Modular Balanced Graphs

by

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Abstract

We say that a graph G is (δ, r)-balanced if the degree of each vertex in G is congruent to $r \pmod{\delta}$ and no two degrees differ by more than δ. In this paper, we give necessary and sufficient conditions for the existence of a (δ, r)-balanced graph with e edges on n vertices. In the case of bipartite graphs where each partition is modular balanced with the same δ but possibly different remainders, Yuceturk gave necessary and sufficient conditions for the existence of such a graph with a list of exceptions in [8] for $\delta = 2$. We state a similar result for $\delta = 3$ and note that the list of exceptions for any higher δ can be found with similar methods. Additionally, we present some partial results from Anti-Ramsey theory which deals with extremal edge colorings of graphs that avoid certain colorings of subgraphs.

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Chapter 1

Modular Balance

1.1 Definitions

Throughout the paper, G will denote a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $\varphi : \mathrm{X} \to \mathbb{Z}$ and let $A \subseteq \mathrm{X}$. φ is *balanced on* A if for any $a, b \in A$, $|\varphi(a) - \varphi(b)| \leq 1$. For a simple graph G, we say that $A \subseteq V(G)$ is *balanced* if the vertex degree function is balanced on A. If $A = V(G)$, we say that G is *balanced* if $V(G)$ is balanced. An integer vector $(a_1, a_2, ..., a_n)$ is *balanced* if $|a_j - a_i| \leq 1$ for all $1 \leq i, j \leq n$. Thus, a graph is balanced if its degree sequence is balanced.

If δ, r are integers such that $0 \leq r \leq \delta - 1$ and n is a positive integer, we say that the integer vector $(a_1, a_2, ..., a_n)$ is *(δ, r)-balanced* if $a_i \equiv r (\text{mod } \delta)$ for $1 \leq i \leq n$ and $|a_j - a_i| \leq \delta$ for all $1 \leq i, j \leq n$. We say that a simple graph G is *(δ, r)-balanced* if the degree sequence of G is (δ, r)-balanced.

Theorem 1.1 *Let δ, r, s be non-negative integers with $r \leq \delta - 1$ and n be a positive integer. There is a (δ, r)-balanced non-negative integer vector $v = (v_1, v_2, v_3, ..., v_n)$ with n coordinates summing to s if and only if:*

$$i) \quad nr \leq s$$

$$ii) \quad nr \equiv s \ (mod\ \delta)$$

Furthermore, if these conditions hold, every such vector consists of:

$$n - (\tfrac{s-nr}{\delta})(mod\ n) \text{ coordinates equal to } \lfloor \tfrac{s-nr}{\delta n} \rfloor \delta + r$$

$$(\tfrac{s-nr}{\delta})(mod\ n) \text{ coordinates equal to } \lceil \tfrac{s-nr}{\delta n} \rceil \delta + r$$

Proof ($\Rightarrow$) Since each coordinate of v must be at least r, $i)$ is clearly necessary. For any coordinate v_i of v, $v_i = r + b_i\delta$ for some non-negative integer b_i, thus:

$$s = \sum_{i=1}^{n} r + b_i\delta = nr + \sum_{i=1}^{n} b_i\delta \equiv nr(mod\ \delta)$$

($\Leftarrow$) Suppose we have δ, r, s, non-negative integers with $r \leq \delta - 1$ and n, a positive integer such that $i)$ and $ii)$ are satisfied.

We will construct our (δ, r)-balanced non-negative integer vector $v = (v_1, v_2, v_3, ..., v_n)$ by first setting each coordinate of v to r. Note that:

$$s = (\tfrac{s-nr}{\delta})\delta + nr = n((\tfrac{s-nr}{\delta n})\delta) + nr$$

Where $\frac{s-nr}{\delta}$ is the non-negative integer (an integer since $nr \equiv s\ (mod\ \delta)$) number of copies of δ that must be added to the coordinates of v to bring the sum up to s. The number of copies of δ that each coordinate receives on average, $\frac{s-nr}{\delta n}$, is not necessarily an integer.

$$n((\lfloor \tfrac{s-nr}{\delta n} \rfloor)\delta) + nr \leq s \leq n((\lceil \tfrac{s-nr}{\delta n} \rceil)\delta) + nr$$

Thus, to maintain (δ, r)-balance, each coordinate of v must get at least $\lfloor \frac{s-nr}{\delta n} \rfloor$ copies of δ and no more than $\lceil \frac{s-nr}{\delta n} \rceil$ copies. So, the only possible entries of v are $\lfloor \frac{s-nr}{\delta n} \rfloor\delta + r$ and $\lceil \frac{s-nr}{\delta n} \rceil\delta + r$.

We can note here that if $\frac{s-nr}{\delta n}$ is an integer, we have equality in the string of inequalities above and each coordinate of v is exactly $(\frac{s-nr}{\delta n})\delta + r$.

For $0 \leq j \leq \lfloor \frac{s-nr}{\delta n} \rfloor$, all coordinates of v must be at least $r + j\delta$ before any can be increased to $r + (j+1)\delta$. Thus, $\frac{s-nr}{\delta}(\text{mod } n)$ coordinates of v will get one more copy of δ than the remaining $n - (\frac{s-nr}{\delta}(\text{mod } n))$ coordinates.

■

Corollary 1.1 *If G is a (δ, r)-balanced graph on n vertices with e edges then:*

$i)\quad nr \leq 2e$

$ii)\quad nr \equiv 2e\ (mod\ \delta)$

$iii)\quad e = \frac{nr}{2} + (\frac{2e-nr}{2\delta})\delta$

Furthermore, the degree sequence of G consists of:

$$n - (\tfrac{2e-nr}{\delta})(mod\ n) \textit{ coordinates equal to } \lfloor \tfrac{2e-nr}{\delta n} \rfloor \delta + r$$
$$(\tfrac{2e-nr}{\delta})(mod\ n) \textit{ coordinates equal to } \lceil \tfrac{2e-nr}{\delta n} \rceil \delta + r$$

Proof Clear since the sum of the degree sequence of G is $2e$. For iii), recall that $2e = s = (\frac{s-nr}{\delta})\delta + nr$.

■

The main question will be, for what values of n, e, δ, and r does there exist a (δ, r)-balanced graph with e edges on n vertices? A necessary condition would be that if r is odd and δ is even then n must be even because all vertex degrees in such a (δ, r)-balanced graph would be odd.

Given integers δ, r such that $0 \leq r \leq \delta - 1$ and positive integer $n > r$, we will first determine the possible numbers of edges with the only requirements being that the resultant degree sequence is (δ, r)-balanced and the degree sum is even. Afterwards, we will determine which of these sequences are actually graphic, that is, which of them are the degree sequence of a simple graph. Throughout, for a given δ, r, n, let m be the integer such that $m\delta + r < n \leq (m+1)\delta + r$.

1.2 Possible Degree Sequences

We will proceed by cases based on the parities of n, δ and r, giving us eight total cases. As mentioned before, the case where n is odd, δ is even, and r is odd produces only sequences with an odd number of odd degrees. There are no (δ, r)-balanced graphs in this case.

The seven remaining cases will be handled by grouping them together as follows:

Group 1: Three remaining cases where δ is even: (n odd, δ even, r even), (n even, δ even, r odd), (n even, δ even, r even)

Group 2: Cases where δ is odd and at least one of n, r is even: (n even, δ odd, r odd), (n even, δ odd, r even), (n odd, δ odd, r even)

Group 3: Case where n, δ, r are all odd.

Within each group, let $[e_0, e_1, e_2, ..., e_{max}]$ be the increasing list of possible edge numbers for that group. That is, the edge numbers whose degree sequence is (δ, r)-balanced and whose degree sum is even.

Group 1: First note that an even δ means that all available degrees will have the same parity. The smallest possible edge number, e_0, will have degree sequence $(\underbrace{r, r, r, ..., r, r}_{n})$. Since at least one of n and r is even, nr is even and $e_0 = \frac{nr}{2}$.

Given a possible edge number e_i with degree sequence $(c\delta + r, c\delta + r, ..., c\delta + r, (c-1)\delta + r, (c-1)\delta + r, ..., (c-1)\delta + r)$ and even degree sum s, we can increase one of the smaller degrees by δ, maintaining the (δ, r)-balance and increasing our degree sum to $s + \delta$, still an even number. Note that this is the smallest increment we can add to the degree sum and maintain (δ, r)-balance. This increases the edge number by $\frac{\delta}{2}$. That is, $e_{i+1} = e_i + \frac{\delta}{2}$.

The largest possible edge number will have degree sequence $(\underbrace{m\delta + r, m\delta + r, ..., m\delta + r}_{n})$, giving us $e_{max} = \frac{nr}{2} + \frac{nm}{2}\delta$.

So, given n, δ, r fitting one of the cases in this group, possible edge numbers will be of the form:

$$e_k = \tfrac{nr}{2} + \tfrac{k}{2}\delta,\ k \in [0, 1, 2, ..., nm]$$

Furthermore, by Corollary 1.1, $\frac{k}{2} = \frac{2e-nr}{2\delta}$ and thus $k = \frac{2e-nr}{\delta}$, the degree sequence for e_k will consist of:

$$n - (k)(\text{mod } n) \text{ entries of degree } \lfloor \tfrac{k}{n} \rfloor \delta + r$$

$$(k)(\text{mod } n) \text{ entries of degree } \lceil \tfrac{k}{n} \rceil \delta + r$$

Group 2: As with Group 1, the smallest possible edge number for Group 2 will be $e_0 = \frac{nr}{2}$ with degree sequence $(\underbrace{r, r, r, ..., r, r}_{n})$ since in each case, at least one of n and r is even.

Suppose we have a possible edge number e_i with degrees $(c-1)\delta + r$ and $c\delta + r$ and even degree sum s. Increasing a single degree by δ will yield a degree sum of $s + \delta$, an odd number since δ is odd. Increasing two degrees by δ will give us a degree sum of $s + 2\delta$, an even number. To maintain (δ, r)-balance, we increase two $(c-1)\delta+r$ entries by δ or, if there is only one $(c-1)\delta + r$ entry, increase that entry and one of the $c\delta + r$ entries by δ as well. Thus, our degree sum must increase by 2δ and our edge number will increase by δ. That is, $e_{i+1} = e_i + \delta$.

In the two cases here where n is even, the largest possible edge number will have degree sequence $(\underbrace{m\delta + r, m\delta + r, ..., m\delta + r}_{n})$. Whether $m\delta + r$ is even or odd, $n(m\delta + r)$ will be an even degree sum. In the lone case where n is odd, if m is even, then $m\delta + r$ is even since δ is odd and r is even. This means $(\underbrace{m\delta + r, m\delta + r, ..., m\delta + r}_{n})$ is still a possible degree sequence. In all of the above instances, the largest degree sum is $n(m\delta + r)$ and the largest edge number is $e_{max} = \frac{nr}{2} + \frac{nm}{2}\delta$.

In the case that n, m are both odd our largest degree sequence will be $(m\delta + r, m\delta + r, ...m\delta + r, (m-1)\delta + r)$ with degree sum $(n-1)(m\delta + r) + (m-1)\delta + r = (nm-1)\delta + nr$ and edge number $e_{max} = \frac{nr}{2} + \frac{nm-1}{2}\delta$. Thus, for all cases in this group we can write the largest edge number as $e_{max} = \frac{nr}{2} + \lfloor \frac{nm}{2} \rfloor \delta$

So, for n, δ, r that fit the conditions of this group, possible edge numbers will be of the form:

$$e_k = \frac{nr}{2} + k\delta,\ k \in [0, 1, 2, ..., \lfloor \frac{nm}{2} \rfloor]$$

Furthermore, Corollary 1.1 tells us that $k = \frac{2e-nr}{2\delta}$ and so $2k = \frac{2e-nr}{\delta}$ and the degree sequence of e_k will consist of:

$$n - (2k)(\text{mod } n) \text{ entries of degree } \lfloor \tfrac{2k}{n} \rfloor \delta + r$$
$$(2k)(\text{mod } n) \text{ entries of degree } \lceil \tfrac{2k}{n} \rceil \delta + r$$

Group 3: For the single case in Group 3, $(\underbrace{r, r, r, ..., r, r}_{n})$ is not a possible degree sequence since n and r are both odd. But $nr + \delta$ is even so we can increase a single degree by δ. Thus, the first possible degree sequence will be $(\delta + r, \underbrace{r, r, ..., r}_{n-1})$ with edge number $e_0 = \frac{nr}{2} + \frac{\delta}{2}$.

Suppose we have a possible edge number e_i with the appropriate degree sequence and even degree sum s. As in Group 2, an odd δ means that we must increase s by 2δ by adding δ to two entries to maintain an even degree sum. This increases the edge number by δ, $e_{i+1} = e_i + \delta$.

If m is odd, $m\delta + r$ is even since δ, r are odd. Thus, $(\underbrace{m\delta + r, m\delta + r, ..., m\delta + r}_{n})$ is the largest possible degree sequence since the degree sum $n(m\delta + r)$ is even. This gives us $e_{max} = \frac{nr}{2} + \frac{nm}{2}\delta = \frac{nr}{2} + \frac{\delta}{2} + \frac{nm-1}{2}\delta$.

If m is even, $m\delta + r$ is odd and thus $(m\delta + r, m\delta + r, ...m\delta + r, (m-1)\delta + r)$ will be our largest possible degree sequence. Here, $e_{max} = \frac{nr}{2} + \frac{nm-1}{2}\delta = \frac{nr}{2} + \frac{\delta}{2} + \frac{nm-2}{2}\delta$.

In either case, we can write $e_{max} = \frac{nr}{2} + \frac{\delta}{2} + \lfloor\frac{nm-1}{2}\rfloor\delta$.

So, if n, δ, r are all odd, our possible edge numbers will be:

$$e_k = \frac{nr}{2} + \frac{\delta}{2} + k\delta = \frac{nr}{2} + (k + \tfrac{1}{2})\delta,\ k \in [0, 1, 2, ..., \lfloor\tfrac{nm-1}{2}\rfloor]$$

By Corollary 1.1, each of these edge numbers implies a degree sequence with the following entries:

$$n - (2k+1)(\text{mod } n) \text{ entries of degree } \lfloor\tfrac{2k+1}{n}\rfloor\delta + r$$
$$(2k+1)(\text{mod } n) \text{ entries of degree } \lceil\tfrac{2k+1}{n}\rceil\delta + r$$

Noting the pattern inherent in the results above, we can state the possible edge numbers for a given n, δ, r as follows.

Theorem 1.2 *Given integers δ, r such that $0 \le r \le \delta - 1$ and positive integer $n > r$ such that if δ is even then at least one of n, r is even, the possible edge numbers of a (δ, r)-balanced graph on n vertices are of the form:*

$$e = \frac{nr}{2} + l\delta$$

$$l = \begin{cases} \frac{k}{2},\ k \in [0,1,2,...,nm] & \text{if } \delta \text{ is even} \\ k,\ k \in [0,1,2,...,\lfloor \frac{nm}{2} \rfloor] & \text{if } \delta \text{ is odd and at least one of } n,r \text{ is even} \\ k + \frac{1}{2},\ k \in [0,1,2,...,\lfloor \frac{nm-1}{2} \rfloor] & \text{if } \delta, n, r \text{ are all odd} \end{cases}$$

Additionally, each e implies a degree sequence consisting of:

$$n - (2l)(mod\ n) \text{ entries of degree } \lfloor \tfrac{2l}{n} \rfloor \delta + r$$

$$(2l)(mod\ n) \text{ entries of degree } \lceil \tfrac{2l}{n} \rceil \delta + r$$

1.3 Realizable Degree Sequences

Theorem 1.3 *Let G be a simple graph on $n = a_1 + a_2$ vertices with a_1 vertices of degree d_1. Call this set of vertices A_1. Let the remaining a_2 vertices have degree d_2 and call this set of vertices A_2. Let x be the number of edges connecting two vertices in A_1. Then the following inequalities hold:*

$$i)\quad 0 \leq x \leq \tfrac{1}{2}a_1(a_1 - 1)$$

$$ii)\quad 0 \leq a_1d_1 - 2x \leq a_1a_2$$

$$iii)\quad 0 \leq \tfrac{1}{2}(a_2d_2 - a_1d_1 + 2x) \leq \tfrac{1}{2}a_2(a_2 - 1)$$

Proof The inequalities are clear from Figure 1.1.

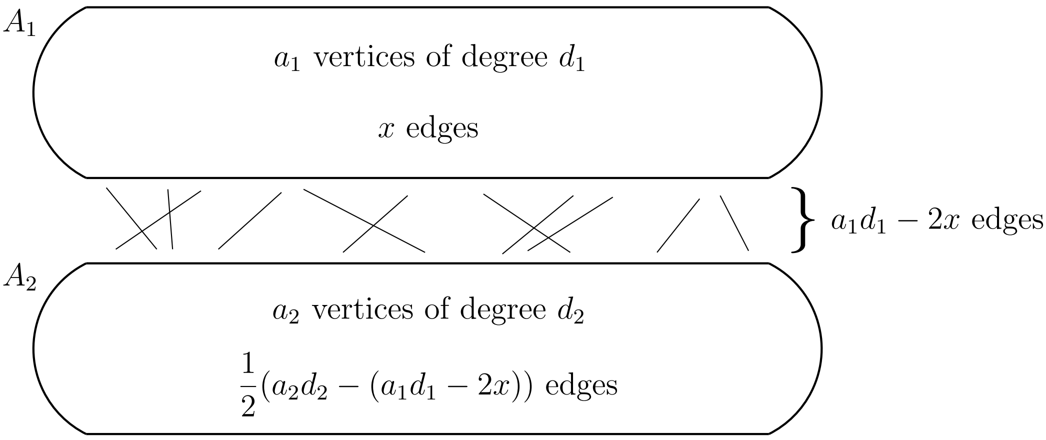

Figure 1.1: Distribution of edges in G

The values in the middle of each string of inequalities are the number of edges i) between vertices in A_1, ii) with one end vertex in A_1 and one end vertex in A_2, and iii) between vertices in A_2. Thus, the left bound in each is clear and the right bound is the maximum number of possible edges of each type.

■

We will use these two lemmas, both proven in [4].

Lemma 1.1 **([4])** *There is a bipartite graph with e edges on bipartition (A, B) where A, B are balanced if and only if $0 \leq e \leq ab$ where $a = |V(A)|$ and $b = |V(B)|$.*

Lemma 1.2 **([4])** *There is a simple, balanced graph on n vertices with e edges if and only if $0 \leq e \leq \binom{n}{2}$.*

Additionally, we will call on the following elementary lemmas relating to balanced integer vectors.

