C-wild knots

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
May 5, 2013

Keywords: Infinite connected sum of knots, Wilder knots, C-wild knots

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Abstract

Here we give a definition of infinite connected sum of tame knots and define a C-wild knot to be an infinite connected sum of tame knots whose wild points form a Cantor set. We further give a classification of C-wild knots in terms of Wilder knots, which are infinite connected sums of tame knots with one wild point.

Acknowledgments

I want to thank Dr. K. Kuperberg, Dr. W. Kuperberg, and Dr. J. Szulga for their help and support.

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Chapter 1

Introduction

In [5], Fox and Harrold gave a complete classification of the Wilder arcs, which were first considered by R.L. Wilder. A Wilder knot is a wild knot with exactly one wild point and can be thought as obtained by identifying the end points of a Wilder arc. Here we consider Wilder knots as an infinite connected sum of tame knots, and show that if doing infinite connected sum in a different way, we can get a wild knot whose wild points form a Cantor set. We call such wild knots C-wild knots and give a classification of these knots in terms of Wilder knots.

In chapter 2, we presented the preliminaries. Starting from section 2.6, all knots are assumed to be oriented and in oriented S^3. In chapter 3, we generalized the concept of connected sum of knots, and infinite connected sum of knots is defined. In chapter 4, we defined Wilder connected sum of knots, a specific way of doing infinite connected sum, considered Wilder Knots as the Wilder connected sum of tame knots, and gave a classification of Wilder knots. In chapter 5, we defined a C-wild knot to be an infinite connected sum of tame knots whose wild points form a Cantor set. Earlier than this, we defined Cantor connected sum of knots, another way of doing infinite connected sum. We showed that every C-wild knot can be obtained by doing Cantor connected sum of tame knots, and gave a classification of C-wild knots based on this.

Chapter 2

Preliminaries

Definition 2.0.1. $\mathbb{R}^n = \{x = (x_1, ..., x_n)\}$ = *the* ***Euclidean space*** *of real n-tuples with the usual norm* $|x| = (\sum {x_i}^2)^{1/2}$ *and metric* $d(x, y) = |x - y|$.

B^n = *the unit* ***n-ball*** *of* $\mathbb{R}^n$ *defined by* $|x| \leq 1$.

$S^n = \partial B^{n+1}$, *the unit* ***n-sphere*** $|x| = 1$.

$I = [0, 1]$ *the unit interval of* $\mathbb{R}^1$.

2.1 Orientation

Definition 2.1.1. *A closed (compact, without boundary)connected n-manifold M is* ***orientable*** *if its nth singular homology group with $\mathbb{Z}$ coefficients $H_n(M) = \mathbb{Z}$. If the connected compact manifold M has nonempty boundary, it is orientable if $H_n(M, \partial M) = \mathbb{Z}$. A choice of one of the two possible generators of $H_n(M)$ resp. $H_n(M, \partial M)$ is called an* ***orientation****, and an orientable manifold together with such a choice is said to be an* ***oriented*** *manifold.*

Lemma 2.1.2. *By restriction any submanifold N (n-dimensional with boundary) of an oriented n-manifold M is oriented. Furthermore, the boundary ∂N of an oriented n-manifold N is oriented by choosing the (n-1)-cycle which is the boundary of the preferred relative n-cycle.*

Definition 2.1.3. *Let M, N be oriented n-manifolds. A homeomorphism $f : M^n \to N^n$ is said to* ***preserve*** *or* ***reverse orientation****, according as the induced homomorphism on the n-th homology carries the preferred generator for M to the preferred generator for N, or to its negative.*

Lemma 2.1.4. *Let M, N be oriented n-manifolds with boundary. Then any homeomorphism from M to N that is an extension of an orientation-preserving (orientation-reversing)* *homeomorphism from a component of the boundary of M to a component of the boundary of N is orientation-preserving (orientation-reversing).*

Lemma 2.1.5. *$H_n(S^n) = \mathbb{Z}$; $H_n(B^n, \partial B^n) = \mathbb{Z}$, for $n \geq 1$.*

2.2 Triangulation

Definition 2.2.1. *A (Euclidean) complex is a collection K of simplexes in $\mathbb{R}^n$, such that (1). K contains all faces of all elements of K. (2)If $\sigma, \tau \in K$, and $\sigma \bigcap \tau \neq \emptyset$, then $\sigma \bigcap \tau$ is a face both of σ and of τ. (3)Every σ in K lies in an open set U which intersects only a finite number of elements of K.*

Definition 2.2.2. *Let K be a complex. A subset X of $|K|$ is a **polyhedron** if there is a subdivision K' of K and $X = |S|$ for some subcomplex S of K'.*

Definition 2.2.3. *A set X is **triangulable** if there is a complex K such that X and $|K|$ are homeomorphic. K is called a **triangulation** of X.*

Definition 2.2.4. *For $n \leq 3$, a **piecewise linear manifold** or **PL manifold** is an n-manifold M with a fixed triangulation. Let K be a fixed triangulation of M, then PL M is $|K|$.*

The following theorems are due to E.E.Moise, see [9] for proofs.

Theorem 2.2.5. *Every triangulated 3-manifold is a combinatorial 3-manifold.*

Theorem 2.2.6 (The triangulation theorem for 3-manifolds)**.** *Every 3-manifold can be triangulated.*

Theorem 2.2.7 (The Hauptvermutung for 3-manifolds)**.** *Let K_1 and K_2 be triangulations of a 3-manifold M. Then there is a subdivision K_i' of K_i for $i = 1, 2$ and a simplicial homeomorphism $\phi : |K_1'| \to |K_2'|$.*

2.3 Knot

Definition 2.3.1. *A subset K of S^3 is a* ***knot*** *if K is homeomorphic with S^1; a subset A of S^3 is an* ***arc*** *if it is homeomorphic with the unit interval $[0, 1]$.*

Definition 2.3.2. *Two oriented knots K and K' of oriented S^3 are* ***equivalent*** *if there is an orientation-preserving homeomorphism of S^3 onto itself that takes K onto K' preserving the orientation.*

Remark 2.3.3. *Depending on the context, by a knot we may also mean an equivalence class of knots.*

Definition 2.3.4. *A* ***unknot****,or* ***trivial knot*** *is a knot equivalent to S^1.*

Definition 2.3.5. *A* ***polygonal knot (arc)*** *in PL S^3 is a knot (arc) that is a polyhedron.*

2.4 Tameness

Definition 2.4.1. *Let K be a complex. A subset X of $|K|$ is* ***tame*** *if there is a homeomorphism $h : |K| \to |K|$ such that $h(X)$ is a polyhedron.*

Definition 2.4.2. *Let K be a complex. A subset X of $|K|$ is* ***locally tame*** *at a point p of X if there is a neighborhood N of p and a homeomorphism h_p of $Cl(N)$ onto a polyhedron such that $h_p(Cl(N) \bigcap X)$ is a subpolyhedron.*

