection property. In fact, this theorem applies to a much more general collection of graphs that might be called "generalized steeples." First, we give a lemma.

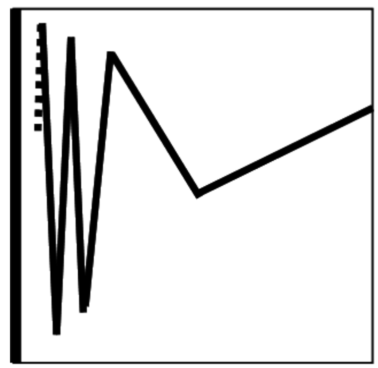

Figure 7.5: f_5

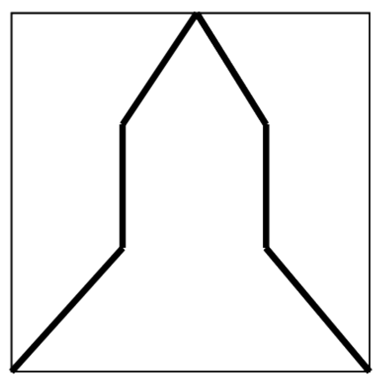

Figure 7.6: f_6

Lemma 7.3. *Suppose A is an arc with endpoints a_1, a_2 and B is an arc with endpoints b_1, b_2. Let $f : A \to 2^B$ be a surjective u.s.c. function that passes the horizontal line test, i.e., $f^{-1}(x)$ is degenerate for each $x \in B$. Then $G(f)$ is an arc with endpoints $\{(f^{-1}(b_1), b_1)\}$ and $\{(f^{-1}(b_2), b_2)\}$.*

Proof. For all $i \geq 2$, let $g_i : A \to 2^A$ be the identity map. Then if $\mathbf{g} = (f, g_2, g_3, g_4, \ldots)$, Theorem 3.4 implies $\varprojlim \mathbf{g}$ is a continuum. Thus, $G(f)$, which is homeomorphic to the projection of $\varprojlim \mathbf{g}$ onto its first two coordinates, is also a continuum. Since f passes the horizontal line test, for a given $x \in B$, $f^{-1}(x)$ is unique. We note that, if $x \in B \setminus \{b_1, b_2\}$, then $G(f) \setminus \{(f^{-1}(x), x)\}$ is the union of two disjoint non-empty sets, $(A \times [b_1, x)) \cap G(f)$ and $(A \times (x, b_2]) \cap G(f)$. Thus, if $x \in B \setminus \{b_1, b_2\}$, $\{(f^{-1}(x), x)\}$ is a cut point of $G(f)$. This means $\{(f^{-1}(b_1), b_1)\}$ and $\{(f^{-1}(b_2), b_2)\}$ are the only non-cut points of $G(f)$, and the proof is complete. □

Theorem 7.4. *Suppose that $a_1, a_2, \ldots, a_n$ is a strictly increasing subset of $[0,1]$ with $a_1 = 0$, $a_n = 1$, and $n \geq 3$. Let $f : [0,1] \to C([0,1])$ be a u.s.c. continuum-valued function with $f(a_i) = 0$ for each odd $i \leq n$ and $f(a_i) = 1$ for each even $i \leq n$. Suppose further that, for each i, $1 \leq i \leq n-1$, $f|_{[a_i, a_{i+1}]}$ is a surjective u.s.c. bonding function that passes the horizontal line test. Then if $f_n = f$ for all positive integers n, $\varprojlim \boldsymbol{f}$ has the full projection property.*

Proof. We will begin by assuming that n is odd, so that $f(1) = 0$. We intend to show that, for each positive integer j, $G(f_1, f_2, \ldots, f_j)$ is an arc with endpoints $(0, 0, \ldots, 0, 0)$ and $(0, 0, \ldots, 0, 1)$ in $\prod_{k=1}^{j+1}[0,1]$. Proceed by induction: by the way f is defined, $G(f_1)$ is an arc with endpoints $(0,0)$ and $(0,1)$. Assume the claim is true for $j-1$, so that $G(f_1, f_2, \ldots, f_{j-1})$ is an arc with endpoints $(0, 0, \ldots, 0, 0)$ and $(0, 0, \ldots, 0, 1)$ in $\prod_{k=1}^{j}[0,1]$.

Let $[0,1]_k$ denote the kth factor space of $\varprojlim \mathbf{f}$. Define $h : [0,1]_{j+1} \to 2^{G(f_1, f_2, \ldots, f_{j-1})}$ by $h(t) = \{(x_1, x_2, \ldots, x_{j-1}, x_j) \in G(f_1, f_2, \ldots, f_{j-1}) | x_j \in f_j(t)\}$. Note that $G(h)$ is homeomorphic to $G(f_1, f_2, \ldots, f_{j-1}, f_j)$, so that $G(h)$ is closed and therefore, h is u.s.c. (For convenience, if $(t, (x_1, x_2, \ldots, x_{j-1}, x_j)) \in G(h)$, we will instead write this ordered pair in the form of its counterpart in $G(f_1, f_2, \ldots, f_{j-1}, f_j)$, i.e., $(x_1, x_2, \ldots, x_{j-1}, x_j, t)$.) Next, note that, for each i, $1 \leq i \leq n-1$, $h|_{[a_i, a_{i+1}]}$ is surjective and passes the horizontal line test, so that $G(h|_{[a_i, a_{i+1}]})$ is an arc (by Lemma 7.3). Moreover, if i is odd, $G(h|_{[a_i, a_{i+1}]})$ is an arc whose endpoints are $(0, 0, \ldots, 0, 0, a_i)$ and $(0, 0, \ldots, 0, 1, a_{i+1})$ in $\prod_{k=1}^{j+1}[0,1]$, so that every other point in $G(h|_{[a_i, a_{i+1}]})$ has $j+1$th coordinate lying strictly between a_i and a_{i+1}. If i is even, $G(h|_{[a_i, a_{i+1}]})$ is an arc whose endpoints are $(0, 0, \ldots, 0, 1, a_i)$ and $(0, 0, \ldots, 0, 0, a_{i+1})$ in $\prod_{k=1}^{j+1}[0,1]$, so that every other point in $G(h|_{[a_i, a_{i+1}]})$ has $j+1$th coordinate lying strictly between a_i and a_{i+1}. So, $G(h) = \bigcup_{i=1}^{n-1} G(h|_{[a_i, a_{i+1}]})$, a union of finitely many arcs. Note that two arcs $G(h|_{[a_k, a_{k+1}]})$ and $G(h|_{[a_m, a_{m+1}]})$, $k < m$, have a point in common iff $k + 1 = m$; in that case, they have only one point in common, namely, $(0, 0, \ldots, 0, 1, a_{k+1})$ if k is odd or $(0, 0, \ldots, 0, 0, a_{k+1})$ if k

is even. This means that $G(h)$ is an arc with endpoints $(0,0,\ldots,0,0,0)$ and $(0,0,\ldots,0,0,1)$ in $\prod_{k=1}^{j+1}[0,1]$. But $G(f_1,f_2,\ldots,f_n)$ is homeomorphic to $G(h)$, so the claim is verified.

Next, we intend to show that $\varprojlim \mathbf{f}$ has the full projection property. To that end, suppose $\pi_n(H)=[0,1]$ for infinitely many positive integers n. Since $f(a_1)=0$ and $f(a_2)=1$, if $\pi_n(H)=[0,1]$ for some $n\geq 2$, then $\{0,1\}\subseteq\pi_{n-1}(H)$. But $\pi_{n-1}(H)$ is connected, so $\pi_{n-1}(H)=[0,1]$; it follows that $\pi_n(H)=[0,1]$ for all positive integers n.

We now consider $\pi_{\{1,2,\ldots,n+1\}}(H)$. Since projection maps are continuous and H is a continuum, $\pi_{\{1,2,\ldots,n+1\}}(H)$ is a subcontinuum of $G(f_1,f_2,\ldots,f_n)$. Since $\{0,1\}\subseteq\pi_{n+1}(H)$, $\pi_{\{1,2,\ldots,n+1\}}(H)$ contains the points $(0,0,\ldots,0,0,0)$ and $(0,0,\ldots,0,0,1)$, the two endpoints of $G(f_1,f_2,\ldots f_n)$. Since $\pi_{\{1,2,\ldots,n+1\}}(H)$ is connected, $\pi_{\{1,2,\ldots,n+1\}}(H)$ must therefore be all of $G(f_1,f_2,\ldots,f_n)$.