Lemma 1.3 *Let $v_1 = (\underbrace{\alpha, \alpha, ..., \alpha}_{i}, \alpha + 1, \alpha + 1, ..., \alpha + 1)$ and $v_2 = (\underbrace{\beta, \beta, ..., \beta}_{j}, \beta - 1, \beta - 1, ..., \beta - 1)$ be balanced integer vectors of length n. Then $v_1 + v_2$ is a balanced integer vector.*

Proof If $i \leq j$, then

$$v_1 + v_2 = (\underbrace{\alpha + \beta, ..., \alpha + \beta}_{i}, \underbrace{\alpha + 1 + \beta, ..., \alpha + 1 + \beta}_{j-i}, \underbrace{\alpha + \beta, ..., \alpha + \beta}_{n-j})$$

which is a balanced vector.

If $i > j$, then

$$v_1 + v_2 = (\underbrace{\alpha + \beta, ..., \alpha + \beta}_{j}, \underbrace{\alpha + \beta - 1, ..., \alpha + \beta - 1}_{i-j}, \underbrace{\alpha + \beta, ..., \alpha + \beta}_{n-i})$$

another balanced vector.

■

Lemma 1.4 *Let $v_1 = (\delta_1, \delta_2, \delta_3, ..., \delta_n)$ be a balanced nonnegative integer vector. If $\sum_{i=1}^{n} \delta_i = \lambda n$ for some integer λ, then $\delta_i = \lambda$ for all i.*

Proof Suppose $\delta_j < \lambda$ for some $1 \leq j \leq n$. Then $\sum_{i=1}^{n} \delta_i \leq \delta_j + \sum_{i=1}^{n-1} \lambda < \lambda n$. Similarly, no δ_j can be greater than λ.

■

Theorem 1.4 *Let a_1, a_2, d_1, d_2, and x be nonnegative integers such that $a_2d_2 - a_1d_1$ is even. If the following system of inequalities holds, then there exists a simple graph consisting of a_1 vertices of degree d_1 and a_2 vertices of degree d_2:*

$$i) \quad 0 \leq x \leq \tfrac{1}{2}a_1(a_1 - 1)$$

$$ii) \quad 0 \leq a_1d_1 - 2x \leq a_1a_2$$

$$iii) \quad 0 \leq \tfrac{1}{2}(a_2d_2 - a_1d_1 + 2x) \leq \tfrac{1}{2}a_2(a_2 - 1)$$

Proof By *ii* and Lemma 1.1, there is a bipartite graph with $a_1d_1 - 2x$ edges on bipartition (A, B) where $|A| = a_1$, $|B| = a_2$, and A, B are both balanced. Form such a graph and call it G_1.

By *i* and Lemma 1.2, there is a simple, balanced graph on a_1 vertices with x edges. Form such a graph on A and call it G_2. Since $a_2d_2 - a_1d_1$ is even and thus $\frac{1}{2}(a_2d_2 - a_1d_1 + 2x)$ is an integer, *iii* and Lemma 1.2 imply that there is a simple, balanced graph on a_2 vertices with $\frac{1}{2}(a_2d_2 - a_1d_1 + 2x)$ edges. Form such a graph on B and call it G_3.

Consider $G = G_1 \bigcup G_2 \bigcup G_3$. By Lemma 1.3, it is possible to form G such that A and B are both balanced in G.

There will be $2x + (a_1d_1 - 2x) = a_1d_1$ edge ends in A. Thus, the degree sequence of A is a balanced vector and the sum of its entries is an integer multiple of the number of entries. By Lemma 1.4, the degree of each of the a_1 vertices in A must be d_1.

Similarly, there will be $(a_2d_2 - a_1d_1 + 2x) + (a_1d_1 - 2x) = a_2d_2$ edge ends in B. Again, we have a balanced degree sequence whose sum is an integer multiple of the number of entries, showing that all a_2 vertices in B have degree d_2.

■

Note that the values for a_1, a_2, d_1, d_2 coming from any possible number of edges from the previous section will yield an even $a_2d_2 - a_1d_1$.

We can isolate the variable x in the system of inequalities appearing in the previous two theorems, giving us the system:

i) $0 \leq x \leq \frac{1}{2}a_1(a_1 - 1)$

ii) $\frac{1}{2}a_1(d_1 - a_2) \leq x \leq \frac{1}{2}a_1d_1$

iii) $\frac{1}{2}(a_1d_1 - a_2d_2) \leq x \leq \frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1d_1 - a_2d_2)$

This system will have a solution for x if each of the nine inequalities created by picking one of the lower bounds and one of the upper bounds has a solution for x. We will write this system as:

$$\left.\begin{array}{c} 0 \\ \frac{1}{2}a_1(d_1 - a_2) \\ \frac{1}{2}(a_1d_1 - a_2d_2) \end{array}\right\} \leq x \leq \left\{\begin{array}{c} \frac{1}{2}a_1(a_1 - 1) \\ \frac{1}{2}a_1d_1 \\ \frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1d_1 - a_2d_2) \end{array}\right.$$

We can now combine the two theorems above, specifying properties of a_1, a_2, d_1, d_2 that will come from the possible edge numbers generated in the previous section.

Theorem 1.5 *Let $a_1, a_2, d_1, d_2, n,$ and e be nonnegative integers such that $a_1 + a_2 = n$ and $a_1d_1 + a_2d_2 = 2e$. Then there exists a simple graph on n vertices with a_1 vertices of degree d_1 and a_2 vertices of degree d_2 if and only if there is an integer solution to the following system of inequalities:*

$$\left.\begin{array}{c} 0 \\ \frac{1}{2}a_1(d_1 - a_2) \\ \frac{1}{2}(a_1d_1 - a_2d_2) \end{array}\right\} \leq x \leq \left\{\begin{array}{c} \frac{1}{2}a_1(a_1 - 1) \\ \frac{1}{2}a_1d_1 \\ \frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1d_1 - a_2d_2) \end{array}\right.$$

Theorem 1.6 *Let $a_1, a_2, d_1, d_2, n,$ and e be nonnegative integers such that $a_1 + a_2 = n$ and $a_1 d_1 + a_2 d_2 = 2e$. If the following system of inequalities has a solution, then it has an integer solution.*

$$\left.\begin{array}{c} 0 \\ \frac{1}{2}a_1(d_1 - a_2) \\ \frac{1}{2}(a_1 d_1 - a_2 d_2) \end{array}\right\} \leq x \leq \left\{\begin{array}{c} \frac{1}{2}a_1(a_1 - 1) \\ \frac{1}{2}a_1 d_1 \\ \frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1 d_1 - a_2 d_2) \end{array}\right.$$

Proof We will verify that in each of the nine strings of inequalities, at least one of the bounds is an integer and thus, if there is a solution, there is an integer solution. We first note that 0 and $\frac{1}{2}a_1(a_1 - 1)$ are always integers and thus any of the inequality strings involving them will have an integer solution, provided the string has a solution.

Since $a_1 d_1 + a_2 d_2 = 2e$, either $a_1 d_1$ and $a_2 d_2$ are both even or they are both odd. Thus, $a_1 d_1 - a_2 d_2$ is even. So, $\frac{1}{2}(a_1 d_1 - a_2 d_2)$ is an integer. Also, $\frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1 d_1 - a_2 d_2)$ is an integer since $\frac{1}{2}a_2(a_2 - 1)$ is an integer. This takes care of all strings of inequalities except for:

$$\frac{1}{2}a_1(d_1 - a_2) \leq x \leq \frac{1}{2}a_1 d_1$$

$\frac{1}{2}a_1 d_1$ is an integer unless a_1 and d_1 are both odd. Suppose this is the case. Then $a_1 d_1$ is odd and so $a_2 d_2$ must also be odd, implying that a_2 is odd. Thus, $d_1 - a_2$ is even and $\frac{1}{2}a_1(d_1 - a_2)$ is an integer.

■

So, we only need to determine whether the strings of inequalities have a solution, equivalent to determining whether the following system of inequalities holds.

$$\left.\begin{array}{c} 0 \\ \frac{1}{2}a_1(d_1 - a_2) \\ \frac{1}{2}(a_1 d_1 - a_2 d_2) \end{array}\right\} \leq \left\{\begin{array}{c} \frac{1}{2}a_1(a_1 - 1) \\ \frac{1}{2}a_1 d_1 \\ \frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1 d_1 - a_2 d_2) \end{array}\right.$$

Theorem 1.7 *If $a_1, a_2, d_1, d_2, n,$ and e are nonnegative integers such that $d_1, d_2 \leq n-1$, $a_1 + a_2 = n$, $a_1 d_1 + a_2 d_2 = 2e$, then the system of inequalities above is equivalent to the system:*

$$a_2 d_2 - a_1 d_1 \leq a_2(a_2 - 1)$$
$$a_1 d_1 - a_2 d_2 \leq a_1(a_1 - 1)$$

Proof The original system includes the nine inequalities:

$$\begin{array}{rl} i) & 0 \leq \frac{1}{2}a_1(a_1-1) \\ ii) & 0 \leq \frac{1}{2}a_1 d_1 \\ iii) & 0 \leq \frac{1}{2}a_2(a_2-1) + \frac{1}{2}(a_1 d_1 - a_2 d_2) \\ iv) & \frac{1}{2}a_1(d_1 - a_2) \leq \frac{1}{2}a_1(a_1-1) \\ v) & \frac{1}{2}a_1(d_1 - a_2) \leq \frac{1}{2}a_1 d_1 \\ vi) & \frac{1}{2}a_1(d_1 - a_2) \leq \frac{1}{2}a_2(a_2-1) + \frac{1}{2}(a_1 d_1 - a_2 d_2) \\ vii) & \frac{1}{2}(a_1 d_1 - a_2 d_2) \leq \frac{1}{2}a_1(a_1-1) \\ viii) & \frac{1}{2}(a_1 d_1 - a_2 d_2) \leq \frac{1}{2}a_1 d_1 \\ ix) & \frac{1}{2}(a_1 d_1 - a_2 d_2) \leq \frac{1}{2}a_2(a_2-1) + \frac{1}{2}(a_1 d_1 - a_2 d_2) \end{array}$$

Since a_1, a_2, d_1, and d_2 are all nonnegative, i, ii, v, $viii$, and ix are clearly true. Consider iv and note that if $a_1 = 0$, the inequality is true. If $a_1 \neq 0$:

$$\frac{1}{2}a_1(d_1 - a_2) \leq \frac{1}{2}a_1(a_1 - 1)$$
$$d_1 - a_2 \leq a_1 - 1$$
$$d_1 \leq a_1 + a_2 - 1$$
$$d_1 \leq n - 1$$

This was one of our assumptions, so iv will be true.

Consider *vi* and note that if $a_2 = 0$, the inequality is true. If $a_2 \neq 0$:

$$\frac{1}{2}a_1(d_1 - a_2) \leq \frac{1}{2}a_2(a_2 - 1) + \frac{1}{2}(a_1d_1 - a_2d_2)$$
$$a_1d_1 - a_1a_2 \leq a_2(a_2 - 1) + a_1d_1 - a_2d_2$$
$$-a_1a_2 \leq a_2(a_2 - 1) - a_2d_2$$
$$-a_1 \leq a_2 - 1 - d_2$$
$$d_2 \leq a_1 + a_2 - 1 = n - 1$$

This was an assumption, and so the inequality is true. We are left with inequalities *iii* and *vii*, which can be rewritten as:

$$a_2d_2 - a_1d_1 \leq a_2(a_2 - 1)$$
$$a_1d_1 - a_2d_2 \leq a_1(a_1 - 1)$$

■

Theorem 1.8 *The two inequalities:*

$$i)\ a_2d_2 - a_1d_1 \leq a_2(a_2 - 1)$$
$$ii)\ a_1d_1 - a_2d_2 \leq a_1(a_1 - 1)$$

are equivalent to the following string of inequalities:

$$a_1d_1 - \binom{a_1}{2} \leq e \leq \binom{a_2}{2} + a_1d_1$$

Proof Inequality i can be rewritten as follows by substituting $2e - a_1d_1$ for a_2d_2.

$$\begin{aligned}(2e - a_1d_1) - a_1d_1 &\leq a_2(a_2 - 1) \\ 2e &\leq a_2(a_2 - 1) + 2a_1d_1 \\ e &\leq \frac{1}{2}a_2(a_2 - 1) + a_1d_1 = \binom{a_2}{2} + a_1d_1\end{aligned}$$

We proceed analogously for inequality ii.

$$\begin{aligned}a_1d_1 - (2e - a_1d_1) &\leq a_1(a_1 - 1) \\ -2e &\leq a_1(a_1 - 1) - 2a_1d_1 \\ e &\geq a_1d_1 - \frac{1}{2}a_1(a_1 - 1) = a_1d_1 - \binom{a_1}{2}\end{aligned}$$

This yields our desired string of inequalities.

■

So, we can combine all of our results in this section and state this theorem on the existence of graphs in which each vertex has one of two possible degrees.

Theorem 1.9 *Let a_1, a_2, d_1, d_2, n, and e be nonnegative integers such that $d_1, d_2 \leq n - 1$, $a_1 + a_2 = n$, and $a_1d_1 + a_2d_2 = 2e$. Then there exists a simple graph on n vertices with a_1 vertices of degree d_1 and a_2 vertices of degree d_2 if and only if:*

$$a_1d_1 - \binom{a_1}{2} \leq e \leq \binom{a_2}{2} + a_1d_1$$

1.4 Main Result

By combining the results of Theorem 1.2 and Theorem 1.9 we can reach the main conclusion of this chapter.

Theorem 1.10 *Let $n, e,$ and δ be positive integers. Let r be an integer such that $0 \leq r < \delta$ and $r < n$. Let m be the integer such that $m\delta + r < n \leq (m+1)\delta + r$. There is a (δ, r)-balanced graph with e edges on n vertices if and only if $e = \frac{nr}{2} + l\delta$ such that:*

$$\begin{aligned} \tfrac{nr}{2} + l\delta \geq &\ (n - (2l)(mod\ n))(\lfloor \tfrac{2l}{n} \rfloor \delta + r) - \tbinom{n-(2l)(mod\ n)}{2} \\ \tfrac{nr}{2} + l\delta \leq &\ \tbinom{(2l)(mod\ n)}{2} + (n - (2l)(mod\ n))(\lfloor \tfrac{2l}{n} \rfloor \delta + r) \end{aligned}$$

$$\textit{where } l = \begin{cases} \frac{k}{2},\ k \in [0, 1, 2, ..., nm] & \textit{if } \delta \textit{ is even} \\ k,\ k \in [0, 1, 2, ..., \lfloor \frac{nm}{2} \rfloor] & \textit{if } \delta \textit{ is odd and at least one of } n, r \textit{ is even} \\ k + \frac{1}{2},\ k \in [0, 1, 2, ..., \lfloor \frac{nm-1}{2} \rfloor] & \textit{if } \delta, n, r \textit{ are all odd} \end{cases}$$

1.5 A Note on Duality

Theorem 1.11 *There is a simple graph G with e edges on $n = a_1 + a_2$ vertices such that a_1 vertices have degree d_1 and a_2 vertices have degree d_2 if and only if there is a simple graph G^* with $\binom{n}{2} - e$ edges on $n = a_1 + a_2$ vertices such that a_1 vertices have degree $(n-1) - d_1$ and a_2 vertices have degree $(n-1) - d_2$.*

Proof This is clear since $G^* = K_n - E(G)$ and $G = K_n - E(G^*)$.

■

Corollary 1.2 *If G is (δ, r)-balanced, then:*

$$\begin{aligned} &i) \quad (n-1) - d_1 \equiv (n-1) - d_2 \equiv ((n-1) - r)(mod\ \delta) \\ &ii) \quad |((n-1) - d_1) - ((n-1) - d_2)| = |d_2 - d_1| \leq \delta \end{aligned}$$

Proof Let d be the degree of a vertex of G. Since G is (δ, r)-balanced, $d = p\delta + r$ for some $0 \leq p \leq m$, so $(n-1) - d = (n-1) - r - p\delta \equiv ((n-1) - r)(\text{mod } \delta)$. This shows i. ii is obvious.

■

Corollary 1.3 *If G is is (δ, r)-balanced, then G^* is $(\delta, (n-1) - r - m\delta)$-balanced.*

Proof By i in the corollary above, the remainder for G^* is $(n-1) - r - \mu\delta$ for some nonnegative integer μ such that:

$$0 \leq (n-1) - r - \mu\delta < \delta$$

Since $m\delta + r \leq n - 1$:

$$(n-1) - r - m\delta = (n-1) - (m\delta + r) \geq 0$$

Also, $(m+1)\delta + r > n - 1$, so:

$$(n-1) - ((m+1)\delta + r) < 0$$

$$(n-1) - (m\delta + r) - \delta < 0$$

$$(n-1) - (m\delta + r) < \delta$$

■

Chapter 2

Balanced Bipartite Graphs

2.1 Bipartite Graphs with Four Degrees

The following theorem, found in [8], gives necessary and sufficient conditions for the existence of a simple, bipartite graph in which each partition consists of vertices of one of two degrees.

Theorem 2.1 *Let* $a_1, a_2, b_1, b_2, d_1, d_2, f_1, f_2$ *be nonnegative integers. There is a simple, bipartite graph on bipartition* (A, B) *such that* A *consists of* a_1 *vertices of degree* d_1 *and* a_2 *vertices of degree* d_2 *and* B *consists of* b_1 *vertices of degree* f_1 *and* b_2 *vertices of degree* f_2 *if and only if*

$$a_1d_1 + a_2d_2 = b_1f_1 + b_2f_2$$

and the following inequalities are all satisfied:

$$\begin{array}{rl} i) & a_1d_1 \le a_1b_1 + b_2f_2 \text{ or, equivalently } b_1f_1 \le a_1b_1 + a_2d_2 \\ ii) & a_1d_1 \le a_1b_2 + b_1f_1 \text{ or, equivalently } b_2f_2 \le a_1b_2 + a_2d_2 \\ iii) & b_1f_1 \le a_2b_1 + a_1d_1 \text{ or, equivalently } a_2d_2 \le a_2b_1 + b_2f_2 \\ iv) & b_2f_2 \le a_2b_2 + a_1d_1 \text{ or, equivalently } a_2d_2 \le a_2b_2 + b_1f_1 \\ v) & a_1 = 0 \text{ or } d_1 \le b_1 + b_2 \\ vi) & a_2 = 0 \text{ or } d_2 \le b_1 + b_2 \\ vii) & b_1 = 0 \text{ or } f_1 \le a_1 + a_2 \\ viii) & b_2 = 0 \text{ or } f_2 \le a_1 + a_2 \end{array}$$

Proof Note that each side of the first equality counts the number of edges and so is necessary. Also note that it can be used to see the equivalent inequalities in conditions i through iv.

