Maximum and minimum degree in iterated line graphs

by

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Abstract

In this thesis we analyze two papers, both by Dr. Stephen G. Hartke and Dr. Aparna W. Higginson, on maximum [2] and minimum [3] degrees of a graph $G$ under iterated line graph operations. Let $\Delta_k$ and $\delta_k$ denote the minimum and the maximum degrees, respectively, of the $k^{th}$ iterated line graph $L^k(G)$. It is shown that if $G$ is not a path, then, there exist integers $A$ and $B$ such that for all $k > A$, $\Delta_{k+1} = 2\Delta_k - 2$ and for all $k > B$, $\delta_{k+1} = 2\delta_k - 2$. 
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Chapter 1

Introduction

The line graph $L(G)$ of a graph $G$ is the graph having edges of $G$ as its vertices, with two vertices being adjacent if and only if the corresponding edges are adjacent in $G$. Please note that all graphs in this discussion are simple. We restrict our discussion to connected graphs. Refer to [4] for basic definitions of graph theory.

One of the most important results in line graphs has been by Beineke, who provides in [1], a new characterization of line graphs in terms of nine excluded subgraphs, also unifying some of the previous characterizations. We provide only the theorem here without the proof.

![Figure 1.1](image.png)

**Theorem 1.1.** A graph $G$ is a line graph of some graph if and only if none of the nine graphs in Figure 1.1 is an induced subgraph of $G$. 
The iterated line graph is defined recursively as $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) = G$.

Let $\Delta$ and $\delta$ be the maximum and the minimum degree, respectively, of a graph $G$. We denote the minimum degree of $L^k(G)$ by $\delta_k$ and the maximum degree by $\Delta_k$. Hartke and Higgins [2] show that if $G$ is not a path, then, there exists an integer $A$, such that, $\Delta_{k+1} = 2\Delta_k - 2$ for all $k > A$. Using similar concepts, they show in [3] that there exists an integer $B$ such that $\delta_{k+1} = 2\delta_k - 2$ for all $k > B$. Rather than focusing on the vertices of minimum and maximum degrees, they observe the behavior of particular kinds of regular subgraphs, of which, the vertices of maximum and minimum degrees form a special case. However, this proves only the existence and the question of tight bounds of $A$ and $B$ is still open.

We now define some notation which will be used throughout the proofs. Neighborhood of a vertex $v$, denoted by $N(v)$, is defined as the set of all vertices adjacent to $v$. Then, if $S$ is a set of vertices of $G$, we use the following notation:-

1. $N(S) = \bigcup_{v \in S} N(v)$
2. $N[S] = N(S) \cup S$
3. $N\langle S \rangle = N(S) \setminus S$

We would first prove a result in Chapter 2 which was used in [2] and [3] without proof. Then, the result for the maximum degree is proved in Chapter 3 and for the minimum degree is proved in Chapter 4.
Chapter 2

An elementary result

In this chapter we will prove that for most graphs, minimum degree is unbounded under line graph iteration. Notice that, if $G$ is not a path, then $\delta_k$ is defined for all $k$. As mentioned in the introduction, all graphs under consideration are simple and we restrict our discussion to connected graphs.

A *leaf* of a graph is a vertex of degree 1.

**Lemma 2.1.** If there exists an integer $A$ such that $\delta_A > 2$, then $\delta_k > 2$ for all $k > A$. Moreover, $\delta_k$ is a strictly increasing sequence for all $k \geq A$, and hence $\lim_{k \to \infty} \delta_k = \infty$.

Proof: Clearly, the minimum possible value of $\delta_{k+1}$ is $2\delta_k - 2$. Now,

\[
\begin{align*}
\delta_A &> 2 \\
2\delta_A &> \delta_A + 2 \\
2\delta_A - 2 &> \delta_A.
\end{align*}
\]

But $2\delta_A - 2$ is the minimum possible value of $\delta_{A+1}$, hence, $\delta_{A+1} > \delta_A$ which implies $\delta_{A+1} > 2$. Now, let $\delta_{A+i} > 2$ for some $i$. Then, following similar set of equations, $\delta_{A+i+1} > \delta_{A+i}$ and $\delta_{A+i+1} > 2$. It follows inductively that $\delta_{k+1} > \delta_k > 2$ for all $k > A$ and therefore $\delta_k$ is a strictly increasing sequence. This also implies that the minimum degree is unbounded under line graph operation.

**Lemma 2.2.** Let $s_k$ be the number of vertices of degree 1 in $L^k(G)$. Then, $\{s_k\}$ is non-increasing.
Proof: Every vertex of degree 1 in a graph $L(G)$ corresponds to an edge in $G$ which is incident with exactly one edge. So, a leaf in $L^k(G)$ corresponds to one leaf in $L^{k-1}(G)$. Also, a leaf in $G$ will give a single leaf under the line graph operation.

**Lemma 2.3.** Let $G$ be a graph which is not a path or a cycle. If $\delta = 2$ then $\lim_{k \to \infty} \delta_k = \infty$.

Proof: A vertex of degree 2 in $L(G)$ will correspond to an edge in $G$ which is incident with exactly two edges. It can either be a leaf or an edge in a path or cycle as shown in the

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \quad \rightarrow \quad \begin{array}{c}
\bullet \rightarrow \bullet \\
\bullet \rightarrow \bullet \\
\bullet \rightarrow \bullet \\
\bullet \rightarrow \bullet \\
\bullet \rightarrow \bullet \\
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\end{array} \]

Figure 2.1

But as $\delta = 2$, $G$ has no leaf. Hence, we only need to consider vertices of degree 2 in $G$.

Now, as $G$ is not a path or a cycle, there exists at least one vertex, say $v$, of degree greater than 2. Also, as $\delta = 2$, $G$ is not a $K_{1,3}$. Let $u$ be a vertex of degree 2 in $G$. As $G$ is connected, there is a path from $u$ to $v$, say $P_0 = (u = y^0_1, y^0_2, ..., y^0_n = v)$, as shown in Figure 2.2. Now, $P_0$ induces a path $P_1 = (y^1_1, y^1_2, ..., y^1_{n-1})$ in $L(G)$ where $d_{L(G)}(y^1_j) \geq 2$ for $1 \leq j \leq n-2$ and $d_{L(G)}(y^1_{n-1}) \geq 3$. Now, let $P_i = (y^i_1, y^i_2, ..., y^i_{n-i})$ with $d_{L(G)}(y^i_j) \geq 2$ for $1 \leq j \leq n-i-1$ and $d_{L(G)}(y^i_{n-i}) \geq 3$. Then $P_i$ induces $P_{i+1}$ in $L^{i+1}(G)$ such that $P_{i+1} = (y^{i+1}_1, y^i_2 + 1, ..., y^{i+1}_{n-i-1})$. 

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Now, for $1 \leq j \leq k - 2$, 

\[
d_L(G)(y^j_2) \geq 2 \\
d_L(G)(y^j_1) + d_L(G)(y^j_1) \geq 2 + 2 \\
d_L(G)(y^j_2) + d_L(G)(y^j_1) - 2 \geq 2 + 2 - 2 \\
d_L(G)(y^{i+1}_1) \geq 2.
\]
and, for $j = k - 1$, 

$$d_{L(G)}(y_{k-1}^i) \geq 2$$

$$d_{L(G)}(y_{k-1}^i) + d_{L(G)}(y_k^i) \geq 2 + 3$$

$$d_{L(G)}(y_{k-1}^i) + d_{L(G)}(y_k^i) - 2 \geq 2 + 3 - 2$$

$$d_{L^1(G)}(y_{k-1}^{i+1}) \geq 3.$$ 

Also, $|P_{i+1}| = |P_i| - 1$. Applying inductively, $P_{n-1} = (y_{i}^{n-1})$ where $d_{L^{n-1}(G)}(y_{i}^{n-1}) \geq 3$ as shown in Figure 2.2, and we get that every vertex of degree 2 will definitely 'disappear' after $n-1$ line graph iterations. Doing this for every vertex of degree 2, there exists an integer $N$ such that $L^N(G)$ has no vertex of degree 2, hence $\delta_N \geq 3$ and we are done from Lemma 2.1. 

□

**Lemma 2.4.** If $G$ is neither a path, cycle nor a $K_{1,3}$, then the minimum degree is unbounded under line graph iteration and moreover, there exists an integer $A$ such that $\lim_{k \to \infty} \delta_k = \infty$ for all $k > A$.