Definition 2.4.3. *A set X is* ***tame*** *in a topological 3-manifold M if M has a triangulation K relative to which X is tame, it then follows that X is tame relative to every triangulation of M by theorem 2.2.7. Otherwise, it is* ***wild****.*

Definition 2.4.4. *Similarly, a subset X of a 3-manifold M is* ***locally tame at a point*** *p of X if M has a triangulation K relative to which X is locally tame at p. If X is locally tame at each point of X, then it is* ***locally tame****.*

Definition 2.4.5. *A point p of X is called a* ***wild point*** *of X if X is not locally tame at p.*

Theorem 2.4.6. *In a 3-manifold, every locally tame set is tame.*

Proof. By theorem 2.2.6, every 3-manifold is triangulable. By [1], every locally tame set is tame in a PL 3-manifold. Then use theorem 2.2.7. □

2.5 Some theorems

Theorem 2.5.1 (Generalized Jordan curve theorem)**.** *If S is homeomorphic to S^{n-1} in S^n, then $S^n - S$ has two components, and S is the boundary of each.*

Proof. See [6]. □

Theorem 2.5.2 (PL Shoenflies theorem)**.** *If S is a PL 2-sphere embedded in PL S^3, then the closure of the complementary components of S are PL 3-cells.*

Proof. See [2] or [9]. □

Theorem 2.5.3 (PL annulus theorem)**.** *If B_1 and B_2 are PL n-cells in PL S^n, with $B_1 \subset Int(B_2)$, then $Cl(B_2 - B_1)$ is PL homeomorphic to $\partial B_1 \times I$.*

Proof. See [7]. □

Lemma 2.5.4. *If C, D are homeomorphic to B^n, then any homeomorphism $h : \partial C \to \partial D$ extends to a homeomorphism $\overline{h} : C \to D$.*

Proof. We can assume that $C = D = B^n$. Then in vector notation, if $x \in \partial B^n$, define $\overline{h}(tx) = th(x)$, $0 \le t \le 1$. □

Definition 2.5.5. *Suppose $B_0, B_1, ..., B_n$ are tame 3-cells in S^3 and for $0 < i \le n$, the balls $B_i \subset Int(B_0)$ are disjoint. Then we call $Cl(B_0 - \bigcup_{i=1}^{n} B_i)$ an* ***n-annulus*** *with boundary $\bigcup S_i$, where $S_i = \partial B_i$.*

Lemma 2.5.6. *Let A, A' be n-annuli in S^3, with boundaries $\bigcup S_i$, $\bigcup S_i'$, and $h : S_0 \to S_0'$ be a homeomorphism. Then h can be extended to a homeomorphism $\overline{h} : A \to A'$ with $h(S_i) = S_i'$.*

Proof. By theorem 2.5.3, this is true for 1-annuli. Suppose it holds for (n-1)-annuli. Let B_i, B_i' be as in the previous definition. By theorem 2.5.3, there are homeomorphisms $f : Cl(B_0 - B_1) \to S^2 \times I$ and $f' : Cl(B_0' - B_1') \to S^2 \times I$ such that $f(S_0) = S^2 \times \{1\}$ and $f'(S_0') = S^2 \times \{1\}$. Let $g = f' \circ h \circ f^{-1}$. Then g is a homeomorphism of $S^2 \times \{1\}$ onto itself, and g can be extend to a homeomorphism $\overline{g} : S^2 \times I \to S^2 \times I$. Let o be the origin of $\mathbb{R}^3$; C be an infinite cone with vertex o, whose intersection with $S^2 \times I$ is a 3-cell C disjoint from $S_2, ..., S_n$, and $\overline{g}(C)$ is disjoint from $S_2', ..., S_n'$. Let D be the 3-cell $Cl(S^2 \times I - C)$, D' be $\overline{g}(D)$, and g_1 be $\overline{g}$ restricted to ∂D. Then by the induction hypothesis, g_1 extends to a homeomorphism $\overline{g_1} : Cl(D - \bigcup\limits_{i=2}^{n} f(B_i)) \to Cl(D' - \bigcup\limits_{i=2}^{n} f'(B_i'))$, mapping $f(S_i)$ to $f'(S_i')$ for $1 < i \leq n$. Define $\widetilde{g} : f(A) \to f'(A')$ by letting $\widetilde{g}$ be $\overline{g}$ on C, be $\overline{g_1}$ on $Cl(D - \bigcup\limits_{i=2}^{n} f(B_i))$. Let $\overline{h}$ be $f'^{-1} \circ \widetilde{g} \circ f$. □

2.6 Connected Sum of Knots

Remark 2.6.1. ***From here on, all knots are assumed to be oriented and in oriented*** S^3.

Definition 2.6.2. *Suppose K is a knot and C is a topological ball in S^3. We say $(C, C \bigcap K)$ is an* ***S ball pair*** *if C is tame and $(C, C \bigcap K)$ is topologically equivalent to the canonical ball pair (B^3, B^1).*

Definition 2.6.3 (connected sum of knots)**.** *Suppose K_i is a tame knot in S^3, $(C_i, C_i \bigcap K_i)$ is an S ball pair, D_i is $Cl(S^3 - C_i)$, and $A_i = D_i \bigcap K_i$ for $i = 1, 2$. By theorem 2.5.2, D_i are 3-cells. Let $f : D_2 \to C_1$ be an orientation-preserving homeomorphism such that $f(\partial A_2) = \partial A_1$ and the simple closed curve $A_1 \bigcup f(A_2)$ is oriented. The* ***connected sum*** *of the knots K_1 and K_2, written $K_1 \sharp K_2$, is the simple closed curve $A_1 \bigcup f(A_2)$ in S^3.*

Remark 2.6.4. *Such an f exists, for there is an orientation-preserving homeomorphism $\partial D_2 \to \partial C_1$ mapping ∂A_2 onto ∂A_1, and it can be extended to get an f by lemma 2.5.4,* *and 2.1.4. And it is easy to see that f can be extended to an orientation-preserving ambient homeomorphism, so the knot type of K_2 is preserved.*

Proposition 2.6.5. *$K_1 \sharp K_2$ is an oriented knot in oriented S^3.*

Proposition 2.6.6. *Connected sum is well-defined for equivalence classes of tame oriented knots in oriented S^3.*

Proof. Here the notation will be as in definition 2.6.3. Let K_1' be a knot equivalent to K_1. Define C_1', D_1', A_1' and f' accordingly. We want to show that $K_1 \sharp K_2$ is equivalent to $K_1' \sharp K_2$. Clearly, $f' \circ f^{-1} : (C_1, f(A_2)) \to (C_1', f'(A_2))$ is an orientation-preserving homeomorphism of pairs. Let $g : (D_1, A_1) \to (D_1', A_1')$ be an orientation-preserving homeomorphism equal to $f' \circ f^{-1}$ on $\partial D_1 = \partial C_1$. Then h equal to $f' \circ f^{-1}$ on C_1, g on D_1 is an desired ambient homeomorphism. □