Finally, let $\mathbf{p}=(p_1,p_2,p_3,\ldots,p_n,\ldots)\in\varprojlim\mathbf{f}$. Since $\pi_{\{1,2,\ldots,n+1\}}(H)=G(f_1,f_2,\ldots,f_n)$ for all n, H contains a point of form $(p_1,p_2,\ldots,p_n,?,?,\ldots)$ for each n. Because $\mathbf{p}$ is the limit point of the set of all such points, and H is closed, $\mathbf{p}\in H$. Hence, $\varprojlim\mathbf{f}\subseteq H$, and $\varprojlim\mathbf{f}=H$. That means $\varprojlim\mathbf{f}$ has the full projection property.

The proof is similar in case 2, where n is even. (In that case, $G(f_1,f_2,\ldots,f_j)$ is an arc with endpoints $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$.) □

Corollary 7.5. *Suppose that f is a u.s.c. function satisfying the hypothesis of Theorem 7.4. Then $\varprojlim \boldsymbol{f}$ is an indecomposable continuum.*

Proof. By Theorem 7.4, $\varprojlim\mathbf{f}$ is a continuum with the full projection property. Since f also satisfies the ϵ-two-pass condition, $\varprojlim\mathbf{f}$ is indecomposable by Theorem 7.2. □

Theorem 7.4 was stated with $f(a_i)=0$ for each odd $i\leq n$ and $f(a_i)=1$ for each even $i\leq n$, but a similar theorem can be proven in the case where $f(a_i)=1$ for each odd $i\leq n$ and $f(a_i)=0$ for each even $i\leq n$. We can also give a version of Theorem 7.4 that holds in a much more general setting:

Theorem 7.6. *For each positive integer n, let X_n be a Hausdorff continuum and let $f_n : X_{n+1} \to 2^{X_n}$ be a surjective u.s.c. function. Suppose that, for each positive integer n, $G = G(f_1, f_2, \ldots, f_n)$ is a continuum and there exist $x_{n+1}, y_{n+1} \in X_{n+1}$ such that G is irreducible between any point in G whose $n+1$ coordinate is x_{n+1} and any point in G whose $n+1$ coordinate is y_{n+1}. Then $\varprojlim \boldsymbol{f}$ is a continuum with the full projection property.*

Proof. Since $G(f_1, f_2, \ldots, f_n)$ is a continuum for each n, $\varprojlim \mathbf{f}$ is a continuum as well. Now suppose H is a subcontinuum of $\varprojlim \mathbf{f}$ with $\pi_n(H) = X_n$ for infinitely many n. We need to show $H = \varprojlim \mathbf{f}$.

Let $\mathbf{p} = (p_1, p_2, \ldots, p_n, \ldots) \in \varprojlim \mathbf{f}$; we need to show $\mathbf{p} \in H$. Let $M = \{n \mid n$ is a positive integer and $\pi_n(H) = X_n\}$. Fix some $n \in M$, $n \geq 2$, and let $x_n, y_n \in X_n$ be such that $G(f_1, f_2, \ldots, f_{n-1})$ is irreducible between each of its points with nth coordinate x_n and each of its points with nth coordinate y_n. We note that, since $\pi_n(H) = X_n$, H contains sequences in $\varprojlim \mathbf{f}$ of form $(x_1, x_2, \ldots, x_{n-1}, x_n, ?, ?, \ldots)$ and $(y_1, y_2, \ldots, y_{n-1}, y_n, ?, ?, \ldots)$. Thus, $\pi_{\{1,2,\ldots,n\}}(H)$ is a subcontinuum of $G(f_1, \ldots, f_{n-1})$ containing $(x_1, x_2, \ldots, x_{n-1}, x_n)$ and $(y_1, y_2, \ldots, y_{n-1}, y_n)$. Hence, by irreducibility, $\pi_{\{1,2,\ldots,n\}}(H) = G(f_1, \ldots, f_{n-1})$. Since $(p_1, p_2, \ldots, p_n) \in G(f_1, \ldots, f_{n-1})$, H therefore contains a point of form $(p_1, p_2, \ldots, p_n, ?, ?, \ldots)$.

This same argument shows H must contain a point of form $(p_1, p_2, \ldots, p_n, ?, ?, \ldots)$ for all $n \in M$. Thus, because H is closed, H contains the limit point of all such points, namely, $\mathbf{p}$ itself. Therefore, $\varprojlim \mathbf{f} \subseteq H$, which says $\varprojlim \mathbf{f} = H$. This means $\varprojlim \mathbf{f}$ has the full projection property. $\square$

An example of a u.s.c. function that satisfies the hypothesis of Theorem 7.4, i.e., a "generalized steeple," may be found in Chapter 10. (See Example 10.9.)

Chapter 8

Inverse Limits on Initial Segments of the Ordinal Numbers

In their book *Inverse Limits: From Continua to Chaos*, Ingram and Mahavier generalize many of their earlier results about inverse limits indexed by the positive integers to inverse limits indexed by more general directed sets [4]. They also give theorems that apply in the special case of inverse limits indexed by a totally ordered directed set. In this section, we will prove analogous theorems in the very special case where the inverse limit's index set is some "long" (i.e., uncountable) initial segment of the ordinals. Our proof techniques will be different than those in [4], however, because we will heavily use transfinite induction. All of our initial theorems here may be thought of as building up to a general theorem "template," i.e., Theorem 8.5.

Let γ be an ordinal. Suppose $\{X_\alpha\}_{\alpha\leq\gamma}$ is a collection of continua and $\mathcal{F} = \{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta\leq\gamma}$ is a collection of surjective u.s.c. functions so that $\forall \alpha < \beta < \eta \leq \gamma$, $f_{\alpha,\beta} \circ f_{\beta,\eta}(x) = f_{\alpha,\eta}(x)$ for all $x \in X_\eta$. Then let us say the functions in $\mathcal{F}$ are *properly composing*. We define $G_\gamma = \{(x_0, x_1, x_2, \ldots, x_\alpha, \ldots, x_\gamma) \in \prod_{\alpha\leq\gamma} X_\alpha \mid x_\alpha \in f_{\alpha,\beta}(x_\beta)\ \forall \alpha < \beta \leq \gamma\}$. A basis for the topology on G_γ is given by $\{O \cap G_\gamma \mid O$ is a basic open subset of $\prod_{\alpha\leq\gamma} X_\alpha\}$. If γ is a limit ordinal, we define $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\} = \{(x_0, x_1, x_2, \ldots, x_\alpha, \ldots) \in \prod_{\alpha<\gamma} X_\alpha \mid x_\alpha \in f_{\alpha,\beta}(x_\beta)\ \forall \alpha < \beta < \gamma\}$. A basis for the topology on $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\}$ is given by $\{O \cap \varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\} \mid O$ is a basic open subset of $\prod_{\alpha<\gamma} X_\alpha\}$. For convenience, we will at times denote $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\}$ by $G_{<\gamma}$.

It is an exercise in transfinite induction (using the surjectivity of the bonding functions $f_{\alpha,\beta}$ and the compactness of the factor spaces X_α) to show that if $\mu < \gamma$, the projection of $G_{<\gamma}$ (or of G_γ) onto the set of all coordinates $\leq \mu$ is G_μ. We also note that, if X_α is a continuum for each $\alpha < \omega$ and $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta<\omega}$ is a collection of properly composing surjective u.s.c. functions, then $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \omega\}$ is in fact a standard u.s.c. inverse limit with a countable index set, i.e., $\varprojlim\{X_\alpha, f_{\alpha,\alpha+1}\}_{\alpha=0}^{\infty}$.

