Colorful Results on Euclidean Distance Graphs and Their Chromatic Numbers

by

Matt Noble

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Approved by

Peter D. Johnson, Chair, Professor of Mathematics and Statistics
Chris Rodger, C. Harry Knowles Professor of Mathematical Sciences
Dean Hoffman, Professor of Mathematics and Statistics

Abstract

In this work, we study Euclidean distance graphs with vertex set $\mathbb{Q}^n$, the n-dimensional rational space. In particular, we deal with the chromatic numbers (and some related parameters) of such graphs when $n = 2$ or $n = 3$. A short history of the topic is given before we approach a few open problems related to the subject. Some of these questions are resolved, either completely or partially. For the problems able to resist our advances, methods are given that hopefully will lead to answers in future work.

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Professionally, I would like to thank the Auburn University Department of Mathematics – in particular, Peter D. Johnson for being a great advisor and a stand-up guy in general.

Personally, I would like to thank Linda, Mark, Seth, and Abby. They know why.

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Chapter 1

Introduction

1.1 Definitions

The idea most fundamental to this work is that of the *distance graph*. Let $(\mathbf{X}, \rho)$ be a metric space where $\rho(x, y)$ denotes the distance between points $x, y \in \mathbf{X}$, and suppose that $D \subset (0, \infty)$. We define the distance graph $G(\mathbf{X}, D)$ as the graph with vertex set $\mathbf{X}$ where $x, y \in \mathbf{X}$ are adjacent if and only if $\rho(x, y) \in D$. We denote by $\chi(\mathbf{X}, D)$ the chromatic number of such a graph – that is to say, the minimum number of colors needed to color $\mathbf{X}$ such that no two adjacent vertices receive the same color. In the parlance associated with the subject, it is said that such a coloring "forbids" the set of distances D. In the case where $D = \{d\}$ for some $d > 0$, we will often refer to the associated graph as a single-distance graph and write $G(\mathbf{X}, d)$ instead of the technically consistent but more awkward-looking $G(\mathbf{X}, \{d\})$.

As is standard, for any set $\mathbf{X}$ and positive integer n, we will write $\mathbf{X}^n$ for the set of all n-tuples whose individual entries are elements of $\mathbf{X}$. Throughout this work, we will continue the tradition of using $\mathbb{R}$ and $\mathbb{Q}$ to denote the fields of real numbers and rational numbers respectively, and $\mathbb{Z}$ to denote the ring of integers. For any $\mathbf{X} \subset \mathbb{R}$, we designate by $\mathbf{X}^+$ all $x \in \mathbf{X}$ such that $x > 0$. One bit of notation that is perhaps non-standard is the description of vectors in the "arrowed bracket" component form instead of the more prevalent "**i**, **j**, **k** unit vector" form. Precisely defined, for any $v_1, ..., v_n \in \mathbb{R}$, $< v_1, ..., v_n >$ will be the vector with the origin as its initial point and $(v_1, ..., v_n)$ as its terminal point.

In almost all publications concerning the chromatic numbers of distance graphs, the distance metric ρ is kept the same throughout the article. Indeed, that will be the case here. Throughout this work, for any distance graph mentioned, the distance function used will be the usual Euclidean distance metric. In other words, if $\mathbf{X} \subset \mathbb{R}^n$ for some positive integer n and $x, y \in \mathbf{X}$ where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, $\rho(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$. Occasionally, it may be convenient to express this distance $\rho(x, y)$ using the alternate notation $|x - y|$.

The above notation and definitions provide a nice starting point for acquaintance with the subject of Euclidean distance graph coloring. Rest assured that they will be omnipresent in this work. However, a few concepts such as the Babai numbers, clique numbers, and some terms from number theory will appear in an individual chapter and then not be needed again. We will save the introduction of these terms until the appropriate time.

1.2 History

The most comprehensive investigation into the history of Euclidean distance graph coloring was done by Soifer in [26]. A full investigation into the research (both past and present) of Euclidean distance graphs on rational points is given in Johnson's compendium work [14]. These are the facts of the case.

In 1950 Edward Nelson, then an undergraduate at the University of Chicago, posed the following question to fellow student John Isbell:

> "What is the minimum number of colors needed to color the real plane such that no two points a distance 1 apart receive the same color?"

The problem has come to be known as the "chromatic number of the plane". In the notation given in the previous section, Nelson's question amounts to determining $\chi(\mathbb{R}^2, 1)$. Nelson himself proved that $4 \leq \chi(\mathbb{R}^2, 1)$ and Isbell, applying some ideas of Hugo Hadwiger [10], showed that $\chi(\mathbb{R}^2, 1) \leq 7$. Amazingly and frustratingly, these bounds have not been improved.

As time marched on, the chromatic number of the plane problem was slowly disseminated through the mathematical landscape. Major figures in its dispersal include Martin

Gardner [9], who made note of the problem in *Scientific American*, the brothers Leo and Willy Moser [20], who provided the simplest proof that $4 \leq \chi(\mathbb{R}^2, 1)$ with the creation of the Moser spindle (see Figure 1.1 at the end of this chapter), and the great Paul Erdős, who served to popularize the subject of Euclidean distance graph coloring with frequent mentions in print and in lectures. In any account, Nelson's problem had become well-known by the late 1960s.

In 1973, Douglas Woodall [27] added a new wrinkle to the fledgling industry of coloring Euclidean distance graphs. In an article primarily concerned with coloring $\mathbb{R}^2$, he included a relatively simple proof showing that $\chi(\mathbb{Q}^2, 1) = 2$. Although he never published anything else on the topic of coloring the rationals, Woodall's result seems to be the first instance of anyone considering the chromatic number of a distance graph with vertex set $\mathbb{Q}^n$ for any positive integer n.

A few years later, Miro Benda and Micha Perles made a much larger contribution to the subject with the creation of *Colorings of Metric Spaces*, the most influential article ever written on coloring $\mathbb{Q}^n$. Note that the last sentence contains the word "creation" instead of "publication" as Benda and Perles' work did not appear in print until the year 2000 [2]. Even as such, copies of the manuscript were widely circulated through the mathematical community, and by the mid-1980s had achieved something of a legendary status (an account of its curious history is given in [16]). Major findings of *Colorings* include the facts that $\chi(\mathbb{Q}^3, 1) = 2$ and $\chi(\mathbb{Q}^4, 1) = 4$. However, the most important element of Benda and Perles' work is the algebraic methods they used to obtain their results. These methods served to inspire Joseph Zaks, Kiran Chilakamarri, and Peter Johnson, each of whom published extensively on the subject and whose work has brought us where we are today.

At present, I would guess that there have been over a hundred scholarly articles written solely on the topic of coloring the rational space $\mathbb{Q}^n$. Yet with all this work, there are still more open questions than there are questions that have been answered. In this dissertation, we will tackle a few of these open questions.

1.3 Outline

Chapter 2 will be an investigation into the chromatic numbers of single-distance graphs with vertex set $\mathbb{Q}^3$. We will address the open question of determining $\chi(\mathbb{Q}^3, d)$ for arbitrary values of d leading to the result that for all odd, positive, square-free integers n whose prime factorization contains no factors congruent to 2 (mod 3), $\chi(\mathbb{Q}^3, \sqrt{2n}) = 4$. As stated, this result may seem somewhat narrow, but combined with previous work it actually goes a long way toward fully resolving this open problem.

In Chapter 3, we study distance graphs of the form $G(\mathbb{Q}^2, D)$ where the set of distances D is of size 2. Sufficient conditions are given on the set D for $\chi(\mathbb{Q}^2, D) = 3$. This answers in the affirmative an existence question posed by Johnson in [14]. We then close the chapter with two additional questions related to such graphs along with a partial solution.

Chapter 4 will consist of the study of two values closely related to the chromatic number of a Euclidean distance graph. We will first define the nth Babai number and the nth clique number of a metric space $(\mathbf{X}, \rho)$ and then use previous results of several different authors to conclude that the second clique number of $\mathbb{Q}^3$ is 6 which in turn implies that the second Babai number of $\mathbb{Q}^3$ is greater than or equal to 6. A few observations are then made concerning a proper coloring of the graph $G(\mathbb{Q}^3, \{2, \sqrt{2}\})$ which hopefully may be used to further increase the lower bound for the second Babai number of $\mathbb{Q}^3$.

Chapter 5 will act as something of a catch-all section. We will begin by revisiting the problem first looked at in Chapter 2, that of determining $\chi(\mathbb{Q}^3, d)$ for an arbitrary distance d, and illustrating the difficulties present when d is of the form $\sqrt{2n}$ for some odd, positive, square-free integer n which contains at least one prime factor congruent to 2 (mod 3). Unfortunately (or fortunately, depending on your point of view), this will necessitate the introduction of several terms and ideas from classical number theory. Once familiar with the basic facts and language of quadratic residues, we will demonstrate how Legendre's Theorem may be used in conjunction with lemmata from graph theory and geometry as an aid to find subgraphs of $G(\mathbb{Q}^3, \sqrt{2n})$ which have chromatic number 4. We close by putting all these pieces together in the form of a search algorithm which hopefully can be used to determine $\chi(\mathbb{Q}^3, \sqrt{10})$.

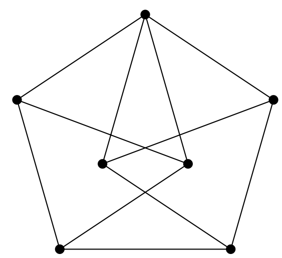

Figure 1.1: The Moser Spindle, a 4-chromatic subgraph of $G(\mathbb{R}^2, 1)$

Chapter 2

Single-distance Graphs in $\mathbb{Q}^3$

2.1 Introduction and Preliminaries

In this chapter, we study single-distance graphs with vertex set $\mathbb{Q}^3$ and address the following question:

$$\text{Given arbitrary distance } d,\ \chi(\mathbb{Q}^3, d) = \underline{\qquad}\ ?$$

Before attempting an answer, we observe a lemma which will considerably lighten our workload.

Lemma 2.1 *For any $q \in \mathbb{Q}^+$, $D \subset (0, \infty)$, and positive integer n, $\chi(\mathbb{Q}^n, D) = \chi(\mathbb{Q}^n, qD)$.*

Proof Consider the function $f : \mathbb{Q}^n \to \mathbb{Q}^n$ where for any $x \in \mathbb{Q}^n$ such that $x = (x_1, ..., x_n)$, $f(x) = (qx_1, ..., qx_n)$. Certainly f is an isomorphism between the vertex set of $G(\mathbb{Q}^n, D)$ and the vertex set of $G(\mathbb{Q}^n, qD)$. For any $x, y \in \mathbb{Q}^n$ and $d > 0$, $|x - y| = d$ if and only if $|f(x) - f(y)| = qd$. Thus the graphs $G(\mathbb{Q}^n, D)$ and $G(\mathbb{Q}^n, qD)$ are isomorphic and we have that $\chi(\mathbb{Q}^n, D) = \chi(\mathbb{Q}^n, qD)$. $\square$

Lemma 2.1 is a relatively simple observation (it was seen without proof in [14], [13], and probably elsewhere as well), but it is of vital importance to the study of distance graphs on rational points. With this in mind, and considering the fact that any distance realized between points of $\mathbb{Q}^3$ is of the form $\sqrt{q}$ for some positive $q \in \mathbb{Q}$, the original question can be completely resolved by answering the following:

$$\text{Given any square-free positive integer } z,\ \chi(\mathbb{Q}^3, \sqrt{z}) = \underline{\qquad}\ ?$$

Of course, for any $d_1, d_2 > 0$ where d_1 and d_2 are not rational multiples of each other, Lemma 2.1 does not apply and we have no immediate guarantee that $\chi(\mathbb{Q}^3, d_1) = \chi(\mathbb{Q}^3, d_2)$. It is an interesting quirk of mathematical history that this fact went seemingly unnoticed for quite some time. As mentioned in the previous chapter, Woodall [27] in 1973 made the first foray into coloring the rationals by showing that $\chi(\mathbb{Q}^2, 1) = 2$. Benda and Perles [2] also focused on the unit distance, showing that $\chi(\mathbb{Q}^3, 1) = 2$ and $\chi(\mathbb{Q}^4, 1) = 4$. Although they did not publish their findings until much later, the results of Benda and Perles work were well-known in the late 1970s. It was not until 1989 that Johnson [17] strayed from the unit-distance pack with the following theorem.

Theorem 2.1 *Let* $D_0 = \{\sqrt{\frac{p}{q}} : p, q$ *are both odd positive integers*$\}$. *Then* $\chi(\mathbb{Q}^3, D_0) = 2$.

Johnson's result is much stronger than what is needed for our current discussion; however it does immediately give us that for any odd positive integer z such that $\sqrt{z}$ is a distance actually realized in $\mathbb{Q}^3$, $\chi(\mathbb{Q}^3, \sqrt{z}) = 2$. In a 1993 paper [6], Chow characterized all single-distance graphs with vertex set $\mathbb{Q}^3$ which could not be properly 2-colored. Crucial to his work was the following lemma for which we provide an alternate proof.

Lemma 2.2 *Let* $n \in \mathbb{Z}^+$ *where* $n \equiv 2 \pmod 4$. *Then* $\chi(\mathbb{Z}^3, \sqrt{n}) \geq 3$.

Proof We will use the elementary graph theory fact that a simple graph has chromatic number greater than or equal to 3 if and only if it contains an odd cycle. Suppose $n = 2d$ for some odd positive integer d. By a well-known characterization of integers representable as the sum of three squares (see [24]), we have that there exist integers a, b, c such that $a^2 + b^2 + c^2 = n$. Since n is even but not divisible by 4, it must be that exactly two of a, b, c are odd. Without loss of generality, we may assume that a and b are both odd and that $a, b, c \geq 0$.

