Upper Bounds On The Coarsening Rates For Some Non-Conserving Equations

by

Nan Jiang

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 4, 2012

Keywords: coarsening, upper bounds, Allen-Cahn equation, Swift-Hohenberg equation,
interpolation inequality, dissipation inequality

Copyright 2012 by Nan Jiang

Approved by

Dmitry Glotov, Assistant Professor of Mathematics
Wenxian Shen, Professor of Mathematics
Yanzhao Cao, Professor of Mathematics

Abstract

In this thesis, we prove one-sided bounds on the coarsening rates for two models of non-conserved curvature driven dynamics by following a strategy developed by Kohn and Otto in [20].

In the first part, we analyze the Allen-Cahn equation in one and two dimensions, with different choices of length scales. The analysis follows the framework of Kohn and Yan in [24]. In the one-dimensional domain, by choosing an H^{-1}-type length scale, our analysis supports the assertion that the coarsening occurs at the rate $t^{1/3}$. In the two-dimensional domain, we consider two types of length scales. First, we obtain the coarsening rate of $t^{1/3}$ using an H^{-1}-type length scale, and then, using another L^2-type length scale yields that the energy decays no faster than the rate $t^{-1/6}$. In all the cases, among the main ingredients, the interpolation inequality requires the most delicate analysis, and the dissipation inequalities are based on basic calculations using Hölder's inequality. An ODE argument is adapted to combine these two components in each case. The well-posedness of the Allen-Cahn equation obtained using fixed point method is presented in the appendix.

For the Swift-Hohenberg equation, we again consider an L^2-type length scale in a two-dimensional domain. The coarsening rate of $t^{1/3}$ rate is established using an interpolation inequality which extends Kohn and Otto's method. This rate is consistent with numerical results as an upper bound on coarsening rates.

Acknowledgments

My special thanks to all the people who helped me, encouraged me and made my time during my PhD so enjoyable at Auburn University. First of All, I would like to thank my advisor Professor Dmitry Glotov for all the support and guidance required to make my work possible and efficient. I have learned from Dr. Glotov not only how to do good mathematical research but also how to organize and write good mathematical papers. Dr. Glotov always gave me the freedom to make my own personal judgement on the path and direction of this thesis.

I am also grateful for the help and direction of members of committee, Professor Wenxian Shen and Professor Yanzhao Cao. Besides the interesting and inspiring courses that I took from them, they also gave me some ideas of how to make my work more integrated. My special thanks go to Professor Konrad Patkowski for his careful and hard work, who read my dissertation many times, and provided a lot of valuable suggestions.

I also want to thank the Department of Mathematics and Statistics for providing such a friendly environment to work in. Particularly thanks to my friends in our department. It has been helpful to study and have discussions with my friends.

Finally, I appreciate the emotional support and encouragements from my parents and friends.

Contents

Abstract ii

Acknowledgments iii

1 Introduction 1

1.1 Allen-Cahn Equation 2

1.2 Swift-Hohenberg Equation 4

1.3 Previous Results 6

1.3.1 Cahn-Hillard Equations 7

1.3.2 Phase-Field Model 8

1.3.3 Epitaxial Growth Model 10

1.3.4 Discrete, Ill-posed Diffusion Equations 11

1.4 Outline of This Thesis 14

2 Allen-Cahn Equation in One-Dimensional Space 17

2.1 Introduction to the Main Result 17

2.2 Preliminary Results 19

2.2.1 Boundedness of Solutions of Allen-Cahn Equations 19

2.2.2 Boundedness of Solutions of Elliptic Allen-Cahn Equations 22

2.3 The Interpolation Inequality 24

2.4 The Dissipation Inequality 35

2.5 Upper Bound on The Coarsening Rate 37

3 Allen-Cahn Equation in Two-Dimensional Space 44

3.1 Energy Decays No Faster than $t^{-1/3}$ 44

3.1.1 Introduction to the Main Result 45

3.1.2 The Interpolation Inequality 46

3.1.3 The Dissipation Inequality 53

3.1.4 Upper Bound on the Coarsening Rate 55

3.2 Energy Decays No Faster than $t^{-1/6}$ 56

3.2.1 Introduction to the Main Result 56

3.2.2 Interpolation Inequality 57

3.2.3 Dissipation Inequality 60

3.2.4 Upper Bound on the Coarsening Rate 61

4 Swift-Hohenberg Equation 66

4.1 Interpolation Inequality 68

4.2 Dissipation Inequality and Upper Bounds on the Coarsening Rates 73

5 Discussion 75

A Well-posedness of Allen-Cahn Equation 78

A.1 Preliminary Results 79

A.2 Existence and Uniqueness of Weak Solution 84

A.3 Existence and Uniqueness of Strong Solutions 91

B Well-posedness of Swift-Hohenberg Equation 94

B.1 Preliminary Results 94

B.2 Existence of Strong Solutions 96

Bibliography 99

Chapter 1

Introduction

In various physical processes, domains that form in multi-stable systems slowly change in time, with the overall pattern becoming coarser. From the physical perspective, our particular point is to model the kinetic behavior for the systems whose spatial structure develops a pattern of domains or clusters that coarsen as time increases. The growth of single-crystal grains in polycrystalline materials, phase separation in alloys, and anti-phase boundary motion in antiferromagnetic materials are some important examples, [32]. For example, as in [5], it is a system in equilibrium, which is quenched from the symmetric (high temperature) phase into the symmetry breaking (low temperature) phase through some phase transition. Once the system sets into the ordered phase, it locally selects one, among all the possible, equilibrium configurations. Different states are chosen at different locations and topological defects in the form of domain walls are created. In the course of time, the patches of ordered regions tend to grow while the density of topological defects diminishes.

It is widely observed that for some coarsening processes described by different equations, some typical length scale that characterizes the distance between the topological defects increases and the length scale behaves as a temporal power law. And the questions will be whether we can find the universal rates for the coarsening. It is difficult to expect all solutions coarsen at the same rate, because in the infinite-time limit the system should typically approach a stable equilibrium and stop coarsening. However, Kohn and Otto's method in [20] provides a effective way to find an upper bound on the coarsening rates. Here in this dissertation, we mainly study the coarsening described by the Allen-Cahn equation and the Swift-Hohenberg equation. So first we introduce these two equations before we go into the details.

1.1 Allen-Cahn Equation

We know that for the scalar ODE

$$\partial_t u = -f(u),$$

with $f : \mathbb{R} \to \mathbb{R}$ and $f \in C^1$, every solution $t \to u(t)$ is monotonic and every bounded solution converges as $t \to \infty$. For the coarsening processes, one of the simplest mathematical models of this behavior arises as a modification of the model

$$\partial_t u(x,t) = -f(u(x,t)), \quad \text{in } [0,1] \times [0,\infty),$$

which is a spatial variation of the above scalar ODE. For any bounded solution, $u_\infty(x) = \lim_{t\to\infty} u(x,t)$ exists for every x, with $f(u_\infty(x)) = 0$ for all x. If f has multiple stable zeros, the limiting state u_∞ is typically non-constant, and the domains will form as time proceeds, corresponding to different limiting values of $u_\infty(x)$.

The Allen-Cahn equation was originally studied by Allen and Cahn in [1]. Our focus is on the parabolic Allen-Cahn equation on domain $\Omega \times [0,\infty)$

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - 2u(1-u^2) = 0, &\quad \text{in } \Omega \times [0,\infty) \\ u(x,t) = 0, &\quad \text{on } \partial\Omega \times [0,\infty) \\ u(x,0) = u_0, &\quad \text{in } \Omega \end{aligned} \tag{1.1}$$

where Ω is an interval I in $\mathbb{R}$ or a square Q in $\mathbb{R}^2$. This PDE corresponds to the gradient flow of the energy

$$E(u) = \frac{1}{2}\int_Q |\nabla u|^2 + (1-u^2)^2 dx,$$

where $\fint$ denotes the spatial average.

We focus on the homogeneous Dirichlet boundary conditions. In the literature, this equation is also considered together with either periodic or homogeneous Neumann boundary conditions. Moreover, for unbounded domains, heteroclinic conditions at infinity are usually imposed. The latter condition ensures that there is at least one transition between phases and it guarantees that the energy has a lower bound. We note that, for the epitaxial growth model, the requirement of periodic boundary condition also ensures a lower bound on the energy. For our model, we could have chosen the Dirichlet boundary conditions $u(0,t) = -1$ and $u(l,t) = 1$ to mimic the heteroclinic condition at infinity, but we note the homogeneous Dirichlet boundary condition yields the same effect of creating interfaces at the boundaries of the bounded domains and it extends naturally to dimension two or higher.

We scale the system and prove a corresponding result in the unit interval $I_1 = [0,1]$. With the length of I denoted by $\frac{1}{\varepsilon} = l$, we define

$$u_\varepsilon(x,t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right).$$

Then u_ε solves the equation

$$\begin{aligned} \partial_t u_\varepsilon - \partial_x^2 u_\varepsilon - \frac{2}{\varepsilon^2} u_\varepsilon(1-u_\varepsilon^2) = 0, &\quad \text{in } I_1 \times [0,\infty) \\ u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, &\quad t > 0 \end{aligned} \tag{1.2}$$

Let $W(u) = \frac{1}{2}(1-u^2)^2$, so that $W'(u) = -2u(1-u^2)$. We observe that $W(u)$ is a double well energy density with equal minima at $u = \pm 1$. As $\varepsilon \to 0$ the solutions u_ε will converge almost everywhere to 1 or -1, [38], [37], [35]. For every t and same initial condition for each $\varepsilon > 0$, the interval I_1 will be partitioned as $I_1 = I_1^1 \cup I_1^{-1} \cup I_1^{\text{rest}}$, where

$$I_1^\delta = \{x \in I_1 | u_\varepsilon(x,t) \to \delta \text{ as } \varepsilon \to 0\},$$

and I_1^{rest} has measure 0. The interface between these two sets corresponds to the grain boundaries.

The only stable states of this system are patternless constant solutions $u = \pm 1$, [9]. The asymptotic behavior of solutions of (1.2) as $t \to \infty$ has been well studied. As stated in [32], for any solution $u(x,t)$, we expect that $u_\infty(x) = \lim_{t\to\infty} u(x,t)$ exists and satisfies the equation of equilibrium:

$$\varepsilon^2 \partial_x^2 u_\varepsilon - W'(u_\varepsilon) = 0.$$

Hence, u_∞ is a stationary solution and for large t, typical solutions will be approximately piece-wise constant in space. In a variety of physical processes, domains that form in multi-stable systems change in time slowly. Similarly the solution to (1.2) changes extremely slowly after reaching a pattern of transition layers developed in a relatively short time. The solution will either grow up to 1 or bring down to -1, decreasing the part of the energy corresponding to the double-well potential.

1.2 Swift-Hohenberg Equation

The equation considered in this section is proposed as a prototypical example of pattern forming systems [10]. It was first derived by Swift and Hohenberg in [36] as a model for pattern-formation equation for a fluid which is thermally convecting. These authors used weakly nonlinear analysis of the Boussinesq equations describing Bénard convection with random thermal fluctuations, as a simple model for the Rayleigh-Bénard instability of roll waves. When spatially periodic patterns emerge in isotropic pattern-formation systems, random initial conditions will lead to patches of patterns with different orientation that are separated by sharp interfaces. The slow dynamics of those interfaces often govern the long-time behavior of systems far from equilibrium.

Here we consider the Swift-Hohenberg equation in the two-dimensional space

$$\begin{aligned} u_t = -(1+\nabla^2)^2 u + \mu u - u^3, &\quad \text{in } Q \times [0,T] \\ u = \frac{\partial u}{\partial v} = 0, &\quad \text{on } \partial Q \times [0,T] \\ u(x,0) = u_0, &\quad \text{in } Q, \end{aligned} \tag{1.3}$$

where $Q \in \mathbb{R}^2$. The Swift-Hohenberg equation describes the nonlinear interaction of plane waves. Most of the pattern forming systems described by the Swift-Hohenberg equation exhibit stationary stripe or roll patters, see for example [19].

We consider u as representing a grayscale image of the temperature at each point, that is, each coordinate has a temperature measurement u associated with that point. Hence, as in Figure 1.1, the image of u represents a set of convection rolls.

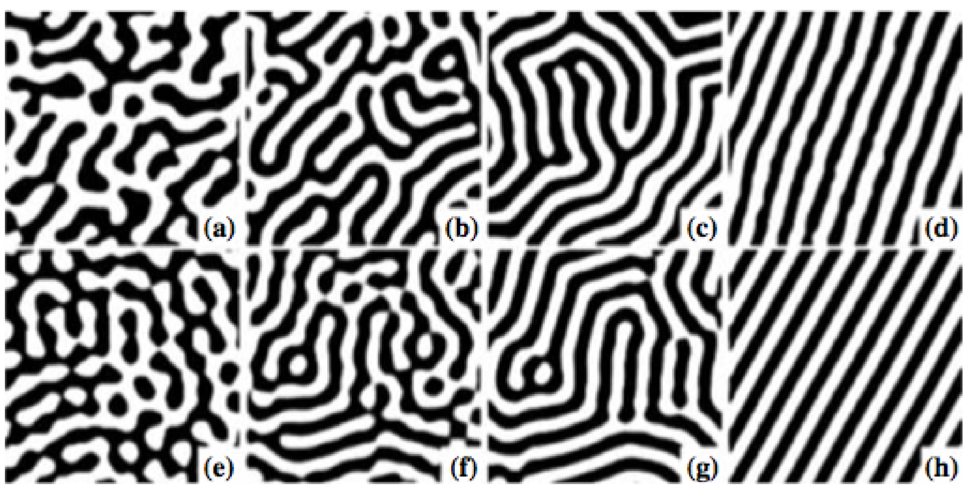

Figure 1.1: Evolution of patterns in time, taken from [21].

We notice that the Swift-Hohenberg equation relates the temporal evolution of the pattern to the spatial structure of the pattern. μ plays the role of a temperature knob, measuring how far the temperature is above the minimum temperature difference required for convection. Therefore, for $\mu < 0$, the heating at the bottom of the fluid is too small to cause convection, while for $\mu > 0$, convection occurs. The term involving the gradient acts to smooth out sharp edges in the pattern.

In Figure 1.1, (a)-(d) are the images obtained from experiments involving the free surface of granular layers at $t = 2, 10, 200, 1000$, where the bright parts correspond to the crests of the free surface and the dark parts corresponds to the troughs of the free surface. As time progressed, they locally align in parallel and create an increasingly ordered pattern and after a long time, a fully ordered striped pattern finally appears. Subfigures (e)-(h) are images obtained from the simulation of the two-dimensional Swift-Hohenberg equation in time for $\mu = 0.2$. We can see that the coarsening dynamics of the striped pattern shows very similar spatiotemporal morphology in both the experiment and numerical calculation.

During the formation of stripes [18], the width of the structure will decay in two stages. When t is small, the linear term in equation (1.3) dominates the system because of the small amplitude of the order parameter u. At this stage, the width decays rapidly. However, in the late-time region, nonlinear term effects emerge and the width decays slowly. Moreover, when the correlation function of the local orientation order parameter is computed in the late-time region in real space, the characteristic length grows algebraically as $L(t) \sim t^z$, while the density $E(t)$ of topological defects decays algebraically as $E(t) \sim t^{-z}$, as in [21], [7], etc.

1.3 Previous Results

The quantitative estimation of the coarsening rates was pioneered by Kohn and Otto in [20]. Their method, originally developed for the Cahn-Hilliard equations, involves the introduction of an auxiliary length scale and establishing and exploiting relations between this length scale and the energy. This method has subsequently been carried out for, among many models, an epitaxial growth model by Kohn and Yan in [24], for a discrete, ill-posed diffusion equation by Esedoḡlu and Slepčev in [13], for a demixing model by Brenier, Otto and Seis in [4], for a phase field model with arbitrarily complicated patterns of phases by Dai and Pego in [12]. We outline the method first as it was implemented originally for the Cahn-Hilliard equation in [20] and then for the epitaxial growth models in [24, 13, 4, 12].

Here we list some previous work with upper bounds on coarsening rates of different models and the outlines of their work. They all follow the method developed by Kohn and Otto in [20], but with specific tools and techniques. All these models have conservation law structures, while the equations that we study have non-conserved curvature driven dynamics.

1.3.1 Cahn-Hillard Equations

For Cahn-Hilliard equations,

$$\frac{\partial m}{\partial t} + \nabla \cdot J = 0,$$

with the associated energy

$$E = \fint \frac{1}{2} \left(|\nabla m|^2 + (1 - m^2)^\gamma \right) dx,$$

where, in the constant mobility case, $\gamma = 2$ and

$$J := -\nabla \frac{\partial E}{\partial m}$$

and, in the degenerate mobility case, $\gamma = 1$ and

$$J := -(1 - m^2)\nabla \frac{\partial E}{\partial m}.$$

Here, $\fint$ denotes the spatial average, and $m \in (-1, 1)$ with $c = \frac{1}{2}(1+m) \in (0, 1)$ standing for the relative concentration of the first species. The focus in [20] is on the case of a "critical mixture", i.e,

$$\fint m dx = 0.$$

A physical scale L appropriate for this model is defined as

$$L := \fint |\nabla^{-1} m| dx := \sup \left\{ \fint m\zeta dx | \zeta \text{ is periodic with } \sup |\nabla \zeta| \leq 1 \right\}.$$

The mathematical interpretation of L is that it is the $(W^{1,\infty})^*$ norm of m. The typical length scale is expected to behave as $L(t) \sim t^{1/3}$ in the constant-mobility Cahn-Hilliard equation and it should behave as $L(t) \sim t^{1/4}$ in the degenerate-mobility case The basic procedure is:

1. In the regime $E \ll 1$, establish an interpolation inequality $EL \gtrsim 1$. We use the notation $\gtrsim$ and $\gg$ throughout the paper as follows: $\alpha \gtrsim \beta$ means $\alpha \geq C\beta$ for some constant $C > 0$, and $\alpha \gg \beta$ means $\frac{\alpha}{\beta}$ is sufficiently large. Thus, this assertion says there exists a constant $C > 0$ such that $EL \geq C$ in the regime where $E \leq \frac{1}{C}$. This is the one-sided version of $EL \sim 1$, since E is the interfacial area density which scales as "1/ length scale L".

2. Find a dissipation inequality between $\dot{E}$ and $\dot{L}$ for each of the constant and degenerate mobility cases.

3. Obtain an upper bound on the coarsening rate by an ODE argument based on the previous two results. The lower bound on energy corresponds to the upper bound on the length scale. To this end, we consider L as an absolutely continuous function of E and rewrite in the dissipation inequality $\dot{L} = \frac{dL}{dE}\dot{E}$. Then an appropriate change of variables yields the desired lower bound on the rate of energy decay.

1.3.2 Phase-Field Model

In 2004, Dai and Pego extended the method of Kohn and Otto and established an upper bound on the coarsening rate for a phase-field model in [12]. The model is given by two equations in a non-dimensional form:

$$\begin{aligned} \varepsilon u_t + \frac{l}{2}\phi_t &= K\Delta u_t, \\ \alpha\varepsilon\phi_t &= \varepsilon\Delta\phi - \frac{1}{\varepsilon}g(\phi) + 2u, \end{aligned} \tag{1.4}$$

where $g(\phi) = G'(\phi) = \phi(\phi^2 - 1)$, and l, K, α are non-dimensional parameters that represent latent heat, thermal diffusivity and a relaxation time, respectively, and ε measures the thickness of the transition layers between two phases $\{\phi \approx +1\}$ the solid phase and $\{\phi \approx -1\}$ the liquid phase, where ε is small and $\varepsilon < \alpha K$. This model describes the solid-liquid phase transition of a pure material in terms of the temperature u and an order parameter ϕ. The special domain is a large cubic cell $Q = [0, a]^n \subset \mathbb{R}^n$ with periodic boundary conditions.

The associated energy is given by

$$E(t) = \fint_Q \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} G(\phi) + \frac{2\varepsilon}{l} u^2 \right),$$

and the length scale is defined as the $H^{-1}-$ norm of $\varepsilon u + \frac{l}{2}\phi$, i.e.,

$$L(t) = \left(\fint_Q |\nabla v|^2 \right)^{1/2},$$

where v is a periodic function that satisfies

$$\Delta v = \varepsilon u + \frac{l}{2}\phi.$$

Following the method of Kohn and Otto, there are three key steps. The first is to find a dissipation relation $|\dot{L}|^2 \leq \frac{KL}{4}(-\dot{E})$, and this can be done by direct calculations. The second key step is to obtain the interpolation inequality. This is done by defining periodic functions ω and ψ that satisfy

$$\Delta \omega = u - \bar{u}, \quad \Delta \psi = \phi - \bar{\phi},$$

where $\bar{u} = \fint u$ and $\bar{\phi} = \fint \phi$, hence, $\nabla v = \varepsilon \nabla \omega + \frac{l}{2} \nabla \psi$. It is shown that,

$$E(t)L(t) \geq \frac{l}{2} L_1(t) E_1(t) - Ca\sqrt{\frac{l\varepsilon}{2}} E(t)^{3/2},$$

where

$$L_1(t) = \left(\fint_Q |\nabla \psi|^2 \right)^{1/2}$$

and

$$E_1(t) = \fint_Q \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} G(\phi) \right).$$

Subsequently, Kohn and Otto's technique can be applied to prove $E_1(t)L_1(t) \geq C$. The observation that $\dot{E}(t) \leq 0$, together with the assumption that $\varepsilon_0 M$ and $\varepsilon_0 a^2 M^3$ are sufficiently small yields $E(t)L(t) \gtrsim 1$ whenever $0 < \varepsilon < \varepsilon_0$ and $E(0) < M$. The third step is the original ODE argument. The dissipation inequality and the interpolation inequality together with the ODE lemma lead directly to the main result, a time-averaged version of the estimate $E(t) \gtrsim t^{-1/3}$.

1.3.3 Epitaxial Growth Model

This method found another application in the epitaxial growth model considered by Kohn and Yan in [24] in a two dimensional domain with periodic boundary conditions, with the square domain $Q \subset \mathbb{R}^2$ as the period cell. The PDE that describes this model is fourth-order and it takes the form

$$u_t + \Delta^2 u + \nabla \cdot (2(1 - |\nabla u|^2)\nabla u) = 0$$

with the associated energy per unit area $Q \subset \mathbb{R}^2$

$$E = \frac{1}{2} \fint_Q |\Delta u|^2 + (1 - |\nabla u|^2)^2.$$

The length scale is taken to be the L^2-norm of u:

$$L = \left(\fint_Q u^2 \right)^{\frac{1}{2}}.$$

Here u represents the deviation of the height of the film surface from its mean 0, hence L is the standard deviation. Numerical simulations and heuristic arguments show that L grows as $t^{1/3}$ and E decays like $t^{-1/3}$. The results in [24] are a weak version of the statement that the system coarsens at the rate of $t^{-1/3}$. The constant in this inequality is independent of domain size. The particular interest is the case when Q is large with side length $\frac{1}{\varepsilon}$, where $\varepsilon > 0$. The most technically delicate point in this paper is the pointwise interpolation inequality. For the continuous function $v(x) = \varepsilon u\left(\frac{x}{\varepsilon}\right)$ with period 1 in each independent variable, the assertion is, with Q_1 denoting the unit square,

$$\left(\fint_{Q_1} \varepsilon|\Delta v|^2 + \varepsilon^{-1}(1 - \nabla v|^2)^2\right)\left(\fint_{Q_1} v^2\right)^{\frac{1}{2}} \geq C,$$

for some constant C.