Proof: From Lemma 2.1 it is sufficient to show that for any graph $G$, as specified, there exists an integer $A$ such that $\delta_A > 2$. As $G$ is neither a path, cycle nor a $K_{1,3}$, there exists an edge, say $e$, such that, $e = xz$ is incident with at least three edges.

Let $\delta(G) = 1$. From Lemma 2.2, the number of leaves is a non-increasing sequence over line graph iteration. Moreover, a leaf in $L(G)$ corresponds to exactly one leaf in $G$. So it would suffice to consider line graph operation on leaves of $G$ and show that it disappears at some iteration.

Let $v$ be incident on a leaf of $G$ such that $d_G(v) = 1$. Then, as $G$ is connected, there is a path $P_0 = (v = y_1^0, y_2^0, ..., y_{n-1}^0 = x, y_n^0 = z)$ from $v$ to the edge $e$ such that $d_G(y_i^0) \geq 2$ for $2 \leq i \leq n-2$, as shown in Figure 2.3. Now, $P_0$ induces a path, say $P_1$, in $L(G)$ such that $P_1 = (y_1^1, y_2^1, ..., y_{n-1}^1)$ where $y_j^1$ corresponds to the edge $y_0^j y_{j+1}^0 \in E(G)$ for $2 \leq j \leq n-1$, as shown in Figure 2.3. Now, as $xz$ is incident with at least three edges, $d_G(y_{n-1}^1) \geq 3$. Also,
\[ y_0^1 = v \]
\[ d_G(y_0^1) = 1 \]
\[ y_n^1 = x \]
\[ y_n^0 = z \]

\[ d_G(y_0^0) \geq 2 \]
\[ d_G(y_0^2) \geq 2 \]
\[ d_G(y_0^{n-2}) \geq 2 \]
\[ y_n^{n-1} = xz \]

\[ dG(y_0^1) = 1 \]
\[ dG(y_0^2) \geq 2 \]
\[ dG(y_0^{n-2}) \geq 2 \]
\[ y_n^{n-1} = xz \]

\[ dL(G)(y_1^1) \geq 1, dL(G)(y_1^2) \geq 2 \text{ for } 2 \leq j \leq n-2 \text{ and } dL(G)(y_{n-1}^1) \geq 3, \text{ as shown in Figure 2.3.} \]

Notice that \(|P_1| = |P_0| - 1.\)

Now, let \(P_i = (y_1^i, y_2^i, \ldots, y_{n-i}^i)\) in \(L^i(G)\), such that, \(d_{L^i(G)}(y_1^i) \geq 1, d_{L^i(G)}(y_j^i) \geq 2 \text{ for } 2 \leq j \leq n-i-1 \text{ and } d_{L^i(G)}(y_{n-i}^i) \geq 3.\) Then \(P_i\) induces a path \(P_{i+1}\) in \(L^{i+1}(G)\) such that \(P_{i+1} = \)
\((y^{i+1}_1, y^{i+1}_2, ..., y^{i+1}_{n-i-1})\) where \(y^{i+1}_j\) corresponds to the edge \(y^i_jy^{i+1}_j\) in \(P_i\) for \(1 \leq j \leq n - i - 1\) as shown in the Figure 2.3.

Now,

\[
d_{L^i(G)}(y^{i}_2) \geq 2
\]
\[
d_{L^i(G)}(y^{i}_2) + d_{L^i(G)}(y^{i}_1) \geq 2 + 1
\]
\[
d_{L^i(G)}(y^{i}_2) + d_{L^i(G)}(y^{i}_j) - 2 \geq 2 + 1 - 2d_{L^{i+1}(G)}(y^{i+1}_j) \geq 1.
\]

Also, for \(2 \leq j \leq k - 2\),

\[
d_{L^i(G)}(y^{i}_j) \geq 2
\]
\[
d_{L^i(G)}(y^{i}_j) + d_{L^i(G)}(y^{i+1}_j) \geq 2 + 2
\]
\[
d_{L^i(G)}(y^{i}_j) + d_{L^i(G)}(y^{i+1}_j) - 2 \geq 2 + 2 - 2
\]
\[
d_{L^{i+1}(G)}(y^{i+1}_j) \geq 2,
\]

and, for \(j = k - 1\),

\[
d_{L^i(G)}(y^{i}_{k-1}) \geq 2
\]
\[
d_{L^i(G)}(y^{i}_{k-1}) + d_{L^i(G)}(y^{i}_k) \geq 2 + 3
\]
\[
d_{L^i(G)}(y^{i}_{k-1}) + d_{L^i(G)}(y^{i}_k) - 2 \geq 2 + 3 - 2
\]
\[
d_{L^{i+1}(G)}(y^{i+1}_{k-1}) \geq 3.
\]

So, \(d_{L^{i+1}(G)}(y^{i+1}_1) \geq 1, d_{L^{i+1}(G)}(y^{i+1}_j) \geq 2\) for \(2 \leq j \leq k - 2\) and \(d_{L^{i+1}(G)}(y^{i+1}_{k-1}) \geq 3\). Also, \(|P_{i+1}| = |P_i| - 1\), then, following inductively starting from \(P_1\) we get that \(P_{n-1} = (y^{n-1}_1)\) where \(d_{L^{n-1}(G)}(y^{n-1}_1) \geq 2\) as shown in the Figure 2.3. Hence, the number of vertices of degree 1 goes down by one.
Let $G$ have $N$ vertices, say $v_1, v_2, ..., v_N$, of degree 1. Then, for every vertex $v_j$ of degree 1 there exists an integer $I_j$ such that there is no vertex of degree 1 in $L^{I_j}(G)$ corresponding to $v_j$. Then, for the integer $I = \max\{I_j \mid 1 \leq j \leq N\}$, there would be no vertex of degree 1 corresponding to any $v_j$. As there is no other way to get degree 1 vertices under line graph operation, $L^{I}(G)$ will have no vertices of degree 1. Also, as $L^k(G)$ is connected for all $k$ we conclude that $\delta_I \geq 2$ and we are done from Lemma 2.1 and Lemma 2.3.
Chapter 3
Maximum degree growth in iterated line graphs

In this chapter it will be shown that for any graph $G$, which is not a path, there exists an integer $D$ such that $\Delta_{k+1} = 2\Delta_k - 2$ for all $k > D$, where $\Delta_k$ is the maximum degree of $L^k(G)$.

If $G$ is a path, then as $G$ is a finite graph, there exists an integer $I$ such that $L^I(G)$ is undefined.

If $G$ is a cycle, then for all $k \in \mathbb{Z}^+$, $\Delta_{k+1} = 2\Delta_k - 2 = 2$.

If $G$ is a $K_{1,3}$, then $L(G)$ is a $K_3$ and hence, for all $k > 1$, $\Delta_{k+1} = 2\Delta_k - 2 = 2$.

Now we have to prove the theorem for any graph $G$ where it is not a path, a cycle or a $K_{1,3}$.

**Definition**: A vertex $v$ is a locally maximum vertex or a l.max. vertex if no vertex in the neighborhood of $v$ has degree greater than that of $v$.

**Definition**: The subgraph of $G$ induced by its l.max. vertices is called the locally maximum subgraph or l.max. subgraph of $G$ and is denoted by $LM(G)$.

**Definition**: A vertex $v \in L^k(G)$ is generated by a vertex $u \in G$ if there is a sequence of vertices $u = v_0, v_1, ..., v_k = v$ such that $v_{i+1} \in L^{i+1}(G)$ corresponds to an edge incident at $v_i \in L^i(G)$. A subgraph $J$ of $L^k(G)$ is generated by a subgraph $H$ of $G$ if, for each vertex $v \in J$, $v$ is generated by a vertex in $H$.

**Lemma 3.1.** All vertices in the same component of $LM(G)$ have the same degree in $G$.

Proof: Let $v$ and $u$ be two vertices in a component of $LM(G)$. Then $v$ and $u$ are l.max. vertices of the graph $G$. As $v \in N(u)$, $d(v) \leq d(u)$ from definition. Similarly, as $u \in N(v)$, $d(u) \leq d(v)$. Hence, $d(u) = d(v)$. 

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Lemma 3.2. The vertices of $L(G)$ corresponding to edges of $G$ incident with the same vertex, say $v$, of $G$, form a clique in $L(G)$. In particular, all the vertices of $LM(L(G))$ generated by $v$ are in the same component of $LM(L(G))$.