Now consider the vector $V =< a, b, c >$. We will create an odd number of vectors, each of whose entries consist of rearrangements or negatives of the entries of V, which together

$$\begin{array}{rl} b & \left\{\begin{array}{c} < a, b, c > \\ \vdots \\ < a, b, c > \end{array}\right. \\ a-1 & \left\{\begin{array}{c} < -b, a, c > \\ \vdots \\ < -b, a, c > \end{array}\right. \\ & \quad < -b, c, a > \\ & + < c, b, a > \\ \cline{2-2} \end{array}$$

Figure 2.1: Summing an odd number of length $\sqrt{n}$ vectors

sum to the zero vector. We begin with b copies of the vector $< a, b, c >$, $a - 1$ copies of the vector $< -b, a, c >$, and single copies of the vectors $< -b, c, a >$ and $< c, b, a >$ as shown in Figure 2.1. The total number of vectors is $a+b+1$ which is odd. Note that the x-component entries of these vectors together sum to c. Note also that the y-component entries of these vectors consist of one "c" entry, an even number of "a" entries, and an even number of "b" entries. Replace half of the "a" entries with "$-a$" and half of the "b" entries with "$-b$". Now the y-component entries together sum to c. Similarly, we may replace $\frac{a+b-2}{2}$ of the z-component "c" entries with "$-c$" and one of the two z-component "a" entries with "$-a$". Now the z-component entries together sum to c as well. So we are left with an odd number of length $\sqrt{n}$ vectors which together sum to $< c, c, c >$. We may easily create an odd number of length $\sqrt{n}$ vectors which together sum to $< -c, -c, -c >$ by multiplying every entry in the previous collection of vectors by -1.

Putting these facts together, we have that for any odd $z \in \mathbb{Z}$, it is possible to create an odd number of length $\sqrt{n}$ vectors that sum to $< zc, zc, zc >$ and whose x-components, y-components, and z-components each have an arbitrarily large number of "a" entries. So use the above process to construct an odd number of length $\sqrt{n}$ vectors that sum to $< ac, ac, ac >$ and whose x-, y-, and z-components each have at least $\frac{c}{2}$ "a" entries. Now replace those $\frac{c}{2}$ "a" entries in the x-, y-, and z-components with "$-a$" giving us an odd number of length $\sqrt{n}$ vectors that sum to $< 0, 0, 0 >$. $\square$

In a 2007 article [13], Johnson, Schneider, and Tiemeyer provided a nice upper bound for the chromatic numbers in question. This bound was achieved by computing the first Babai number of $\mathbb{Q}^3$, but we intend to delay the introduction of the Babai numbers until Chapter 4 and rephrase their result in terms more appropriate to the matter at hand with the following theorem.

Theorem 2.2 *For any $d > 0$, $\chi(\mathbb{Q}^3, d) \leq 4$.*

In light of Lemmas 2.1 and 2.2 and Theorems 2.1 and 2.2, the original question is narrowed down to the following:

Given any odd, positive, square-free integer p,
does $\chi(\mathbb{Q}^3, \sqrt{2p}) = 3$ or does $\chi(\mathbb{Q}^3, \sqrt{2p}) = 4$?

The main result of this chapter is that for all such values of p whose prime factorization contains no factors congruent to $2 \pmod 3$, $\chi(\mathbb{Q}^3, \sqrt{2p}) = 4$. We obtain this conclusion by proving two slightly stronger results on coloring $\mathbb{Z}^3$. We will need the help of the following lemmata.

Lemma 2.3 *Let $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b, c) = 1$ and $a + b + c \equiv 0 \pmod 2$. Let V be the group of vectors generated by $< \pm a, \pm b, \pm c >$, $< \pm a, \pm c, \pm b >$, $< \pm b, \pm a, \pm c >$, $< \pm b, \pm c, \pm a >$, $< \pm c, \pm a, \pm b >$, $< \pm c, \pm b, \pm a >$ under the usual vector addition. Then $V = \{< x, y, z >: x, y, z \in \mathbb{Z}$ and $x + y + z \equiv 0 \pmod 2\}$.*

Proof Clearly $V \subseteq \{< x, y, z >: x, y, z \in \mathbb{Z}$ and $x+y+z \equiv 0 \pmod 2\}$. Since $\gcd(a, b, c) = 1$ and $a + b + c \equiv 0 \pmod 2$, it must be the case that exactly two of a, b, c are odd. Note that if $< v_1, v_2, v_3 > \in V$ and σ is any permutation of the set $\{v_1, v_2, v_3\}$, then $< \pm\sigma(v_1), \pm\sigma(v_2), \pm\sigma(v_3) > \in V$. Note also that for any even integers m, n, and p, $< ma, 0, 0 >$, $< nb, 0, 0 >$, $< pc, 0, 0 > \in V$. Since $\gcd(a, b, c) = 1$, there exist integers r, s, t such that $ra + sb + tc = 1$. But this gives us $< 2ra, 0, 0 > + < 2sb, 0, 0 > + < 2tc, 0, 0 > =$

$< 2,0,0 >$. Thus $< \pm 2,0,0 >$, $< 0,\pm 2,0 >$, $< 0,0,\pm 2 > \in V$. So given any vector $< x,y,z >$ satisfying $x,y,z \in \mathbb{Z}$ and $x+y+z \equiv 0 \pmod 2$, we can select $< 0,0,0 >$ or an appropriate vector from those used to generate V and add to it some combination of $< \pm 2,0,0 >$,$< 0,\pm 2,0 >$,$< 0,0,\pm 2 >$ to construct $< x,y,z >$. $\square$

Lemma 2.4 *A square-free positive integer n can be represented as $n = a^2+ab+b^2$ for some $a,b \in \mathbb{Z}$ if and only if n contains no prime factor congruent to $2\,(mod 3)$.*

Ionascu [12] attributes this result to Euler. However, the author's own efforts to track down its genesis were ultimately unsuccessful. In any case, this lemma appears to be a fairly well-known fact and can be gleaned from the material presented in most beginning number theory textbooks. See Chapter 4 of [4] for a thorough treatment on representations of integers using binary quadratic forms.

Lemma 2.5 *Let n be a square-free positive integer which contains no prime factor congruent to $2\,(mod 3)$. Then there exists $a,b,c \in \mathbb{Z}$ such that*
$a^2+b^2+c^2 = 2n$ and $a+b+c = 0$.

Proof Given some n as described above, by Lemma 2.4 there exists $a,b \in \mathbb{Z}$ such that
$a^2+ab+b^2 = n$.
$\Rightarrow 2a^2+2ab+2b^2 = 2n$
$\Rightarrow a^2+b^2+(-a-b)^2 = 2n$.
Now if we let $c = -a-b$ we have that $a^2+b^2+c^2 = 2n$ and $a+b+c = 0$. $\square$

2.2 Results

Throughout this section, we will designate by S the set of all odd, positive, square-free integers whose prime factorization consists solely of factors congruent to $1\,(\mathrm{mod}\,3)$.

Theorem 2.3 *For every $s \in S$, $\chi(\mathbb{Z}^3, \sqrt{2s}) = 4$.*

Proof Let $s \in S$. As $\chi(\mathbb{Q}^3, \sqrt{2s}) \leq 4$ [13], clearly $\chi(\mathbb{Z}^3, \sqrt{2s}) \leq 4$ as well. By Lemma 2.5 there exists $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = 2s$ and $a + b + c = 0$. Thus the vectors $< a, b, c >$, $< b, c, a >$, and $< c, a, b >$ each have length $\sqrt{2s}$ and together sum to the zero vector. We can use these vectors to create the equilateral triangles in Figure 2.2, each with side length $\sqrt{2s}$ and vertices in $\mathbb{Z}^3$.

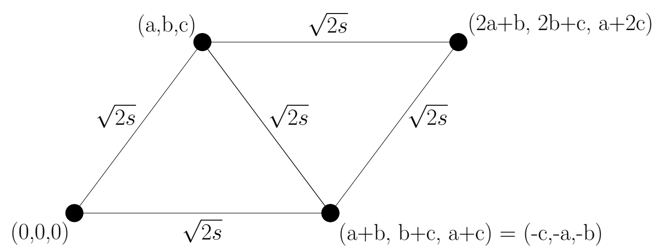

Figure 2.2: Pair of triangles in $G(\mathbb{Z}^3, \sqrt{2s})$

We will now assume there exists some proper 3-coloring of $G(\mathbb{Z}^3, \sqrt{2s})$ and obtain a contradiction. The above diagram shows that in any proper 3-coloring of $G(\mathbb{Z}^3, \sqrt{2s})$, the vertices $(0, 0, 0)$ and $(2a + b, 2b + c, a + 2c)$ must receive the same color. Furthermore, any arrow in $\mathbb{Z}^3$ representing the vector $< 2a + b, 2b + c, a + 2c >$ must have initial and terminal point colored the same color. By permuting the entries of (a, b, c) and $(a + b, b + c, a + c)$ using the same permutation or by replacing corresponding entries of (a, b, c) and $(a+b, b+c, a+c)$ with the negatives of those entries, it can be seen that each of the vectors $< \pm(2a+b), \pm(2b+c), \pm(a+2c) >$, $< \pm(2a+b), \pm(a+2c), \pm(2b+c) >$, $< \pm(2b+c), \pm(2a+b), \pm(a + 2c) >$, $< \pm(2b + c), \pm(a + 2c), \pm(2a + b) >$, $< \pm(a + 2c), \pm(2a + b), \pm(2b + c) >$, $< \pm(a + 2c), \pm(2b + c), \pm(2a + b) >$ must also have initial and terminal point colored the same color. Let V be the group of $\mathbb{Z}^3$ vectors generated under vector addition by these vectors. In any proper 3-coloring of $G(\mathbb{Z}^3, \sqrt{2s})$, any vector in V must have initial and terminal point colored the same color. Note that $(2a + b)^2 + (2b + c)^2 + (a + 2c)^2 = 6s$ and since $6s \equiv 2 \pmod 4$, it must be the case that exactly two of $(2a + b), (2b + c), (a + 2c)$ are odd. Also, since $6s$ is square-free, $\gcd(2a + b, 2b + c, a + 2c) = 1$. Then by Lemma 2.3, $V = \{< x, y, z >: x, y, z \in \mathbb{Z}$ and $x + y + z \equiv 0 \pmod 2\}$. This means that $< a, b, c > \in V$

and consequently that $(0,0,0)$ and (a,b,c) must be colored the same color which is our desired contradiction. Hence $\chi(\mathbb{Z}^3, \sqrt{2s}) = 4$. $\square$

It would be nice to complete the proof of our main result by focusing on the chromatic number of $G(\mathbb{Z}^3, \sqrt{6s})$ but that simply will not work, as evidenced by the following intermediate result.

Theorem 2.4 *For every $s \in S$, $\chi(\mathbb{Z}^3, \sqrt{6s}) = 3$.*

Proof Let $s \in S$. By Lemma 2.5, there exist $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 + c^2 = 6s$ and $a+b+c = 0$. Just as in the proof of Theorem 2.3, we can use the vectors $< a, b, c >$, $< b, c, a >$, and $< c, a, b >$ to create a 3-cycle in $G(\mathbb{Z}^3, \sqrt{6s})$, thus ensuring that $\chi(\mathbb{Z}^3, \sqrt{6s}) \geq 3$. Let $X, Y \in \mathbb{Z}^3$ where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. Let $\Delta_i = x_i - y_i$ for $i \in \{1, 2, 3\}$ and suppose that X and Y are adjacent in $G(\mathbb{Z}^3, \sqrt{6s})$. Then $(\Delta_1)^2 + (\Delta_2)^2 + (\Delta_3)^2 = 6s$. This means that $(\Delta_1)^2 + (\Delta_2)^2 + (\Delta_3)^2 \equiv 0 \pmod 3$ but $(\Delta_1)^2 + (\Delta_2)^2 + (\Delta_3)^2 \not\equiv 0 \pmod 9$, which in turn implies that $\Delta_i \not\equiv 0 \pmod 3$ for $i \in \{1, 2, 3\}$. (Here we are using the very basic fact that given any integer n such that $n \not\equiv 0 \pmod 3$, $n^2 \equiv 1 \pmod 3$.)
So to properly 3-color $G(\mathbb{Z}^3, \sqrt{6s})$, we need only use $\varphi : \mathbb{Z}^3 \to \{0, 1, 2\}$ where for every $X = (x_1, x_2, x_3)$, $\varphi(X) = x_1 \pmod 3$. $\square$

Theorem 2.5 *For every $s \in S$, $\chi(\mathbb{Z}^3, 3\sqrt{6s}) = 4$.*

Proof Let $s \in S$. Again, by the results of [13] we clearly have that $\chi(\mathbb{Z}^3, 3\sqrt{6s}) \leq 4$. By Lemma 2.4 and 2.5, there exist $a, b, c \in \mathbb{Z}$ such that $a^2 + ab + b^2 = 3s$, $a^2 + b^2 + c^2 = 6s$, and $a + b + c = 0$. From this we get two facts. First notice that $(3a)^2 + (3b)^2 + (3c)^2 = 54s$ implying that the vertices $(0,0,0)$ and $(3a, 3b, 3c)$ are adjacent in $G(\mathbb{Z}^3, 3\sqrt{6s})$. Secondly, the following points define an equilateral triangle with side length $3\sqrt{6s}$ and vertices in $\mathbb{Z}^3$ as given in Figure 2.3.

Note: This triangle is obtained by a slight modification to the equilateral triangle parameterizations given in [12].

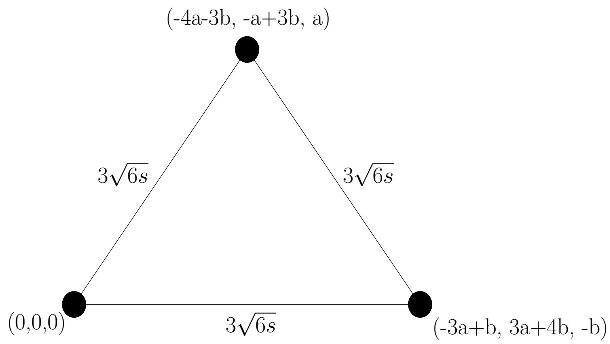

Figure 2.3: Alternate triangle parameterization in $G(\mathbb{Z}^3, 3\sqrt{6s})$

We can extend this figure into three equilateral triangles each with side length $3\sqrt{6s}$ and vertices in $\mathbb{Z}^3$ as in Figure 2.4.