First, we will show that the existence of such a graph is equivalent to a particular system of inequalities having a solution. We will then show that the system is equivalent to conditions i through $viii$. Assume we have a simple, bipartite graph on bipartition (A, B) such that A consists of a_1 vertices of degree d_1 and a_2 vertices of degree d_2 and B consists of b_1 vertices of degree f_1 and b_2 vertices of degree f_2.

Let A_1 be the set of vertices of degree a_1 and A_2 be the set of vertices of degree a_2 in A. Similarly, we have B_1 and B_2. Suppose there are x edges between A_1 and B_1. Then the distribution of edges can be summarized as follows:

	B_1	B_2
A_1	x	$a_1d_1 - x$
A_2	$b_1f_1 - x$	$a_2d_2 - b_1f_1 + x = b_2f_2 - a_1d_1 + x$

Table 2.1: Distribution of Edges

Furthermore, we get the following inequalities based on the maximum and minimum number of possible edges of each type.

$$0 \le x \le a_1b_1$$

$$0 \le a_1d_1 - x \le a_1b_2$$

$$0 \le b_1f_1 - x \le a_2b_1$$

$$0 \le a_2d_2 - b_1f_1 + x \le a_2b_2$$

Isolating x in each of them gives us the system of sixteen inequalities:

$$\left.\begin{array}{c} 0 \\ a_1d_1 - a_1b_2 \\ b_1f_1 - a_2b_1 \\ b_1f_1 - a_2d_2 \end{array}\right\} \le x \le \left\{\begin{array}{c} a_1b_1 \\ a_1d_1 \\ b_1f_1 \\ a_2b_2 - a_2d_2 + b_1f_1 \end{array}\right.$$

Now, suppose that we have nonnegative integers $a_1, a_2, b_1, b_2, d_1, d_2, f_1, f_2$ and disjoint sets of vertices A_1, A_2, B_1, B_2 such that $|A_1| = a_1$, $|A_2| = a_2$, $|B_1| = b_1$, $|B_2| = b_2$. Also, suppose that there is a solution, x, to the system above.

We can use the construction found in [4] to form a simple, bipartite graph G_1 with x edges between A_1 and B_1 such that both parts are balanced. Similarly we can form a simple, bipartite graph G_2 with $a_1d_1 - x$ vertices between A_1 and B_2 so that parts are balanced.

If we consider $G_1 \bigcup G_2$, we can arrange the vertices such that A_1 will be balanced by Lemma 1.3 and each degree in the degree sequence of A_1 will be d_1 by Lemma 1.4. We can construct additional bipartite graphs from A_2 to B_1 and B_2 with $b_1f_1 - x$ and $a_2d_2 - b_1f_1 + x$ edges respectively. As before, the lemmas guarantee we can arrange vertices to get the desired graph with four degrees.

Again, suppose that we have nonnegative integers $a_1, a_2, b_1, b_2, d_1, d_2, f_1, f_2$. We will now show that the system of sixteen inequalities below is equivalent to conditions i through $viii$ stated in the theorem.

$$\left.\begin{array}{c} 0 \\ a_1d_1 - a_1b_2 \\ b_1f_1 - a_2b_1 \\ b_1f_1 - a_2d_2 \end{array}\right\} \leq \left\{\begin{array}{c} a_1b_1 \\ a_1d_1 \\ b_1f_1 \\ a_2b_2 - a_2d_2 + b_1f_1 \end{array}\right.$$

Starting with the inequalities that have 0 on the left, the following are all clear.

$$0 \leq a_1b_1$$
$$0 \leq a_1d_1$$
$$0 \leq b_1f_1$$

The last one is equivalent to condition iv.

$$\begin{aligned} 0 &\leq a_2b_2 - a_2d_2 + b_1f_1 \\ a_2d_2 &\leq a_2b_2 + b_1f_1 \end{aligned}$$

Moving to the inequalities with $a_1d_1 - a_1b_2$ on the left, the first is equivalent to condition v.

$$a_1d_1 - a_1b_2 \leq a_1b_1$$
$$a_1d_1 \leq a_1b_1 + a_1b_2$$
$$d_1 \leq b_1 + b_2$$

The second is clear.

$$a_1d_1 - a_1b_2 \leq a_1d_1$$
$$0 \leq a_1b_2$$

The third is equivalent to condition ii.

$$a_1d_1 - a_1b_2 \leq b_1f_1$$
$$a_1d_1 \leq a_1b_2 + b_1f_1$$

If we recall that $a_1d_1 + a_2d_2 = b_1f_1 + b_2f_2$, the fourth is equivalent to condition $viii$.

$$a_1d_1 - a_1b_2 \leq a_2b_2 - a_2d_2 + b_1f_1$$
$$a_1d_1 + a_2d_2 \leq a_2b_2 + b_1f_1 + a_1b_2$$
$$b_1f_1 + b_2f_2 \leq a_2b_2 + b_1f_1 + a_1b_2$$
$$b_2f_2 \leq a_2b_2 + a_1b_2$$
$$f_2 \leq a_2 + a_1$$

Now, the inequalities with $b_1f_1 - a_2b_1$ on the left. The first is equivalent to condition *vii*.

$$b_1f_1 - a_2b_1 \leq a_1b_1$$
$$b_1f_1 \leq a_1b_1 + a_2b_1$$
$$f_1 \leq a_1 + a_2$$

The second is equivalent to condition *iii*.

$$b_1f_1 - a_2b_1 \leq a_1d_1$$
$$b_1f_1 \leq a_1d_1 + a_2b_1$$

The third is clear.

$$b_1f_1 - a_2b_1 \leq b_1f_1$$
$$0 \leq a_2b_1$$

The fourth is equivalent to condition *vi*.

$$b_1f_1 - a_2b_1 \leq a_2b_2 - a_2d_2 + b_1f_1$$
$$-a_2b_1 \leq a_2b_2 - a_2d_2$$
$$a_2d_2 \leq a_2b_1 + a_2b_2$$
$$d_2 \leq b_1 + b_2$$

Finally, consider the inequalities with $b_1f_1 - a_2d_2$ on the left. The first is equivalent to condition i, the last of the conditions we needed to show.

$$\begin{aligned} b_1f_1 - a_2d_2 &\leq a_1b_1 \\ b_1f_1 &\leq a_1b_1 + a_2d_2 \end{aligned}$$

Noting that $a_1d_1 + a_2d_2 = b_1f_1 + b_2f_2$, the second is clear.

$$\begin{aligned} b_1f_1 - a_2d_2 &\leq a_1d_1 \\ b_1f_1 &\leq a_1d_1 + a_2d_2 \\ b_1f_1 &\leq b_1f_1 + b_2f_2 \\ 0 &\leq b_2f_2 \end{aligned}$$

The third is clear.

$$\begin{aligned} b_1f_1 - a_2d_2 &\leq b_1f_1 \\ 0 &\leq a_2d_2 \end{aligned}$$

To complete the proof, the fourth is clear as well.

$$\begin{aligned} b_1f_1 - a_2d_2 &\leq a_2b_2 - a_2d_2 + b_1f_1 \\ 0 &\leq a_2b_2 \end{aligned}$$

■

2.2 Modular Balanced Bipartite Graphs

Let a, b be positive integers, e be a nonnegative integer and let r_a, r_b, δ be integers such that $0 \leq r_a, r_b \leq \delta - 1$. We say that the bipartite graph G with e edges on bipartition (A, B), with $|A| = a$ and $|B| = b$, is (δ, r_a, r_b)-*balanced* if:

i) For all $u \in A$, $d_G(u) \equiv r_a (\text{mod } \delta)$ and furthermore for all $v \in A$, $|d_G(u) - d_G(v)| \leq \delta$

ii) For all $u \in B$, $d_G(u) \equiv r_b (\text{mod } \delta)$ and furthermore for all $v \in B$, $|d_G(u) - d_G(v)| \leq \delta$

Lemma 2.1 *Let a, b be positive integers, e be a nonnegative integer and let r_a, r_b, δ be integers such that $0 \leq r_a, r_b \leq \delta - 1$. If G is a simple, bipartite graph with e edges on bipartition (A, B), with $|A| = a$ and $|B| = b$ such that G is (δ, r_a, r_b)-balanced, then A consists of:*

$$a - l_a \text{ vertices of degree } \delta q_a + r_a$$

$$l_a \text{ vertices of degree } \delta q_a + r_a + \delta$$

$$\text{where } q_a = \left\lfloor \frac{e - r_a a}{\delta a} \right\rfloor \text{ and } l_a = \frac{e - r_a a}{\delta} (mod\, a)$$

Also, B consists of:

$$b - l_b \text{ vertices of degree } \delta q_b + r_b$$

$$l_b \text{ vertices of degree } \delta q_b + r_b + \delta$$

$$\text{where } q_b = \left\lfloor \frac{e - r_b b}{\delta b} \right\rfloor \text{ and } l_b = \frac{e - r_b b}{\delta} (mod\, b)$$

Proof The proof follows from Theorem 1.1. If G is (δ, r_a, r_b)-balanced, then $e = r_a a + \delta\nu$ for some nonnegative integer ν. Let $0 \leq q_a$ and $0 \leq l_a \leq a - 1$ be the integers such that $\nu = a q_a + l_a$. That is, q_a is the number of times that every vertex in A has had its degree increased by δ from r_a and l_a is the number of vertices whose degrees are increased by an additional δ.

So $e = \delta a q_a + \delta l_a + r_a a$ and we have:

$$\begin{aligned} q_a &= \frac{e - r_a a - \delta l_a}{\delta a} \\ &= \frac{e - r_a a}{\delta a} - \frac{l_a}{a} \\ &= \left\lfloor \frac{e - r_a a}{\delta a} \right\rfloor \end{aligned}$$

Also, we have:

$$\begin{aligned} l_a &= \frac{e - r_a a}{\delta} - a q_a \\ &= \frac{e - r_a a}{\delta} (mod\, a) \end{aligned}$$

We have simliar results for B.

■

If A and B are disjoint sets, $K_{A,B}$ is the complete bipartite graph on bipartition (A, B). The *bipartite complement* of a bipartite graph G on bipartition (A, B) with edge set E is the bipartite graph $G^{'}$ on (A, B) with edge set $E^{'} = E(K_{A,B}) \backslash E$.

We can note here that if G is (δ, r_a, r_b)-balanced on (A, B), then $G^{'}$ is $(\delta, r_a^{'}, r_b^{'})$-balanced where $r_a + r_a^{'} \equiv b \pmod{\delta}$, $r_b + r_b^{'} \equiv a \pmod{\delta}$, $|A| = a$, and $|B| = b$.

The following lemma outlines necessary conditions for the existence of (δ, r_a, r_b)-balanced graphs.

Lemma 2.2 *Let a, b, δ be positive integers, e be a nonnegative integer, and $r_a, r_b, r_a^{'}, r_b^{'} \in \{0, 1, ..., \delta - 1\}$ such that $r_a + r_a^{'} \equiv b$ (mod δ) and $r_b + r_b^{'} \equiv a$ (mod δ). If there is a simple, (δ, r_a, r_b)-balanced bipartite graph with e edges on (A, B) where $|A| = a$ and $|B| = b$, then:*

$$r_a a \leq e \leq ab - r_a^{'} a$$
$$r_b b \leq e \leq ab - r_b^{'} b$$

and all five quantities above are congruent (mod δ).

Proof $e = r_a a + \delta\nu$ for some nonnegative ν so $r_a a \le e$ and the two are congruent (mod δ). Also, $ab - r_a^{'} a = a(b - r_a^{'}) \equiv r_a a$ (mod δ). If a simple, bipartite graph G is (δ, r_a, r_b)-balanced, then $G^{'}$ is $(\delta, r_a^{'}, r_b^{'})$-balanced. Thus, $r_a^{'} a \le ab - e$ and we can get $e \le ab - r_a^{'} a$. Similarly, we can get the desired conditions for B. ■

2.2.1 $\delta = 2$

With certain exceptions, the conditions in Lemma 2.2 are also sufficient for $\delta = 2$. The result, originally in [8], is stated here with an additional class of exceptions overlooked by the author and the omission of several erroneous exceptions.

Theorem 2.2 *Let a, b be positive integers, e be a nonnegative integer, and $r_a, r_b, r_a^{'}, r_b^{'} \in \{0, 1\}$ such that $r_a + r_a^{'} \equiv b$ (mod 2) and $r_b + r_b^{'} \equiv a$ (mod 2). There is a simple, $(2, r_a, r_b)$-balanced bipartite graph with e edges on (A, B) where $|A| = a$ and $|B| = b$ if and only if:*

$$r_a a \le e \le ab - r_a^{'} a$$
$$r_b b \le e \le ab - r_b^{'} b$$

and all five quantities above are congruent (mod 2) with the following exceptions (and the analogous exceptions of Class II with A and B reversed):

Exception Class I:

$$a \ge 2,\ r_a = 0,\ r_a^{'} = b(mod\ 2)$$
$$b \ge 2,\ r_b = 0,\ r_b^{'} = a(mod\ 2)$$
$$e = 2$$

Exception Class I:*

$$a \geq 2,\ r_a = b(mod\ 2),\ r_a^{'} = 0$$
$$b \geq 2,\ r_b = a(mod\ 2),\ r_b^{'} = 0$$
$$e = ab - 2$$

Exception Class II:

$$a = 2,\ r_a = (m+1)(mod\ 2),\ r_a^{'} = (b - r_a)(mod\ 2)$$
$$b \geq 4,\ r_b = 0,\ r_b^{'} = 0$$
$$e = 2m,\ where\ m \in [2, 3, 4, ..., b-2]$$

This was done by translating the problem of (δ, r_a, r_b)-balanced graphs into the problem of bipartite graphs with four degrees and applying Theorem 2.1.

2.2.2 $\delta = 3$

We will now state the main result of this chapter, a result similar to Theorem 2.2 with list of exceptions for $\delta = 3$.

Theorem 2.3 *Let a, b be positive integers, e be a nonnegative integer, and $r_a, r_b, r_a^{'}, r_b^{'} \in \{0, 1, 2\}$ such that $r_a + r_a^{'} \equiv b\ (mod\ 3)$ and $r_b + r_b^{'} \equiv a\ (mod\ 3)$. There is a simple, $(3, r_a, r_b)$-balanced bipartite graph with e edges on (A, B) where $|A| = a$ and $|B| = b$ if and only if:*

$$r_a a \leq e \leq ab - r_a^{'} a$$
$$r_b b \leq e \leq ab - r_b^{'} b$$

and all five quantities above are congruent (mod 3) with the following exceptions (and the analogous exceptions with A and B reversed) where the necessary conditions are not sufficient:

Exception Class I:

$$a \geq 4,\ r_a = 0,\ r_a^{'} = 0$$

$$b = 6,\ r_b = 1,\ r_b^{'} = (a-1)(mod\ 3)$$

$$e = 9$$

Exception Class I:*

$$a = 6,\ r_a = (b-1)(mod\ 3),\ r_a^{'} = 1$$

$$b \geq 4,\ r_b = 0,\ r_b^{'} = 0$$

$$e = 6b - 9$$

Exception Class II:

$$a \geq 6,\ r_a = 0,\ r_a^{'} = 1$$

$$b = 4,\ r_b = 0,\ r_b^{'} = a(mod\ 3)$$

$$e = 15$$

Exception Class II:*

$$a = 4,\ r_a = b(mod\ 3),\ r_a^{'} = 0$$

$$b \geq 6,\ r_b = 1,\ r_b^{'} = 0$$

$$e = 4b - 15$$

Exception Class III:

$$a = 5,\ r_a = 1,\ r_a^{'} = 1$$

$$b = 5,\ r_b = 1,\ r_b^{'} = 1$$

$$e = 11$$

Exception Class III:*

$$a = 5,\ r_a = 1,\ r_a^{'} = 1$$

$$b = 5,\ r_b = 1,\ r_b^{'} = 1$$

$$e = 14$$

Exception Class IV:

$$a \geq 3,\ r_a = 0,\ r_a^{'} = b(mod\ 3)$$

$$b \geq 3,\ r_b = 0,\ r_b^{'} = a(mod\ 3)$$

$$e = 3$$

Exception Class IV:*

$$a \geq 3,\ r_a = b(mod\ 3),\ r_a^{'} = 0$$

$$b \geq 3,\ r_b = a(mod\ 3),\ r_b^{'} = 0$$

$$e = ab - 3$$

Exception Class V:

$$a \geq 3,\ r_a = 0,\ r_a^{'} = b(mod\ 3)$$

$$b \geq 3,\ r_b = 0,\ r_b^{'} = a(mod\ 3)$$

$$e = 6$$

Exception Class V:*

$$a \geq 3,\ r_a = b(mod\ 3),\ r_a^{'} = 0$$

$$b \geq 3,\ r_b = a(mod\ 3),\ r_b^{'} = 0$$

$$e = ab - 6$$

Exception Class VI:

$$a \geq 4,\ r_a = 0,\ r_a^{'} = 0$$

$$b = 3,\ r_b \neq m(mod\ 3),\ r_b^{'} = (a - r_b)(mod\ 3)$$

$$e = 3m,\ where\ m \in [2, 3, 4, ..., a-2]$$

Proof Let a, b be positive integers, e be a nonnegative integer, and $r_a, r_b, r_a^{'}, r_b^{'} \in \{0, 1, 2\}$ such that $r_a + r_a^{'} \equiv b \pmod 3$ and $r_b + r_b^{'} \equiv a \pmod 3$.

(⇒) This implication is clear by Lemma 2.2.

(⇐) Suppose:

$$r_a a \leq e \leq ab - r_a^{'} a$$
$$r_b b \leq e \leq ab - r_b^{'} b$$

and all five quantities above are congruent (mod 3).

