Triangulations and Simplex Tilings of Polyhedra

by

Braxton Carrigan

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 4, 2012

Keywords: triangulation, tiling, tetrahedralization

Approved by

Andras Bezdek, Chair, Professor of Mathematics
Krystyna Kuperberg, Professor of Mathematics
Wlodzimierz Kuperberg, Professor of Mathematics
Chris Rodger, Don Logan Endowed Chair of Mathematics

Abstract

This dissertation summarizes my research in the area of Discrete Geometry. The particular problems of Discrete Geometry discussed in this dissertation are concerned with partitioning three dimensional polyhedra into tetrahedra. The most widely used partition of a polyhedra is triangulation, where a polyhedron is broken into a set of convex polyhedra all with four vertices, called tetrahedra, joined together in a face-to-face manner. If one does not require that the tetrahedra to meet along common faces, then we say that the partition is a tiling.

Many of the algorithmic implementations in the field of Computational Geometry are dependent on the results of triangulation. For example computing the volume of a polyhedron is done by adding volumes of tetrahedra of a triangulation. In Chapter 2 we will provide a brief history of triangulation and present a number of known non-triangulable polyhedra. In this dissertation we will particularly address non-triangulable polyhedra.

Our research was motivated by a recent paper of J. Rambau [20], who showed that a nonconvex twisted prisms cannot be triangulated. As in algebra when proving a number is not divisible by 2012 one may show it is not divisible by 2, we will revisit Rambau's results and show a new shorter proof that the twisted prism is non-triangulable by proving it is non-tilable. In doing so, we identify a new family of nonconvex non-tilable polyhedra, which we call an altered right prism. Furthermore we will perturb the vertices of a regular dodecahedron in a twisted manner resulting in a non-tilable polyhedra. Also for completeness, we describe two concrete non-triangulable polyhedra which can be tiled with tetrahedra.

From observations made about the provided non-triangulable polyhedra, we started to systematically study extensions of surface triangulations of convex polyhedra. Among others we proved that if each vertex of a convex polyhedron is adjacent to no more than

three non-triangular faces, then for every surface triangulation one can perturb the vertices of the polyhedron so that the resulting polyhedron is combinatorially equivalent to the given surface triangulation.

Acknowledgments

I would like to thank all of my committee members for their suggestions and support in completing this text. Mostly I am indebted to Dr. Andras Bezdek for the hours he spent discussing the contents of this dissertation. His guidance through this process will forever leave a mark on my career. I could not have completed my degree without his assistance.

I am also indebted to my entire family for their continued loving support. Specifically I wish to thank my wife Catie for her love, support, and patience while I pursued my Ph.D. I am truly blessed to have a multitude of friends, family, and colleagues who have continually given me strength to pursue my goals.

Table of Contents

List of Illustrations

Chapter 1

Introduction

Geometry is a subject as old as mathematics itself. Most of the early theorems and proofs of mathematics were a result of studying point and line configurations. With the development of abstract geometries such as topology and differential geometries, much of the classical concrete problems of geometry seemed to be neglected. While some of the great mathematicians such as Cauchy, Gauss, and Kepler dabbled in such topics, it appeared that the geometries of the Greeks and Euclid were only used for teaching purposes. With the insurgence of technology and the necessity for algorithmic solutions, came a rebirth of research interest into the more concrete geometric structures and so the intuitive geometric questions posed by Thue and Minkowski gave birth to the subject of Discrete Geometry. Later L. Fejes Tóth [10] and [11] formalized this field of mathematics which he initially called "Intuitive Geometry". Currently W. Kuperberg and others are translating Fejes Tóth's German monograph [11] into English. Many other monographs and survey papers have since been written in an attempt to compile the up to date results of the field. Some of the larger compilations are by P. Argawal and J. Pach [18], C. Rogers [21], and P, Brass, W. Moser and J. Pach [4].

By the mid 20^{th} century many mathematicians were looking for answers to the simply stated geometric questions of optimization and packing problems and it would not be long until we would find useful application for the results of these early discrete geometers. In the 1970's the growing areas of Computer Science such as Computer Graphics, Robotics, Algorithmic Design, Geographic Information Systems, Coding Theory and others spawned the area of Computational Geometry. Some recent introductory texts which have compiled some of the newest results in this field are [8] and [3]. It is worth noting one of the earliest monographs in this field was compiled by H. Edelsbrunner [9].

Today Discrete Geometry and Computational Geometry are often combined as the two areas study similar problems, in fact they frequently use results from one another. This dissertation contains four results, each associated with the area of Discrete and Computational Geometry, which constitutes my research during the period of 2010-2012. We will provide formal definitions and theorems throughout the text, but the purpose of this introduction is to present the motivation for choosing this topic and informally present our results.

The specific problems we will discuss concern three dimensional polyhedra and their partitions into tetrahedra. A partition is called a triangulation if the tetrahedra are joined together in a face-to-face manner. If one does not require that the tetrahedra meet along common faces, then we say that the partition is a tiling. We will also consider extensions of surface triangulations to triangulations (tilings resp.).

My research started by reading a recent paper of Rambau [20], who showed that twisted prisms cannot be triangulated. It did not take long to realize that triangulation is an extensively studied area with large amounts of literature and many applications. For example computing the volume of a polyhedron is done by adding volumes of tetrahedra of a triangulation.

Three dimensional problems usually are more difficult than their planar counterparts, and in the area of triangulation there are surprisingly many differences between the planar and the three dimensional versions. For example everybody knows that every polygon can be triangulated, moreover the number of triangles is the same in each triangulation. Neither of these hold for triangulations of polyhedra, and thus the characterization of non-triangulable polyhedra is a very difficult open problem.

We aim to describe new families of non-triangulable polyhedra. Before presenting our original results, we will provide a brief overview of the history of triangulation and discuss seven known examples of non-triangulable polyhedra. In particular Rambau [20] showed that no triangulation of a prism uses the set of cyclic diagonals along the lateral faces to prove that the twisted prism cannot be triangulated. After reading the details of Rambau's paper, it is natural to ask if there exist a tiling of a twisted prism. This question motivated the research which is summarized in this dissertation. The dissertation contains the following new results.

- Examples 8 and 9 are non-triangulable polyhedra which can be tiled by tetrahedra.
- Theorem 3.4 gives an infinite family of non-tilable polyhedra.
- The family of tilings not only includes the family of triangulations, but also allows a different approach when proving non-existence. In fact we revisited Rambau's corollary on twisted prisms and present a new and shorter proof.
- We will further generalize our technique to present another family of non-tilable polyhedra in Theorem 3.8, further demonstrating the non triviality of tiling by tetrahedra.
- From the observations of the techniques presented in showing a polyhedron is non-triangulable, we will show how negative results of extendable surface triangulations can produce non-triangulable polyhedra.
- We will provide both positive and negative results of extendable surface triangulations on Archimedean and Johnson Solids.

Chapter 2

Brief History of Triangulation and Terminology

Triangulation is of great concern and has many applications, in fact Devadoss and O'Rourke [8] state that " triangulations are the prime factorizations of polygons, alas without the benefit of the "Fundamental Theorem of Arithmetic" guaranteeing unique factorization." This statement alludes to the difficulties of dealing with the non-unique nature of triangulation and its importance to the field of geometry. Triangulation has become such a useful partitioning that entire chapters of texts are devoted to the topic and there is even a monograph by J. De Loera, J. Rambau, and F. Santos [7] providing a detailed history of the algorithms and applications surrounding triangulation.

Definition 1. *A triangulation of a point configuration* $\boldsymbol{A}$ *in the Euclidean n space, denoted as* $\mathbb{R}^n$*, is a collection of d-simplices, all of whose vertices are points in* $\boldsymbol{A}$ *and satisfies the following two properties:*
1. The union of all the simplices equals conv($\boldsymbol{A}$*). (Union Property)*
2. Any pair of the simplices intersect in a common face (possibly empty). (Intersection Property)

Throughout the text we will partition n-dimensional polytopes, where the 2 dimensional polytope is called a polygon and the 3 dimensional polytope is called a polyhedron. A polygon is a closed region whose boundary is a polygonal chain with no self intersections. We call the boundary segments edges, and the points where the edges meet will be called vertices. Unless otherwise stated we will require that two adjacent line segments are not collinear. A polyhedron is the three dimensional analogue of a polygon which is an enclosed region of $\mathbb{R}^3$ bounded by finitely many polygons so that if two polygons intersect, they do so along a common edge or vertex. In a polyhedron no two adjacent faces are coplanar. Therefore a polytope is the n-dimensional analogue of a polygon which is an enclosed region of $\mathbb{R}^n$ bounded by finitely many $n-1$ dimensional polytopes joined edge to edge, where an edge is the $n-2$ dimensional boundary of the $n-1$ dimensional polytope boundaries, so that if two $n-1$ dimensional boundary polytopes intersect, they do so along a common boundary.

2.1 History of Triangulation

There exist many algorithms which attempt to triangulate a point set and each algorithm must first depend on finding the convex hull, which has been shown to have complexity $O(n \log n)$. (We will briefly discuss complexity in a later section of this chapter.) Surprisingly the complexity of triangulating a point set has the same complexity as finding its convex hull. In this introduction we will present techniques used for triangulation, yet we will not give the details of such algorithms or complexity arguments.

Many triangulation algorithms use a step called *lifting* (Figure 2.1) and as an extension *bending*.

- A *lift* of a point configuration $\mathbf{A}$ is constructed by assigning a height $\omega_i > 0$ to every $a_i \in \mathbf{A}$, thus giving a point configuration $\hat{A} \in \mathbb{R}^{m+1}$.

- The *lower envelope* of a lift is the face structure of the convex hull of $\hat{A}$ seen from below (the $\mathbb{R}^m$ plane).

- A triangulation of a point configuration $\mathbf{A} \in \mathbb{R}^m$ is *regular* if it is a projection of a lower envelope of a lifting of $\mathbf{A}$.

Bending is similar to lifting yet involves assigning heights to a subset of the points while leaving the complement at height 0. Different envelopes are considered depending on the application of the bend.

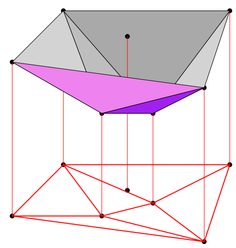


Figure 2.1: A triangulation determined by a lower envelope of a lift

One should notice that points in the interior of the convex hull are not necessarily vertices of a simplex of the triangulation. A triangulation is *full* if every point of the point set is a vertex of at least one simplex in the triangulation.

One of the most frequently used regular triangulation is the Delauney Triangulation which can be found by lifting the point set onto the $\mathbb{R}^{m+1}$ paraboloid, then projecting the lower envelope of the lift back onto the $\mathbb{R}^m$ space. Interestingly the Delauney Triangulation is the dual of the Voronoi Diagram.

Definition 2. *For every point p of a point set S, we define the Voronoi Region to be all points $x \in \mathbb{R}^m$ such that $|x - p| \leq |x - q|$ for all $q \in S$.*
The Voronoi Diagram is the collection of all points in $\mathbb{R}^m$ which have more than one nearest neighbor, or the boundaries of the Voronoi Regions.

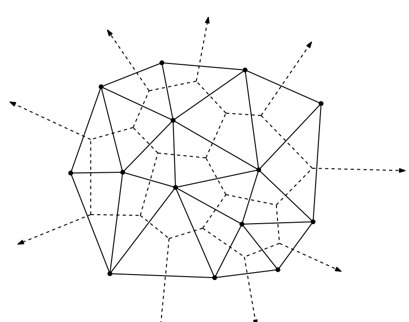


Figure 2.2: Delauney triangulation and Voronoi diagram

Our new results will concern triangulations of polyhedra in $\mathbb{R}^3$. A *triangulation* of a polyhedron P is a partition into finitely many non-overlapping tetrahedra joined face-to-face. In our original results we restrict ourselves to triangulations where the vertices of each tetrahedron are a subset of the vertex set of P. Since we will be concerned with triangulations of polyhedra, all triangulations will be full.

Two-Dimensional Triangulations

Let us first investigate the intersection property of triangulation of a point set in $\mathbb{R}^2$. In Figure 2.3 we notice the shaded region is a triangle, but the neighboring triangles do not share the entire edge, and thus the configuration is not a triangulation. We may wish to refer to the shaded region as a quadrilateral to emphasize it has four vertices of existing triangles in the partitioning along its boundaries. This is a situation when we wish not to have the angle between two adjacent edges of a polygon be 180^o.

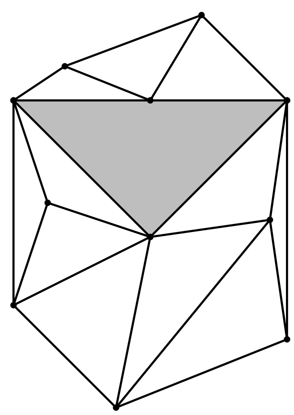


Figure 2.3: A partition into triangles which is not a triangulation

To avoid ambiguity with such configurations, it is common to consider the point set to be in general position which in this context means that no three points are collinear. Since our results are concerned with triangulating polyhedra it will not be possible to have a vertex lying on the edge of a triangle, so this issue will not arise.

There exist many interesting combinatorial results concerning triangulations in $\mathbb{R}^2$, of which many hinge on the concept of flips. Two triangulations are *flips* of one another if the same partitioning is obtained by deleting a diagonal of a convex quadrilateral (or interior vertex of a triangle with its incident edges) from each of the triangulations. (In $\mathbb{R}^3$ almost triangulations are obtained from deleting common faces, edges, or a vertex.)