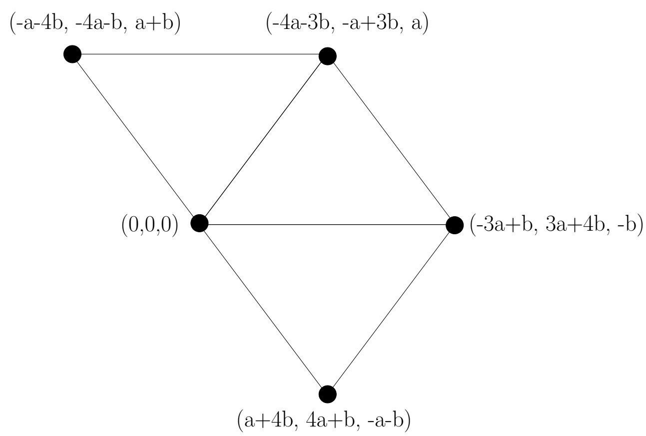

Figure 2.4: Three triangles in $G(\mathbb{Z}^3, 3\sqrt{6s})$

Let V_1 be the vector with initial point $(a + 4b, 4a + b, -a - b)$ and terminal point $(-4a - 3b, -a + 3b, a)$. Let V_2 be the vector with initial point $(-3a + b, 3a + 4b, -b)$ and terminal point $(-a - 4b, -4a - b, a + b)$. Using the same strategy as that in the proof of Theorem 2.3, we will again suppose that $\chi(\mathbb{Z}^3, 3\sqrt{6s}) = 3$ and obtain a contradiction. In any proper 3-coloring of $G(\mathbb{Z}^3, 3\sqrt{6s})$ the initial and terminal point of V_1 must be colored the same color and the initial and terminal point of V_2 must be colored the same color. Writing V_1 and V_2 in component form we have that

$$V_1 =< -5a - 7b, -5a + 2b, 2a + b >$$

$$V_2 =< 2a - 5b, -7a - 5b, a + 2b >$$

a	b	Vector whose entries are each not congruent to $0 \pmod 9$
$1 \pmod 9$	$1 \pmod 9$	V_2
$1 \pmod 9$	$4 \pmod 9$	V_1
$1 \pmod 9$	$7 \pmod 9$	V_2
$4 \pmod 9$	$1 \pmod 9$	V_2
$4 \pmod 9$	$4 \pmod 9$	V_1
$4 \pmod 9$	$7 \pmod 9$	V_1
$7 \pmod 9$	$1 \pmod 9$	V_1
$7 \pmod 9$	$4 \pmod 9$	V_2
$7 \pmod 9$	$7 \pmod 9$	V_1
$2 \pmod 9$	$2 \pmod 9$	V_1
$2 \pmod 9$	$5 \pmod 9$	V_2
$2 \pmod 9$	$8 \pmod 9$	V_1
$5 \pmod 9$	$2 \pmod 9$	V_1
$5 \pmod 9$	$5 \pmod 9$	V_1
$5 \pmod 9$	$8 \pmod 9$	V_2
$8 \pmod 9$	$2 \pmod 9$	V_2
$8 \pmod 9$	$5 \pmod 9$	V_1
$8 \pmod 9$	$8 \pmod 9$	V_1

Table 2.1: List of congruences modulo 9

We know that $a^2 + ab + b^2 = 3s$. This implies that either a and b are both congruent to 1 $\pmod 3$ or a and b are both congruent to $2 \pmod 3$. Therefore, since $a \equiv b \pmod 3$, the individual entries of V_1 and of V_2 are each congruent to $0 \pmod 3$. We now desire to show that at least one of V_1 and V_2 has all three entries not congruent to $0 \pmod 9$. This can be done through simple inspection. Table 2.1 lists all possible congruences modulo 9 for a and b along with a vector, either V_1 or V_2, whose individual entries are each not congruent to 0 $\pmod 9$.

With this information in mind, and also considering that V_1 and V_2 both have length $9\sqrt{2s}$, it must be the case that at least one of V_1, V_2 can be written as $< 3x, 3y, 3z >$ for some $x, y, z \in \mathbb{Z}$ where $x + y + z \equiv 0 \pmod 2$. Note also that $\gcd(x, y, z) = 1$, as s is odd and square-free. Again using the same ideas as in the proof of Theorem 2.3, we have that in any proper 3-coloring of $G(\mathbb{Z}^3, 3\sqrt{6s})$ each of the following vectors must have initial and terminal point colored the same color: $< \pm 3x, \pm 3y, \pm 3z >$, $< \pm 3x, \pm 3z, \pm 3y >$, $< \pm 3y, \pm 3x, \pm 3z >$, $< \pm 3y, \pm 3z, \pm 3x >$, $< \pm 3z, \pm 3x, \pm 3y >$, $< \pm 3z, \pm 3y, \pm 3x >$.

Let V be the group of vectors generated by those listed above under vector addition. By applying Lemma 2.3, we have that $V = \{< 3m, 3n, 3p >: m, n, p \in \mathbb{Z}$ and $m + n + p \equiv 0 \pmod 2\}$. Any vector in V must have initial and terminal point colored the same color. But $< 3a, 3b, 3c > \in V$ and $(0, 0, 0)$ is adjacent to $(3a, 3b, 3c)$ in $G(\mathbb{Z}^3, 3\sqrt{6s})$. Thus $\chi(\mathbb{Z}^3, 3\sqrt{6s}) = 4$. □

Theorem 2.3 and Theorem 2.5, along with the previously mentioned results of [13], imply the main result of this chapter.

2.3 Main Result

To establish the main result of this chapter, we need only do a little bookkeeping with the results of the previous sections.

Theorem 2.6 *Let p be any positive, square-free integer which contains no prime factor congruent to* $2 \pmod 3$. *Then* $\chi(\mathbb{Q}^3, \sqrt{2p}) = 4$.

Proof If p is not divisible by 3, we have

$$\begin{aligned} 4 &\geq \chi(\mathbb{Q}^3, \sqrt{2p}) && \text{(Theorem 2.2)} \\ &\geq \chi(\mathbb{Z}^3, \sqrt{2p}) \\ &= 4. && \text{(Theorem 2.3)} \end{aligned}$$

If p is divisible by 3, we have

$$
\begin{aligned}
4 &\geq \chi(\mathbb{Q}^3, \sqrt{2p}) && \text{(Theorem 2.2)} \\
&= \chi(\mathbb{Q}^3, 3\sqrt{2p}) && \text{(Lemma 2.1)} \\
&\geq \chi(\mathbb{Z}^3, 3\sqrt{2p}) && \\
&= 4. && \text{(Theorem 2.3)}
\end{aligned}
$$

$\square$

Chapter 3

Two-distance Graphs in $\mathbb{Q}^2$

3.1 Introduction

As mentioned in Chapter 1, the best source of information on all things pertaining to coloring the rationals is Johnson's compendium on the subject [14]. It is in fact a question posed in [14] that served as the impetus for the work done in this chapter. Johnson asks if there exists a set $D \subset (0, \infty)$, $|D| = 2$, such that $\chi(\mathbb{Q}^2, D) = 3$. We naturally refer to such D as a two-distance set. Before we delve into an answer, a little historical perspective is needed. If the set D is restricted to containing a single distance, then pretty much everything is known about coloring $\mathbb{Q}^2$. Woodall became an unexpected pioneer in the field of coloring Euclidean distance graphs on rational points with his 1973 result [27] that $\chi(\mathbb{Q}^2, 1) = 2$. This result was extended by Abrams and Johnson in 2001 [1], where they show that for any d realized as a distance in $\mathbb{Q}^2$, $\chi(\mathbb{Q}^2, d) = 2$. Furthermore, Abrams and Johnson use this fact to conclude that for any two-distance set D, $\chi(\mathbb{Q}^2, D) \leq 4$. Although some results stand out (see [17] and [18]), work done calculating the exact value $\chi(\mathbb{Q}^2, D)$ for various two-distance sets D has been somewhat sparse. In this chapter, however, we will answer Johnson's question in the affirmative and make a beginning attempt at characterizing all two-distance sets D such that $\chi(\mathbb{Q}^2, D) = 3$.

3.2 Preliminaries

Before the statement of our main result, it will be helpful to rehash a few past results pertaining to $\chi(\mathbb{Q}^2, D)$. First of all, it is important to know which distances are actually realized between points of $\mathbb{Q}^2$. The answer comes from a well-known theorem of Euler.

Theorem 3.1 *A positive integer n can be written as $n = a^2 + b^2$ for $a, b \in \mathbb{Z}$ if and only if in the prime factorization of n, factors congruent to $3 \pmod 4$ each appear to an even degree.*

We have the following immediate corollary.

Corollary 3.1 *A distance d is realized between points of $\mathbb{Q}^2$ if and only if $d = \sqrt{\frac{p}{q}}$ for relatively prime $p, q \in \mathbb{Z}^+$ such that in the prime factorization of p and in the prime factorization of q, factors congruent to $3 \pmod 4$ each appear to an even degree.*

Proof Suppose distance d is realized between points of $\mathbb{Q}^2$ and let $d = \sqrt{\frac{p}{q}}$ for relatively prime $p, q \in \mathbb{Z}^+$. Then qd is realized as a distance between points of $\mathbb{Q}^2$, which in turn implies the existence of $x_1, x_2 \in \mathbb{Q}$ such that $(x_1)^2 + (x_2)^2 = pq$. Write $x_1 = \frac{a}{c}$ and $x_2 = \frac{b}{c}$ for $a, b, c \in \mathbb{Z}$. Then $a^2 + b^2 = pqc^2$. Applying Theorem 3.1 and using the fact that p, q are relatively prime, we have that any prime factor congruent to $3 \pmod 4$ must divide either p or q an even number of times.

Conversely, suppose that p, q are relatively prime positive integers such that in the prime factorization of each, factors congruent to $3 \pmod 4$ each appear an even number of times. Then $pq = a^2 + b^2$ for some $a, b \in \mathbb{Z} \Rightarrow \frac{p}{q} = (\frac{a}{q})^2 + (\frac{b}{q})^2 \Rightarrow d = \sqrt{\frac{p}{q}}$ is a distance realized in $\mathbb{Q}^2$. $\square$

It is also useful to recall Theorem 2.1 from the previous chapter. For any $q \in \mathbb{Q}^+$ and set of distances D, the graphs $G(\mathbb{Q}^2, D)$ and $G(\mathbb{Q}^2, qD)$ are isomorphic, and thus $\chi(\mathbb{Q}^2, D) = \chi(\mathbb{Q}^2, qD)$. So when attempting to determine $\chi(\mathbb{Q}^2, D)$ for arbitrary two-distance sets D, we really need only consider sets of the form $D = \{\sqrt{z_1}, \sqrt{z_2}\}$ where z_1, z_2 are both positive integers and $\gcd(z_1, z_2)$ is square-free.

Lastly, we have the following theorem from Jungreis, Reid, and Witte [18].

Theorem 3.2 *Let $D \subset \mathbb{R}^+$. No 2-coloring of $\mathbb{Q}^2$ forbids the distances D if and only if there are $d_1, d_2 \in D$ such that*

(a) *each of* d_1, d_2 *occurs as a distance between rational points in the plane and*

(b) *there exist* $p, q \in \mathbb{Z}^+$ *such that* $\frac{d_1}{d_2} = \sqrt{\frac{p}{q}}$ *and* $p + q$ *is odd.*

Rephrasing this in terms more appropriate to the matter at hand, we obtain the following corollary.

Corollary 3.2 *Let* $D = \{\sqrt{z_1}, \sqrt{z_2}\}$ *for* $z_1, z_2 \in \mathbb{Z}^+$ *where* $\gcd(z_1, z_2)$ *is square-free and* $\sqrt{z_1}, \sqrt{z_2}$ *are both realized as distances in* $\mathbb{Q}^2$*. Suppose either*

(a) *one of* z_1, z_2 *is odd and the other even or*

(b) *one of* z_1, z_2 *is congruent to* $2 \pmod 4$ *and the other congruent to* $0 \pmod 4$*.*

Then $\chi(\mathbb{Q}^2, D) \geq 3$*.*

3.3 Main Result

Theorem 3.3 *Let* $D = \{\sqrt{p_1}, \sqrt{p_2}\}$ *where* $p_1, p_2 \in \mathbb{Z}^+$ *and* $\gcd(p_1, p_2)$ *is square-free. Suppose that each of the following is true.*

(a) *In the prime factorization of* p_1 *and in the prime factorization of* p_2*, factors congruent to* $3 \pmod 4$ *each appear to an even degree.*

(b) *Either one of* p_1, p_2 *is odd and the other even or one of* p_1, p_2 *is congruent to* $2 \pmod 4$ *and the other congruent to* $0 \pmod 4$*.*

(c) $p_1 \equiv p_2 \pmod 3$*.*

Then $\chi(\mathbb{Q}^2, D) = 3$*.*

Proof By Corollary 3.1 and Corollary 3.2, hypotheses (a) and (b) imply that $\chi(\mathbb{Q}^2, D) \geq 3$. In order to show the existence of a proper 3-coloring of $G(\mathbb{Q}^2, D)$, we will invoke the help of the famed de Bruijn-Erdős Theorem [3].

For any infinite graph G with finite chromatic number, there exists a finite subgraph H of G such that $\chi(G) = \chi(H)$.