Proof: It follows from the definition of line graphs that the vertices of $L(G)$, corresponding to the edges of $G$ that share a vertex, will be adjacent to each other. \hfill \square

Lemma 3.3. If $w$ is a l.max. vertex of $L(G)$, then $w$ corresponds to an edge $e$ in $G$ such that at least one end of $e$, say $v$, is l.max. in $G$ and the other end of $e$, say $u$, has the maximum degree among the neighbors of $v$ in $G$.

Proof: Assume that neither $v$ nor $u$ is a l.max. vertex. Let $d_G(v) \geq d_G(u)$. Then, as $v$ is not a l.max. vertex, there exists a vertex $y \in N(v)$ such that $d_G(y) > d_G(v)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3.1}
\caption{Figure 3.1}
\end{figure}

Now, the edge $vy$ of $G$ corresponds to a vertex $vy$ of $L(G)$, adjacent to $w$ as shown in the Figure 3.1. Also,

$$d_G(v) \geq d_G(u).$$

But, as $d_G(y) > d_G(v),

$$d_G(v) + d_G(y) - 2 > d_G(u) + d_G(v) - 2$$

$$d_{L(G)}(vy) > d_{L(G)}(w).$$
contradicting that $w$ is a l.max. vertex of $L(G)$.

Hence, no such $y$ exists, implying that $v$ is a l.max. vertex of $G$.

Now, let there exist a vertex $z \in N(v)$ such that $d_G(z) > d_G(u)$.

Then the edge $vz$ of $G$ corresponds to a vertex $vz$ adjacent to $w$ in $L(G)$ as shown in the Figure 3.2.

But,

$$d_G(z) > d_G(u)$$

$$d_G(z) + d_G(v) - 2 > d_G(u) + d_G(v) - 2$$

$$d_{L(G)}(vz) > d_{L(G)}(w),$$

contradicting that $w$ is a l.max. vertex of $L(G)$. Hence, no such $z$ exists, implying that $u$ has the maximum degree in $N(v)$.

**Lemma 3.4.** Let $v$ be an isolated vertex of $LM(G)$.

(a) If $v$ has any neighbor of the same degree as that of $v$, then, $v$ generates no l.max. vertices of $L(G)$.
(b) If all neighbors of \( v \) have degree less than that of \( v \), and \( u \) is such a neighbor, then the edge \( uv \) corresponds to a l.max. vertex of \( L(G) \) if and only if \( u \) has the maximum degree among the neighbors of \( v \) and for all \( z \in N(u) \setminus \{v\} \), \( d_G(z) \leq d_G(v) \).

Proof:

\[
\begin{align*}
G & \quad \rightarrow \quad L(G) \\
\quad \text{Figure 3.3}
\end{align*}
\]

(a) As \( u \) is not a l.max. vertex of \( G \), there exists a vertex \( z \) adjacent to \( u \), such that, \( d_G(z) > d_G(u) = d_G(v) \). Then, \( u \) and \( z \) generate a vertex \( uz \) adjacent to \( w \), generated by \( v \) and \( u \), as shown in Figure 3.3. Now, \( d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 > d_G(u) + d_G(v) - 2 = d_{L(G)}(w) \), therefore, the edge \( vu \) does not correspond to a l.max. vertex of \( L(G) \), for any \( u \) with \( d_G(u) = d_G(v) \). Hence, by Lemma 3.3, \( v \) does not generate a l.max. vertex of \( L(G) \).

(b) Let there exist a vertex \( z \in N(u) \setminus \{v\} \) such that \( d_G(z) > d_G(v) \). Then the edge \( uz \) corresponds to a vertex \( uz \) in \( L(G) \) adjacent to a vertex \( w \), which corresponds to the edge \( uv \) in \( G \), as shown in Figure 3.3. Now, \( d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 > d_G(u) + d_G(v) - 2 = d_{L(G)}(w) \), therefore, \( w \) will not be a l.max. vertex.

Now, let, for all \( z \in N(u) \setminus \{v\} \), \( d_G(z) \leq d_G(v) \). Then,

\[
\begin{align*}
d_G(u) + d_G(z) - 2 & \leq d_G(u) + d_G(v) - 2, \\
d_{L(G)}(uz) & \leq d_{L(G)}(w),
\end{align*}
\]

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where \( w \) corresponds to the edge \( uv \) of \( G \). Therefore, the edge \( uv \) corresponds to a l.max. vertex of \( L(G) \).

Moreover, if \( uz \) is a l.max. vertex, it would be adjacent to \( w \) implying that the number of components will not increase.

**Lemma 3.5.** Let \( C \) be a component of \( LM(G) \) which is not a single vertex.

a) If \( v_1 \) and \( v_2 \) are adjacent vertices in \( C \), then the vertex \( w \in L(G) \), corresponding to the edge \( v_1v_2 \), is a l.max. vertex.

b) If \( u \in N(C) \), then no edge joining \( u \) to a vertex in \( C \) corresponds to a l.max. vertex of \( L(G) \).

Proof:

\[
\text{Figure 3.4}
\]

a) Let \( e' = v_1v_2 \) be an edge in \( C \). Let \( w \in L(G) \) be the vertex corresponding to \( e' \). Then, any neighbor \( x \) of \( w \) will correspond to an edge \( e \), in \( G \), incident at either \( v_1 \) or \( v_2 \). Let \( e \) be incident at \( v_1 \) and some vertex \( z \in N(v_1) \), as shown in the Figure 3.4. Then, as \( v_1 \) is a l.max. vertex,

\[
d_G(z) \leq d_G(v_1) \\
d_G(z) + d_G(v_2) - 2 \leq d_G(v_1) + d_G(v_2) - 2
\]
From Lemma 3.1, \( d_G(v_1) = d_G(v_2) \),

\[
d_G(z) + d_G(v_1) - 2 \leq d_G(v_1) + d_G(v_2) - 2
\]

\[
d_L(G)(x) \leq d_L(G)(w),
\]

hence, \( w \) is a l.max. vertex.

\[
\begin{array}{c}
\text{Figure 3.5}
\end{array}
\]

b) As \( u \in N(C) \), it is adjacent to a vertex, say \( v_1 \), in \( C \). As \( C \) is not a single vertex, there exists a vertex \( v_2 \in C \) adjacent to \( v_1 \). Let \( w \) be the vertex in \( L(G) \) corresponding to the edge \( v_1v_2 \) and let \( r \) be the common degree of vertices in \( C \). Then, \( d_{L(G)}(w) = 2r - 2 \).

Now, the edge \( uv_1 \) corresponds to a vertex \( x \) adjacent to \( w \) in \( L(G) \), as shown in the Figure 3.5. Also, \( d_{L(G)}(x) = d_G(u) + r - 2 \) and as \( v_1 \) is a l.max. vertex, we get that \( d_G(u) \leq r \).

If \( d_G(u) < r \), then,

\[
d_G(u) + r - 2 < r + r - 2 \\
d_{L(G)}(x) < d_{L(G)}(w),
\]
hence, $x$ can not be a l.max. vertex.

If $d_G(u) = r$ then as $u$ is not a l.max. vertex, there exists a vertex $z \in N(u) \setminus \{v_1\}$ such that $d_G(z) > d_G(u)$. Then, the edge $uz$ corresponds to a vertex $y$ in $L(G)$, adjacent to $x$ as shown in Figure 3.5. Now,

$$d_G(z) > d_G(u)$$
$$d_G(z) + d_G(u) - 2 > d_G(u) + d_G(u) - 2$$
$$d_G(z) + d_G(u) - 2 > d_G(u) + r - 2$$
$$d_{L(G)}(y) > d_{L(G)}(x),$$

and hence, $x$ can not be a l.max. vertex. □

**Corollary 3.1:** It follows from Lemma 3.5 that $L(C)$ is a component of $LM(L(G))$.

**Corollary 3.2:** If $C$ is a single vertex, then from Lemma 3.4 it generates at most one component of $LM(L(G))$. Otherwise, if $C$ is not a single vertex, then every vertex of $C$ generates a l.max. vertex from Lemma 3.5(a). As the line graph operation preserves connectivity, $C$ will generate at most one component of $LM(L(G))$. Hence, in either case, $C$ generates at most one component.

**Lemma 3.6.** There exists an integer $A$ such that for all $k > A$, every component of $LM(L^k(G))$ generates exactly one component of $LM(L^{k+1}(G))$.