We will first translate the problem of a $(3, r_a, r_b)$-balanced bipartite graph to the problem of a bipartite graph with four degrees. Let

$$q_a = \left\lfloor \frac{e - r_a a}{3a} \right\rfloor, \; q_b = \left\lfloor \frac{e - r_b b}{3b} \right\rfloor$$

and

$$l_a = \frac{e - r_a a}{3} (mod\, a), \; l_b = \frac{e - r_b b}{3} (mod\, b)$$

By Lemma 2.1, if a $(3, r_a, r_b)$-balanced bipartite graph exists, it would have the following values for $a_1, a_2, d_1, d_2, b_1, b_2, f_1, f_2$, where, as in Theorem 2.1, A consists of a_1 vertices of degree d_1 and a_2 vertices of degree d_2 and B consists of b_1 vertices of degree f_1 and b_2 vertices of degree f_2:

$a_1 = a - l_a$ $d_1 = 3q_a + r_a$	$b_1 = b - l_b$ $f_1 = 3q_b + r_b$
$a_2 = l_a$ $d_2 = 3q_a + r_a + 3$	$b_2 = l_b$ $f_2 = 3q_b + r_b + 3$

Table 2.2: Translation to Bipartite Graph with Four Degrees

We will now verify that the necessary inequalities hold for the exceptions but no such graphs exist. For Class I, we have:

$$0 \leq 9 \leq 6a \checkmark (\text{since } a \geq 4)$$

and:

$$\text{and } 6 \leq 9 \leq 6a - 6r_b^{'} = 6(a - r_b^{'}) \checkmark (\text{since } a - r_b^{'} \geq 2)$$

All five terms are congruent to 0 (mod 3). However, we get the following values for $a_1, a_2, d_1, d_2, b_1, b_2, f_1, f_2$ and condition *iv* from Theorem 2.1 does not hold:

$a_1 = a - 3$ $d_1 = 0$	$b_1 = 5$ $f_1 = 1$
$a_2 = 3$ $d_2 = 3$	$b_2 = 1$ $f_2 = 4$

Table 2.3: $a_2 d_2 = 9 > 3 + 5 = a_2 b_2 + b_1 f_1$

Obviously, such a graph can't exist since B has a vertex whose degree is larger than the number of vertices of nonzero degree in A.

For Class I*, the inequalities are:

$$6r_a \leq 6b - 9 \leq 6b - 6 \checkmark \text{(since } b \geq 4)$$

and:

$$0 \leq 6b - 9 \leq 6b. \checkmark \text{(since } b \geq 4)$$

Again, all terms are congruent to 0 (mod 3). Since these would be the bipartite complements of Class I, such graphs do not exist.

For Class II:

$$0 \leq 15 \leq 4a - a = 3a \checkmark \text{(since } a \geq 6)$$

and:

$$0 \leq 15 \leq 4a - 4r_b^{'} = 4(a - r_b^{'}) \checkmark \text{(since } a - r_b^{'} \geq 4)$$

Since $a \equiv r_b^{'} \pmod 3$, all the terms are congruent to 0 (mod 3). However, with the following values, condition *iv* from Theorem 2.1 does not hold.

$a_1 = a - 5$ $d_1 = 0$	$b_1 = 3$ $f_1 = 3$
$a_2 = 5$ $d_2 = 3$	$b_2 = 1$ $f_2 = 6$

Table 2.4: $a_2 d_2 = 15 > 5 + 9 = a_2 b_2 + b_1 f_1$

As with Class I, there is a vertex in B with higher degree than the number of vertices of nonzero degree in A.

Class II*, the bipartite complements of Class II:

$$4r_a \leq 4b - 15 \leq 4b \checkmark (\text{since } b \geq 6)$$

and:

$$b \leq 4b - 15 \leq 4b \checkmark (\text{since } b \geq 6)$$

To see congruence (mod 3), note that $r_a = b \pmod 3$ and so $4r_a = r_a + 3r_a \equiv b \pmod 3$, $4b - 15 = b + (3b - 15) \equiv b \pmod 3$, and $4b = b + 3b \equiv b \pmod 3$.

For Class III, both necessary inequalities are $5 \leq 11 \leq 20$ and all terms are congruent (mod 3). However:

$a_1 = 3$ $d_1 = 1$	$b_1 = 3$ $f_1 = 1$
$a_2 = 2$ $d_2 = 4$	$b_2 = 2$ $f_2 = 4$

Table 2.5: $a_2 d_2 = 8 > 4 + 3 = a_2 b_2 + b_1 f_1$

For Class III*, both inequalities are $5 \leq 14 \leq 20$ and all terms are congruent (mod 3). Since the graph from Class III does not exist, neither does this one.

Class IV satisfies the inequalities since:

$$0 \leq 3 \leq ab - ar_a' = a(b - r_a') \checkmark (\text{since } a \geq 3 \text{ and } b - r_a' \geq 1$$

and:

$$0 \leq 3 \leq ab - br_b' = b(a - r_b') \checkmark (\text{since } b \geq 3 \text{ and } a - r_b' \geq 1$$

Since $b - r_a' \equiv 0 \pmod 3$ and $a - r_b' \equiv 0 \pmod 3$, all the terms are congruent to 0 (mod 3). However:

$a_1 = a - 1$ $d_1 = 0$	$b_1 = b - 1$ $f_1 = 0$
$a_2 = 1$ $d_2 = 3$	$b_2 = 1$ $f_2 = 3$

Table 2.6: $a_2d_2 = 3 > 1 + 0 = a_2b_2 + b_1f_1$

Clearly, no such simple, bipartite graphs exist.

For Class IV*, the bipartite complements of Class IV:

$$r_a a \leq ab - 3 \leq ab \checkmark (\text{since } r_a a \leq 2a \leq 3a - 3 \leq ab - 3)$$

and:

$$r_b b \leq ab - 3 \leq ab \checkmark (\text{since } r_b b \leq 2b \leq 3b - 3 \leq ab - 3)$$

Since $r_a \equiv b \pmod 3$ and $r_b \equiv a \pmod 3$, the terms are all congruent to $ab \pmod 3$. Again, such graphs do not exist.

Class V:

$$0 \leq 6 \leq ab - ar_a^{'} = a(b - r_a^{'}) \checkmark (b - r_a^{'} \geq 2 \text{ since if } b = 3,\ r_a^{'} = 0)$$

and:

$$0 \leq 6 \leq ab - br_b^{'} = b(a - r_b^{'}) \checkmark (ba - r_b^{'} \geq 2 \text{ since if } a = 3,\ r_b^{'} = 0)$$

Since $r_a^{'} \equiv b \pmod 3$ and $r_b^{'} \equiv a \pmod 3$, all terms are congruent to $0 \pmod 3$. However, condition iv of Theorem 2.1 is not satisfied.

$a_1 = a - 2$ $d_1 = 0$	$b_1 = b - 2$ $f_1 = 0$
$a_2 = 2$ $d_2 = 3$	$b_2 = 2$ $f_2 = 3$

Table 2.7: $a_2d_2 = 6 > 4 + 0 = a_2b_2 + b_1f_1$

As we have seen in previous classes, there are vertices in one part with degree larger than the number of vertices of nonzero degree in the other part.

Since graphs of Class V do not exist, neither do those of Class V*. To see that the necessary conditions hold:

$$r_a a \leq ab - 6 \leq ab \checkmark (r_a a \leq 2a \leq ab - 6 \text{ since if } b = 3,\ r_a = 0)$$

and:

$$r_b b \leq ab - 6 \leq ab \checkmark (r_b b \leq 2b \leq ab - 6 \text{ since if } a = 3,\ r_b = 0)$$

Since $r_a \equiv b \pmod 3$ and $r_b \equiv a \pmod 3$, the terms are all congruent to $ab \pmod 3$.

Finally, for Class VI, our necessary inequalities are satisfied since $2 \leq m \leq a - 2$.

$$0 \leq 3m \leq 3a \checkmark$$

and:

$$3r_b \leq 3m \leq 3a - 3r_b^{'} = 3(a - r_b^{'}) \checkmark$$

Note that:

$$l_b = \frac{e - r_b b}{3} (mod b) = \frac{3m - 3r_b}{3} (mod 3) = (m - r_b)(mod 3)$$

Thus, if $r_b = (m+1)(\text{mod } 3)$, we have:

$a_1 = a - m$ $d_1 = 0$	$b_1 = 2$ $f_1 < m$
$a_2 = m$ $d_2 = 3$	$b_2 = 1$ $f_2 < m + 3$

Table 2.8: $a_2 d_2 = 3m = m + 2m > a_2 b_2 + b_1 f_1$

If $r_b = (m+2)(\text{mod } 3)$, we have:

$a_1 = a - m$ $d_1 = 0$	$b_1 = 1$ $f_1 < m$
$a_2 = m$ $d_2 = 3$	$b_2 = 2$ $f_2 < m + 3$

Table 2.9: $a_2 d_2 = 3m = 2m + m > a_2 b_2 + b_1 f_1$

Now, we will verify that the conditions of Theorem 2.1 hold for cases other than our exceptions. We first note that the numbers of edge ends in each part are equal because:

$$\begin{aligned} a_1 d_1 + a_2 d_2 &= (a - l_a)(3q_a + r_a) + (l_a)(3q_a + r_a + 3) \\ &= 3aq_a + 3l_a + ar_a - 3l_a q_a + 3l_a q_a - l_a r_a + l_a r_a \\ &= 3aq_a + 3l_a + ar_a \\ &= e \end{aligned}$$

$b_1 f_1 + b_2 f_2 = e$ by a similar argument.

Condition v says:

$$d_1 \leq b_1 + b_2$$

$$3q_a + r_a \leq b$$

Since $3q_a = \dfrac{e - r_a a - 3l_a}{a} = \dfrac{e}{a} - r_a - \dfrac{3l_a}{a}$ and $e \leq ab$, we can get:

$$\begin{aligned} 3q_a + r_a &= \frac{e}{a} - r_a - \frac{3l_a}{a} + r_a \\ &= \frac{e}{a} - \frac{3l_a}{a} \\ &\leq b - \frac{3l_a}{a} \\ &\leq b \end{aligned}$$

Condition vii can be shown in the same manner.

To show the remaining conditions of Theorem 2.1 we will proceed by cases. First suppose $a_2 = 0 = b_2$. Thus conditions $ii, iii, iv, vi, viii$ are automatically satisfied. To see condition i:

$$a_1 d_1 \leq a_1 b_1 + b_2 f_2$$
$$a_1 d_1 \leq a_1 b_1$$
$$d_1 \leq b_1$$
$$3q_a + r_a \leq b$$

which was shown previously.

Now assume that one of a_2 and b_2 is zero while the other is nonzero. Say, $a_2 = 0$ and $b_2 \neq 0$. Thus, iii, iv, vi are automatically satisfied and we have shown v and vii already. Note that i reduces to vii:

$$b_1 f_1 \leq a_1 b_1 + a_2 d_2$$
$$b_1 f_1 \leq a_1 b_1$$
$$f_1 \leq a_1$$

Also, ii reduces to $viii$:

$$b_2 f_2 \leq a_1 b_2 + a_2 d_2$$
$$b_1 f_2 \leq a_1 b_2$$
$$f_2 \leq a_1$$

Thus, it only remains to show *viii* for this case. That is, we need:

$$f_2 \leq a_1 + a_2$$
$$f_2 \leq a$$
$$3q_b + r_b + 3 \leq a$$

Since $3q_b = \frac{e - r_b b - 3l_b}{b} = \frac{e}{b} - r_b - \frac{3l_b}{b}$, $e \leq ab$, and $b_2 = l_b \neq 0$ we have:

$$\begin{aligned}
3q_b + r_b + 3 &= \frac{e}{b} - r_b - \frac{3l_b}{b} + r_b + 3 \\
&= \frac{e}{b} - \frac{3l_b}{b} + 3 \\
&\leq a - \frac{3l_b}{b} + 3 \\
&< a + 3 \\
&\leq a + 2
\end{aligned}$$

So, $a \geq 3q_b + r_b + 1$ and *viii* is satisfied unless $a = 3q_b + r_b + 1$ or $a = 3q_b + r_b + 2$. Suppose $a = 3q_b + r_b + 1$. Since $r_b + r_b^{'} \equiv a \pmod 3$, $r_b^{'} = 1$. But then:

$$\begin{aligned}
3q_b + r_b + 3 &= \frac{e}{b} - \frac{3l_b}{b} + 3 \\
&\leq (a - r_b^{'}) - \frac{3l_b}{b} + 3 && (e \leq ab - r_b^{'} b) \\
&= a + 2 - \frac{3l_b}{b} \\
&< a + 2 && (b_2 = l_b \neq 0)
\end{aligned}$$

This is a contradiction of the assumption that $a = 3q_b + r_b + 1$.

Now suppose $a = 3q_b + r_b + 2$ and thus $r_b^{'} = 2$. Then:

$$\begin{aligned}
3q_b + r_b + 3 &= \frac{e}{b} - \frac{3l_b}{b} + 3 \\
&\leq (a - r_b^{'}) - \frac{3l_b}{b} + 3 && (e \leq ab - r_b^{'}b) \\
&= a + 1 - \frac{3l_b}{b} \\
&< a + 1 && (b_2 = l_b \neq 0)
\end{aligned}$$

This is a contradiction of the assumption that $a = 3q_b + r_b + 2$. This completes the case.

Finally, assume that $a_2 = l_a$ and $b_2 = l_b$ are both nonzero. Note that vi and $viii$ reduce to $3q_a + r_a + 3 \leq b$ and $3q_b + r_b + 3 \leq a$ which can both be shown in the same manner as condition $viii$ was verified in the previous case.

To verify i, we need to show:

$$\begin{aligned}
a_1 d_1 &\leq a_1 b_1 + b_2 f_2 \\
(a - l_a)(3q_a + r_a) &\leq (a - l_a)(b - l_b) + l_b(3q_b + r_b + 3) \\
(a - l_a)(3q_a + r_a - b + l_b) &\leq l_b(3q_b + r_b + 3) \\
(a - l_a)(l_b - (b - 3q_a - r_a)) &\leq l_b(3q_b + r_b + 3)
\end{aligned}$$

Since $(a - l_a), l_b, 3q_b + r_b + 3$ are all positive, the inequality is automatically true if $l_b \leq b - 3q_a - r_a$. So, assume $l_b > b - 3q_a - r_a = b - (3q_a + r_a)$.

$$
\begin{aligned}
a_1d_1 &\leq a_1b_1 + b_2f_2 \\
(a - l_a)(3q_a + r_a) &\leq (a - l_a)(b - l_b) + l_b(3q_b + r_b + 3) \\
3aq_a + ar_a - l_a(3q_a + r_a) &\leq ab - al_b - bl_a + l_al_b + l_b(3q_b + r_b + 3) \\
3aq_a + ar_a - l_a(3q_a + r_a) + 3l_a - 3l_a &\leq ab - al_b - bl_a + l_al_b + l_b(3q_b + r_b + 3) \\
(3aq_a + ar_a + 3l_a) - l_a(3q_a + r_a + 3) &\leq ab - al_b - bl_a + l_al_b + l_b(3q_b + r_b + 3) \\
e - l_a(3q_a + r_a + 3) &\leq ab - al_b - bl_a + l_al_b + l_b(3q_b + r_b + 3)
\end{aligned}
$$

$$e + l_a(b - (3q_a + r_a + 3)) + l_b(a - (3q_b + r_b + 3)) - l_al_b \leq ab$$

Since $b - (3q_a + r_a + 3) < b - (3q_a + r_a) < l_b$, we have that $l_a(b - (3q_a + r_a + 3)) < l_al_b$ and thus:

$$
\begin{aligned}
e + l_a(b - (3q_a + r_a + 3)) + l_b(a - (3q_b + r_b + 3)) - l_al_b &< e + l_al_b + l_b(a - (3q_b + r_b + 3)) - l_al_b \\
&= e + l_b(a - (3q_b + r_b + 3))
\end{aligned}
$$

Here we can use the fact that $e = (b - l_b)(3q_b + r_b) + l_b(3q_b + r_b + 3)$ and get:

$$
\begin{aligned}
e + l_b(a - (3q_b + r_b + 3)) &= (b - l_b)(3q_b + r_b) + l_b(3q_b + r_b + 3) + l_b(a - (3q_b + r_b + 3)) \\
&= (b - l_b)(3q_b + r_b) + al_b
\end{aligned}
$$

And since $3q_b + r_b \leq a$ as proved in *vii*:

$$
\begin{aligned}
(b - l_b)(3q_b + r_b) + al_b &\leq (b - l_b)a + al_b \\
&= ab
\end{aligned}
$$

This shows condition i.

To show condition ii, we need to show that:

$$a_1 d_1 \leq a_1 b_2 + b_1 f_1$$
$$(a - l_a)(3q_a + r_a) \leq (a - l_a)l_b + (b - l_b)(3q_b + r_b)$$
$$(a - l_a)(3q_a + r_a - l_b) \leq (b - l_b)(3q_b + r_b)$$

Note that if $3q_a + r_a \leq l_b$, the inequality is satisfied since $a - l_a$, $b - l_b$, and $3q_b + r_b$ are all nonnegative. So, assume $3q_a + r_a + 1 \geq l_b$. We need to show:

$$(a - l_a)(3q_a + r_a) \leq (a - l_a)l_b + (b - l_b)(3q_b + r_b)$$
$$3aq_a + ar_a - l_a(3q_a + r_a) \leq al_b - l_a l_b + 3bq_b + br_b - l_b(3q_b + r_b)$$
$$3aq_a + ar_a - l_a(3q_a + r_a) + 3l_a - 3l_a \leq al_b - l_a l_b + 3bq_b + br_b - l_b(3q_b + r_b) + 3l_b - 3l_b$$
$$(3aq_a + ar_a + 3l_a) - l_a(3q_a + r_a + 3) \leq al_b - l_a l_b + (3bq_b + br_b + 3l_b) - l_b(3q_b + r_b + 3)$$

Here we can note that $3aq_a + ar_a + 3l_a = e = 3bq_b + br_b + 3l_b$, so we need to show:

$$-l_a(3q_a + r_a + 3) \leq al_b - l_a l_b - l_b(3q_b + r_b + 3)$$
$$l_a l_b + l_b(3q_b + r_b + 3) \leq al_b + l_a(3q_a + r_a + 3)$$

We assumed that $3q_a + r_a + 1 \geq l_b$, so $l_a l_b < l_a(3q_a + r_a + 3)$. Also, we know from $viii$ that $3q_b + r_b + 3 \leq a$. Thus, we have:

$$\begin{aligned} l_a l_b + l_b(3q_b + r_b + 3) &\leq l_a(3q_a + r_a + 3) + l_b(3q_b + r_b + 3) \\ &\leq l_a(3q_a + r_a + 3) + al_b \end{aligned}$$

This shows ii. The proof of iii is analogous, switching parts A and B.