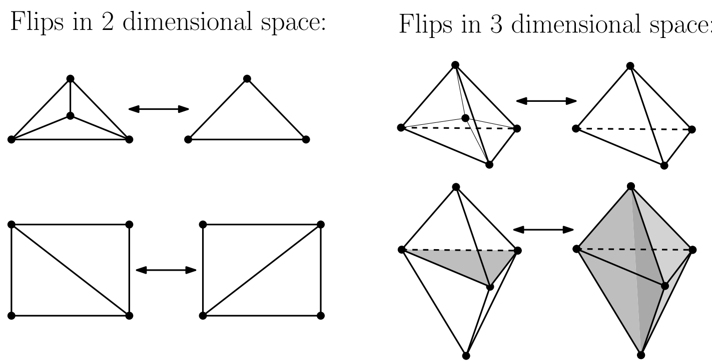


Figure 2.4: Flips in two and three dimensional space

In two dimensional space it is common to use a *graph of triangulations* (Figure 2.5) where each node is a triangulation and the edges between nodes represent a flip. C. Lawson [14] showed that the flip graph is connected. Therefore implying that any two triangulations can be transformed to one another through a series of flips, which has not been shown for higher dimensions.

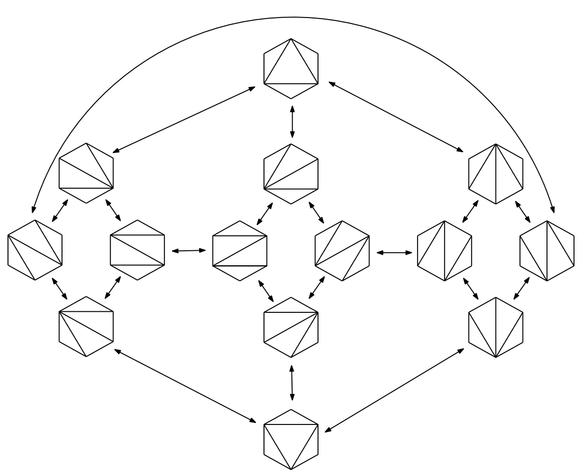


Figure 2.5: Graph of triangulations of a hexagon

The connectedness of the flip graph can be helpful in finding combinatorial results in $\mathbb{R}^2$ on both point configurations and polygons. Two important combinatorial results are:

Theorem 2.1. *Let T be triangulation of a point configuration $\boldsymbol{A} \in \mathbb{R}^2$, where n is the number of points in $\boldsymbol{A}$ and c is the number of points in $\boldsymbol{A}$ lying on the boundary of conv($\boldsymbol{A}$), then T contains $2n - c - 2$ triangles.*

Theorem 2.2. *The number of triangulations of a convex n-gon is the $(n-2)^{th}$ Catalan number, where the n^{th} Catalan number is defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Triangulation in Higher Dimension

While the graphs of triangulations have been highly useful in $\mathbb{R}^2$, the process of combinatorially classifying triangulations becomes difficult in $\mathbb{R}^3$. The most elementary counting problems, such as how many tetrahedra are included in a triangulation, are daunting in higher dimensional spaces. As seen in Figure 2.6, not every triangulation of a polyhedron consist of the same number of tetrahedra.

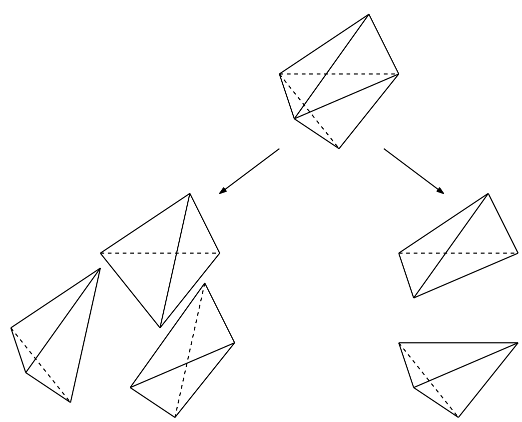


Figure 2.6: Triangulations of different cardinality

The non-uniqueness provides difficulties for combinatorial results in higher dimensional spaces, yet there do exist some results concerning the number and the existence of triangulations. Four such results are listed below along with an important conjecture.

Goodman and Pach [12] utilized the concept of bending to show:

Theorem 2.3. *If A and B are disjoint convex polytopes in $\mathbb{R}^n$, then the closure of $Conv\{A \cup B\} - A - B$ can be triangulated.*

and

Theorem 2.4. *If A and B are convex polytopes in $\mathbb{R}^n$, with $B \subset A$, then the closure of $A - B$ can be triangulated.*

The same year Sleator, Tarjan, and Thurston [23] also provided some combinatorial results for triangulations in $\mathbb{R}^3$.

Theorem 2.5. *Let A be a point configuration in $\mathbb{R}^3$, then there exists a triangulation of A with no more than $2v - 7$ tetrahedra, where v is the number of vertices in $conv(A)$.*

and

Corollary 2.6. *Every polyhedra of $\mathbb{R}^3$, with $n \geq 13$ vertices, has a triangulation (with the introduction of new vertices, called Steiner points) with at most $2n - 10$ tetrahedra.*

The proof of Theorem 2.5 uses a fan argument which we will discuss later in the text when finding extendable surface triangulations. From these results Sleator, Tarjan, and Thurston [23] conjecture:

Conjecture 2.7. *For every three-dimensional polyhedron with $n \geq 13$ vertices, there exist no triangulation with less than $2n - 10$ tetrahedra and its flip graph contains a hamiltonian cycle, which contain every vertex of the flip graph.*

2.2 Complexity and NP-Complete

Questions which can be answered with a simple yes or no, such as "can a particular polyhedron be triangulated?" are called decision problems. Decision problems can be answered by an algorithm which through a series of questions, or steps, arrives at the yes or no answer. Algorithms are measured for effectiveness by their time complexity, or how long it will take the algorithm to answer the question. Time is measured by the number of steps an algorithm takes to arrive at an answer. Given a variable n of the decision problem, if the algorithm solves the question in fewer than a constant multiple of a function $f(n)$ steps, then we say the algorithm's complexity is $O(f(n))$. Specifically we say an algorithm is in polynomial time, if its complexity is $O(n^k)$ for some whole number k.

One can study algorithms that run either on deterministic or nondeterministic computers. A deterministic computer is the typical computer which goes through a series of yes and no questions until it determines a final solution. A non-deterministic computer is allowed unlimited parallel computing meaning it has the ability to explore the options of yes and no simultaneously and do so at every step.

A decision problem is in the complexity class P if there exists an algorithm operating on a deterministic computer which can arrive at the answer in polynomial time. A decision problem is in the complexity class NP if there exists an algorithm operating on a non-deterministic computer which can arrive at the answer in polynomial time and a deterministic computer can check the answer in polynomial time. So it is obvious that $P \subset NP$.

Within the class of NP problems we call a decision problem NP-complete if we can show it is NP and all NP problem can be reduced to it in polynomial time. Therefore if we can reduce a decision problem in polynomial time to a known NP-complete problem, then it is NP-complete. The first NP-complete problem that all others can be reduced to in polynomial time was given by Steve Cook in 1971. If someone can find an algorithm operating on a deterministic machine which solves any NP-complete problem in polynomial time, then this algorithm would solve all NP problems on a deterministic machine in polynomial time concluding that $P = NP$, one of the well known unsolved problems in mathematics today.

Although significant progress has been made in triangulations of higher dimensions, the most compelling result appeared in 1992 by Ruppert and Seidel [22].

Theorem 2.8. *It is NP-Complete to decide whether a given three-dimensional polyhedron can be triangulated without using additional Steiner points.*

2.3 Non-Triangulable Polyhedra

The results of this dissertation are motivated by one of the unsolved problem in [8]. O'Rourke and Devadoss ask to, "Find characteristics that determine whether or not a polyhedron is tetrahedralizable. Even identifying a large natural class of tetrahedralizable polyhedra would be interesting." (*O'Rourke and Devadoss use the synonymous term tetrahedralizable for triangulable.*)

It was first shown in 1911 by Lennes [15] that not all three-dimensional polyhedra are triangulable. We will provide seven other known examples of non-triangulable polyhedra in this section.

Example 1 (Schönhardt [24])

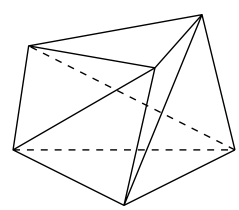


Figure 2.7: Schönhardt's twisted triangular prism

One of the most frequently quoted and simplest examples was given by Schönhardt [24] in 1927. Schönhardt made a nonconvex twisted triangular prism (Figure 2.7) by rotating the top face of a right triangular prism so that a set of cyclic diagonals became edges with interior dihedral angles greater than 180^o.

Claim: Schönhardt's twisted triangular prism cannot be triangulated.

Proof: Every diagonal of the polyhedron lies outside the polyhedron. Therefore any tetrahedron containing four vertices of the twisted triangular prism will contain at least one edge lying outside the polyhedron. ■

Example 2 (Bagemihl [1])

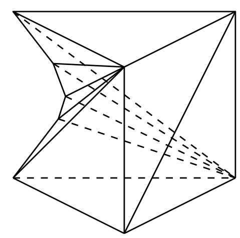


Figure 2.8: Bagemihl's polyhedron

In 1948, Bagemihl [1] modified Schönhardt's idea to construct a nonconvex polyhedron with $n \geq 6$ vertices. Figure 2.8 is constructed by replacing one of the twisted vertical edges from Schönhardt's twisted triangular prism with a concave curve and placed $n - 6$ vertices along the curve so that the interior dihedral angles of an edge between these new vertices and the vertices of the twisted triangular prism is greater than 180^o.

Claim: Bagemihl's generalization cannot be triangulated.

Proof: If a triangulation exists, then the top triangular face must be a face of some tetrahedron of the triangulation. For every vertex v, not on the top face, there is a diagonal from v to some vertex on the top face which lies outside the polyhedron. Therefore there is no tetrahedron from the vertex set with the top face as a boundary lying inside the polyhedron. ■

Example 3 (Ruppert and Seidel[22])

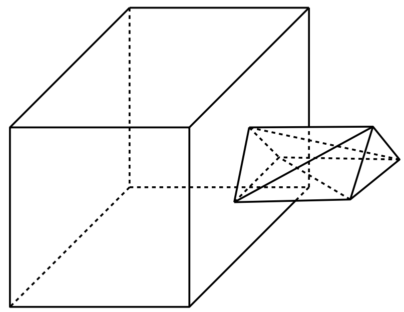


Figure 2.9: Attaching Schönhardt's twisted prism to a cube

Another idea of creating non-triangulable polyhedron with large number of vertices was presented by Ruppert and Seidel [22]. They attached a copy of a non-triangulable polyhedron to another polyhedron. Figure 2.9 shows a polyhedron where a copy of Schönhardt's nonconvex twisted triangular prism, called a niche, is attached to a face of a cube. Although the union of a cube and Schönhardt's nonconvex twisted triangular prism (Figure 2.9) is not a polyhedron by our definition, this is sometimes considered a polyhedron by those wishing to define a polyhedron to be homeomorphic to a ball. To satisfy our definition we could attach a niche to other polyhedron along a common face so that the desired properties still hold.

Claim: If the base of the niche is small enough then Ruppert's and Seidel's polyhedron cannot be triangulated.

Proof: It can be arranged that those vertices of the Schöhardt prism which do not lie on the face of the cube do not see any vertex of the cube. Since each diagonal to the non-attached base of the triangular prism lies outside the polyhedron, then there must exist a tetrahedron contained inside the non-convex twisted triangular prism. We know from Example 1 this is not possible, so no set of tetrahedra triangulates the union. ■

Example 4 (Jessen [13])

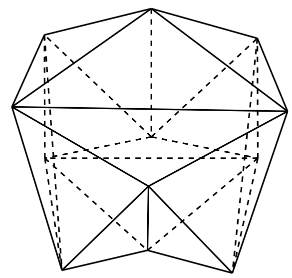


Figure 2.10: Jessen's orthogonal icosahedron

The orthogonal icosahedron (Figure 2.10) was discovered in 1967 by Jessen [13] while looking for an answer to a rigidity problem. To construct the Jessen polyhedron (Figure 2.10), first take three pairwise orthogonal rectangles all sharing a common center with side length in a ratio of $1 : \sqrt{2}$, so that no two edges from different rectangles intersect. Now take the convex hull of the set of 12 vertices of the three rectangles, forming an irregular icosahedron, and remove the six tetrahedra which have two adjacent triangles along an edge of length 1.

Claim: Jessen's polyhedron cannot be triangulated.

Proof: As shown in the claim of Bagemihl's generalization, a polyhedron P cannot be triangulated if there exists a face F such that for every vertex $v \in P$, $v \notin F$, there is a vertex $w \in F$ where the diagonal $\overline{vw}$ is not contained completely inside P. Any triangular face created by the removal of a tetrahedron is such that any vertex not on this face has a diagonal between this vertex and one of the vertices of the face protruding outside the polyhedron. ■

Example 5 (Thurston et al. [19])

Thurston's polyhedron (Figure 2.11), attributed to Thurston by Paterson and Yao [19], is made from removing 18 non-intersecting square prisms, six from each pair of parallel faces, from the cube. Although this is not considered a polyhedron by our definition, we still find the construction relevant to triangulations of polyhedra and we can slightly modify this shape

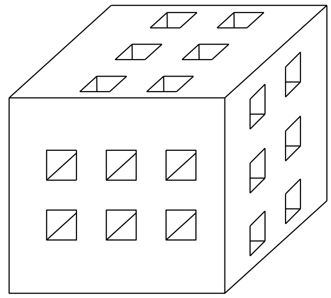


Figure 2.11: Thurston's polyhedron

to become a polyhedron. It is important to note that this polyhedron was independently discovered by several people including W. Kuperberg, Holden, and Seidel.