The use of this theorem in coloring Euclidean distance graphs is certainly not a new idea, but it is of particular importance to the argument that follows, so perhaps an explanation of its use here may prove.......useful. By the previously mentioned results of [1], we have that $\chi(\mathbb{Q}^2, D)$ is finite. So let H be a finite subgraph of $G(\mathbb{Q}^2, D)$ such that $\chi(H) = \chi(\mathbb{Q}^2, D)$. Let $P_1, ..., P_m$ be the vertices of H where $P_i = (\frac{a_i}{c_i}, \frac{b_i}{d_i})$ for $a_i, b_i, c_i, d_i \in \mathbb{Z}$ and $i \in \{1, ..., m\}$. Let $n = c_1 \cdots c_m d_1 \cdots d_m$ be the product of all the denominators of the individual entries of $P_1, ..., P_m$. We now create a new distance graph H^* with vertices $nP_1, nP_2, ..., nP_m$ where vertices are adjacent if and only if they are distance $n\sqrt{p_1}$ or $n\sqrt{p_2}$ apart. The graphs H and H^* are isomorphic and thus have the same chromatic number. Note however that each point $nP_1, nP_2, ..., nP_m$ is a point of $\mathbb{Z}^2$. So in order to show the existence of a proper 3-coloring of $G(\mathbb{Q}^2, D)$, it suffices to show the existence of a proper 3-coloring of $G(\mathbb{Z}^2, \{n\sqrt{p_1}, n\sqrt{p_2}\})$ for each $n \in \mathbb{Z}^+$. We will divide the task into three cases and use an induction argument similar to that presented by Burkert in [5]. Note that we do not have to worry about the case where p_1 and p_2 are both congruent to $0 \pmod 3$ as hypothesis (a) precludes that possibility.

Case 1 Suppose $p_1 \equiv p_2 \equiv 1 \pmod 3$ and 3 does not divide n.

Let $X, Y \in \mathbb{Z}^2$ where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. Let $\Delta_i = x_i - y_i$ for $i \in \{1, 2\}$ and suppose that X and Y are adjacent in $G(\mathbb{Z}^2, \{n\sqrt{p_1}, n\sqrt{p_2}\})$. Then it must be the case that

$$(\Delta_1)^2 + (\Delta_2)^2 = p_1 n^2 \text{ or } (\Delta_1)^2 + (\Delta_2)^2 = p_2 n^2.$$

We now use the elementary number theory fact that for any $z \in \mathbb{Z}$ such that $z \not\equiv 0 \pmod 3$, $z^2 \equiv 1 \pmod 3$. Since 3 does not divide n and $p_1 \equiv p_2 \equiv 1 \pmod 3$, both $p_1 n^2$ and $p_2 n^2$ are congruent to $1 \pmod 3$. This implies that exactly one of Δ_1 and Δ_2 is congruent to

$0 \pmod 3$. So to forbid distances $n\sqrt{p_1}$ and $n\sqrt{p_2}$ we need only color $\mathbb{Z}^2$ with the function $\tau : \mathbb{Z}^2 \to \{0,1,2\}$ where for all $X \in \mathbb{Z}^2$ of the form $X = (x_1, x_2)$, $\tau(X) = x_1 + x_2 \pmod 3$.

Case 2 Suppose $p_1 \equiv p_2 \equiv 2 \pmod 3$ and 3 does not divide n.

Let $X, Y \in \mathbb{Z}^2$ be as previously described and suppose that X and Y are adjacent in $G(\mathbb{Z}^2, \{n\sqrt{p_1}, n\sqrt{p_2}\})$. Again we have that

$$(\Delta_1)^2 + (\Delta_2)^2 = p_1 n^2 \text{ or } (\Delta_1)^2 + (\Delta_2)^2 = p_2 n^2.$$

Since 3 does not divide n and $p_1 \equiv p_2 \equiv 2 \pmod 3$, it must be that $(\Delta_1)^2 + (\Delta_2)^2 \equiv 2 \pmod 3$. This implies that neither Δ_1 nor Δ_2 are congruent to $0 \pmod 3$. So here we need only color $\mathbb{Z}^2$ with the function $\tau : \mathbb{Z}^2 \to \{0,1,2\}$ where for all $X \in \mathbb{Z}^2$ of the form $X = (x_1, x_2)$, $\tau(X) = x_1 \pmod 3$.

Case 3 Suppose 3 divides n.

Our plan here will be to set up an induction argument on the number of times 3 divides n. The base step of the induction was already taken care of in Cases 1 and 2. Suppose $n = 3^k c$ for $k \geq 1$ and $c \in \mathbb{Z}^+$ where 3 does not divide c. Furthermore, suppose that for each $m \in \mathbb{Z}$ such that $0 \leq m \leq k-1$, we can 3-color $\mathbb{Z}^2$ to simultaneously forbid distances $3^m c\sqrt{p_1}$ and $3^m c\sqrt{p_2}$. For every $X \in \mathbb{Z}^2$, again using the form $X = (x_1, x_2)$, we will write X as $X = 3(q_1, q_2) + (r_1, r_2)$ where $(q_1, q_2) \in \mathbb{Z}^2$ and $r_1, r_2 \in \{0,1,2\}$. Then we color X with the color that (q_1, q_2) would receive when coloring $\mathbb{Z}^2$ to forbid distances $\frac{n\sqrt{p_1}}{3}$ and $\frac{n\sqrt{p_2}}{3}$.

Why does this coloring do the trick? Suppose X and Y are adjacent in $G(\mathbb{Z}^2, \{n\sqrt{p_1}, n\sqrt{p_2}\})$. Again, we have that

$$(\Delta_1)^2 + (\Delta_2)^2 = p_1 n^2 \text{ or } (\Delta_1)^2 + (\Delta_2)^2 = p_2 n^2.$$

Since 3 divides n, both Δ_1 and Δ_2 are congruent to $0 \pmod 3$. Writing $X = 3(q_1, q_2) + (r_1, r_2)$ and $Y = 3(s_1, s_2) + (t_1, t_2)$ as described above, we have that $(r_1, r_2) = (t_1, t_2)$. Let d denote the distance between X and Y. Then

$$\begin{aligned}
d &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\
&= \sqrt{(3q_1 + r_1 - 3s_1 - t_1)^2 + (3q_2 + r_2 - 3s_2 - t_2)^2} \\
&= \sqrt{(3q_1 - 3s_1)^2 + (3q_2 - 3s_2)^2} \\
&= 3\sqrt{(q_1 - s_1)^2 + (q_2 - s_2)^2}
\end{aligned}$$

This implies that the distance between (q_1, q_2) and (s_1, s_2) is either $\frac{n\sqrt{p_1}}{3}$ or $\frac{n\sqrt{p_2}}{3}$. By our induction hypothesis there exists a 3-coloring of $\mathbb{Z}^2$ to forbid distances $\frac{n\sqrt{p_1}}{3}$ and $\frac{n\sqrt{p_2}}{3}$. Let $\phi : \mathbb{Z}^2 \to \{0, 1, 2\}$ be such a coloring. We color the vertices of $G(\mathbb{Z}^2, \{n\sqrt{p_1}, n\sqrt{p_2}\})$ with a function $\psi : \mathbb{Z}^2 \to \{0, 1, 2\}$ such that $\psi(X) = \phi(q_1, q_2)$ and $\psi(Y) = \phi(s_1, s_2)$. Since $\phi(q_1, q_2) \neq \phi(s_1, s_2)$, X and Y must receive different colors. Thus $\chi(\mathbb{Q}^2, \{\sqrt{p_1}, \sqrt{p_2}\}) = 3$. $\square$

3.4 Further Work

Theorem 3.3, along with pretty much everything else in this chapter, appears in [23]. At the end of that article, the following two questions were raised.

Question 1: Does there exist a two-distance set D such that $\chi(\mathbb{Q}^2, D) = 3$ and qD does not fit the hypotheses of Theorem 3.3 for any $q \in \mathbb{Q}^+$?

Question 2: Does there exist a two-distance set $D = \{\sqrt{z_1}, \sqrt{z_2}\}$ for some $z_1, z_2 \in \mathbb{Z}^+$ such that $\chi(\mathbb{Z}^2, D) \neq \chi(\mathbb{Q}^2, D)$?

An answer to either of these questions remains elusive. However, it is worth noting that if the answer to Question 1 is no, then the answer to Question 2 is yes. To display this connection between the two, we will consider the chromatic numbers of the graphs $G(\mathbb{Z}^2, \{1, \sqrt{26}\})$ and $G(\mathbb{Q}^2, \{1, \sqrt{26}\})$.

Theorem 3.4 $\chi(\mathbb{Z}^2, \{1, \sqrt{26}\}) = 3$.

Proof The points $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$, $(4,0)$, $(5,0)$, $(5,1)$ are the vertices of a 7-cycle in $G(\mathbb{Z}^2, \{1, \sqrt{26}\})$. We now again use the fact that any graph containing an odd cycle is not bipartite, and therefore cannot be properly 2-colored. So certainly $\chi(\mathbb{Z}^2, \{1, \sqrt{26}\}) \geq 3$. Now observe that for any points $Z_1, Z_2 \in \mathbb{Z}^2$, Z_1 and Z_2 are adjacent in $G(\mathbb{Z}^2, \{1, \sqrt{26}\})$ if and only if they constitute the initial and terminal points of one of the following vectors: $< \pm 1, 0 >$, $< 0, \pm 1 >$, $< \pm 1, \pm 5 >$, $< \pm 5, \pm 1 >$.

To properly 3-color $G(\mathbb{Z}^2, \{1, \sqrt{26}\})$ we execute a tiling of $\mathbb{Z}^2$ with the tile given in Figure 3.1, where R denotes the color red, G the color green, and B the color blue.

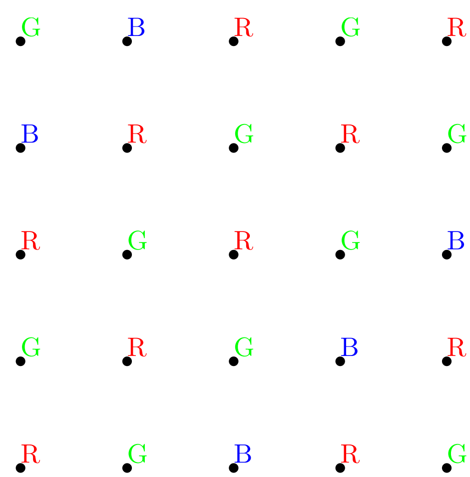

Figure 3.1: Tile leading to a proper 3-coloring of $G(\mathbb{Z}^2, \{1, \sqrt{26}\})$

In such a coloring, it is easily seen that any arrow representing the vector $< \pm 1, 0 >$ or the vector $< 0, \pm 1 >$ will have initial and terminal points colored different colors. Any arrow representing the vector $< 0, \pm 5 >$ or the vector $< \pm 5, 0 >$ will have initial point and terminal point colored the same color. Thus the distances 1 and $\sqrt{26}$ are forbidden by the coloring, and we have that $\chi(\mathbb{Z}^2, \{1, \sqrt{26}\}) = 3$. $\square$

Suppose the answer to Question 1 is in fact no. Since $1 \not\equiv 26 \pmod 3$, we would have that $\chi(\mathbb{Q}^2, \{1, \sqrt{26}\}) = 4$. We have just shown that $\chi(\mathbb{Z}^2, \{1, \sqrt{26}\}) = 3$ so Question 2 must be answered in the affirmative.

Chapter 4

The Second Babai Number and Second Clique Number of $\mathbb{Q}^3$

4.1 Introduction

In this chapter, we set aside the study of chromatic numbers of Euclidean distance graphs in the traditional sense, and instead focus our discussion on two closely related values – the nth Babai number and the nth clique number of a metric space $(\mathbf{X}, \rho)$. Introduced by Abrams and Johnson in 2001 [1], the nth Babai number $B_n(\mathbf{X})$ is defined as

$$B_n(\mathbf{X}) = \sup\{\chi(\mathbf{X}, D) \text{ where } D \subset (0, \infty) \text{ and } |D| = n\}.$$

We now adopt the convention of using $\omega(\mathbf{X}, D)$ to denote the clique number of $G(\mathbf{X}, D)$. In other words, $\omega(\mathbf{X}, D)$ designates the largest integer m such that the complete graph K_m is a subgraph of $G(\mathbf{X}, D)$. The nth clique number $C_n(\mathbf{X})$ is then defined as

$$C_n(\mathbf{X}) = \sup\{\omega(\mathbf{X}, D) \text{ where } D \subset (0, \infty) \text{ and } |D| = n\}.$$

It is worth noting that when $\mathbf{X} \subset \mathbb{R}^m$ for some $m \in \mathbb{Z}^+$ and ρ is the usual Euclidean metric on $\mathbb{R}^m$, in the definitions of both $B_n(\mathbf{X})$ and $C_n(\mathbf{X})$ the "sup" is a "max"; $B_n(\mathbf{X})$ is finite, and therefore so is $C_n(\mathbf{X})$ [1].

In 2009 [15], Johnson and Tiemeyer investigated the second Babai number and second clique number of $\mathbb{Q}^3$, concluding that $B_2(\mathbb{Q}^3) \geq 5$ and $C_2(\mathbb{Q}^3) \geq 5$. Here we will better their efforts, showing that $C_2(\mathbb{Q}^3) = 6$ which of course implies that $B_2(\mathbb{Q}^3) \geq 6$. We do this by recalling a 1960s result [8] of Einhorn and Schoenberg on certain configurations of points in $\mathbb{R}^3$ and then showing which of those configurations can be embedded in $\mathbb{Q}^3$. Recent results

[12] of Ionascu are then used to show exactly which distances d_1, d_2 result in $G(\mathbb{Q}^3, \{d_1, d_2\})$ containing a copy of K_6 as a subgraph. We conclude with a few observations which could possibly be used in future work to further increase the lower bound for $B_2(\mathbb{Q}^3)$.

4.2 Main Results

The following theorem is a summary of a few of the results found in Einhorn and Schoenberg's 1966 work [8].

Theorem 4.1 *Up to rotation, translation, and scaling, there are exactly six unique sets of six points in $\mathbb{R}^3$ with the property that only two pairwise distances are realized in the set. Furthermore, there is no set of seven points in $\mathbb{R}^3$ with the same property.*

A visual representation of the six configurations is given in Table 4.1.