1.3.4 Discrete, Ill-posed Diffusion Equations

For the coarsening phenomena in discrete, ill-posed diffusion equations, upper bounds on the coarsening rate can also be found using a similar framework. Discrete, ill-posed diffusion equations arise in the methods of granular flow, image processing, population dynamics, and many other applications. The specific equation studied by Esedoḡlu and Slepčev in [13] is as follows

$$v_t = \Delta R(v) = R'(v)\Delta v + R''(v)|\nabla v|^2,$$

where $R(\xi) : \mathbb{R} \to \mathbb{R}$ with $R'(\xi) < 0$ for all $|\xi|$ large enough. The unknown function v is defined on the unit-space lattice $\mathbb{L} = \{1, 2, \ldots, N\}^d$, where d is the dimension. The l^2 scalar product takes the form

$$v \cdot w = \sum_{q \in \mathbb{L}} v_q w_q.$$

Denote by $\mathcal{L} = \{v : \mathbb{L} \to \mathbb{R}\}$ all real-valued lattice configurations, and by $\mathcal{P} = \{v : \mathbb{L} \to [0, \infty)\}$ only the nonnegative configurations.

On the set $\mathcal{Z} = \{v \in \mathcal{L} : \bar{v} = 0\}$, where $\bar{v}$ is the average value of v defined as

$$\bar{v} = \overline{\sum_{q \in \mathbb{L}}} v_q = \frac{1}{|\mathbb{L}|} \sum_{q \in \mathbb{L}} v_q,$$

the discrete H^{-1} norm is introduced in the following way: given $s \in \mathcal{Z}$ there exists a unique, up to a constant, solution p of the discrete Poisson equation

$$-\Delta p = s.$$

Define the H^{-1} inner product by

$$\langle s_1, s_2 \rangle = \sum_{q \in \mathbb{L}} (\nabla^+ p_1)_q \cdot (\nabla^+ p_2)_q,$$

where ∇^+ is the forward finite difference, $\nabla^+ v = (\partial_1^+ v, \ldots, \partial_d^+ v)$ for $(\partial_i^+ v)_q = v_{q+e_i} - v_q$. Integration by parts gives $\langle s_1, s_2 \rangle = s_1 \cdot p_2 = p_1 \cdot s_2$. Then for $s \in \mathcal{Z}$, the H^{-1} norm is defined by

$$||s|| = \sup_{\xi \neq \text{const}} \frac{s \cdot \xi}{\sqrt{\sum_{q \in \mathbb{L}} |\nabla^+ \xi_q|^2}}. \tag{1.5}$$

We next outline the proof of the interpolation and dissipation inequalities following [13]. The energy is defined as

$$E(v) := \sum_{q \in \mathbb{L}} f(v_q),$$

for $v \in \mathcal{P}$ and $f' = R$, and the associated length scale is defined as

$$L = \frac{1}{\sqrt{|\mathbb{L}|}} ||v - \bar{v}||,$$

Then, by (1.5),

$$L = \sup_{\xi \neq \text{const}} \frac{\overline{\sum}_{q \in \mathbb{L}} (v_q - \bar{v}) \xi_q}{\sqrt{\overline{\sum}_{q \in \mathbb{L}} |\nabla^+ \xi_q|^2}}.$$

The dissipation inequality $(\dot{L})^2 \le -\dot{\overline{E}}$ can be easily obtained as follows:

$$\frac{dL^2}{dt} = 2L\dot{L} = \frac{2}{|\mathbb{L}|}\langle \dot{v}, v - \bar{v}\rangle,$$

so that

$$\dot{L} = \frac{\langle \dot{v}, v-\bar{v}\rangle}{|\mathbb{L}|\cdot L} = \frac{\sqrt{|\mathbb{L}|}\langle \dot{v}, v-\bar{v}\rangle}{|\mathbb{L}|||v-\bar{v}||},$$

combining the Cauchy-Schwarz inequality

$$\langle \dot{v}, v-\bar{v}\rangle^2 \le ||\dot{v}||^2||v-\bar{v}||^2,$$

with

$$\langle \dot{v}, \dot{v}\rangle = -\langle -\Delta R(v), \dot{v}\rangle = -R(v)\cdot\dot{v} = -f'(v)\cdot\dot{v} = -\nabla E(v)\cdot\dot{v},$$

we have

$$\dot{L}^2 \le \frac{1}{|\mathbb{L}|}\langle \dot{v}, \dot{v}\rangle = -\frac{1}{|\mathbb{L}|}\nabla E(v)\cdot\dot{v} = -\dot{\overline{E}},$$

where we used the chain rule in the last equality.

To prove the interpolation inequality, for $\rho > 0$ and $\alpha \in [0,1)$, define

$$F_\alpha(z) = \begin{cases} 0 & \text{if } 0 \le z \le \rho \\ z^\alpha & \text{if } z > \rho \end{cases}$$

The assumptions $f \ge \mu F_\alpha$ for some $\mu > 0$ and $\bar{v} > \rho$ are made to to avoid zero energy density. Suppose the associated energy $\overline{E} = \dfrac{E(v)}{|\mathbb{L}|}$ satisfies

$$\overline{E} < \frac{\mu(\bar{v}-\rho)^{2-\alpha}}{72\bar{v}^{2(1-\alpha)}},$$

Then by choosing an appropriate ξ, it can be proved that, when $d = 1$,

$$\overline{E}L^{1-\alpha} \geq \frac{1}{24}\mu(\bar{v} - \rho)^{\frac{3(1-\alpha)}{2}+1}\bar{v}^{-\frac{3(1-\alpha)}{2}},$$

and, when $d \geq 2$,

$$\overline{E}L^{2(1-\alpha)} \geq \frac{1}{24}\mu d^{-(1-\alpha)}(\bar{v} - \rho)^{3-2\alpha}\bar{v}^{-(1-\alpha)}.$$

Hence, in general, the interpolation inequality can be written as $\overline{E}L^{\beta} \geq \theta$ for some explicitly defined $\beta > 0$ and $\theta > 0$.

1.4 Outline of This Thesis

Our analysis follows the work of Kohn and Yan, and the work of Kohn and Otto. Specifically, we prove that the energy averaged in time decays no faster than a power law.

The energy that corresponds to the Allen-Cahn equation is

$$E = \fint \frac{1}{2}(|\nabla u|^2 + (1 - u^2)^2)dx. \tag{1.6}$$

This energy plays an important role in physics and has been well studied in [3]. In addition to the energy, we introduce an auxiliary length scale. Our main result is that the energy in the parabolic Allen-Cahn equation decays no faster than $t^{-1/3}$ or $t^{-1/6}$ with constant coefficients depending on the size of the domain in various ways.

The energy that corresponds to Swift-Hohenberg equation is

$$E(t) = \fint_Q \frac{1}{2}|(1 + \nabla^2)u|^2 + \frac{1}{4}u^2 - \frac{1}{2}\mu u^2 + \frac{1}{4}\mu^2 dx.$$

Our main result is that the energy in the Swift-Hohenberg equation decays no faster than $t^{-1/3}$.

We expect that the energy $E(t)$ is concentrated mostly on the interfaces and has the same dimension as the H^{-1}-type physical length scale $L(t)$ so that $E(t) \sim 1/L(t)$. The interpolation inequality is a one-sided version of this relation and it takes the form of $EL \geq C$ for some positive constant C. And for the L^2-type length scale, we establish a similar version of this relation $E^2(t) \sim 1/L(t)$. The interfacial area decreases, that is, $\dot{E} \leq 0$, because the motion is surface-energy-driven. However, we need a more accurate energy-dissipating structure of the dynamics. This refined structure is obtained in the form of the dissipation inequality. Finally, the interpolation and the dissipation inequalities are employed in the proof of an ODE lemma from which an upper bound on the time-averaged coarsening rate follows.

Here is the basic strategy :

1. We assemble the well-posedness of the Allen-Cahn and Swift-Hohenberg equations from various sources in the appendix. Specifically, for the Allen-Cahn equation, preliminaries include establishing various bounds for solution u, which can be done by standard parabolic estimates using the maximum principle.

2. The most important part is to obtain the dissipation inequality as well as the interpolation inequality. The dissipation inequality can be obtained by an elementary method. We use two different strategies to prove the interpolation inequality according to different length scales:

(a) First, obtain a uniform lower bound of the energy and then refine this lower bound to relate it to the length scale, as in section 2.3 and section 3.1.2;

(b) An alternative way to relate E to L is to separate the L^2 norm of u into two parts: one for E and the other for L, since we have

$$1 - \fint u^2 dx = \fint (1-u^2)dx \leq (\fint (1-u^2)^2 dx)^{1/2} \lesssim E^{1/2},$$

for the Allen-Cahn equation, and

$$\mu - \fint u^2 dx = \fint (\mu - u^2) dx \leq (\fint (\mu - u^2)^2 dx)^{1/2} \lesssim E^{1/2},$$

for the Swift-Hohenberg equation. The right-hand-sides of both inequalities decay to zero as $E \ll 1$. This technique is implemented in section 3.2.3 and section 4.1.

3. Various versions of the ODE lemma from Kohn and Otto's paper are given. Two distinct versions appear in section 2.5 and section 3.2.4.

Chapter 2

Allen-Cahn Equation in One-Dimensional Space

In this chapter, we look at the upper bound on the coarsening rates for the Allen-Cahn equation in the one-dimensional space.

2.1 Introduction to the Main Result

We consider the solutions of the parabolic Allen-Cahn equation in the domain $I \times [0, \infty)$

$$\begin{aligned} \partial_t u - \partial_x^2 u - 2u(1-u^2) &= 0, \quad \text{in } I \times [0,\infty) \\ u(0,t) = u(l,t) &= 0, \quad t > 0 \\ u(x,0) &= u_0(x), \quad \text{in } I \end{aligned} \tag{2.1}$$

where $I = [0, l] \subset \mathbb{R}$ and u_0 is bounded. Because we focus on the upper bounds on the coarsening rates of Allen-Cahn equation, we will give the well-posedness results in Appendix A.

First, let us look at the process of domain wall formation that occurs for this system and how coarsening by domain wall motion and annihilation can be described. We expect that starting with a bounded initial condition, u rapidly approaches 1 where $u > 0$, and -1 where $u < 0$ at the initial stage of coarsening. Domain walls or transition layers form between these domains at positions corresponding roughly to zeros in the initial data. Fusco and Hale [17] developed a rigorous geometric theory of domain wall dynamics. Their idea for a geometric description of these slow dynamics is to describe solutions containing N domain walls in terms of an $N-$dimensional manifold of "metastable" states in X. Using a restricted gradient flow approach, given N domain walls initially located at given positions

$h_1 < h_2 < \cdots < h_N$ in $(0,1)$, the positions will evolve in time according to equations well-approximated by exponentially small nearest neighbor interactions:

$$\partial_t h_j = 12\epsilon \left(e^{-\frac{h_{j+1}-h_j}{\epsilon}} - e^{-\frac{h_j - h_{j-1}}{\epsilon}} \right),$$

where $h_0 = -h_1$ and $h_{N+1} - 1 = 1 - h_N$ are obtained by reflection through boundaries.

Equation (1.2) corresponds to the gradient flow of the energy

$$E(u) = \frac{1}{2} \fint_I |\partial_x u|^2 + (1-u^2)^2 dx, \tag{2.2}$$

where $\fint$ denotes averaging over the interval.

To construct the length scale, we let

$$v(x) = \int_0^x u(z) dz,$$

and define

$$L = \left(\fint_I v^2(x) dx \right)^{\frac{1}{2}} = \left(\frac{1}{|I|} \int_I u(z) dz \right)^{\frac{1}{2}}, \tag{2.3}$$

where $|I|$ stands for the length of interval I. In our proof we will employ that $u_\varepsilon \in H_0^1(I_1)$ for each fixed time, according to the well-posedness in Appendix A.

We now present our main results. We state a special case as Theorem 2.1 and then the general case as Theorem 2.2.

Theorem 2.1. *Suppose the initial energy is E_0 and the initial length scale is L_0. Then we have*

$$\fint_0^T E^2 dt \gtrsim \fint_0^T (t^{-\frac{1}{3}})^2 dt \quad \text{for} \quad T \gg L_0^3 \gg 1 \gg E_0.$$

Here, we use $\fint$ to denote averaging over the time interval $[0,T]$. This theorem states that $E \gtrsim t^{-1/3}$ in a L^2 time-averaged sense and it also holds for some time average of the other norms of E, as well as E replaced by $E^\theta L^{-(1-\theta)}$.

Theorem 2.2. *Suppose the initial energy is E_0 and the initial length scale is L_0. For any $0 \le \theta \le 1$, suppose r satisfies $r < 3, \theta r > 1$ and $(1-\theta)r < 2$. Then we have*

$$\fint_0^T E^{\theta r} L^{-(1-\theta)r} dt \gtrsim \fint_0^T (t^{-\frac{1}{3}})^r dt \quad for \quad T \gg L_0^3 \gg 1 \gg E_0.$$

Both of the two inequalities above in Theorem 2.1 and Theorem 2.2, respectively, depend on the size of the domain I, and the specific dependence will be given in the ODE Lemma 2.13 later.

Notice that when $\theta = 1$, it permits $1 < r < 3$, and the minimum possible θ permitted is $\frac{1}{3}$. The conclusion of the theorem is strongest when θ and r are smallest, i.e., for values close to the curve $\theta r = 1$. Indeed, if the estimate holds for a given $r_0 < 3$ then it holds for all r between r_0 and 3 by an application of Jensen's inequality, and if the estimate holds for a given $\theta_0 < 1$, then it holds for all $\theta > \theta_0$ by an application of the interpolation inequality, [20].

Theorems 2.1 and 2.2 are valid in the two-dimensional domain with the energy and length scale defined in (3.2) and (3.3). The two-dimensional analogs of these two results will be proved in chapter 3.

2.2 Preliminary Results

In this section, we show the boundedness of solutions of the elliptic and parabolic Allen-Cahn equations, which will be used in the following proof for interpolation and dissipation inequalities.

2.2.1 Boundedness of Solutions of Allen-Cahn Equations

In this section, first we prove that the solution of the parabolic Allen-Cahn equation (A.1) is uniformly bounded in domain Q, where we suppose the domain is $Q = [0, l]^n$ and $Q_1 = [0, 1]^n$ with $n = 1$ or 2. We prove this by pointwise parabolic estimates. We will use the boundedness to prove dissipation inequality for a L^2 length scale for the two-dimensional space case.

The technique for the proof of the following lemma appears in [3] in the context of the Ginzburg-Landau equation.

Lemma 2.3. *Let u be a solution of (A.1) with a bounded initial condition, then for $t \geq \varepsilon^2$ and $x \in Q_1 = [0, 1]$,*

$$|u(x,t)| \leq \sqrt{2}.$$

Proof. By the dilation scaling as above, the space domain is $Q = [0, 1/\varepsilon]^n$. Set $\sigma(x,t) := |u(x,t)|^2 - 1$ and multiply equation (A.1) by u, to get

$$\frac{d\sigma}{dt} - \Delta\sigma + 2|\nabla u|^2 + 4\sigma(\sigma + 1) = 0. \tag{2.4}$$

Now consider the ODE

$$y'(t) + 4y(t)(y(t) + 1) = 0 \tag{2.5}$$

which is the space independent version of (2.4). We verify directly that

$$y_0(t) = \frac{e^{-4t}}{1 - e^{-4t}}$$

is a solution for equation (2.5) for $t > 0$. Let $\tilde{\sigma}(x,t) = y_0(t)$, then

$$\frac{d\tilde{\sigma}}{dt} - \Delta\tilde{\sigma} + 4\tilde{\sigma}(\tilde{\sigma} + 1) = 0. \tag{2.6}$$

Subtracting (2.4) from (2.6) gives us

$$\frac{d}{dt}(\tilde{\sigma} - \sigma) - \Delta(\tilde{\sigma} - \sigma) + 4(\tilde{\sigma} - \sigma)(\tilde{\sigma} + \sigma + 1) = 2|\nabla u|^2 \geq 0. \tag{2.7}$$

Since $1+\sigma+\tilde{\sigma} = |u|^2+\tilde{\sigma} \geq 0$, then, by the maximum principle in [14]

$$\tilde{\sigma}(x,t) - \sigma(x,t) \geq 0 \quad \text{for all } t \geq 0 \text{ and } x \in Q.$$

Therefore,

$$\sigma(x,t) \leq y_0(t), \quad \text{for all } t > 0 \text{ and } x \in Q,$$

so that

$$|u(x,t)|^2 = \sigma(x,t) + 1 \leq 2, \quad \text{for all } t \geq \frac{1}{4} \text{ and } x \in Q.$$

□

Remark 2.4. *We can also notice that*

$$|\nabla u_\varepsilon(x,t)| \leq \frac{C}{\varepsilon}, \quad |\frac{du_\varepsilon}{dt}(x,t)| \leq \frac{C}{\varepsilon^2}.$$

Indeed, $|u(x,t)| \leq \sqrt{2}$ *for* $t \geq \frac{1}{4}$*, we have*

$$|u(1-u^2)| \leq \sqrt{2} \quad \textit{for } t \geq \frac{1}{4}.$$

Let $p > 2$ *be fixed. It follows from the standard regularity theory for the linear heat equation that for each compact set* $\mathcal{F} \subset Q \times [\frac{1}{4}, \infty)$ *we have*

$$||\frac{du}{dt}||_{L^p(\mathcal{F})} \leq C(\mathcal{F}) \quad \textit{and} \quad ||\Delta u||_{L^p(\mathcal{F})} \leq C(\mathcal{F})$$

In particular, by Sobolev embedding and the L^∞ *bound for* u *we have*

$$||u||_{\mathcal{C}^{0,\alpha}(I \times [\frac{1}{2}, \infty))} \leq C,$$

where $\alpha = \frac{1}{2}(1 - \frac{1}{p})$. *Moreover,*

$$||u(1-u^2)||_{\mathcal{C}^{0,\alpha}(I\times[\frac{1}{2},\infty))} \leq C.$$

By the $\mathcal{C}^{0,\alpha}$ *regularity theory, we have*

$$||u||_{\mathcal{C}^{0,\alpha}(I\times[1,\infty))} \leq C.$$

Hence,

$$|\nabla u_\varepsilon(x,t)| \leq \frac{C}{\varepsilon}, \quad |\frac{du_\varepsilon}{dt}(x,t)| \leq \frac{C}{\varepsilon^2} \quad \textit{for } t \geq \varepsilon^2 \textit{ and } x \in Q.$$

2.2.2 Boundedness of Solutions of Elliptic Allen-Cahn Equations

In this section we will establish uniform bound on the solution to the elliptic Allen-Cahn equation and its derivative. This will be used to prove that, in the one-dimensional case, the energy is uniformly bounded from below. The boundedness of energy will be used to prove the interpolation inequality.

We consider the solution of equation

$$\begin{aligned} -\Delta u_\varepsilon - \frac{2}{\varepsilon^2} u_\varepsilon (1-u_\varepsilon^2) &= 0, \quad \text{in} \quad Q \\ u_\varepsilon &= 0, \quad \text{on} \quad \partial Q, \end{aligned} \tag{2.8}$$

and define

$$E_\varepsilon(u_\varepsilon) = \int_Q \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2.$$

If $u_\varepsilon \in H_0^1(Q)$ is a minimizer of $E_\varepsilon(u)$, then u_ε is a solution to (2.8). Indeed,

$$i(\tau) = E_\varepsilon(u+\tau v) = \int_Q \varepsilon(\nabla u + \tau\nabla v)^2 + \frac{1}{\varepsilon}(1-(u+\tau v)^2)^2.$$

Then

$$i'(\tau) = \int_Q 2\varepsilon(\nabla u + \tau\nabla v)\nabla v - \frac{4}{\varepsilon}(1-(u+\tau v)^2)(u+\tau v)v.$$

Integration by parts and evaluation at $\tau = 0$ yields

$$i'(0) = -2\varepsilon \int_Q (\Delta u + \frac{2}{\varepsilon^2}u(1-u^2))v = -2\varepsilon(\Delta u + \frac{2}{\varepsilon^2}u(1-u^2), v)_{L^2},$$

so that the critical points of the energy satisfy equation (2.8).

The next lemma shows us that the solution of the equation (2.8) and the gradient of the solution are bounded from above.

Lemma 2.5. *Let u_ε be a solution of (2.8). Then $|u_\varepsilon| \leq 1$ and $|\nabla u_\varepsilon| \leq \dfrac{C}{\varepsilon}$ on Q.*

Proof. First, we observe that equation (2.8) has the weak form

$$\int_Q \nabla u_\varepsilon \cdot \nabla v - \frac{2}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon v dx = 0, \tag{2.9}$$

for all $v \in H_0^1(Q)$. Note that the boundary conditions are incorporated into (2.9).

Now, denote $Q^+ = \{|u_\varepsilon| > 1\}$ and let $v = (\operatorname{sgn} u_\varepsilon)(|u_\varepsilon| - 1)_+$, where $x_+ = \max(x, 0)$. Then $v \in H_0^1(Q)$, i.e., v is a test function for (2.9). We first compute

$$\nabla v = (\operatorname{sgn} u_\varepsilon)\nabla|u_\varepsilon|$$

in Q^+. Substituting v into (2.9), we have

$$\int_{Q^+} |\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon^2}(|u_\varepsilon|^2 - 1)|u_\varepsilon|(|u_\varepsilon| - 1)dx = 0,$$

that is,

$$\int_{Q^+} |\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon^2}(|u_\varepsilon| - 1)^2|u_\varepsilon|(|u_\varepsilon| + 1)dx = 0, \tag{2.10}$$

since the integrand of (2.10) is non-negative, we must have meas $Q^+ = 0$, and therefore, $|u_\varepsilon| \le 1$ almost everywhere in Q.

Second, let

$$v_\varepsilon = u_\varepsilon - w,$$

where v_ε is the solution of the equation

$$-\Delta v_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \text{on} \quad Q$$
$$v_\varepsilon = 0 \quad \text{on} \quad \partial Q,$$

and w is the solution to the equation

$$-\Delta w = 0 \quad \text{on} \quad Q$$
$$w = 0 \quad \text{on} \quad \partial Q.$$

Then, it follows from the elliptic estimates as appearing in [2] and the fact that $|u_\varepsilon| \le 1$that

$$||\nabla v_\varepsilon||_{L^\infty} \le \frac{C}{\varepsilon} ||v_\varepsilon||_{L^\infty} \le \frac{C}{\varepsilon} (||u_\varepsilon||_{L^\infty} + ||w||_{L^\infty}) \le \frac{C}{\varepsilon}.$$

□

2.3 The Interpolation Inequality

In this section, we prove that

$$EL \gtrsim 1 \quad \text{when} \quad E \ll 1, \tag{2.11}$$

where E and L are defined in (2.2) and (2.3), and the constants implicit in (2.11) are independent of the size of I.

With the scaling as in the previous section

$$u_\varepsilon(x,t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

u_ε solves the equation

$$\begin{aligned} \partial_t u_\varepsilon - \partial_x^2 u_\varepsilon - \frac{2}{\varepsilon^2} u_\varepsilon (1-u_\varepsilon^2) = 0, \quad \text{in } I_1 \times [0,\infty) \\ u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, \quad t > 0. \end{aligned} \tag{2.12}$$

Let

$$v_\varepsilon(x) = \int_0^x u_\varepsilon(z)dz = \varepsilon v\left(\frac{x}{\varepsilon}\right). \tag{2.13}$$

Then E and L may be rewritten as,

$$E = \frac{1}{2} \fint_{I_1} \varepsilon^2 |\partial_x u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 dx,$$

and

$$L = \frac{1}{\varepsilon}\left(\fint_{I_1} v_\varepsilon^2 dx\right)^{\frac{1}{2}},$$

respectively. Correspondingly, and (2.11) becomes

$$\left(\fint_{I_1} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 dx\right)\left(\fint_{I_1} v_\varepsilon^2 dx\right)^{\frac{1}{2}} \gtrsim 1$$

when

$$\fint_{I_1} \varepsilon^2 |\partial_x u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 dx \ll 1.$$

The last statement is a consequence of the following theorem.