Corollary 2.6.7. *Let K_i be equivalence classes of tame knots, then*

1. $K_1 \sharp K_2 = K_2 \sharp K_1$
2. $(K_1 \sharp K_2) \sharp K_3 = K_1 \sharp (K_2 \sharp K_3)$.
3. $K_1 \sharp K_2 = K_1$, *where K_2 is the unknot.*

2.7 Knot Factorization

Definition 2.7.1. *A **prime knot** is a non-trivial tame knot which is not the connected sum of two non-trivial tame knots. Knots that are not prime are said to be **composite**.*

Theorem 2.7.2. *There exist infinitely many inequivalent prime knots.*

Proof. See [3]. □

Definition 2.7.3 (decomposing sphere system for a tame knot)**.** *Let S_j, $1 \leq j \leq m$, be disjoint tame 2-spheres embedded in S^3, bounding $2m$ balls B_i, $1 \leq i \leq 2m$. If B_i contains only the s balls $B_{l(1)}, ..., B_{l(s)}$ as proper subsets, $R_i = (B_i - \bigcup_{q=1}^{s} Int(B_{l(q)})$ is called **the domain***

*R_i. The spheres S_j are said to be **decomposing** with respect to a tame knot K in S^3 if the following conditions are fulfilled:*

1. Each sphere S_j meets K transversely in two points.

*2. The knot K_i, which is the union of the arc $A_i = K \bigcap R_i$, oriented as K, and arcs on the boundary of R_i, is prime. K_i is called a **prime factor** of K **determined** by B_i.*

*We call $\mathcal{S} = \{S_j | 1 \leq j \leq m\}$ a **decomposing sphere system** with respect to K; if K is prime we put $\mathcal{S} = \emptyset$.*

Remark 2.7.4. *K_i does not depend on the choice of the arcs on ∂R_i.*

Definition 2.7.5. *Two decomposing sphere systems $\mathcal{S} = \{(S_j, K)\}$ and $\mathcal{S}' = \{(S_l{}', K)\}$ are called **equivalent** if they define the same (unordered) factors K_i.*

Theorem 2.7.6. *Any non-trivial tame knot K can be decomposed into a finite number of prime knots $K = K_1 \sharp K_2 \sharp ... \sharp K_n$. Furthermore, the decomposition is unique up to order. That is, if $K = K_1 \sharp K_2 \sharp ... \sharp K_n = K_1' \sharp K_2' \sharp ... \sharp K_m'$ are two decompositions, then $n = m$ and $K_i = K_{l(i)}'$ for some permutation l of $\{1, 2, ..., n\}$.*

Proof. W.l.o.g, we can assume that K is a polygonal knot in PL S^3. The first part of the theorem is an easy consequence of the additivity of the genus of PL knots.The uniqueness of decomposition is proved by showing that any two decomposing sphere systems $\mathcal{S} = \{(S_j, K)\}$, $\mathcal{S}' = \{(S_l{}', K)\}$ are equivalent. For a detailed proof, please refer to [3]. $\square$

Chapter 3

Generalizing Some Concepts

3.1 Infinite Connected Sum of Knots

Definition 3.1.1. *Let K be a knot in S^3. Suppose $B_0, B_1, ..., B_n$ are tame balls whose boundary meets K transversely in two points. For $0 < i \leq n$, the balls $B_i \subset Int(B_0)$ are disjoint. Let $\widetilde{K}$ be the knot which is the union of the arcs $K \bigcap Cl(B_0 - \bigcup_{i=1}^{n} B_i)$, oriented as K, and arcs on the boundary of $Cl(B_0 - \bigcup_{i=1}^{n} B_i)$, oriented in the way that $\widetilde{K}$ is oriented. Then $\widetilde{K}$ is called the knot* ***determined*** *by $Cl(B_0 - \bigcup_{i=1}^{n} B_i)$ and K.*

Recall that $Cl(B_0 - \bigcup_{i=1}^{n} B_i)$ is an n-annulus as defined in definition 2.5.5. We define a ***0-annulus*** *to be a tame ball in S^3. The word* ***annulus*** *may refer to a k-annulus for any $k \geq 0$.*

The notations are consistent from 3.1.2 to 3.1.6.

Definition 3.1.2. *Let K be a knot in S^3. Then K is the* ***connected sum*** *of tame knots if*
(1) there is a finite or countable sequence $\mathcal{S} = \{S_j\}$ of disjoint tame 2-spheres embedded in S^3, with each S_j meets K transversely in two points.
(2) if $\{B_i\}$ is a countable sequence of tame 3-cells such that $\{\partial B_i\}$ is a subsequence of $\{S_j\}$ and $B_{i+1} \subset Int(B_i)$, then $\bigcap B_i$ is a point.
(3) let B be a ball bounded by some element of $\mathcal{S}$, then there are at most a finite number of balls whose boundaries are elements in $\mathcal{S}$, say, $C_1, ..., C_n$, that are outermost in B, in the sense that $C_i \subset Int(B)$ and there does not exist a ball $D \subset Int(B)$ bounded by some $S_k \in \mathcal{S}$ such that $C_i \subset Int(D)$.
(4) the notation here is as in (3). $Cl(B - \bigcup_{i=1}^{n} C_i)$ determines a non-trivial tame knot, which

*is called a **factor** of K.*

*If $\{S_j\}$ is a countable sequence, we say K is the **infinite connected sum** of tame knots. $\mathcal{S} = \{S_j\}$ is called a **decomposing sphere system** for K. If in (3) the tame knot is also prime, then we call $\mathcal{S} = \{S_j\}$ a **prime decomposing sphere system** for K. We may also call a prime decomposing sphere system a decomposing sphere system.*

Remark 3.1.3. *Note that (2) implies that if B is a ball bounded by some element of $\mathcal{S}$, and $C \subset Int(B)$ is another ball bounded by some element of $\mathcal{S}$, then there is a ball $D \subset Int(B)$ whose boundary is an element of $\mathcal{S}$ such that $C \subset Int(D)$ and D is outermost in B. Thus this excludes the case where there are infinitely many balls whose boundaries are elements in $\mathcal{S}$ contained in $Int(B)$, but no one is outermost in B.*

Remark 3.1.4. *In the next two chapters, we will give examples of constructing infinite connected sum of knots with tame knots.*

Proposition 3.1.5. *If $\{S_j\}$ is a countable, then $S^3 - \bigcup S_j$ is the union of a countable collection of disjoint open annuli $\{A_i\}$ and a closed set W of totally disconnected points, such that the boundaries of $Cl(A_i)$ are elements of $\mathcal{S}$ and W is the set of wild points of K. Moreover, each closed annulus $Cl(A_i)$ determines a non-trivial tame knot.*

Proof. By conditions (1),(2),(3)(see remark 3.1.3), $S^3 - \bigcup S_j$ contains the union of a countable collection of disjoint open annuli $\{A_i\}$ such that the boundaries of $Cl(A_i)$ are elements of $\mathcal{S}$. Let $W = S^3 - \bigcup S_j - \bigcup A_i$. By (4), each closed annulus determines a non-trivial tame knot. By (3) and (2), there is a countable sequence $\{B_n\}$ of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\{S_j\}$, $B_{n+1} \subset Int(B_n)$, and $\bigcap B_n$ is a point p. Then p is not in any S_j, for the spheres are disjoint, nor in any A_i. So $W \neq \emptyset$.