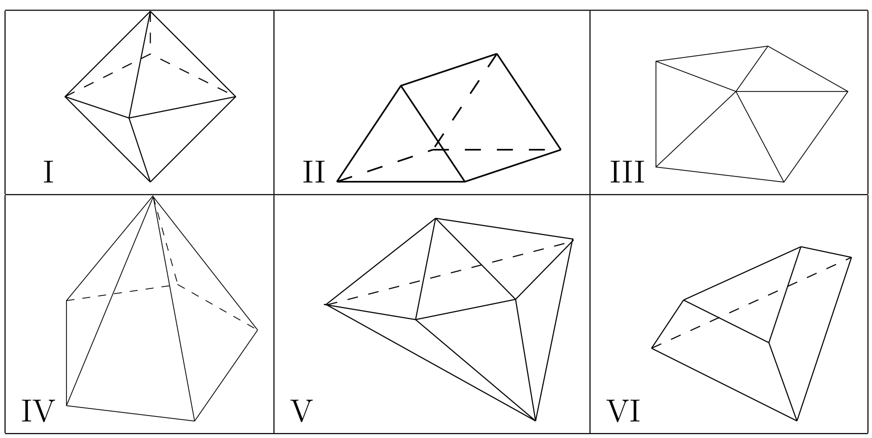

Table 4.1: The six possible two-distance configurations in $\mathbb{R}^3$

Einhorn and Schoenberg describe the six configurations in detail, but for our purposes the following abbreviated descriptions will suffice. Configuration I is a regular octahedron. Configuration II is a triangular prism with three square faces. Configurations III, IV, V, and VI each consist of six-point subsets of the twelve points that form the vertices of a regular icosahedron.

Theorem 4.2 *Configuration I can be embedded in $\mathbb{Q}^3$.*

Proof We need only observe that the points $(0,0,0),(1,1,0),(1,0,1),\ (1,0,-1),(2,0,0)$, and $(1,-1,0)$ constitute the vertices of a regular octahedron. $\square$

To show the impossibility of embedding Configurations III, IV, V, and VI in $\mathbb{Q}^3$, we will make use of a classical geometric argument illustrated by the following lemma.

Lemma 4.1 *In any regular pentagon, the diagonal length divided by the side length is the golden ratio* $\frac{1+\sqrt{5}}{2}$.

Proof By an easy calculation, each interior angle of a regular pentagon measures $108°$. Suppose a given regular pentagon has side length s. Label four consecutive vertices A, B, C, and D and insert two line segments – one with endpoints A and C and another with endpoints B and D. Label the point of intersection of these two line segments E and fill in all relevant angle measures as in Figure 4.1.

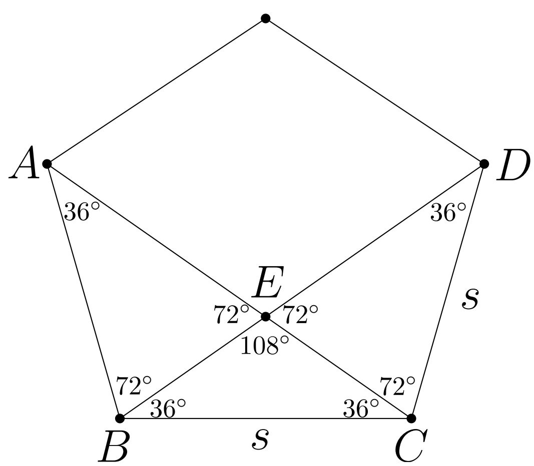

Figure 4.1: Regular pentagon with diagonals

Suppose line segment BE has length d. Then line segment EC has length d as well. Note also that line segment ED has length s, as the points E, D, and C form the vertices of an isosceles triangle. As seen in Figure 4.2, triangles BEC and DCB are similar.

This allows us to set up the following proportion.

$$\frac{s}{d} = \frac{d+s}{s}$$

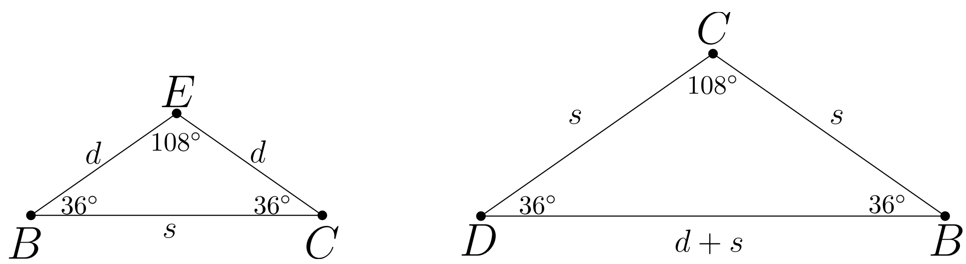

Figure 4.2: Similar triangles

$$\Rightarrow d^2 + ds - s^2 = 0$$

Solving for d, we find that

$$d = \frac{-s + \sqrt{s^2 - (4)(-s^2)}}{2}$$

$$\Rightarrow d = \frac{-s + s\sqrt{5}}{2}$$

$$\Rightarrow d + s = \frac{s + s\sqrt{5}}{2}$$

$$\Rightarrow \frac{d+s}{s} = \frac{1+\sqrt{5}}{2}$$

Thus the ratio of the diagonal length divided by the side length is $\dfrac{1+\sqrt{5}}{2}$. □

Theorem 4.3 *Configurations III, IV, V, and VI cannot be embedded in* $\mathbb{Q}^3$.

Proof As described by Einhorn and Schoenberg (and which may be apparent from Table 4.1 above), Configurations III, IV, V, and VI each have at least three of their six points constituting three of the five vertices of a regular pentagon. Note that any distance realized between points of $\mathbb{Q}^3$ is of the form $\sqrt{q}$ for some $q \in \mathbb{Q}^+$. Thus the ratio of two distances both realized between points of $\mathbb{Q}^3$ must be of the form $\sqrt{q}$ for some $q \in \mathbb{Q}^+$. If any of

Configurations III, IV, V, or VI could be embedded in $\mathbb{Q}^3$, we would by Lemma 4.1 have two distances each realized between points of $\mathbb{Q}^3$ whose ratio is $\frac{1+\sqrt{5}}{2}$. $\square$

To demonstrate the impossibility of embedding Configuration II in $\mathbb{Q}^3$, we will employ the following lemmata. Lemma 4.2 is a restatement of Chow's 1993 result [6] and was seen in Chapter 2. Lemma 4.3 is a result of Ionascu's [11] recent work counting equilateral triangles in $\mathbb{Z}^3$. Although at first glance the subject of enumerating triangles may seem to have little in common with coloring distance graphs, Ionascu's lemma has in fact proven very useful to the general study of Euclidean distance graphs with vertex set $\mathbb{Q}^3$.

Lemma 4.2 *Let $z \in \mathbb{Z}^+$. Then $\chi(\mathbb{Z}^3, \sqrt{z}) \geq 3$ if and only if in the prime factorization of z, 2 appears to an odd degree.*

Lemma 4.3 *Let T be an equilateral triangle with vertices in $\mathbb{Z}^3$. Then there exist $a, b, c, d \in \mathbb{Z}$ such that T is contained in a plane with normal vector $< a, b, c >$ and $a^2 + b^2 + c^2 = 3d^2$.*

Theorem 4.4 *Configuration II cannot be embedded in $\mathbb{Q}^3$.*

Proof Suppose Configuration II can be embedded in $\mathbb{Q}^3$. Then a similar triangular prism can be embedded in $\mathbb{Z}^3$. Let T_1 be the "top" equilateral triangle of such a triangular prism and let T_2 be the "bottom" equilateral triangle. Let $\sqrt{z}$ be the side length of each of these triangles. Then there exists vertex $V_1 \in T_1$ and vertex $V_2 \in T_2$ such that V_1 and V_2 are distance $\sqrt{z}$ apart. If we let $< a, b, c >$ denote the vector with initial point V_1 and terminal point V_2, we have that $a^2 + b^2 + c^2 = z$ and $a, b, c \in \mathbb{Z}$. But since vector $< a, b, c >$ is normal to the plane containing T_1, it must be that $z = 3d^2$ for some $d \in \mathbb{Z}$. Thus 2 must appear to an even degree in the prime factorization of z. This contradicts Lemma 4.2, however, as the existence of triangles T_1 and T_2 clearly imply that $\chi(\mathbb{Q}^3, \sqrt{z}) \geq 3$. $\square$

Of course, Theorem 4.1 and Theorem 4.2 imply the following.

Theorem 4.5 *$C_2(\mathbb{Q}^3) = 6$ and $B_2(\mathbb{Q}^3) \geq 6$.*

4.3 Further Work

Abrams and Johnson note in [1] that for any $X \subset \mathbb{R}^n$, $B_1(X) \leq m$ implies that $B_2(X) \leq m^2$. In Chapter 2, Theorem 2.2 gives that for any $d > 0$, $\chi(\mathbb{Q}^3, d) \leq 4$. Combining these facts with the work done in the previous section, we have that $6 \leq B_2(\mathbb{Q}^3) \leq 16$. Clearly this result leaves much room for improvement. If one wished to sharpen these bounds, it seems the greatest chance of success would lie in attempting to increase the lower bound – in other words, for some $d_1, d_2 > 0$, find a 7-chromatic subgraph of $G(\mathbb{Q}^3, \{d_1, d_2\})$. Of course, this may ultimately be a futile task. It may very well be the case that $B_2(\mathbb{Q}^3) = 6$ and any attempt at finding such a subgraph is doomed from the start. But still, it cannot hurt to try.

Thinking along these lines, it is natural to ask at this point what values of d_1 and d_2 result in the graph $G(\mathbb{Q}^3, \{d_1, d_2\})$ containing a copy of the complete graph K_6. Furthermore, exactly how do these copies of K_6 appear as subgraphs of $G(\mathbb{Q}^3, \{d_1, d_2\})$? We answer these questions with the following two theorems, whose methods of proof mimic the arguments presented in [11].

Theorem 4.6 *The graph $G(\mathbb{Q}^3, \{d_1, d_2\})$ contains K_6 as a subgraph if and only if $\{d_1, d_2\} = \{q\sqrt{2}, 2q\}$ for some $q \in \mathbb{Q}^+$.*

Proof The six points given in the proof of Theorem 4.2 constitute a copy of K_6 in $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$. Recalling Lemma 2.1, we have that for any $q \in \mathbb{Q}^+$ and any $D \subset (0, \infty)$, the graphs $G(\mathbb{Q}^3, D)$ and $G(\mathbb{Q}^3, qD)$ are isomorphic. Thus for any $q \in \mathbb{Q}^+$, $G(\mathbb{Q}^3, \{q\sqrt{2}, 2q\})$ contains K_6 as a subgraph.

Conversely, suppose that $G(\mathbb{Q}^3, \{d_1, d_2\})$ contains K_6 as a subgraph. From the previous section, we know that those six points which form the vertices of this copy of K_6 must also form the vertices of a regular octahedron, say of edge length s. Note that this gives $\{d_1, d_2\} = \{s, s\sqrt{2}\}$. We now label the vertices of such an octahedron as given in Figure 4.3.

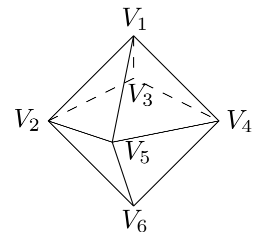

Figure 4.3: Regular octahedron with labeled vertices

V_1, V_2, V_3 form the vertices of an equilateral triangle of side length s which we will call T_1. V_4, V_5, V_6 also form the vertices of an equilateral triangle of side length s which we will call T_2. Now let V_7 and V_8 be the circumcenters of T_1 and T_2 respectively. Let u be the vector with initial point V_7 and terminal point V_8. Then $u =< a, b, c >$ for some $a, b, c \in \mathbb{Q}$. After a little computation, we find that u has length $\sqrt{\frac{2}{3}}s$ and is normal to the plane containing T_1. By slightly varying Lemma 4.3, we have that $\frac{2}{3}s^2 = a^2 + b^2 + c^2 = 3d^2 \Rightarrow 2s^2 = 9d^2$ for some $d \in \mathbb{Q}$. This implies that s must be of the form $q\sqrt{2}$ for some $q \in \mathbb{Q}^+$. $\square$

Theorem 4.7 *Any three points in $\mathbb{Q}^3$ constituting the vertices of an equilateral triangle of side length $\sqrt{2}$ also make up three of the six vertices of some copy of K_6 in the graph $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$.*

Proof Let $v_1, v_2, v_3 \in \mathbb{Q}^3$ constitute the vertices of an equilateral triangle of side length $\sqrt{2}$. Call this triangle T. Let c be the circumcenter of triangle T and note that $c \in \mathbb{Q}^3$ as well. Form a new equilateral triangle T' by rotating T $180°$ about point c such that T and T' lie in the same plane. Label the vertices of T' v_1', v_2', v_3' as shown in Figure 4.4. If we let u_1 denote the vector with initial point v_1 and terminal point c, and let u_2 denote the vector with initial point v_1 and terminal point v_1', we have that $2u_1 = u_2$. Thus $v_1' \in \mathbb{Q}^3$. Similarly, $v_2', v_3' \in \mathbb{Q}^3$ as well.