Claim: Thurston's polyhedron cannot be triangulated.

Proof: We say that a point in a polyhedron "sees" another point in the polyhedron if the line segment between the two points is contained inside the polyhedron. We observe that each point of a tetrahedron can see each of the tetrahedron's vertices. If a polyhedron contains a point which does not see at least four non-coplanar vertices of the polyhedron, then it cannot be contained in a tetrahedron from a triangulation. In Thurston's polyhedron, the center of the cube does not see any vertex of the polyhedron, thus obviously not in the interior of a tetrahedron of a triangulation. ■

Example 6 (Chazelle [5])

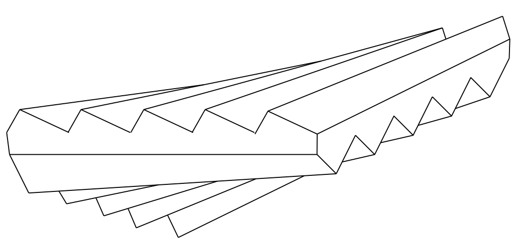


Figure 2.12: Chazelle's polyhedron

In an attempt to create triangulation algorithms, computational geometers have introduced the use of Steiner points, or new vertices on existing edges, to allow each polyhedron to be triangulated. Although our aim is to triangulate polyhedra without introducing new vertices, the results of such a process can be useful.

Chazelle started with a rectangular prism oriented with one edge along the z-axis containing the origin. Let wedges on the bottom face be parallel to the x-axis, and wedges on the top face be parallel to the y-axis. Each wedge's edge is within epsilon of the hyperbolic parapaloid $z = xy$ so that no two wedges intersect. The polyhedra obtained from deleting all the wedges from the rectangular prism is the Chazelle Polyhedron (Figure 2.12).

Claim: Chazelle's polyhedron cannot be triangulated.

Proof: A surprising connection was found by Eppstein between our problem and the problem of finding a lower bound on the number of convex pieces into which any polyhedron of n vertices can be partitioned. Chazelle [5] constructed a polyhedron which can not be partitioned into fewer then $O(n^2)$ convex polyhedra. Coincidently his polyhedron spans at most $O(n)$ tetrahedra, which excludes the existence of a triangulation. ■

Example 7 (Rambau [20])

Rambau [20] discovered another generalization of the Schönhardt twisted triangular prism, which he calls the nonconvex twisted prism (Schönhardt prism).

Let C_n be a convex polygon with n vertices, where the vertices of C_n are labeled clockwise as $v_1, v_2, ..., v_n$.

The *right prism over* C_n (Figure 2.14) is $P_{C_n} = \text{conv}\{(C_n \times \{0\}) \cup (C_n \times \{1\})\}$.

To construct the nonconvex twisted prism, pick a point O in the interior of C_n and rotate C_n clockwise about O by ϵ. Label the vertices of $C_n(\epsilon)$, $v_1(\epsilon), v_2(\epsilon), ..., v_n(\epsilon)$, corresponding to the vertices of C_n. The **convex twisted prism over** C_n is $P_{C_n}(\epsilon) = \text{conv}\{(C_n \times \{0\}) \cup (C_n(\epsilon) \times \{1\})\}$.

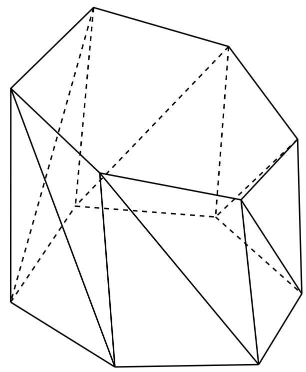


Figure 2.13: Nonconvex twisted prism S_{C_6}

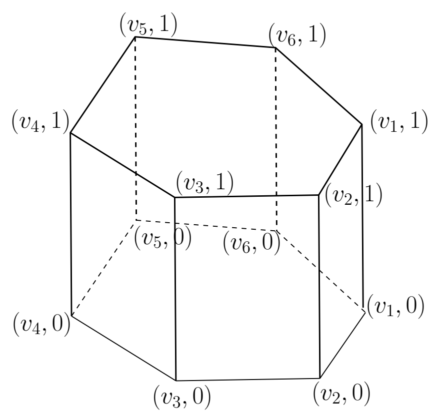


Figure 2.14: Right prism P_{C_6}

The **non-convex twisted prism over** C_n (Figure 2.13) is:

$S_{C_n} = P_{C_n}(\epsilon)$ - conv$\{(v_i,0),(v_{i+1},0),(v_i(\epsilon),1),(v_{i+1}(\epsilon),1)\}$, for all $i \in (1, n)$ taken modulo n.

Rambau [20] proves:

Theorem 2.9. *No right prism P_{C_n}, for $n \geq 3$, admits a triangulation that uses the cyclic diagonals $\{(v_i,0),(v_{i+1},1)\}$.*

Which implies

Corollary 2.10. *For all $n \geq 3$ and all sufficiently small $\epsilon > 0$, the non-convex twisted prism S_{C_n} cannot be triangulated.*

Since our new results are closely related to Rambau's results we wish to discuss the techniques used in his proof.

Theorem 2.9 Proof Overview

Assume P_{C_n} is triangulated so that each cyclic diagonal is an edge of at least one tetrahedron. Rambau observes that every tetrahedron in the triangulation of P_{C_n} contains at least one vertex on each base of the prism. Therefore Rambau is able to view each tetrahedron of the triangulation in a cross section of P_{C_n}. So he chooses a hyperplane parallel to the base which intersects the prism near $(C_n \times \{1\})$.

Rambau also observes that the intersection of the hyperplane and the prism, a copy of C_n which he calls S_n, is subdivided into regions called mixed cells. The mixed cells are the intersections of the hyperplane and the tetrahedra from the triangulation. There are three types of mixed cells in the subdivision of S_n. A tetrahedron containing three vertices from the bottom and one vertex from the top intersects the hyperplane in a small triangle, called a short mixed triangle, and a tetrahedron containing three vertices from the top and one vertex from the bottom intersects the hyperplane in a large triangle, called a long mixed triangle. He labels the boundaries of each short triangle as short edges and large triangles as long edges. Any tetrahedron which has two vertices on the top and two vertices from the bottom will intersect the hyperplane in a parallelogram, called a mixed parallelogram, with two parallel short edges and two parallel long edges. Furthermore, each boundary of a mixed cell is parallel to an edge or diagonal of S_n. Also the intersection property of triangulation provides that mixed cells intersect each other along an entire edge.

Each parallelogram is adjacent to at least one short triangle and one long triangle. Also Rambau notices that the edges of S_n are partitioned into one short edge and one long edge which correspond to the diagonal on a lateral face used by the triangulation. Therefore since we are assuming the cyclic diagonals are edges of tetrahedra, we know the mixed edges alternate along the perimeter of S_n to correspond with the cyclic diagonals.

For every edge of S_n there is a vertex of the mixed subdivision on the edge separating the long mixed edge and short edge. So we assume the mixed subdivision of S_n contains these edges. He starts along the short mixed edge on a boundary edge, and knowing the two neighboring edges along the boundary are long edges the mixed cell containing this edge has a vertex in the interior of S_n. Now he orders the halfplanes, bounded by diagonals in the projection of $(C_n \times \{0\})$ onto S_n at each short edge on the perimeter of S_n, as positive if it contains the short edge and negative otherwise. Then he looks at the cells on the positive side of of each diagonal and shows that no mixed parallelogram is on the positive side of both its short edges. Finally by the connectedness of the subdivision through adjacent mixed cells, he finds a contradiction that at least one mixed parallelogram is on the positive side of both its short edges. Therefore no triangulation of P_{C_n} uses the set of cyclic diagonals.$\blacksquare$

Chapter 3

Tiling by Simplices

3.1 Terminology

For our new results, we will introduce new concepts which have been alluded to by previous results, yet we wish to formally define such concepts in this dissertation. First we will introduce the concept of *tiling by simplices*, which weakens the intersection property of *triangulation*.

Definition 3. *A* ***tiling by simplices*** *of a point configuration* $\boldsymbol{A} \in \mathbb{R}^d$ *is a collection of d-simplices, all of whose vertices are points in* $\boldsymbol{A}$*, which satisfies the following two properties:*
1. The union of all the simplices equals conv($\boldsymbol{A}$*). (Union Property)*
2. The intersection of any two simplices (possibly empty) is a subset of a $\mathbb{R}^{d-1}$ *space. (Intersection Property)*

Specifically in $\mathbb{R}^3$, a *tiling by tetrahedra* of a polyhedron P is a partition into finitely many tetrahedra such that the intersection of two tetrahedra (possibly empty) is a subset of a plane.

Remark 1. *Figure 3.1 describes a tiling of the cube which is not a triangulation.*

Definition 4. *A* ***surface triangulation*** *of a polyhedron* P *is the triangulation of the faces of* P*. We will denote the set of diagonals as* $\bar{P}$ *and the set of triangles bounded by* $\bar{P}$ *and the edges of* P *as* $\hat{P}$*.*

Definition 5. *We say a surface triangulation of a polyhedron* P *is* ***extendable*** *if there exist a triangulation of* P *where every* $t \in \hat{P}$ *is a face of a tetrahedron of the triangulation.*

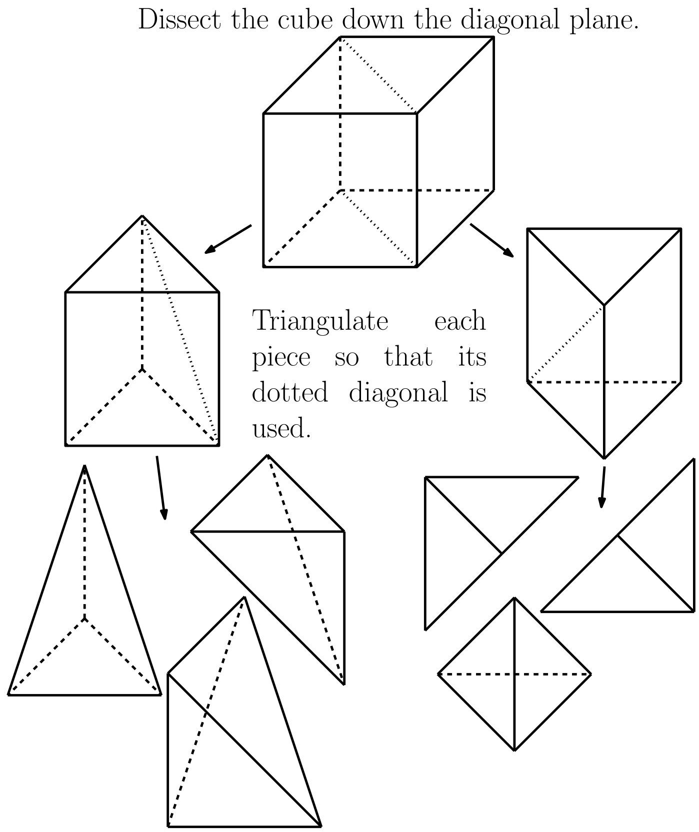


Figure 3.1: Tiling a cube

Definition 6. *A* ***(Schönhardt type) realization*** $\tilde{P}$*, of a surface triangulation of a convex polyhedron* P*, is a polyhedron which is constructed from moving vertices of* P *within an* ϵ*-neighborhood such that every* $d \in \bar{P}$ *is an edge of* $\tilde{P}$ *with a concave interior dihedral angle and every edge of* P *is an edge of* $\tilde{P}$ *with a convex interior dihedral angle. Thus every* $t \in \hat{P}$ *becomes a face of* $\tilde{P}$*.*

Remark 2. *The Schönhardt twisted triangular trism (Figure 2.7 on page 12) is a Schönhardt type realization of a surface triangulation on the right triangular prism.*

Remark 3. *In Chapter* 4*, we will classify a set of polyhedra where every surface triangulation can be realized.*

As previously stated, we wish to re-prove Corollary 2.10, yet we will make an even stronger claim that S_{C_n} cannot be tiled by tetrahedra. Figure 3.1 clearly shows that a tiling by tetrahedra exists for P_{C_4}, which is not a triangulation. Furthermore, this shows that there exists such a tiling which uses the cyclic diagonals of the cube. We will present a new approach for showing a polyhedron is non-triangulable by showing it is unable to be tiled by tetrahedra. We will also demonstrate how this technique can be used for other polyhedra. Since Rambau's technique only applies if a hyperplane intersects every tetrahedron from the triangulation, we will provide a family of non-tilable polyhedra in Theorem 3.8 where a hyperplane would not intersect every tetrahedra of a triangulation. Let us first provide two examples of non-triangulable polyhedra which can be tiled by tetrahedra.

3.2 Non-Triangulable Polyhedra which can be Tiled by Tetrahedra

Theorem 3.1. *There exist a polyhedron which is not triangulable, but can be tiled by tetrahedra.*

Proof We will present Examples 8 and 9 as such polyhedra.