Theorem 8.1. *Let γ be an ordinal. Suppose $\{X_\alpha\}_{\alpha\leq\gamma}$ is a collection of continua and $\{f_{\alpha,\beta} : X_\beta \to C(X_\alpha)\}_{\alpha<\beta\leq\gamma}$ is a collection of properly composing surjective u.s.c. continuum-valued functions. Then G_γ is a continuum.*

Proof. By Theorem 3.3, $G_{<\omega}$ is a continuum. Thus, G_γ is a continuum for each finite γ, since G_γ is the projection of $G_{<\omega}$ onto the set of all coordinates $\leq \gamma$. So, it remains to prove the theorem for all $\gamma \geq \omega$.

We proceed by transfinite induction on γ. Suppose the theorem holds for each $\rho < \gamma$; we must show the theorem holds for γ. Since $\gamma \geq \omega$, we know that $\gamma = \mu + n$ for some limit ordinal μ and integer $n \geq 0$.

For a given $\rho < \mu$, let H_ρ be the set of all points

$\mathbf{x} = (x_0, x_1, x_2, \ldots, x_\rho, \ldots, x_\mu, x_{\mu+1}, \ldots, x_{\mu+n})$ in $\prod_{\alpha\leq\mu+n} X_\alpha$

so that $x_\alpha \in f_{\alpha,\beta}(x_\beta)$ for all $\alpha < \beta \leq \rho$, $x_\alpha \in f_{\alpha,\mu+k}(x_{\mu+k})$ for all $\alpha \leq \rho$, $0 \leq k \leq n$, and $x_{\mu+k} \in f_{\mu+k,\mu+j}(x_{\mu+j})$ for all $0 \leq k < j \leq n$. We intend to show that H_ρ is a continuum.

If $A = \{\alpha \mid \alpha \leq \mu + n\}$, let us define a function $h : A \to A$ as follows: $h(\alpha) = \alpha$ if $\alpha \leq \rho$ or $\rho + n + 2 \leq \alpha < \mu$; $h(\rho + k + 1) = \mu + k$ for $0 \leq k \leq n$; $h(\mu + k) = \rho + k + 1$ for $0 \leq k \leq n$. (We note that h simply exchanges the ordinal $\rho+k+1$ with the ordinal $\mu+k$ for $0 \leq k \leq n$ and fixes all other ordinals.) Now, for each $\alpha \leq \gamma$, let $Y_\alpha = X_{h(\alpha)}$. Also, for each $\alpha < \beta \leq \rho + n + 1$, let $g_{\alpha,\beta} : Y_\beta \to C(Y_\alpha)$ be given by $g_{\alpha,\beta} = f_{h(\alpha),h(\beta)}$. Then, because of the way H_ρ was defined, the collection of functions $\{g_{\alpha,\beta} : Y_\beta \to C(Y_\alpha)\}_{\alpha<\beta\leq\rho+n+1}$ is a properly composing collection. Thus, if we let $\widetilde{G}_{\rho+n+1} = \{(y_\alpha)_{\alpha\leq\rho+n+1} \in \prod_{\alpha\leq\rho+n+1} Y_\alpha \mid y_\alpha \in g_{\alpha,\beta}(y_\beta)$ for all $\alpha < \beta \leq \rho + n + 1\}$, then since $\rho + n + 1 < \gamma$, the inductive hypothesis applies to $\widetilde{G}_{\rho+n+1}$. Hence, $\widetilde{G}_{\rho+n+1}$ is a continuum. Finally, if $B = \{\alpha \mid \alpha \leq \rho$ or $\mu \leq \alpha \leq \mu + n\}$, then it is easily seen that $\pi_B(H_\rho)$ is homeomorphic to $\widetilde{G}_{\rho+n+1}$. So $\pi_B(H_\rho)$ is a continuum. But then, because H_ρ is homeomorphic to $\pi_B(H_\rho) \times \prod_{\alpha\leq\gamma,\alpha\notin B} X_\alpha$, a product of continua, we conclude that H_ρ is a continuum.

We note that, if $\rho < \eta < \mu$, then H_ρ contains H_η, so that $\{H_\rho\}_{\rho<\mu}$ is a monotonic collection of continua.

Claim: $G_{\mu+n} = \bigcap_{\rho<\mu} H_\rho$.

Justification: Since $G_{\mu+n} \subseteq H_\rho$ for each $\rho < \mu$, and $\{H_\rho\}_{\rho<\mu}$ is a monotonic collection, we have $G_{\mu+n} \subseteq \bigcap_{\rho<\mu} H_\rho$. On the other hand, if $\mathbf{x} \in \bigcap_{\rho<\mu} H_\rho$, then there are three cases. For any $\rho < \mu$ and $\mu \leq \eta \leq \mu + n$, we have $\mathbf{x} \in H_\rho$, so that $x_\rho \in f_{\rho,\eta}(x_\eta)$. For any $\mu \leq \rho < \eta \leq \mu + n$, since $\mathbf{x} \in H_1$, $x_\rho \in f_{\rho,\eta}(x_\eta)$. Finally, for any $\rho < \eta < \mu$, since $\mathbf{x} \in H_\eta$, $x_\rho \in f_{\rho,\eta}(x_\eta)$. All cases are accounted for, so $\mathbf{x} \in G_{\mu+n}$. Therefore, $G_{\mu+n} = \bigcap_{\rho<\mu} H_\rho$.

Finally, we conclude that, since $G_{\mu+n} = \bigcap_{\rho<\mu} H_\rho$ is the intersection of a monotonic collection of continua, $G_{\mu+n} = G_\gamma$ is a continuum. Thus, the proof is complete. □

Theorem 8.2. *Suppose γ is a limit ordinal, $\{X_\alpha\}_{\alpha<\gamma}$ is a collection of continua and $\{f_{\alpha,\beta} : X_\beta \to C(X_\alpha)\}_{\alpha<\beta<\gamma}$ is a collection of properly composing surjective u.s.c. continuum-valued functions. Then $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\}$ is a continuum.*

Proof. For each $\rho < \gamma$, let $K_\rho = \{(x_\alpha)_{\alpha<\gamma} \in \prod_{\alpha<\gamma} X_\alpha \mid x_\alpha \in f_{\alpha,\beta}(x_\beta)\ \forall \alpha < \beta \leq \rho\}$. Then for each ρ, K_ρ is homeomorphic to $G_\rho \times \prod_{\rho+1\leq\alpha<\gamma} X_\alpha$. By Theorem 8.1, G_ρ is a continuum; thus, K_ρ is homeomorphic to a product of continua and hence, K_ρ is a continuum. We also note that $\{K_\rho\}_{\rho<\gamma}$ is a monotonic collection of continua. But then $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\} = \bigcap_{\rho<\gamma} K_\rho$, which is a continuum. □

Theorem 8.3. *Let γ be an ordinal. Suppose $\{X_\alpha\}_{\alpha\leq\gamma}$ is a collection of continua and $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta\leq\gamma}$ is a collection of properly composing surjective u.s.c. functions. Suppose further that for each $\alpha < \beta \leq \gamma$ and each $x_\alpha \in X_\alpha$, $f^{-1}_{\alpha,\beta}(x_\alpha)$ is a non-empty, connected set. Then G_γ is a continuum.*

Proof. Theorem 3.4 implies that the theorem is true for each finite γ. So, it remains to prove the theorem for all $\gamma \geq \omega$. The rest of the argument is identical to the one used to prove Theorem 8.1. □

Theorem 8.4. *Suppose γ is a limit ordinal, $\{X_\alpha\}_{\alpha<\gamma}$ is a collection of continua and $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta<\gamma}$ is a collection of properly composing surjective u.s.c. functions. Suppose further that for each $\alpha < \beta < \gamma$ and each $x_\alpha \in X_\alpha$, $f^{-1}_{\alpha,\beta}(x_\alpha)$ is a non-empty, connected set. Then $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\}$ is a continuum.*

Proof. It suffices to use an argument analogous to the proof of Theorem 8.2. □

We note that, in the proof of Theorem 8.1, all that was needed to start the transfinite induction was the fact that the theorem was true for each finite n. Thus, instead of assuming that each function was continuum-valued, we could have just as well assumed that each function satisfied some property $\mathcal{P}$ that causes each G_n (with n finite) to be a continuum. This observation leads us to the following "theorem template":

Theorem 8.5. *Suppose the following is true: "Let n be a positive integer. Suppose $\{X_\alpha\}_{\alpha\leq n}$ is a collection of continua and $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta\leq n}$ is a collection of properly composing surjective u.s.c. bonding functions each with property $\mathcal{P}$. Then G_n is a continuum."*

Then the following is true: "Let $\gamma \geq \omega$. Suppose $\{X_\alpha\}_{\alpha\leq\gamma}$ is a collection of continua and $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta\leq\gamma}$ is a collection of properly composing surjective u.s.c. bonding functions each with property $\mathcal{P}$. Then G_γ is a continuum."