Let P be the plane containing T (and also T'). We now form a new equilateral triangle T'' by translating T' by a vector of length $\frac{2}{\sqrt{3}}$ perpendicular to plane P. The vertices of T'' together with the vertices of T constitute a copy of K_6 in the graph $G(\mathbb{R}^3, \{\sqrt{2}, 2\})$. It remains to be seen that the vertices of T'' lie in $\mathbb{Q}^3$. To accomplish this, we need only

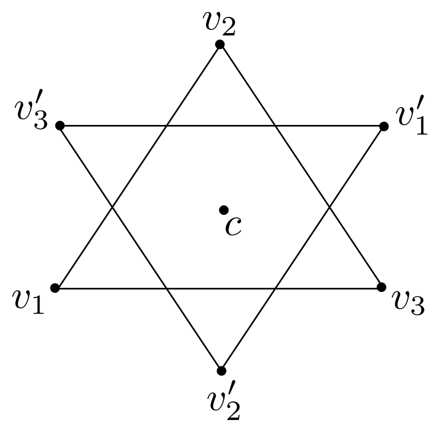

Figure 4.4: Pair of triangles with vertices in $\mathbb{Q}^3$

show that a length $\frac{2}{\sqrt{3}}$ vector with initial point in $\mathbb{Q}^3$ and perpendicular to P has terminal point in $\mathbb{Q}^3$ as well. By Lemma 4.3, P has a normal vector of the form $< a, b, c >$ where $a^2 + b^2 + c^2 = 3d^2$ and $a, b, c, d \in \mathbb{Z}$. Multiply this vector by the scalar $\frac{2}{3d}$. The resulting vector $< \frac{2a}{3d}, \frac{2b}{3d}, \frac{2c}{3d} >$ has length $\sqrt{\frac{4a^2+4b^2+4c^2}{9d^2}} = \frac{2}{3}\sqrt{\frac{3d^2}{d^2}} = \frac{2}{\sqrt{3}}$. $\square$

So it appears that the "easiest" way of increasing the lower bound of $B_2(\mathbb{Q}^3)$ lies in constructing an argument that will increase the lower bound of $\chi(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ from 6 up to 7. This line of thought could be flawed for at least two different reasons. As mentioned earlier, it could be the case that $B_2(\mathbb{Q}^3) = 6$ and efforts to find a 7-chromatic subgraph of $G(\mathbb{Q}^3, \{d_1, d_2\})$ for any distances d_1 and d_2 are simply a waste of time. We also have no guarantee that $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ has the largest chromatic number among all two-distance graphs with vertex set $\mathbb{Q}^3$ even though $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ has the largest clique number among such graphs. It could be the case that $\chi(\mathbb{Q}^3, \{\sqrt{2}, 2\}) = 6$ but for some $\{d_1, d_2\} \neq \{\sqrt{2}, 2\}$, $\chi(\mathbb{Q}^3, \{d_1, d_2\}) > 6$. However, even if our search is in vain for either of these reasons, the eventual production of a proper 6-coloring of $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ would be of some interest in its own right.

Applying Lemma 2.1 and Theorems 2.1 and 2.6 from Chapter 2, we have that $\chi(\mathbb{Q}^3, 2) = 2$ and that $\chi(\mathbb{Q}^3, \sqrt{2}) = 4$. Let $\phi_1 : \mathbb{Q}^3 \to \{1, 2\}$ be a proper 2-coloring of $G(\mathbb{Q}^3, 2)$ and $\phi_2 : \mathbb{Q}^3 \to \{1, 2, 3, 4\}$ be a proper 4-coloring of $G(\mathbb{Q}^3, \sqrt{2})$. We can properly 8-color the graph $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ with the function $\phi_3 : \mathbb{Q}^3 \to \{11, 12, 13, 14, 21, 22, 23, 24\}$ where for all $q \in \mathbb{Q}^3$, $\phi_3(q) = \phi_1(q)\phi_2(q)$. So as it stands, $6 \leq \chi(\mathbb{Q}^3, \{\sqrt{2}, 2\}) \leq 8$. For the rest of this chapter, we will assume that $\chi(\mathbb{Q}^3, \{\sqrt{2}, 2\}) = 6$ and make a few observations on the conditions a proper 6-coloring of $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ must satisfy. Hopefully, in future work we may use these observations to obtain a contradiction, thus showing that $B_2(\mathbb{Q}^3) > 6$.

Theorem 4.8 *Suppose $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ has been properly 6-colored and T is an equilateral triangle with side length $\sqrt{2}$ and vertices in $\mathbb{Q}^3$. Suppose a new triangle T' is formed by translating T by a length $\frac{4}{\sqrt{3}}$ vector normal to the plane containing T. Then the vertices of T and the vertices of T' must be colored with the same three colors.*

Proof Let triangles T and T' be as described above. By Theorem 4.7, there exists a triangle T'' such that the vertices of T together with the vertices of T'' constitute the vertices of a copy of K_6 in $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$. The vertices of T' along with the vertices of T'' also form the vertices of a copy of K_6 in $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$. Thus in any proper 6-coloring of $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$, the vertices of T and the vertices of T' must be colored with the same three colors. $\square$

Theorem 4.9 *Let P be a plane containing an equilateral triangle with vertices in $\mathbb{Q}^3$, and let $p_1, p_2 \in P \cap \mathbb{Q}^3$ where $|p_1 - p_2| = \sqrt{6}$. Then there exist $q_1, q_2 \in P \cap \mathbb{Q}^3$ such that p_1, q_1, q_2 and p_2, q_1, q_2 are each the vertices of an equilateral triangle of side length $\sqrt{2}$.*

Proof Without loss of generality, assume that $p_1 = (0, 0, 0)$ and $p_2 = (x, y, z)$. Let v be the vector lying in plane P with initial point $(\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$, having length $\frac{\sqrt{2}}{2}$, and orthogonal to vector $< x, y, z >$. As given in Figure 4.5, let (x_0, y_0, z_0) be the terminal point of v. We need only show that $(x_0, y_0, z_0) \in \mathbb{Q}^3$.

By Lemma 4.3, P has a normal vector of the form $< a, b, c >$ where $a^2 + b^2 + c^2 = 3d^2$ for some $a, b, c, d \in \mathbb{Z}$. If we let $< n_1, n_2, n_3 >$ be the cross-product of vectors $< a, b, c >$ and $< x, y, z >$, we have that $v = s < n_1, n_2, n_3 >$ for some scalar s. We now need only show that $s \in \mathbb{Q}$. Since $< a, b, c >$ and $< x, y, z >$ are orthogonal, we have that $| < n_1, n_2, n_3 > | = | < a, b, c > || < x, y, z > |$. Thus $n_1^2 + n_2^2 + n_3^2 = (3d^2)(6) = 18d^2$. Since $|v| = \frac{\sqrt{2}}{2}$, we have that $s^2(n_1^2 + n_2^2 + n_3^2) = \frac{1}{2} \Rightarrow s^2 = \frac{1}{2(18d^2)} \Rightarrow s = \pm\frac{1}{6d} \Rightarrow s \in \mathbb{Q}$. $\square$

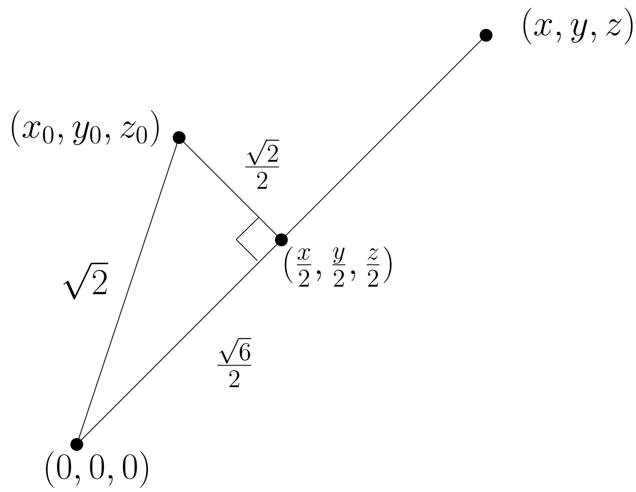

Figure 4.5: Triangle in $\mathbb{Q}^3$ with labeled sides and vertices

Theorem 4.10 *Suppose the graph $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ has been properly 6-colored. Let P be a plane containing an equilateral triangle of side length $\sqrt{2}$ with vertices in $\mathbb{Q}^3$, and let $p \in P \cap \mathbb{Q}^3$. Then at least one of the following must be true:*

(*i*) *Every rational point lying on the circle of radius $\sqrt{6}$ centered at p and lying in plane P must be colored the same color as p.*

(*ii*) *For any $n \in \mathbb{Z}^+$, every point a distance $\frac{4n}{\sqrt{3}}$ from p along a vector normal to plane P must be colored the same color as p.*

Proof Let C be the circle of radius $\sqrt{6}$ centered at p and lying in plane P, and suppose that (i) is false. Then there exists $q \in C \cap \mathbb{Q}^3$ such that p and q are colored with different colors. By Theorem 4.9, there exist points $m_1, m_2 \in P \cap \mathbb{Q}^3$ such that p, m_1, m_2 and q, m_1, m_2 each form the vertices of equilateral triangles of side length $\sqrt{2}$. Let T_1 be the triangle with vertices p, m_1, m_2 and let T_2 be the triangle with vertices q, m_1, m_2. Translate each of these triangles by the same length $\frac{4}{\sqrt{3}}$ vector normal to plane P. By Theorem 4.8, we have that the vertices of this translate of T_1 must be colored with the same three colors as the vertices of T_1, and the vertices of this translate of T_2 must be colored with the same three colors as the vertices of T_2. This implies that the translate of p must be colored the same color as p. Repeating this argument, we have that (ii) must be true. $\square$

Theorem 4.10 gives a fairly strong condition that any proper 6-coloring of $G(\mathbb{Q}^3, \{\sqrt{2}, 2\})$ must satisfy. As mentioned earlier, it is our hope that with a little ingenuity, it can be used in future work to show that $B_2(\mathbb{Q}^3) \geq 7$.

Chapter 5

Unfinished Business Concerning Single-distance Graphs in $\mathbb{Q}^3$

5.1 Introduction

We now again turn our attention to single-distance graphs with vertex set $\mathbb{Q}^3$ and address the problem of determining $\chi(\mathbb{Q}^3, d)$ for an arbitrary $d > 0$. Considering the results of [6], [13], and [17], along with the work done in Chapter 2, a complete resolution of this question can be reached by answering the following:

> Given any odd, positive, square-free integer n which contains at least one prime factor congruent to 2 (mod 3), does $\chi(\mathbb{Q}^3, \sqrt{2n}) = 3$ or does $\chi(\mathbb{Q}^3, \sqrt{2n}) = 4$?

For the sake of brevity, throughout this chapter we will designate by K the set of all odd, positive, square-free integers whose prime factorization contains at least one prime factor congruent to $2 \pmod 3$. Recall the result from Chapter 2, in which it is shown that for any odd, positive, square-free integer n containing no prime factor congruent to $2 \pmod 3$, $\chi(\mathbb{Q}^3, \sqrt{2n}) = 4$. The proof presented was heavily dependent on parameterizations of equilateral triangles with vertices in $\mathbb{Q}^3$, which were in turn used to generate 4-chromatic subgraphs of $G(\mathbb{Q}^3, \sqrt{2n})$. Unfortunately, these methods fall flat on their faces when attempting to determine $\chi(\mathbb{Q}^3, \sqrt{2k})$ for any $k \in K$, as the graph $G(\mathbb{Q}^3, \sqrt{2k})$ is triangle-free. This fact is immediately seen as a corollary to Ionascu's work in [11]; however we will include an alternate proof in the pages ahead. Our proof will not have the benefit of economy of words (it is actually a bit longer than Ionascu's proof), but in the process we will develop a lemma useful in search algorithms presented later in this chapter. Hopefully, these algorithms may be employed to eventually find 4-chromatic subgraphs of $G(\mathbb{Q}^3, \sqrt{2k})$ and put this troublesome question to rest.

Before proceeding, we must make note of a few terms and theorems from the realm of number theory. All of this information may be found in Chapters 3, 4, and 6 of [22] or Chapters 3 and 5 of [19], or really any introductory number theory text worth its salt.

5.2 Number Theory Background

In the subject of number theory, we will be primarily concerned with the topic of *quadratic residues*. Given two integers a and b, a is said to be a quadratic residue of b if there exists an integer k such that $k^2 \equiv a \pmod{b}$. If no such k exists, we say a is a quadratic non-residue of b. Very frequently the word "quadratic" is dropped in print as it will be clear from context. Volume after volume has been written on the subjects of residues and congruences, but for the purposes of our discussion we really need only concern ourselves with a few basic facts:

1) An integer a is a non-residue of a square-free integer b if and only if there exists a prime factor c of b such that a is a non-residue of c.

2) For any integers a, b, c with c prime, if a is a non-residue of c, and b is a residue of c where $b \not\equiv 0 \pmod{c}$, ab is a non-residue of c.

Along with these observations, we will use two major theorems. Theorem 5.1 was proved by Adrien Marie Legendre in the late eighteenth century. Theorem 5.2 is the Law of Quadratic Reciprocity and is the most fundamental result of classical number theory. It was first proved by Gauss in his groundbreaking 1804 manuscript *Disquisitiones Arithmeticae*.

Theorem 5.1 *(Legendre's Theorem) Let a, b, c be non-zero integers, not each positive or each negative, and suppose that abc is square-free. Then the equation*

$$ax^2 + by^2 + cz^2 = 0$$

has a non-trivial rational solution (x, y, z) if and only if each of the following are satisfied:

(i) $-ab$ is a quadratic residue of c

(ii) $-ac$ *is a quadratic residue of* b

(iii) $-bc$ *is a quadratic residue of* a*.*

Theorem 5.2 *(Law of Quadratic Reciprocity) Let* $p, q > 2$ *with* p *and* q *both prime. If at least one of* p *and* q *is congruent to* $1 \pmod 4$*, then* p *is a quadratic residue of* q *if and only if* q *is a quadratic residue of* p*. If both* p *and* q *are congruent to* $3 \pmod 4$*,* p *is a quadratic residue of* q *if and only if* q *is a quadratic non-residue of* p*.*

The Law of Quadratic Reciprocity is usually presented with the following supplement, of which we will make some use.