Example 8

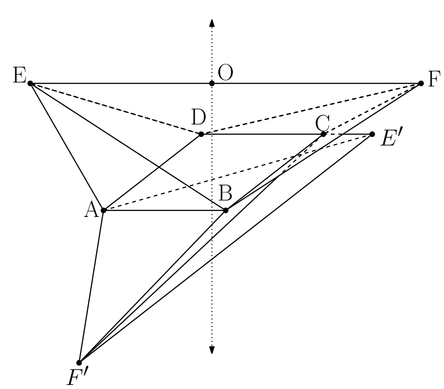


Figure 3.2: A non-triangulable polyhedron which can be tiled with tetrahedra

Start with a horizontal unit square Q. Let A, B, C and D be the vertices of Q in counterclockwise order when we look down at the square from above. Let the point O be over Q at unit distance from its vertices. Next add to this arrangement a segment EF, whose midpoint is O, has length 4, and is parallel to AB (assume E is closer to A than to

B). Rotate this segment clockwise (i.e. opposite to the order of the vertices A, B, C and D) around the vertical line through O by a small angle ϵ. Let P be a non-convex polyhedron bounded Q and by six triangles EAB, EBF, BFC, CDF, EFC, and EDA.

Finally let P' be the image of P under the reflection around the plane of Q followed by a 90° rotation around the vertical line containing O. Label the images of E and F as E' and F' respectively.

First notice that P is triangulable as it is the union of the tetrahedra $EABD, EBDF$ and $DBCF$. Since the same holds for P' we have that the union of P and P' can be tiled by tetrahedra.

Next we show that the union of P and P' is not triangulable. Since neither E nor F can see the vertices E' and F', we have that any triangulation of the union is the union of triangulations of P and P'. The polyhedron P was constructed so that the dihedral angles corresponding to the edges EB and FD are concave, therefore the diagonals AF and EC lie outside of P. It is easy to see that the triangles ABC and ACD cannot be faces of disjoint tetrahedra contained in P, thus diagonal BD must be an edge of at least one tetrahedron in any triangulation of P. A similar argument applied for P' gives that the diagonal AC is an edge of at least one tetrahedron in any triangulation of P'. Thus the union of P and P' is not triangulable.

and

Example 9

Start with a unit cube with faces labeled Top, Bottom, Left, Right, Front, and Back and translate the Top to the right by a distance greater than 1. Label the vertices as in Figure 3.4. Move vertices W and Y along the lines AW and CY respectively up by ϵ and vertices X and Z along the line XZ away from one another by ϵ. Now let P be the polyhedron

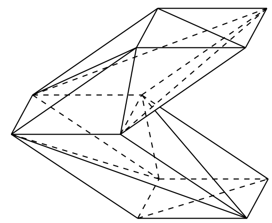


Figure 3.3: A non-triangulable polyhedron which can be tiled with tetrahedra

bounded by the square $ABCD$ and the ten triangles ABW, BWX, BXY, BCY, CDY, DYZ, DWZ, ADW, WXZ and XYZ.

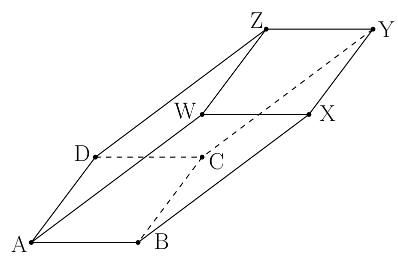


Figure 3.4: Labeling the translated cube

Finally let P' be the image of P under the rotation of 180^o about the line through the midpoints of AD and BC. Label the images of W, X, Y, and Z as W', X', Y', and Z' respectively.

First notice that P is triangulable as it is the union of the tetrahedra $ABDW$, $BWXZ$, $BCDY$, $BDWZ$, $BDYZ$ and $BXYZ$. Since the same holds for P' we have that the union of P and P' can be tiled by tetrahedra.

Next we show that the union of P and P' is not triangulable. Since the vertices W, X, Y, nor Z cannot see the vertices W', X', Y', and Z', we have that any triangulation of the union is the union of triangulations of P and P'. The polyhedron P was constructed so that the dihedral angles corresponding to the edges BW, BY, DW, and DY are concave, therefore the diagonals AX, AZ, CX, and CZ lie outside of P.

Assume the triangles ABC and ACD are faces of disjoint tetrahedra contained in P, then the fourth vertex of each tetrahedra respectively is W or Y. Since the tetrahedra $ABCW$ and $ACDY$ do not intersect along a common face, but do intersect in a plane, they cannot both be in the triangulation. A similar argument is made for $ABCY$ and $ACDW$, therefore the fourth vertex of the tetrahedra containing ABC and ACD respectively is the same.

By symmetry let's assume $ABCW$ and $ACDW$ are tetrahedra of the triangulation. The tetrahedron containing XYZ as a face either has vertex B or D as its fourth vertex, but $XYZB$ and $XYZD$ intersect both $ABCW$ and $ACDW$. Therefore diagonal AC cannot be an edge of a tetrahedron in a triangulation of P, so diagonal BD must be an edge of at least one tetrahedron in any triangulation of P. A similar argument applied for P' yields that the diagonal AC is an edge of at least one tetrahedron in any triangulation of P'. Therefore the union of P and P' is not triangulable. ■

Remark 4. *A non-triangulable polyhedron is tilable only if it contains at least four coplanar vertices where no three are incident with a common face.*

Since S_{C_n} does not contain 4 coplanar points, for sufficiently small ϵ, where no three are incident with a common face, Remark 4 implies that no tiling exists.

Throughout the remainder of the text, we will often look at tetrahedra contained inside a polyhedron and determine if any two interior tetrahedra intersect. If two tetrahedra intersect in more than a plane, we can conclude that both tetrahedra cannot be in a tiling by tetrahedra.

Lemma 3.2. *Let two tetrahedra T_O and T_B share an edge e and contain two coplanar faces t_O and t_B respectively on a plane P. If there exists a plane $Q \neq P$ containing e such that the fourth vertex O of T_O is in the open halfplane bounded by Q containing t_B and the fourth vertex B of T_B is in the open halfplane bounded by Q containing t_O, then T_O and T_B overlap.*

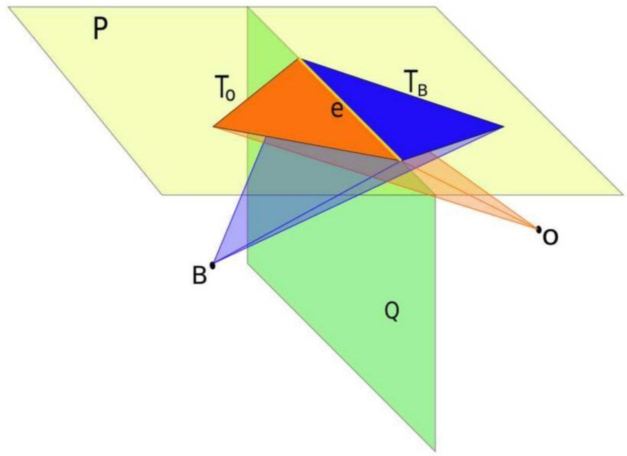


Figure 3.5: Intersecting tetrahedra

Lemma 3.2 can simply be proven by noticing that the interior dihedral angle of T_O at e and the interior dihedral angle of T_B at e sum to greater than 180^o.

Since each face of a tetrahedron $t \in T$ is a triangle, we say T induces a surface triangulation. Rambau used this observation in proving Corollary 2.10 by using Theorem 2.9. We will also use this observation when considering which tetrahedron a particular surface triangle belongs.

Definition 7. *An **ear** is a triangle in a triangulation of a polygon P with exactly two of its edges being edges of P. The vertex incident with these two edges will be the ear vertex.*

Theorem 3.3. *(Meisters [17]) For $n > 3$, every triangulation of a polygon has at least 2 ears.*

It is common to view each triangulation as a tree by letting each triangle be represented by a dual vertex where two dual vertices are adjacent if the corresponding triangles share an edge. In this dual tree each ear is a leaf. We will borrow the terminology of pruning a leaf, to prune ears of a triangulation.

Definition 8. *An ear E is* ***pruned*** *by deleting the ear from the triangulation, leaving the edge which was not an edge of P as an edge of $P' = P - E$. In doing so, we delete the ear vertex from the polygon.*

3.3 Non-Tilable Polyhedra

Definition 9. *We construct a polyhedron which will have two horizontal faces (bottom base and upper base) and several side faces. Let the bottom base be a convex polygonn C_n on n vertices labeled clockwise as $b_1, b_2, ..., b_n$. Define l_i to be the line containing edge $\overline{b_i b_{i+1}}$ (indices taken modulo n). Now we will call the closed area bounded by the lines l_i, l_{i-1}, and l_{i-2}, which contains $\overline{b_{i-1} b_i}$ but does not contain C_n, region R_i (Figure 3.6). (Region R_i may be infinite if l_i and l_{i-2} are parallel or intersect on the same side of l_{i-1} as the polygon.)*
Now define the upper base as the convex polygon $U_n = conv\{u_i, u_2, ..., u_n\}$, where $u_i \in R_i$.
Let $B'_{C_n} = conv\{(C_n \times \{0\}) \cup (U_n \times \{1\})\}$,
and B_{C_n} (Figure 3.7) $= B'_{C_n} - conv\{(b_i,0),(b_{i+1},0),(u_{i+1},1),(u_{i+2},1)\}$, for all $i \in \{1, 2, \ldots, n\}$ taken modulo n.

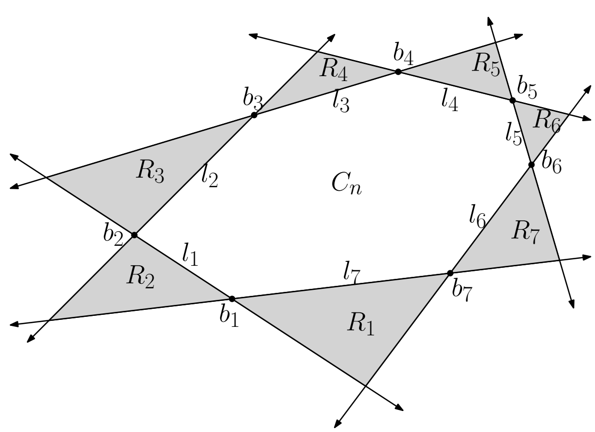


Figure 3.6: All regions R_i for C_7

Theorem 3.4. *The non-convex polyhedron B_{C_n} cannot be tiled with tetrahedra.*

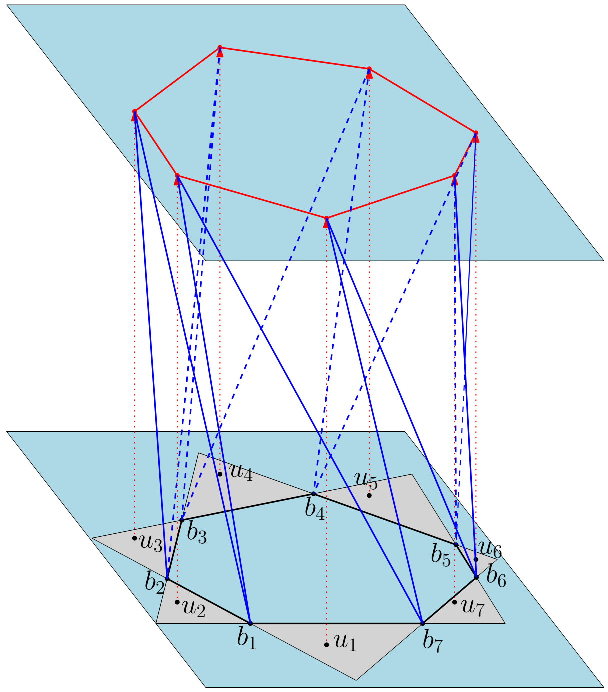


Figure 3.7: Polyhedron B_{C_7}

Proof Assume a set of simplices (tetrahedra) S tiles B_{C_n}. The tiling by S induces a triangulation of $(U_n \times \{1\})$, which we will call T. Now, for every $t \in T$ there exists exactly one $s \in S$ such that t is a face of s. Obviously, the fourth vertex of s must be a vertex of $(C_n \times \{0\})$.

Define a *sub-polygon* to be the convex hull of a subset of the vertices of a polygon. Let P be the set of sub-polygons of U_n such that every edge of a sub-polygon $p \in P$ is an edge of some $t \in T$.

Let e be an edge of p and let t be a triangle of T having e as an edge and inside p. We will say p is a *separator* if every point b_i in the open halfplane, bounded by the line containing e, which does not contain p cannot be in a tetrahedron of S with t as a face.

Let $P' \subseteq P$ so that every $p' \in P'$ is separating. P' is not empty since U_n is a separating sub-polygon. A minimal separating sub-polygon is a sub-polygon with the fewest vertices. Let $m \in P'$ be a minimal separating sub-polygon with n vertices. If $n > 3$, then there is a

$t \in T$ which is an ear of m. Let d be the edge of t which is not an edge of m. Observe that there exists a triangle $t' \in T$ which has d as an edge and is contained in m.

Remark 5. *The construction of U_n yields the property that the line containing the diagonal $\overline{u_i u_j}$ (for $i < j$) bounds two open halfplanes such that the halfplane containing the vertices u_k for $i < k < j$ also contains the vertices b_m for $i \leq m < j$ and no other vertices from the polygon C_n.*

Let Q be the plane through d perpendicular to $U_n \times \{1\}$. Since m is separating, we can conclude by Lemma 3.2 that t' cannot be in a tetrahedron with any $(b_i, 0)$ where b_i is in the open halfplane, bounded by the line containing d, which contains t. Therefore we can prune t so that $m - t$ is a separating sub-polygon. Since $m - t$ has fewer vertices than m, m is not a minimal separating sub-polygon. Therefore we can conclude that the minimal separating sub-polygon is a triangle.