Theorem 2.6 (Interpolation)**.** *There is a constant $c_* > 0$ with the property that for any function $u_\varepsilon \in H_0^1(I_1)$ and any $\varepsilon > 0$,*

$$\left(\fint_{I_1} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2\right)\left(\fint_{I_1} v_\varepsilon^2\right)^{\frac{1}{2}} + \fint_{I_1} \varepsilon^2|\partial_x u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 \geq c_*.$$

In the next few lemmas, we establish various lower bounds on the energy E_ε that lay the groundwork for proving Theorem 2.6. In the first lemma, we show that E_ε is uniformly bounded below.

Lemma 2.7. *Define*

$$E_\varepsilon(u_\varepsilon) = \int_{I_1} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2,$$

where I_1 is the unit interval in $\mathbb{R}$. There exist constants $a_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ and any $u_\varepsilon \in H_0^1(I_1)$, we have

$$E_\varepsilon(u_\varepsilon) \geq a_0.$$

Proof. We first prove this lemma for solutions. We claim specifically that, when u_ε^0 is a solution to equation (2.8), there exist $\mu > 0$ and $\varepsilon_0 > 0$ such that if

$$\frac{1}{\varepsilon}\int_{I_1} (1-|u_\varepsilon^0|^2)^2 \leq \mu, \quad \text{with } \varepsilon < \varepsilon_0, \tag{2.14}$$

then

$$|u_\varepsilon^0(x)| \geq \frac{1}{2} \quad \forall x \in I_1.$$

This is a contradiction with the regularity and the homogeneous Dirichlet boundary conditions.

Next, we prove this claim. By Lemma 2.5, we have

$$|\partial_x u_\varepsilon^0| \leq \frac{C}{\varepsilon},$$

where C does not depend on ε. Therefore,

$$|u_\varepsilon^0(x) - u_\varepsilon^0(y)| \le \frac{C}{\varepsilon}|x-y|, \quad \forall x, y \in I_1.$$

Assume, by contradiction, that $|u_\varepsilon^0(x_0)| < \dfrac{1}{2}$ for some $x_0 \in I_1$. Then,

$$|u_\varepsilon^0(x) - u_\varepsilon^0(x_0)| \le \frac{C}{\varepsilon}|x-x_0|,$$

and

$$|u_\varepsilon^0(x)| \le \frac{1}{2} + \frac{C}{\varepsilon}\rho \quad \text{in} \quad I_1 \cap J_\rho(x_0),$$

where J_ρ is an interval of length 2ρ with $\rho \le 1$.

We choose ρ in such a way that $\dfrac{C}{\varepsilon}\rho = \dfrac{1}{4}$, i.e., $\rho = \dfrac{\varepsilon}{4C}$, so that

$$1 - |u_\varepsilon^0(x)| \ge \frac{1}{4} \quad \text{in} \quad Q_1 \cap J_\rho(x_0),$$

and, consequently,

$$(|u_\varepsilon^0(x)|^2 - 1)^2 \ge \frac{1}{16} \quad I_1 \cap J_\rho(x_0).$$

Also,

$$\text{meas}(I_1 \cap J_h) \ge h \quad \forall x \in I_1 \text{ and } \forall h \le 1.$$

Hence,

$$\int_{I_1} (1 - |u_\varepsilon|^2)^2 \ge \int_{I_1 \cap J_\rho(x_0)} (1 - |u_\varepsilon^0|^2)^2 \ge \frac{\varepsilon}{64C}.$$

Therefore,

$$\frac{1}{\varepsilon}\int_{I_1} (1 - |u_\varepsilon^0|^2)^2 \ge \frac{1}{64C}.$$

Let $\mu < \dfrac{1}{64C}$ and $\varepsilon_0 = 4C$ so that we arrive at a contradiction with (2.14). Hence, there exists $a_0 > 0$ such that $E_\varepsilon(u_\varepsilon^0) \ge a_0$. In particular, this lower bound is valid for the minimizers of the energy E_ε.

Finally, suppose u_ε is any function. Let u_ε^0 be a minimizer of E_ε and, therefore, also a solution of (2.8). Then

$$E_\varepsilon(u_\varepsilon) \geq E_\varepsilon(u_\varepsilon^0) \geq a_0.$$

□

The next lemma is based on a result from Modica [29]. It guarantees compactness in L^2 of a sequence u_{ε_j} with uniformly bounded energy by taking advantage of the polynomial structure of the nonlinear part of the energy $W(u) = (1-u^2)^2$.

Lemma 2.8. *Suppose $\{\varepsilon_j\}$ is a sequence such that $\varepsilon_j \to 0$ and $\{u_{\varepsilon_j}\}$ is a sequence for which $E_{\varepsilon_j}(u_{\varepsilon_j})$ is uniformly bounded. Then there exists a subsequence of $\{\varepsilon_j\}$, without loss of generality also denoted by $\{\varepsilon_j\}$, such that $\{u_{\varepsilon_j}\}$ converges to a function u_0 in $L^2(I_1)$ as $j \to \infty$.*

Proof. Recall that $I_1 = [0,1]$ and fix $\varepsilon > 0$.

Define

$$\phi(t) = \int_0^t |1-s^2|ds \quad \text{and} \quad w_\varepsilon(x) = \phi(u_\varepsilon(x)).$$

Since there exist $t_0 > 1, c_1 > 0$ and $c_2 > 0$ such that

$$c_1 t^4 \leq (1-t^2)^2 \leq c_2 t^4, \quad \text{for } t \geq t_0,$$

we see that

$$\phi(t) \leq \int_0^{t_0} |1-s^2|ds + \int_{t_0}^t \sqrt{c_2} s^2 ds \leq \int_0^{t_0} |1-s^2|ds + \frac{\sqrt{c_2}}{3} t^3.$$

Then, for some $c_3 > 0$ and $c_4 > 0$, we have

$$\phi(t) \leq c_3 + c_4(1-t^2)^2, \quad \text{for } t \geq t_0.$$

Therefore,

$$\int_{I_1} w_\varepsilon dx = \int_{I_1} \phi(u_\varepsilon)dx \leq c_3 + c_4 \int_{I_1} (1-u_\varepsilon^2)^2 dx \leq c_3 + c_4 \varepsilon E_\varepsilon(u_\varepsilon),$$

and we conclude that $\{w_\varepsilon\}$ is bounded in $L^1(I_1)$. On the other hand,

$$\partial_x w_\varepsilon(x) = \phi'(u_\varepsilon(x))\partial_x u_\varepsilon,$$

and

$$\begin{aligned}\int_{I_1} |\partial_x w_\varepsilon| dx &= \int_{I_1} |1-u_\varepsilon^2||\partial_x u_\varepsilon| dx \\ &\leq \frac{1}{2}\int_{I_1} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 dx \\ &= \frac{1}{2} E_\varepsilon(u_\varepsilon) \\ &\leq c_5,\end{aligned}$$

for some $c_5 > 0$, so the compactness yields that there is a sequence $\{\varepsilon_h\}$ of positive numbers converging to 0 such that $\{w_\varepsilon\}$ converges in $L^1(I_1)$ to a function w_0.

Now, let ψ be the inverse function of ϕ and define $u_0(x) = \psi(w_0(x))$. We have $\phi'(t) = |1-t^2| \geq \sqrt{c_1} t_0^2$ for every $t \geq t_0$. Hence ψ is Lipschitz continuous on $[\phi(t_0), \infty)$ and uniformly continuous on the entire real line. It follows that $u_{\varepsilon_j} = \psi \circ w_{\varepsilon_j}$ converges in measure on I to u_0 as $j \to \infty$. Since

$$\int_{I_1} u_{\varepsilon_j}^4 dx \leq t_0^4 + \frac{1}{c_1}\int_{I_1} |1-u_{\varepsilon_j}^2(x)|^2 dx \leq t_0^4 + \frac{\varepsilon_j}{c_1} E_{\varepsilon_j}(u_{\varepsilon_j}),$$

that is, $\{u_\varepsilon\}$ is bounded in $L^4(I_1)$, hence, $\{u_\varepsilon\}$ converges to u_0 in $L^2(I_1)$.

□

In the next lemma, we refine the result in Lemma 2.7 and claim that the bound may be taken to be any constant c with $c \leq \dfrac{1}{2}$ as long as the length scale is sufficiently small.

Lemma 2.9. *Let $E_\varepsilon(u_\varepsilon)$ be as in Lemma 2.7 and v_ε be defined as in equation (2.13). For any $c \le \frac{1}{2}$, there exists $\gamma > 0$ such that for all $\varepsilon \le 1$, if $\int_{I_1} v_\varepsilon^2 \le \gamma$, then $E_\varepsilon(u_\varepsilon) \ge c$.*

Proof. We prove this lemma by contradiction. Suppose, for some $c \le \frac{1}{2}$, there exist sequences $\{v_{\varepsilon_j}\}$ and $\{\varepsilon_j\}$ such that

$$\int_{I_1} v_{\varepsilon_j}^2 \le \frac{1}{j} \quad \text{but} \quad E_{\varepsilon_j}(v_{\varepsilon_j}) < c. \tag{2.15}$$

If $\liminf_j \varepsilon_j = 0$, by the compactness result in Lemma 2.8, we know that $\{u_{\varepsilon_j}\}$ is relatively compact in $L^2(I_1)$. If $\liminf_j \varepsilon_j > 0$, we choose a subsequence, without loss of generality, also denoted as $\{\varepsilon_j\}$, such that $\inf \varepsilon_j > 0$, so that $\{\int_{I_1} |\partial_x u_{\varepsilon_j}|^2\}$ is bounded, hence, $\{u_{\varepsilon_j}\}$ is pre-compact in $L^2(I_1)$. In both cases, for a further subsequence, $u_{\varepsilon_j} \to u_\infty$ in $L^2(I_1)$ for some u_∞ in $L^2(I_1)$.

On the other hand, (2.15) implies $\lim_{j\to\infty} v_{\varepsilon_j} = v_\infty = 0$, therefore $\partial_x v_\infty = u_\infty = 0$. But by compactness of $\{u_{\varepsilon_j}\}$ and Fatou's lemma,

$$1 = \int_{I_1} (1 - u_\infty^2)^2 \le \liminf_j \int_{I_1} (1 - u_{\varepsilon_j}^2)^2 \le \liminf_j c\varepsilon_j \le \frac{1}{2}.$$

This contradiction shows that the lemma is true. □

In the proof of the conclusion of Theorem 2.6, namely,

$$(\text{energy density }) \cdot (\text{ length scale }) \gtrsim 1,$$

we note that the most difficult case is when $\varepsilon \to 0$. Before proving Theorem 2.6, we need to establish the following proposition first. It states that when the length scale is bounded above by some constant depending on the length of the interval, the energy has a lower bound.

This will be used to prove Theorem 2.6 by contradiction by employing the following technique. Mainly, we assume the length scale is small, and we divide the domain into a mesh of subintervals. On most of the subintervals, the length scale in relatively small. At the same time, by Lemma 2.9, the energy is large on those subintervals where the length scale is small. The proposition below gives a rigorous basis for this argument.

Proposition 2.10. *For any $c_0 \leq \frac{1}{2}$, there exists a constant $c_1 > 0$ with the following property. Consider any interval $I \subset \mathbb{R}$ with length l and any $u_\varepsilon \in H_0^1(I)$ satisfying*

$$\int_I v_\varepsilon^2 dx \leq c_1 l^3,$$

where v_ε is defined as in (2.13).

Then we have

$$\text{Case } A: \quad \int_I \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq c_0 \quad \text{if} \quad l \geq \varepsilon, \tag{2.16}$$

$$\text{Case } B: \quad \int_I \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq \frac{c_0 l}{\varepsilon} \quad \text{if} \quad l \leq \varepsilon, \tag{2.17}$$

Proof. For Case A, we define $u_l(x) = u_\varepsilon(lx)$. Then, with

$$v_l(x) = \int_0^x u_l(z)dz = \int_0^x u_\varepsilon(lz)dz = \frac{1}{l}v_\varepsilon(lx),$$

(2.16) is equivalent to proving that for any $c_0 \leq \dfrac{1}{2}$, there exists c_1 such that if

$$\frac{\varepsilon}{l} \leq 1, \quad \int_{I_1} v_l^2 \leq c_1,$$

then

$$\int_{I_1} \left(\frac{\varepsilon}{l}\right) |\partial_x u_l|^2 + \left(\frac{l}{\varepsilon}\right)(1-u_l^2)^2 \geq c_0.$$

Since $\dfrac{\varepsilon}{l} \leq 1$, this is exactly the result of Lemma 2.9.

We prove case B by contradiction. Suppose for some $c_0 \leq \frac{1}{2}$, there exist sequences $\varepsilon_k, I_k, l_k, u_{\varepsilon_k}$ satisfying

$$\begin{cases} \frac{\varepsilon_k}{l_k} \geq 1, \\ \frac{1}{l_k^3} \int_{I_k} v_{\varepsilon_k}^2 \to 0, \\ \int_{I_k} \varepsilon_k^2 |\partial_x u_{\varepsilon_k}|^2 + (1 - u_{\varepsilon_k}^2)^2 < c_0 l_k. \end{cases} \tag{2.18}$$

Using the same scaling as in case A, $u_k(x) = u_{\varepsilon_k}(l_k x)$ and $v_k(x) = \frac{1}{l_k} v_{\varepsilon_k}(lx)$, (2.18) becomes

$$\int_{I_1} v_k^2 \to 0, \quad \int_{I_1} \left(\frac{\varepsilon_k}{l_k}\right)^2 |\partial_x u_k|^2 + (1 - u_k^2)^2 < c_0.$$

Since $\frac{\varepsilon_k}{l_k} \geq 1$, we can use the same argument as in the proof of Lemma 2.9 to arrive at a contradiction. □

Now we can prove the main theorem based on the previous results.

Proof of Theorem 2.6. : We prove this theorem by contradiction. Assume that there exist sequences $\varepsilon_k, u_{\varepsilon_k}$ such that

$$\begin{aligned} \left(\fint_{I_1} \varepsilon_k |\partial_x u_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k}(1 - u_{\varepsilon_k}^2)^2\right) \left(\fint_{I_1} v_{\varepsilon_k}^2\right)^{\frac{1}{2}} & \\ + \fint_{I_1} \varepsilon_k^2 |\partial_x u_{\varepsilon_k}|^2 + (1 - u_{\varepsilon_k}^2)^2 \to 0. & \end{aligned} \tag{2.19}$$

Case 1: Suppose $\liminf_k \varepsilon_k > 0$, then in the second term of (2.19), we have

$$\fint_{I_1} |\partial_x u_{\varepsilon_k}|^2 \to 0 \quad \text{and} \quad \fint_{I_1} u_{\varepsilon_k}^2 \to 1. \tag{2.20}$$

But, by the Poincaré inequality,

$$\fint_{I_1} u_{\varepsilon_k}^2 \leq C \fint_{I_1} |\partial_x u_{\varepsilon_k}|^2 \to 0.$$

This is a contradiction with (2.20).

Case 2: Suppose $\liminf_k \varepsilon_k = 0$ but $\int_{I_1} v_{\varepsilon_k}^2$ is bounded away from 0. Without loss of generality, suppose $\varepsilon_k \to 0$. The convergence of the first term in (2.19) to 0 gives us

$$\fint_{I_1} \varepsilon_k |\partial_x u_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k}(1-u_{\varepsilon_k}^2)^2 \to 0,$$

is in a contradiction with Lemma 2.7.

Case 3: Suppose $\lim \varepsilon_k = 0$ and $\lim \int_{I_1} v_{\varepsilon_k}^2 = 0$. We use Proposition 2.10 to obtain a contradiction. Fix c_0 and drop the subscript k to simplify the notation, and we write $u_\varepsilon = u_{\varepsilon_k}, v_\varepsilon = v_{\varepsilon_k}$. Define

$$\delta = \left(\fint_{I_1} v_\varepsilon^2\right)^{\frac{1}{2}}.$$

For any integer $N > 1$, we partition the unit interval I_1 into N subintervals of length $\omega = \dfrac{1}{N}$. The value of N will be determined later. If

$$\int_{I_\omega} v_\varepsilon^2 \leq c_1 \omega^3,$$

by applying Proposition 2.10 to I_ω, we have

$$\int_{I_\omega} \varepsilon |\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq c_0 \quad \text{if} \quad \omega \geq \varepsilon,$$

or

$$\int_{I_\omega} \varepsilon |\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq \frac{c_0 \omega}{\varepsilon} \quad \text{if} \quad \omega \leq \varepsilon.$$

The choice of N depends on the relation between ε and δ.

Alternative 1: Suppose $\varepsilon \gg \delta$. Then we choose N such that $\omega \approx \sqrt{\delta\varepsilon}$, that is, $\varepsilon \gg \omega \gg \delta$. For any N line segment I_ω of length ω, we say $I_\omega \in \mathcal{A}$ if $\int_{I_\omega} v_\varepsilon^2 \geq c_1 \omega^3$ and let $|\mathcal{A}|$ be the number of line segments in $\mathcal{A}$. Since

$$|\mathcal{A}|c_1\omega^3 \leq \sum_{I_\omega \in \mathcal{A}} \int_{I_\omega} v_\varepsilon^2 \leq \int_{I_1} v_\varepsilon^2 = \delta^2,$$

we have

$$|\mathcal{A}| \leq \frac{\delta^2}{c_1\omega^3} \ll \frac{1}{\omega} = N.$$

Therefore, the relation

$$\int_{I_\omega} v_\varepsilon^2 \leq c_1\omega^3$$

holds on most intervals. Since $\varepsilon \gg \omega$, for $I_\omega \notin \mathcal{A}$, i.e., when

$$\int_{I_\omega} v_\varepsilon^2 \leq c_1\omega^3,$$

Proposition 2.10 gives

$$\int_{I_\omega} \varepsilon^2|\partial_x u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 \geq c_0\omega.$$

Summing over all of these line segments, we have

$$\int_{I_1} \varepsilon^2|\partial_x u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 \geq \sum_{I_\omega \notin \mathcal{A}} \int_{I_\omega} \varepsilon|\partial_x u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 \geq c_0 \sum_{I_\omega \notin \mathcal{A}} \omega \gtrsim 1$$

This is a contradiction with (2.19).

Alternative 2: Suppose $\delta \gtrsim \varepsilon$. In this case, we choose N and $\omega = \dfrac{1}{N}$ such that $\omega = M\delta$, where M is a constant to be chosen later. We notice that $\omega \gtrsim \delta \gtrsim \varepsilon$. Again, considering N line segments I_ω, and we say $I_\omega \in \mathcal{A}$ if $\int_{I_\omega} v_\varepsilon^2 \geq c_1\omega^3$. Since

$$|\mathcal{A}|c_1\omega^3 \leq \sum_{I_\omega \in \mathcal{A}} \int_{I_\omega} v_\varepsilon^2 \leq \int_{I_1} v_\varepsilon^2 = \delta^2,$$

then we have

$$|\mathcal{A}| \leq \frac{\delta^2}{c_1\omega^3} = \frac{1}{c_1M^2}\frac{1}{\omega} = \frac{N}{c_1M^2},$$

Now, we choose M such that $M^2 c_1 > 2$. This choice guarantees that $|\mathcal{A}| < \frac{1}{2}\frac{1}{\omega} = \frac{N}{2}$, so at least half of the line segments I_ω are not in $\mathcal{A}$.

Recalling that we have $\omega \gtrsim \varepsilon$, Proposition 2.10 yields

$$\int_{I_\omega} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq c_0,$$

for each $I_\omega \notin \mathcal{A}$. Summing over all of these line segments gives us

$$\int_{I_1} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq \sum_{I_\omega \notin \mathcal{A}} \int_{I_\omega} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq \frac{c_0}{2\omega} \gtrsim \frac{1}{\omega}$$

Therefore,

$$\left(\fint_{I_1} \varepsilon|\partial_x u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2\right)\left(\fint_{I_1} v_\varepsilon^2\right)^{\frac{1}{2}} \gtrsim \frac{\delta}{\omega} \gtrsim 1.$$

This contradiction with (2.19) completes the proof of the theorem. □

2.4 The Dissipation Inequality

The dissipation inequality proved in this section provides the second ingredient for the ODE lemma in Section 2.5. It relates the rates of change with respect to the time of the energy and of the length scale. There are two critical points in the dissipation inequality. First, we notice that $\dot{E} \leq 0$, that is, the energy is decreasing because the motion of the interfaces or transition layers is surface energy driven. Second, a refinement of this inequality involves $\dot{L}$ which is controlled by $\dot{E}$ because coarsening requires motion which dissipates energy.

Lemma 2.11 (Dissipation). *Suppose u is a solution of (2.1) and again let E and L be defined as in (2.2) and (2.3), respectively. Then*

$$(\dot{L})^2 \leq -|I|^2 \dot{E}.$$

Proof. We first derive an expression for the time derivative of the energy:

$$\begin{aligned}\dot{E} &= \frac{1}{2}\fint_I 2\partial_x u \cdot \partial_t \partial_x u + 2(1-u^2)(-2u \cdot \partial_t u) dx \\ &= -\fint_I \partial_x^2 u \cdot \partial_t u - 2u(1-u^2)\partial_t u dx \\ &= \fint_I -(\partial_x^2 u + 2u(1-u^2))\partial_t u dx \\ &= -\fint_I (\partial_t u)^2 dx.\end{aligned}$$

Next, we turn to the time derivative of the length scale. Considering its square

$$L^2 = \fint_I v^2 dx,$$

we obtain

$$2L\dot{L} = \frac{dL^2}{dt} = \partial_t \fint_I v^2 dx = 2\fint_I v\,\partial_t v\,dx \le 2\left(\fint_I v^2 dx\right)^{\frac{1}{2}}\left(\fint_I (\partial_t v)^2 dx\right)^{\frac{1}{2}}. \tag{2.21}$$

On the other hand,

$$\partial_t v(x) = \int_0^x \partial_t u(z) dz,$$

so that,

$$\fint_I (\partial_t v)^2 dx = \frac{1}{|I|}\int_I \left(\int_0^x \partial_t u(z) dz\right)^2 dx. \tag{2.22}$$

By Hölder's inequality

$$\int_0^x \partial_t u(z) dz \le x^{\frac{1}{2}}\left(\int_0^x (\partial_t u(z))^2 dz\right)^{\frac{1}{2}} \le |I|^{\frac{1}{2}}\left(\int_0^x (\partial_t u(z))^2 dz\right)^{\frac{1}{2}},$$

and (2.22) becomes

$$\fint_I (\partial_t v)^2 dx = \int_I \int_0^x (\partial_t u(z))^2 dz dx \le |I|^2 \fint_I (\partial_t u(x))^2 dx$$

Hence, (2.21) gives us

$$2L\dot{L} \leq 2|I|L\left(\fint_I (\partial_t u)^2 dx\right)^{\frac{1}{2}} = 2|I|L(-\dot{E})^{\frac{1}{2}},$$

i.e.,

$$(\dot{L})^2 \leq |I|^2(-\dot{E}).$$

□

2.5 Upper Bound on The Coarsening Rate

The next lemma is an ODE argument of Kohn and Otto [20]. Our proof carefully traces the dependence of the coarsening rate on the size of the domain, and makes precise change of the variables required for the specific dissipation inequality listed in the hypothesis of the lemma.