Proof: Let $c_k$ be the number of components of $LM(L^k(G))$. From Corollary 3.2, $\{c_k\}$ is a non-increasing sequence. But as $c_k$ is a non-negative number for all $k$, there exists an integer $A$, such that $c_k$ is constant for all $k > A$. □

We now define new notation which would be followed in the rest of this chapter. Let $C_{A+1}$ be a component of $LM(L^{A+1}(G))$ where $A$ is the integer from Lemma 3.6. Inductively, for each $k > A$, let $C_{k+1}$ be the component of $L^{k+1}(G)$ generated by $C_k$. Let $r_k$ be the common
degree of vertices in $C_k$. We can further choose $A$ to be sufficiently large so that $\delta_k > 2$ for all $k > A$ from Lemma 2.1.

**Lemma 3.7.** Let $u \in N(C_D)$ be adjacent to a vertex $v_D \in C_D$, where $D$ is an integer greater than $A$. Let $y \in L^{D+1}(G)$ correspond to the edge $uv$ of $L^D(G)$, so $y \in N[C_{D+1}]$.

(a) If $C_D$ is not a single vertex, so $y \in N(C_{D+1})$, and,

$$r_{D+1} - d_{L^{D+1}(G)}(y) = r_D - d_{L^D(G)}(u).$$

(b) In case $C_D$ is a single vertex, then,

$$r_{D+1} - d_{L^{D+1}(G)}(y) < r_D - d_{L^D(G)}(u)$$

Proof:

(a) From Lemma 3.5(a), if $C_D$ has an edge then it generates $C_{D+1}$, as shown in Figure 3.6, and $r_{D+1} = 2r_D - 2$.

Also, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. So,

$$r_{D+1} - d_{L^{D+1}(G)}(y) = (2r_D - 2) - (d_{L^D(G)}(u) + r_D - 2)$$

$$r_{D+1} - d_{L^{D+1}(G)}(y) = r_D - d_{L^D(G)}(u).$$
(b) Suppose \( y \in N(C_{D+1}) \). Again, \( d_{L^{D+1}(G)}(y) = d_{L^{D}(G)}(u) + r_D - 2 \). Let \( x \) be a vertex of largest degree in \( N(v_D) \) such that the edge \( xv_D \) corresponds to a l.max. vertex \( v_{D+1} \) in \( C_{D+1} \) from Lemma 3.6. Such a vertex \( x \) exists from Lemma 3.3 and as \( C_{D+1} \) is non-empty, we have,

\[
r_{D+1} = d_{L^{D}(G)}(x) + r_D - 2.
\]

As \( C_D \) is a single vertex, from Lemma 3.4(a), \( d_{L^{D}(G)}(x) < r_D \) as \( C_D \) generates \( C_{D+1} \) and hence,

\[
d_{L^{D}(G)}(x) + r_D - 2 < r_D + r_D - 2
\]

\[
r_{D+1} < 2r_D - 2
\]

\[
r_{D+1} - d_{L^{D+1}(G)}(y) < (2r_D - 2) - d_{L^{D+1}(G)}(y)
\]

But, since \( d_{L^{D+1}(G)}(y) = d_{L^{D}(G)}(u) + r_D - 2 \), we have,

\[
r_{D+1} - d_{L^{D+1}(G)}(y) < (2r_D - 2) - (d_{L^{D}(G)}(u) + r_D - 2)
\]

\[
r_{D+1} - d_{L^{D+1}(G)}(y) < r_D - d_{L^{D}(G)}(u)
\]

Now, suppose \( y \in C_{D+1} \). Then \( r_{D+1} - d_{L^{D+1}(G)}(y) = 0 \). Also, as \( C_D \) is a single vertex, \( r_D - d_{L^{D}(G)}(u) \neq 0 \) as otherwise \( C_D \) will not generate a component. Hence, \( r_{D+1} - d_{L^{D+1}(G)}(y) < r_D - d_{L^{D}(G)}(u) \).

\[ \square \]

**Lemma 3.8.** If \( u \in N(C_k) \) then \( u \) generates a vertex \( y \in N[C_{k+1}] \).
Proof: As \( u \in N(C_k) \), \( u \) is adjacent to a vertex \( v \in C_k \). Let the edge \( uv \) correspond to the vertex \( y \in L^{k+1}(G) \). If \( C_k \) has an edge, from Lemma 3.5(a) \( v \) generates a vertex in \( C_{k+1} \). Also, if \( C_k = \{ v \} \), as \( k > A \), \( v \) generates every vertex in \( C_{k+1} \). So, there exists a vertex \( w \in C_{k+1} \) generated by \( v \). Now, the edges in \( L^k(G) \) corresponding to \( y \) and \( w \), are incident at the vertex \( v \). Hence, \( y \) is adjacent to the vertex \( w \) in \( L^{k+1}(G) \), implying that, if \( y \) is a l.max. vertex then \( y \in C_{k+1} \) or else \( y \in N(C_{k+1}) \).

Let \( N(C_B) = \{ u_1, u_2, \ldots, u_n \} \). Then from Lemma 3.8, for every \( 1 \leq j \leq n \), \( u_j \) generates a vertex, say \( y^1_j \), in \( N[C_{B+1}] \).

Now, if \( y^i_j \) is a vertex in \( N(C_{B+i}) \), then from Lemma 3.8, it generates a vertex, say \( y^{i+1}_j \), in \( N[C_{B+i+1}] \). Otherwise, if \( y^i_j \) is a vertex in \( C_{B+i} \), then from Lemma 3.5(a), it generates a vertex, say \( y^{i+1}_j \), in \( C_{B+i+1} \). It follows inductively that \( u_j \) generates a sequence of vertices \( (u_j = y^0_j, y^1_j, y^2_j, y^3_j, \ldots) \) where \( y^i_j \in N[C_{B+i}] \) and, moreover, \( y^i_j \in C_{B+i} \) for all \( i > I \) if \( y^I_j \in C_{B+i} \) for some integer \( I \).

Then we define a function \( f(u_j, i) : N(C_B) \rightarrow \mathbb{R} \) by \( f(u_j, i) = r_{B+i} - d_{L^{B+i}(G)}(y^i_j) \) where \( i \in \mathbb{Z}^+ \). Clearly \( f(u_j, i) \) is non-negative and from Lemma 3.7 it is a non-increasing function of \( i \). Also, if \( C_{B+i} \) is a single vertex and \( y^i_j \in N(C_{B+i}) \), then, from Lemma 3.4(a), \( f(u_j, i) \) can not equal to zero because otherwise \( C_{B+i} \) will not generate a component.

**Theorem 3.1.** Let \( G \) be a simple connected graph. Let \( C_A \) be a component of \( LM(L^A(G)) \). Then, there are a finite number of integers \( k > A \), such that \( C_k \), generated by \( C_A \), is a single vertex.

Proof: The proof is by contradiction. Let us assume that there are an infinite number of integers \( k > A \) such that \( C_k \) is a single vertex. Then we prove the following series of lemmas.

**Lemma 3.9.** If \( u_1 \in N(C_B) \) generates \( (y^0_1, y^1_1, y^2_1, y^3_1, \ldots) \), then there exists an integer \( I \) such that \( y^I_1 \in C_{B+I} \).
Proof: We prove this by contradiction. Let \( y^i_1 \in N(C_{B+i}) \) for all \( i \). The function \( f(u_1, i) \) is non-increasing and decreases when \( C_{B+i} \) is a single vertex. As there are infinite number of integers \( k > A \) such that \( C_k \) is a single vertex, there are infinite integers \( i \) such that \( C_{B+i} \) is a single vertex as \( B > A \). Hence, from Lemma 3.7(b) there exists an integer \( D > B \) such that \( f(u_1, D - B) = 0 \).

Now, if \( C_D \) is a single vertex, then as \( y^i_1 \in N(C_{B+i}) \) for all \( i \), it follows that \( f(u_1, D - B) \) can not be zero and we have a contradiction. Otherwise, if \( C_D \) has an edge, then let \( E \) be the smallest integer greater than \( D \) such that \( C_E \) is a single vertex. From Lemma 3.7(a), \( f(u_1, E - B) = f(u_1, D - B) = 0 \), and we again have a contradiction. \( \square \)

**Lemma 3.10.** If \( u_1 \in N(C_B) \) then there exists an integer \( D \geq B \) such that \( u_1 \) generates \( y^D_B \in N(C_D) \) where \( C_D \) is a single vertex and \( d_{L^D(G)}(y^D_B) \) is maximum in \( N(C_D) \).