Some $t = \{u_x, u_y, u_z\} \in T$ is a minimal separating sub-polygon. Since t is separating, for every b_i outside of t, t is not in a tetrahedron with $(b_i, 0)$. By Remark 5, the only vertices which can exist inside t are b_x, b_y, or b_z, but the segments $\overline{(b_i, 0)(u_i, 1)}$ lie outside B_{C_n}. Therefore no set of tetrahedra tiles B_{C_n}. ■

A closer look at the proof yields that Remark 5 is the only observation necessary of U_n for the proof. Thus we will define a particular alteration A_{C_n} of a prism.

Let C_n be the same convex polygon defined in B_{C_n}. Let $A_n = \text{conv}\{a_1, a_2, ..., a_n\}$, where the line containing the diagonal $\overline{a_i a_j}$ (for $i < j$) bounds two open halfplanes such that the halfplane containing the vertices a_k for $i < k < j$ also contains the vertices b_m for $i \leq m < j$ from the polygon C_n. Let $A'_{C_n} = \text{conv}\{(C_n \times \{0\}) \cup (A_n \times \{1\})\}$. The *nonconvex altered prism over C_n* is $A_{C_n} = A'_{C_n} -$ conv$\{(b_i,0),(b_{i+1},0),(a_i,1),(a_{i+1},1)\}$, for all $i \in (1, n)$ taken modulo n.

Corollary 3.5. *The non-convex altered prism A_{C_n} cannot be tiled by tetrahedra, hence it also non-triangulable.*

Relationship between S_{C_n} and A_{C_n}

Let us first make an observation about how the center of rotation used to construct $C_n(\epsilon)$ relates to Remark 5. Consider a line segment $\overline{ac}$ and place a point b on this segment between a and c. Define the line l to be the line perpendicular to $\overline{ac}$ through the point c. Assume we rotate all points about a center O by some small angle of rotation to the points a', b', and c' respectively. If O does not lie in the same open halfplane, bounded by l, as a and b, then for some Δ and all rotation by α, $0 < \alpha < \Delta$, about O the line containing the segment $\overline{a'c'}$ bounds two halfplanes one of which contains both a and b. Thus we will define the halfplane, bounded by l, containing a and b as the $c+$ halfplane, and the other as the $c-$ halfplane.

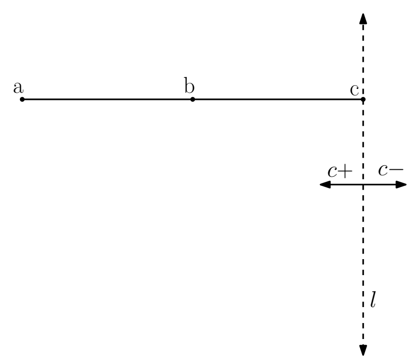


Figure 3.8: Halfplanes $c+$ and $c-$

It is easy to see that there is a convex polygon C_n where no rotational center yields the observations made in Remark 5 between C_n and $C_n(\epsilon)$. Such an example is provided on the coordinate plane in Figure 3.9.

We notice for small rotations if the center of rotation lies on a point with an x-coordinate greater than or equal to 0, then the diagonal $\overline{(-1,1)(-3,1-\epsilon)}$ will not satisfy Remark 5. Similarly, if the center of rotation lies on a point with an x-coordinate less than or equal to 0, the diagonal $\overline{(1,-1)(3,-1+\epsilon)}$ will not satisfy Remark 5. Therefore, there is no center of rotation which will satisfy Remark 5.

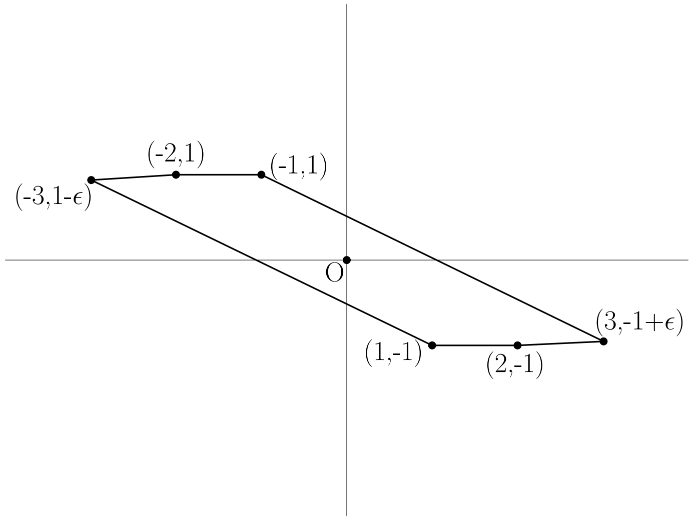


Figure 3.9: Polygonal base with no rotational center

This provides that if we wish to show that the results of Rambau hold in the case of tiling by tetrahedra, we must alter our proof technique to satisfy all polyhedra in S_{C_n}. We make the observation that in Lemma 3.2 we need not have plane Q be perpendicular to plane P, the lemma holds for any plane Q containing the edge E where $Q \neq P$.

Corollary 3.6. *(Analogue of Rambau's Corollary)*
For all $n \geq 3$ and all sufficiently small $\epsilon > 0$, the non-convex twisted prism S_{C_n} cannot be triangulated.

Proof It suffices to show that for any C_n, there exists a sufficiently small ϵ such that for any diagonal $\overline{(v_i,1)(v_j,1)}$ (for $i < j$) of $C_n(\epsilon)$ there is a plane Q containing the diagonal $\overline{(v_i,1)(v_j,1)}$ which bounds two open halfspaces such that the halfspace containing the vertices $(v_k,1)$ for $i < k < j$ also contains the vertices $(v_m,0)$ for $i \leq m < j$ and no other vertices from the polygon $C_n \times \{0\}$. When constructing $C_n(\epsilon)$ we must consider the planes through each diagonal. Now, for any rotational center O, there is some angle of rotation α_{ij} where the diagonal $\overline{v_i(\alpha_{ij})v_j(\alpha_{ij})}$ lies on a line parallel to the diagonal $\overline{v_{i-1}v_{j-1}}$. Thus, for every S_{C_n} constructed by a rotation about O by $0 < \epsilon_{ij} < \alpha_{ij}$ there exists a plane Q satisfying the conditions of Lemma 3.2 for the diagonal $\overline{(v_i, 1)(v_j, 1)}$. It follows that if we let $\alpha = \min\{\alpha_{ij} | i, j \in (1, 2, 3, ..., n)\}$, then for $0 < \epsilon < \alpha$ S_{C_n}, constructed from rotation by ϵ about the center O, cannot be tiled by tetrahedra. ■

We notice that the results of Schönhardt, Bagemihl and Jessen provide polyhedra where no diagonal is contained inside the polyhedron, thus cannot be tiled by tetrahedra. Also figures producing an interior point which cannot be seen by any vertex, such as Thurston's polyhedron, cannot be tiled by tetrahedra. The results of Rambau do not give such a conclusion, yet, as shown previously, S_{C_n} cannot be tiled by tetrahedra. With these results it is natural for one to ask if there exist other polyhedra which cannot be tiled. By a generalization of the technique in Theorem 3.4, we will show that a Schönhardt type realization of a surface triangulation of the Regular Dodecahedron cannot be tiled by tetrahedra. To do so, we will first revisit Lemma 3.2 and state a more general lemma of two intersecting tetrahedra.

Lemma 3.7. *Let two tetrahedra T_A and T_B share an edge e, where e is an edge of triangles A and B from the tetrahedra respectively. Let a be the fourth vertex of T_A and b be the fourth vertex of T_B. If b is in the intersection containing T_A of the the halfspace bounded by the plane containing A and the halfspace bounded by the plane containing a and e, then T_A and T_B do not have disjoint interiors.*

The proof of Lemma 3.7 is simply concluded by noticing a similar relationship as in Lemma 3.2. The sum of the dihedral angles formed in T_A and T_B at the edge e is greater than the dihedral angle formed at e by A and B.

Remark 6. *When implementing Lemma 3.7 we must consider each possible tetrahedra separately, instead of considering the orthogonal plane, since A and B are no longer assumed to be coplanar.*

Nonconvex Twisted Dodecahedron

Similar to the construction of the Schönhardt Prism, we will start with a well known convex polyhedron. Consider the regular dodecahedron DH oriented with one pentagonal face at height 0, called P_0, and its parallel face at height 1,called P_1. The remaining ten vertices are equally partitioned at two heights between 0 and 1. For simplicity, we will label these two heights a and b, $a < b$.

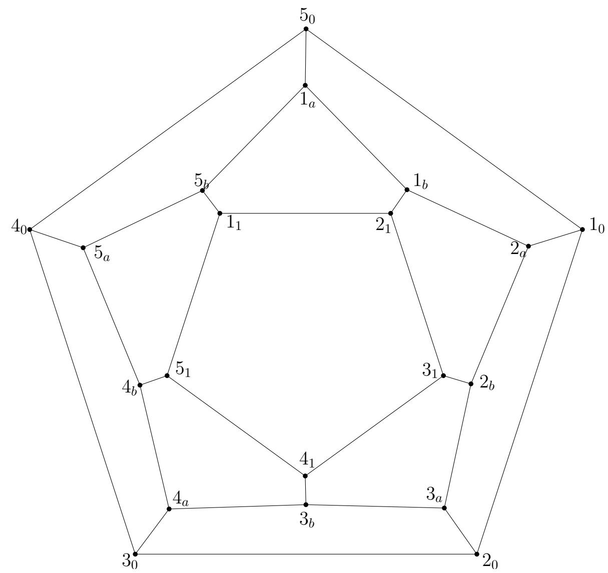


Figure 3.10: Edge graph of DH

We will first label all the vertices of the regular dodecahedron. For this construction we will only be using labels from the set $\{1, 2, 3, 4, 5\}$, thus all labels will be taken modulo 5. We will also be using clockwise and counterclockwise orientation viewed from the above perspective. Label P_0 with vertices $\{1_0, 2_0, 3_0, 4_0, 5_0\}$ in a clockwise manner, such that $\overline{i_0(i+1)_0}$ is an edge. Then let the five vertices adjacent to P_0 be $P_a = \{1_a, 2_a, 3_a, 4_a, 5_a\}$, where $\overline{i_0(i+1)_a}$ is an edge. Now let the five vertices adjacent to P_1 be $P_b = \{1_b, 2_b, 3_b, 4_b, 5_b\}$, so that $\overline{i_a i_b}$ and $\overline{i_b(i+1)_a}$ are edges. Finally let the five vertices of P_1 be $\{1_1, 2_1, 3_1, 4_1, 5_1\}$, such that $\overline{i_b(i+1)_1}$ is an edge. So the regular dodecahedron is $DH = conv\{P_0, P_1, P_a, P_b\}$ (Figure 3.10)

Now to create the nonconvex twisted dodecahedron $DH(\epsilon)$ (Figure 3.11) we will rotate P_0 about the center point counterclockwise by an angle $\beta \leq \epsilon$, and P_1 about the center point clockwise by an angle $\tau \leq \epsilon$. So the bottom face is rotated by β in one direction and the top face is rotated by τ in the opposing direction where the orientation of each rotation is viewed from above the dodecahedron. Now we will take the convex hull of the 20 points minus the convex hull of each set of five points which was the face of DH, with the exception of the top and bottom faces.

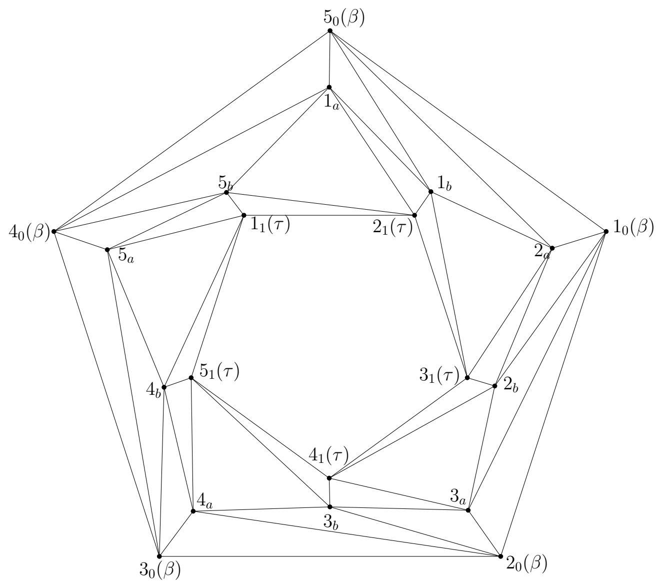


Figure 3.11: Edge graph of $DH(\epsilon)$

Remark 7. *We will choose $\epsilon < 18^o$, so that the top and bottom faces are not translates of each other.*

So $P_0(\beta) = \{1_0(\beta), 2_0(\beta), 3_0(\beta), 4_0(\beta), 5_0(\beta)\}$

and $P_1(\tau) = \{1_1(\tau), 2_1(\tau), 3_1(\tau), 4_1(\tau), 5_1(\tau)\}$

Definition 10. *The* ***nonconvex twisted dodecahedron*** *(Figure 3.11) is*
$DH(\epsilon) = conv\{P_0(\beta), P_1(\tau), P_a, P_b\} - conv\{i_0(\beta), (i+1)_0(\beta), (i+1)_a, (i+1)_b, (i+2)_a\} - conv\{i_1(\tau), (i+1)_1(\tau), (i-1)_b, i_b, i_a\}$ *for all $i \in \{1, 2, 3, 4, 5\}$ (taken modulo 5).*

Remark 8. *The diagonals of the form* $\overline{i_n i_x}$ *for* $i \in \{1,2,3,4,5\}$*,* $n \in \{0,1\}$*, and* $x \in \{a,b\}$ *lie outside* $DH(\epsilon)$*.*

Theorem 3.8. *For* $0 < \epsilon < 18^o$ *the nonconvex twisted dodecahedron* $DH(\epsilon)$ *cannot be tiled by tetrahedra without new vertices.*

Proof Outline

- Assume a tiling of $DH(\epsilon)$ exists and induces a triangulation on $P_1(\tau)$ by triangles A, B, and C.