If $p \in W$, then there is a countable sequence $\{B_n\}$ of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\{S_j\}$, $p \in B_{n+1} \subset Int(B_n)$, and by (2), $\bigcap B_n = p$. So $\bigcap(B_n \bigcap K) = p$, and hence $p \in K$. And p is a wild point of K, for there are infinitely many knots converging to

p. Conversely, if p is a wild point of K, then p is not in any S_j or A_i by (2),(3),(4). Hence $p \in W$. So W is the set of wild points of K.

W is closed in K, for the set of points at which K is locally tame is open. Since any two distinct points in W are separated by two disjoint balls, W is totally disconnected. □

The following lemma can be obtained from the proof of proposition 3.1.5.

Lemma 3.1.6. *If $\{B_n\}$ is a countable sequence of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\{S_j\}$, $B_{n+1} \subset Int(B_n)$, then $\bigcap B_n$ is a wild point of K. If p is a wild point of K, then there is a countable sequence $\{B_n\}$ of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\{S_j\}$, $B_{n+1} \subset Int(B_n)$, and $\bigcap B_n = p$.*

3.2 Some Lemmas

Lemma 3.2.1. *Let K be a tame knot in S^3, $\alpha \subset K$ be an arc, and $\widehat{K}$ be a factor of K. Then there is a tame 3-cell C such that $K \bigcap C = \alpha$ and the knot determined by C is $\widehat{K}$.*

Proof. Since $\widehat{K}$ is a factor of K, there is a tame 3-cell D such that the knot determined by D and K is $\widehat{K}$. Let $D \bigcap K = \beta$. Since K is tame, there is an orientation-preserving homeomorphism $h : (S^3, K) \to (S^3, K)$ such that $h(\beta) = \alpha$. Let $C = h(D)$. □

Lemma 3.2.2. *Let the notation be defined as in 3.1.1, and assume that $\widetilde{K}$ is a non-trivial tame knot. Let $\widehat{K}$ be a factor of $\widetilde{K}$. Let B be $Cl(S^3 - B_0)$ or an element in $\{B_1, ..., B_n\}$. Then there is a a tame 3-cell C such that $B \subset Int(C)$, $\partial C \bigcap \partial B_i = \emptyset$ for $0 \le i \le n$, and the knot determined by $Cl(C - B)$ and K is $\widehat{K}$.*

Proof. W.l.o.g, assume that B is $Cl(S^3 - B_0)$. Let β be $\widetilde{K} \bigcap \partial B$, and α be a subarc of $\widetilde{K}$ such that $\alpha \bigcap \partial B_i = \emptyset$ for $1 \le i \le n$ and $\beta \subset Int(\alpha)$. By lemma 3.2.1,there is a tame 3-cell D such that $\widetilde{K} \bigcap D = \alpha$ and the knot determined by D and $\widetilde{K}$ is $\widehat{K}$. W.o.l.g, we can assume that ∂D and ∂B_i are polyhedra in PL S^3 and ∂D is in general position with ∂B_i, $0 \le i \le n$. Then replace ∂D by a sphere S that is disjoint from each ∂B_i, obtained from ∂D by a finite number of pushes (ambient homeomorphisms with compact support) [1]. Let C be the closure of the component of S that contains B. $\square$

Remark 3.2.3. *Let the notation be as in lemma 3.2. Let M be the union of elements in a subset of $\{Cl(S^3 - B_0)\} \cup \{B_1, ..., B_n\}$. With an argument similar to that in proposition 3.2, we can show that there is a a tame 3-cell C such that $M \subset Int(C)$, $\partial C \bigcap \partial B_i = \emptyset$ for $0 \leq i \leq n$, and the knot determined by $Cl(C - M)$ and K is $\widehat{K}$.*

[1] See [2] for definitions and details.

Chapter 4

The Wilder Knots

4.1 Wilder Connected Sum

Definition 4.1.1 (Wilder connected sum of knots)**.** *Suppose K_1 is a non-trivial tame knot in S^3, $C_1, C_2, ...$ is a sequence of tame 3-cells such that $C_{i+1} \subset Int(C_i)$, $\bigcap C_i = p \in K_1$, and $(C_i, C_i \bigcap K_1)$ is an S ball pair for each i. Let $\{K_i\}$ be a sequence of non-trivial tame knots in S^3, and $(C_{ij}, C_{ij} \bigcap K_i)$ be an S ball pair, with $C_{i1} \bigcap C_{i2} = \emptyset$ for $i = 2, 3, ...,$ $j = 1, 2$. Define $M_i = Cl(S^3 - C_{i1} \bigcup C_{i2})$, $N_i = Cl(C_{i-1} - C_i)$, for $i \geq 2$, and $M_1 = N_1 = Cl(S^3 - C_1)$. By theorem 2.5.3, M_i, N_i are homeomorphic to $S^2 \times I$ for $i \geq 2$. Let α_{ij} be the two arcs $M_i \bigcap K_i$, β_{ij} be the two arcs $N_i \bigcap K_1$, $i \geq 2$, where α_{i1} is the arc exiting ∂C_{i1} and entering ∂C_{i2}, β_{i1} is the arc exiting ∂C_{i-1} and entering ∂C_i. Define $f_i : M_i \to N_i$ to be an orientation-preserving homeomorphism such that $f_i(\partial C_{i1}) = \partial C_{i-1}$, and for fixed ij, f_i maps the end points of α_{ij} to the end points of β_{ij}, for $i = 2, 3, ...,$ $j = 1, 2$. Let $\alpha_1 = M_1 \bigcap K_1$. The **Wilder connected sum** of $\{K_i\}$, written $K_1 \sharp K_2 \sharp ...$, is the simple closed curve $\alpha_1 \cup \bigcup f_i(\alpha_{ij}) \cup \{p\}$ in S^3.*

Remark 4.1.2. *To see the f_i defined above exist, let S_1, S_2 be the two components of the boundary of $A = S^2 \times I$ and $p_{ij} \in S_i$, $q_{ij} \in S_i$, $i = 1, 2, j = 1, 2$ are eight points. W.l.o.g, we can assume that p_{ij} should be mapped to q_{ij}. It is easy to see that there are orientation-preserving (OP) and orientation-reversing (OR) homeomorphisms of S_1 onto S_1 taking p_{1j} to q_{1j}. Such an OPH (ORH) can be extended to an OPH (ORH), say h, of A onto itself. Then keep the points on S_1 fixed and slide $h(p_{2j})$ on S_2 till it is mapped to q_{2j}. Also, each f_i can be extended to an OP ambient homeomorphism, and so the knot type of each K_i is preserved.*