Before proceeding with the ODE lemma, we first prove that L is absolutely continuous when viewed as a function of E. This fact will be invoked in the proof of the ODE Lemma 2.13.

Proposition 2.12. *Suppose E and L are continuously differentiable on $[0,T]$ and*

$$(\dot{L})^2 \leq \dot{E}, \quad \text{on } [0,T]. \tag{2.23}$$

Then

$$L \circ E^{-1} : E([0,T]) \to \mathbb{R}$$

is absolutely continuous, i.e.,

for any $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, if* $0 \le s_1 < t_1 < \cdots < s_n < t_n \le T$ *satisfy*

$$\sum_{k=1}^{n} |E(t_k) - E(s_k)| < \delta, \text{ then}$$
$$\sum_{k=1}^{n} |L(t_k) - L(s_k)| < \varepsilon. \tag{2.24}$$

Proof. Claim: there exists $h_0 > 0$ such that for any $t \in [0, T]$ and any $h \in [0, h_0]$ with $t + h \in [0, T]$,

$$(L(t+h) - L(t))^2 \le 4(E(t+h) - E(t))h.$$

Proof of claim: Assume that there exist $\{t_k\} \subset [0, T]$ and $h_k \to 0$ with $t_k + h_k \in [0, T]$ such that

$$(L(t_k + h_k) - L(t_k))^2 > 4(E(t_k + h_k) - E(t_k))h_k,$$

or, equivalently,

$$\left(\frac{L(t+h) - L(t)}{h_k}\right)^2 > \frac{4(E(t+h) - E(t))}{h_k},$$

without loss of generality, $t_k \to t \in [0, T]$. Taking the limit and using the continuity of $\dot{E}$ and $\dot{L}$, we get

$$(\dot{L}(t))^2 > 2\dot{E}(t),$$

a contradiction with (2.23).

To prove (2.24), let $\varepsilon > 0$ and take $\delta = \dfrac{\varepsilon^2}{4T}$. Suppose $0 < s_1 < t_1 < \cdots < s_n < t_n \le T$ satisfy

$$\sum_{k=1}^{n} |E(t_k) - E(s_k)| < \delta.$$

For each k such that $t_k - s_k > h_0$, let $s_{k,j} = s_k + jh_0$, $j = 0, 1, \ldots, n_k - 1$, where $n_k = \lceil \frac{t_k - s_k}{h_0} \rceil$ so that $s_k = s_{k,0} < s_{k,1} < \cdots < s_{k,n_k-1} < s_{k,n_k} = t_k$. Then, by claim

$$\begin{aligned}
\sum_{k=1}^{n} |L(t_k) - L(s_k)| &\leq \sum_{k=1}^{n} \sum_{j=1}^{n_k} |L(s_{k,j}) - L(s_{k,j-1})| \\
&\leq 2 \sum_{k=1}^{n} \sum_{j=1}^{n_k} ((E(s_{k,j}) - E(s_{k,j-1}))(s_{k,j} - s_{k,j-1}))^{1/2} \\
&\leq 2 \left(\sum_{k=1}^{n} \sum_{j=1}^{n_k} E(s_{k,j}) - E(s_{k,j-1}) \right)^{1/2} \left(\sum_{k=1}^{n} \sum_{j=1}^{n_k} s_{k,j} - s_{k,j-1} \right)^{1/2} \\
&= 2 (\sum_{k=1}^{n} E(t_k) - E(s_k))^{1/2} (\sum_{k=1}^{n} (t_k - s_k))^{1/2} \\
&\leq 2\sqrt{T\delta} \\
&= \varepsilon.
\end{aligned}$$

□

Next, we prove the ODE lemma.

Lemma 2.13 (ODE)**.** *For any $0 \leq \theta \leq 1$ and $|I| \gtrsim 1$, suppose r satisfies: $r < 3$, $\theta r > 1$ and $(1-\theta)r < 2$. Then $EL \gtrsim 1$ and $(\dot{L})^2 \leq |I|^2(-\dot{E})$ imply*

$$\int_0^T E^{\theta r} L^{-(1-\theta)r} dt \gtrsim K T^{-\frac{r}{3}} \quad for \quad T \gg L_0^3 \gg 1 \gg E_0, \tag{2.25}$$

where $K = \left(\frac{1}{|I|} \right)^{\frac{2r}{3}}$.

Proof. E is a monotone function of time due to the differential inequality $(\dot{L})^2 \leq |I|^2(-\dot{E})$. Indeed, since $\dot{E} \leq -\frac{\dot{L}^2}{|I|^2} < 0$, E is decreasing. Moreover, by Proposition 2.12, L is an absolutely continuous function of E, and we can write the dissipation inequality as

$$\left(\frac{dL}{de} \right)^2 (\dot{E})^2 \leq |I|^2 |\dot{E}|. \tag{2.26}$$

Here, the lower case e is used as an independent variable corresponding to the energy in order to distinguish it from $E = E(t)$.

From (2.26), we have

$$\left(\frac{dL}{de}\right)^2 |\dot{E}| \leq |I|^2. \tag{2.27}$$

When $\dot{E} = 0$, (2.27) is still true trivially. Multiplying (2.27) by any function $f(E(t))$ and integrating in time on the interval $[0, T]$ gives

$$\int_0^T f(E(t))dt \geq \frac{1}{|I|^2} \int_{E(T)}^{E(0)} f(e) \left(\frac{dL}{de}\right)^2 de.$$

Taking $f = e^{\theta r} L^{-(1-\theta)r}$ and writing $E_0 = E(0)$, $E_T = E(T)$, we then have

$$\int_0^T E^{\theta r}(t) L^{-(1-\theta)r}(t)dt \geq \frac{1}{|I|^2} \int_{E_T}^{E_0} e^{\theta r} L(e)^{-(1-\theta)r} \left(\frac{dL}{de}\right)^2 de \tag{2.28}$$

for all $T > 0$.

Now we estimate the right hand side of (2.28). Consider the change of variables

$$\hat{e} = \frac{1}{1-\theta r} e^{1-\theta r} \text{ and } \hat{L} = \frac{1}{1 - \frac{(1-\theta)r}{2}} L^{1-\frac{(1-\theta)r}{2}}.$$

The hypotheses

$$\theta r > 1, \quad (1-\theta)r < 2$$

imply

$$\theta > \frac{1}{3} \tag{2.29}$$

which will be used later. Since

$$\left(\frac{dL}{de}\right)^2 de = \left(\frac{d\hat{L}}{d\hat{e}}\right)^2 \left(\frac{dL}{d\hat{L}}\right)^2 \left(\frac{d\hat{e}}{de}\right) d\hat{e}$$

and

$$\frac{d\hat{e}}{de} = e^{-\theta r}, \quad \frac{d\hat{L}}{dL} = L^{-\frac{(1-\theta)r}{2}},$$

the integral in the right hand side of (2.28) can be written as:

$$\int_{E_T}^{E_0} \left(\frac{de}{d\hat{e}}\right) \left(\frac{d\hat{L}}{dL}\right)^2 \left(\frac{dL}{de}\right)^2 de = \int_{\hat{E}_T}^{\hat{E}_0} \left(\frac{d\hat{L}}{d\hat{e}}\right)^2 d\hat{e}.$$

The last integral is bounded below by the minimum over all functions $\hat{L}(\hat{e})$ with the boundary conditions

$$\hat{L}(\hat{E}_0) = \frac{1}{1-\frac{(1-\theta)r}{2}}(L(0))^{1-\frac{(1-\theta)r}{2}}, \quad \hat{L}(\hat{E}_T) = \frac{1}{1-\frac{(1-\theta)r}{2}}(L(T))^{1-\frac{(1-\theta)r}{2}}.$$

To simplify the notations we denote these end conditions by $\hat{L}_0$ and $\hat{L}_T$, respectively. By the Dirichlet principle, the extremal $\hat{L}$ is linear in $\hat{e}$, so we have

$$\int_0^T E^{\theta r} L^{-(1-\theta)r} dt \geq \frac{1}{|I|^2} \frac{(\hat{L}_T - \hat{L}_0)^2}{\hat{E}_0 - \hat{E}_T}. \tag{2.30}$$

When T is chosen so that

$$L(T) \geq 2L(0),$$

the right side of (2.30) can be controlled since

$$\hat{L}_T - \hat{L}_0 \gtrsim \hat{L}_T \quad \text{and} \quad \hat{E}_0 - \hat{E}_T \leq -\hat{E}_T.$$

Therefore,

$$\int_0^T E^{\theta r} L^{-(1-\theta)r} dt \gtrsim \frac{1}{|I|^2} \frac{\hat{L}_T^2}{-\hat{E}_T} = \frac{1}{|I|^2} L_T^{2-(1-\theta)r} E_T^{\theta r - 1}.$$

Rewriting the right hand side as

$$L_T^{2-(1-\theta)r} E_T^{\theta r - 1} = [E_T^{\theta} L_T^{-(1-\theta)}]^{r-3} [E_T L_T]^{3\theta - 1},$$

we conclude, using $EL \gtrsim 1$ and (2.29), that

$$\int_0^T E^{\theta r} L^{-(1-\theta)r} dt \gtrsim \frac{1}{|I|^2} [E_T^\theta L_T^{-(1-\theta)}]^{r-3} \tag{2.31}$$

provided that $L(T) \geq 2L(0)$. Introducing

$$h(T) := \int_0^T E^{\theta r} L^{-(1-\theta)r} dt,$$

we can write (2.31) as $h \gtrsim \frac{1}{|I|^2}(h')^{\frac{r-3}{r}}$, so we have:

$$h^{\frac{r}{3-r}}(T) h'(T) \gtrsim \left(\frac{1}{|I|}\right)^{\frac{2r}{3-r}}, \tag{2.32}$$

provided $L(T) \geq 2L(0)$. Here we used $r < 3$.

The previous method does not work when $L(T) < 2L(0)$, but, for such T, we have

$$E(T) \gtrsim L(T)^{-1} \gtrsim L_0^{-1},$$

which implies

$$E^\theta(T) L^{-(1-\theta)}(T) = (E(T)L(T))^\theta L^{-1}(T) \gtrsim L_0^{-1}.$$

Thus

$$h'(T) \gtrsim L_0^{-r} \text{ if } L(T) < 2L_0. \tag{2.33}$$

Combining (2.32) and (2.33), using $r < 3$, we have:

$$\frac{d}{dt}(h + L_0^{3-r})^{\frac{3}{3-r}} \sim (h(t) + L_0^{3-r})^{\frac{r}{3-r}} h'(t) \gtrsim \left(\frac{1}{|I|}\right)^{\frac{2r}{3-r}}$$

for all $t > 0$. Indeed, when $L(T) \geq 2L(0)$, by (2.32),

$$\frac{d}{dt}(h + L_0^{3-r})^{\frac{3}{3-r}} \sim (h(t) + L_0^{3-r})^{\frac{r}{3-r}} h'(t) \gtrsim h(t)^{\frac{r}{3-r}} h'(T) \gtrsim \left(\frac{1}{|I|}\right)^{\frac{2r}{3-r}}$$

and, when $L(T) < 2L(0)$, by (2.33),

$$\frac{d}{dt}(h + L_0^{3-r})^{\frac{3}{3-r}} \sim (h(t) + L_0^{3-r})^{\frac{r}{3-r}} h'(t) \gtrsim L_0^r h'(t) \gtrsim 1 \gtrsim \left(\frac{1}{|I|}\right)^{\frac{2r}{3-r}}$$

for $|I| \gtrsim 1$.

Integration in time gives

$$h(T) + L_0^{3-r} \gtrsim \left(\frac{1}{|I|}\right)^{\frac{2r}{3}} T^{\frac{3-r}{3}}$$

for all $T > 0$.

Restricting attention to $T^{\frac{3-r}{3}} \gg L_0^{3-r}$, this becomes

$$\int_0^T E^{\theta r} L^{-(1-\theta)r} dt = h(T) \gtrsim \left(\frac{1}{|I|}\right)^{\frac{2r}{3}} T^{\frac{3-r}{3}} \quad for\ T \gg L_0^3$$

which is precisely (2.25) as we need. □

Chapter 3

Allen-Cahn Equation in Two-Dimensional Space

In this chapter, we consider the solutions of the parabolic Allen-Cahn equation on the domain $Q \times [0, \infty)$

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - 2u(1-u^2) = 0, &\quad \text{in } Q \times [0,\infty) \\ u(x,t) = 0, &\quad \text{on } \partial Q \times [0,\infty) \\ u(x,0) = u_0(x), &\quad \text{in } Q \end{aligned} \tag{3.1}$$

where Q is a square in $\mathbb{R}^2$ and here we suppose u_0 is uniformly bounded.

This PDE corresponds to the gradient flow of the energy

$$E(u) = \frac{1}{2} \fint_Q |\nabla u|^2 + (1-u^2)^2 dx. \tag{3.2}$$

3.1 Energy Decays No Faster than $t^{-1/3}$

Here again we want to obtain a rigorous and time-averaged upper bound on coarsening rates for the Allen-Cahn equation in two-dimensional space. In this section, we establish an upper bound on the coarsening rate $t^{1/3}$. The volume-averaged free energy E is a decreasing function of time and it scales as the reciprocal to the length scale L which is dual to E. Our result indicates the one-sided $L \sim t^{1/3}$ coarsening rate, that is, in a time-averaged sense, it is impossible for solutions to coarsen at a rate faster than the expected power law. The interfacial area decreases, that is, $\dot{E} \leq 0$, however, we need a more accurate energy-dissipating structure of the dynamics. This refined structure is obtained in the form of a dissipation inequality. Finally, the interpolation and the dissipation inequalities are combined in the proof of an ODE Lemma 3.7 which is similar to Lemma 2.13 in Section 2.5, from which an upper bound on the time-averaged coarsening rate follows.

3.1.1 Introduction to the Main Result

We generalize the length scale introduced in the one-dimensional case and define

$$L = \frac{||u||_{H^{-1}(Q)}}{|Q|^{1/2}}. \tag{3.3}$$

We define H^{-1} norm as in [31] or [11], namely for each $u \in H_0^1(Q)$, there exits unique solution p of the Poisson equation

$$\begin{aligned} -\Delta p = u, &\quad \text{in } Q \\ p = 0, &\quad \text{on } \partial Q \end{aligned}$$

such that $p \in H_0^1(Q)$. We define the H^{-1}-inner product of u_1 and u_2 as the L^2-inner product of the gradients of the corresponding solutions of the Poisson equations:

$$\langle u_1, u_2 \rangle_{H^{-1}} = \langle \nabla p_1, \nabla p_2 \rangle_{L^2},$$

so that

$$||u||_{H^{-1}(Q)} = ||\nabla p||_{L^2(Q)}.$$

The main results have the statements that are the same as those in the one-dimensional case and we obtain the same upper bound on the coarsening rate. As above, a special case is stated the same as Theorem 2.1 and the general case is stated the same as Theorem 2.2, but with the energy and length scale defined by (3.2) and (3.3), respectively, also with the two-dimensional domain and different dependence on the size of the domains. And the dependence on the size of the domain Q will be provided in Lemma 3.7.

3.1.2 The Interpolation Inequality

In this section, we prove that

$$EL \gtrsim 1 \quad \text{when} \quad E \ll 1, \tag{3.4}$$

where E and L are defined in (3.2) and (3.3), The constants implicit in (3.4) are independent of the size of Q.

We scale the system and prove the corresponding result for the unit square Q_1. With the side length of Q denoted by $\frac{1}{\varepsilon}$, we define

$$p_\varepsilon(x,t) = \varepsilon^2 p\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

then for $u_\varepsilon(x) = -\Delta p_\varepsilon(x)$ and $u(x) = -\Delta p(x)$, we have

$$u_\varepsilon(x,t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right).$$

Hence, u_ε solves the equation

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon - \frac{2}{\varepsilon^2} u_\varepsilon (1 - u_\varepsilon^2) = 0, &\quad \text{in } Q_1 \times [0, \infty) \\ u_\varepsilon(x,t) = 0, &\quad \text{on } \partial Q_1 \times [0, \infty) \end{aligned} \tag{3.5}$$

The quantities E and L may be rewritten as,

$$E = \frac{1}{2} \fint_{Q_1} \varepsilon^2 |\nabla u_\varepsilon|^2 + (1 - u_\varepsilon^2)^2 dx,$$

and

$$\begin{aligned}
L &= \frac{1}{|Q|^{1/2}}||u||_{H^{-1}(Q)} \\
&= \frac{1}{|Q|^{1/2}}\left(\int_Q |\nabla p(x)|dx\right)^{\frac{1}{2}} \\
&= \frac{1}{|Q|^{1/2}}\left(\int_{Q_1} \frac{1}{\varepsilon^4}|\nabla p_\varepsilon(x)|^2 dx\right)^{\frac{1}{2}} \\
&= \frac{1}{\varepsilon}||u_\varepsilon||_{H^{-1}(Q_1)}
\end{aligned}$$

respectively, and (3.4) becomes

$$\left(⨍_{Q_1} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 dx\right)||u_\varepsilon||_{H^{-1}(Q_1)} \gtrsim 1$$

when

$$⨍_{Q_1} \varepsilon^2|\nabla u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 dx \ll 1$$

The last statement is a consequence of the following theorem.

Theorem 3.1 (Interpolation)**.** *There is a constant $c_* > 0$ with the following property: For any $u_\varepsilon \in H_0^1(Q_1)$ and any $\varepsilon > 0$,*

$$\left(⨍_{Q_1} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 dx\right)||u_\varepsilon||_{H^{-1}(Q_1)} + ⨍_{Q_1} \varepsilon^2|\nabla u_\varepsilon|^2 + (1-u_\varepsilon^2)^2 dx \geq c_*.$$

As in Section 2.3, we start by establishing a uniform lower bound on energy E_ε. The proof is the same as that of Lemma 2.7, therefore we state the following lemma without proof.

Lemma 3.2. *Define*

$$E_\varepsilon(u_\varepsilon) = \int_{Q_1} \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2$$

where Q_1 is the unit square in $\mathbb{R}^2$. There exit constants $a_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ and any $u_\varepsilon \in H_0^1(Q_1)$, we have

$$E_\varepsilon(u_\varepsilon) \geq a_0.$$

The next lemma claiming compactness in $L^2(Q)$ of the set of admissible functions with uniformly bounded energy is the same as Lemma 2.8 of one-dimensional case as well, and we omit its proof.

Lemma 3.3. *Suppose $E_\varepsilon(u_\varepsilon)$ is uniformly bounded and $\{\varepsilon_j\}$ is a sequence such that $\varepsilon_j \to 0$, then there exist a subsequence $\{\varepsilon_j\}$ of positive numbers such that $\{u_{\varepsilon_j}\}$ converges to a function u_0 in $L^2(Q)$ as $j \to \infty$.*

The following lemma and proposition share the techniques of their proofs with Lemma 2.9 and Proposition 2.10, respectively, but the use of a different length scale warrants separate proofs.

Lemma 3.4. *Let $E_\varepsilon(u_\varepsilon)$ be as in Lemma 3.2. For any $c \leq \frac{1}{2}$, there exists $\gamma > 0$ such that for all $\varepsilon \leq 1$, if $||u_\varepsilon||_{H^{-1}(Q_1)} \leq \gamma$, then $E_\varepsilon(u_\varepsilon) \geq c$.*

Proof. We prove this lemma by contradiction. Suppose, for some $c \leq \frac{1}{2}$, there exist sequences $\{u_{\varepsilon_j}\}$ and $\{\varepsilon_j\}$ such that

$$||u_{\varepsilon_j}||_{H^{-1}(Q_1)} \leq \frac{1}{j} \quad \text{but} \quad E_{\varepsilon_j}(u_{\varepsilon_j}) < c. \tag{3.6}$$

If $\liminf_j \varepsilon_j = 0$, by the compactness theorem in Lemma 3.3, we know that $\{u_{\varepsilon_j}\}$ is relatively compact in L^2. If $\liminf_j \varepsilon_j > 0$, we choose a subsequence, without loss of generality, also denoted as $\{\varepsilon_j\}$, such that $\inf \varepsilon_j > 0$, so that $\{\int_{Q_1} |\nabla u_{\varepsilon_j}|^2\}$ is bounded, hence, $\{u_{\varepsilon_j}\}$ is pre-compact in L^2. In both cases, for a further subsequence, $u_{\varepsilon_j} \to u_0$ in L^2 for some u_0 in L^2.

On the other hand, (3.6) implies $\lim_{j\to\infty} u_{\varepsilon_j} = u_0 = 0$, since $||\nabla p_{\varepsilon_j}||_{L^2} \to 0$ as $j \to \infty$, i.e., $\lim_{j\to\infty} \nabla p_{\varepsilon_j} = \nabla p_0 = 0$, and $u_0 = -\Delta p_0 = 0$. But by compactness of $\{u_{\varepsilon_j}\}$ and Fatou's lemma,

$$1 = \int_{Q_1} (1-u_0^2)^2 \leq \liminf_j \int_{Q_1} (1-u_{\varepsilon_j}^2)^2 \leq c\varepsilon_j \leq \frac{1}{2}.$$

This contradiction shows that the lemma is true. □

Proposition 3.5. *For any $c_0 \leq \frac{1}{2}$, there exists a constant $c_1 > 0$ with the following property. Consider any square $Q \subset \mathbb{R}^2$ with side length l and any $u_\varepsilon \in H_0^1(Q)$ satisfying*

$$||u_\varepsilon||_{H^{-1}(Q)} \leq c_1|Q|.$$

Then we have

$$\textit{Case A}: \quad \int_Q \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq c_0|Q|^{1/2} \quad \textit{if} \quad |Q|^{1/2} \geq \varepsilon, \tag{3.7}$$

$$\textit{Case B}: \quad \int_Q \varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1-u_\varepsilon^2)^2 \geq \frac{c_0|Q|}{\varepsilon} \quad \textit{if} \quad |Q|^{1/2} \leq \varepsilon, \tag{3.8}$$

Proof. For Case A, we define $u(x) = u_\varepsilon(lx)$, or, equivalently, $u_\varepsilon(x) = u\left(\frac{x}{l}\right)$, and suppose without loss of generality that Q is centered at the origin. Then (3.7) is equivalent to proving that for any $c_0 \leq \frac{1}{2}$, there exists c_1 such that if

$$\frac{\varepsilon}{|Q|^{1/2}} \leq 1, \quad ||u||_{H^{-1}(Q_1)} \leq c_1,$$

then,

$$\int_{Q_1} \left(\frac{\varepsilon}{|Q|^{1/2}}\right) |\nabla u|^2 + \left(\frac{|Q|^{1/2}}{\varepsilon}\right) (1-u^2)^2 \geq c_0.$$

Since $\frac{\varepsilon}{|Q|^{1/2}} \leq 1$, this is exactly the result of Lemma 3.4.

We prove Case B by contradiction. Suppose for some $c_0 \leq \frac{1}{2}$, there exists sequences $\varepsilon_k, Q_k, l_k, u_{\varepsilon_k}$ satisfying

$$\begin{cases} \dfrac{\varepsilon_k}{|Q_k|^{1/2}} \geq 1, \\ \dfrac{1}{|Q_k|}||u_{\varepsilon_k}||_{H^{-1}(Q_k)} \to 0, \\ \int_{Q_k} \varepsilon_k^2 |\nabla u_{\varepsilon_k}|^2 + (1-u_{\varepsilon_k}^2)^2 < c_0 |Q_k|. \end{cases} \tag{3.9}$$

Using the same scaling as above, $u_k(x) = u_{\varepsilon_k}(l_k x)$, (3.9) becomes

$$||u_k||_{H^{-1}(Q_1)} \to 0, \qquad \int_{Q_1} \left(\frac{\varepsilon_k}{|Q_k|^{1/2}}\right)^2 |\nabla u_k|^2 + (1-u_k^2)^2 < c_0.$$

Since $\dfrac{\varepsilon_k}{|Q_k|^{1/2}} \geq 1$, then we can use the same argument as in the proof of Lemma 3.4 to arrive at a contradiction. □

We have all the necessary tools to prove the main result for the two-dimensional case.