Proof: From Lemma 3.9 there exists an integer \( I \) such that \( u_1 \) generates \( y^I_1 \in C_{B+I} \). Let \( I \) be the smallest such integer. Hence, \( y^I_1 \in N(C_{B+I-1}) \). From Lemma 3.5, if \( C_{B+I-1} \) has an edge then \( y^I_1 \) cannot generate a vertex in \( C_{B+I} \). Hence, \( C_{B+I-1} \) is a single vertex. Also, from Lemma 3.3, \( d_{L^{B+1-I}(G)}(y^I_1) \) is maximum in \( N(C_{B+I-1}) \). \( \square \)

**Lemma 3.11.** If \( u_1 \in N(C_B) \) where \( C_B \) is not a single vertex, then, \( d_{L^B(G)}(u_1) \neq r_B \).

Proof: Assume that \( d_{L^B(G)}(u_1) = r_B \) and hence, \( f(u_1, 0) = 0 \). But as \( f(u_1, j) \) is non-negative and non-increasing, \( f(u_1, j) = 0 \) for all \( j \). But, from Lemma 3.10, there exists an integer \( D \geq B \) such that \( u_1 \) generates \( y^D_B \in N(C_D) \) where \( C_D \) is a single vertex with \( f(u_1, D - B) = 0 \), which is a contradiction. \( \square \)

**Corollary 3.3.** From Lemma 3.4(a) and Lemma 3.11, if \( u \in N(C_k) \) then \( d_{L^k(G)}(u_1) \neq r_k \).

**Lemma 3.12.** Let \( C_B = \{ v_B \} \) and \( u_1, u_2, ..., u_n \) be vertices of equal degree in \( N(C_B) \) such that \( d_{L^B(G)}(u_i) \) is maximum in \( N(C_B) \). Then, \( u_i \) generates a vertex \( v_i \in C_{B+1} \) for all \( 1 \leq i \leq n \). Moreover, \( u_1, u_2, ..., u_n \) generate \( l \).max. vertices which induce a complete subgraph in \( C_{B+1} \).
Proof: As \( C_B \) generates \( C_{B+1} \), from Lemma 3.3 there exists an integer \( I \in [1, n] \) such that \( u_I \) generates a vertex \( v_{B+1} \in C_{B+1} \). Let there be some \( J \neq I \) such that \( u_J \) does not generate any vertex in \( C_{B+1} \). Then, from Lemma 3.8, \( u_J \) generates a vertex, say \( u \), in \( N(C_{B+1}) \). Now, 
\[
    r_{B+1} = L_{B+1}(u_{B+1}) = L_B(u_I) + r_B - 2 = L_B(u_J) + r_B - 2 = L_{B+1}(u)
\]
which is a contradiction from Corollary 3.3 and hence no such \( J \) exists.

So, all \( u_1, u_2, \ldots, u_n \) generate \( \text{max} \) vertices, say \( v_1, v_2, \ldots, v_n \), in \( C_{B+1} \) such that \( v_i \) corresponds to the edge \( u_iv_B \) in \( L^B(G) \). As all the corresponding edges share the vertex \( v_B \), the vertices \( v_1, v_2, \ldots, v_n \) induce a complete subgraph. \( \square \)

**Lemma 3.13.** Let \( u_1, u_2 \in N(C_B) \) with \( L_B(u_1) = L_B(u_2) \). Furthermore, let \( u_i \) generate the sequence \((u_1 = y_1^0, y_1^1, y_1^2, y_1^3, \ldots)\) and \( u_2 \) generate the sequence \((u_2 = y_2^0, y_2^1, y_2^2, y_2^3, \ldots)\). Then, \( L_{B+i}(y_1^i) = L_{B+i}(y_2^i) \) for all \( i \in \mathbb{Z}^+ \) and either \( y_1^i, y_2^i \in C_{B+i} \) or \( y_1^i, y_2^i \in N(C_{B+i}) \).

Proof: For \( i = 1 \),
\[
    L_{B+1}(y_1^1) = L_B(u_1) + r_B - 2 = L_B(u_2) + r_B - 2 = L_{B+1}(y_2^1).
\]

If \( C_B \) has an edge, then \( y_1^1, y_2^1 \in N(C_{B+1}) \) from Lemma 3.5(b) as \( u_1, u_2 \in N(C_B) \), otherwise, \( C_B \) is a single vertex. If \( L_B(u_1) = L_B(u_2) \) is maximum in \( N(C_B) \), then \( y_1^1, y_2^1 \in C_{B+1} \) from Lemma 3.12. On the other hand, if \( L_B(u_1) = L_B(u_2) \) is not maximum in \( N(C_B) \), then \( y_1^1, y_2^1 \in N(C_{B+1}) \).

Let, for \( i = n \), \( L_{B+n}(y_1^n) = L_{B+n}(y_2^n) \) and either \( y_1^n, y_2^n \in C_{B+n} \) or \( y_1^n, y_2^n \in N(C_{B+n}) \).

Now, if \( y_1^n, y_2^n \in C_{B+n} \) then from Lemma 3.5(a), \( y_1^{n+1}, y_2^{n+1} \in C_{B+n+1} \) and \( L_{B+n+1}(y_1^{n+1}) = L_{B+n+1}(y_2^{n+1}) = r_{B+n+1} \).

Otherwise \( y_1^n, y_2^n \in N(C_{B+n}) \). If \( C_{B+n} \) has an edge, from Lemma 3.5(b) we get that
\[ y_1^{n+1}, y_2^{n+1} \in N(C_{B+n+1}). \] Then,

\[
d_{L_{B+n+1}(G)}(y_1^{n+1}) = d_{L_{B+n}(G)}(y_1^n) + r_{B+n} - 2
\]

\[
= d_{L_{B+n}(G)}(y_2^n) + r_{B+n} - 2
\]

\[
= d_{L_{B+n+1}(G)}(y_2^{n+1}).
\]

But, if \( y_1^n, y_2^n \in N(C_{B+n}) \) and \( C_{B+n} \) is a single vertex, then, if \( d_{L_{B+n}(G)}(y_1^n) = d_{L_{B+n}(G)}(y_2^n) \) is maximum in \( N(C_{B+n}) \), and then from Lemma 3.12, \( y_1^n \) and \( y_2^n \) generate \( y_1^{n+1} \) and \( y_2^{n+1} \), respectively, in \( C_{B+n+1} \). Else, if \( d_{L_{B+n}(G)}(y_1^n) = d_{L_{B+n}(G)}(y_2^n) \) is not maximum in \( N(C_{B+n}) \), then from Lemma 3.3, \( y_1^{n+1} \) and \( y_2^{n+1} \) are in \( N(C_{B+n+1}) \) and \( d_{L_{B+n+1}(G)}(y_1^{n+1}) = d_{L_{B+n}(G)}(y_1^n) + r_{B+n} - 2 = d_{L_{B+n}(G)}(y_2^n) + r_{B+n} - 2 = d_{L_{B+n+1}(G)}(y_2^{n+1}). \]

**Lemma 3.14.** If \( u_1, u_2, ..., u_n \in N(C_B) \) with \( d_{L_{B}(G)}(u_i) = d_{L_{B}(G)}(u_j) \), then there exists an integer \( E > B \) such that \( u_1, u_2, ..., u_n \) generate vertices \( y_1^{E-B}, y_2^{E-B}, ..., y_n^{E-B} \in C_E \) which form a clique.

Proof: From Lemma 3.10 and Lemma 3.13, there exists an integer \( D \geq B \) such that \( u_j \) generates \( y_j^{D-B} \in N(C_D) \), \( 1 \leq j \leq n \), where \( C_D \) is a single vertex, say \( v_D \), and \( d_{L_{D}(G)}(y_j^{D-B}) \) is maximum in \( N(C_D) \). Then, from Lemma 3.12, \( y_j^{D-B} \) for \( 1 \leq j \leq n \) induce a complete subgraph in \( C_{D+1} \).

**Lemma 3.15.** There exists an integer \( E > A \) such that \( C_{E-1} \) has exactly one edge.

Proof: Pick an integer \( B > A \) such that \( C_B = \{v_B\} \). Such an integer exists from our assumption that there are infinite integers \( k > A \) where \( C_k \) is a single vertex. Then, as
δ_A > 2, we have, from Lemma 2.1,

\[ \delta_B > 2 \]
\[ -\delta_B < -2 \]
\[ r_B - \delta_B + 1 < r_B - 2 + 1 \]
\[ r_B - \delta_k + 1 < r_B \]

Now, there are \( r_B \) neighbors of \( v_B \) with \( r_B - \delta_B + 1 \) possible unique degrees. Hence, by Pigeonhole principle, there exists at least two vertices \( u_1, u_2 \in N\langle C_B \rangle \) such that \( d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2) \). Then from Lemma 3.14, \( u_1, u_2 \) will induce an edge in \( C_D \) for some \( D > B \). But, as there are infinite integers such that \( C_k \) is a single vertex, there exists an integer \( E > D \) such that \( C_E \) is a single vertex. Let \( E \) be the smallest such integer. Then, \( C_{E-1} \) will have exactly one edge and the lemma is proved.

\[ \square \]

**Lemma 3.16.** There exists an integer \( E \) such that \( C_E = \{v_E\} \) and there are three vertices \( u_1, u_2, u_3 \in N\langle C_E \rangle \) with equal degree.