Proposition 4.1.3. *$K_1 \sharp K_2 \sharp ...$ is an oriented knot in oriented S^3.*

Proposition 4.1.4. *Wilder connected sum is well-defined for equivalence classes of tame oriented knots in oriented S^3.*

Proof. Suppose $\{K_i'\}$ is a sequence of tame knots such that K_i' is equivalent to K_i for each i. Define M_i', N_i', α_{ij}', β_{ij}', f_i', α_1' accordingly. Let $h_1 : (N_1, \alpha_1) \to (N_1', \alpha_1')$ be an orientation-preserving homeomorphism(OPH). Suppose we have got an OPH $h_n : (\bigcup_{i=1}^{n} N_i, \alpha_1 \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{2} f_i(\alpha_{ij})) \to (\bigcup_{i=1}^{n} N_i', \alpha_1' \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{2} f_i'(\alpha_{ij}'))$. Then ${f_{n+1}'}^{-1} h_n f_{n+1} : \partial C_{(n+1)1} \to \partial C'_{(n+1)1}$ is an OPH that maps $K_{n+1} \cap \partial C_{(n+1)1}$ to $K'_{n+1} \cap \partial C'_{(n+1)1}$, and it can be extended to an OPH $g : M_{n+1} \to M'_{n+1}$, which maps $K_{n+1} \cap \partial C_{(n+1)2}$ to $K'_{n+1} \cap \partial C'_{(n+1)2}$ as shown in the previous remark. Then g can be modified to get an OPH $\widetilde{g} : (M_{n+1}, \bigcup_{j=1}^{2} \alpha_{(n+1)j}) \to (M'_{n+1}, \bigcup_{j=1}^{2} \alpha'_{(n+1)j})$. Define h_{n+1} by letting $h_{n+1} = h_n \cup f'_{n+1} \widetilde{g} {f_{n+1}}^{-1}$. Finally, define h to be h_n on $\bigcup_{i=1}^{n} N_i$ for each n, and $h(p) = p'$. $\square$

4.2 The Wilder Knots

Definition 4.2.1. *A knot K in S^3 is a* ***Wilder knot*** *if it is an infinite connected sum of tame knots with one wild point.*

Remark 4.2.2. *Intuitively, we can think a Wilder knot as a simple closed curve with a sequence of tame knots tied in it convergent to a point p in it, possibly from each side of p. See figure 4.1.*

Definition 4.2.3. *A wild knot is* ***mildly wild*** *if it is the union of two tame arcs.*

Proposition 4.2.4. *A Wilder knot is a mildly wild knot.*

Remark 4.2.5. *There is a mildly wild knot with one wild point that is not a Wilder knot. See [8].*

Proposition 4.2.6. *Let $\{K_i\}$ be a sequence of prime knots in S^3. The Wilder connected sum $K_1 \sharp K_2 \sharp ...$ is a Wilder knot. Conversely, every Wilder knot is the Wilder connected sum of a sequence of prime knots.*

Proof. The first part is clear. Let K be an infinite connected sum of tame knots with one wild point p. Let $\mathcal{S} = \{S_j\}$ be a decomposing sphere system for K. By lemma 3.1.6, there is exactly one countable sequence $\{B_n\}$ of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\{S_j\}$, $B_{n+1} \subset Int(B_n)$, and $\bigcap B_n$ is a point, which should be p. By definition 3.1.2, the knots determined by $Cl(S^3 - B_1)$ and $Cl(B_n - B_{n+1})$ are tame. By theorem 2.7.6, every tame knot can be factored into prime knots. Then the result follows from lemma 3.2. □

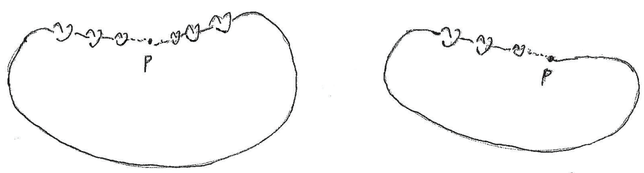

Figure 4.1: Wilder knot

Definition 4.2.7. *A prime knot type is an **infinitely occurring prime** if it appears infinite times in (a decomposition of) a wild knot, otherwise, a **finitely occurring prime**. We say two wild knots **have the same list of prime knots** if and only if they have the same set of finitely and infinitely occurring primes, with each finitely occurring prime appearing exactly the same number of times. We say two wild knots **almost have the same list of prime knots** if and only if they differ by a finite number of finitely occurring primes.*

When working on the following theorem, we referred to the paper [5] by Fox and Harrold.

Theorem 4.2.8. *Two Wilder knots K and K' are equivalent if and only if they have the same list of prime knots.*

Proof. Suppose K and K' are equivalent. By proposition 4.2.6, K can be factored into prime knots. We show that K is uniquely factored into prime knots, and it would follow that K and K' have exactly the same list of prime knots. Assume that K has n type-π knots in a decomposition. Take a tame 2-sphere S that separates the wild point p of K from the n type-π knots of K, and intersects K transversely in two points. Denote the closure of the complementary component of S that does not contain p as D_1, and the closure of the other complementary component of S as D_2. Let K_1 be the knot determined by D_1. Suppose in a different decomposition, K has m type-π knots. Take a tame 2-sphere $S' \subset Int(D_2)$ that separates the wild point p of K from the m type-π knots of K and intersects K transversely in two points. Let D'_1 be the closure of the complementary component of S' that does not contain p, and K'_1 be the knot determined by D'_1. Then K_1 is a factor of K'_1, and hence m can not be less than n, for otherwise K'_1 would not be uniquely factored. By symmetry, $m = n$. If K has infinitely many type-π knots in one decomposition, then for each n, K has no less than n type-π knots in a different decomposition, so K has infinitely many type-π knots in the other decomposition.