Proof of Theorem 3.1. We prove this theorem by contradiction. We suppose there exists a sequence $u_{\varepsilon_k}, \varepsilon_k$ such that

$$\begin{aligned} \left(\fint_{Q_1} \varepsilon_k |\nabla u_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k}(1-u_{\varepsilon_k}^2)^2\right) ||u_{\varepsilon_k}||_{H^{-1}(Q_1)} \\ + \fint_{Q_1} \varepsilon_k^2 |\nabla u_{\varepsilon_k}|^2 + (1-u_{\varepsilon_k}^2)^2 \to 0. \end{aligned} \tag{3.10}$$

Case 1: Suppose $\liminf_k \varepsilon_k > 0$, then, in the second term of (3.10), we have

$$\fint_{Q_1} |\nabla u_{\varepsilon_k}|^2 \to 0 \quad \text{and} \quad \fint_{Q_1} u_{\varepsilon_k}^2 \to 1. \tag{3.11}$$

By the Poincaré inequality

$$\fint_{Q_1} u_{\varepsilon_k}^2 \leq C \fint_{Q_1} |\nabla u_{\varepsilon_k}|^2 \to 0.$$

This is a contradiction with (3.11).

Case 2: Suppose $\liminf_k \varepsilon_k = 0$ but $||u_{\varepsilon_k}||_{H^{-1}(Q_1)}$ is bounded away from 0. Without loss of generality, suppose $\varepsilon_k \to 0$. The convergence of the first term in (3.10) to 0 gives us

$$\fint_{Q_1} \varepsilon_k^2 |\nabla u_{\varepsilon_k}|^2 + (1 - u_{\varepsilon_k}^2)^2 \to 0,$$

in a contradiction with Lemma 3.2.

Case 3: Suppose $\liminf_k \varepsilon_k = 0$ and $\liminf_k ||u_{\varepsilon_k}||_{H^{-1}(Q_1)} = 0$. We use Proposition 3.5 to obtain a contradiction. Fix c_0 and, dropping the subscript k to simplify the notation, we write $u_{\varepsilon_k} = u_\varepsilon$. Define

$$\delta = ||u_\varepsilon||_{H^{-1}(Q_1)}.$$

For any integer $N > 1$, we partition the unit square Q_1 into N^2 squares of side length $\omega = \dfrac{1}{N}$. The value of N will be determined later. If

$$||u_\varepsilon||_{H^{-1}(Q_\omega)} \leq c_1 \omega^2,$$

by applying Proposition 3.5 to Q_ω, we have

$$\int_{Q_\omega} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - u_\varepsilon^2)^2 \geq c_0 |Q_\omega|^{1/2} \quad \text{if} \quad |Q_\omega|^{1/2} = \omega \geq \varepsilon$$

or

$$\int_{Q_\omega} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - u_\varepsilon^2)^2 \geq \frac{c_0 |Q_\omega|}{\varepsilon} \quad \text{if} \quad |Q_\omega|^{1/2} = \omega \leq \varepsilon$$

The choice of N depends on the relation between ε and δ.

Alternative 1: Suppose $\varepsilon \gg \delta$. Then we choose N such that $\omega \approx \sqrt{\varepsilon\delta}$ and therefore $\varepsilon \gg \omega \gg \delta$. For the N^2 squares Q_ω, we say $Q_\omega \in \mathcal{A}$ if $||u_\varepsilon||_{H^{-1}(Q_\omega)} \geq c_1 |Q_\omega| = c_1 \omega^2$ and let

$|\mathcal{A}|$ be the number of squares in $\mathcal{A}$. Since $\delta \ll \omega = \dfrac{1}{N} < 1$ and

$$|\mathcal{A}| c_1^2 \omega^4 \leq \sum_{Q_\omega \in \mathcal{A}} ||u_\varepsilon||^2_{H^{-1}(Q_\omega)} \leq ||u_\varepsilon||^2_{H^{-1}(Q_1)} = \delta^2,$$

we have

$$|\mathcal{A}| \leq \frac{\delta^2}{c_1^2 \omega^4} \ll \frac{1}{\omega^2} = N.$$

Therefore, the inequality

$$||u_\varepsilon||_{H^{-1}(Q_\omega)} \leq c_1 \omega^2$$

holds on most, in particular on at least half, of the squares. Since $\varepsilon \gg \omega$, for $Q_\omega \notin \mathcal{A}$, i.e, when

$$||u_\varepsilon||_{H^{-1}(Q_\omega)} \leq c_1 \omega^2,$$

Proposition 3.5 gives

$$\int_{Q_\omega} \varepsilon^2 |\nabla u_\varepsilon|^2 + (1 - u_\varepsilon^2)^2 \geq c_0 |Q_\omega|$$

Summing over all these squares, we have

$$\int_{Q_1} \varepsilon^2 |\nabla u_\varepsilon|^2 + (1 - u_\varepsilon^2)^2 \geq \sum_{Q_\omega \notin \mathcal{A}} \int_{Q_\omega} \varepsilon |\nabla u_\varepsilon|^2 + (1 - u_\varepsilon^2)^2 \geq c_0 \sum_{Q_\omega \notin \mathcal{A}} |Q_\omega| \gtrsim 1$$

This is a contradiction with (3.10).

Alternative 2: Suppose $\delta \gtrsim \varepsilon$. Then we choose N and $\omega = \frac{1}{N}$ such that $\omega = M\delta$, where M is a constant to be chosen later. Again, consider the N^2 squares Q_ω, we say $Q_\omega \in \mathcal{A}$ if $||u_\varepsilon||_{H^{-1}(Q_\omega)} \geq c_1 \omega^2$. Since

$$|\mathcal{A}| c_1^2 \omega^4 \leq \sum_{Q_\omega \in \mathcal{A}} ||u_\varepsilon||^2_{H^{-1}(Q_\omega)} \leq ||u_\varepsilon||^2_{H^{-1}(Q_1)} = \delta^2,$$

we have

$$|\mathcal{A}| \leq \frac{\delta^2}{c_1^2 \omega^4},$$

and $\delta = \dfrac{\omega}{M} \leq \dfrac{1}{M}$. Then we estimate

$$|\mathcal{A}| < \frac{1}{M^2 c_1^2} \frac{1}{\omega^2}.$$

Now, we choose M such that $M^2 c_1^2 \geq 2$. This guarantees that $|\mathcal{A}| < \dfrac{1}{2}\dfrac{1}{\omega^2} = \dfrac{N}{2}$, so at least half of the squares Q_ω are not in $\mathcal{A}$.

Considering that we have $\omega \gtrsim \delta \gtrsim \varepsilon$, by proposition 3.5, for each $Q_\omega \notin \mathcal{A}$,

$$\int_{Q_\omega} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - u_\varepsilon^2)^2 \geq c_0 |Q_\omega|^{1/2}.$$

Summing over all these squares gives us

$$\int_{Q_1} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - u_\varepsilon^2)^2 \geq \sum_{Q_\omega \notin \mathcal{A}} \int_{Q_\omega} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - u_\varepsilon^2)^2 \geq \frac{c_0 |Q_\omega|^{1/2}}{2\omega^2} \gtrsim \frac{1}{\omega}.$$

Therefore,

$$\left(\fint_{Q_1} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}(1 - u_\varepsilon^2)^2 \right) ||u_\varepsilon||_{H^{-1}(Q_1)} \gtrsim \frac{\delta}{\omega} \gtrsim 1.$$

This contradiction with (3.10) completes the proof. □

3.1.3 The Dissipation Inequality

Lemma 3.6 (Dissipation). *Suppose u is a solution of (3.1) and again let E and L be defined as in (3.2) and (3.3), respectively. Then*

$$(\dot{L})^2 \lesssim -|Q|\dot{E}$$

Proof. As in Lemma 2.11, by replacing u_x by ∇u, we again have that integration by parts yields

$$\begin{aligned} \dot{E} &= \frac{1}{2}\fint 2\nabla u \cdot \nabla u_t + 2(1-u^2)(-2u \cdot u_t)dx \\ &= -\fint u_t^2 dx, \end{aligned} \tag{3.12}$$

where the boundary integrals are all vanishing due to the boundary conditions.

Next, we compute

$$2L\dot{L} = \frac{dL^2}{dt} = \frac{2}{|Q|}\langle u, \dot{u}\rangle_{H^{-1}} \leq \frac{2}{|Q|}||u||_{H^{-1}(Q)}||\dot{u}||_{H^{-1}(Q)}.$$

Therefore,

$$\dot{L}^2 \leq \frac{||\dot{u}||^2_{H^{-1}(Q)}}{|Q|}. \tag{3.13}$$

Let q be the solution to equation

$$\begin{aligned} -\Delta q &= \dot{u}, \quad \text{in } Q \\ q &= 0, \quad \text{on } \partial Q. \end{aligned}$$

Then

$$\begin{aligned} ||\nabla q||^2_{L^2(Q)} =& ||\dot{u}||^2_{H^{-1}(Q)} = \int_Q q\dot{u}dx \\ \leq& ||q||_{L^2(Q)}||\dot{u}||_{L^2(Q)} \\ \leq& \frac{\varepsilon}{2}||q||^2_{L^2(Q)} + \frac{2}{\varepsilon}||\dot{u}||^2_{L^2(Q)}. \end{aligned} \tag{3.14}$$

By the Poincaré inequality,

$$||q||_{L^2(Q)} \lesssim |Q|^{1/2}||\nabla q||_{L^2(Q)}.$$

Thus (3.14) becomes

$$||\nabla q||^2_{L^2(Q)} \lesssim \frac{\varepsilon}{2}|Q| \cdot ||\nabla q||^2_{L^2(Q)} + \frac{2}{\varepsilon}||\dot{u}||^2_{L^2(Q)}.$$

Choosing $\varepsilon = \dfrac{1}{|Q|}$ in the inequality above, we have

$$||\dot{u}||_{H^{-1}(Q)} = ||\nabla q||_{L^2(Q)} \lesssim |Q|^{1/2}||\dot{u}||_{L^2(Q)}. \tag{3.15}$$

Combining (3.12), (3.13) and (3.15), we get

$$\dot{L}^2 \lesssim ||\dot{u}||^2_{L^2(Q)} = -|Q|\dot{E},$$

so that

$$(\dot{L})^2 \lesssim -|Q|\dot{E},$$

where the dependence on the system size included in the right hand side. □

3.1.4 Upper Bound on the Coarsening Rate

Now we can start to prove the ODE lemma by interpolation inequality and dissipation inequality as before.

Lemma 3.7 (ODE)**.** *For any $0 \leq \theta \leq 1$ and $|Q| \gtrsim 1$, suppose r satisfies : $r < 3$, $\theta r > 1$ and $(1-\theta)r < 2$. Then $EL \gtrsim 1$ and $(\dot{L})^2 \leq |Q|(-\dot{E})$ imply*

$$⨍_0^T E^{\theta r} L^{(1-\theta)r} dt \gtrsim \left(\frac{1}{|Q|}\right)^{\frac{r}{3}} T^{-\frac{r}{3}} \quad for \quad T \gg L_0^3 \gg 1 \gg E_0. \tag{3.16}$$

This ODE lemma is similar to Lemma 2.13 and hence we just give the statement without a detailed proof. The only difference is the power of the domain size. In this lemma, we have $|Q|$ and in Lemma 2.13 we have $|I|^2$, so we only need to adjust the corresponding power in the statement of this lemma.

3.2 Energy Decays No Faster than $t^{-1/6}$

In this section, we will use another length scale which will improve our result in the sense that it allows us to see that the energy is dissipated at a slower rate. By choosing the new length scale, our interpolation inequality transforms into $E^2L \gtrsim 1$, so that $L \sim E^{-2}$, and when the energy decays as $t^{-1/6}$, the coarsening rate for length scale is $t^{1/3}$. The one-sided version of this statement that we prove here is that the energy decays no faster than $t^{-1/6}$.

3.2.1 Introduction to the Main Result

The auxiliary length scale that is employed in our analysis takes the form

$$L = \fint_Q |W(u)|dx, \tag{3.17}$$

where we define

$$W(u) = \int_0^u |1 - s^2|ds.$$

Here, we use $\fint$ to denote averaging over the spatial domain.

Now we state the main result which follows immediately from Lemma 3.12.

Theorem 3.8. *For the initial energy E_0 and initial length scale L_0 satisfying $T \gg L_0^4 \gg 1 \gg E_0$, it holds that*

$$\fint_0^T E^3 dt \gtrsim \fint_0^T (t^{-\frac{1}{4}})^2 dt.$$

As in the one-dimensional case, this theorem follows from a more general and stronger statement that appears in the next theorem.

Theorem 3.9. *For any $0 \leq \theta \leq 1$, suppose r satisfies $r < 4$, $\theta r > \dfrac{4}{3}$ and $(1-\theta)r < \dfrac{8}{3}$. Then we have*

$$\fint_0^T E^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r} dt \gtrsim \fint_0^T t^{-\frac{r}{4}} dt \quad \text{for} \quad T \gg L_0^4 \gg 1 \gg E_0.$$

3.2.2 Interpolation Inequality

For the interpolation inequality, here we only use the definition of E and L in the interfacial regime $E \ll 1$, without considering the Allen-Cahn dynamics.

Lemma 3.10 (Interpolation)**.** *If E and L are defined as in (3.2) and (3.17), respectively, then*

$$E^2 L \gtrsim k \quad \textit{for} \quad E \ll 1,$$

where the constant number in the inequality $k = \dfrac{1}{|Q|}$.

Proof. Define

$$W(u) := \int_0^u |1 - t^2| dt,$$

so that

$$\frac{\partial W}{\partial u} = |1 - u^2|.$$

Then, by Cauchy's inequality,

$$⨍ |\nabla (W(u))| dx = ⨍ |\nabla u| \frac{\partial W}{\partial u} dx \leq \frac{1}{2} ⨍ |\nabla u|^2 + (\frac{\partial W}{\partial u})^2 dx = E.$$

This inequality was introduced by Modica and Mortola in [28].

We next introduce a smooth mollifier φ that is radially symmetric, non-negative, and supported in the unit ball with $\int_{\mathbb{R}^2} \varphi = 1$. Let the subscript ε denote the convolution with the kernel

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2} \varphi \left(\frac{x}{\varepsilon} \right),$$

that is,

$$u_\varepsilon = u * \varphi_\varepsilon.$$

The L^2 norm of u may be split up as follows

$$⨍ u^2 dx \lesssim ⨍ (u - u_\varepsilon)^2 dx + ⨍ u_\varepsilon^2 dx. \tag{3.18}$$

This inequality holds for any $\varepsilon > 0$ and the precise value of ε will be selected later in the proof.

The first term in (3.18) is estimated in terms of the energy:

$$
\begin{aligned}
\fint (u - u_\varepsilon)^2 dx &\leq \sup_{|h|\leq\varepsilon} \fint (u(x) - u(x+h))^2 dx \\
&\lesssim \sup_{|h|\leq\varepsilon} \fint |W(u(x)) - W(u(x+h))| dx \\
&\lesssim \varepsilon \fint |\nabla(W(u))| dx.
\end{aligned} \tag{3.19}
$$

For the second term in (3.18), we note that for any $\varepsilon > 0$, since

$$
\begin{aligned}
|u_\varepsilon(x)| &= \left| \int \varphi_\varepsilon(x-y) u(y) dy \right| \\
&= \frac{1}{\varepsilon^2} \int \varphi\left(\frac{x-y}{\varepsilon}\right) u(y) dy \\
&\leq \int \varphi_\varepsilon^{1/2}(x-y) \varphi_\varepsilon^{1/2}(x-y) |u(y)| dy \\
&\leq \left(\int \varphi_\varepsilon(x-y) dy \right)^{1/2} \left(\int \varphi_\varepsilon(x-y) |u(y)|^2 dy \right)^{1/2} \\
&= \left(\int \varphi_\varepsilon(x-y) |u(y)|^2 dy \right)^{1/2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\fint u_\varepsilon^2(x) dx &\leq \fint \int \varphi_\varepsilon(x-y) |u(y)|^2 dy dx \\
&= \fint \int \varphi_\varepsilon(x-y) |u(x)|^2 dx dy \\
&= \frac{1}{\varepsilon^2} \int |u(x)|^2 \left(\fint \varphi\left(\frac{x-y}{\varepsilon}\right) dy \right)^2 dx \\
&\leq \frac{1}{\varepsilon^2} \int |u(x)|^2 dx.
\end{aligned}
$$

Thus,

$$\begin{aligned}\fint u_\varepsilon^2 dx &\leq \frac{|Q|}{\varepsilon^2}\fint |u|^2 dx \\ &\lesssim \frac{|Q|}{\varepsilon^2}\fint W(u) dx.\end{aligned} \tag{3.20}$$

The last inequality holds since $|u|^2 \lesssim W(u)$ for any u without considering the Allen-Cahn dynamics.

Combining (3.18), (3.19) and (3.20), we have

$$\begin{aligned}\fint u^2 dx &\lesssim \varepsilon \fint |\nabla(W(u))| dx + \frac{|Q|}{\varepsilon^2}\fint W(u) dx \\ &\leq \varepsilon E + \frac{|Q|}{\varepsilon^2} L.\end{aligned}$$

We now select $\varepsilon = \left(\frac{|Q|L}{E}\right)^{1/3}$, with $E \ll 1$, to obtain

$$\varepsilon E + \frac{|Q|}{\varepsilon^2} L = |Q|^{1/3} L^{1/3} E^{2/3} + |Q|^{1/3} L^{1/3} E^{2/3} = 2|Q|^{1/3} L^{1/3} E^{2/3}.$$

Hence

$$\fint u^2 \lesssim |Q|^{1/3} E^{2/3} L^{1/3}.$$

We also have

$$1 - \fint u^2 dx = \fint (1-u^2) dx \leq \left(\fint (1-u^2)^2 dx\right)^{1/2} \leq E^{1/2}.$$

Therefore,

$$1 \lesssim |Q|^{1/3} E^{2/3} L^{1/3} + E^{1/2}$$

which gives us Lemma 3.10 for $E \ll 1$. □

3.2.3 Dissipation Inequality

The next lemma concerns with the rate at which L can change in relation to E and the rate of change of E.

Lemma 3.11 (Dissipation)**.** *Suppose u is a solution of (3.1) and again let E and L be defined as in (3.2) and (3.17), respectively. Then*

$$(\dot{L})^2 \lesssim -\dot{E}E.$$

Proof. As in Lemma 2.11, by replacing u_x with ∇u, we again have

$$\begin{aligned}\dot{E} &= \frac{1}{2}⨍ 2\nabla u \cdot \nabla u_t + 2(1-u^2)(-2u \cdot u_t)dx \\ &= -⨍ u_t^2 dx.\end{aligned}$$

Since

$$L = ⨍ W(u)dx,$$

$$\frac{d}{dt}L = ⨍ u_t|1-u^2|dx,$$

so that

$$\begin{aligned}|\dot{L}| &\le ⨍ |1-u^2||u_t|dx \\ &\le \left(⨍ u_t^2 dx\right)^{1/2} \left(⨍ (1-u^2)^2 dx\right)^{1/2} \\ &\le |\dot{E}|^{1/2}|E|^{1/2}.\end{aligned}$$

Hence, we have Lemma 3.11. □

3.2.4 Upper Bound on the Coarsening Rate

A variant of the following lemma appears in [20]. The version that we prove here employs the interpolation inequality proved in Lemma 3.10.

Lemma 3.12 (ODE)**.** *For any* $0 \le \theta \le 1$ *and* $|Q| \gtrsim 1$*, suppose* r *satisfies:* $r < 4$*,* $\theta r > \frac{4}{3}$ *and* $(1-\theta)r < \frac{8}{3}$*. Then* $E^2 L \gtrsim k$ *and* $(\dot{L})^2 \lesssim E(-\dot{E})$ *imply*

$$\int_0^T E^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r} dt \gtrsim k^{\frac{r(3\theta-1)}{4}} T^{-\frac{r}{4}} \quad \textit{for} \quad T \gg L_0^4 \gg 1 \gg E_0, \tag{3.21}$$

where the constant number in the inequality $k = \frac{1}{|Q|}$.

Proof. The energy E is a monotone function of time due to the differential inequality $(\dot{L})^2 \lesssim E(-\dot{E})$. Indeed, since $\dot{E} \lesssim -\frac{\dot{L}^2}{E} < 0$, E is decreasing. Moreover, using a technique similar to the one used in the proof of Proposition 2.12, we can see that L is an absolutely continuous function of E, and we can write the dissipation inequality as

$$\left(\frac{dL}{de}\right)^2 (\dot{E})^2 \lesssim E|\dot{E}|. \tag{3.22}$$

Here, the lower case e is used for the energy as an independent variable to distinguish it from $E = E(t)$. From (3.22), we have

$$\frac{1}{E}\left(\frac{dL}{de}\right)^2 |\dot{E}| \lesssim 1. \tag{3.23}$$

When $\dot{E} = 0$, (3.23) is still true trivially. Multiplying (3.23) by any function $f(E(t))$ and integrating in time on the interval $[0, T]$ gives

$$\int_0^T f(E(t))dt \gtrsim \int_{E(T)}^{E(0)} \frac{f(e)}{e}\left(\frac{dL}{de}\right)^2 de.$$

Taking $f = e^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r}$ and writing $E_0 = E(0)$, $E_T = E(T)$, we then have

$$\int_0^T E^{\frac{3}{2}\theta r}(t) L^{-\frac{3}{4}(1-\theta)r}(t)dt \gtrsim \int_{E_T}^{E_0} e^{\frac{3}{2}\theta r - 1} L^{-\frac{3}{4}(1-\theta)r} \left(\frac{dL}{de}\right)^2 de \tag{3.24}$$

for all $T > 0$.