Proof: Let \( \delta'_k \) denote the minimum degree in \( N\langle C_k \rangle \). Then,

\[ \delta'_k = \delta'_{k-1} + r_{k-1} - 2. \]
Now, pick an integer \( B \) such that \( C_B \) has exactly one edge. Such an integer exists from Lemma 3.15. Then, \( r_{B+1} = 2r_B - 2 \) from Lemma 3.5(a). But,

\[
\begin{align*}
\delta_B' &> 2 \\
\delta_B' + r_B - 1 &> 2 + r_B - 1 \\
r_B - 1 &> 2 + r_B - 1 - \delta_B' \\
r_B - 1 &> 2 + 2r_B - 1 - r_B - \delta_B' + 2 - 2 \\
r_B - 1 &> (2r_B - 2) - (r_B + \delta_B' - 2) + 1 \\
r_B - 1 &> r_{B+1} - \delta_{B+1}' + 1 \\
2r_B - 2 &> 2(r_{B+1} - \delta_{B+1}' + 1) \\
r_{B+1} &> 2(r_{B+1} - \delta_{B+1}' + 1)
\end{align*}
\]

Now, \( r_{B+1} \) is the number of neighbors of \( v_{B+1} \) and as \( v_{B+1} \) is a l.max. vertex, we have that \((r_{B+1} - \delta_{B+1}' + 1)\) is the number of possible unique values of degree of a neighbor of \( v_{B+1} \). Also, because \( C_B \) has exactly one edge, \( C_{B+1} = \{v_{B+1}\} \), and therefore, from Pigeonhole principle, there exist at least three vertices of equal degree in \( N(C_{B+1}) \).

Continuing rest of the proof of Theorem 3.1:

Now, from Lemma 3.16 and Lemma 3.14, there exists an integer \( F > E \) such that \( C_F \) contains a \( K_3 \). Hence, from Lemma 3.5(a), \( C_k \) contains \( K_3 \) all \( k > F \) which contradicts that there are infinite number of integers \( k > A \) such that \( C_k \) is a single vertex. Hence, there are finite values of \( k > A \) where \( C_k \) is a single vertex and there exists an integer \( I \), such that \( C_{I+i} \), generated by \( C_k \), has at least one edge, for all \( i \).

So, from Theorem 3.1, for each component \( C_B^j \) of \( LM(L_B(G)) \) where \( B > A \), there exists an integer \( I_j > B \) such that \( C_{I_j+i}^j \) generated by \( C_B^j \) has at least one edge for all \( i \). Suppose \( LM(L_B^B(G)) \) has \( N \) components. Using this reasoning for all components \( C_B^j, 1 \leq j \leq N \), there exists an integer \( D = \text{max}\{I_j \mid 1 \leq j \leq N\} \), such that every component of \( LM(L_B^D+i(G)) \) has at least one edge for all \( i \).
Clearly, the vertices of maximum degree of any graph $G$ are also l.max. vertices and hence are components of $LM(G)$. But every component of $LM(L^{D+i})$ has edges for all $i$, hence, every vertex of maximum degree is adjacent to at least one vertex of maximum degree, and so, $\Delta_k = 2\Delta_{k-1} - 2$ for all $k > D$. 
Chapter 4
Minimum degree growth in iterated line graphs

In this chapter it will be shown that for any graph $G$, which is not a path, there exists an integer $D$ such that $\delta_{k+1} = 2\delta_k - 2$ for all $k > D$, where $\delta_k$ is the minimum degree of $L^k(G)$.

Note that, most of the lemmas for this proof parallel the lemmas, proved in Chapter 3, with the inequalities reversed. However, a different line of reasoning is used in the second half of the proof to contradict a theorem similar to Theorem 3.1. If $G$ is a path, then as $G$ is a finite graph, there exists an integer $I$ such that $L^I(G)$ is undefined.

If $G$ is a cycle, then for all $k \in \mathbb{Z}^+$, $\delta_{k+1} = 2\delta_k - 2 = 2$.

If $G$ is a $K_{1,3}$, then $L(G)$ is a $K_3$ and hence, for all $k > 1$, $\delta_{k+1} = 2\delta_k - 2 = 2$.

Now we have to prove the theorem for any graph $G$ where it is not a path, a cycle or a $K_{1,3}$.

**Definition:** A vertex $v$ is a *locally minimum vertex* or a *l.min. vertex* if no vertex in the neighborhood of $v$ has degree smaller than that of $v$.

**Definition:** The subgraph of $G$ induced by its l.min. vertices is called the *locally minimum subgraph* or *l.min. subgraph* of $G$ and is denoted by $lm(G)$.

**Lemma 4.1.** All vertices in the same component of $lm(G)$ have the same degree in $G$.

Proof: Let $v$ and $u$ be two vertices in a component of $lm(G)$. Then $v$ and $u$ are l.min. vertices of the graph $G$. As $v \in N(u)$, $d(v) \geq d(u)$ from definition. Similarly, as $u \in N(v)$, $d(u) \geq d(v)$. Hence, $d(u) = d(v)$. \qed
Lemma 4.2. If \( w \) is a l.min. vertex of \( L(G) \), then \( w \) corresponds to an edge \( e \) in \( G \) such that at least one end of \( e \), say \( v \), is l.min. in \( G \) and the other end of \( e \), say \( u \), has the smallest degree among the neighbors of \( v \) in \( G \).

Proof: Assume that neither \( v \) nor \( u \) is a l.min. vertex. Let \( d_G(v) \leq d_G(u) \). Then, as \( v \) is not a l.min. vertex, there exists a vertex \( y \in N(v) \) such that \( d_G(y) < d_G(v) \).

Now, the edge \( vy \) of \( G \) corresponds to the vertex \( vy \) of \( L(G) \), adjacent to \( w \), as shown in the Figure 4.1. Also,

\[
d_G(v) \leq d_G(u).
\]

But, as \( d_G(y) < d_G(v) \),

\[
d_G(v) + d_G(y) - 2 < d_G(u) + d_G(v) - 2,
\]

\[
d_{L(G)}(vy) < d_{L(G)}(w),
\]

contradicting that \( w \) is a l.min. vertex of \( L(G) \), hence, no such \( y \) exists, implying that \( v \) is a l.min. vertex of \( G \).

Now, let there exist a vertex \( z \in N(v) \) such that \( d_G(z) < d_G(u) \).

Then the edge \( vz \) of \( G \) corresponds to the vertex \( vz \) adjacent to \( w \) in \( L(G) \), as shown.
in the Figure 4.2.

But,

\[
\begin{align*}
  d_G(z) &< d_G(u) \\
  d_G(z) + d_G(v) - 2 &< d_G(u) + d_G(v) - 2 \\
  d_{L(G)}(vz) &< d_{L(G)}(w),
\end{align*}
\]

contradicting that \( w \) is a l.min. vertex of \( L(G) \). Hence, no such \( z \) exists, implying that \( u \) has the minimum degree in \( N(v) \).

Lemma 4.3. Let \( v \) be an isolated vertex of \( LM(G) \).

(a) If \( v \) has any neighbor of the same degree as that of \( v \), then, \( v \) generates no l.min. vertices of \( L(G) \).

(b) If all neighbors of \( v \) have degree greater than that of \( v \), and \( u \) is such a neighbor, then the edge \( uv \) corresponds to a l.min. vertex of \( L(G) \) if and only if \( u \) has the minimum degree among the neighbors of \( v \), and for all \( z \in N(u) \setminus \{v\} \), \( d_G(z) \geq d_G(v) \).

Proof:

(a) Let \( u \) be a neighbor of \( v \) such that \( d_G(u) = d_G(v) \). As \( u \) is not a l.min. vertex of \( G \), there exists a vertex \( z \) adjacent to \( u \), such that, \( d_G(z) < d_G(u) = d_G(v) \). Then,
the edge \( uz \) of \( G \) corresponds to a vertex \( uz \) in \( L(G) \), adjacent to a vertex \( w \), which in turn corresponds to the edge \( uv \) of \( G \), as shown in Figure 4.3. Now, \( d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 < d_G(u) + d_G(v) - 2 = d_{L(G)}(w) \).