Conversely, suppose K and K' have the same list of prime knots. By proposition 4.2.6, $K = K_1\sharp K_2\sharp...$ and $K' = K'_1\sharp K'_2\sharp...$, where $\{K_i\}$ and $\{K'_i\}$ are two sequences of prime knots. Since Wilder knots are uniquely decomposed into prime knots, $K = K_1\sharp K_2\sharp...$ and $K' = K'_1\sharp K'_2\sharp...$ have the same list of prime knots. If we can show that there is a decomposing sphere system for K such that $K = K'_1\sharp K'_2\sharp...$, then K and K' are equivalent by proposition 4.1.4. Let p be the wild point of K. Fix an n, there is a tame 2-sphere S_1 that meets K transversely in two points and separates $K'_1, K'_2, ..., K'_n$ from p. Using lemma 3.2 repeatedly, there are tame 3-cells $C_1, C_2, ..., C_n$ with $C_{i+1} \subset Int(C_i)$, $S_1 \subset Int(C_n)$, and the knot determined by $S^3 - Int(C_1)$ is K'_1, the knot determined by $C_i - Int(C_{i+1})$ is K'_{i+1}, $1 \leq i < n$. Next we take a tame 2-sphere S_2 that meets K transversely in two points and separates

$K'_{n+1}, K'_{n+2}, ..., K'_{2n}$ from p. If S_1 separates $K'_{n+1}, K'_{n+2}, ..., K'_{2n}$ from p, we take $S_2 = S_1$. By lemma 3.2, there are tame 3-cells $C_{n+1}, C_{n+2}, ..., C_{2n}$ with $C_{i+1} \subset Int(C_i)$, $S_2 \subset Int(C_{2n})$, and the knot determined by $C_i - Int(C_{i+1})$ is K'_{i+1} for $n \leq i < 2n$. Continuing in this way, we can find a sequence of tame 3-cells $C_1, C_2, ...$ such that $C_{i+1} \subset Int(C_i)$, and $\bigcap C_i = p$, and the knot determined by $S^3 - Int(C_1)$ is K'_1, the knot determined by $C_i - Int(C_{i+1})$ is K'_{i+1}. $\square$

Chapter 5

C-wild Knots

5.1 Cantor Connected Sum

Definition 5.1.1 (Cantor connected sum of knots)**.** *Suppose $K_{01} = K_0$ is a non-trivial tame knot in S^3, C_{ij}, $i = 1, 2, ...,$ $j = 1, 2, ..., 2^i$ is a sequence of tame 3-cells such that C_{1j} are disjoint in S^3 and $C_{(i+1)(2j-1)}$, $C_{(i+1)2j}$ are disjoint subsets of $Int(C_{ij})$, $(C_{ij}, C_{ij} \bigcap K_1)$ is an S ball pair for each ij, and $diamC_{ij} \to 0$ as $i \to \infty$. Let $\{K_{ij}\}$ be a sequence of non-trivial tame knots in S^3, and $(C_{ijk}, C_{ijk} \bigcap K_{ij})$ be disjoint S ball pairs for fixed ij, $i = 1, 2, ...,j = 1, 2, ..., 2^i$,$k = 0, 1, 2$. Define $M_{ij} = Cl(S^3 - \bigcup_{k=0}^{2} C_{ijk})$, $N_{ij} = Cl(C_{ij} - C_{(i+1)(2j-1)} \bigcup C_{(i+1)2j})$, for $i \geq 1$, and $M_0 = N_0 = Cl(S^3 - C_{11} \bigcup C_{12})$. By lemma 2.5.6, M_{ij}, N_{ij} are homeomorphic for $i \geq 1$. Let α_{ijk} be the three arcs $M_{ij} \bigcap K_{ij}$, β_{ijk} be the three arcs $N_{ij} \bigcap K_0$, where α_{ij0} is the arc exiting ∂C_{ij0}, α_{ij2} is the arc entering ∂C_{ij0}, β_{ij0} is the arc exiting ∂C_{ij}, β_{ij2} is the arc entering ∂C_{ij}. Define $f_{ij} : M_{ij} \to N_{ij}$ to be an orientation-preserving homeomorphism such that $f_{ij}(\partial C_{ij0}) = \partial C_{ij}$, and for fixed ijk, f_{ij} maps the end points of α_{ijk} to the end points of β_{ijk}, for $i = 1, 2, ...,$ $j = 1, 2, ..., 2^i$, $k = 0, 1, 2$. Let $\alpha_0 = M_0 \bigcap K_0$. Let $C_i = \bigcup_j C_{ij}$. The **Cantor connected sum** of $\{K_{ij}\}$, is the simple closed curve $\alpha_0 \cup \bigcup f_{ij}(\alpha_{ijk}) \cup \bigcap C_i$ in S^3.*

Remark 5.1.2. *With lemma 2.5.6 and an argument similar to that in remark 4.1.2, we can show that f_{ij} defined above exist and can be extended to an OP ambient homeomorphism.*

Proposition 5.1.3. *The Cantor connected sum is an oriented knot in oriented S^3.*

Proposition 5.1.4. *Cantor connected sum is well-defined for equivalence classes of tame oriented knots in oriented S^3.*

Proof. Let $M_i = \bigcup_j M_{ij}$, $N_i = \bigcup_j N_{ij}$, $f_i = \bigcup_j f_{ij}$. Then use an argument analogous to the proof of 4.1.4. □

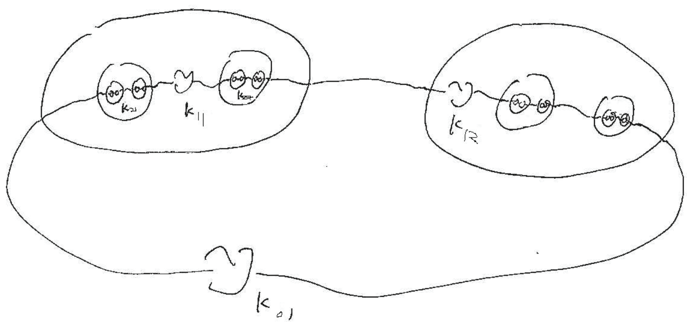

Figure 5.1: Cantor connected sum

5.2 C-wild knots

Definition 5.2.1. *A* ***Cantor set*** *is a compact metrizable space that is totally disconnected and has no isolated points.*

Definition 5.2.2. *A* ***C-wild knot*** *is an infinite connected sum of tame knots whose wild points form a Cantor set.*

Proposition 5.2.3. *Let K be a C-wild knot and W be the set of wild points of K. Then $K - W$ is a countable union of open tame arcs $\{A_i\}$ such that $Cl(A_i)$ is a tame arc whose end points are wild points of K.*

Proposition 5.2.4. *A C-wild knot is the Cantor connected sum of a sequence* $\{K_{ij}\}$ *of prime knots, where* $i = 0, 1, ...,$ $j = 1, 2, ..., 2^i$. *Conversely, the Cantor connected sum of a sequence of prime knots is a C-wild knot.*

Proof. The second part is clear. The first part is clear by proposition 5.2.3, theorem 2.7.6 and lemma 3.2. There is a detailed proof in theorem 5.3.7. □

Proposition 5.2.5. *A C-wild knot can be and is uniquely decomposed into prime knots.*

Proof. The existence part is by proposition 5.2.4. With an argument similar to that in the proof of proposition 4.2.8, we can show that every C-wild knot is uniquely decomposed into prime knots. □

Definition 5.2.6. *Let* K *be a C-wild knot. Let* C_{ij}, $i = 1, 2, ...,$ $j = 1, 2, ..., 2^i$ *be a sequence of tame 3-cells such that* C_{1j} *are disjoint in* S^3 *and* $C_{(i+1)(2j-1)}$, $C_{(i+1)2j}$ *are disjoint subsets of* $Int(C_{ij})$. *We say* $\mathcal{C} = \{\partial C_{ij}\}$ *is a* ***Cantor decomposing sphere system (Cdds)*** *for* K *if* $\mathcal{C}$ *is a decomposing sphere system for* K.