So, it only remains to show condition *iv*. To do this, we need to show:

$$b_2 f_2 \leq a_2 b_2 + a_1 d_1$$
$$l_b(3q_b + r_b + 3) \leq l_a l_b + (a - l_a)(3q_a + r_a)$$

or:

$$a_2 d_2 \leq a_2 b_2 + b_1 f_1$$
$$l_a(3q_a + r_a + 3) \leq l_a l_b + (b - l_b)(3q_b + r_b)$$

These inequalities can be reduced to:

$$l_b(3q_b + r_b + 3 - l_a) \leq (a - l_a)(3q_a + r_a)$$

and:

$$l_a(3q_a + r_a + 3 - l_b) \leq (b - l_b)(3q_b + r_b)$$

So, if either $3q_b + r_b + 3 \leq l_a$ or $3q_a + r_a + 3 \leq l_b$, the inequalities are satisfied.

We know from *viii* that $3q_b + r_b + 3 \leq a$. So, we have:

$$\begin{aligned} l_b(3q_b + r_b + 3) &\leq a l_b \\ &= l_a l_b + (a - l_a) l_b \end{aligned}$$

So, if $l_b \leq 3q_a + r_a$, the inequalities are satisfied. Similarly, if $l_a \leq 3q_b + r_b$, they are satisfied as well. That is, we only have problems when:

$$3q_a + r_a + 1 \leq l_b \leq 3q_a + r_a + 2$$

and:

$$3q_b + r_b + 1 \leq l_a \leq 3q_b + r_b + 2$$

Case 1: Suppose $l_a = 3q_b + r_b + 2$ and $l_b = 3q_a + r_a + 1$. We need to show:

$$\begin{aligned}
b_2 f_2 &\leq a_2 b_2 + a_1 d_1 \\
l_b(3q_b + r_b + 3) &\leq l_a l_b + (a - l_a)(3q_a + r_a) \\
(3q_a + r_a + 1)(3q_b + r_b + 3) &\leq (3q_b + r_b + 2)(3q_a + r_a + 1) + (a - l_a)(3q_a + r_a) \\
3q_a + r_a + 1 &\leq (a - l_a)(3q_a + r_a) \\
1 &\leq (a - l_a - 1)(3q_a + r_a) \\
1 &\leq (a - l_a - 1)(l_b - 1)
\end{aligned}$$

Since $a \geq 3q_b + r_b + 3 = l_a + 1$ by condition *viii*, $a - l_a - 1 \geq 0$. Also, we assumed $l_b \geq 1$. Thus, the inequality is true unless $a - l_a - 1 = 0$ or $l_b - 1 = 0$.

Suppose $l_b - 1 = 0$. That is, $l_b = 3q_a + r_a + 1 = 1$ and so $q_a = r_a = 0$. Consider the number of edges, e, calculated based on both A and B.

$$\begin{aligned}
3aq_a + 3l_a + ar_a &= 3q_b + 3l_b + br_b \\
3l_a &= 3q_b + 3l_b + br_b && (q_a = r_a = 0) \\
3(3q_b + r_b + 2) &= 3q_b + 3l_b + br_b \\
3(3q_b + r_b) + 6 &= b(3q_b + r_b) + 3 && (l_b = 1) \\
3 &= (b - 3)(3q_b + r_b)
\end{aligned}$$

Since $b - 3$ is an integer and $3q_b + r_b$ is a nonnegative integer, their product can be 3 only when one of those terms is 3 and one of them is 1.

If $b - 3 = 3$ and $3q_b + r_b = 1$, then $b = 6$, $q_b = 0$, and $r_b = 1$. Thus $l_a = 3q_b + r_b + 2 = 3$ and $a \geq 4$. Also, $e = 3l_a = 9$. These are the Class I exceptions.

If $b - 3 = 1$ and $3q_b + r_b = 3$, then $b = 4$, $q_b = 1$, and $r_b = 0$. Thus $l_a = 5$, $a \geq 6$, and $e = 3l_a = 15$. This gives us the Class II exceptions.

Suppose $a - l_a - 1 = 0$. Again, consider the number of edges, e, calculated as the number of edge ends in A.

$$\begin{aligned} 3aq_a + 3l_a + ar_a &= 3aq_a + 3(a-1) + ar_a \\ &= a(3q_a + r_a + 3) - 3 \\ &= a(l_b + 2) - 3 \end{aligned}$$

Now, beginning with e calculated from B:

$$\begin{aligned} 3q_b + 3l_b + br_b &= b(3q_b + r_b) + 3l_b \\ &= b(a-3) + 3l_b \\ &= ab - 3(b - l_b) \end{aligned}$$

So, we have:

$$\begin{aligned} a(l_b + 2) - 3 &= ab - 3(b - l_b) \\ a(l_b + 2) - ab &= 3 - 3(b - l_b) \\ 2a - a(b - l_b) &= 3 - 3(b - l_b) \\ 2a - 3 &= (a-3)(b - l_b) \\ \frac{2a-3}{a-3} &= b - l_b \end{aligned}$$

Note that if $a = 3$ then $l_a = 3q_b + r_b + 2 = a - 1 = 2$, and thus $q_b = r_b = 0$. But then, equating our two edge counts:

$$3aq_a + 3l_a + ar_a = 3q_b + 3l_b + br_b$$
$$9q_a + 3l_a + 3r_a = 3l_b$$
$$3q_a + l_a + r_a = l_b$$
$$3q_a + 2 + r_a = l_b$$

This contradics our assumption for l_b.

The only positive integers, a, for which $\frac{2a-3}{a-3}$ yields a positive integer for $b - l_b$ are $a = 4$ and $a = 6$.

If $a = 6$, then $\frac{2a-3}{a-3} = b - l_b = 3$ and $e = ab - 3(b - l_b) = ab - 9 = 6b - 9$. This gives us the Class I* exceptions, the bipartite complements of Class I.

If $a = 4$, then $\frac{2a-3}{a-3} = b - l_b = 5$ and $e = ab - 3(b - l_b) = ab - 15 = 6b - 15$. These are the Class II* exceptions, the bipartite complements of Class II.

Case 2: Suppose $l_a = 3q_b + r_b + 1$ and $l_b = 3q_a + r_a + 1$. We need to show:

$$b_2 f_2 \leq a_2 b_2 + a_1 d_1$$
$$l_b(3q_b + r_b + 3) \leq l_a l_b + (a - l_a)(3q_a + r_a)$$
$$(3q_a + r_a + 1)(3q_b + r_b + 3) \leq (3q_b + r_b + 1)(3q_a + r_a + 1) + (a - l_a)(3q_a + r_a)$$
$$2(3q_a + r_a + 1) \leq (a - l_a)(3q_a + r_a)$$
$$2 \leq (a - l_a - 2)(3q_a + r_a)$$
$$2 \leq (a - l_a - 2)(l_b - 1)$$

We know that $a \geq 3q_b + r_b + 3 = l_a + 2$ from condition *viii* of Theorem 2.1 and $l_b \geq 1$ by assumption. Thus both terms on the right are at least 0 and the inequality is satisfied unless $a - l_a - 2 = 0$, $l_b - 1 = 0$, or both are equal to 1.

Suppose that both $a - l_a - 2 = 1$ and $l_b - 1 = 1$. Thus $l_a = a - 3$ and $l_b = 2$, implying $q_a = 0$ and $r_a = 1$. Starting with our two calculations of e:

$$\begin{aligned} 3aq_a + 3l_a + ar_a &= 3q_b + 3l_b + br_b \\ 3l_a + a &= b(3q_b + r_b) + 6 \\ 3(3q_b + r_b) + 3 + a &= b(3q_b + r_b) + 6 \\ a - 3 &= (b-3)(3q_b + r_b) \\ l_a &= (b-3)(l_a - 1) \end{aligned}$$

Since l_a is a nonnegative integer, this can only be true when $l_a = 2$ and $b - 3 = 2$. So, $b = 5$, $q_b = 0$, $r_b = 1$, $a = l_a + 3 = 5$, and $e = 3l_a + a = 11$. This is the Class III exception.

Suppose $l_b - 1 = 0$. Then $l_b = 3q_a + r_a + 1 = 1$ and so $q_a = r_a = 0$.

$$\begin{aligned} e &= 3aq_a + 3l_a + ar_a \\ &= 3l_a \\ &= 3(3q_b + r_b + 1) \\ &= 3(3q_b + r_b) + 3 \end{aligned}$$

Also:

$$\begin{aligned} e &= 3bq_b + 3l_b + br_b \\ &= b(3q_b + r_b) + 3 \end{aligned}$$

Thus:

$$\begin{aligned} 3(3q_b + r_b) + 3 &= b(3q_b + r_b) + 3 \\ 0 &= (b-3)(3q_b + r_b) \end{aligned}$$

So, we have problems if $b = 3$ or $3q_b + r_b = 0$.

If $3q_b + r_b = 0$, then $q_b = r_b = 0$, $l_a = 3q_b + r_b + 1 = 1$, and $e = 3l_a = 3$. This gives us the exceptions of Class IV.

Suppose $b = 3$. Recall that we already know $r_a = 0$ and $e = 3l_a$. Also, in this case we know that $a \geq l_a + 2$. Furthermore:

$$l_b = 1 = \frac{e - r_b b}{3}(mod b) = \frac{3l_a - r_b 3}{3}(mod 3) = (l_a - r_b)(mod 3)$$

This means that $r_b = (l_a - 1)(\text{mod } 3)$. So we have found a subset of the Class VI exceptions, specifically those where $r_b = (l_a - 1)(\text{mod } 3)$. Note that if $l_a = 1$ or $a = 3$, we get a Class IV exception.

Suppose $a - l_a - 2 = 0$ and thus $l_a = a - 2$ and $a = l_a + 2 = 3q_b + r_b + 3$.

$$\begin{aligned} e &= 3aq_a + 3l_a + ar_a \\ &= 3aq_a + 3(a-2) + ar_a \\ &= a(3q_a + r_a + 3) - 6 \\ &= a(l_b + 2) - 6 \end{aligned}$$

Also:

$$\begin{aligned} e &= 3bq_b + 3l_b + br_b \\ &= b(3q_b + r_b) + 3l_b \\ &= b(a-3) + 3l_b \\ &= ab - 3(b - l_b) \end{aligned}$$

So, we have:

$$a(l_b + 2) - 6 = ab - 3(b - l_b)$$

$$a(l_b + 2) - ab = 6 - 3(b - l_b)$$

$$a(l_b - b) + 2a = 6 - 3(b - l_b)$$

$$(3 - a)(b - l_b) = 6 - 2a$$

Meaning that either $b - l_b = \dfrac{6-2a}{3-a} = 2$ or $a = 3$.

If $b - l_b = 2$, then $e = ab - 3(b - l_b) = ab - 6$. This gives us Exception Class V*, the bipartite complements of Class V.

If $a = 3$, we have $l_a = 1$, $q_b = r_b = 0$ and we get the analogous Class VI exceptions as we did above with A and B reversed.

Case 3: Suppose $l_a = 3q_b + r_b + 2$ and $l_b = 3q_a + r_a + 2$. We need to show:

$$b_2 f_2 \leq a_2 b_2 + a_1 d_1$$

$$l_b(3q_b + r_b + 3) \leq l_a l_b + (a - l_a)(3q_a + r_a)$$

$$(3q_a + r_a + 2)(3q_b + r_b + 3) \leq (3q_b + r_b + 2)(3q_a + r_a + 2) + (a - l_a)(3q_a + r_a)$$

$$3q_a + r_a + 2 \leq (a - l_a)(3q_a + r_a)$$

$$2 \leq (a - l_a - 1)(3q_a + r_a)$$

$$2 \leq (a - l_a - 1)(l_b - 2)$$

Similar to previous cases, the inequality is satisfied unless $a - l_a - 1 = 0$, $l_b - 2 = 0$, or both are equal to 1.

Suppose $a - l_a - 1 = 1$ and $l_b - 2 = 1$. Since $l_b = 3q_a + r_a + 2 = 3$, $q_a = 0$ and $r_a = 1$.

$$\begin{aligned}
3aq_a + 3l_a + ar_a &= 3q_b + 3l_b + br_b \\
3l_a + a &= b(3q_b + r_b) + 9 \\
3(3q_b + r_b) + 6 + a &= b(3q_b + r_b) + 9 \\
a - 3 &= (b - 3)(3q_b + r_b) \\
l_a - 1 &= (b - 3)(l_a - 2)
\end{aligned}$$

Since $l_a - 1$ is a nonnegative integer, this is only true when $l_a - 1 = 2$ and $b - 3 = 2$. Thus, $l_a = 3$, $q_b = 0$, $r_b = 1$, $b = 5$, $a = l_a + 2 = 5$, and $e = 3l_a + a = 14$. This is the Class III* exception, the bipartite complement of Class III.

Suppose $l_b = 3q_a + r_a + 2 = 2$. Then $q_a = r_a = 0$.

$$\begin{aligned}
e &= 3aq_a + 3l_a + ar_a \\
&= 3l_a \\
&= 3(3q_b + r_b + 2) \\
&= 3(3q_b + r_b) + 6
\end{aligned}$$

Also:

$$\begin{aligned}
e &= 3bq_b + 3l_b + br_b \\
&= b(3q_b + r_b) + 6
\end{aligned}$$

So we have:

$$\begin{aligned}
3(3q_b + r_b) + 6 &= b(3q_b + r_b) + 6 \\
0 &= (b - 3)(3q_b + r_b)
\end{aligned}$$

Thus, we have problems if $b = 3$ or $3q_b + r_b = 0$.

If $3q_b + r_b = 0$, $q_b = r_b = 0$, $l_a = 3q_b + r_b + 2 = 2$, and $e = 3l_a = 6$. These are the Class V exceptions.

Suppose $b = 3$. We've already assumed that $r_a = 0$ and $e = 3l_a$ where $l_a \geq 2$. Since $l_b = 2$, $r_b = (l_a - 2)(\text{mod } 3)$, filling in the remaining entries in Class VI. Note that if $a = 3$ or $l_a = a - 1$, we get an exception from Class IV*.

Suppose $a - l_a - 1 = 0$. Then $l_a = a - 1$ and $a = l_a + 1 = 3q_b + r_b + 3$.

$$\begin{aligned} e &= 3aq_a + 3l_a + ar_a \\ &= 3aq_a + 3(a-1) + ar_a \\ &= a(3q_a + r_a + 3) - 3 \\ &= a(l_b + 1) - 3 \end{aligned}$$

Also:

$$\begin{aligned} e &= 3bq_b + 3l_b + br_b \\ &= b(3q_b + r_b) + 3l_b \\ &= b(a-3) + 3l_b \\ &= ab - 3(b - l_b) \end{aligned}$$

So:

$$\begin{aligned} a(l_b + 1) - 3 &= ab - 3(b - l_b) \\ a(l_b + 1) - ab &= 3 - 3(b - l_b) \\ a(l_b - b) + a &= 3 - 3(b - l_b) \\ (3 - a)(b - l_b) &= 3 - a \end{aligned}$$

So, either $b - l_b = 1$ or $a = 3$.

If $b - l_b = 1$, then $e = ab - 3(b - l_b) = ab - 3$ and we get the Class IV* exceptions. These are the bipartite complements of Exception Class IV.

If $a = 3$, we get the analogous exceptions of those found previously with A and B reversed.

■

2.2.3 $\delta \geq 4$

By examining the proof of Theorem 2.3, we can see that it will translate to an arbitrary value for δ until we need to show condition *iv* from Theorem 2.1 when both $a_2 = l_a$ and $b_2 = l_b$ are nonzero. This is where the exceptions are found for $\delta = 3$ in Theorem 2.3.

To verify condition *iv* from Theorem 2.1, we need to show one of these equivalent conditions:

$$b_2 f_2 \leq a_2 b_2 + a_1 d_1$$
$$a_2 d_2 \leq a_2 b_2 + b_1 f_1$$

For an arbitrary δ, Lemma 2.1 says this will mean we need to show one of the following:

$$l_b(\delta q_b + r_b + \delta) \leq l_a l_b + (a - l_a)(\delta q_a + r_a)$$
$$l_a(\delta q_a + r_a + \delta) \leq l_a l_b + (b - l_b)(\delta q_b + r_b)$$

Analagous to the case of $\delta = 3$, these will be satisfied unless both of the following are true:

$$\delta q_a + r_a + 1 \leq l_b \leq \delta q_a + r_a + (\delta - 1)$$

and:

$$\delta q_b + r_b + 1 \leq l_a \leq 3q_b + r_b + (\delta - 1)$$

This means we will have $\binom{\delta}{2}$ combinations of l_a and l_b to consider. So, for larger values of δ, we can predict that our list of exceptions in any result similar to Theorem 2.2 or Theorem 2.3 will grow significantly, but will be found in the same way as in the $\delta = 3$ case.

So, we can combine this observation with Lemma 2.1 to state the following, general result.

Conjecture 2.1 *Let a, b be positive integers, e be a nonnegative integer and let $r_a, r_b, r_a', r_b', \delta$ be integers such that $0 \leq r_a, r_b, r_a', r_b' \leq \delta - 1$, $r_a + r_a' \equiv b \pmod{\delta}$, and $r_b + r_b' \equiv a \pmod{\delta}$. There is a simple, bipartite, (δ, r_a, r_b)-balanced graph G with e edges on bipartition (A, B), with $|A| = a$ and $|B| = b$ such that A consists of:*

$$a - l_a \text{ vertices of degree } \delta q_a + r_a$$
$$l_a \text{ vertices of degree } \delta q_a + r_a + \delta$$
$$\text{where } q_a = \left\lfloor \frac{e - r_a a}{\delta a} \right\rfloor \text{ and } l_a = \frac{e - r_a a}{\delta} (mod\, a)$$

and B consists of:

$$b - l_b \text{ vertices of degree } \delta q_b + r_b$$
$$l_b \text{ vertices of degree } \delta q_b + r_b + \delta$$
$$\text{where } q_b = \left\lfloor \frac{e - r_b b}{\delta b} \right\rfloor \text{ and } l_b = \frac{e - r_b b}{\delta} (mod\, b)$$

if and only if the following inequalities are satisfied and all five quantities are congruent (mod δ):

$$r_a a \leq e \leq ab - r_a' a$$
$$r_b b \leq e \leq ab - r_b' b$$

with a list of exceptions that can be found by examining the $\binom{\delta}{2}$ combinations of l_a and l_b such that:

$$\delta q_a + r_a + 1 \leq l_b \leq \delta q_a + r_a + (\delta - 1)$$

and:

$$\delta q_b + r_b + 1 \leq l_a \leq 3 q_b + r_b + (\delta - 1)$$

Chapter 3

Anti-Ramsey Numbers

3.1 Definitions

Throughout the chapter, G will denote a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. C_n and P_n will denote the cycle and path on n vertices, respectively. C_n^+ and C_n^{++} will denote the cycle on n vertices with one and two pendant edges, respectively. Note that, for our purposes, the two pendent edges in C_n^{++} can be pendant to two different vertices or to the same vertex.