- We will use Lemma 3.7 to show that at least one triangle from $P_1(\tau)$ is in a tetrahedron with a vertex from $P_0(\beta)$.

- Using a case analysis and Lemma 3.7 we will show that there is a triangular face which can only be in a tetrahedron which contains a diagonal lying outside $DH(\epsilon)$ for every combination where a triangle of $P_1(\tau)$ is in a tetrahedron with a vertex from $P_0(\beta)$, contradicting that such a tiling exists.

Detailed Proof

As in theorem 3.4, we will assume there exist a set of simplices (tetrahedra) S which tiles $DH(\epsilon)$, and consider the induced triangulation T of $P_1(\tau)$. Since $P_1(\tau)$ is a pentagon, having only one unique triangulation, and all vertices of $P_1(\tau)$ are vertex transitive, we can assume, without loss of generality, that

$$T = \{(1_1(\tau), 2_1(\tau), 3_1(\tau)), (1_1(\tau), 3_1(\tau), 4_1(\tau)), (1_1(\tau), 4_1(\tau), 5_1(\tau))\}.$$

For simplicity we will let triangle $(1_1(\tau), 2_1(\tau), 3_1(\tau)) = A$, triangle $(1_1(\tau), 3_1(\tau), 4_1(\tau)) = B$, and triangle $(1_1(\tau), 4_1(\tau), 5_1(\tau)) = C$ shown in Figure 3.12. We will also refer to a tetrahedron containing these triangles as faces by $\{X, p\}$ where $X \in \{A, B, C\}$ and p is some vertex of $DH(\epsilon)$ not on $P_1(\tau)$.

First we will show there exist an $s \in S$ where $t \in T$ is a face of s and the fourth vertex of s is a vertex of $P_0(\beta)$.

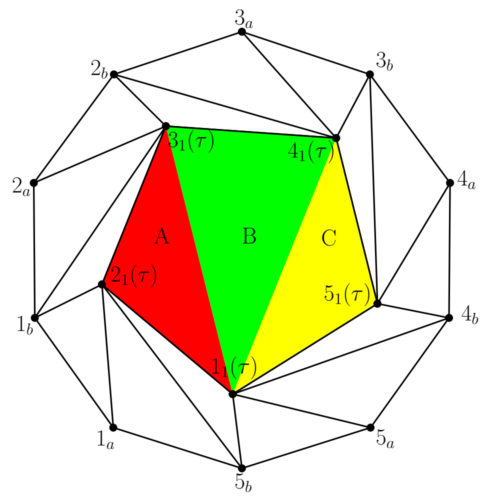


Figure 3.12: Top view of $DH(\epsilon)$

Assume each tetrahedra containing three points from $P_1(\tau)$ does not contain a fourth point from $P_0(\beta)$. Recall that triangle A cannot be in a tetrahedron with vertices from the set $\{1_a, 1_b, 2_a, 2_b, 3_a, 3_b\}$. If triangle B is in a tetrahedron with 2_a or 2_b, then by Lemma 3.7 triangle A cannot be in a tetrahedron with vertices from the set $\{4_a, 4_b, 5_a, 5_b\}$. Thus reaching a contradiction that triangle A is not in a tetrahedron with a vertex from $P_0(\beta)$.

Similarly if triangle B is in a tetrahedron with 5_a or 5_b, then by Lemma 3.7 triangle C cannot be in a tetrahedron with vertices from the set $\{2_a, 2_b, 3_a, 3_b\}$. Thus reaching a contradiction that triangle C is not in a tetrahedron with a vertex from $P_0(\beta)$.

Therefore triangle B cannot be in a tetrahedron with vertices of P_a or P_b, so at least one triangle in T must be in a tetrahedra with a vertex from $P_0(\beta)$.

Now we will show no tetrahedron of the tiling can be of the form $\{X, z_0(\beta)\}$ for $X \in \{A, B, C\}$. We will do so by taking a case analysis of each possibility for a tetrahedron $s \in S$.

Case 1: $s = \{B, z_0(\beta)\}$

Case A: Assume $z = 2$, so $s = \{B, 2_0(\beta)\}$.

Since face $\{3_1(\tau), 4_1(\tau), 2_b\}$ shares and edge with triangle B we can see by Lemma 3.7 that the tetrahedron of S having $\{3_1(\tau), 4_1(\tau), 2_b\}$ as a face can only have its fourth vertex be from the set $\{3_a, 3_b, 2_0(\beta)\}$, but by construction all tetrahedra of these constraints contain a diagonal lying outside $DH(\epsilon)$. ※

Case B: Assume $z = 1, 3, 4$,or 5

We will find a contradiction for $z = 5$, so $s = \{B, 5_0(\beta)\}$. The other three will follow with symmetry and similar arguments.

Since triangle B and triangle A share edge $\overline{1_1(\tau)3_1(\tau)}$, then Lemma 3.7 provides that the tetrahedron of S having triangle A as a face can only have it's fourth vertex from the set $\{1_0(\beta), 5_0(\beta), 1_a, 1_b, 2_a, 2_b\}$, but by construction A cannot be in a tetrahedron with the vertices from the set $\{1_a, 1_b, 2_a, 2_b\}$.

Thus if we assume the tetrahedron $\{A, 1_0(\beta)\} \in S$, then by a similar argument as in Case 1-A the tetrahedra of S having $\{2_1(\tau), 3_1(\tau), 1_b\}$ as a face, will have its fourth vertex be from the set $\{2_a, 2_b, 1_0(\beta)\}$ but by construction all tetrahedra of these constraints contain a diagonal ling outside $DH(\epsilon)$. Similarly if we assume the tetrahedron $\{A, 5_0(\beta)\} \in S$, then by a similar argument as in Case 1-A the tetrahedron of S having $\{1_1(\tau), 2_1(\tau), 5_b\}$ as a face, will have its fourth vertex be from the set $\{1_a, 1_b, 5_0(\beta)\}$ but by construction all tetrahedra of these constraints contain a diagonal ling outside $DH(\epsilon)$.※

Case 2: $s = \{A, z_0(\beta)\}$ (*By symmetry a similar argument can be made for triangle C.*)

Case A: We have seen that $z \neq 1$ or 5 by the argument in Case 1-B.

Case B: Assume $z = 2$

Since face $\{2_1(\tau), 3_1(\tau), 1_b\}$ shares an edge with triangle A, then Lemma 3.7 along with the construction yields that the tetrahedron of S containing $\{2_1(\tau), 3_1(\tau), 1_b\}$ as a face must have as its fourth vertex $2_0(\beta)$. Since face $\{3_1(\tau), 1_b, 2_a\}$ shares an edge with face $\{2_1(\tau), 3_1(\tau), 1_b\}$, Lemma 3.7 provides that the tetrahedron of S containing $\{3_1(\tau), 1_b, 2_a\}$ as a face has as it's fourth vertex be from the set $\{1_0(\beta), 2_b, 2_0(\beta), 3_a\}$, but by construction all tetrahedra of these constraints contain a diagonal lying outside $DH(\epsilon)$. ※

Case C: Assume $z = 3$ or 4

We will find a contradiction for $z = 3$, so $s = \{A, 3_0(\beta)\}$. The case of $z = 4$ will follow from a similar argument.

Recall that case 1 showed triangle B cannot be in a tetrahedron with a vertex of $P_0(\beta)$. Since triangle A and triangle B share and edge Lemma 3.7 and the construction provides that the tetrahedron of S containing B as a face has as its fourth vertex 5_a.

Since triangle B and triangle C share an edge and B is in a tetrahedron with vertex 5_a, then by Lemma 3.7, all tetrahedra not intersecting $\{B, 5_a\}$ and containing C as a face has a diagonal lying outside $DH(\epsilon)$.※

So we have shown that for any set of tetrahedra S which tiles $DH(\epsilon)$, there is at least one induced triangle of $P_1(\tau)$ in a tetrahedron with a vertex in $P_0(\beta)$ and that there is no tetrahedron with a triangle of $P_1(\tau)$ and a fourth vertex from $P_0(\beta)$. Therefore there exist no set of tetrahedra which tiles $DH(\epsilon)$. ■

3.4 Open Problem

The result for $DH(\epsilon)$ motivates a generalization just as Schönhardt example motivated Rambau's generalization.

Notice that the position of P_0 and P_1 in DH is the same as the bases of the right pentagonal anti-prism. So we will define a n-gonal pentaprism as a polyhedron with two bases in the same position as the right n-sided anti-prism and bounded on the sides by $2n$ pentagonal lateral faces.

Definition 11. *Let R_n be a regular n-gon. Let R_0 be a copy of R_n at height 0 and R_1 be a copy of R_n at height 1 so that conv$\{R_0, R_1\}$ is the right n-sided anti-prism. Let the plane* *containing R_0 be P_0 and the plane containing R_1 be P_1. If δ is the interior dihedral angle between a base and a lateral face of the right n-sided anti-prism, then let α be such that $\delta < \alpha < 180$.*

Label the edges of R_0 as e_i for $i \in \{2, 3, \cdots, n+1\}$ and define the plane containing e_i which forms an angle above R_0 of measure α with R_0 to be P_i. Similarly, label the edges of R_1 as e_j for $j \in \{n+2, n+3, \cdots, 2n+1\}$ and define the plane containing e_j which forms an angle below R_1 of measure α with R_1 to be P_j.

Now let H_i be the halfspace bounded by P_i for $i \in \{0, 1, \cdots, 2n+1\} = I$ containing the right n-sided anti-prism. The ***n-sided pentaprism*** *is $PP_n = \cap_{i \in I} H_i$.*

Remark 9. *$DH = PP_5$ for $\alpha = \arccos(\frac{-1}{\sqrt{5}})$.*

We pose the open problem: Is the nonconvex twisted PP_n, as described by the construction of $DH(\epsilon)$, tilable by tetrahedra for all $n > 3$?

It follows from the proof of Theorem 3.8 that for $n = 3$ or 4, the non-convex twisted pentaprism cannot be tiled. However, for arbitrarily n, the regular n-gon has many non-isomorphic triangulation, thus a more sophisticated method is needed.

Chapter 4

Extendable Surface Triangulation

In the previous chapter we discussed triangulable or tilable polyhedra while alluding to surface triangulations. We wish to formalize the relationship between these concepts.

Definition 12. *A surface triangulation of a given polyhedron is called* ***extendable*** *to a triangulation (to a tiling respectively) if the polyhedron has a triangulation (tiling respectively) such that each triangle of the surface triangulation is a face of one of the participating tetrahedra. In this case we also say that the set of tetrahedra which triangulate (or tile respectively) the polyhedron* ***induces*** *the given surface triangulation.*

It is hard to tell which chain of thinking led the different authors to construct their non-triangulable polyhedra (Examples 1-6), however a close look at the polyhedra can reveal some common key features which lead to further non-triangulable polyhedra. If one starts with a specific symmetrical convex polyhedron and finds a non-extendable surface triangulation, then there might be a chance that one can perturb the vertices of the polyhedron so that the new polyhedron is non-triangulable. We will make this perturbing idea more precise.

Theorem 4.1. *There is a suffeciently small ϵ so that an ϵ perturbed surface triangulation is triangulable if and only if the surface triangulation is extendable to a triangulation.*

The proof of Theorem 4.1 hinges solely of the fact that the set of tetrahedra are joined face-to-face. It is natural to assume the analogous holds true for extending surface triangulations to a tiling by tetrahedra. However, we notice that the tetrahedra are not joined face-to-face in a tiling and thus the analogue is not true.