Theorem 5.2.7. *Two C-wild knots* K *and* K' *are equivalent if and only if there are prime Cantor decomposing sphere systems* $\{\partial C_{ij}\}$ *and* $\{\partial C'_{ij}\}$ *for* K *resp.* K' *such that*

1. K *and* K' *have the same list of prime knots.*

2. fix ij, *the C-wild knots* J_{ij} *and* J'_{ij} *determined by* C_{ij} *and* C'_{ij} *respectively almost have the same list of prime knots.*

Proof. Suppose K and K' are equivalent. W.l.o.g, we can assume that $K = K'$ as sets. Let W be the set of wild points of K. Let $\{\partial C_{ij}\}$ and $\{\partial C'_{ij}\}$ be two Cdds for K such that $C_{ij} \bigcap W = C'_{ij} \bigcap W$. By proposition 5.2.5, K is uniquely decomposed into prime knots. And it is easy to see that for fix ij, the C-wild knots J_{ij} and J'_{ij} determined by C_{ij} and C'_{ij} respectively almost have the same list of prime knots.

Conversely, suppose there are Cdds $\{\partial C_{ij}\}$ and $\{\partial C'_{ij}\}$ for K resp. K' satisfying conditions 1,2, and K and K' are the Cantor connected sum of $\{K_{ij}\}$ resp. $\{K'_{ij}\}$. We want to find a Cdds ∂D_{ij} for K such that K is the Cantor connected sum of $\{K'_{ij}\}$, then K would be equivalent to K' by proposition 5.1.4. By 1, 2, for a large enough n, K'_{01} and the prime knots appearing in J'_{1j} but not in J_{1j}, for $j = 1, 2$, would appear in the tame knot determined by $Cl(S^3 - \bigcup_{j=1}^{2^n} C_{nj})$. By remark 3.2.3, we can find D_{1j} such that the knot determined by D_{1j} is J'_{1j} and the knot determined by $Cl(S^3 - \bigcup D_{1j})$ is K'_{01}, $j = 1, 2$. Next we want to find D_{21} and D_{22} such that the knot determined by D_{2j} is J'_{2j} and the knot determined by $Cl(D_{11} - (D_{21} \cup D_{22}))$ is K'_{11}, $j = 1, 2$. If all the prime knots appearing in $J_{n1} \cup J_{n2} \cup ... \cup J_{n2^{n-2}}$ also appear in J'_{21}, and all the prime knots appearing in $J_{n(2^{n-2}+1)} \cup ... \cup J_{n2^{n-1}}$ also appear in J'_{22} then we can find D_{21} and D_{22} using remark 3.2.3. If not, Since J_{2j} and J'_{2j}, almost have the same list of prime knots, for m large enough, all the prime knots appearing in $J_{m1} \cup J_{m2} \cup ... \cup J_{m2^{m-2}}$ also appear in J'_{21}, and all the prime knots appearing in $J_{m(2^{m-2}+1)} \cup ... \cup J_{m2^{m-1}}$ also appear in J'_{22}. Then we can find D_{21} and D_{22} by applying remark 3.2.3. Continuing this way, we can find a desired Cdds $\{\partial D_{ij}\}$ for K. $\square$

5.3 Wilder Knots in C-wild Knots

Definition 5.3.1. *Let K be a C-wild knot and p be a wild point of K. Let $\mathcal{S} = \{S_k\}$ be a decomposing sphere system for K. A Wilder knot is called* ***a Wilder knot (relative to $\mathcal{S}$) in K with wild point*** *p if it is the union of subarcs of K and arcs on a subsequence of $\{S_k\}$ and its wild point is p.*

Proposition 5.3.2. *The notations are defined as in definition 5.3.1. Then there is a Wilder knot in K with wild point p.*

Proof. By lemma 3.1.6, there is a countable sequence $\{B_n\}$ of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\mathcal{S} = \{S_k\}$, $B_{n+1} \subset Int(B_n)$, and $\bigcap B_n = p$. We can assume that B_{n+1} is outermost in B_n. Let M_{n+1} be the union of all the balls that are outermost in B_n, and whose boundaries are elements of $\mathcal{S}$. Let K_n be the tame knot determined by $Cl(B_n - M_{n+1})$. Then the infinite connected sum of $\{K_n\}$ is a desired Wilder knot. $\square$

Remark 5.3.3. *Note that there are infinitely many inequivalent Wilder knots in K with wild point p.*

Theorem 5.3.4. *Two C-wild knots K and K' are equivalent if and only if there are prime Cantor decomposing sphere systems $\{\partial C_{ij}\}$ and $\{\partial C'_{ij}\}$ for K resp. K' such that*

1. K and K' have the same list of prime knots.

2. let J_{ij} and J'_{ij} be the C-wild knots determined by C_{ij} and C'_{ij} respectively. Fix ij and let ω be a Wilder knot in J_{ij}. Then there is a Wilder knot γ in J'_{ij} such that all but finitely many prime factors of ω are prime factors of γ.

Proof. Suppose K and K' are equivalent. W.l.o.g, we can assume that $K = K'$ as sets. Let W be the set of wild points of K. Let $\{\partial C_{ij}\}$ and $\{\partial C'_{ij}\}$ be two Cdds for K such that $C_{ij} \cap W = C'_{ij} \cap W$. Let ω be a Wilder knot in $J_{i_0 j_0}$ with wild point p. By lemma 3.1.6, there is a countable sequence $\{B_n\}$ of tame 3-cells such that $\{\partial B_n\}$ is a subsequence of $\{\partial C_{ij}\}$, B_{n+1} is outermost in B_n, and $\bigcap B_n = p$. Suppose $\partial B_n \cap \omega \neq \emptyset$. Let ω_n be the tame knot determined by $Cl(B_n - B_{n+1})$ and ω. Assume that ω is the Wilder connected sum of $\{\omega_n\}$. Further assume that $C_{i_0 j_0} = C_{11}$ and $B_n = C_{n1}$ as sets. For each $n \geq 2$, take m_n large enough so that ω_{n-1} is a factor of the tame knot determined by $Cl(C_{(n-1)1} - C_{n1} - \bigcup_{j=2^{m_n-n}+1}^{2^{m_n-n+1}} C'_{m_n j})$ and K. Let γ be the knot that is the union of subarcs of J'_{11} and arcs on the boundaries of $\{C'_{m_n j}\}$.