A *k-edge-coloring* of G is a labeling of the edges of G with elements of a set of colors S where $|S| = k$ and each color in S is used on at least one edge of G. Throughout the paper, we will refer to an edge-coloring simply as a *coloring*. A *rainbow coloring* of G is a coloring that labels each edge of G with a different color. G is said to be *totally multicolored* in this coloring. At the other end of the spectrum, a *monochromatic coloring* of G is a coloring that labels each edge of G with the same color. G is said to be *monochromatic* in this coloring.

Given a set of graphs $G_1, ..., G_k$, the k-color *ramsey number* for this set of graphs, denoted $R(G_1, ..., G_k)$, is the minimum integer n such that any k-edge-coloring of K_n must contain a monochromatic copy of G_i in the color i for some i.

In [6], Ramsey proved that for any such set of graphs, the ramsey number is finite. In general, Ramsey numbers themselves have proven difficult to find but have led to several useful generalizations. One of these, the anti-ramsey numbers, will be the focus of this chapter.

Given graphs $H \subseteq G$, the *anti-ramsey number*, denoted $ar(G, H)$, is the maximum number of colors, k, such that there exists an edge-coloring of G with exactly k colors in which every copy of H has at least two edges labeled the same color. That is, the coloring yields no totally multicolored copy of H.

Given graphs $K, H \subseteq G$, the *sub-anti-ramsey number*, denoted $sar(K, G, H)$, is the maximum number of colors, k, such that if $\kappa > k$ colors are used on any copy of K, then there is a totally multicolored copy of H in G.

Claim If any coloring of G using exactly l colors produces a multicolored copy of H, then any coloring using more than l colors produces a totally multicolored copy of H.

Proof Assume we have colored the edges of G with $\lambda > l$ colors. Denote this color set as $[1, 2, 3, ..., l, l+1, ..., \lambda]$. Recolor all edges colored from the set $[l+1, ..., \lambda]$ with the color l. Thus we have a coloring with exactly l colors, which we assumed produces a totally multicolored H. At most one of the edges in this copy of H was recolored from the original coloring and if it was, its original color does not appear on any other edge in this copy. Thus, in the original coloring, this copy of H was also totally multicolored.

■

In this paper (and indeed in most anti-ramsey results), G will be a large complete graph. We will give partial investigations of the cases where H is a cycle or a path.

The *Turán number* of a graph H is the maximum number of edges a simple graph on n vertices can have without having a subgraph isomorphic to H, denoted $ex(n, H)$.

Claim ([1]) $ar(K_n, H) \leq ex(n, H)$.

Proof Color the edges of K_n with $\lambda > ex(n, H)$ colors. Pick one edge of each color and call this subgraph G'. G' has $\lambda > ex(n, H)$ edges and so therefore must contain a copy of H. Since G' is totally multicolored, this copy of H is totally multicolored.

■

Thus, the Turán numbers form an upper bound for the anti-ramsey numbers. However, it is often not a very useful one.

3.2 Cycles With Pendant Edges

The anti-ramsey number for cycles was conjectured in [1] by Erdős, Simonovits, and Sós and later proven by Montellano-Ballesteros and Neumann-Lara in [5]. We use one specific case that was proven in the original paper.

Theorem 3.1 **([1])** $ar(K_n, C_3) = n - 1$

Proof Order and name the vertices of K_n as $[v_1, v_2, v_3, ..., v_{n-1}, v_n]$ and suppose we have the set of colors $[1, 2, 3, ..., n-2, n-1]$. For each edge $v_i v_j$ where $i < j$, color the edge with color i. We will thus use all $n - 1$ colors and a triangle with vertices v_i, v_j, v_k where $i < j < k$ will have two edges colored i. This shows that $ar(K_n, C_3) \geq n - 1$.

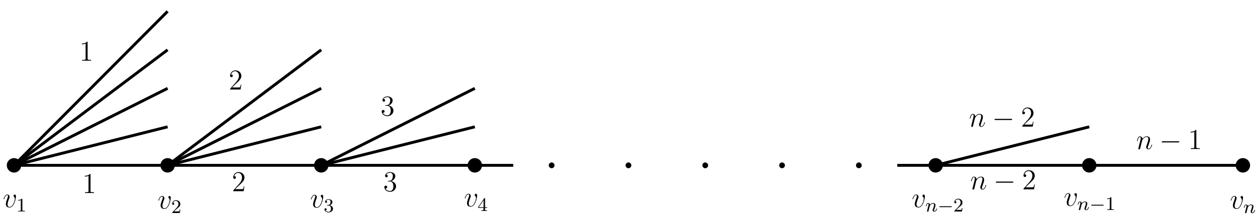

Figure 3.1: $(n - 1)$-coloring of K_n containing no rainbow C_3

Now, color the edges of K_n with n colors and pick one edge of each color. Call this subgraph G'. G' has n edges on at most n vertices so it must contain a cycle. Let C be the smallest cycle in G'. If C is a triangle, then we have a totally multicolored C_3 and we're done. So, assume C is a cycle of length $l > 3$. Call the vertices of C $[c_1, c_2, ..., c_l]$. Pick a pair of nonadjacent vertices in C, c_α, c_β where $\alpha < \beta$, and consider the edge between them. If the edge $c_\alpha c_\beta$ is colored δ, then δ may occur in C but it can only occur at most once since G' and thus C is totally multicolored. It can occur in the subgraph of C induced by $[c_1, c_2, ..., c_{\alpha-1}, c_\alpha, c_\beta, c_{\beta+1}, ..., c_{l-1}, c_l]$ or the subgraph of C induced by $[c_\alpha, c_{\alpha+1}, ..., c_{\beta-1}, c_\beta]$ but not both. Without loss of generality, suppose that δ does not occur in the subgraph induced by $[c_\alpha, c_{\alpha+1}, ..., c_{\beta-1}, c_\beta]$. Thus, we can form a smaller, totally multicolored cycle C' by taking the the edges induced by $[c_\alpha, c_{\alpha+1}, ..., c_{\beta-1}, c_\beta]$ and adding the edge $c_\alpha c_\beta$.

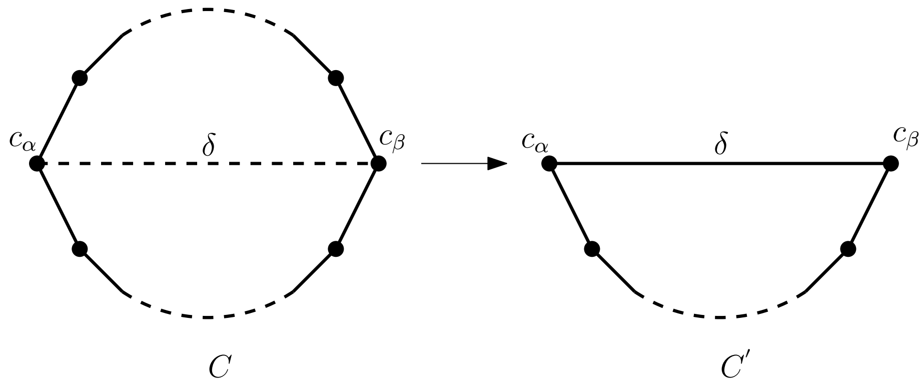

Figure 3.2: Forming C' from C

We can repeat this process until we arrive at a totally multicolored C_3.

■

Theorem 3.2 $ar(K_n, C_3^+) = n - 1$

Proof The coloring used in the preceding theorem will work again here to establish that $ar(K_n, C_3^+) \geq n - 1$.

Color the edges of K_n with n colors and pick one edge of each color. Call this graph G'. G' has n edges on at most n vertices so it must contain a cycle. Let C be the smallest cycle in G'. If C is a triangle, then either it has a pendant edge in C and we are done or $G' - C$ (having $n-3$ edges on at most $n-3$ vertices) must contain another triangle. Pick any edge between these two totally multicolored triangles. This edge along with the triangle in which its color does not appear is a totally multicolored C_3^+.

If C is not a triangle, proceed as in the proof of the previous conjecture to reduce C to a C_3. Let v_1, v_2, v_3, v_4, v_5 be the vertices in C such that the edge v_2v_4 was the edge that reduced the cycle to the triangle $v_2v_3v_4$.

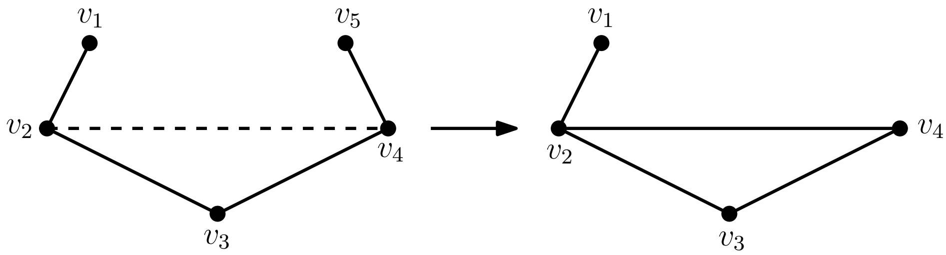

Figure 3.3: Reducing to a totally multicolored C_3^+

The color of v_2v_4 can occur on at most one of v_1v_2 and v_4v_5. Without loss, assume that it does not occur on v_1v_2. Thus, the triangle $v_2v_3v_4$ along with the edge v_1v_2 is a totally multicolored C_3^+.

■

The following, more general result was proven in [3].

Theorem 3.3 **([3])** $ar(K_n, C_m^+) = ar(K_n, C_m)$ *for* $n \geq m + 1$

We now consider two pendant edges. As mentioned before, it is immaterial whether the two edges are pendant to different vertices or not. Whereas adding one pendant edge did not affect the anti-ramsey number, adding two gives us the flexibility to create a coloring with a larger number of colors.

Theorem 3.4 $ar(K_n, C_3^{++}) \geq n > ar(K_n, C_3^+) = ar(K_n, C_3)$

Proof Order and name the vertices of K_n as $[v_1, v_2, v_3, ..., v_{n-1}, v_n]$. Color the edges of the triangle with vertices v_1, v_2, and v_3 with colors 1, 2, and 3. Call this triangle T. For edges v_iv_j where $4 \leq i < j \leq n$, color the edge with color i, bringing us to $n-1$ colors used. Call this subgraph S. Finally, color the remaining edges, those connecting T and S, with color n.

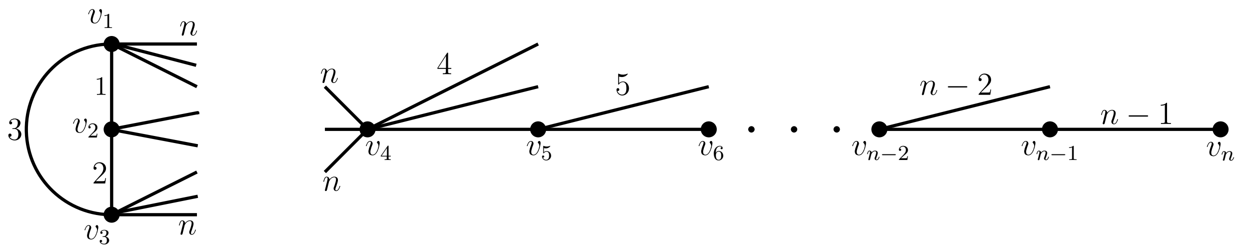

Figure 3.4: n-coloring of K_n containing no rainbow C_3^{++}

By the argument used in the preceding proofs, there can be no totally multicolored triangle within S and clearly there can not be one using edges between T and S. Thus, T is the only totally multicolored triangle in our coloring. Since all edges pendant to T are the same color, there can be no totally multicolored C_3^{++}.

■

And again, the more general result was proven by Gorgol in [3].

Theorem 3.5 **([3])** $ar(K_n, C_m^{++}) > ar(K_n, C_m)$

3.3 Paths

We begin with the following theorem, which states that the only way to avoid a totally multicolored P_4 is to use two or fewer colors. Adding as few as one edge of a third color produces a path on three edges where each of our three colors is used.

Theorem 3.6 $ar(K_n, P_4) = 2$ *for* $n \geq 5$

Proof Obviously, $ar(K_n, P_4) \geq 2$. So, assume we have colored K_n with colors $1, 2, 3$ and pick one edge of each color. Call this graph G'. If G' is a path, we're done. The other possibilities are as follows.

Case 1: G' is three disjoint edges. Let v_k and v_k^* be the vertices on the edge colored k for $1 \leq k \leq 3$. To avoid a totally multicolored $P_4 \subset G$, the edge v_1v_2 must be colored 1 or 2. Without loss, we can assume 2. Then v_2v_3 must be colored 2 for if v_2v_3 is colored 3, $v_1^*v_1v_2v_3$ is totally multicolored and if v_2v_3 is colored 1, $v_3^*v_3v_2v_1$ is totally multicolored.

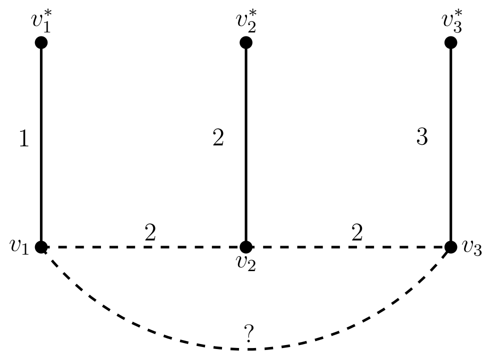

Figure 3.5: G' is three disjoint edges

However, this means that no matter what color we use on edge v_1v_3, we have formed a totally multicolored P_4. If v_1v_3 is colored 1, then $v_2v_1v_3v_3^*$ is totally multicolored. If v_1v_3 is colored 2, then $v_1^*v_1v_3v_3^*$ is totally multicolored. If v_1v_3 is colored 3, then $v_2v_3v_1v_1^*$ is totally multicolored.

Case 2: G' is a P_3 and a disjoint edge. Let the P_3 have vertices v_1, v_2, and v_3 with v_1v_2 colored 1 and v_2v_3 colored 2. Let the disjoint edge be v_4v_5, colored 3. To avoid a totally multicolored P_4 the edge v_3v_4 must be colored 2 and similarly v_1v_5 must be colored 1. However, this would form a totally multicolored P_4 with edges v_3v_4, v_4v_5, v_1v_5.

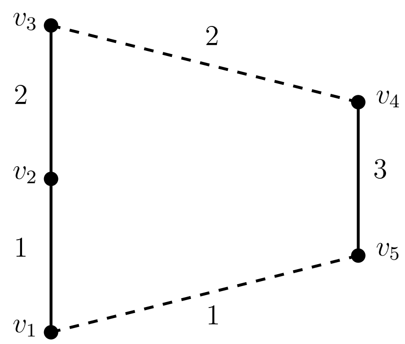

Figure 3.6: G' is a P_3 and a disjoint edge

Case 3: G' is a triangle. Call the vertices of G' v_1, v_2, and v_3. Assume edge v_1v_2 is colored 1, v_2v_3 is colored 2, and v_3v_1 is colored 3. To avoid a totally multicolored P_4, any edge connecting v_2 to $G - G'$ must be colored 3. Call one such edge e_1. Similarly, any edge, call it e_2 connecting v_3 to $G - G'$ must be colored 1. If $n \geq 5$, it is possible to pick e_1 and e_2 such that they do not share the same vertex in $G - G'$. In this case, the path consisting of edges e_1, e_2, and v_2v_3 is totally multicolored.

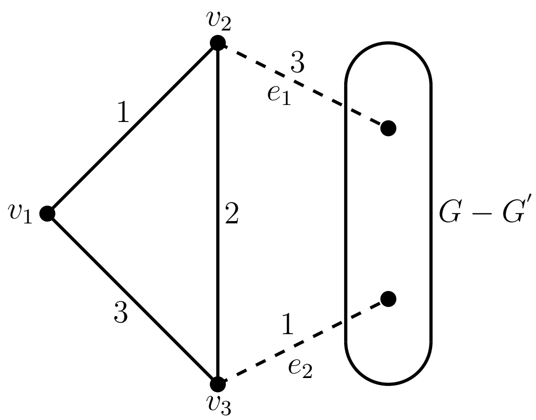

Figure 3.7: G' is a triangle

Case 4: G' is a 3-star. Call the leaf vertices v_1, v_2, and v_3. Call the center v_4. Assume that edge v_4v_i is colored i for $1 \leq i \leq 3$. To avoid a totally multicolored P_4, the edge v_2v_3 must be colored 1. Also, any edge (call it e_1) connecting v_2 to $G - G'$ must be colored 2. However, if both of those are true, then e_1, v_2v_3, and v_3v_4 form a totally multicolored P_4.

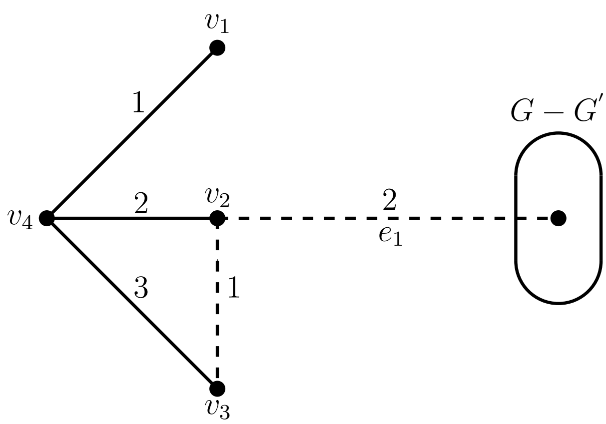

Figure 3.8: G' is a 3-star

■

If $n = 4$, it is easy to verify that the 3-edge-coloring in Figure 3.9 produces no totally multicolored P_4.

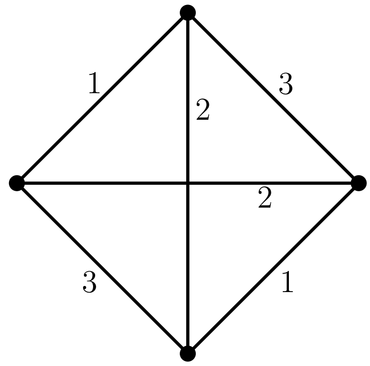

Figure 3.9: 3-coloring of K_4 containing no totally multicolored P_4

Lemma 3.1 *The only connected simple graphs whose longest path is a P_3 are stars and C_3.*

Proof Suppose a graph contains a P_3 as a subgraph. To avoid forming a P_4, the only possible edge connected to either endpoint is the edge joining them, forming a C_3. Since a C_3 with a pendant edge contains a P_4, there can not be any more edges in the graph if the edge joining the endpoints is included.