As an example of how to use this template, consider the inverse limit $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \gamma\}$ obtained when all the factor spaces X_α are the same continuum $[0,1]$, there is a single surjective u.s.c. bonding function $f : [0,1] \to 2^{[0,1]}$, and also $f \circ f = f$ (so that the functions are properly composing). The property $\mathcal{P}$ could be, "f is the union of two distinct continuous functions g and h, at least one of which is surjective." Corollary 4.5 implies that each G_n (n finite) is a continuum, so that (by Theorem 8.5) G_γ is also a continuum for each $\gamma \geq \omega$. Thus, it is possible to generalize Corollary 4.5 to the case of a "long" inverse limit. Indeed, we note that many of the indecomposability theorems from the previous chapters have analogues in the case of a "long" inverse limit, provided that the u.s.c. bonding functions are properly composing. (For samples of bonding functions $f : [0,1] \to 2^{[0,1]}$ satisfying $f \circ f = f$, see

Examples 10.10 and 10.11 in Chapter 10. We also discuss the long inverse limit spaces produced using these functions.)

Chapter 9

Two-Sided Inverse Limits

We turn now to another special case of an inverse limit on a totally ordered directed set. Suppose that for each integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. Then we define $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ to be the inverse limit space consisting of all points of form $\mathbf{x} = (x_i)_{i\in\mathbb{Z}} = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots)$, where $x_i \in f_i(x_{i+1})$ for each integer i, and a basis for the topology on the space is

$\{O \cap \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}} | \, O$ is basic open in $\prod_{i\in\mathbb{Z}} X_i\}$.

We will often call the space $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ a "two-sided" inverse limit, as opposed to the corresponding "one-sided" inverse limit indexed by the positive integers, $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. If each f_i is a continuous function, then the two-sided inverse limit is clearly homeomorphic to the standard one-sided one. However, if each f_i is u.s.c., then $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ may be different from $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. In this chapter, we investigate the relationship between the two-sided inverse limit and the ordinary one-sided one. The issue of indecomposability will play a role here as well.

Let us begin with two basic theorems that provide a sufficient condition for compactness and connectedness of the two-sided inverse limit. The proofs of these theorems are straightforward, but may be found in Chapter 5 of [14].

Theorem 9.1. *Suppose that, for each integer i, X_i is a compact Hausdorff space and $f_i : X_{i+1} \to 2^{X_i}$ is u.s.c. Then $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ is non-empty and compact.*

Theorem 9.2. *Suppose that, for each integer i, X_i is a Hausdorff continuum, $f_i : X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and for each x in X_{i+1}, $f_i(x)$ is connected. Then $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ is a Hausdorff continuum.*

The following examples show how the two-sided inverse limit may be different from the one-sided one.

Example 9.3. *For each integer i, let $X_i = [0,1]$ and let the graph of $f_i : [0,1] \to 2^{[0,1]}$ consist of the straight line segments joining $(0,0)$ to $(1,0)$ and $(0,0)$ to $(1,1)$. (This bonding function comes from Example 131 in [4]; see Figure 5.1.)*

First we consider the two-sided inverse limit. Let A_z be the set of all two-sided sequences of form $(\ldots, 0, 0, x, x, x, \ldots)$, where the leftmost x appears in the zth coordinate and $x \in [0,1]$. We note that A_z is an arc for each integer z. Let $A = \{(\ldots, x, x, x, \ldots)|\ x \in [0,1]\}$, so that A is also an arc. Thus, $(\bigcup_{z\in\mathbb{Z}} A_z) \cup A = \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$, and $(\bigcap_{z\in\mathbb{Z}} A_z) \cap A = (\ldots, 0, 0, 0, \ldots)$, a single point. Thus, $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ is a fan.

However, this fan is not homeomorphic to the fan given by the corresponding one-sided inverse limit, $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. For, as we will show, $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ contains a limit arc while $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ does not.

Consider the arc A in $\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ given by $\{(\ldots, x, x, x, \ldots)|\ x \in [0,1]\}$. We will prove that A consists entirely of limit points of $(\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}) \setminus A$. To that end, let $O = (\prod_{i\in\mathbb{Z}} O_i) \cap \varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}$ be a basic open set containing some point $\mathbf{x} = (\ldots, x, x, x, \ldots)$ of A, where $x \in [0,1]$ and $O_i = [0,1]$ for all but finitely many i. If for each i, $O_i = X_i$, then clearly O contains points not in A. So suppose O is a proper subset of the space. Since $O_i = [0,1]$ for all but finitely many i, there must be some least integer j for which $O_j \subsetneq X_j$, and some greatest integer k for which $O_k \subsetneq X_k$. If $x \neq 0$, then the sequence $(\ldots, 0, 0, \ldots, 0, x, x, x, \ldots)$, where the leftmost x lies in the jth coordinate, clearly lies in O. If $x = 0$, then the sequence $(\ldots, 0, 0, \ldots, 0, 1, 1, \ldots)$, where the leftmost 1 lies in the $k+1$th coordinate, must lie in O. Either way, O must contain a point in $(\varprojlim\{X_i, f_i\}_{i\in\mathbb{Z}}) \setminus A$, and thus, A is a limit arc.

On the other hand, the one-sided inverse limit $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$ contains no limit arc. To see this, let B be any arc that is a subset of $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, and let $\mathbf{x} = (0, 0, \ldots, 0, x, x, \ldots)$, where $x \neq 0$, be any non-endpoint of B. (Assume without loss of generality that the leftmost

x appearing in the sequence $\mathbf{x}$ lies in the jth coordinate.) Then, by the way the inverse limit is defined, there is some $[a,b] \subseteq (0,1)$ so that B contains an arc $C = \{\mathbf{y} = (0,0,\ldots,0,y,y,\ldots) \mid$ the leftmost y of the sequence $\mathbf{y}$ lies in the jth coordinate, and $y \in [a,b]\}$ that contains $\mathbf{x}$ as a non-endpoint. If $O_1 = [0,a/2), O_2 = [0,a/2), \ldots, O_{j-1} = [0,a/2)$, and $O_j = (a,b)$, then $(O_1 \times O_2 \times \cdots \times O_{j-1} \times O_j \times [0,1] \times \cdots) \cap (\varprojlim\{X_i, f_i\}_{i=1}^{\infty})$ is open, contains $\mathbf{x}$, but misses $(\varprojlim\{X_i, f_i\}_{i=1}^{\infty}) \setminus B$ entirely.

(See Figure 9.1.)

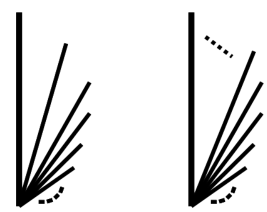

Figure 9.1: The spaces from Example 9.3. Left: one-sided inverse limit; right: two-sided.

Example 9.4. *For each integer i, let $X_i = [0,1]$ and let $f_i : [0,1] \to 2^{[0,1]}$ be defined by the graph consisting of the following straight line segments:*

i. For each integer $n \geq 0$, the segment joining the points $(1 - \frac{1}{2^n}, 1 - \frac{1}{2^n})$ and $(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$.

ii. For each integer $n \geq 0$, the segment joining the points $(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$ and $(1 - \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}})$. (See Figure 9.2.)

In this example, the one-sided inverse limit is just a (countable) union of countably many n-cells for each positive integer n, and thus, does not contain a Hilbert cube. However, the two-sided inverse limit does contain a Hilbert cube because it contains all points of form $(\ldots, d, \frac{7}{8}, c, \frac{3}{4}, b, \frac{1}{2}, a, 0, 0, \ldots)$, where $a \in [0, \frac{1}{2}], b \in [\frac{1}{2}, \frac{3}{4}], c \in [\frac{3}{4}, \frac{7}{8}]$, etc. Thus, the one-sided and two-sided inverse limits are not homeomorphic.