Theorem 4.2. *There exist extendable surface triangulations to a tiling by tetrahedra, where an ϵ perturbation of the surface triangulations is non-tilable.*

Remark 10. *A polyhedron with only triangular faces is a ϵ perturbation of itself, thus if tilable then there is a polyhedra which it is an ϵ perturbed surface triangulation which is extendable.*

Proof The simplest example of such a formation is to consider the tiling of a cube shown in Figure 4.1. Since the cyclic diagonals are a subset of the induced surface triangulation, then the perturbation of the twist described by Rambau [20] would produce a corresponding tiling, yet Theorem 3.4 showed that the twisted cube is not tilable. ■

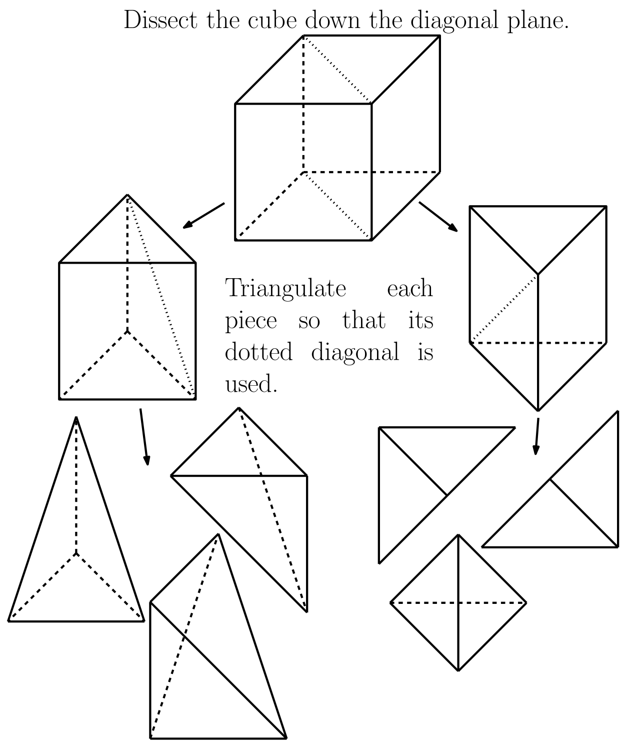


Figure 4.1: Tiling of a cube

So we ask, why is this so? We will assume we have an extended tiling of the surface triangulation of cyclic diagonals on the cube. Notice that the bottom face is triangulated into two triangles which must be connected to two opposing vertices of the top face as shown by the two tetrahedron in Figure 4.2. It is obvious by Lemma 3.2 that the two tetrahedra will overlap for any small twist.

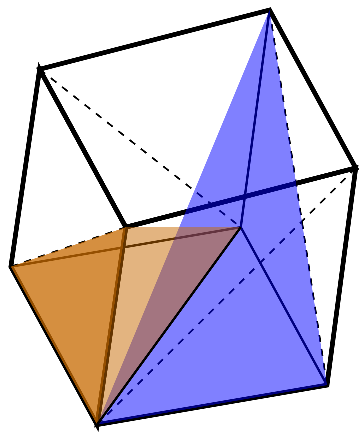


Figure 4.2: Extended tiling which crosses

4.1 Realizing Surface Triangulations

Let us discuss a perturbation in a planar setting. Assume a given polygon P has its edges partitioned into line segments by placing new vertices along the edges. Some (or all) edges may contain no new vertices, thus not partitioned. The introduction of new vertices will result in a degenerate polygon, where some pairs of adjacent edges are collinear. Let $\epsilon > 0$ be a sufficiently small positive number. Making a degenerate polygon *semi-concave* is to transform (Figure 4.3) its vertices so that the vertices are repositioned within their ϵ neighborhood and all degenerate internal angles (180^o) of the given polygon become concave. If no new vertices are added in the partitioning, then there is no need to move any of the vertices. Otherwise one can get a desired semi-concave polygon by replacing each edge of P with a slightly bent concave polygonal arc with the new vertices from that edge.

One can say that the role of the sufficiently small ϵ in the above definition is the same as saying one wants to perturb the vertices slightly. The $\mathbb{R}^3$ variant of this problem is less trivial and not an analogue of the above definition as we will not add new vertices to edges, but rather new edges on the faces of a polyhedron. Let us start with a proper definitions:

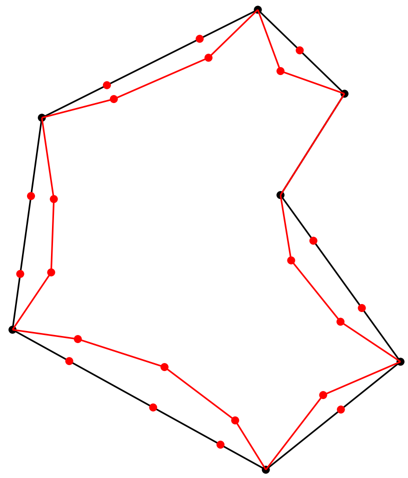


Figure 4.3: Transforming a degenerate polygon into a semi-concave polygon

Definition 13. *Let $\hat{P}$ be a surface triangulation of a given polyhedron P. $\hat{P}$ can be viewed as a degenerated polyhedron. Let $\epsilon > 0$ be a sufficiently small positive number. Making a degenerate polyhedron **semi-concave** is to transform its vertices so that the vertices are repositioned within their ϵ neighborhood and the dihedral angles between coplanar adjacent faces become concave. The resulting polyhedron will be called the **realized** semi-concave polyhedron of the surface triangulation of P. We will also refer to the transformation as realizing a surface triangulations.*

Theorem 4.3. *If each vertex of a convex polyhedron is adjacent to no more than three non-triangular faces, then every surface triangulation of the polyhedron can be realized.*

Proof Let P be the convex polyhedron satisfying the conditions of Theorem 4.3 and let $\hat{P}$ be a surface triangulation of P. Recall $\bar{P}$ is the set of diagonals of the surface triangulation. Color all the triangular faces of P blue, and every triangle of $\hat{P}$ red (except those which are faces of P). Also color every edge of P black and every diagonal of $\bar{P}$ white. Consider the degenerate polyhedra bounded by the blue and red triangles; the interior dihedral angle at every white edge is 180^o and every blue triangular face is bounded by three black edges, whose dihedral angles are convex.

It is easy to see that there exist a sufficiently small ϵ so that dragging any vertex of P along with its incident edges to a point inside an ϵ neighborhood will result in all the convex dihedral angles remaining convex, and the concave dihedral angles remaining concave. Let v be a vertex of P such that it is an ear vertex (Definition 7) of at least one of the faces triangulated by $\hat{P}$. The dihedral angles corresponding to the white edge of an ear can become convex or concave depending on the perturbation of its ear vertex. Our job is to find a perturbation which makes the dihedral angles of the white edge concave.

We will distinguish between two cases:

Case 1: If vertex v is an ear vertex on every non triangular incident face:

Let v be the ear vertex of ears E_1, E_2 and E_3 on faces F_1, F_2 and F_3 respectively. (If fewer than three non triangular faces exist, disregard the extra ears and faces.) We will drag v along with its incident edges of P to a point inside of the ϵ neighborhood of v which is not coplanar with F_1, F_2 or F_3. Furthermore we will pick the point so that each white edge of E_1, E_2, and E_3 becomes concave.

Each plane containing F_i bounds two hemispheres of the ϵ neighborhood of v, one which contains all the points which if v is dragged to will make the white edge of E_i a concave interior angle and the other contains the points which will make a convex interior angle. All three faces contain v, so the intersection of the hemispheres causing concave interior angles at the white edges is not empty, thus such a point exists.

Case 2: If vertex v is not an ear vertex of at least one of its non triangular incident faces.

Assume v is not an ear vertex of the non triangular faces F_1 and F_2. (If only one non triangular faces exist, disregard F_2) We will drag v along with its incident edges of P to a point inside of the ϵ neighborhood of v which is coplanar with F_1 and F_2 and causes the white edge of every ear, where v is an ear vertex, to become concave. The line which is the intersection of the planes containing F_1 and F_2 contains such a point. Notice every white edge along F_1 and F_2 remains degenerate.

In both cases, change the color of the white edge(s), which are now concave, to black and the red ear(s), which are now faces, to blue. The new polyhedra (possibly still degenerate) has at least one less red triangle and at least one less white edge. This implies that the above process, beginning with choosing a vertex v which is incident to a red ear, can be repeated finitely many times resulting with a polyhedron with only blue triangular faces and concave dihedral angles at each of the original diagonals of $\bar{P}$. ∎

Remark 11. *With the process shown in Theorem 4.3, each diagonal along a face P which crosses a diagonal of $\bar{P}$ becomes outside the realized polyhedron.*

4.2 Polyhedra with Regular Polygonal Faces

We will now attempt to find non-extendable surface triangulations where the realized semi-concave polyhedron is a non-triangulable polyhedron. We will begin our search by investigating the surface triangulations of polyhedra composed of regular polygonal faces, since each meets the conditions of Theorem 4.3

Platonic Solids and Prisms

Theorem 2.9 clearly defined a partial surface triangulation of the right prism which cannot be extended.

Theorem 4.4. *The 5 Platonic solids are divided into two classes depending on the extendibility of their surface triangulations:*

A) Every surface triangulation is extendable to a triangulation.

B) There exist at least one surface triangulation which is not extendable to a triangulation.

Proof It is obvious that there is only one surface triangulation of the regular tetrahedron, octahedron, and icosahedron, which is the empty set of diagonals, thus the surface triangulation of each of these is extendable. However, we have discussed in length the statement of Theorem 2.9 showing the surface triangulation of the cyclic diagonals of a prism is not extendable, therefore there exists a surface triangulation of the cube (and every right prism) which is not extendable. With the relationship given in Theorem 4.1, we can also conclude from Theorem 3.8 that the surface triangulation of the regular dodecahedron which can be realized to $DH(\epsilon)$ is not extendable. ■

Archimedean and Johnson Solids

We will explicitly study the surface triangulations of the Archimedean solids. We have provided the names, vertex labeling and an image (taken from http://en.wikipedia.org/wiki/Archimedean_solid) of each Archimedean Solid. As a bi-product, we will mention results concerning some Johnson solids.

Theorem 4.5. *The 13 Archimedean solids are divided into four classes depending on the extendibility of their surface triangulations:*

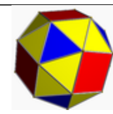
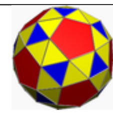
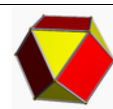
A) Every surface triangulation is extendable to a triangulation.

B) Every surface triangulation can be extended to a tiling by tetrahedra, but it is unknown if each surface triangulation can be extended to a triangulation.

C) There exist at least one surface triangulation which is not extendable to a triangulation.

D) It is unknown if each surface triangulation can be extended to a triangulation.

Class A

1. Snub cube	(3,3,3,3,4)	
2. Snub dodecahedron	(3,3,3,3,5)	
3. Cuboctahedron	(3,4,3,4)	

Definition 14. *A fan is a triangulation such that all simplices contain a common vertex. We will call the common vertex the fan vertex.*

Remark 12. *To show that all convex polyhedra contain a triangulation it is sufficient to create a fan at one of the vertices. In a fan triangulation of a convex polyhedron, we notice that the triangulation induces a fan triangulation at that vertex on every adjacent face.*

Proof of Class A If a surface triangulation of a convex polyhedron contains a vertex which is a fan of each of its adjacent faces, then by Remark 12 it is extendable. A fan triangulation is the set of all tetrahedra created by taking the three vertices of every triangle of the surface triangle, except those containing the fan vertex, and adjoining it with the fan vertex.

Since each triangulation of a quadrilateral and pentagon are fans we will first consider the Archimedean solids which only contain triangles, squares, and pentagons. Not all such polyhedra with this property must be triangulated with a fan vertex, but if every square and pentagon is only adjacent to triangular faces, then every surface triangulation contains a fan vertex. The snub cube and snub dodecahedron have this property, thus belong to class A. It is worth noting that there are forty four Johnson solids which have this property. You can find a list of these Johnson solids in the appendix.

Although the cuboctahedron has previously been shown to be in class A [7], Figure 4.4 shows the only unique surface triangulation without a fan vertex. It would suffice to find the extension of this surface triangulation, however, we wish to present a new method of finding an extension of all surface triangulation simultaneously.

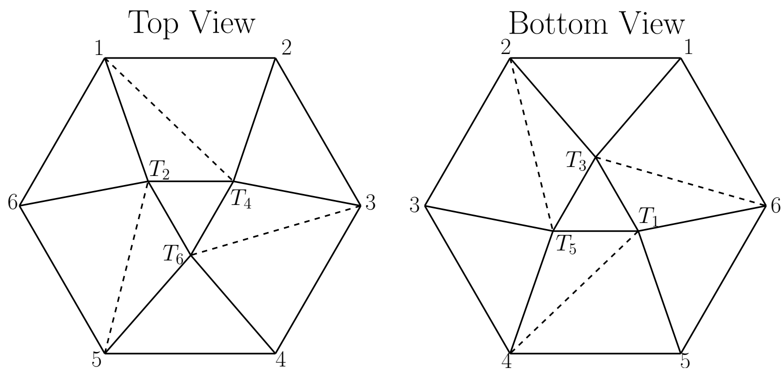
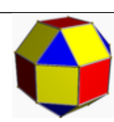


Figure 4.4: Vertex labeling of the cuboctahedron

Consider the vertices labeled as in Figure 4.4 and remove the convex hull of the point set $\{T_1, T_2, T_3, T_4, T_5, T_6\}$, which is a triangular anti-prism. Now we can break the remaining volume into the six congruent pieces which are the convex hulls of points sets $\{1,2,T_2,T_3,T_4\}$, $\{2,3,T_3,T_4,T_5\}$, $\{3,4,T_4,T_5,T_6\}$, $\{4,5,T_1,T_5,T_6\}$, $\{5,6,T_1,T_2,T_6\}$, and $\{1,6,T_1,T_2,T_3\}$. Notice each piece is connected to two others by common triangles, also each shares a triangular face with the anti-prism. Now we should notice that the six congruent pieces all have exactly one square face from the exterior of the cuboctahedron, so they can be triangulated with either diagonal of the square and by the fan argument each surface triangulation is extendable. ∎

Class B

4) Rhombicuboctahedron	(3,4,4,4)	

Proof of Class B

Remark 13. *If we want to tile a polyhedron, it suffices to slice the shape by a plane(s) which creates no new vertices and consider tiling the remaining pieces. The issue with this method when attempting to triangulate the polyhedron is that the triangulation of each piece must induce the same surface triangulation on the cross section.*

First observe that on the rhombicuboctahedron there are three sets of cyclic squares, where each is the convex hull of a octagonal prism. Also notice each pair of cyclic squares intersect on two opposing parallel faces.