So, the edge \( uv \) cannot correspond to a l.min. vertex of \( L(G) \) for any \( u \) with \( d_G(u) = d_G(v) \). Hence, by Lemma 4.2, \( v \) does not generate a l.min. vertex in \( L(G) \).

(b) Let there exist a vertex \( z \in N(u) \setminus \{v\} \) such that \( d_G(z) < d_G(v) \). Then, the edge \( uz \) corresponds to a vertex \( uz \) in \( L(G) \) which is adjacent to a vertex \( w \), which in turn corresponds to the edge \( vu \) of \( G \), as shown in Figure 4.3. Now, \( d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 < d_G(u) + d_G(v) - 2 = d_{L(G)}(w) \).

Therefore, the edge \( uv \) will not correspond to a l.min. vertex of \( L(G) \). Now, let, for all \( z \in N(u) \setminus \{v\} \), \( d_G(z) \geq d_G(v) \). Then,

\[
d_G(u) + d_G(z) - 2 \geq d_G(u) + d_G(v) - 2
\]

\[
d_{L(G)}(uz) \geq d_{L(G)}(w),
\]

where \( w \) corresponds to the edge \( vu \). Therefore, \( w \) would be a l.min. vertex. Moreover, if \( uz \) is a l.min. vertex, it would be adjacent to \( w \) implying that the number of components will not increase. \( \square \)

**Lemma 4.4.** Let \( C \) be a component of \( \text{lm}(G) \) which is not a single vertex.

a) If \( v_1 \) and \( v_2 \) are adjacent vertices in \( C \), then the vertex \( w \in L(G) \), corresponding to the edge \( v_1v_2 \), is a l.min. vertex.
b) If \( u \in N(C) \), then no edge joining \( u \) to a vertex in \( C \) corresponds to a l.min. vertex of \( L(G) \).

Proof:

\[ d_G(z) \geq d_G(v_1), \]
\[ d_G(z) + d_G(v_2) - 2 \geq d_G(v_1) + d_G(v_2) - 2. \]

From Lemma 4.1, \( d_G(v_1) = d_G(v_2) \),

\[ d_G(z) + d_G(v_1) - 2 \geq d_G(v_1) + d_G(v_2) - 2, \]
\[ d_{L(G)}(x) \geq d_{L(G)}(w). \]

Hence, \( w \) is a l.min. vertex.
b) As $u \in N(C)$, it is adjacent to a vertex, say $v_1$, in $C$. As $C$ is not a single vertex, there exists a vertex $v_2 \in C$ adjacent to $v_1$. Let $w$ be the vertex in $L(G)$ corresponding to the edge $v_1v_2$ and let $r$ be the common degree of vertices in $C$. Then, $d_{L(G)}(w) = 2r - 2$.

Now, the edge $uv_1$ corresponds to a vertex $x$ adjacent to $w$, as shown in Figure 4.5. Also, $d_{L(G)}(x) = d_G(u) + r - 2$ and as $v_1$ is a l.min. vertex, we get that $d_G(u) \geq r$.

If $d_G(u) > r$, then,

$$d_G(u) + r - 2 > r + r - 2,$$

$$d_{L(G)}(x) > d_{L(G)}(w),$$

and $x$ can not be a l.min. vertex.

Otherwise, $d_G(u) = r$. But as $u$ is not a l.min. vertex, there exists a vertex $z \in N(u) \setminus \{v_1\}$ such that $d_G(z) < d_G(u)$. Then, the edge $uz$ of $G$ corresponds to a vertex $y$ adjacent to $x$, as shown in Figure 4.5. Now,

$$d_G(z) < d_G(u),$$

$$d_G(z) + d_G(u) - 2 < d_G(u) + d_G(u) - 2,$$

$$d_G(z) + d_G(u) - 2 < d_G(u) + r - 2,$$

$$d_{L(G)}(y) < d_{L(G)}(x),$$

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and hence, $x$ can not be a l.min. vertex.

**Corollary 4.1:** It follows from Lemma 4.4 that $L(C)$ is a component of $lm(L(G))$.

**Corollary 4.2:** If $C$ is a single vertex, then from Lemma 4.3 it generates at most one component of $lm(L(G))$. Otherwise, if $C$ is not a single vertex, every vertex of $C$ generates a l.min. vertex from Lemma 4.4(a). As the line graph operation preserves connectivity, $C$ will generate at most one component of $lm(L(G))$. Hence, in either case, $C$ generates at most one component.

**Lemma 4.5.** There exists an integer $A$ such that for all $k > A$, every component of $lm(L^k(G))$ generates exactly one component of $lm(L^{k+1}(G))$.

Proof: Let $c_k$ be the number of components of $lm(L^k(G))$. From Corollary 4.2, $\{c_k\}$ is a non-increasing sequence. But as $c_k$ is a non-negative number for every $k$, there exists an integer $A$, such that $c_k$ is constant for all $k > A$.

We now define new notation which would be followed in the rest of this chapter. Let $C_{A+1}$ be a component of $lm(L^{A+1}(G))$ where $A$ is the integer from Lemma 4.5. Inductively, for each $k > A$, let $C_{k+1}$ be the component of $L^{k+1}(G)$ generated by $C_k$. Let $r_k$ be the common degree of vertices in $C_k$. We can further choose $A$ to be sufficiently large so that $\delta_k > 2$ for all $k > A$, from Lemma 2.1.

**Lemma 4.6.** Let $u \in N(C_D)$ be adjacent to a vertex $v \in C_D$, where $D$ is an integer greater than $A$. Further, let $y \in L^{D+1}(G)$ correspond to the edge $uv$ of $L^D(G)$, so $y \in N[C_{D+1}]$. Then the following holds.

(a) If $C_D$ is not a single vertex, then

$$d_{L^{D+1}(G)}(y) - r_{D+1} = d_{L^D(G)}(u) - r_D$$
(b) Otherwise, if $C_D$ is a single vertex, then,

$$d_{L^{D+1}(G)}(y) - r_{D+1} < d_{L^D(G)}(u) - r_D$$

Proof:

(a) From Lemma 4.4(a), if $C_D$ has an edge then it generates $C_{D+1}$, as shown in Figure 4.6, and $r_{D+1} = 2r_D - 2$.

Also, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. So,

$$d_{L^{D+1}(G)}(y) - r_{D+1} = (d_{L^D(G)}(u) + r_D - 2) - (2r_D - 2)$$

$$d_{L^{D+1}(G)}(y) - r_{D+1} = d_{L^D(G)}(u) - r_D.$$

(b) Suppose $y \in N(C_{D+1})$. Again, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. Let $x$ be a vertex of smallest degree in $N(v_D)$ such that the edge $xv_D$ corresponds to a l.min. vertex, say
v_{D+1}, in C_{D+1} from Lemma 4.5. Such a vertex x exists from Lemma 4.2 and because C_{D+1} is non-empty. Then,

\[ r_{D+1} = d_{L^D(G)}(x) + r_D - 2. \]

Since C_D is a single vertex, hence from Lemma 4.3(a), \( d_{L^D(G)}(x) > r_D \) as C_D generates C_{D+1}. So,

\[ d_{L^D(G)}(x) + r_D - 2 > r_D + r_D - 2, \]
\[ r_{D+1} > 2r_D - 2, \]
\[ r_{D+1} - d_{L^{D+1}(G)}(y) > (2r_D - 2) - d_{L^{D+1}(G)}(y). \]

But, \( d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2 \), therefore,

\[ r_{D+1} - d_{L^{D+1}(G)}(y) > (2r_D - 2) - (d_{L^D(G)}(u) + r_D - 2), \]
\[ r_{D+1} - d_{L^{D+1}(G)}(y) > r_D - d_{L^D(G)}(u), \]
\[ d_{L^{D+1}(G)}(y) - r_{D+1} < d_{L^D(G)}(u) - r_D. \]

Now, suppose \( y \in C_{D+1} \). Then \( d_{L^{D+1}(G)}(y) - r_{D+1} = 0 \). But as \( C_D \) is a single vertex, \( d_{L^D(G)}(u) - r_D \neq 0 \), as otherwise \( C_D \) will not generate a component. Hence \( d_{L^{D+1}(G)}(y) - r_{D+1} < d_{L^D(G)}(u) - r_D. \)

**Lemma 4.7.** If \( u \in N(C_k) \) then \( u \) generates a vertex \( y \in N[C_{k+1}]. \)

Proof: As \( u \in N(C_k) \), \( u \) is adjacent to a vertex \( v \in C_k \). Let the edge \( uv \) correspond to the vertex \( y \in L^{k+1}(G) \). If \( C_k \) has an edge, from Lemma 4.4(a) \( v \) generates a vertex in \( C_{k+1} \). Also, if \( C_k = \{v\} \), as \( k > A \), \( v \) generates every vertex in \( C_{k+1} \). Then, there exists a vertex \( w \in C_{k+1} \) generated by \( v \). Now, the edges in \( L^k(G) \) corresponding to \( y \) and \( w \), are incident at the vertex \( v \). Hence, \( y \) is adjacent to the vertex \( w \) in \( L^{k+1}(G) \), implying that, if \( y \) is a
Let $N(C_B) = \{u_1, u_2, \ldots, u_n\}$. Then, from Lemma 4.7, for every $1 \leq j \leq n$, $u_j$ generates a vertex, say $y^1_j$, in $N[C_{B+1}]$.