We note that, in Example 9.3, although the one-sided and two-sided inverse limits are not homeomorphic, at least the one-sided inverse limit may be embedded in the two-sided

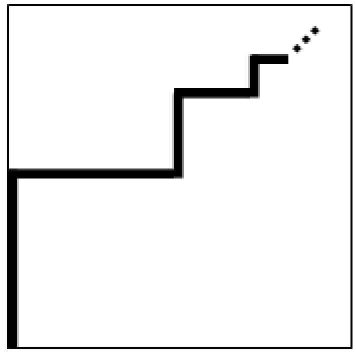

Figure 9.2: The bonding function from Example 9.4

one. However, we will soon see that this need not be the case in general. First, we need the following two theorems.

Theorem 9.5. *Suppose* $f : [0,1] \to 2^{[0,1]}$ *is a surjective u.s.c. function such that* f^{-1} *is a continuous map from* $[0,1]$ *to* $[0,1]$*. Then the two-sided inverse limit* $\varprojlim\{[0,1], f\}_{i\in\mathbb{Z}}$ *is homeomorphic to the one-sided inverse limit* $\varprojlim\{[0,1], f^{-1}\}_{i=1}^{\infty}$*.*

Proof. Since f^{-1} is continuous, $\varprojlim\{[0,1], f^{-1}\}_{i=1}^{\infty}$ and $\varprojlim\{[0,1], f^{-1}\}_{i\in\mathbb{Z}}$ are homeomorphic. So, it will suffice to show that $\varprojlim\{[0,1], f\}_{i\in\mathbb{Z}}$ is homeomorphic to $\varprojlim\{[0,1], f^{-1}\}_{i\in\mathbb{Z}}$. If $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \in \varprojlim\{[0,1], f^{-1}\}_{i\in\mathbb{Z}}$, let us define $h : \varprojlim\{[0,1], f^{-1}\}_{i\in\mathbb{Z}} \to \varprojlim\{[0,1], f\}_{i\in\mathbb{Z}}$ by $h((\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)) = (\ldots, x_2, x_1, x_0, x_{-1}, x_{-2}, \ldots)$. It is easy to see that h is a well-defined function, and that h is one-to-one and surjective. h is continuous because the preimage via h of any basic open set $\prod_{i\in\mathbb{Z}} O_i$ (where all but finitely many O_i are proper subsets of $[0,1]$) intersected with $\varprojlim\{[0,1], f\}_{i\in\mathbb{Z}}$ is $\prod_{i\in\mathbb{Z}} O_{-i}$ intersected with $\varprojlim\{[0,1], f^{-1}\}_{i\in\mathbb{Z}}$, which is open. An analogous argument shows that h^{-1} is continuous, so h is a homeomorphism. □

Theorem 9.6. *Suppose* $f : [0,1] \to 2^{[0,1]}$ *is a surjective u.s.c. function that passes the horizontal line test, i.e.,* $f^{-1}(x)$ *is degenerate for each* $x \in [0,1]$*. Then* $\varprojlim\{[0,1], f\}_{i=1}^{\infty}$ *(abbreviated by* $\varprojlim \boldsymbol{f}$*) is an arc.*

Proof. Since f is surjective and u.s.c., $\varprojlim \mathbf{f}$ is non-empty, non-degenerate and compact. Moreover, since $f^{-1}(x)$ is connected for each $x \in [0,1]$, Theorem 3.4 applies and $\varprojlim \mathbf{f}$ is a continuum. We will show $\varprojlim \mathbf{f}$ is an arc by showing it has exactly two non-cut points.

By assumption, $f^{-1}(x)$ is degenerate for each $x \in [0,1]$. Hence, because f is surjective, $\{(x, f^{-1}(x), f^{-2}(x), \ldots) | x \in [0,1]\}$ is the set of all points in $\varprojlim \mathbf{f}$. We note that each point $(x, f^{-1}(x), f^{-2}(x), \ldots)$ with $x \in (0,1)$ is a cut point of $\varprojlim \mathbf{f}$, since the set $\varprojlim \mathbf{f} \setminus \{(x, f^{-1}(x), f^{-2}(x), \ldots)\}$ may be separated by the two disjoint open sets $[0,x) \times \prod_{i=2}^{\infty}[0,1]$ and $(x,1] \times \prod_{i=2}^{\infty}[0,1]$. Every non-degenerate continuum contains at least two non-cut points; since each point $(x, f^{-1}(x), f^{-2}(x), \ldots)$ with $x \in (0,1)$ is a cut point of $\varprojlim \mathbf{f}$, only the points $(0, f^{-1}(0), \ldots)$ and $(1, f^{-1}(1), \ldots)$ are non-cut points of $\varprojlim \mathbf{f}$. That is, $\varprojlim \mathbf{f}$ is a continuum with exactly two non-cut points, so that $\varprojlim \mathbf{f}$ must be an arc. $\square$

Example 9.7. *Let $f : [0,1] \to 2^{[0,1]}$ be the inverse of the Henderson map.*

To analyze this example, we must first consider the Henderson map, a continuous function described in [2] and pictured in Figure 9.3. It is well-known that the one-sided inverse limit with the Henderson map as its single bonding function is the pseudo-arc, a hereditarily indecomposable continuum. Now let $f : [0,1] \to 2^{[0,1]}$ be the inverse of the Henderson map, as pictured in Figure 9.4. Since the Henderson map is continuous, it is also u.s.c., implying that its inverse, f, is u.s.c. as well. f is also surjective. Now, being a continuous function, the Henderson map passes the vertical line test; therefore, its inverse, f, passes the horizontal line test. That means, by Theorem 9.6, the one-sided inverse limit $\varprojlim\{[0,1], f\}_{i=1}^{\infty}$ is an arc. However, f^{-1} is the Henderson map, a continuous function from $[0,1]$ to $[0,1]$; thus, by Theorem 9.5, the two-sided inverse limit $\varprojlim\{[0,1], f\}_{i \in \mathbb{Z}}$ is homeomorphic to the one-sided inverse limit $\varprojlim\{[0,1], f^{-1}\}_{i=1}^{\infty}$, which is the pseudo-arc. Therefore, the one-sided inverse limit $\varprojlim\{[0,1], f\}_{i=1}^{\infty}$ is an arc, but the two-sided inverse limit $\varprojlim\{[0,1], f\}_{i \in \mathbb{Z}}$ is the pseudo-arc. It is noteworthy that, since the pseudo-arc is a hereditarily indecomposable continuum, the one-sided inverse limit cannot be embedded into the two-sided one (or vice-versa) in this example.

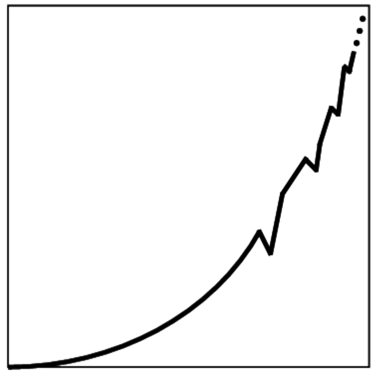

Figure 9.3: The Henderson map

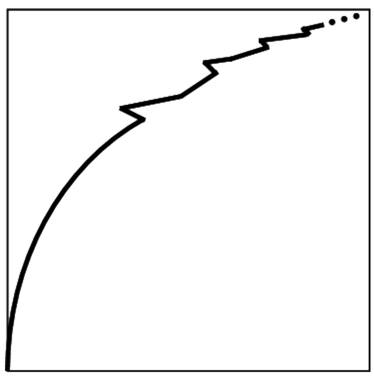

Figure 9.4: The bonding function f from Example 9.7

So, we have seen how different one-sided and two-sided u.s.c. inverse limits may be. Yet the following theorem shows that, in the case of inverse limits with a single bonding function, the one-sided inverse limit is a continuum iff the two-sided inverse limit is a continuum.