Now we estimate the right hand side of (3.24). Consider the change of variables

$$\hat{e} = \frac{1}{2 - \frac{3}{2}\theta r} e^{2-\frac{3}{2}\theta r} \quad \text{and} \quad \hat{L} = \frac{1}{1 - \frac{3(1-\theta)r}{8}} L^{1-\frac{3(1-\theta)r}{8}}.$$

The hypotheses

$$\theta r > \frac{4}{3}, \quad (1-\theta)r < \frac{8}{3}$$

imply

$$\theta > \frac{1}{3} \tag{3.25}$$

which will be used later. Since

$$\left(\frac{dL}{de}\right)^2 de = \left(\frac{d\hat{L}}{d\hat{e}}\right)^2 \left(\frac{dL}{d\hat{L}}\right)^2 \left(\frac{d\hat{e}}{de}\right) d\hat{e}$$

and

$$\frac{d\hat{e}}{de} = e^{1-\frac{3}{2}\theta r}, \quad \frac{d\hat{L}}{dL} = L^{-\frac{3(1-\theta)r}{8}},$$

we can write the right hand side of (3.24) as:

$$\int_{E_T}^{E_0} \left(\frac{de}{d\hat{e}}\right) \left(\frac{d\hat{L}}{dL}\right)^2 \left(\frac{dL}{de}\right)^2 de = \int_{\hat{E_T}}^{\hat{E}_0} \left(\frac{d\hat{L}}{d\hat{e}}\right)^2 d\hat{e}$$

which is bounded below by the minimum over all functions $\hat{L}(\hat{e})$ with boundary conditions

$$\hat{L}(\hat{E_0}) = \frac{1}{1 - \frac{3(1-\theta)r}{8}} (L(0))^{1-\frac{3(1-\theta)r}{8}}, \quad \hat{L}(\hat{E_T}) = \frac{1}{1 - \frac{3(1-\theta)r}{8}} (L(T))^{1-\frac{3(1-\theta)r}{8}}.$$

To simplify the notations we denote these boundary conditions by $\hat{L}_0$ and $\hat{L}_T$, respectively. The extremal $\hat{L}$ is a linear function of $\hat{e}$, so we have:

$$\int_0^T E^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r} dt \gtrsim \frac{(\hat{L}_T - \hat{L}_0)^2}{\hat{E}_0 - \hat{E}_T}. \tag{3.26}$$

When T is such that

$$L(T) \geq 2L(0),$$

the right side of (3.26) can be controlled since

$$\hat{L}_T - \hat{L}_0 \gtrsim \hat{L}_T \quad \text{and} \quad \hat{E}_0 - \hat{E}_T \leq -\hat{E}_T,$$

so

$$\int_0^T E^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r} dt \gtrsim \frac{\hat{L}_T^2}{-\hat{E}_T} = L_T^{2-\frac{3}{4}(1-\theta)r} E_T^{\frac{3}{2}\theta r - 2}.$$

Rewriting the right hand side as

$$L_T^{2-\frac{3}{4}(1-\theta)r} E_T^{\frac{3}{2}\theta r - 2} = [E_T^{\frac{3}{2}\theta} L_T^{-\frac{3}{4}(1-\theta)}]^{r-4} [E_T^2 L_T]^{3\theta - 1},$$

we conclude, using $E^2 L \gtrsim k$ and (3.25), that

$$\int_0^T E^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r} dt \gtrsim [E_T^{\frac{3}{2}\theta} L_T^{-\frac{3}{4}(1-\theta)}]^{r-4} k^{3\theta-1} \tag{3.27}$$

provided $L(T) \geq 2L(0)$. Introducing

$$h(T) := \int_0^T E^{\frac{3}{2}\theta r} L^{-\frac{3}{4}(1-\theta)r} dt,$$

we can write (3.27) as

$$h \gtrsim (h')^{\frac{r-4}{r}} k^{3\theta - 1},$$

so we have:

$$h^{\frac{r}{4-r}}(T)h'(T) \gtrsim k^{\frac{r(3\theta-1)}{4-r}} \tag{3.28}$$

provided $L(T) \geq 2L(0)$. Here we used $r < 4$.

The previous method does not work when $L(T) < 2L(0)$, but, for such T, we have

$$E^2(T) \gtrsim L(T)^{-1} \gtrsim L_0^{-1},$$

which implies

$$E^{\frac{3}{2}\theta}(T)L^{-\frac{3}{4}(1-\theta)}(T) = (E^2(T)L(T))^{\frac{3}{4}\theta}L^{-\frac{3}{4}}(T) \gtrsim k^{\frac{3}{4}\theta}L_0^{-\frac{3}{4}}.$$

Thus

$$h'(T) \gtrsim k^{\frac{3}{4}\theta r}L_0^{-\frac{3}{4}r} \quad \text{if} \quad L(T) < 2L_0. \tag{3.29}$$

Combining (3.28) and (3.29), using $r < 4$, we have:

$$\frac{d}{dt}(h + L_0^{\frac{3(4-r)}{4}})^{\frac{4}{4-r}} \sim (h(t) + L_0^{\frac{3(4-r)}{4}})^{\frac{r}{4-r}}h'(t) \gtrsim k^{\frac{r(4\theta-2)}{4-r}}$$

for all $t > 0$. Indeed, for $L(T) \geq 2L(0)$, using (3.28), we have

$$\frac{d}{dt}(h + L_0^{\frac{3(4-r)}{4}})^{\frac{4}{4-r}} \sim (h(t) + L_0^{\frac{3(4-r)}{4}})^{\frac{r}{4-r}}h'(t) \gtrsim h^{\frac{r}{4-r}}(T)h'(T) \gtrsim k^{\frac{r(3\theta-1)}{4-r}},$$

and for $L(T) < 2L(0)$, by (3.29),

$$\frac{d}{dt}(h + L_0^{\frac{3(4-r)}{4}})^{\frac{4}{4-r}} \sim (h(t) + L_0^{\frac{3(4-r)}{4}})^{\frac{r}{4-r}}h'(t) \gtrsim L_0^{\frac{3r}{4}}h'(T) \gtrsim k^{\frac{3\theta r}{4}}.$$

By assumptions, $\theta r > \dfrac{4}{3}$ and $|Q| \gtrsim 1$, we have

$$k^{\frac{r(3\theta-1)}{4-r}} < k^{\frac{3\theta r}{4}}.$$

Integration in time gives

$$h(T) + L_0^{\frac{3(4-r)}{4}} \gtrsim k^{\frac{r(3\theta-1)}{4}} T^{\frac{4-r}{4}}$$

for all $T > 0$.

Restricting attention to $T^{\frac{4-r}{4}} \gg L_0^{4-r}$, this becomes

$$\int_0^T E^{\theta r} L^{(1-\theta)r} dt = h(T) \gtrsim k^{\frac{r(3\theta-1)}{4}} T^{\frac{4-r}{4}} \quad \text{for} \quad T \gg L_0^4$$

which is precisely (3.21) as we need. □

As stated above, we establish the upper bounds on coarsening rates. Therefore obtaining the slower rates of the decay of the energy corresponds to an improvement. This improvement needs to be considered in conjunction with the dependence on the size of the domain in the coefficient appearing with the power of t in each case. For the length scale $L = ||u||_{H^{-1}(Q)}$, it is $|Q|^{-r/3}$ and, for the length scale $L = \fint W(u)dx$, it is $|Q|^{-r(3\theta-1)/4}$.

Chapter 4

Swift-Hohenberg Equation

We study the coarsening of two-dimensional oblique stripe patterns of the Swift-Hohenberg equation. We expect the models to exhibit isotropic coarsening with a single characteristic length scale with the growth in time governed by a power law. Coarsening in the Swift-Hohenberg equation has been studied numerically, in [7]. Several numerical methods have been used to describe the pattern dynamics of the Swift-Hohenberg equation. Cross and Newell [8] proposed that higher order gradient terms in the phase equation would control the dynamics and suggested a growth rate $t^{1/3}$ in [8]. Then Cross and Hohenberg [6] suggested $t^{1/4}$ as an alternative. Elder, Viñals and Grant obtained numerical results in [15] showing the length scale increasing with time as $t^{1/4}$ when the equation has a noise term (corresponding to a finite temperature thermodynamic system), and they observed a slower growth rate consistent with $t^{1/5}$ without the noise term.

In this section, we establish a one-sided version of this result, that is, an upper bound on the coarsening rate of the Swift-Hohenberg equation, and prove the system will coarsen no faster than a power law. As in the previous sections, we apply Kohn and Otto's method in [20] to a properly chosen length scale and energy. Again, to focus on the coarsening rates, we provide the well-posedness of Swift-Hohenberg equation in Appendix B.

We consider the Swift-Hohenberg equation

$$u_t = -(1 + \nabla^2)^2 u + \mu u - u^3, \tag{4.1}$$

with a variational formulation in terms of the energy functional

$$E(t) = \fint_Q \left(\frac{1}{2}|(1 + \nabla^2)u|^2 + \frac{1}{4}u^4 - \frac{1}{2}\mu u^2 + \frac{1}{4}\mu^2 \right) dx. \tag{4.2}$$

Here the potential function F_μ is given by

$$F_\mu(u) = \frac{1-\mu}{2}u^2 + \frac{1}{4}u^4 + \frac{1}{4}\mu^2,$$

since by (4.2),

$$\begin{aligned} E(t) =& \fint_Q \left(\frac{1}{2}|(1 + \nabla^2)u|^2 + \frac{1}{4}(u^2 - \mu)^2 \right) dx \\ =& \fint_Q \left(\frac{1}{2}|\nabla^2 u|^2 - |\nabla u|^2 + \frac{1}{4}(u^2 - \mu)^2 + \frac{1}{2}u^2 \right) dx. \end{aligned} \tag{4.3}$$

Note that F_μ is normalized so that $F_\mu(0) = \frac{1}{4}\mu^2$. The constant μ is viewed as an eigenvalue parameter. and if $\mu > 0$, the potential function is a double well potential with the minima located at $u = \pm\sqrt{\mu}$.

We also notice that the Swift-Hohenberg equation defines a gradient flow so that

$$u_t = -\frac{\partial E}{\partial u}.$$

As long as the minima of (4.2) are isolated, $u(x,t) \to U(x)$ as $t \to \infty$. Using this variational formulation, it would be possible to predict that stationary solutions are stable by showing that they are minima of (4.2).

We define a correlation length scale by

$$L(t) = \left(\fint_Q u^2 dx \right)^{1/2}. \tag{4.4}$$

We will use the idea of Kohn and Otto's method in [20] to obtain interpolation inequality $EL \gtrsim 1$ and dissipation inequality $\dot{L}^2 \lesssim -\dot{E}$, and then, by the ODE lemma, to find an upper bound on the coarsening rate. Now we state the main result which follows immediately from Lemma 4.5.

Theorem 4.1. *For initial energy E_0 and initial length scale L_0 satisfying $T \gg L_0^3 \gg 1 \gg E_0$, it holds that*

$$\fint_0^T E^2 dt \gtrsim \fint_0^T (t^{-\frac{1}{3}})^2 dt.$$

A more general and stronger statement appears in the next theorem.

Theorem 4.2. *For any $0 \leq \theta \leq 1$, suppose r satisfies $r < 3$, $\theta r > 1$ and $(1-\theta)r < 2$. Then we have*

$$\fint_0^T E^{\theta r} L^{(1-\theta)r} dt \gtrsim \fint_0^T t^{-\frac{r}{3}} dt \quad for \quad T \gg L_0^3 \gg 1 \gg E_0.$$

4.1 Interpolation Inequality

Lemma 4.3 (Interpolation)**.** *For E and L defined in (4.2) and (4.4) respectively, provided there exists $c_0 > 0$ such that $\mu > c_0$, then*

$$EL \gtrsim 1 \quad when \quad E \ll 1,$$

where the constant number implied in the inequality depends on the system size $|Q|$.

Proof. The proof is similar to that of Lemma 1 in [20], however, we need a somewhat different treatment since we have different energy and length scale. For completeness and also for the purpose of tracking the constants, we here reproduce the details.

Writing $\mu = (\mu - u^2) + u^2$, we first focus on the first term in the right hand side and estimate by the first equation in (4.3)

$$\fint (\mu - u^2) dx \leq \left(\fint (\mu - u^2)^2 dx \right)^{1/2} \leq 2E^{1/2}. \tag{4.5}$$

Next, we need to estimate $\fint u^2 dx$ in terms of L and E.

We define

$$W(u) = \int_0^u |\mu - t^2| dt,$$

so that

$$\frac{\partial W}{\partial u} = |\mu - u^2|.$$

By Young's inequality

$$\fint |\nabla(W(u))| dx = \fint |\nabla u| \frac{\partial W}{\partial u} \leq \fint \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{2\epsilon} \left| \frac{\partial W}{\partial u} \right|^2 dx, \tag{4.6}$$

for any $\epsilon > 0$. By the Poincaré inequality and using the boundary condition, we have

$$\begin{aligned}
\fint |\nabla u|^2 dx &= \fint \Delta u \cdot u dx \\
&\leq \frac{1}{2\delta} \fint |\Delta u|^2 dx + \frac{\delta}{2} \fint u^2 dx \\
&\leq \frac{1}{2\delta} \fint |\Delta u|^2 dx + \frac{\delta C|Q|}{2} \fint \nabla u^2 dx,
\end{aligned}$$

choosing $\delta = \dfrac{1}{C|Q|}$,

$$\fint |\nabla u|^2 dx \leq C|Q| \fint |\Delta u|^2 dx. \tag{4.7}$$

Furthermore,

$$\begin{aligned}
\fint |\nabla u|^2 dx &= \frac{2C|Q|}{1 - 2C|Q|} \left(\fint \frac{1}{2} \frac{1}{C|Q|} |\nabla u|^2 dx - \fint |\nabla u|^2 dx \right) \\
&\leq \frac{2C|Q|}{1 - 2C|Q|} \left(\fint \frac{1}{2} |\Delta u|^2 dx - \fint |\nabla u|^2 dx \right) \\
&= C^* \fint \frac{1}{2} |\Delta u|^2 - |\nabla u|^2 dx,
\end{aligned}$$

where $C^* = \dfrac{2C|Q|}{1 - 2C|Q|}$ is close to 1 when $|Q|$ is large.

Then (4.6) becomes

$$\fint |\nabla(W(u))|dx \leq \fint \frac{\epsilon C^*}{2}\left(\frac{1}{2}|\Delta u|^2 - |\nabla u|^2\right) + \frac{1}{2\epsilon}(u^2-\mu)^2 dx,$$

by choosing $\epsilon = \frac{2}{\sqrt{C^*}}$, we have

$$\begin{aligned}\fint |\nabla(W(u))|dx &\leq \sqrt{C^*}\fint \frac{1}{2}|\Delta u|^2 - |\nabla u|^2 + \frac{1}{4}(u^2-\mu)^2 dx \\ &\leq \sqrt{C^*}\fint \frac{1}{2}|\Delta u|^2 - |\nabla u|^2 + \frac{1}{4}(u^2-\mu)^2 + \frac{1}{2}u^2 dx \\ &= \sqrt{C^*}E.\end{aligned}$$

We will use a smooth mollifier η which is radically symmetric, non-negative and supported in the unit ball with $\int_{\mathbb{R}^2}\eta = 1$. Let the subscript δ denote the convolution with the kernel

$$\eta_\delta(\cdot) = \frac{1}{\delta^2}\eta\left(\frac{\cdot}{\delta}\right).$$

The parameter δ will be determined later. Now we split the L^2-norm $\fint u^2 dx$ into two parts:

$$\fint u^2 dx \leq 2\fint (u-u_\delta)^2 dx + 2\fint u_\delta^2 dx. \tag{4.8}$$

Note that

$$|u_1 - u_2|^2 \leq 8|W(u_1) - W(u_2)|$$

for all u_1 and u_2, therefore we obtain the following estimate

$$\begin{aligned}\fint (u-u_\delta)^2 dx &\leq \sup_{|h|\leq\delta}\fint (u(x)-u(x+h))^2 dx \\ &\leq 8\sup_{|h|\leq\delta}\fint |W(u(x)) - W(u(x+h))|dx \\ &\leq 8\delta\fint |\nabla(W(u))|dx \\ &\leq 8\sqrt{2C^*}\delta E.\end{aligned}$$

For the second term of (4.8), we separately deal with $|u_\delta|$ being either small or large:

$$\fint u_\delta^2 = \fint u_\delta^2 - \min\{u_\delta^2, \mu^2\} dx + \fint \min\{u_\delta^2, \mu^2\} dx. \tag{4.9}$$

Since $F(u) = u^2 - \min\{u^2, \mu^2\}$ is a convex function in u, by a version of Jensen's inequality and the fact that $\int \eta(x)dx = 1$, we have

$$F(u_\delta(x)) = F\left(\int \eta(y)u(x-\delta y)dy\right) \leq \int \eta(y)F(u(x-\delta y))dy.$$

The first term of (4.9) becomes

$$\begin{aligned}
\fint u_\delta^2 - \min\{u_\delta^2, \mu^2\} &\leq \fint \int \eta(y)F(u(x-\delta y))dydx \\
&= \int \eta(y) \fint [u^2(x-\delta y) - \min\{u^2(x-\delta y), \mu^2\}]dxdy \\
&= \fint u^2(x) - \min\{u^2(x), \mu^2\}dx \\
&\leq \frac{1}{4}(\mu - u^2)^2 dx \\
&\leq E.
\end{aligned}$$

For the second term of (4.9), we have

$$\fint \min\{u_\delta^2, \mu^2\}dx \leq \mu \fint |u_\delta| dx.$$

By duality,

$$\fint |u_\delta| dx = \sup\{\fint u_\delta(x)\xi(x)dx : \xi \text{ is } Q\text{−periodic and } |\xi(x)| \leq 1 \text{ a.e.}\}.$$

Consider ξ that is $Q-$periodic with $|\xi(x)| \leq 1$ a.e., and write

$$\xi_\delta(x) = \int \frac{1}{\delta^2}\eta\left(\frac{x-y}{\delta}\right)\xi(y)dy.$$

Hence,

$$\nabla \xi_\delta(x) = \frac{1}{\delta} \int \frac{1}{\delta^2} \nabla \eta \left(\frac{x-y}{\delta} \right) \xi(y) dy = \frac{1}{\delta} \int \nabla \eta(y) \xi(x - \delta y) dy,$$

and thus

$$\sup |\nabla \xi_\delta| \leq \kappa \frac{1}{\delta} \sup |\xi| \leq \frac{\kappa}{\delta},$$

where $\kappa = \int |\nabla \eta| dx$. Therefore,

$$\begin{aligned} \fint \nabla u_\delta(x) \xi(x) dx &= \fint \nabla u(x) \xi_\delta(x) dx \\ &= \fint -u(x) \nabla \xi_\delta(x) dx \\ &\leq \left(\fint u^2 dx \right)^{1/2} \left(\fint |\nabla \xi_\delta|^2 \right)^{1/2} \\ &\leq \frac{\kappa}{\delta} L. \end{aligned}$$

Therefore, taking the superemum over all ξ, we get

$$\fint |u_\delta| dx \leq C|Q|^{1/2} \fint |\nabla u_\delta| dx \leq C|Q|^{1/2} \frac{\kappa}{\delta} L.$$

Combining all the above estimates, we have

$$\fint u^2 dx \leq 16\sqrt{2C^*} \delta E + 2E + 2\mu C|Q|^{1/2} \frac{\kappa}{\delta} L.$$

Taking $\delta = \left(\frac{C|Q|^{1/2} \mu \kappa L}{8\sqrt{2C^*} E} \right)^{1/2}$ to minimize the right hand side over δ, we get

$$\fint u^2 dx \leq \tilde{C}(EL)^{1/2} + 2E,$$

where $\tilde{C} = 8(2C|Q|^{1/2} \mu \kappa \sqrt{2C^*})^{1/2}$. Combining with estimate (4.5), we obtain

$$\mu \leq \tilde{C}(EL)^{1/2} + 2E + 2E^{1/2},$$

which yields Lemma 4.3 for $E \ll 1$. $\square$

4.2 Dissipation Inequality and Upper Bounds on the Coarsening Rates

Next lemma relates the rate at which L can change when energy is dissipated to the rate of change of the energy. We find a dissipation relation that bounds the growth rate in terms of a suitable measure of length scale.

Lemma 4.4 (Dissipation)**.** *Suppose u is a solution of (1.3) and again let E and L be defined as in (4.2) and (4.4), respectively. Then*

$$(\dot{L})^2 \leq -\dot{E}.$$

Proof. From (4.3), we have

$$\begin{aligned}
\dot{E} &= \fint \nabla^2 u \cdot \nabla^2 u_t - 2\nabla u \cdot \nabla u_t + (u^2 - \mu + 1)(u \cdot u_t) dx \\
&= \fint \nabla^4 u \cdot u_t + 2\nabla^2 u \cdot u_t + u(u^2 - \mu + 1) u_t dx \\
&= \fint \left(\nabla^4 u + 2\nabla^2 u + u(u^2 - \mu + 1)\right) u_t dx \\
&= -\fint u_t^2 dx.
\end{aligned}$$

Since

$$L^2 = \fint |u|^2 dx,$$

then

$$\begin{aligned}
2L\dot{L} &= 2\fint u u_t dx \\
&\leq 2 \left(\fint |u|^2 dx\right)^{1/2} \left(\fint |u_t|^2 dx\right)^{1/2} \\
&= 2L(-\dot{E})^{1/2}
\end{aligned}$$

so that

$$|\dot{L}|^2 \leq -\dot{E}$$

□

Hence, we have Lemma 4.5. Since the proof of this lemma will be very similar with the proof of Lemma 2.13, we omit the details.

Lemma 4.5 (ODE)**.** *For any* $0 \leq \theta \leq 1$, *suppose* r *satisfies:* $r < 3$, $\theta r > 1$ *and* $(1-\theta)r < 2$. *Then* $EL \gtrsim 1$ *and* $(\dot{L})^2 \leq (-\dot{E})$ *imply*

$$\fint_0^T E^{\theta r} L^{(1-\theta)r} dt \gtrsim T^{-\frac{r}{3}} \quad for \quad T \gg L_0^3 \gg 1 \gg E_0, \tag{4.10}$$

where the constant number implied in the above inequality depends on the system size $|Q|$.

This lemma has a proof similar to that of Lemma 2.13 but with different dependence on the size of the domain. We apply this lemma with the interpolation inequality depending on the size of the domain instead of the dissipation inequality. By calculations similar to Lemma 2.13, we have $K = \tilde{C}^{\frac{2\theta r(3-\theta)}{3}}$, where $\tilde{C} = 8(2C|Q|^{1/2}\mu\kappa\sqrt{2C^*})^{1/2}$ in Lemma 4.3.

Chapter 5

Discussion

Our accomplishment is the time-averaged lower bounds for energy E which corresponds to the time-averaged upper bounds on the coarsening rates for the Allen-Cahn equation and the Swift-Hohenberg equations. The lower bounds on the coarsening rates would depend on the geometry of the domain and cannot be established using the technique that we employed. It will be nice to find pointwise-in-time bounds for both the energy E and the length scale L, but it will require some new ideas.

We can see that the equations to which Kohn and Otto's method has been applied previously, including the Cahn-Hilliard equations, epitaxial growth model, phase-field model, discrete ill-posed diffusion equations, and many other models we did not list in this dissertation, all have conservation law structures in the equations. On the other hand, both of the equations that we have studied, namely, the Allen-Cahn equation and the Swift-Hohenberg equation, are non-conserving. In general, the equations with a conservation law structure can be written as

$$\frac{du}{dt} = -\nabla \cdot J.$$

This structure provides certain advantages. For example, when proving the dissipation inequality for the Cahn-Hilliard equation, this property can be used for integration by parts to have

$$\int_{t_1}^{t_2} \fint \frac{du}{dt} \zeta dxdt = \int_{t_1}^{t_2} \fint J \cdot \nabla \zeta dxdt \leq \int_{t_1}^{t_2} \fint |J| dxdt, \quad \text{where } \zeta \text{ is periodic and } \sup |\nabla \zeta| \leq 1.$$

This seems to be an important step in the proof. The same technique is also used in the proof of the dissipation inequality for discrete, ill-posed diffusion equations,

$$\langle \frac{du}{dt}, s\rangle_{H^{-1}} = -\langle -\Delta R(u), s\rangle_{H^{-1}} = -R(u)\cdot s = -\nabla E(u)\cdot s.$$

On the other hand, if the equation has a conservation law structure, then the mean value $\bar{u} = 0$ can be naturally maintained. This property is used in the proofs of interpolation inequality for both epitaxial growth model and phase-field model. However, since the equations that we are looking at have no natural conservation law structure, we have to find some other methods to approach the results that we need. In the proof of the interpolation inequality of the Allen-Cahn equation, we followed the frameworks of Kohn and Yan's method, but the details have been proved by different methods. For example, to obtain the lower bound on the energy density, we used the idea that the solution to an elliptic the Allen-Cahn equation will give the minimum value of energy density which has a lower bound by regularity. We also adjust the interpolation inequality to be $E^2L \gtrsim 1$ for the two-dimensional Allen-Cahn equation with a length scale

$$L = \fint W(u)dx,$$

with

$$W(u) = \int_0^u |1-s^2|ds.$$

This different interpolation inequality, combined with the dissipation inequality, improves the coarsening rates and the upper bound for the energy decay. But we will be very interested in looking for some universal and geometry-independent upper bounds on coarsening rates for non-conserving equations. This may require some new techniques.