Conversely, suppose there are prime Cdds $\{\partial C_{ij}\}$ and $\{\partial C'_{ij}\}$ for K resp. K' such that conditions 1, 2 are satisfied. Suppose that there are infinitely many prime factors of $J_{i_0 j_0}$ that are not prime factors of $J'_{i_0 j_0}$. Then we can find a Wilder knot ω in $J_{i_0 j_0}$ such that infinitely many prime factors of ω are not prime factors of $J'_{i_0 j_0}$. But this contradicts condition 2. So condition 2 of proposition 5.2.7 holds. So K and K' are equivalent. □

Definition 5.3.5. *A **cyclic order on a set X** is a relation, written $[a, b, c]$, that satisfies the following axioms:*

Cyclicity: If $[a, b, c]$ then $[b, c, a]$

Asymmetry: If $[a, b, c]$ then not $[c, b, a]$

Transitivity: If $[a, b, c]$ and $[a, c, d]$ then $[a, b, d]$

Totality: If a, b, and c are distinct, then either $[a, b, c]$ or $[c, b, a]$

Definition 5.3.6. *A function between two cyclically ordered sets $f : X \to Y$ is called **order-preserving** if $[a, b, c]$ implies $[f(a), f(b), f(c)]$.*

Theorem 5.3.7. *Two C-wild knots K and K' with prime decomposing sphere systems $\mathcal{S} = \{S_k\}$ resp. $\mathcal{S}' = \{S'_k\}$ are equivalent if and only if*

1. K and K' have the same list of prime knots.

2. let W, W' be the set of wild points of K resp. K'. Then there is an order-preserving bijection $f : W \to W'$ such that if p, q are the end points of a tame subarc of K, then $f(p)$, $f(q)$ are the end points of a tame subarc of K'.

3. suppose ω is a Wilder knot in K with wild point p. Then there is a Wilder knot γ with wild point $f(p)$ in K' such that all but finitely many prime factors of ω are prime factors of γ.

Proof. Assume conditions 1, 2, 3. If $x \in W$, then x' denote $f(x) \in W'$. Let V be the set of wild points of K that are end points of tame subarcs of K.

Step 1. find a prime Cdds for K.

Let $p_i, q_i \in W$ be the end points of a tame subarc α_i of K for $i = 1, 2$ such that $[p_1, p_2, q_2]$ and $[p_1, q_2, q_1]$. Let C_{11} be a tame 3-cell that intersects K transversely in two points on $Int(\alpha_1)$ resp. $Int(\alpha_2)$ such that $p_1, p_2 \in Int(C_{11})$, and C_{12} be a tame 3-cell that intersects K transversely in two points on $Int(\alpha_1)$ resp. $Int(\alpha_2)$ such that $q_1, q_2 \in Int(C_{12})$. Moreover, C_{11} and C_{12} are disjoint, and the knot determined by $Cl(S^3 - (C_{11} \cup C_{12}))$ is prime, which is possible by lemma 3.2. Let $\alpha_3 \subset Int(C_{11})$ be a tame subarc of K with end points $p_3, q_3 \in V$ such that $[p_1, p_3, q_3]$ and $[p_1, q_3, p_2]$. Let $C_{21} \subset Int(C_{11})$ be a tame 3-cell that intersects K transversely in two points on $Int(\alpha_1)$ resp. $Int(\alpha_3)$ such that $p_1, p_3 \in Int(C_{21})$, and $C_{22} \subset Int(C_{11})$ be a tame 3-cell that intersects K transversely in two points on $Int(\alpha_3)$

resp. $Int(\alpha_2)$ such that $q_3, p_2 \in Int(C_{22})$. Moreover, C_{21} and C_{22} are disjoint, and the knot determined by $Cl(C_{11} - (C_{21} \cup C_{22}))$ is prime. Continuing in this way, we can find a prime Cdds $\{\partial C_{ij}\}$ for K.

Step 2. find a corresponding prime Cdds for K'.

By condition (2), $p'_i, q'_i \in W'$ are the end points of a tame subarc α'_i of K' for $i = 1, 2$ such that $[p'_1, p'_2, q'_2]$ and $[p'_1, q'_2, q'_1]$. Let C'_{11} be a tame 3-cell that intersects K' transversely in two points on $Int(\alpha'_1)$ resp. $Int(\alpha'_2)$ such that $p'_1, p'_2 \in Int(C'_{11})$... Continuing in this way, we can find a prime Cdds $\{\partial C'_{ij}\}$ for K'.

Step 3. show that prime Cdds $\mathcal{C} = \{\partial C_{ij}\}$ for K, $\mathcal{C}' = \{\partial C'_{ij}\}$ for K' satisfy conditions 1, 2 of proposition 5.3.4.

Condition 1 of 5.3.4 is true by proposition 5.2.5, and condition 1 of this theorem. Now we prove condition 2 of 5.3.4. Let J_{ij} and J'_{ij} be the C-wild knots determined by C_{ij} and C'_{ij} respectively. Let ω be a Wilder knot in $J_{i_0j_0}$ with wild point p. By lemma 3.1.6, there is a countable sequence $\{C_m\}$ of tame 3-cells such that $\{\partial C_m\}$ is a subsequence of $\{\partial C_{ij}\}$, $C_{m+1} \subset Int(C_m)$, and $\bigcap C_m = p$. We can assume that C_{m+1} is outermost in C_m, $\partial C_m \bigcap \omega \neq \emptyset$, and ∂C_1 is also an element of $\mathcal{S}$. Let ω_m be the tame knot determined by $Cl(C_m - C_{m+1})$ and ω. Then we can find $B_{m1}, ..., B_{mn_m}$ whose boundaries are elements of $\mathcal{S}$ such that ω_m is a factor of the tame knot determined by $Cl(C_m - C_{m+1} - \bigcup_{t=1}^{n_m} B_{mt})$ and K. Let $\overline{\omega}$ be the knot that is the union of subarcs of K and arcs on the boundaries of C_1 and $\{B_{mt}\}$, where $1 \leq t \leq n_m$, $m \geq 1$. Then $\overline{\omega}$ is a Wilder knot relative to $\mathcal{S}$ in K with wild point p such that all but a finite number of prime factors of ω are prime factors of $\overline{\omega}$. By condition 3, there is a Wilder knot $\overline{\gamma}$ relative to $\mathcal{S}'$ with wild point p' in K' such that all but finitely many prime factors of $\overline{\omega}$ are prime factors of $\overline{\gamma}$. Then we can find a Wilder knot γ relative to $\mathcal{C}'$ with wild point p' in K' such that all but finitely many prime factors of $\overline{\gamma}$ are prime factors of γ. By step 2, p' is in $C'_{i_0j_0}$. So we can assume that γ is in $J'_{i_0j_0}$. So conditions 1, 2 of 5.3.4 are satisfied and hence K and K' are equivalent.

Conversely, assume that K and K' with prime dds $\mathcal{S} = \{S_k\}$ resp. $\mathcal{S}' = \{S'_k\}$ are equivalent. Condition 1 is by proposition 5.2.5. Condition 2 follows immediately. Condition 3 can be done similarly as in Step 3. $\square$

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