Theorem 9.8. *Suppose X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective u.s.c. function. Then $\varprojlim\{X, f\}_{i=1}^{\infty}$ is a continuum iff $\varprojlim\{X, f\}_{i\in\mathbb{Z}}$ is a continuum.*

Proof. If $\varprojlim\{X, f\}_{i\in\mathbb{Z}}$ is a continuum, then the projection of $\varprojlim\{X, f\}_{i\in\mathbb{Z}}$ onto the $1, 2, 3, \ldots$ coordinates, $\varprojlim\{X, f\}_{i=1}^{\infty}$, must be a continuum as well. On the other hand, suppose $\varprojlim\{X, f\}_{i=1}^{\infty}$ is a continuum. For each $j \in \mathbb{Z}$, let $K_j = \{\mathbf{x} \in \prod_{i\in\mathbb{Z}} X \mid x_i \in f_i(x_{i+1})\ \forall i \geq j\}$. Note that, for a given $j \in \mathbb{Z}$, K_j is homeomorphic to $(\prod_{i\in\mathbb{Z}, i<j} X) \times \varprojlim\{X, f\}_{i=j}^{\infty}$. However, $\varprojlim\{X, f\}_{i=j}^{\infty}$ is homeomorphic to $\varprojlim\{X, f\}_{i=1}^{\infty}$, a continuum; thus, K_j is homeomorphic to a product of continua and hence, K_j is a continuum. Now if $j < m$, then clearly $K_j \subseteq K_m$.

So, $\{K_j\}_{j\in\mathbb{Z}}$ is a monotonic collection of continua. This means $\bigcap_{j\in\mathbb{Z}} K_j$ is a continuum. However, $\bigcap_{j\in\mathbb{Z}} K_j = \varprojlim\{X, f\}_{i\in\mathbb{Z}}$, so the proof is complete. $\square$

Theorem 9.8 leads us to ask the following question: If the two-sided inverse limit with the single bonding function f is an indecomposable continuum, is the corresponding one-sided inverse limit also an indecomposable continuum? In light of Example 9.7, the answer is no. However, the converse is true, as we see in the following theorem:

Theorem 9.9. *Suppose X is a Hausdorff continuum and $f : X \to 2^X$ is a surjective u.s.c. bonding function so that $\varprojlim\{X, f\}_{i=1}^{\infty}$ is an indecomposable continuum. Then $\varprojlim\{X, f\}_{i\in\mathbb{Z}}$ is also an indecomposable continuum.*

Proof. Suppose by way of contradiction that $\varprojlim\{X, f\}_{i\in\mathbb{Z}} = H \cup K$, a union of two proper subcontinua. If S is a subset of $\prod_{i\in\mathbb{Z}} X$, we denote by $\pi_{\geq j}(S)$ the projection of S onto coordinates $j, j+1, j+2, \ldots$. Let us note that whenever S is a proper subcontinuum of $\varprojlim\{X, f\}_{i\in\mathbb{Z}}$, there must exist some integer j for which $\pi_{\geq j}(S)$ is a proper subset of $\varprojlim\{X, f\}_{i=j}^{\infty}$. For, otherwise, $\pi_{\geq j}(S) = \varprojlim\{X, f\}_{i=j}^{\infty}$ for all $j \in \mathbb{Z}$, and then, since S is closed, it may be shown that $S = \varprojlim\{X, f\}_{i\in\mathbb{Z}}$. That would be a contradiction; therefore, there exists some small enough integer j such that $\pi_{\geq j}(H)$ and $\pi_{\geq j}(K)$ are both proper subcontinua of $\varprojlim\{X, f\}_{i=j}^{\infty}$. Next, we note that since $\varprojlim\{X, f\}_{i\in\mathbb{Z}} = H \cup K$, it must follow that $\varprojlim\{X, f\}_{i=j}^{\infty} = \pi_{\geq j}(H) \cup \pi_{\geq j}(K)$. Thus, $\varprojlim\{X, f\}_{i=j}^{\infty}$ has been shown to be decomposable; however, $\varprojlim\{X, f\}_{i=j}^{\infty}$ is homeomorphic to $\varprojlim\{X, f\}_{i=1}^{\infty}$, which is indecomposable. (Contradiction.) $\square$

Using this result, it is easy to adjust the indecomposability theorems from the previous chapters to apply to the case of two-sided inverse limits. Indeed, for each example of an indecomposable one-sided inverse limit with a single u.s.c. bonding function, the two-sided version is an indecomposable continuum as well.

Chapter 10

Illustrative Examples

We now present various examples of inverse limits with u.s.c. functions $f_i : [0,1] \to 2^{[0,1]}$. Note that, in each example, the graphs described are always closed, so that (by Theorem 3.1) the resulting bonding functions are automatically u.s.c. (Some of these examples, or similar versions of them, were also presented in [13] or [4].)

Example 10.1. *Let the graph of* $f : [0,1] \to 2^{[0,1]}$ *consist of straight line segments joining points* $(0,0)$ *to* $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{4})$ *to* $(\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{4}, \frac{1}{4})$ *to* $(\frac{1}{2}, 0)$, *and* $(\frac{1}{2}, 0)$ *to* $(1,1)$. *(See Figure 10.1.)*

By Theorem 4.1, the inverse limit with the single bonding function f, $\varprojlim \mathbf{f}$, is a decomposable continuum. Possible choices of open sets U and V (as mentioned in the theorem) are indicated in the diagram.

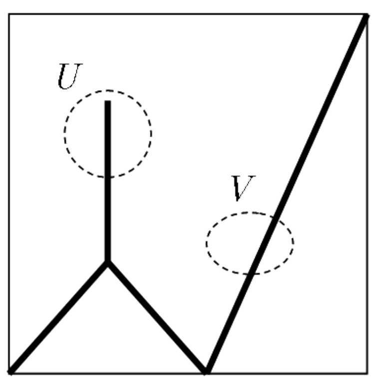

Figure 10.1: The graph of the function f from Example 10.1

Example 10.2. *For each positive integer* i, *let* $f_i : [0,1] \to 2^{[0,1]}$ *be the standard tent map, except for some* $f_j : [0,1] \to 2^{[0,1]}$, *whose graph consists of the standard tent map together with the line segment joining points* $(0,0)$ *and* $(0, \frac{1}{2})$. *(See Figure 10.2.)*

By Theorem 4.1, $\varprojlim \mathbf{f}$ is a decomposable continuum. Were f_j the standard tent map, of course, the inverse limit would be indecomposable. It is striking that such a small adjustment to just one bonding function can drastically alter the decomposability of the inverse limit. (Indeed, adding to the tent map the vertical line segment that joins $(0,0)$ to $(0,\epsilon)$ for any $\epsilon > 0$ would have had the same effect.) This example shows yet again the difficulty in finding a general "subsequence" theorem for u.s.c. inverse limits. (See also Example 3 in [3].)

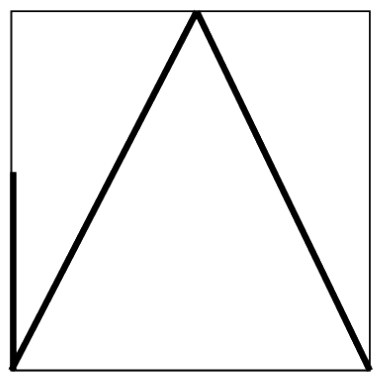

Figure 10.2: The graph of the function f_j from Example 10.2

Example 10.3. *Let the graph of* $f : [0,1] \to 2^{[0,1]}$ *be given by the straight line segments joining* $(0,0)$ *to* $(\frac{1}{3},\frac{1}{3})$*, from* $(\frac{1}{3},\frac{1}{3})$ *to* $(\frac{1}{3},\frac{2}{3})$*, from* $(\frac{1}{3},\frac{2}{3})$ *to* $(\frac{2}{3},\frac{2}{3})$*, and from* $(\frac{2}{3},\frac{2}{3})$ *to* $(1,1)$*. (See Figure 10.3.)*

Then $(\frac{1}{3},\frac{1}{3},\frac{1}{3},\ldots) \in \varprojlim \mathbf{f}$, $f^{-1}(\frac{1}{3}) = \frac{1}{3}$, and $U = [0,1] \times (\frac{1}{3},\frac{2}{3})$ is an open subset of $[0,1] \times [0,1]$ with $G(f) \cap U \subseteq \{\frac{1}{3}\} \times f(\frac{1}{3})$. Thus, Theorem 4.3 applies and $\varprojlim \mathbf{f}$ is a decomposable continuum.