Theorem 5.3 *-1 is a quadratic residue of a prime* $p > 2$ *if and only if* $p \equiv 1 \pmod 4$*.*

5.3 Non-existence of Triangles in $G(\mathbb{Q}^3, \sqrt{2k})$

Lemma 5.1 *Let* $P_1, P_2 \in \mathbb{Q}^3$ *where the vector with initial point* P_1 *and terminal point* P_2 *is given by* $< a, b, c >$ *where* $c \neq 0$*. Then for any* $q \in \mathbb{Q}^+$*, there exists* $P_3 \in \mathbb{Q}^3$ *with* $|P_1 - P_3| = |P_2 - P_3| = \sqrt{q}$ *if and only if the following equation has a non-trivial rational solution* (x, y, z)*:*

$$\left(1 + \frac{a^2}{c^2}\right) x^2 + \left(1 + \frac{b^2}{a^2 + c^2}\right) y^2 - \left(q - \frac{a^2 + b^2 + c^2}{4}\right) z^2 = 0.$$

Proof Without loss of generality, we may assume that $P_1 = (0, 0, 0)$ and $P_2 = (a, b, c)$. The set of all points distance $\sqrt{q}$ from both P_1 and P_2, if such points exist, is given by the circle of radius $r = \sqrt{q - \frac{a^2+b^2+c^2}{4}}$, centered at point $(\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$, and having normal vector $< a, b, c >$. Here, by normal vector we just mean a vector normal to the plane containing the circle. Call this circle C. There exists $P_3 \in \mathbb{Q}^3$ with $|P_1 - P_3| = |P_2 - P_3| = \sqrt{q}$ if and only if there exists a rational point on C. To make computation easier, we will translate each point of C by the vector $< \frac{-a}{2}, \frac{-b}{2}, \frac{-c}{2} >$. This translate of C is a circle centered at the origin, which we will designate C'. Certainly C' contains a rational point if and only if C contains a rational point. Let $(x_0, y_0, z_0) \in C'$. Then we have that

$$x_0{}^2 + y_0{}^2 + z_0{}^2 = r^2 \tag{5.1}$$

and

$$ax_0 + by_0 + cz_0 = 0. \tag{5.2}$$

Rewriting Equation 5.2 as $z_0 = \frac{-ax_0 - by_0}{c}$ and substituting into equation 5.1 we find that

$$\left(1 + \frac{a^2}{c^2}\right) x_0{}^2 + \left(1 + \frac{b^2}{c^2}\right) y_0{}^2 + \frac{2abx_0y_0}{c^2} = r^2. \tag{5.3}$$

We now make a linear transformation which will in effect eliminate the x_0y_0 term. Let $x_0 = u + mv$ where $m = \frac{-ab}{a^2+c^2}$ and $y_0 = v$. Then Equation 5.3 becomes

$$\left(1 + \frac{a^2}{c^2}\right)(u^2 + 2muv + m^2v^2) + \left(1 + \frac{b^2}{c^2}\right) v^2 + \frac{2ab}{c^2}uv + \frac{2ab}{c^2}mv^2 = r^2. \tag{5.4}$$

Collecting like terms, we may rewrite Equation 5.4 as

$$\left(1 + \frac{a^2}{c^2}\right) u^2 + \left[\left(1 + \frac{a^2}{c^2}\right)\left(\frac{-ab}{a^2 + c^2}\right)^2 + 1 + \frac{b^2}{c^2} + \left(\frac{2ab}{c^2}\right)\left(\frac{-ab}{a^2 + c^2}\right)\right] v^2 = r^2. \tag{5.5}$$

Now we let w denote the coefficient before the v^2 term and begin the arduous process of simplifying. We have

$$w = \left[\left(\frac{a^2 + c^2}{c^2}\right)\left(\frac{a^2b^2}{(a^2 + c^2)^2}\right) + \frac{c^2(a^2 + c^2)^2}{c^2(a^2 + c^2)^2} + \frac{b^2(a^2 + c^2)^2}{c^2(a^2 + c^2)^2} - \frac{2a^2b^2(a^2 + c^2)}{c^2(a^2 + c^2)^2}\right]$$

$$\Rightarrow w = \left[\frac{a^4b^2 + a^2b^2c^2 + a^4c^2 + 2a^2c^4 + c^6 + a^4b^2 + 2a^2b^2c^2 + b^2c^4 - 2a^4b^2 - 2a^2b^2c^2}{c^2(a^2+c^2)^2}\right]$$

$$\Rightarrow w = \left[\frac{c^2(a^2b^2 + a^4 + 2a^2c^2 + c^4 + b^2c^2)}{c^2(a^2+c^2)^2}\right]$$

$$\Rightarrow w = \frac{(a^2+c^2)(b^2) + (a^2+c^2)^2}{(a^2+c^2)^2}$$

$$\Rightarrow w = \frac{a^2+b^2+c^2}{a^2+c^2} = 1 + \frac{b^2}{a^2+c^2}.$$

So Equation 5.5 can be rewritten as

$$\left(1 + \frac{a^2}{c^2}\right)u^2 + \left(1 + \frac{b^2}{a^2+c^2}\right)v^2 - r^2 = 0. \tag{5.6}$$

Now let $u = \frac{x}{z}$ and $v = \frac{y}{z}$. Equation 5.6 can then be rewritten further as

$$\left(1 + \frac{a^2}{c^2}\right)x^2 + \left(1 + \frac{b^2}{a^2+c^2}\right)y^2 - \left(q - \frac{a^2+b^2+c^2}{4}\right)z^2 = 0 \tag{5.7}$$

which has a non-trivial rational solution if and only if Equation 5.3 has a rational solution. □

One may find the stipulation that $c \neq 0$ to be limiting, but that is not the case. For any $a, b, c \in \mathbb{Q}$ not all zero and any permutation σ of the set $\{a, b, c\}$, the initial and terminal points of the $\mathbb{Q}^3$ vector $< a, b, c >$ have a rational point distance d from each if and only if the initial and terminal points of the vector $< \sigma(a), \sigma(b), \sigma(c) >$ have a rational point distance d from each. So given any points $P_1, P_2 \in \mathbb{Q}^3$, we can easily use Theorem 5.1 to decide if the equation in the statement of Lemma 5.1 has a non-trivial rational solution.

Lemma 5.2 *There exist infinitely many values of θ, $0 < \theta < 2\pi$, such that a rotation of the plane $\mathbb{R}^2$ about the origin and through the angle θ is an isometry that maps $\mathbb{Q}^2$ onto itself.*

Proof Clearly such a rotation is an isometry. It is well-known that rational points are dense on the unit circle, so we have infinitely many choices for $(r, s) \in \mathbb{Q}^2$ such that $r^2 + s^2 = 1$. This gives $r = \cos\theta$ and $s = \sin\theta$ for some angle θ. Now suppose $(a, b) \in \mathbb{Q}^2$ is mapped in the described rotation to the point (x, y) as given in Figure 5.1.

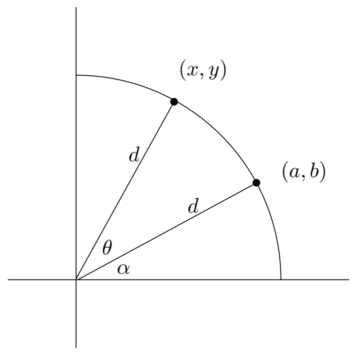

Figure 5.1: Point (a, b) with its image in a rational rotation of $\mathbb{R}^2$

We immediately extract from the above figure the relationships $\frac{a}{d} = \cos\alpha$, $\frac{b}{d} = \sin\alpha$, $\frac{x}{d} = \cos(\alpha + \theta)$, and $\frac{y}{d} = \sin(\alpha + \theta)$. Applying basic trigonometric identities, we have $\frac{x}{d} = \cos\alpha\cos\theta - \sin\alpha\sin\theta \Rightarrow \frac{x}{d} = \frac{a}{d}r - \frac{b}{d}s \Rightarrow x = ar - bs$. Similarly, we have $\frac{y}{d} = \sin\alpha\cos\theta + \cos\alpha\sin\theta \Rightarrow \frac{y}{d} = \frac{b}{d}r + \frac{a}{d}s \Rightarrow y = br + as$. Thus $(x, y) \in \mathbb{Q}^2$. $\square$

Theorem 5.4 *Let $P_1, P_2 \in \mathbb{Q}^3$, and let $|P_1 - P_2| = \sqrt{2k}$ for some $k \in K$. Then there does not exist $P_3 \in \mathbb{Q}^3$ with $|P_1 - P_3| = |P_2 - P_3| = \sqrt{2k}$.*

Proof Without loss of generality, assume that $P_1 = (0, 0, 0)$ and $P_2 = (\frac{a}{q}, \frac{b}{q}, \frac{c}{q})$ where $a, b, c, q \in \mathbb{Z}$. In light of Lemma 5.2, we may assume that a, b, c are each non-zero. Suppose there exists $P_3 \in \mathbb{Q}^3$ where P_3 is distance $\sqrt{2k}$ from both P_1 and P_2. This implies there exists a point $P_4 \in \mathbb{Q}^3$ distance $q\sqrt{2k}$ from each of the points $(0, 0, 0)$ and (a, b, c). Then by Lemma 5.1 the following equation has a non-trivial rational solution:

$$\left(1+\frac{a^2}{c^2}\right)x^2+\left(1+\frac{b^2}{a^2+c^2}\right)y^2-\left(a^2+b^2+c^2-\frac{a^2+b^2+c^2}{4}\right)z^2=0. \tag{5.8}$$

If we let $x=\frac{c(a^2+b^2+c^2)}{q(a^2+c^2)}x_0$ and $z=2z_0$, the above equation becomes

$$\left(\frac{(a^2+b^2+c^2)^2}{q^2(a^2+c^2)}\right){x_0}^2+\left(\frac{a^2+b^2+c^2}{a^2+c^2}\right)y^2-3\left(a^2+b^2+c^2\right){z_0}^2=0. \tag{5.9}$$

We may now multiply both sides of this equation by (a^2+c^2) and divide both sides by $(a^2+b^2+c^2)$ to obtain

$$2k{x_0}^2+y^2-3(a^2+c^2){z_0}^2=0 \tag{5.10}$$

which must also have a non-trivial rational solution.

Let m be a prime factor of k congruent to $2 \pmod 3$. We have that $2q^2k=a^2+b^2+c^2$. Write $a=a_1m^\alpha$, $b=b_1m^\beta$, $c=c_1m^\gamma$ where $a_1,b_1,c_1\in\mathbb{Z}$ are each not divisible by m and α,β,γ are each non-negative integers. Note that we may apply any permutation to $\{a,b,c\}$ in Equation 5.10 and the resulting equation must also have a non-trivial rational solution. With this in mind, we may without loss of generality assume that $\alpha\le\beta\le\gamma$. Rewrite the previous equation as $2q^2k=a_1^2m^{2\alpha}+b_1^2m^{2\beta}+c_1^2m^{2\gamma}$ and divide both sides by $m^{2\alpha}$. We are left with $\frac{2q^2k}{m^{2\alpha}}=a_1^2+b_1^2m^{2(\beta-\alpha)}+c_1^2m^{2(\gamma-\alpha)}$, and note here that $\frac{2q^2k}{m^{2\alpha}}\in\mathbb{Z}$ and is divisible by m. If $\gamma>\alpha$, we have that $-(a_1^2+c_1^2m^{2(\gamma-\alpha)})=(b_1m^{\beta-\alpha})^2-\frac{2q^2k}{m^{2\alpha}}$ implying that the square-free part of $-(a^2+c^2)$ is a residue of m but is also not divisible by m. If $\alpha=\beta=\gamma$, we are left with $-(a_1^2+c_1^2)=b_1^2-\frac{2q^2k}{m^{2\alpha}}$. Note here that m does not divide $(a_1^2+c_1^2)$ as that would imply $m|b_1$, and thus the square-free part of $-(a^2+c^2)$ must again be a residue of m but not divisible by m.

Let d be the square-free part of (a^2+c^2). For any integers u and v, $u^2+v^2\equiv 0 \pmod 3$ implies $3|u$ and $3|v$. If $3|d$ we would have that $3|a$ and $3|c$ which in turn implies that d is the square-free part of $(\frac{a}{3})^2+(\frac{c}{3})^2$. Repeating this argument, we would have that $3^t|a$ and

$3^t|c$ for all positive integers t. So as it stands, it must be the case that 3 does not divide d. Now applying Legendre's Theorem and the basic facts of the previous section, we have that $3d$ must be a residue of m. The preceding arguments show that $-d$ is a residue of m, so we are left with -3 being a residue of m as well. We will consider two cases and apply the Law of Quadratic Reciprocity to obtain a contradiction.

If $m \equiv 1 \pmod 4$, we have that -1 is a residue of m. Now applying the Law of Quadratic Reciprocity, it must be that 3 and m are each residues of each other. This is a contradiction, however, as $m \equiv 2 \pmod 3$ is a non-residue of 3. If $m \equiv 3 \pmod 4$, we have that -1 is a non-residue of m. Again the Law of Quadratic Reciprocity gives us that $m \equiv 2 \pmod 3$ must be a residue of 3 – the same contradiction. □

5.4 An Algorithmic Search for 4-chromatic Subgraphs of $G(\mathbb{Q}^3, \sqrt{2k})$

Digressions aside, we now return to the issue at the beginning of this chapter – that of determining $\chi(\mathbb{Q}^3, \sqrt{2k})$ for some $k \in K$. This problem may be resolved in one of two ways. We could either exhibit a proper 3-coloring of the graph $G(\mathbb{Q}^3, \sqrt{2k})$ or show the existence of a 4-chromatic subgraph of $G(\mathbb{Q}^3, \sqrt{2k})$. Theorem 5.4 gives us that $G(\mathbb{Q}^3, \sqrt{2k})$ is triangle-free, but this fact in itself has no bearing on $\chi(\mathbb{Q}^3, \sqrt{2k})$. In 1955, Jan Mycielski [21] showed the existence of triangle-free graphs of arbitrary chromatic number. His proof consisted of the construction of a sequence of triangle-free graphs (unsurprisingly now referred to as the Mycielskian graphs) with the property that each graph in the sequence has chromatic number one greater than the previous graph. For our purposes, the most notable of these graphs is the third Mycielskian, sometimes called the the Grötzsch graph. It is the unique smallest triangle-free graph of chromatic number 4 [7], and is given in Figure 5.2. With this information in mind and considering the fact that at present no one has shown the existence of single $d > 0$ such that $\chi(\mathbb{Q}^3, d) = 3$, it seems our greatest chance of success in resolving the aforementioned problem lies in the latter approach.

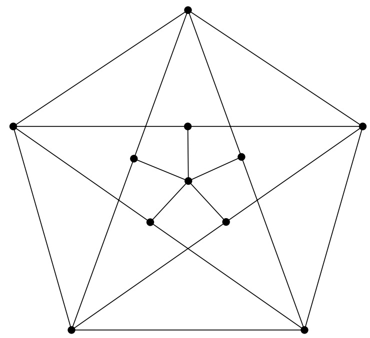

Figure 5.2: The Mycielski-Grötzsch graph

In the upcoming pages we will describe an algorithm that can potentially be used to construct 4-chromatic subgraphs of $G(\mathbb{Q}^3, \sqrt{2k})$ for $k \in K$. In doing so we will stray from the standard format of mathematical writing – that of stating relevant lemmata, stating and proving a theorem, expounding upon the result, and then repeating the process. Instead we will illustrate our algorithm as it pertains to a specific example – the graph $G(\mathbb{Q}^3, \sqrt{10})$. Along the way we will occasionally break to sample a needed result from graph theory or number theory.