In this thesis, only two equations which are non-conserving were selected to study the upper bounds on coarsening rates, but there are many energy-driven dynamic systems that describe coarsening models in material science. For example, a natural extension would be to look at a more general equation modeling epitaxial thin film growth in [22],

$$\begin{aligned} u_t + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g, &\quad \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, &\quad \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0, &\quad \text{in } Q \end{aligned}$$

where n denotes the outward pointing unit normal to the boundary $\partial\Omega$ of the domain Ω.

While the energy associated with this equation appears in the introduction, a different choice of length scale alternative to the L^2-norm of u employed in [24] may have the potential to yield more precise upper bounds on the coarsening rate. For example, the Hessian may be a candidate for such a choice of length scale. While the general framework of Kohn and Otto [20] may be followed, the implementation would require different techniques from what we have used.

Appendix A

Well-posedness of Allen-Cahn Equation

Here we prove the existence and uniqueness of the weak and strong solutions of the Allen-Cahn equation in a bounded domain $\Omega \in \mathbb{R}^n$, where $n = 1, 2$. The proofs appear in [34] and cover the case of a more general non-linear term. For the completeness of the presentation, we adapt the theorems from [34] to apply specifically to our models.

We consider the parabolic Allen-Cahn equation on domain $Q \times [0, \infty)$

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - 2u(1-u^2) = 0, &\quad \text{in } Q \times [0,\infty) \\ u(x,t) = 0, &\quad \text{on } \partial Q \times [0,\infty) \\ u(x,0) = u_0(x), &\quad \text{in } Q \end{aligned} \tag{A.1}$$

where Q is the square in $\mathbb{R}^2$ with side length l.

Define $W(u) = \dfrac{1}{2}(1-u^2)^2$, and let

$$w(u) = -W'(u) = 2u(1-u^2).$$

Notice that $w(s)$ is a C^1 function which satisfies the bounds

$$-1 - 2|s|^4 \leq w(s)s \leq 1 - |s|^4, \tag{A.2}$$

and

$$w'(s) \leq 2, \tag{A.3}$$

both for all $s \in \mathbb{R}$. Indeed, $-1 - 2s^4 \leq w(s)s = 2s^2 - 2s^4 \leq 1 - s^4$ and $w'(s) = 2 - 6s^2 \leq 2$.

We proceed to define the sense in which equation (A.1) holds. We say $u \in L^2(0,T;H_0^1(Q))$ with $\frac{du}{dt} \in L^{4/3}(0,T;H^{-1}(Q))$ is a *weak* solution of (A.1) if for any $v \in L^4(0,T;H_0^1(Q))$, we have

$$\langle \frac{du}{dt}, v\rangle + a(u,v) = \langle w(u), v\rangle \tag{A.4}$$

for almost every $t \in [0,T]$, where the bilinear form of $a(\cdot,\cdot)$ is defined as

$$a(u,v) = \langle \Delta u, v\rangle = \sum_{j=1}^{m} (D_j u, D_j v),$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(Q)$ and we use $\langle v^*, u\rangle$ to denote the pairing between an element $v^* \in H^{-1}(Q)$ and an element $u \in H_0^1(Q)$, that is, there is an element $v \in H_0^1(Q)$ such that

$$\langle v^*, u\rangle = (v,u)_{H_0^1(Q)}.$$

A.1 Preliminary Results

In what follows, we write L^2 for $L^2(Q)$, H_0^1 for $H_0^1(Q)$, H^{-1} for $H^{-1}(Q)$, etc. To prove the existence of solutions, the technique is essentially to construct a sequence that is weakly convergent and show that the limit is a solution. We will need some strong convergence of u_n and a weak version of the dominated convergence theorem [34].

Before proving the compactness theorem, we will need the following lemma, which is a special case of Ehrling's lemma.

Lemma A.1. *For each $\epsilon > 0$ there exists a constant c_ϵ such that*

$$||u||_{L^2} \le \epsilon ||u||_{H_0^1} + c_\epsilon ||u||_{H^{-1}}, \quad \textit{for all } u \in H_0^1.$$

Proof. We prove this by contradiction, so suppose there exists $\epsilon > 0$ such that for each $n \in \mathbb{Z}^+$ there is a u_n with

$$||u_n||_{L^2} \geq \epsilon ||u_n||_{H_0^1} + n||u_n||_{H^{-1}}.$$

Let $v_n = \dfrac{u_n}{||u_n||_{H_0^1}}$, then

$$||v_n||_{L^2} \geq \epsilon + n||v_n||_{H^{-1}}. \tag{A.5}$$

Since $H_0^1 \subset\subset L^2$ and v_n is bounded in H_0^1 with norm 1, v_n is also bounded in L^2. By (A.5), it follows that $||v_n||_{H^{-1}} \to 0$ as $n \to \infty$. However, $H_0^1 \subset\subset L^2$ and $v_n \to v$ in L^2 imply that $v = 0$ which contradicts (A.5). □

We can now prove the compactness theorem.

Theorem A.2. *Suppose u_n is a sequence that is uniformly bounded in $L^2(0,T;H_0^1)$, and $\dfrac{du_n}{dt}$ is uniformly bounded in $L^{4/3}(0,T;H^{-1})$. Then there is a subsequence that converges strongly in $L^2(0,T;L^2)$.*

Proof. Since H_0^1 is reflexive, so is $L^2(0,T;H_0^1)$. Since u_n is bounded in $L^2(0,T;H_0^1)$, using Alaoglu compactness theorem, there is a subsequence u_n such that

$$u_n \rightharpoonup u \quad \text{in} \quad L^2(0,T;H_0^1).$$

Next, we show that

$$v_n = u_n - u \to 0 \quad \text{in} \quad L^2(0,T;L^2).$$

To this end, we first establish that $v_n \to 0$ in $L^2(0,T;H^{-1})$ which is sufficient to guarantee that $v_n \to 0$ in $L^2(0,T;L^2)$.

Indeed, Lemma A.1 shows that for each $\epsilon > 0$ there exists a c_ϵ such that

$$||v_n||^2_{L^2(0,T;L^2)} \leq \epsilon ||v_n||^2_{L^2(0,T;H_0^1)} + c_\epsilon ||v_n||^2_{L^2(0,T;H^{-1})},$$

and since v_n is bounded in $L^2(0,T;H_0^1)$,

$$||v_n||^2_{L^2(0,T;L^2)} \leq \epsilon C + c_\epsilon ||v_n||^2_{L^2(0,T;H^{-1})}.$$

If $v_n \to 0$ in $L^2(0,T;H^{-1})$, then

$$\limsup_{n\to\infty} ||v_n||^2_{L^2(0,T;L^2)} \leq \epsilon C$$

for any $\epsilon > 0$. Hence

$$\lim_{n\to\infty} ||v_n||^2_{L^2(0,T;L^2)} = 0.$$

To prove that $v_n \to 0$ in $L^2(0,T;H^{-1})$, we observe that for $v_n \in H^1(0,T;H^{-1})$, we have $v_n \in C([0,T];H^{-1})$ by Theorem 5.9.2 in [14], and

$$\begin{aligned} \max_{0\leq t\leq T} ||v_n||_{H^{-1}} &\leq C||v_n||_{H^1(0,T;H^{-1})} \\ &\leq C(||v_n||_{L^2(0,T;H_0^1)} + ||\frac{dv_n}{dt}||_{L^2(0,T;H^{-1})}) \leq M. \end{aligned} \tag{A.6}$$

Denoting $\dot{v} = \dfrac{d}{dt}v$, we integrate equation

$$v_n(t) = v_n(w) - \int_t^w \dot{v}_n(\tau)d\tau,$$

with respect to w from t to $t+s$ to get

$$\begin{aligned} v_n(t) &= \frac{1}{s}\left(\int_t^{t+s} v_n(w)dw - \int_t^{t+s}\int_t^w \dot{v}_n(\tau)d\tau dw\right) \\ &= A_n + B_n, \end{aligned}$$

where

$$A_n = \frac{1}{s}\int_t^{t+s} v_n(w)dw, \quad B_n = -\int_t^{t+s}\int_t^w \dot{v}_n(\tau)d\tau dw.$$

Now take $\xi > 0$ and estimate

$$\begin{aligned}||B_n||_{H^{-1}} &\leq \int_s^{t+s} ||\frac{dv_n}{dt}||_{H^{-1}} dw \\ &\leq s^{1/4} \left(\int_t^{t+s} ||\frac{dv_n}{dt}||_{H^{-1}}^{4/3} dw \right)^{3/4} \\ &\leq s^{1/4} ||\frac{dv_n}{dt}||_{L^{4/3}(0,T;H^{-1})}.\end{aligned}$$

We choose s such that

$$||B_n||_{H^{-1}} \leq \frac{\xi}{2}. \tag{A.7}$$

For this value of s, notice that

$$\int_t^{t+s} v_n(w)dw \rightharpoonup 0 \quad \text{in} \quad H_0^1. \tag{A.8}$$

Indeed, if χ is the indicator function of $[t, t+s]$ and $\phi \in H^{-1}$, then $\chi\phi$ is an element of $L^2(0,T;H^{-1})$ and

$$\int_0^T \langle v_n(t), \chi\phi \rangle dt = \int_t^{t+s} \langle v_n(t), \phi \rangle dt = \langle \int_t^{t+s} v_n(t)dt, \phi \rangle.$$

Since $v_n \rightharpoonup 0$ in $L^2(0,T;H_0^1)$, then (A.8) follows. Hence $A_n \rightharpoonup 0$ in H_0^1, then $A_n \to 0$ in H^{-1}. Therefore, for n large enough we have

$$||A_n||_{H^{-1}} \leq \frac{\xi}{2}$$

which together with (A.7) gives

$$||v_n||_{H^{-1}} \leq \xi.$$

Since $v_n(t) \to 0$ in H^{-1} and $v_n(t)$ is bounded in H^{-1} for almost every $t \in [0,T]$ by (A.6). Lebesgue's dominated convergence theorem gives $v_n \to 0$ in $L^2(0,T;H^{-1})$ and thus completes the proof. □

Next we will prove a weak version of the dominated convergence theorem stating that if a sequence $\{u_j\}$ is bounded in L^p and converges pointwise, then $u_j \rightharpoonup g$ in L^p. Although this lemma will be used to prove the existence of solutions to the Allen-Cahn equation with the case $p = \frac{4}{3}$, we prove a more general version for any $p > 1$.

Lemma A.3. *Let Ω be a bounded open set in $\mathbb{R}^m$ and let u_j be a sequence of functions in $L^p(\Omega)$ with*

$$||u_j||_{L^p(\Omega)} \leq C \quad \textit{for all} \quad j \in \mathbb{Z}^+.$$

If $u \in L^p(\Omega)$ and $u_j \to u$ almost everywhere then $u_j \rightharpoonup u$ in $L^p(\Omega)$.

Proof. Let

$$E_n = \{x | x \in \Omega, |u_j(x) - u(x)| \leq 1 \text{ for all } j \geq n\}.$$

These sets E_n increase with n and the measure of E_n increases to the measure of Ω as $n \to \infty$ since $u_j \to u$ almost everywhere.

Let Φ_n be the set of functions in $L^q(\Omega)$, where q is the Hölder conjugate of p, with support in E_n. Let $\Phi = \bigcup_{n=1}^{\infty} \Phi_n$. We can see that Φ is dense in E_n. For $\phi \in L^q(\Omega)$, take $\phi_n = \chi[E_n]\phi$, where $\chi[E]$ is the characteristic function of E. Then, since $\phi_n \to \phi$ almost everywhere and $|\phi_n(x)| \leq |\phi(x)|$, Lebesgue's dominated convergence theorem gives $\phi_n \to \phi$ in $L^q(\Omega)$.

If we take $\phi \in \Phi$, then $\phi \in \Phi_{n_0}$ for some n_0 and, for $j \geq n_0$, we have

$$|\phi(x)(u_j(x) - u(x))| \leq |\phi(x)|.$$

Lebesgue's dominated convergence theorem yields

$$\int_\Omega \phi(u_j - u)dx \to 0 \quad \text{as} \quad j \to \infty.$$

By the density of Φ in $L^q(\Omega)$, for $v \in L^q(\Omega)$, given $\delta > 0$, choose $\phi \in \Phi$ such that

$$||v - \phi||_{L^q(\Omega)} < \frac{\delta}{4C}$$

and N such that

$$\int_\Omega \phi(u_j - u)dx < \frac{\delta}{2} \quad \text{for all} \quad j \geq N.$$

Then

$$\int_\Omega (u_j - u)(v - \phi + \phi)dx < 2C(\frac{\delta}{4C}) + \frac{\delta}{2} = \delta,$$

which shows that $u_j \rightharpoonup u$ in $L^p(\Omega)$. □

A.2 Existence and Uniqueness of Weak Solution

We will obtain a solution by using the basis $\{w_j\}$ of eigenfunctions of the Laplacian to approximate the equation by systems of ODEs. We prove the existence and uniqueness of the approximations of (A.1) using the corresponding results for ODEs. Then the Alaoglu compactness theorem will guarantee the existence of a weak limit. Finally, the limit is shown to satisfy (A.1).

Theorem A.4. *Equation (A.1) has a unique weak solution given by (A.4): for any $T > 0$ given $u(0) = u_0 \in L^2(Q)$ there exists a solution u with*

$$u \in L^2(0, T; H_0^1(Q)) \cap L^4(Q \times (0, T)), \quad u \in C^0([0, T]; L^2(Q)),$$

and $u_0 \mapsto u(t)$ is continuous on $L^2(Q)$. Equation (A.1) holds as an equality in the space $L^{4/3}(0, T; H^{-1}(Q))$.

Proof. Let the functions $w_k = w_k(x)$, $k = 1, 2, \cdots$ be eigenfunctions of $-\Delta$. It is known that $\{w_k\}_{k=1}^\infty$ are smooth and form an orthogonal basis of $H_0^1(Q)$. For a fixed $n > 0$, define

P_n as the orthogonal projection in L^2 onto the span of $\{w_k\}_{k=1}^n$:

$$P_n u = \sum_{j=1}^{n} (u, w_j) w_j.$$

We will look for the $n-$dimensional approximation

$$u_n(t) = \sum_{j=1}^{n} u_{nj}(t) w_j,$$

of the solution u, with the coefficients $u_{nj}(t)$ $(0 \leq t \leq T, 1 \leq j \leq n)$ that satisfy

$$u_{nj}(0) = (u_n(0), w_j) = (u_0, w_j)$$

and

$$(\frac{du_n}{dt}, w_j) + (-\Delta u_n, w_j) = (w(u_n), w_j), \quad j = 1, \ldots, n$$

that is,

$$\frac{du_{nj}}{dt} - \Delta u_{nj} = (w(u_n), w_j).$$

We could also write this in a vector form as

$$\frac{du_n}{dt} - \Delta u_n = P_n w(u_n), \quad u_n(0) = P_n u_0. \tag{A.9}$$

Since the nonlinearity in (A.9) is locally Lipschitz, then the finite-dimensional system has a unique solution on some finite time interval.

Multiplying (A.9) by u_n and integrating over Q, we get

$$\frac{1}{2}\frac{d}{dt}||u_n||^2_{L^2(Q)} + (-\Delta u_n, u_n) = (P_n w(u_n), u_n). \tag{A.10}$$

Since

$$(-\Delta u_n, u_n) = ||\nabla u_n||^2_{L^2(Q)},$$

and $||u_n||^2_{H^1_0(Q)}$ is equivalent to $||\nabla u_n||^2_{L^2(Q)}$, we have

$$(P_n w(u_n), u_n) = (w(u_n), P_n u_n) = (w(u_n), u_n) \leq \int_Q 1 - |u_n|^4 dx,$$

then (A.10) becomes

$$\frac{1}{2}\frac{d}{dt}||u_n||^2_{L^2(Q)} + ||u_n||^2_{H^1_0(Q)} + \int_Q |u_n|^4 dx \leq |Q|.$$

Integrating both sides in time from 0 to T gives

$$\frac{1}{2}||u_n(T)||^2_{L^2(Q)} + ||u_n||^2_{L^2(0,T;H^1_0(Q))} + ||u_n||^4_{L^4(Q\times(0,T))} \leq \frac{1}{2}||u_0||^2_{L^2(Q)} + T|Q|.$$

With the time interval being finite, the domain being bounded, and $u_0 \in L^2(Q)$, we see that

$$\begin{aligned} &u_n \text{ is uniformly bounded in } L^\infty(0,T;L^2(Q));\\ &u_n \text{ is uniformly bounded in } L^2(0,T;H^1_0(Q));\\ &u_n \text{ is uniformly bounded in } L^4(Q\times(0,T)). \end{aligned} \tag{A.11}$$

Since $|w(s)| \leq \beta(1+|s|^3)$,

$$\begin{aligned} ||w(u_n)||^{4/3}_{L^{4/3}(Q\times(0,T))} =& \int_0^T \left(\int_Q |w(u_n)|^{4/3} dx\right) dt\\ \leq& \beta^{4/3}\int_0^T \left(\int_Q (1+|u_n|^3)^{4/3}\right) dt\\ \leq& \int_0^T C\left(\int_Q |u_n|^4 + 1 dx\right) dt. \end{aligned}$$

We have $w(u_n)$ is uniformly bounded in $L^{4/3}(Q\times(0,T))$.

Next, note also that, by the Sobolev embedding theorem,

$$H^1(Q) \subset L^4(Q),$$

that is, $v \in H_0^1(Q)$ implies that $v \in L^4(Q)$. Then, it follows that, if $u \in L^{4/3}(Q)$, the dual of $L^4(Q)$, then $u \in H^{-1}(Q)$, the dual of $H_0^1(Q)$, so that

$$L^{4/3}(Q) \subset H^{-1}(Q).$$

Therefore, $L^2(0,T;H^{-1}(Q))$ and $L^{4/3}(0,T;L^{4/3}(Q))$ are continuously embedded in the space $L^{4/3}(0,T;H^{-1}(Q))$. It follows by (A.9) that we have

$$\frac{du_n}{dt} \text{ is uniformly bounded in } L^{4/3}(0,T;H^{-1}(Q)). \tag{A.12}$$

By Banach-Alaoglu weak-* compactness theorem, we can extract a convergent subsequence $\{u_n\}$ converging weakly in the following spaces

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(Q)); \\ u_n &\rightharpoonup u \text{ in } L^4(Q \times (0,T)); \\ w(u_n) &\rightharpoonup \chi \ \text{ in } L^{4/3}(Q \times (0,T)), \end{aligned} \tag{A.13}$$

for some $\chi \in L^{4/3}(Q \times (0,T))$.

Furthermore, since u_n is uniformly bounded in $L^2(0,T;H_0^1(Q))$ by (A.11), and $\dfrac{du_n}{dt}$ is uniformly bounded in $L^{4/3}(0,T;H^{-1}(Q))$ by (A.12), and the following chain of embeddings holds $H_0^1(Q) \subset\subset L^2(Q) \subset H^{-1}(Q)$ with $H_0^1(Q)$ being reflexive, by Theorem A.2, we can extract a further subsequence such that

$$u_n \to u \text{ in } L^2(0,T;L^2(Q)).$$

Actually, we need $P_n w(u_n) \rightharpoonup \chi$ in $L^{4/3}(Q \times (0,T))$. Therefore we write

$$\begin{aligned}&\int_{Q\times(0,T)} (P_n w(u_n) - \chi)\varphi dxdt \\ &= \int_{Q\times(0,T)} (w(u_n) - \chi)\varphi dxdt - \int_{Q\times(0,T)} Q_n w(u_n)\varphi dxdt,\end{aligned}$$

for all $\varphi \in L^4(Q \times (0,T))$, where $Q_n = I - P_n$. The first terms tends to zero by (A.13). For the second term, let

$$\varphi = \sum_{j=1}^{n} \alpha_j(t)\varphi_j$$

with $\alpha_j \in L^4(0,T)$ and $\varphi_j \in C_c^\infty(Q)$. Such functions φ are dense in $L^4(Q \times (0,T))$ and the following identity holds for them

$$\int_{Q\times(0,T)} Q_n w(u_n) \left(\sum_{j=1}^{n} \alpha_j(t)\varphi_j\right) dxdt = \int_{Q\times(0,T)} w(u_n) \left(\sum_{j=1}^{n} \alpha_j(t) Q_n \varphi_j\right) dxdt.$$

Since $P_n u \to u$ in $L^4(Q)$ for $u \in L^4(Q)$, we have $Q_n u \to 0$ in $L^4(Q)$, that is, $Q_n \varphi_j \to 0$ in $L^4(Q)$ for each j. Hence we have the convergence $P_n w(u_n) \rightharpoonup \chi$ in $L^{4/3}(Q \times (0,T))$ as required.

It follows that every term in (A.9) converges in the weak-$*$ topology of the dual space of $V = L^2(0,T;H_0^1(Q)) \cap L^4(Q \times (0,T))$ which is $V^* = L^2(0,T;H^{-1}(Q)) + L^{4/3}(Q \times (0,T))$. Then

$$\frac{du}{dt} - \Delta u = \chi$$

holds in $L^2(0,T;H^{-1}(Q)) + L^{4/3}(Q \times (0,T)) \subset L^{4/3}(0,T;H^{-1})$.

It remains to show that $\chi = w(u)$. Since $u_n \to u$ in $L^2(Q\times(0,T))$, there is a subsequence u_{n_j} such that $u_{n_j}(x,t) \to u(x,t)$ for a.e. $(x,t) \in Q \times (0,T)$. It follows, using the continuity of w, that $w(u_{n_j}(x,t)) \to w(u(x,t))$ for a.e $(x,t) \in Q \times (0,T)$. With the uniform bound on $w(u_{n_j}) \in L^{4/3}(Q \times (0,T))$, we deduce that $w(u_{n_j}) \rightharpoonup w(u)$ in $L^{4/3}(Q \times (0,T))$ by Lemma A.3. By the uniqueness of the limit, $w(u) = \chi$.

To prove the continuity of $u(t)$ from $[0,T]$ into $L^2(Q)$, notice that $u \in L^2(0,T;H_0^1(Q)) \cap L^4(Q \times (0,T))$ and that

$$\frac{du}{dt} = \Delta u + w(u) \in L^2(0,T;H^{-1}(Q)) + L^{4/3}(Q \times (0,T)).$$

By extending u outside $[0,T]$ by zero and setting $u_m = (u)_{\frac{1}{m}}$, a mollified version of u with respect to variable t, we can approximate u by a sequence $u_m \in C^1([0,T];H_0^1(Q))$ which converges to u in the sense that

$$u_m \to u \text{ in } L^2(0,T;H^1(Q));$$
$$\frac{du_m}{dt} \to \frac{du}{dt} \text{ in } L^2(0,T;H^{-1}(Q)).$$

Then for any t_0,

$$||u_m(t)||^2_{L^2(Q)} = ||u_m(t_0)||^2_{L^2(Q)} + 2\int_{t_0}^{t} \langle \frac{d}{ds}u_m(s), u_m(s)\rangle ds.$$

Choosing t_0 such that

$$||u_m(t_0)||^2_{L^2(Q)} = \frac{1}{T}\int_0^T ||u_m(t)||^2_{L^2(Q)} dt,$$

we estimate

$$|\langle \frac{d}{dt}u_m(s), u_m(s)\rangle| \le ||\frac{du_m(s)}{dt}||_{H^{-1}(Q)}||u_m(s)||_{H_0^1(Q)},$$

to obtain,

$$||u_m(t)||^2_{L^2(Q)} \le \frac{1}{T}\int_0^T ||u_m(t)||^2_{L^2(Q)} dt + 2\int_0^T ||\frac{du_m}{dt}||_{H^{-1}(Q)}||u_m||_{H_0^1(Q)} dt.$$

Hence,

$$\sup_{t\in[0,T]} ||u_m(t)||^2_{L^2(Q)} \le C(||u_m||_{L^2(0,T;H_0^1(Q))} + ||\frac{du_m}{dt}||_{L^2(0,T;H^{-1}(Q))}),$$

where the constant C depends on T.