Also, let us note that neither Theorem 4.1 nor Theorem 4.2 applies in this case. For, if U is any open subset of $G(f)$, $G(f) \setminus U$ is either not the graph of a u.s.c. function from $[0,1]$ into $2^{[0,1]}$ or it is the graph of a u.s.c. function that maps a point $(\frac{1}{3})$ to a disconnected set.

Example 10.4. *Let* $f : [0,1] \to 2^{[0,1]}$ *be defined by the graph that is the union of the traditional tent map and the reflection of the tent map about the line* $y = \frac{1}{2}$*. (See Figure 10.4.)*

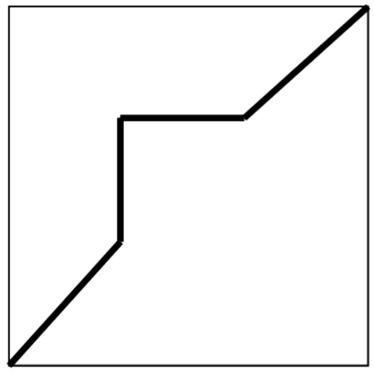

Figure 10.3: The graph of the function f from Example 10.3

Then by Theorem 4.4 (or by Corollary 4.5), $\varprojlim \mathbf{f}$ is a decomposable continuum. This space appears to contain a fan-like structure of bucket handle continua.

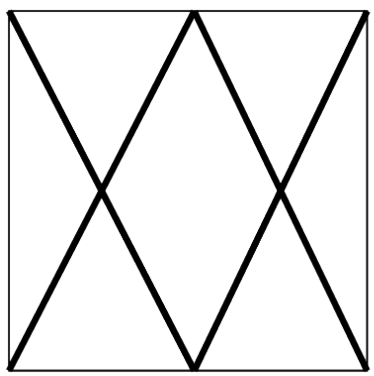

Figure 10.4: The graph of the function f from Example 10.4

Example 10.5. *Let the graph of $f : [0,1] \to 2^{[0,1]}$ be given by drawing straight line segments from $(0,0)$ to $(\frac{1}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{3})$ to $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{3}, \frac{2}{3})$ to $(\frac{1}{2}, 1)$, and then drawing the reflection of this figure about the line $x = \frac{1}{2}$. (See Figure 10.5.)*

Then f is a steeple with turning point $a = \frac{1}{2}$. So, by Theorem 5.4, $\varprojlim \mathbf{f}$ is homeomorphic to the bucket handle and thus, $\varprojlim \mathbf{f}$ is indecomposable.

Example 10.6. *Let the graph of $f : [0,1] \to 2^{[0,1]}$ be given by drawing straight line segments from $(0,0)$ to $(0, \frac{1}{3})$, $(0, \frac{1}{3})$ to $(\frac{1}{2}, 1)$ and then drawing the reflection of this figure about the line $x = \frac{1}{2}$. (See Figure 10.6.)*

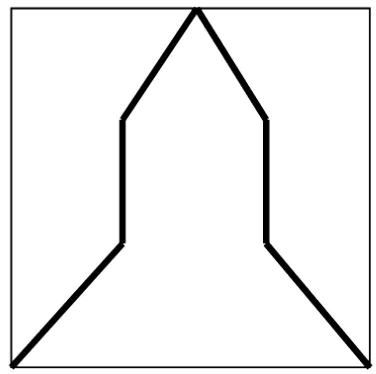

Figure 10.5: The graph of the function f from Example 10.5

Then $\varprojlim \mathbf{f}$ is decomposable by Theorem 4.1. (The open set U from Theorem 4.1 could be $[0, \frac{1}{4}) \times [0, \frac{1}{4})$.) Note that f is *not* a steeple function because $f(0)$ does not equal $\{0\}$.

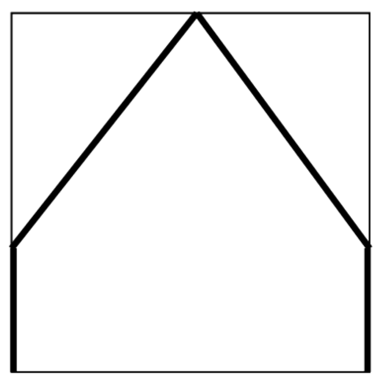

Figure 10.6: The graph of the function f from Example 10.6

Example 10.7. *Consider the u.s.c. function $f : [0,1] \to 2^{[0,1]}$ mentioned in Chapter 5 whose graph is topologically equivalent to a $sin(\frac{1}{x})$ curve. Specifically, let the graph of f consist of the following straight line segments:*

i. For each odd integer $n \geq 1$, the segment joining the points $(\frac{1}{2^n}, 0)$ and $(\frac{1}{2^{n-1}}, 1)$.

ii. For each even integer $n \geq 2$, the segment joining the points $(\frac{1}{2^n}, 1)$ and $(\frac{1}{2^{n-1}}, 0)$.

iii. The vertical line segment joining the points $(0,0)$ and $(0,1)$.

(See Figure 5.2.)

Then, by Theorem 5.2, $\varprojlim \mathbf{f}$ is an indecomposable continuum. The projection of $\varprojlim \mathbf{f}$ onto the first three factor spaces (i.e., inside $[0,1]^3$) is a countable sequence of $sin(\frac{1}{x})$ curves joined end to end and limiting to a $sin(\frac{1}{x})$ curve on the back face of the cube. (See Figure 10.7.) We note that $\varprojlim \mathbf{f}$ has a structure reminiscent of the indecomposable continua constructed by Michel Smith in [12].

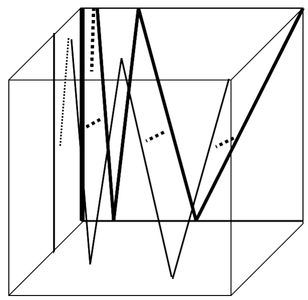

Figure 10.7: The projection of the inverse limit from Example 10.7 onto its first three factor spaces

Example 10.8. *Let the graph of* $f : [0,1] \to 2^{[0,1]}$ *be given by the u.s.c. function shown in Figure 10.8.*

We note that the set $A = \{0, 1\}$ is non-empty, closed, and nowhere dense; moreover, $f(0) = f(1) = [0,1]$. Let B be the subset of $[0,1] \setminus \{0,1\}$ consisting of points where $f|_{[0,1]\setminus\{0,1\}}$ is not differentiable. Note that B is closed in $[0,1] \setminus \{0,1\}$ and also nowhere dense. Next, note that $f|_{[0,1]\setminus(\{0,1\}\cup B)}$ is an open mapping. Furthermore, for each interval of form $(0,c)$ in $[0,1]$ and each y in the interval $(0,1)$, there exists some $x \in ([0,1] \setminus \{0,1\})$ with $f(x) = y$. The same statement is true for each interval of form $(c,1)$. Thus, the hypothesis of Theorem 6.14 is satisfied, and $\varprojlim \mathbf{f}$ is an indecomposable continuum.

Example 10.9. *Let* $f : [0,1] \to 2^{[0,1]}$ *be given by squeezing two copies of the steeple graph in Figure 10.5 into* $[0,1] \times [0,1]$, *as shown in Figure 10.9.*

If we take $a_1 = 0, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = \frac{3}{4}$, and $a_5 = 1$, then the hypothesis of Theorem 7.4 is satisfied and thus, by Corollary 7.5, $\varprojlim \mathbf{f}$ is an indecomposable continuum.