If that edge is not included, there can be arbitrarily many edges attached to the center vertex of the P_3, forming a larger star. Any two of these edges form a P_3 and the same rule applies as above. If there were more than 2 edges in our star, joining any of the leaves would create a C_3 with a pendant edge.

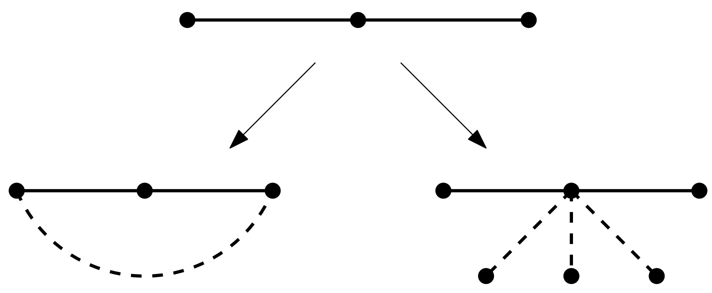

Figure 3.10: Two ways to add edges to a P_3 without forming a P_4

■

Theorem 3.7 $ar(K_n, P_5) = n$

Proof Pick one vertex of K_n and call it v. Color the $n - 1$ edges between v and $K_n - v$ with different colors. Color all edges in $K_n - v$ with another color. Thus, our coloring uses n colors and any P_5 can only use two edges between v and $K_n - v$ and so must use two edges in $K_n - v$, guaranteeing that it is not totally multicolored. This shows that $ar(K_n, P_5) \geq n$.

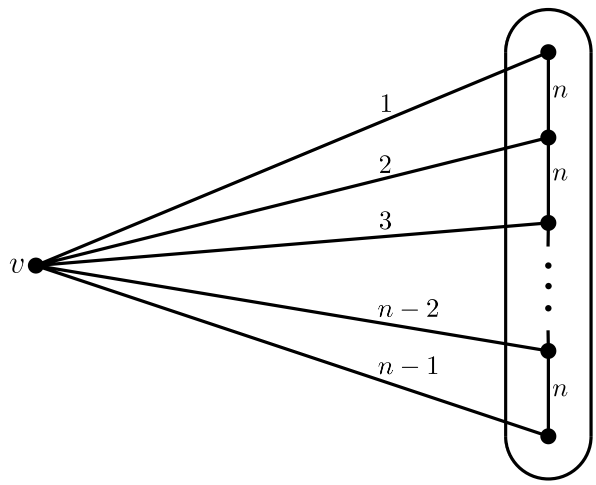

Figure 3.11: n - coloring of K_n that contains no totally multicolored P_5

Color the edges of K_n with $n + 1$ colors and pick one edge of each color and call the graph with these edges G. G has more edges than vertices and so, if G is not connected, at least one of its components must have more edges than vertices. Call one such component C.

Consider the longest path in C. Since $|E(C)| > |V(C)|$, the longest path can not be a P_2 for in that case C is a P_2 which has more vertices than edges. Nor can the longest path be a P_3 by the lemma above since neither stars nor C_3 have more edges than vertices. If the longest path in C is a P_5, we are done, so assume that the longest path in C is a P_4.

If there is another component of G, call it C^*, that also contains a P_4, then Figure 3.12 and Table 3.1 show that the coloring must contain a totally multicolored P_5.

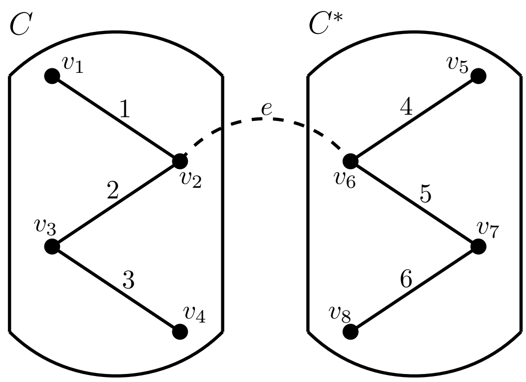

Figure 3.12: Two components of G containing a P_4

Color of e	Totally Multicolored Path Created
$1, \geq 5$	$v_4v_3v_2v_6v_5$
$2, 3, 4$	$v_1v_2v_6v_7v_8$

Table 3.1: Totally Multicolored Paths Created By Coloring Edge e in Figure 3.12

If there is another component of G, again call it C^*, that contains a C_3, then Figure 3.13 and Table 3.2 show that the coloring must contain a totally multicolored P_5.

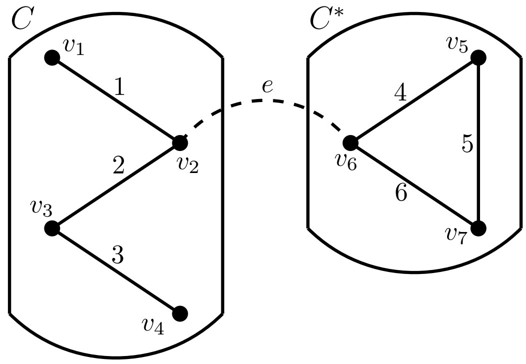

Figure 3.13: Component of G containing a P_4 and component containing a C_3

Color of e	Totally Multicolored Path Created
$1, \geq 5$	$v_4v_3v_2v_6v_5$
$2, 3, 4$	$v_1v_2v_6v_7v_5$

Table 3.2: Totally Multicolored Paths Created By Coloring Edge e in Figure 3.13

So, assume that no other component of G contains a P_4 or a C_3. Thus, any other component C^* of G must be a k-star, where $1 \leq k$. $|E(C^*)| < |V(C^*)|$ for any such component. But $|E(G)| > |V(G)|$ so not only does C have more edges than vertices, it has to have at least two more edges than vertices to make up for the other components.

If C has 4 vertices, it must be a K_4. Then, Figure 3.14 and Table 3.3 show that the coloring must contain a totally multicolored P_5.

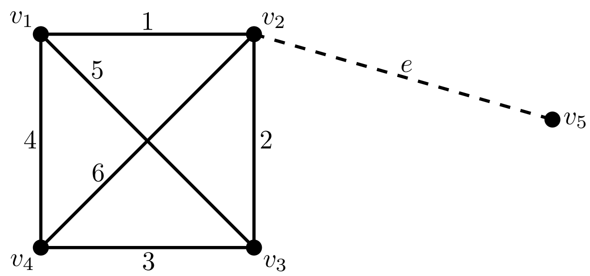

Figure 3.14: C is a K_4

Color of e	Totally Multicolored Path Created
$2, \geq 5$	$v_3v_4v_1v_2v_5$
$1, 3$	$v_3v_1v_4v_2v_5$
4	$v_4v_3v_1v_2v_5$

Table 3.3: Totally Multicolored Paths Created By Coloring Edge e in Figure 3.14

So, assume C has $m \geq 5$ vertices and at least $m + 2 \geq 7$ edges. Consider the longest cycle in C. If C contains a C_i for $i \geq 5$, then C contains a totally multicolored P_5.

If the longest cycle in C is a C_4, then the fifth vertex has to be connected to the C_4, forming a P_5.

So assume that the longest cycle in C is a triangle. Call the vertices of one such triangle v_1, v_2, v_3. The remaining vertices of C, $v_4, v_5, ..., v_m$ must be connected to the triangle. Label the remaining vertices so that v_4 is connected via the edge v_3v_4. Consider a fifth vertex, v_5. If v_5v_1 is an edge in C, then $v_5v_1v_2v_3v_4$ is a totally multicolored P_5. Similarly, if v_5v_2 is an edge in C, then $v_5v_2v_1v_3v_4$ is a totally multicolored P_5. If v_5v_4 is an edge in C, then $v_5v_4v_3v_2v_1$ is totally multicolored. Thus, all vertices $v_5, ..., v_m$ can only be connected to v_3.

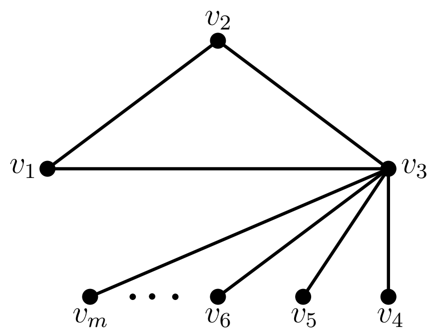

Figure 3.15: Longest cycle C is a triangle

However, this will give us only $m = |V(C)|$ edges, which is not enough. Thus, there must be a P_5 in C.

■

Lemma 3.2 $sar(K_5, K_n, P_6) \leq 7$

Proof Assume $n \geq 6$ and consider one copy of $K_5 \subseteq K_n$. Call this copy K_5^*. Color the edges of K_5^* with at least 8 unique colors and pick 8 edges, each of which having a different color, leaving two edges possibly colored using duplicate colors. These two edges either share a vertex or they do not.

First, assume the two edges share a vertex, v_1. Let v_6 be a vertex in $K_n - K_5^*$ and consider the edge v_6v_1. Let c be the color of v_6v_1. By considering Figure 3.16, we can note that because of the path $v_6v_1v_3v_2v_4v_5$, $c \in [8, 1, 7, 3]$ or a totally multicolored P_6 is formed. Similarly, the paths $v_6v_1v_3v_4v_2v_5$ and $v_6v_1v_4v_5v_2v_3$ necessitate that c is also in the color sets $[8, 2, 7, 4]$ and $[6, 3, 4, 1]$ respectively. However, these three color sets share no common element and thus, regardless of c, a totally multicolored P_6 is formed.

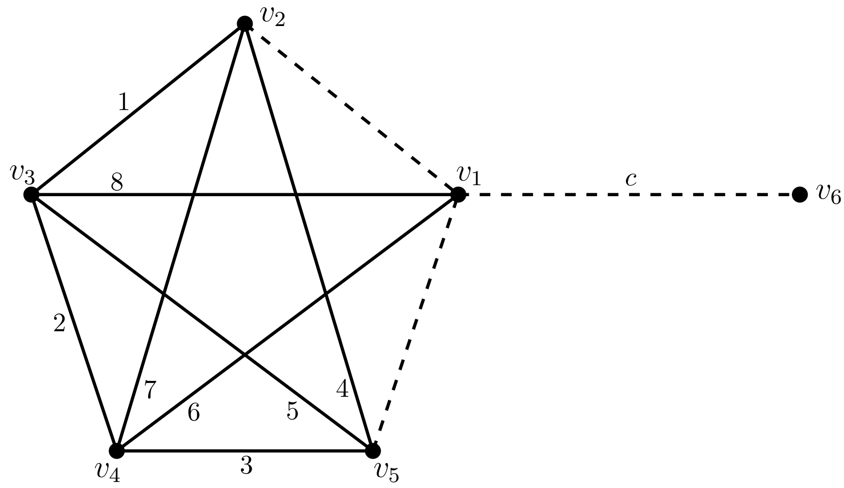

Figure 3.16: Coloring of K_5^* where duplicate colors share a vertex

Assume that the two edges do not share a vertex. Again, Let v_6 be a vertex in $K_n - K_5^*$ and consider the edge v_6v_1. Let c be the color of v_6v_1. Consider the paths $v_6v_1v_4v_2v_3v_5$, $v_6v_1v_5v_2v_3v_4$, and $v_6v_1v_5v_3v_4v_2$ in Figure 3.17. As above, the respective color sets $[4, 5, 3, 6]$, $[1, 8, 3, 2]$, and $[1, 6, 2, 5]$ share no common element and a totally multicolored P_6 is formed regardless of c.

■

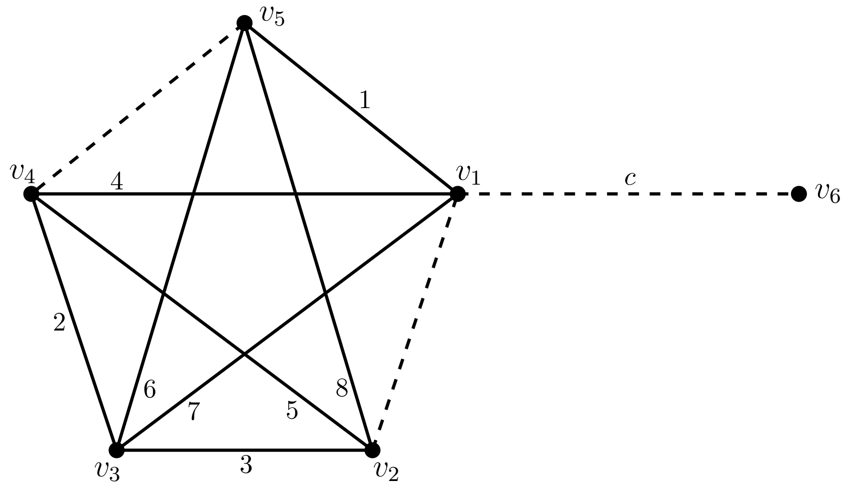

Figure 3.17: Coloring of K_5^* where duplicate colors do not share a vertex

Lemma 3.3 *For $n \geq 6$, if the edges of a copy of $K_5 \subseteq K_n$ are colored with exactly 7 colors such that one edge of each color can be chosen such that the duplicate edges left do not all share a vertex, then there is a totally multicolored copy of P_6 in K_n.*

Proof There are three possibilities for the three duplicate edges. In each case, call the vertices of the K_5 v_1, v_2, v_3, v_4, v_5 and consider another vertex v_6.

Case 1: The three edges form a P_4. By considering Figure 3.18 and Table 3.4, we can see that no matter what color we use on edge v_1v_6, a totally multicolored P_6 is formed.

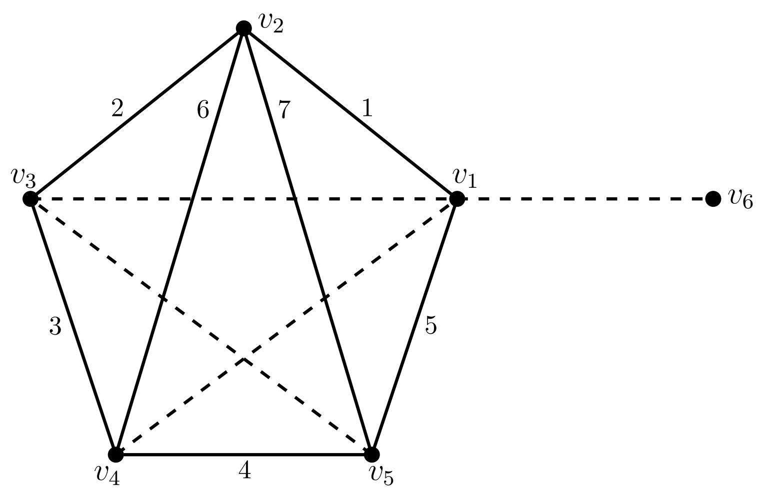

Figure 3.18: Duplicate edges form a P_4

Color of v_1v_6	Totally Multicolored Path Created
≥ 5	$v_6v_1v_2v_3v_4v_5$
1,2,4	$v_6v_1v_5v_2v_4v_3$
3	$v_6v_1v_5v_4v_2v_3$

Table 3.4: Totally Multicolored Paths Created By Coloring Edge v_1v_6 in Figure 3.18

Case 2: The three edges form a triangle. Figure 3.19 and Table 3.5 illustrate that any color on edge v_1v_6 creates a totally multicolored P_6.

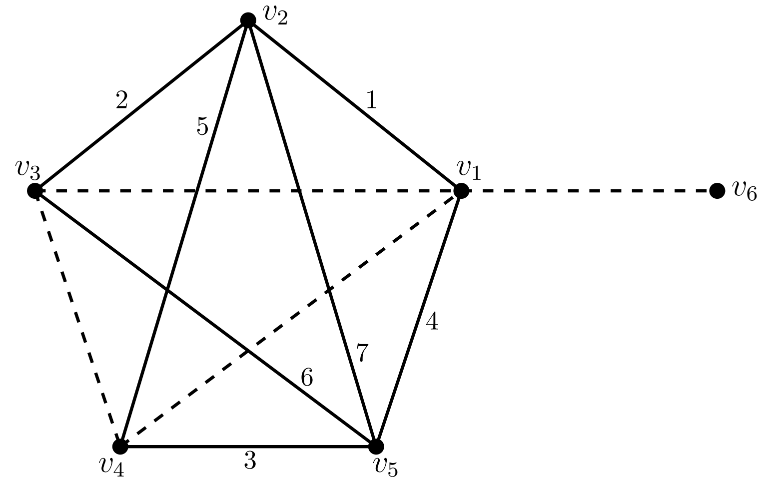

Figure 3.19: Duplicate edges form a triangle

Color of v_1v_6	Totally Multicolored Path Created
1, ≥ 6	$v_6v_1v_5v_4v_2v_3$
4,5	$v_6v_1v_2v_3v_5v_4$
2	$v_6v_1v_2v_4v_5v_3$
3	$v_6v_1v_5v_3v_2v_4$

Table 3.5: Totally Multicolored Paths Created By Coloring Edge v_1v_6 in Figure 3.19

Case 3: The three edges form a P_3 and a disjoint edge. Figure 3.20 and Table 3.6 complete the argument.

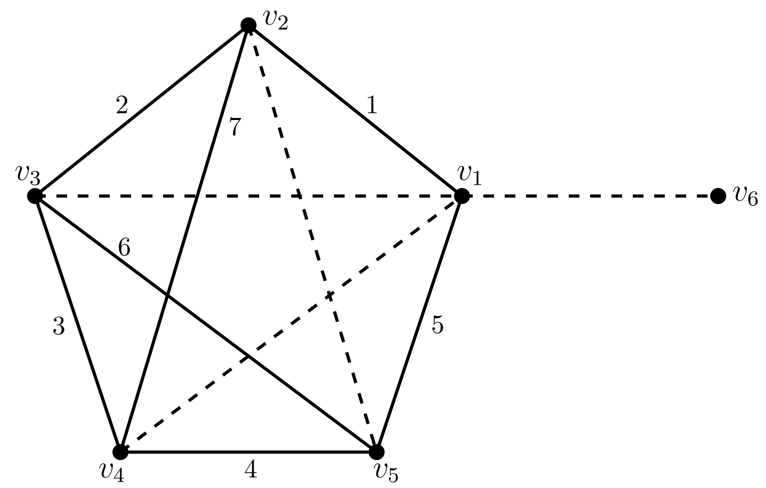

Figure 3.20: Duplicate edges form a P_3 and a disjoint edge

Color of v_1v_6	Totally Multicolored Path Created
≥ 5	$v_6v_1v_2v_3v_4v_5$
1,3,4	$v_6v_1v_5v_3v_2v_4$
2	$v_6v_1v_2v_4v_3v_5$

Table 3.6: Totally Multicolored Paths Created By Coloring Edge v_1v_6 in Figure 3.20

■

Lemma 3.4 *If a coloring of K_5 with exactly 7 colors is such that the three duplicate edges left after picking one edge of each color always form a $3-star$, then the coloring is as follows: Color each edge of some $K_4 \subseteq K_5$ with unique colors and all the remaining edges with the seventh color.*

Proof Assume we have colored the edges of K_5 with exactly 7 colors and picked one edge of each color, calling this set of edges E, such that the remaining duplicate edges form a $3 - star$. Call the vertex at the center of the star v and let the color of the fourth edge incident with v be 7.