We begin by noting that the points $(0,0,0), (3,1,0), (2,1,3), (2,0,0), (1,0,3)$ form the vertices of a 5-cycle in the graph $G(\mathbb{Q}^3, \sqrt{10})$. Let $C_1, ..., C_5$ be circles such that C_1 is the set of all points distance $\sqrt{10}$ from both $(1,0,3)$ and $(3,1,0)$, C_2 is the set of all points distance $\sqrt{10}$ from both $(0,0,0)$ and $(2,1,3)$, C_3 is the set of all points distance $\sqrt{10}$ from both $(3,1,0)$ and $(2,0,0)$, C_4 is the set of all points distance $\sqrt{10}$ from both $(2,1,3)$ and $(1,0,3)$, and C_5 is the set of all points distance $\sqrt{10}$ from both $(2,0,0)$ and $(0,0,0)$. A depiction of these circles and their orientation to the vertices of the previously listed 5-cycle is given in Figure 5.3.

Please note at this point that Figure 5.3 is meant only as an aid to help visualize a graph we are attempting to construct. As depicted, it may seem that the points $(0,0,0), (3,1,0), (2,1,3), (2,0,0), (1,0,3)$ form the vertices of a regular pentagon in $\mathbb{Q}^3$. That is impossible, however, as evidenced by Lemma 4.1 from the previous chapter.

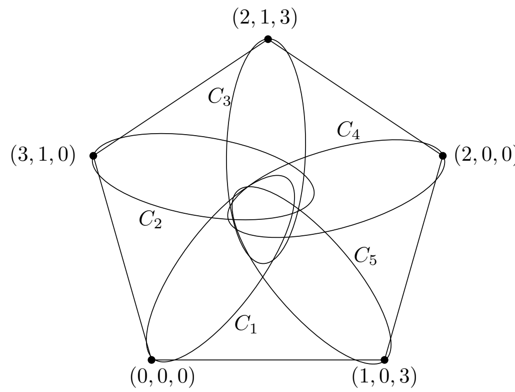

Figure 5.3: Circles in $\mathbb{R}^3$ along with a 5-cycle in $G(\mathbb{Q}^3, \sqrt{10})$

A better description of $C_1, ..., C_5$ can be found by listing the center and radius of each circle along with a vector normal to the plane containing each circle.

C_1 is centered at $(2, \frac{1}{2}, \frac{3}{2})$, has radius $\frac{\sqrt{26}}{2}$, and normal vector $< 2, 1, -3 >$.

C_2 is centered at $(1, \frac{1}{2}, \frac{3}{2})$, has radius $\frac{\sqrt{26}}{2}$, and normal vector $< 2, 1, 3 >$.

C_3 is centered at $(\frac{5}{2}, \frac{1}{2}, 0)$, has radius $\frac{\sqrt{38}}{2}$, and normal vector $< 1, 1, 0 >$.

C_4 is centered at $(\frac{3}{2}, \frac{1}{2}, 3)$, has radius $\frac{\sqrt{38}}{2}$, and normal vector $< 1, 1, 0 >$.

C_5 is centered at $(1, 0, 0)$, has radius 3, and normal vector $< 1, 0, 0 >$.

We now desire a characterization of the rational points on each of these circles. This can be found by applying a well-known number theory result, a proof of which can be found in [19], [25], or many other texts which place an emphasis on the study of Diophantine equations.

Lemma 5.3 *Let* $ax^2 + bxy + cy^2 + dx + ey + f = 0$ *be the equation of a conic where* $a, b, c, d, e, f \in \mathbb{Q}$. *If the conic contains one rational point, it contains infinitely many.*

Nagell goes further in [22], giving a parameterization of the rational solutions to the above equation.

Lemma 5.4 *Let* $ax^2 + bxy + cy^2 + dx + ey + f = 0$ *be the equation of a conic where* $a, b, c, d, e, f \in \mathbb{Q}$. *Suppose* (ξ, η) *is a rational point on the conic. Additional rational points* (x, y) *on the conic may be parameterized as*

$$x = \frac{-d - a\xi - b\eta - (2c\eta + e)t + c\xi t^2}{a + bt + ct^2}, \quad y = \frac{a\eta - (2a\xi + d)t - (b\xi + c\eta + e)t^2}{a + bt + ct^2}$$

where the parameter t runs through all the rational numbers. The only rational point not obtained through this parameterization (should it actually exist on the conic) is the point $(\xi, \frac{-b\xi - c\eta - e}{c})$ and is found by letting t approach ∞.

Now suppose (x, y, z) is a point on C_1. Then the following two equations must be satisfied:

$$(x-2)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{3}{2}\right)^2 = \frac{13}{2} \tag{5.11}$$

and

$$2(x-2) + \left(y - \frac{1}{2}\right) - 3\left(z - \frac{3}{2}\right) = 0. \tag{5.12}$$

If we rewrite Equation 5.12 as $z = \frac{2x+y}{3}$ and then substitute into Equation 5.11, we may then simplify to obtain

$$13x^2 + 4xy + 10y^2 - 54x - 18y = 0 \tag{5.13}$$

whose rational solutions are characterized by Lemma 5.4. We have that $(0, 0, 0)$ is a rational point on C_1, or in other words $x = 0$, $y = 0$ is a solution to Equation 5.13. Applying Lemma 5.4, we find additional rational points on C_1 parameterized as

$$\left(\frac{54 + 18t_1}{13 + 4t_1 + 10t_1^2}, \frac{54t_1 + 18t_1^2}{13 + 4t_1 + 10t_1^2}, \frac{36 + 30t_1 + 6t_1^2}{13 + 4t_1 + 10t_1^2}\right) \text{ for } t_1 \in \mathbb{Q}.$$

We may now repeat this process to find rational points on circles $C_2, ..., C_5$, also in terms of a rational parameter. After some work, we find that additional rational points on C_2 are given by

$$\left(\frac{13 + 42t_2 + 30t_2^2}{5 + 12t_2 + 10t_2^2}, \frac{9 + 6t_2 - 8t_2^2}{5 + 12t_2 + 10t_2^2}, \frac{-2t_2 + 6t_2^2}{5 + 12t_2 + 10t_2^2}\right) \text{ for } t_2 \in \mathbb{Q}.$$

Additional rational points on C_3 are given by

$$\left(\frac{6-6t_3+2t_3^2}{2+t_3^2}, \frac{6t_3+t_3^2}{2+t_3^2}, \frac{6+2t_3-3t_3^2}{2+t_3^2}\right) \text{ for } t_3 \in \mathbb{Q}.$$

Additional rational points on C_4 are given by

$$\left(\frac{2+6t_4+2t_4^2}{2+t_4^2}, \frac{2-6t_4}{2+t_4^2}, \frac{-2t_4+6t_4^2}{2+t_4^2}\right) \text{ for } t_4 \in \mathbb{Q}.$$

And finally, additional rational points on C_5 are given by

$$\left(1, \frac{-3+3t_5^2}{1+t_5^2}, \frac{-6t_5}{1+t_5^2}\right) \text{ for } t_5 \in \mathbb{Q}.$$

If we can carefully select $t_1, ..., t_5 \in \mathbb{Q}$ yielding rational points $P_1 \in C_1, ..., P_5 \in C_5$ with the property that there exists $P_6 \in \mathbb{Q}^3$ distance $\sqrt{10}$ from each of $P_1, ..., P_5$, we would in effect have embedded the Mycielski-Grötzsch graph in $G(\mathbb{Q}^3, \sqrt{10})$, allowing us to claim $\chi(\mathbb{Q}^3, \sqrt{10}) = 4$ and wipe our hands of the whole mess. That said, finding such $t_1, ..., t_5$ seems very difficult – for that matter, they may not even exist in the first place. Instead we will attempt to embed a similar graph in $G(\mathbb{Q}^3, \sqrt{10})$. To see that this graph also has chromatic number 4, it will be best at this point to assume that $\chi(\mathbb{Q}^3, \sqrt{10}) = 3$ and proceed by way of contradiction. We first make note of a lemma.

Lemma 5.5 *Let C be an odd cycle which has been properly 3-colored – say with colors red, green, and blue. Then there exists vertices v_1, v_2, v_3 of C such that v_1 is colored red and is adjacent to vertices colored green and blue, v_2 is colored green and is adjacent to vertices colored red and blue, and v_3 is colored blue and is adjacent to vertices colored red and green.*

Proof Suppose C is described as above, and each red vertex is adjacent to either two green vertices or two blue vertices. We can then recolor each red vertex blue or green (whichever is necessary) so that C is properly colored using only two colors. This is a contradiction, however, as any odd cycle has chromatic number 3. Similarly, there must be a green vertex adjacent to red and blue vertices and a blue vertex adjacent to red and green vertices. $\square$

Supposing that $G(\mathbb{Q}^3, \sqrt{10})$ has been properly 3-colored – again say with the colors red, green, and blue, Lemma 5.5 implies that of the circles $C_1, ..., C_5$, one has each of its rational points colored red, one has each of its rational points colored green, and one has each of its rational points colored blue. We will describe such circles as being monochromatic. Now let $\mathcal{C}_1 = \{C_1, C_2, C_3\}$, $\mathcal{C}_2 = \{C_2, C_3, C_4\}$, $\mathcal{C}_3 = \{C_3, C_4, C_5\}$, $\mathcal{C}_4 = \{C_4, C_5, C_1\}$, and $\mathcal{C}_5 = \{C_5, C_1, C_2\}$. After a little consideration, it is evident that for some $i \in \{1, 2, 3, 4, 5\}$, $\mathcal{C}_i$ consists of three monochromatic circles – one red, one blue, and one green.

We begin by considering $\mathcal{C}_1$, and attempting to find rational points $q_1 \in C_1$, $q_2 \in C_2$, $q_3 \in C_3$ such that there exists $q_4 \in \mathbb{Q}^3$ such that $|q_1 - q_4| = |q_2 - q_4| = |q_3 - q_4| = \sqrt{10}$. This q_4 would imply that $\mathcal{C}_1$ cannot consist of three monochromatic circles. If we are successful in our search, we would then repeat the process for $\mathcal{C}_2, ..., \mathcal{C}_5$. Given points q_1, q_2, q_3 as described above, determining whether such a q_4 exists is a relatively simple matter. We illustrate the needed steps below.

1) Begin by selecting $t_1, t_2, t_3 \in \mathbb{Q}$.

2) Plug these values into the parameterizations given earlier to obtain rational points $q_1 \in C_1, q_2 \in C_2$, and $q_3 \in C_3$.

3) Let P_1 and P_2 be the planes consisting of all points equidistant from q_1, q_2 and q_1, q_3 respectively. If P_1 and P_2 happen to be parallel (in other words, if the points q_1, q_2, q_3 are collinear), return to Step 1 and select new values of t_1, t_2, t_3.

4) Let L be the line of intersection of P_1 and P_2, and let (x_0, y_0, z_0) be any rational point on L. Such a point must exist as the circumcenter of the triangle with vertices q_1, q_2, q_3 is in $\mathbb{Q}^3$ and also on L. Let $< v_1, v_2, v_3 >$ be the cross product of vectors $\overrightarrow{q_1 q_2}$ and $\overrightarrow{q_1 q_3}$, and note that $v_1, v_2, v_3 \in \mathbb{Q}$.

5) Parameterize L as $x = x_0 + v_1 s$, $Y = y_0 + v_2 s$, $z = z_0 + v_3 s$.

6) Supposing $q_1 = (x_1, y_1, z_1)$, solve the following quadratic equation for s:

$$(x_0 + v_1 s - x_1)^2 + (y_0 + v_2 s - y_1)^2 + (z_0 + v_3 s - z_1)^2 = 10. \tag{5.14}$$

If this equation has a rational solution for s, the desired q_4 exists, and we conclude that $\mathcal{C}_1$ cannot consist of three monochromatic circles.

5.5 Concluding Remarks

It seems incredibly unlikely that one would "arbitrarily" choose rational numbers t_1, t_2, and t_3 and luckily stumble upon a rational solution for Equation 5.14. However, $\mathbb{Q}^3$ is countable, so we may put it in one-to-one correspondence with the natural numbers and institute a computer search to attempt to find a triple (t_1, t_2, t_3) leading to a rational s. This method of search cannot be exhaustive, but as we are only interested in showing the existence of a single rational solution, the idea seems promising.

We can increase the efficiency of our search algorithm by first narrowing down the list of rational triples (t_1, t_2, t_3) that could possibly yield a rational solution to Equation 5.14. To do this, we put $\mathbb{Q}^2$ into one-to-one correspondence with the natural numbers and then for $(t_1, t_2) \in \mathbb{Q}^2$, use Lemma 5.1 in conjunction with Legendre's Theorem to decide whether the points q_1 and q_2 given by the previous parameterizations have a rational point q_4 distance $\sqrt{10}$ from each. If they do not, we can for each $p \in \mathbb{Q}$ rule out (t_1, t_2, p) as possibly leading to a rational s.

Should our search prove successful, we would of course be interested in using the same ideas to determine $\chi(\mathbb{Q}^3, \sqrt{2k})$ for other k values. Additional complications could arise. We began our study of $G(\mathbb{Q}^3, \sqrt{10})$ with a 5-cycle which was seemingly plucked out of thin air. As far as we know, however, it is unknown whether or not for each $k \in K$ the graph $G(\mathbb{Q}^3, \sqrt{2k})$ contains a 5-cycle. Empirical evidence suggests that it does. In practice it has been very easy to construct such 5-cycles. In fact, we conjecture that the graph $G(\mathbb{Z}^3, \sqrt{2k})$ contains a 5-cycle for each $k \in K$. Although it only springs up as a side issue in our current topic, resolution of this question would be a worthwhile goal in future research.

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