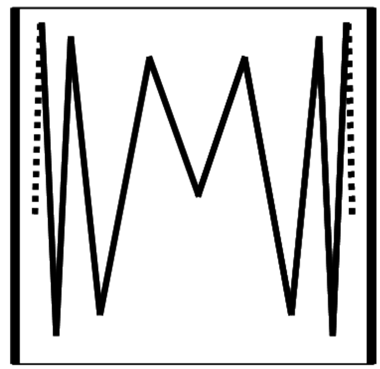

Figure 10.8: The graph of the function f from Example 10.8

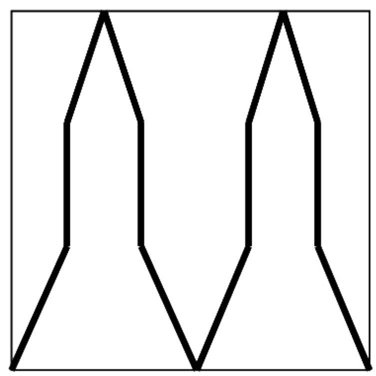

Figure 10.9: The graph of the function f from Example 10.9

Finally, we give some illustrative examples of inverse limits on various kinds of linearly ordered sets.

Example 10.10. *Let the graph of* $g : [0, 1] \to 2^{[0,1]}$ *be the union of the line segment joining the points* $(0, 0)$ *and* $(0, 1)$ *and the line segment joining the points* $(0, 0)$ *and* $(1, 1)$.

We note that $g^2(x) = g(x)$ for each $x \in [0, 1]$; that is, $g \circ g = g$. So, if $X_\alpha = [0, 1]$ for each $\alpha < \omega_1$, and $g_{\alpha,\beta} = g$ for each $\alpha < \beta < \omega_1$, we know that the collection of functions $\{g_{\alpha,\beta} : X_\beta \to C(X_\alpha)\}_{\alpha<\beta<\omega_1}$ is a properly composing collection of surjective u.s.c. functions and the "long" inverse limit is well-defined. Direct inspection reveals that the "short" inverse limit of g (denoted by $\varprojlim\{[0, 1], g, \omega\}$) is a fan with countably many legs and one limit arc. The long inverse limit of g, i.e., $\varprojlim\{[0, 1], g, \omega_1\}$, is a fan with uncountably many legs, where every leg corresponding to a limit ordinal is a limit arc of the legs corresponding to that ordinal's predecessors. (See Figure 10.10).

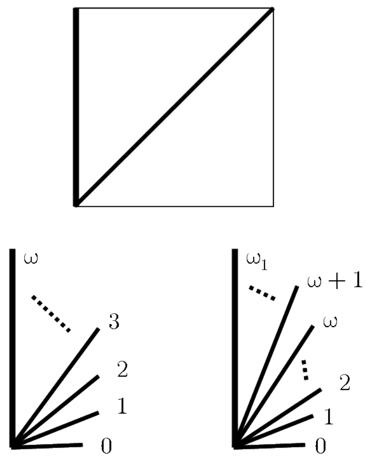

Figure 10.10: Counter-clockwise from top: The graph of g, the short inverse limit, the long inverse limit

Example 10.11. *Let the graph of* $h : [0,1] \to 2^{[0,1]}$ *be the union of the line segment joining the points* $(0,0)$ *and* $(0,1)$ *and the line segment joining the points* $(0,1)$ *and* $(1,1)$.

Then $h^2(x) = h(x)$ for each $x \in [0,1]$; i.e., $h \circ h = h$. Thus, as in Example 10.10, the long inverse limit (with $h_{\alpha,\beta} = h$ for all $\alpha < \beta < \omega_1$) is well-defined. The short inverse limit, $\varprojlim\{[0,1], h, \omega\}$, is homeomorphic to $[0,1]$; on the other hand, the long inverse limit, $\varprojlim\{[0,1], h, \omega_1\}$, is homeomorphic to the compactified long line, L. (See Figure 10.11.) We note also that, if the index set for the inverse limit with bonding function h is $[0,1]$, it may be shown directly that $\varprojlim\{[0,1], h, [0,1]\}$ is homeomorphic to the Lexicographic Arc.

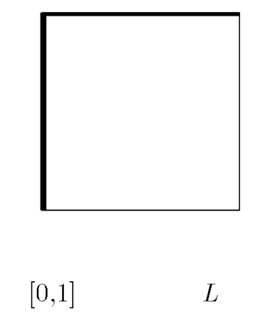

Figure 10.11: Counter-clockwise from top: The graph of h, the short inverse limit, the long inverse limit

Chapter 11

Possibilities For Future Research

Our mission has been to find necessary and/or sufficient conditions for decomposability or indecomposability of u.s.c. inverse limit spaces. The results we presented in [15], [16], and [13] helped pave the way for other mathematicians' work on this topic (e.g., in [5] and [17]). With this dissertation, we have further developed the theory of u.s.c. inverse limits and indecomposability. In particular, we have sought conditions for indecomposability that are very easy to check, simply by observing some basic characteristics of the bonding functions. We have given many such conditions, but plenty of work still remains to be done.

One of the major open problems in the theory of u.s.c. inverse limits is finding sufficient and/or necessary conditions, stated in terms of the bonding functions f_i, for $\varprojlim \mathbf{f}$ to be a continuum. As more such conditions are discovered, we will hopefully be able to modify these conditions to obtain new information about the decomposability or indecomposability of the inverse limit. For example, Van Nall's paper [11] contains many results that should be helpful to us, as well as the wealth of material in the work of Ingram and Mahavier [4]. In general, any new theorem about how u.s.c. inverse limits can generate continua may potentially lead to a similar theorem about u.s.c. inverse limits generating indecomposable (or decomposable) continua.

Thus far, we have mostly considered just the property of indecomposability; what about hereditary indecomposability? We know that there exist continuous bonding functions $f : [0, 1] \to [0, 1]$ such that $\varprojlim \mathbf{f}$ is a hereditarily indecomposable continuum (e.g., f could be the Henderson map). However, hereditarily indecomposable continua produced as inverse limits with u.s.c. bonding functions $f : [0, 1] \to 2^{[0,1]}$ that are not singleton-valued (i.e., cannot be identified with continuous functions) still remain to be studied. We have shown that if f is the inverse of the Henderson map, then the two-sided inverse limit $\varprojlim\{[0,1], f\}_{i\in\mathbb{Z}}$ is the pseudo-arc. However, we do not yet know of a non-singleton-valued surjective u.s.c. bonding function $f : [0,1] \to 2^{[0,1]}$ such that the one-sided inverse limit $\varprojlim\{[0,1], f\}_{i=1}^{\infty}$ is the pseudo-arc. (If such an f exists, it cannot be continuum-valued, for then $G(f)$ would contain some set of form $\{a\} \times [b,c], b < c$, implying that the inverse limit contains an arc.)

Aside from questions about indecomposability, the more general kinds of inverse limits seen in Chapters 8 and 9 are interesting in their own right and deserve to be studied further. The difficulty in obtaining examples of the "long" inverse limits described in Chapter 8 is finding collections of functions $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta<\omega_1}$ that compose properly, so that the long inverse limit $\varprojlim\{X_\alpha, f_{\alpha,\beta}, \omega_1\}$ is well-defined. We have noted that if $X_\alpha = X$ for each α, the surjective u.s.c. function $f : X \to 2^X$ satisfies $f = f \circ f$, and $f_{\alpha,\beta} = f$ for each $\alpha < \beta < \omega_1$, then $\{f_{\alpha,\beta} : X_\beta \to 2^{X_\alpha}\}_{\alpha<\beta<\omega_1}$ is automatically a properly composing collection of functions. Thus, it is of interest to find necessary and sufficient conditions for a u.s.c. function $f : X \to 2^X$ (or even just $f : [0,1] \to 2^{[0,1]}$) to satisfy $f \circ f = f$.

As far as the two-sided u.s.c. inverse limits are concerned, it would be helpful to have more theorems about the relationship between one-sided and two-sided inverse limits. For example, are there conditions on the bonding function $f : [0,1] \to 2^{[0,1]}$ that would guarantee the one-sided and two-sided inverse limits are (non)homeomorphic? Or that one space can be embedded into the other? Also, what topological spaces are homeomorphic to some two-sided inverse limit with a single surjective u.s.c. bonding function $f : [0,1] \to 2^{[0,1]}$? What spaces are homeomorphic to a two-sided u.s.c. inverse limit, but *not* a one-sided one?

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