We will attempt to break the rhombicuboctahedron into three pieces by removing one of these octagonal prisms as shown in Figure 4.5.

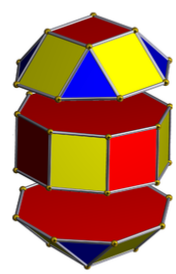


Figure 4.5: Slicing the rhombicuboctahedron

Before doing this we must first show that each piece can be tiled by tetrahedra. So by Theorem 3.4 we need that the octagonal prism not be surface triangulated by the cyclic diagonals. Since there are three of these it is easily seen that all three cyclic squares cannot be triangulated by cyclic diagonals simultaneously, thus every surface triangulation will contain at least one octagonal prism not containing the cyclic diagonals along its lateral faces. Therefore we can cut the rhombicuboctahedron as shown in Figure 4.5 and tile each piece with tetrahedra. ■

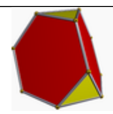
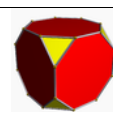
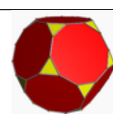
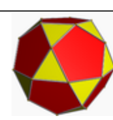
It is obvious that this method does not provide a triangulation of the rhombicuboctahedron since either octagonal cut may contain different triangulations on its cross sections, yet if we can show that every surface triangulation of the square cupola (the top and bottom pieces in Figure 4.5) can be extended, then we will be able to triangulate each octagonal cross section in the same manner.

Open Problem

Is every surface triangulation of a square cupola extendable to a triangulation?

Remark 14. *This will also prove that each surface triangulation of the square orthobicupola and square gyrobicupola is extendable.(Notice this is not the case for the elongated square gyrobicupola. Although it can be broken into three pieces as the rhombicuboctohedron, the octagonal prism may be triangulated on the surface by the cyclic diagonals.)*

Class C

5)Truncated tetrahedron	(3,6,6)	
6) Truncated cube	(3,8,8)	
7) Truncated dodecahedron	(3,10,10)	
8) Icosidodecahedron	(3,5,3,5)	

Proof of Class C

For the truncated tetrahedron, we consider the partial surface triangulation of three hexagonal faces shown in Figure 4.6.

Notice the top face is a triangle {1,2,3} and the bottom face is a hexagon {A,B,C,D,E,F}, and the other three vertices are labeled by there adjacent vertices.

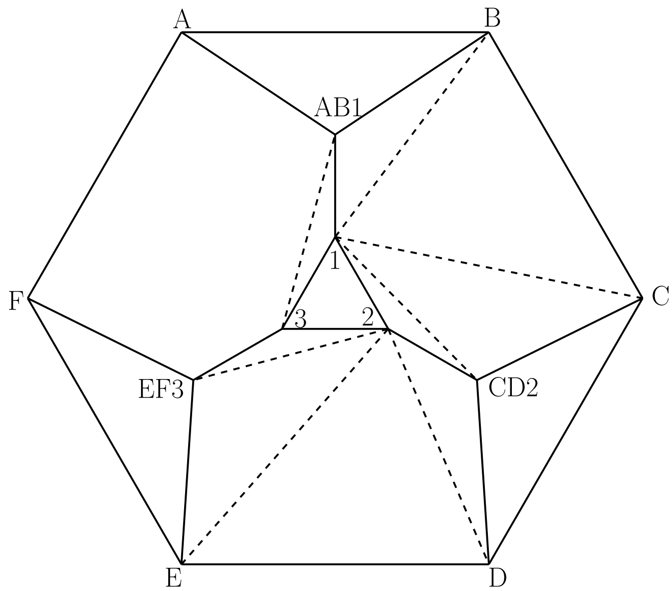


Figure 4.6: Edge graph of a partial surface triangulation of the truncated tetrahedron

We will again consider the abstract case, and show no tiling by tetrahedra induces such a surface triangulation. Now let us focus on the surface triangle {B,C,1}. If we assume a tiling by tetrahedra exists, we can consider which vertex will be in a tetrahedron with triangle {B,C,1}. It is obvious the the vertices {A,F,2,AB1,CD2} are not the fourth vertex. Now if we prescribe a surface triangulation on the bottom hexagon {A,B,C,D,E,F} which contains diagonal $\overline{AC}$ or $\overline{CF}$, the triangle {B,C,1} must be a face of tetrahedra {B,C,1,3} in the tiling (or triangulation) of the truncated tetrahedron.

Now we can form a similar argument that the surface triangle {D,E,2} must belong to the tetrahedra {D,E,1,2} by prescribing a surface triangulation contain diagonal $\overline{CE}$. Now we reach a contradiction as tetrahedra {B,C,1,3} and {D,E,1,2} overlap.※

Therefore any surface triangulation containing the diagonals $\overline{B1}$, $\overline{C1}$, $\overline{(CD2)1}$, $\overline{D2}$, $\overline{E2}$, $\overline{(EF3)2}$, $\overline{(AB1)3}$, $\overline{CE}$ and $\overline{AC}$ or $\overline{CF}$ cannot be extended.

Similar arguments can be made for the other three polyhedra in this class. In each polyhedron we use a similar technique of investigating triangles from a surface triangulation which are all adjacent to a common triangular face and use diagonals on other faces to force which tetrahedra contains the surface triangles incident with the common triangular face. An edge graph for such a surface triangulation of the truncated cube is provided in Figure 4.7. (We will not provide the diagram of the truncated dodecahedron or the icosidodecahedron, because it looses its visual appeal with the distortion needed to draw the faces in an edge graph.)

We have color coded the diagram so that each face diagonal (dotted curves) restricts vertices that can be in a tetrahedron with the corresponding colored face triangle. We have also colored each vertex to show it can be in a tetrahedron with the triangle of the same color. The crux of the argument is to recognize that no two of the given colored triangles can both be in a tetrahedron with another vertex from their shared triangular face. Then we arbitrarily choose two of the colored triangles to be in a tetrahedron with a fourth vertex elsewhere, and find that any combination of their fourth vertices respectively will cause the two tetrahedra to overlap. ■

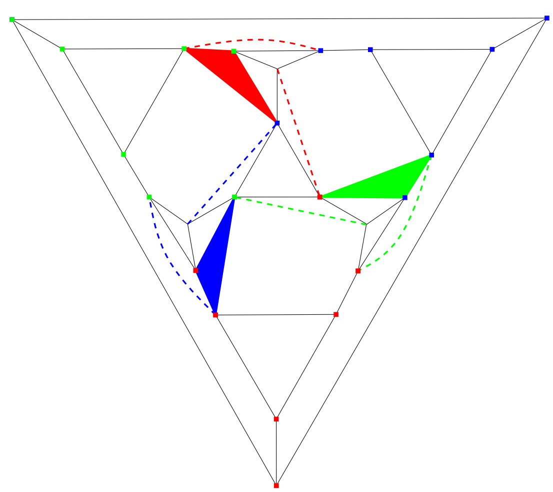
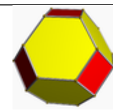
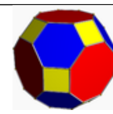
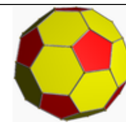
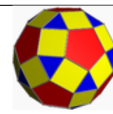
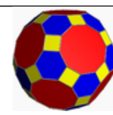


Figure 4.7: Edge graph of a partial surface triangulation of the truncated cube

Obviously we need to consider more surface diagonals in the cases of the truncated dodecahedron and the icosidodecahedron.

Class D

9) Truncated octahedron	(4,6,6)	
10) Great rhombicuboctahedron	(4,6,8)	
11) Truncated icosahedron	(5,6,6)	
12) Rhombicosidodecahedron	(3,4,5,4)	
13) Great rhombicosidodecahedron	(4,6,10)	

These polyhedra cannot be broken into smaller shapes to be triangulated, since their is no plane cutting the polyhedron into two pieces without creating new vertices. The cyclic surface triangle method shown for class C also does not seem to work as the rhombicosidodecahedron is the only shape containing triangular faces, and it seems as if having more than three non-triangular faces adjacent to a triangular face will hinder the contradiction reached with the other four polyhedra where this technique proved fruitful.

Bibliography

[1] F. Bagemihl, *On indecomposable polyhedra*, American Math. Monthly **55**, (1948), 411-413.

[2] A. Below, J.A. De Loera and J. Richter-Gebert, *Finding minimal triangulations of convex 3-polytopes is NP-hard*, Preceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms, (San Francisco 2000), ACM, New York, (2000), 65-66.

[3] M. de Berg, O. Cheong, M. van Kreveld and M. Overmars, *Computational Geometry: Algorithms and Applications*, Third Edition, Springer, (2008).

[4] P. Brass, W. Moser and J. Pach, Research Problems in Discrete Geometry, Springer, 2005.

[5] B. Chazelle, *Convex partitions of polyhedra: a lower bound and worse-case optimal algorithm*, SIAM Journal of Computation **13**, No. 3, (1984), 488-507.

[6] B. Chazelle and L. Palios, *Triangulating a nonconvex polytope*, Discrete and Computational Geometry **5**, (1990), 129-138.

[7] J.A. De Loera, J. Rambau and F. Santos, Triangulations: structures for algorithms and applications, Algorithms and Computation in Mathematics **25**, Springer, 2010.

[8] S. Devadoss and J. O'Rourke, Discrete and Computational Geometry, Princeton University Press, 2011.

[9] H. Edelsbrunner, Algorithms in Combinatorial Geometry, Springer-Verlag, New York, 1987.

[10] L. Fejes Tóth, Regular Figures, Pergamon Press, New York, 1964.

[11] L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Second Edition, Springer-Verlag, New York, 1972.

[12] J.E. Goodman and J. Pach, *Cell decomposition of polytopes by bending*, Israel J. Math. **64**, 1988, 129-138.

[13] B. Jessen, *Orthogonal icosahedron*, Nordisk Mat. Tidskr. **15**, (1978), 165-170.

[14] C. Lawson, *Transforming triangulations*, Discrete Math. **3**, (1972), 365-372.

[15] N.J. Lennes, *Theorems on the simple finite polygon and polyhedron*, American Journal of Mathematics **33**, (1911), 37-62.

[16] J. O'Rourke, Computational Geometry in C, Second Edition, Cambridge University Press, 1998.

[17] G.H. Meisters, *Polygons have ears*, American Mathematical Monthly, (June/July 1975),(648-651).

[18] J. Pach and P.K. Agarwal, Comdinatorial Geometry, A Wiley-Interscience Publication, New York, 1995.

[19] M. S. Paterson and F.F. Yao, *Binary partitions with applications to hidden-surface removal and solid modeling*, Proc. 5th ACM Symp. Comp. Geom., (1989), 23-32.

[20] J. Rambau, *On a generalization of Schönhardt's polyhedron*, Combinatorial and Computational Geometry **52**, (2005), 501-516.

[21] C.A. Rogers, Packing and Covering, Cambridge University Press, New York, 1964.

[22] J. Ruppert and R. Seidel, *On the difficulty of triangulating three-dimensional nonconvex polyhedra*, Discrete and Computaional Geometry, **7** issue 3, (1992), 227-253.

[23] D.D. Sleator, R.E. Tarjan and W.P. Thirston, *Rotation distance, triangulations, and hyperbolic geometry*, Journal of the AMS, **1**, (1988), 647-681.

[24] E. Schönhardt, *Über die Zerlegung von Dreieckspolyedern in Tetraeder*, Mathematics Annalen **89**, (1927), 309-312.

Appendices

Appendix A

Johnson Solids Containing Fan Vertices in every Surface Triangulation

1. Square Pyramid
2. Pentagonal Pyramid
3. Elongated Triangular Pyramid
4. Elongated Square Pyramid
5. Elongated Pentagonal Pyramid
6. Gyroelongated Square Pyramid
7. Gyroelongated Pentagonal Pyramid
8. Triangular Dipyramid
9. Pentagonal Dipyramid
10. Elongated Triangular Dipyramid
11. Elongated Square Dipyramid
12. Elongated Pentagonal Dipyramid
13. Gyroelongated Square Dipyramid
14. Gyroelongated Triangular Cupola
15. Gyroelongated Square Cupola
16. Gyroelongated Pentagonal Cupola
17. Gyroelongated Triangular Bicupola
18. Gyroelongated Square Bicupola
19. Gyroelongated Pentagonal Bicupola
20. Gyroelongated Pentagonal Cupolarotunda
21. Augmented Triangular Prism
22. Biaugmented Triangular Prism
23. Triaugmented Triangular Prism
24. Augmented Pentagonal Prism
25. Biaugmented Pentagonal Prism
26. Augmented Hexagonal Prism
27. Parabiaugmented Hexagonal Prism
28. Metabiaugmented Hexagonal Prism
29. Triaugmented Hexagonal Prism
30. Augmented Dodecahedron
31. Parabiaugmented Dodecahedron
32. Metabiaugmented Dodecahedron

33. Triaugmented Dodecahedron
34. Metabidiminished Icosahedron
35. Tridiminished Icosahedron
36. Augmented Tridiminished Icosahedron
37. Snub Disphenoid
38. Snub Square Antiprism
39. Sphenocorona
40. Augmented Sphenocorona
41. Sphenomegacorona
42. Hebesphenomegacorona
43. Disphenocingulum
44. Triangular Hebesphenorotunda