Pick one of the duplicate edges and call it e. Assume e is colored c. If $c \neq 7$, the edge in E that is colored c is not incident with v. Replace the edge colored c in E with e. We have now chosen one edge of each color such that the remaining duplicate edges do not form a $3 - star$. Thus, each of our original three duplicate edges must have been colored 7, giving us the coloring above.

■

Corollary 3.1 *If a copy of $K_5 \subseteq K_n$, call it K_5^*, is colored as described in the lemma above, there can be no colors on the the edges between K_5^* and the rest of K_n that do not appear in K_5^* without creating a totally multicolored P_6.*

Proof Consider Figure 3.21. If any edge between v_1 and $K_n - K_5^*$ is colored 8, then a P_6 composed of that edge and any Hamilton path on K_5^* starting at v_1 is totally multicolored. If an edge between any other vertex of K_5^* and $K_n - K_5^*$ is colored 8, then a P_6 composed of that edge and a Hamilton path on K_5^* starting at that vertex and ending at v_1 is totally multicolored.

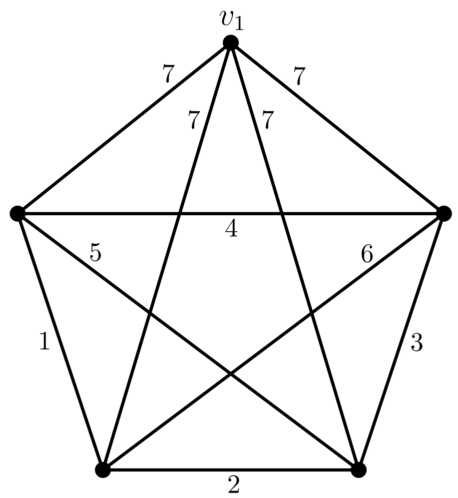

Figure 3.21: K_5 colored with exactly 7 colors such that the three duplicate edges left after picking one edge of each color always form a $3 - star$

■

Theorem 3.8 $ar(K_n, P_6) = n + 1$

Proof Pick one vertex of K_n and call it v. Color the $n-1$ edges between v and $K_n - v$ with different colors. Color all edges in $K_n - v$ with a further two colors. Thus, our coloring uses $n+1$ colors and any P_6 can only use two edges between v and $K_n - v$ and so must use three edges in $K_n - v$, guaranteeing that it is not totally multicolored. This shows that $ar(K_n, P_6) \geq n + 1$.

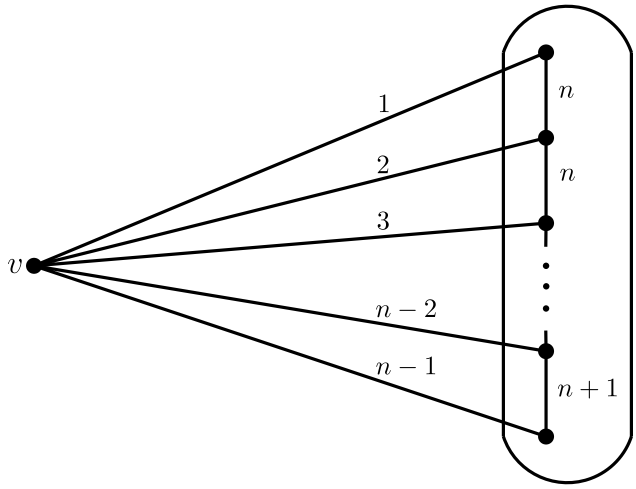

Figure 3.22: $(n+1)$ - coloring of K_n that contains no totally multicolored P_6

Now, assume we have colored the edges of K_n with at least $n+1$ colors such that the coloring does not produce a totally multicolored P_6. Since $ar(K_n, P_5) = n$, the coloring must produce a totally multicolored P_5. Call this path $P = v_1v_2v_3v_4v_5$. And let the edge v_iv_{i+1} be colored i for $1 \leq i \leq 4$. Let Φ be the K_5 induced by $V(P)$. Let K be the subgraph induced by $K_n - V(P)$ and label the vertices of K as $v_6, v_7, ..., v_n$. Let C be the set of colors $[5, 6, 7, ..., n+1, ...]$. See Figure 3.23.

Our first goal will be to show that there are at most $n-5$ colors from C that occur outside of Φ We will consider the ways that a vertex in K can be incident with two or more edges colored with different colors from C. Note that if any edge between v_1 or v_5 and K is colored from C, we have a totally multicolored P_6.

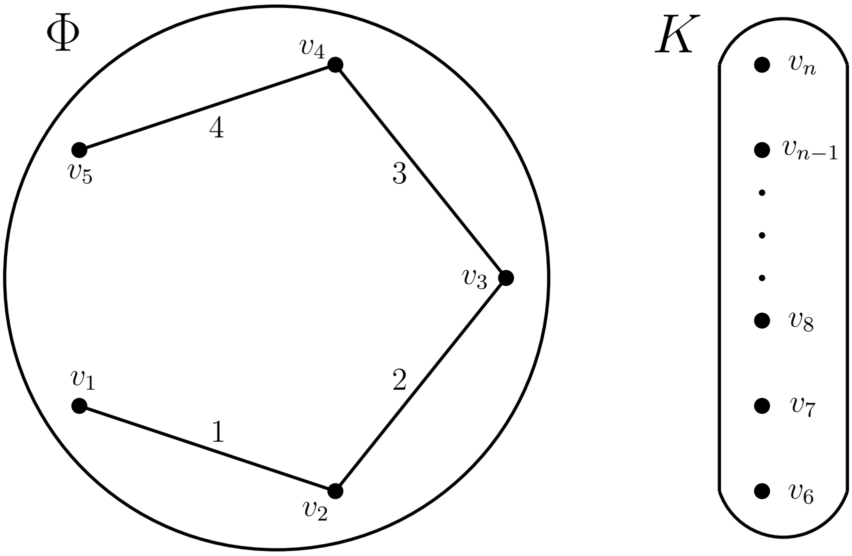

Figure 3.23: $P \subset \Phi$ and K

Case 1: Let v_m be a vertex in K and suppose v_mv_3 and either v_mv_2 or v_mv_4 (let us suppose it is v_mv_2) are colored with different colors from C, say 5 and 6. However, this would mean that $v_1v_2v_mv_3v_4v_5$ would be totally multicolored, contradicting our assumption that there is no totally multicolored P_6 in our coloring. See Figure 3.24.

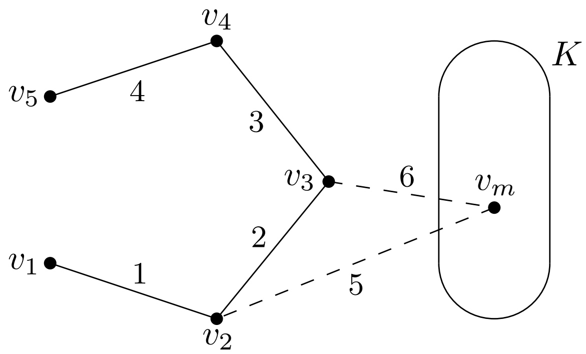

Figure 3.24: Case 1

Case 2: Let v_m and v_μ be vertices in K. Suppose that v_mv_μ and either v_mv_2 or v_mv_4 (let us suppose it is v_mv_2) are colored with different colors from C, again 5 and 6. Then $v_\mu v_mv_2v_3v_4v_5$ would be a totally multicolored P_6, a contradiction. See Figure 3.25

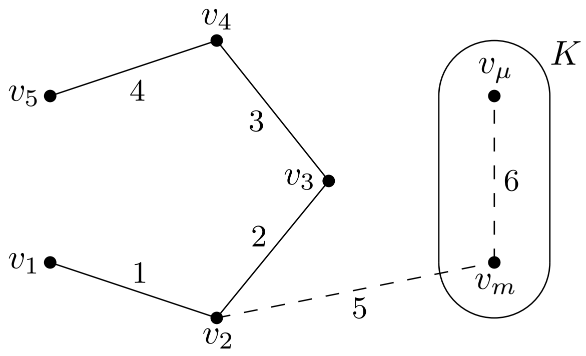

Figure 3.25: Case 2

Case 3: Suppose v_m, v_μ, and v_ν are all vertices in K and suppose that $v_m v_\mu$ and $v_m v_\nu$ are both colored with different colors from C. Let $v_m v_\nu$ be colored 5 and $v_m v_\mu$ be colored 6. Consider the edge $v_1 v_\nu$. We know that any color from C will create a totally multicolored P_6. Furthermore, coloring the edge 3 or 4 will make $v_3 v_2 v_1 v_\nu v_m v_\mu$ totally multicolored. Thus, the edge can only be colored 1 or 2.

Similarly, $v_5 v_\mu$ must be colored either 3 or 4. However, if $v_1 v_\nu$ is colored 2, this would make $v_2 v_1 v_\nu v_m v_\mu v_5$ totally multicolored. So, $v_1 v_\nu$ must be colored 1 and by the same argument $v_5 v_\mu$ can only be colored 4.

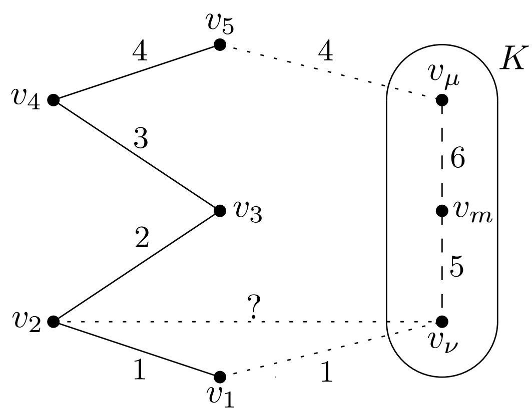

Figure 3.26: Case 3

But, with the enforced colors as shown in Figure 3.26, any color at all on the edge $v_2 v_\nu$ will create a totally multicolored P_6 as shown by Table 3.7.

Color	Totally Multicolored Path Created
1	$v_5v_4v_3v_2v_\nu v_m$
2	$v_1v_2v_\nu v_mv_\mu v_5$
3	$v_1v_2v_\nu v_mv_\mu v_5$
4	$v_4v_3v_2v_\nu v_mv_\mu$
5	$v_1v_\nu v_2v_3v_4v_5$
≥ 6	$v_5v_4v_3v_2v_\nu v_m$

Table 3.7: Totally Multicolored Paths Created By Coloring Edge v_2v_ν in Figure 3.26

Case 4: Let v_m be a vertex in K. Suppose v_mv_2 and v_mv_4 are colored with different colors from C, say v_mv_2 is colored 5 and v_mv_4 is colored 6. Consider the edge v_1v_5. No matter what its color, a totally multicolored P_6 is formed as shown in Table 3.8.

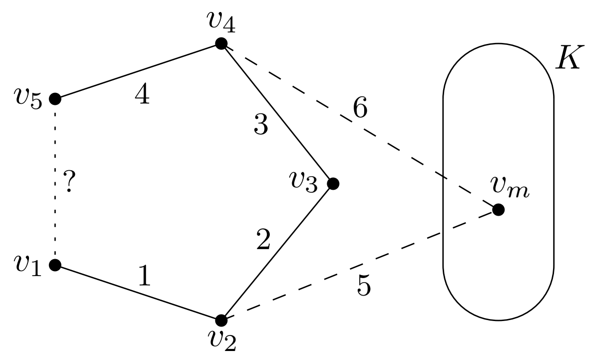

Figure 3.27: Case 4

Color	Totally Multicolored Path Created
1	$v_mv_2v_3v_4v_5v_1$
2	$v_mv_2v_1v_5v_4v_3$
3	$v_mv_4v_5v_1v_2v_3$
4	$v_mv_4v_3v_2v_1v_5$
5	$v_mv_4v_3v_2v_1v_5$
≥ 6	$v_mv_2v_3v_4v_5v_1$

Table 3.8: Totally Multicolored Paths Created By Coloring Edge v_1v_5 in Figure 3.27

Case 5: Suppose that v_μ and v_m are vertices in K and suppose that v_3v_m is colored 5 and v_mv_μ is colored 6. If $n \geq 8$, any other edge from v_μ to an additional vertex in K, v_ν, must be colored 6. We know from case 3 that no such edge can be colored with any color from C other than 6. Furthermore, it is easy to see that if $v_\mu v_\nu$ is colored 1 or 2 then $v_\nu v_\mu v_m v_3 v_4 v_5$ is totally multicolored. Similarly, if $v_\mu v_\nu$ is colored 3 or 4 then $v_\nu v_\mu v_m v_3 v_2 v_1$ is totally multicolored.

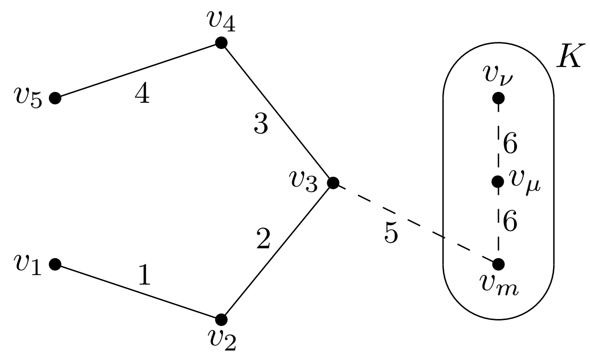

Figure 3.28: Every vertex in K must be incident to an edge colored 6

This means that every vertex in K is incident to an edge colored 6. From previous cases, this means that the only way for any vertex v_ν in K other than v_m to be incident with more than one color from C is for v_3v_ν to be colored from C. If v_3v_ν is colored 5 for all v_ν in K, then 5 and 6 are the only colors from C that occur outside of Φ.

So assume that for some vertex v_ν in K other than v_m, the edge v_3v_ν is colored with a color from C other than 5 or 6. Let v_3v_m be colored 5, v_3v_ν be colored 7, and v_mv_μ, $v_\nu v_\rho$ be colored 6, where v_ρ is an additional vertex in K. Note that it may be the case that v_μ and v_ρ are the same vertex. Also, it is possible that $v_\mu = v_\nu$ or $v_\rho = v_m$. Consider the edge v_1v_5. Figure 3.29 and Table 3.9 show that any color on edge v_1v_5 creates a totally multicolored P_6.

Color	Totally Multicolored Path Created
$3, 4, \geq 7$	$v_\mu v_m v_3 v_2 v_1 v_5$
$1, 2, 5$	$v_\rho v_\nu v_3 v_4 v_5 v_1$
6	$v_m v_3 v_2 v_1 v_5 v_4$

Table 3.9: Totally Multicolored Paths Created By Coloring Edge v_1v_5 in Figure 3.29

So, in this case it is possible for a vertex in K to be incident with two unique colors from C, but then those are the only colors from C that occur outside of Φ. For $n = 6$ this

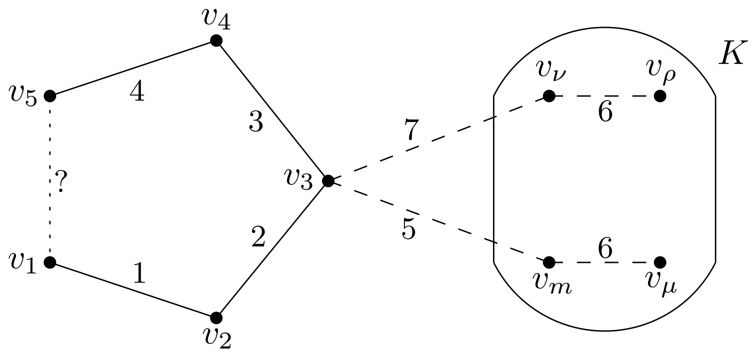

Figure 3.29: Case 5

case can not happen and for $n \geq 7$, $2 \leq n-5$. In every other case, any vertex in K can add at most one color from C to our total palette of colors. Thus, there are at most $n-5$ colors from C that occur outside of Φ.

We assumed at least $n+1$ colors were used, so there must be at least six additional colors on Φ. Furthermore, the lemmas above show that there can be at most seven colors on Φ without creating a totally multicolored P_6. These colors will be $1, 2, 3, 4$, and either two or three colors from C.

If we use exactly six colors on Φ, then we have at most $(n-5)+6 = n+1$ colors in total.

If we use seven colors, the corollary above states that all of the colors on the edges between K and Φ must occur in Φ to avoid a totally multicolored P_6. That is, these seven colors are the only colors that appear outside of K. By Case 3 earlier in this proof, each vertex within K can be incident with at most one edge within K that is colored from C. This means that at most $\lfloor \frac{n-5}{2} \rfloor$ colors from C can occur within K. So we can have at most $\lfloor \frac{n-5}{2} \rfloor + 7$ colors, which is at most $n+1$ for $n \geq 6$.

■

Simonovits and Sós proved the following general result for paths in [7].

Theorem 3.9 **([7])** *There exists a constant c such that if $t \geq 5$, $n > ct^2$, then*

$$ar(K_n, P_{2t+3+\varepsilon}) = tn - \binom{t+1}{2} + 1 + \varepsilon$$

where ε *is either* 0 *or* 1.

The specific statement and proof restricts us to sufficiently long paths but the authors note that the cases when $t \leq 4$ are also true but required more cases to be evaluated and were thus omitted. Note that this result confirms that $ar(K_n, P_6) = n + 1$ as conjectured.

The extremal coloring used is as follows. Partition the vertices of K_n into sets $A = [a_1, a_2, ..., a_t]$ and $B = [b_1, b_2, ..., b_{n-t}]$. Color each edge in the subgraph induced by A and each edge with one end in A and one end in B with a different color and color the edges of the subgraph induced by B with $1 + \varepsilon$ colors. Note that this is an extension of the colorings used in the previous theorems.

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