Since u_m is a Cauchy sequence in $L^2(0,T;H_0^1(Q))$ and $\frac{du_m}{dt}$ is a Cauchy sequence in the space $L^2(0,T;H^{-1}(Q))$, it follows that u_m is a Cauchy sequence in $C^0([0,T];L^2(Q))$ and therefore $u \in C^0([0,T];L^2(Q))$.

Next we need to show that $u(0) = u_0$. Choose some $\varphi \in C^1([0,T];H_0^1(Q) \cap L^4(Q))$ with $\varphi(T) = 0$. We note that, in particular, $\varphi \in L^2(0,T;H_0^1(Q)) \cap L^4(Q \times (0,T))$. Integrating by parts the following equation in the variable t, we have

$$\int_0^T -\langle u, \varphi' \rangle + (-\Delta u, \varphi) ds = \int_0^T \langle w(u(s)), \varphi \rangle ds + (u(0), \varphi(0)).$$

Performing the same step in the Galerkin approximation yields

$$\int_0^T -\langle u_n, \varphi' \rangle + (-\Delta u_n, \varphi) ds = \int_0^T \langle P_n w(u_n(s)), \varphi \rangle ds + (u_n(0), \varphi(0)).$$

Since $u_n(0) = P_n u_0 \to u_0$, we take limits to have

$$\int_0^T -\langle u, \varphi' \rangle + (-\Delta u, \varphi) ds = \int_0^T \langle w(u(s)), \varphi \rangle ds + (u_0, \varphi(0)).$$

Hence, $u(0) = u_0$.

To prove uniqueness and continuous dependence, let u_0 and v_0 be in $L^2(Q)$ and consider $h(t) = u(t) - v(t)$. Then,

$$\frac{dh}{dt} - \Delta h = w(u) - w(v), \quad h(0) = u_0 - v_0,$$

and multiplying by h and integrating over Q gives

$$\frac{1}{2}\frac{d}{dt}||h||_{L^2(Q)}^2 + ||h||_{H_0^1(Q)}^2 = (w(u) - w(v), h).$$

Note that we have $w'(s) \leq 2$, for all $s \in \mathbb{R}$, so

$$\begin{aligned}(w(u)-w(v),h) &= \int_Q (w(u(x))-w(v(x)))(u(x)-v(x))dx \\ &= \int_Q \left(\int_{v(x)}^{u(x)} w'(s)ds \right) (u(x)-v(x))dx \\ &\leq \int_Q 2|u(x)-v(x)|^2 dx \\ &= 2||h||^2_{L^2(Q)}.\end{aligned}$$

We therefore obtain

$$\frac{1}{2}\frac{d}{dt}||h||^2_{L^2(Q)} \leq 2||h||^2_{L^2(Q)},$$

and by Gronwall's inequality

$$||u(t)-v(t)||_{L^2(Q)} \leq e^{2t}||u_0-v_0||_{L^2(Q)}.$$

Hence we have uniqueness if $u_0 = v_0$ and continuous dependence on initial conditions otherwise. □

A.3 Existence and Uniqueness of Strong Solutions

Now, we will show how increasing the regularity of u_0 results in more regular solutions. In particular, if $u_0 \in H_0^1 \cap L^p$, then $u(t)$ is in this space for all $t \geq 0$, and the solutions are continuous into H_0^1. We call such solutions strong solutions. The uniqueness follows from the previous theorem, since a strong solution is automatically a weak solution.

Theorem A.5. *If $u_0 \in H_0^1(Q) \cap L^4(Q)$, then there exists a unique strong solution*

$$u(t) \in C^0([0,T]; H_0^1(Q)) \cap L^\infty(0,T; L^4(Q)) \cap L^2(0,T; D(A)),$$

where $A = -\Delta$ and $D(A)$ is the domain of A.

Proof. We take the inner product of ordinary differential system

$$\frac{du_n}{dt} + Au_n = P_n w(u_n), \quad u_n(0) = P_n u_0, \tag{A.14}$$

with Au_n to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla u_n||^2_{L^2(Q)} + ||Au_n||^2_{L^2(Q)} &= -\int_Q P_n w(u_n)\Delta u_n dx \\ &= -\int_Q w(u_n)\Delta u_n dx \\ &= \int_Q w'(u_n)|\nabla u_n|^2 dx,\end{aligned}$$

using the boundary condition $u_n = 0$ on ∂Q for integration by parts and the fact that $w(0) = 0$.

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}||\nabla u_n||^2_{L^2(Q)} + ||Au_n||^2_{L^2(Q)} \leq 2||\nabla u_n||^2_{L^2(Q)}.$$

Integrating both sides from 0 to T gives

$$\frac{1}{2}||\nabla u_n(T)||^2_{L^2(Q)} + \int_0^T ||Au_n(s)||^2_{L^2(Q)} ds \leq \int_0^T ||\nabla u_n||^2_{L^2(Q)} dt + \frac{1}{2}||\nabla u_n(0)||^2_{L^2(Q)},$$

and so u_n is uniformly bounded in $L^2(0, T; D(A))$ and $L^\infty(0, T; H^1_0(Q))$, and we already know from the previous proof that $u_n \in L^2(0, T; H^1_0(Q))$.

We now multiply (A.14) by $\frac{du_n}{dt}$ and integrate over Q. Simplifying the last term in the resulting identity as follows

$$(P_n w, \frac{du_n}{dt}) = (w, P_n \frac{du_n}{dt}) = (w, \frac{du_n}{dt}),$$

yields

$$\left\|\frac{du_n}{dt}\right\|^2_{L^2(Q)} + \frac{1}{2}\frac{d}{dt}||\nabla u_n||^2_{L^2(Q)} = \frac{d}{dt}\int_Q \mathcal{W}(u_n)dx,$$

where $\mathcal{W}(s) = \int_0^s w(t)dt$. Integrating the above equation from 0 to t gives us

$$\int_0^t \left\|\frac{du_n}{dt}\right\|^2_{L^2(Q)} ds + \frac{1}{2}||\nabla u_n(t)||^2_{L^2(Q)} - \int_Q \mathcal{W}(u_n(t))dx$$
$$\leq \frac{1}{2}||\nabla u_0||^2_{L^2(Q)} + \int_Q \mathcal{W}(u_n(0))dx,$$

and using

$$-1 - \frac{3}{4}|s|^4 \leq \mathcal{W}(s) \leq 1 - \frac{1}{4}|s|^4,$$

we have

$$\int_0^t \left\|\frac{du_n}{dt}\right\|^2_{L^2(Q)} ds + \frac{1}{2}||\nabla u_n||^2_{L^2(Q)} + \frac{1}{4}\int_Q |u_n(t)|^4 dx$$
$$\leq 2|Q| + \frac{1}{2}||\nabla u_0||^2_{L^2(Q)} + \frac{3}{4}\int_Q |u_0|^4 dx.$$

Hence, $\frac{du_n}{dt}$ is uniformly bounded in $L^2(0,T;L^2(Q))$ and u_n is uniformly bounded in the space $L^\infty(0,T;L^4(Q))$.

Therefore we can extract the appropriate subsequence such that

$$u \in L^\infty(0,T;L^4(Q)), u \in L^2(0,T;D(A)), \frac{du}{dt} \in L^2(0,T;L^2(Q)),$$

which implies that $u \in C^0([0,T],H^1_0(Q))$.

□

Appendix B

Well-posedness of Swift-Hohenberg Equation

In this chapter, we will show the existence of strong solutions to equation (1.3). And we also show that all the solutions are bounded for uniformly bounded initial conditions. Our results are revised from [26] and [27], therefore, we omit the detailed proofs for the theorems.

B.1 Preliminary Results

First, we establish some notations that will be used in the following results.

1. Weighted norms. Define the norm

$$||u||_{p,\rho} = \left(\int_Q |u(x)|^p \rho(x) dx \right)^{1/p}, \tag{B.1}$$

where $\rho : \mathbb{R}^2 \to (0, \infty)$ is a suitable weight with $|\nabla \rho(x)| \leq \rho_0 \rho(x)$ for some $\rho_0 < \infty$ and $\rho_1 = \int_{\mathbb{R}^2} \rho(x) < \infty$.

2. Uniformly local spaces $L^p_{lu}(Q)$. Define the norm

$$||u||_{p,lu} = \sup\{||u||_{p,T_y\rho} : y \in \mathbb{R}^2\},$$

where $T_y\rho = \rho(x-y)$ is the translated weight and the translations are continuous with respect to the norm above. By the definition above,

$$||u||_{p,T_y\rho} = \left(\int_Q \rho(x-y)|u(x)|^p dx \right)^{1/p}.$$

3. Space $\tilde{L}^p_{lu}(Q)$ and associated Sobolev spaces $\tilde{W}^{s,p}_{lu}(Q)$. Define

$$\tilde{L}^p_{lu}(Q) = \{u \in L^p_{loc}(Q) : ||u||_{p,lu} < \infty\}$$
$$\tilde{W}^{s,p}_{lu}(Q) = \{u \in \tilde{L}^p_{lu}(Q) : D^q u \in \tilde{L}^p_{lu}(Q) \quad \forall q \in \mathbb{N}^d_0 \text{ with } q_1 + \cdots + q_d \leq s\}$$

for integers s.

4. Uniformly local Sobolev spaces

$$W^{s,p}_{lu}(Q) = \text{closure of } C^\infty_{bdd}(Q) \text{ in } \tilde{W}^{s,p}_{lu}(Q),$$

where $C^\infty_{bdd}(Q)$ is the set of all C^∞ functions which have all derivatives bounded in Q. This construction ensures the density of $W^{s+1,p}_{lu}(Q)$ in $W^{s,p}_{lu}(Q)$ for bounded domain.

5. Weighted Hilbert space. Define

$$W^{2,2}_\rho(Q) = \{u \in L^2_\rho(Q) : \nabla u, \nabla^2 u \in L^2_\rho(Q)\},$$

where

$$L^2_\rho(Q) = \{u \in L^2_{loc}(Q) : ||u||_\rho \leq \infty\}.$$

We start with an elementary result about the lower bound on the H^k norms.

Lemma B.1. *For any weight ρ with $|\nabla\rho(x)| \leq \rho_0\rho$ and $\lambda > 0$, we have*

$$||\nabla^k u||^2_{2,\rho} \geq 2\lambda||\nabla^{k-1}u||^2_{2,\rho} - \lambda(\rho_0^2 + \lambda)||\nabla^{k-2}u||^2_{2,\rho},$$

for any $k \geq 2$.

B.2 Existence of Strong Solutions

Here we intend to use the theorem proved by Levermore and Oliver in [25] for the existence of solutions to the Swift-Hohenberg equation.

Theorem B.2. *Let X, Y and Z be Banach spaces with $Y \subset Z \subset X$ and let $T(t) = e^{At}|_{t \geq 0}$ be a semigroup on X with*

$$\begin{aligned} ||T(t)u||_Y &\leq ct^{-\alpha}||u||_X, \quad \text{for all } u \in X, \\ ||T(t)u||_Y &\leq ct^{-\beta}||u||_Z, \quad \text{for all } u \in Z, \end{aligned} \tag{B.2}$$

for $t \in (0, 1]$.

Moreover, let $N : Y \to Z$ be locally Lipschitz with

$$||N(u_1) - N(u_2)||_Z \leq c(||u_1||_Y^{2\sigma} + ||u_2||_Y^{2\sigma})||u_1 - u_2||_Y, \quad \text{for all } u_1, u_2 \in Y, \tag{B.3}$$

for some $\sigma > 0$, and without loss of generality we assume $N(0) = 0$.

Assume the exponents α, β, σ satisfy

$$\begin{aligned} 0 \leq \beta &< 1, \\ 0 \leq (2\sigma + 1)\alpha &< 1, \\ \beta + 2\sigma\alpha &< 1, \end{aligned} \tag{B.4}$$

then for every $M > 0$ there exists a time $T(M) > 0$ such that for every initial condition $u_0 \in X$ with $||u_0||_X \leq M$ there exists a unique solution $u \in C([0, T], X) \cap C([0, T], Y)$ to

$$u(t) = T(t)u_0 + \int_0^t T(t - \tau)N(u(\tau))d\tau. \tag{B.5}$$

In addition, the mapping from u_0 to u is locally Lipschitz continuous from X to $C([0, T], X)$.

Notice that (B.5) is a fixed point equation in Y norm. First, by contraction mapping theorem, we can prove that there exists a unique solution $u \in E([0,T])$ to the fixed point problem (B.5). Then by the continuous embedding $Z \subset X$ and the continuity of the semigroup $T(t)$, we can show that $u \in C([0,T], X)$.

In order to use this theorem to the Swift-Hohenberg equation, we need to construct an analytic semigroup $T(t) = e^{At}|_{t\geq 0}$ for the linear part of the equation. Here, we define

$$Au = -(1+\nabla^2)^2 u = -\nabla^4 u - 2\nabla^2 u - u. \tag{B.6}$$

Theorem B.3. *Define the operator $A : D(A) \subset L^p_{lu}(Q) \to L^p_{lu}(Q)$ by (B.6) for $p \in [2,\infty)$ with $D(A) = W^{4,p}_{lu} \cap \{$ boundary conditions $\}$.*

Then for all admissible domains Q the resolvent $(A-\lambda I)^{-1} : L^p_{lu}(Q) \to D(A)$ exists and satisfies the estimate

$$||(A-\lambda I)^{-1}u||_{p,lu} \leq \frac{C}{\lambda - R}||u||_{p,lu}, \quad \textit{for all } u \in L^p_{lu}(Q),$$

where R depends only on ρ_0 as defined in equation (B.1) and C depends only on p.

To prove this theorem, consider the boundary value problem

$$\nabla^4 u + 2\nabla^2 u + \gamma u = f,$$

for some γ which will be determined later, with the associated weak form

$$B[u,v] = \int_Q \rho f v dx, \quad \text{for all } v \in C_0^\infty(Q),$$

where

$$B[u,v] = \int_Q \rho\nabla^2 u \nabla^2 v + 2\nabla^2 u \nabla v \nabla \rho + v\nabla^2 u \nabla^2 \rho + 2\rho v \nabla^2 u + \rho\gamma u v dx.$$

For any $f \in L^2_{lu}(Q)$ we can use Lax-Milgram theorem in the weighted Hilbert space $W^{2,2}_\rho(Q)$ and this yields a unique solution $u \in W^{2,2}_\rho(Q)$. Finally, Hille-Yosida theorem with operator $A_0 u = -\nabla^4 u - 2\nabla^2 u$, that is, $f = (\gamma I - A_0)u$, will complete the proof of this theorem.

Next we can apply Theorem B.2 and Theorem B.3 to prove the existence of the solutions to the Swift-Hohenberg equation. Theorem B.3 shows that $A = -\nabla^4 - 2\nabla^2 - 1$ is the infinitesimal generator of an analytic semigroup $T(t) = e^{At}|_{t\geq 0}$.

Theorem B.4. *For each admissible domain $Q \subset \mathbb{R}^2$ and boundary conditions, and all initial conditions $u_0 \in L^p_{lu}(Q)$ there is a unique strong solution $u(t) = T(t)u_0$.*

Furthermore, for fixed weight $\rho(x) = e^{-|x|}$, there is a constant C such that for all admissible domains $Q \subset \mathbb{R}^2$ and all initial conditions $u_0 \in L^p_{lu}(Q)$ we have

$$\limsup_{t\to\infty} ||u(t)||_{1,4,lu} \leq C.$$

We can construct an absorbing set in $W^{3,d}_{lu}(Q)$ by the above conclusion. Let $B_0 = \{u \in W^{3,p}_{lu}(Q) : ||u||_{3,d,lu} \leq 2C\}$, then

$$B_{abs}(Q) = \cup_{t>0} T(t)(B_0) \subset W^{3,d}_{lu}(Q)$$

is a bounded, invariant set, since the union can also be taken over a finite time interval. The above estimates imply that every bounded set in $L^p_{lu}(Q)$ with $p > d$ is absorbed in finite time into B_{abs}. Moreover, we define

$$C_{abs} = \sup\{||u||_{1,4,lu} : \text{ there exists } Q \text{ admissible } : u \in B_{abs}(Q)\},$$

$$C_\infty = \sup\{||u||_\infty : \text{ there exists } Q \text{ admissible } : u \in B_{abs}(Q)\}$$

to have universal constants to estimate the norms in B_{abs}.

Bibliography

[1] S. M. Allen and J. W. Cahn; "A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening", Acta Met. 27(1979), 1085-1095.

[2] F. Bethuel, H. Brezis and F. Hélein; "Asymptotics of the minimization of a Ginzburg-Landau functional", Calc. Var. 1(1993), 123-148.

[3] F. Bethuel, G, Orlandi and D. Smets; "Convergence of the parabolic Ginzburg-Landau equation to motion by mean cutvature", Ann. Math 163(2006), 37-163.

[4] Y. Brenier, F. Otto and C, Seis; "Upper bounds on coarsening rates in demixing binary viscous liquids", SIAM J. Math. Anal. 43(2011), pp. 114-134.

[5] L. F. Cugliandolo; " Topics in coarsening phenomena", Lecture notes. Fundamental Problems in Statistical Physics XII, Leuven, Aug 30 - Sept 12, 2009.

[6] M. C. Cross and P. C. Hohenberg; "Pattern formation outside of equilibrium", Rev. Mod. Phys. 65(3) (1994), 851-1112.

[7] M. C. Cross and D. I. Meiron; "Domain coarsening in systems far from equilibrium", Phys. Rev. Lett. 75 (1995), no.11, 2152-2155.

[8] M. C. Cross and A. C. Newell; "Convection patterns in large aspect ratio systems", Physica 10D(1984), 299-328.

[9] J. Carr and R. L. Pego; "Metastable Patterns in Solutions of $\partial_t v = \varepsilon^2 u_{xx} - f(u)$", Cumm. Pure Appl. Math. XLII(1989), 523-576.

[10] P. G. Daniels and E. Jhugeroo; "Pattern formation in rectangular planform containers", Q. JI Mech. Appl. Math, Vol. 65. No. 1 (2011), 161-180.

[11] R. Dautray and J.-L. Lions; "Mathematical analysis and numerical methods for science and technology, Vol. 2, functional and variational methods", Springer-Verlag, (1988).

[12] S. Dai and R. L. Pego; "An upper bound on the coarsening rate for mushy zones in a phase-field model", Interfaces and Free Boundaries 7(2005), 187-197.

[13] S. Esedoḡlu and D. Slepčev; "Refined upper bounds on the coarsening rate of discrete, ill-posed diffusion equations", Nonlinearity 21(2008), 2759-2776.

[14] L. C. Evans; "Partial differential equations", American. Math. Soc. (2010), 369-370.

[15] K. R. Elder, J. Viñals and M. Grant; "Ordering dynamics in the two-dimensional stochastic Swift-Hohenberg equation", Phys. Rev. Lett. 68(1992), no. 20, 3024-3027.

[16] A. Fridman; "Partial differential equations", Holt, Rinehart and Winston INC., 1969.

[17] G. Fusco and J. K. Hale; "Slow-motion manifolds, dormant instability, and singular perturbations", J. Dynam. Differential Equations, 1 (1989), 75-94.

[18] J. R. Gomez-Solano and D. Boyer; "Coarsening in potential and nonpotential models of oblique stripe patterns" , Phys. Rev. E 76, 041131(2007) 1-12.

[19] M. Haragus and A. Scheel; "Interface between rolls in the Swift-Hohenberg equation", Int. J. Dyn. Syst. Differ. Equ. 1 (2007), no. 2, 89-97.

[20] R. Kohn and F. Otto; "Upper bounds on coarsening rates", Comm. Math. Phys 229(2002), 375-395.

[21] K. Kim and H. K. Park; "Coarsening dynamics of striped patterns in thin granular layers under vertical vibration", Phys.Rev. Ltee. 88 (2002), no.20, 204303-1-204303-4.

[22] B. B. King, O. Stein and M. Winkler; "A fourth-order parabolic equation modeling epitaxial thin film growth", J. Math. Anal. Appl. 286(2003), 459-490.

[23] T. Kato; "Perturbation theory for linear operators" , Springer-Verlag, 2nd ed., 1976.

[24] R. Kohn and X. Yan; "Upper bound on the coarsening rate for an epitaxial growth model", Comm. Pure Appl. Math 56(2003), no. 11, 1549-1564.

[25] C. D. Levermore and M. Oliver; "Dynamical systems and probabilistic methods in partial differential equations" Ch. The complex Ginzburg-Landau equation as a model problem, Lect. Appl. Math., Am. Math. Soc, 31(1996), 141-190.

[26] D. Lloyd; "Localised solutions of partial dierential Equations", Ph.D Thesis, Department of Engineering Mathematics, University of Bristol, 2005, 107-125.

[27] A. Mielke; "The complex Ginzberg-Landau equation on large and unbounded domains: sharper bounds and attractors", Nonlinearity, 10(1997), 199-222.

[28] L. Modica and S. Mortola; "Un esempio di $\Gamma-$convergenza", Boll. U.M.I. 14-B(1977), 285-299.

[29] L. Modica; "The gradient theory of phase transitions and the minimal interface criterion", Arch. Rat. Mech. Anal 98(1987), 123-142.

[30] A. Mielke and G. Schneider; "Attractors for modulation equations on unbounded domains-existence and comparison", Nonlinearity, 8(1995), no. 5, 743-768.

[31] S. Osher, A. Sole and L. Vese; "Image decomposition and restoration using total variation minimization and the H^{-1} norm", Multiscale Model. Simul. (2003) Vol. 1, No. 3, 349-370.

[32] R. L. Pego; "Lectures on Dynamics in Models of Coarsening and Coagulation", Lectures on Dynamics in Models of Coarsening and Coagulation, World Scientific (Singapore) 2007.

[33] M. Renardy and R. C. Rogers; "An introduction to partial differential equations", Springer-Verlag, and, ed., 2003.

[34] J. C. Robinson; "Infinite-dimensional dynamical systems", Cambridge University Press (2001), 221-230.

[35] P. Sternberg; "Vector-valued local minimizers of nonconvex variational problems", Rocky Mountain J. Math. Volume 21, Number 2 (1991), 799-807.

[36] J. Swift and P. J. Hohenberg; "Hydrodynamic fluctuations at the convective instability", Phys. Rev. A, 15 (1977), 319-328.

[37] M. S. Trumper; "Relaxation of the curve shortening flow via the parabolic Ginzburg-Landau equation", Calculus of Variations and Partial Differential Equations, 31(3), 359-386.

[38] M. S. Trumper; "Existence of a solution to a vector-valued Allen-Cahn equation with a three well potential", Indiana Math. J. 58(2009), No.1, 213-267.