Now, if $y^i_j$ is a vertex in $N(C_{B+i})$, then from Lemma 4.7, it generates a vertex, say $y^{i+1}_j$ in $N[C_{B+i+1}]$. Otherwise, if $y^i_j$ is a vertex in $C_{B+i}$, then from Lemma 4.4(a), it generates a vertex, say $y^{i+1}_j$ in $C_{B+i+1}$. It follows inductively that $u_j$ generates a sequence of vertices $(u_j = y^0_j, y^1_j, y^2_j, y^3_j, \ldots)$ where $y^i_j \in N[C_{B+i}]$ and, moreover, $y^i_j \in C_{B+i}$ for all $i > I$ if $y^I_j \in C_{B+i}$ for some integer $I$.

Then, we define a function $f(u_j, i) : N(C_B) \rightarrow \mathbb{R}$ by $f(u_j, i) = d_{L^B+L^A(G)}(y^i_j) - r_{B+i}$ where $i \in \mathbb{Z}^+$. Clearly $f(u_j, i)$ is non-negative and from Lemma 4.6 it is a non-increasing function of $i$. Also, if $C_{B+i}$ is a single vertex and $y^i_j \in N(C_{B+i})$, then, from Lemma 4.3(a), $f(u_j, i)$ can not equal to zero as otherwise $C_{B+i}$ will not generate a component.

**Theorem 4.1.** Let $G$ be a simple and connected graph. Let $C_A$ be a component of $lm(L^A(G))$. Then, there are a finite number of integers $k > A$, such that $C_k$, generated by $C_A$, is a single vertex.

Proof: The proof is by contradiction. Let us assume that there are infinite number of integers $k > A$ such that $C_k$ is a single vertex. Then we prove the following series of lemmas.

**Lemma 4.8.** If $u_1 \in N(C_B)$ generates $(y^0_1, y^1_1, y^2_1, y^3_1, \ldots)$, then there exists an integer $I$ such that $y^I_1 \in C_{B+1}$.

Proof: We prove this by contradiction. Let $y^i_1 \in N(C_{B+i})$ for all $i$. The function $f(u_1, i)$ is non-increasing and decreases when $C_{B+i}$ is a single vertex. As there are infinite number of integers $k > A$ such that $C_k$ is a single vertex, there are infinite integers $i$ such that $C_{B+i}$ is a single vertex as $B > A$. Hence, from Lemma 4.6(b) there exists an integer $D > B$ such that $f(u_1, D - B) = 0$.

Now, if $C_D$ is a single vertex, then as $y^i_1 \in N(C_{B+i})$ for all $i$, $f(u_1, D - B)$ can not be
zero and we have a contradiction. Otherwise, if \( C_D \) has an edge, then let \( E \) be the smallest integer greater than \( D \) such that \( C_E \) is a single vertex. From Lemma 4.6(a), \( f(u_1, E - B) = f(u_1, D - B) = 0 \), and we again have a contradiction.

**Lemma 4.9.** If \( u_1 \in N(C_B) \) then there exists an integer \( D \geq B \) such that \( u_1 \) generates \( y_1^{D-B} \in N(C_D) \) where \( C_D \) is a single vertex and \( d_{L^D(G)}(y_1^{D-B}) \) is minimum in \( N(C_D) \).

**Proof:** From Lemma 4.8 there exists an integer \( I \) such that \( u_1 \) generates \( y_I^I \in C_{B+I} \). Let \( I \) be the smallest such integer. Then, \( y_I^{I-1} \in N(C_{B+I-1}) \). From Lemma 4.4, if \( C_{B+I-1} \) has an edge then \( y_I^{I-1} \) cannot generate a vertex in \( C_{B+I} \). Hence, \( C_{B+I-1} \) is a single vertex. Also, from Lemma 4.2, \( d_{L^{B+I-1}(G)}(y_1^{I-1}) \) is minimum in \( N(C_{B+I-1}) \). \( \square \)

**Lemma 4.10.** If \( u_1 \in N(C_B) \) where \( C_B \) is not a single vertex, then, \( d_{L^B(G)}(u_1) \neq r_B \).

**Proof:** Assume that \( d_{L^B(G)}(u_1) = r_B \) and hence, \( f(u_1, 0) = 0 \). But as \( f(u_i, j) \) is non-negative and non-increasing, \( f(u_1, j) = 0 \) for all \( j \). But, from Lemma 4.9, there exists an integer \( D \geq B \) such that \( u_1 \) generates \( y_1^{D-B} \in N(C_D) \) where \( C_D \) is a single vertex with \( f(u_1, D - B) = 0 \), which is a contradiction. \( \square \)

**Corollary 4.3.** From Lemma 4.3(a) and Lemma 4.10, if \( u \in N(C_k) \) then \( d_{L^k(G)}(u) \neq r_k \).

**Lemma 4.11.** Let \( C_B = \{v_B\} \) and \( u_1, u_2, \ldots, u_n \) be vertices of equal degree in \( N(C_B) \) such that \( d_{L^B(G)}(u_i) \) is minimum in \( N(C_B) \). Then, \( u_i \) generates a vertex \( v_i \in C_{B+1} \) for all \( 1 \leq i \leq n \).

Moreover, \( u_1, u_2, \ldots, u_n \) generate l.min. vertices which induce a complete subgraph in \( C_{B+1} \).

**Proof:** As \( C_B \) generates \( C_{B+1} \), from Lemma 4.2 there exists an integer \( I \in [1, n] \) such that \( u_I \) generates a vertex in \( C_{B+1} \). Let there be some \( J \neq I \) such that \( u_J \) does not generate any vertex \( v \in C_{B+1} \). Then, from Lemma 4.7 it follows that \( u_J \) generates a vertex, say \( u \), in \( N(C_{B+1}) \). Now, \( r_{B+1} = d_{L^{B+1}(G)}(v_{B+1}) = d_{L^B(G)}(u_I) + r_B - 2 = d_{L^B(G)}(u_J) + r_B - 2 = d_{L^{B+1}(G)}(u) \) which is a contradiction from Corollary 4.3 and hence no such \( J \) exists.

So, all \( u_1, u_2, \ldots, u_n \) generate l.min. vertices, say \( v_1, v_2, \ldots, v_n \), in \( C_{B+1} \) such that \( v_i \) corresponds
Lemma 4.12. Let $u_1, u_2 \in N(C_B)$ with $d_{LB(G)}(u_1) = d_{LB(G)}(u_2)$. Let $u_1$ generate the sequence $(u_1 = y_1^1, y_1^2, y_1^3, \ldots)$ and $u_2$ generate the sequence $(u_2 = y_2^0, y_2^1, y_2^2, y_2^3, \ldots)$. Then, $d_{LB+i(G)}(y_1^i) = d_{LB+i(G)}(y_2^i)$ for all $i \in \mathbb{Z}^+$ and either $y_1^i, y_2^i \in C_{B+i}$ or $y_1^i, y_2^i \in N(C_{B+i})$.

Proof: For $i = 1$,

$$
\begin{align*}
  d_{LB+1(G)}(y_1^1) &= d_{LB(G)}(u_1) + r_B - 2 \\
  &= d_{LB(G)}(u_2) + r_B - 2 \\
  &= d_{LB+1(G)}(y_2^1)
\end{align*}
$$

If $C_B$ has an edge, then $y_1^1, y_2^1 \in N(C_{B+1})$ from Lemma 4.4(b) as $u_1, u_2 \in N(C_B)$.

Otherwise, $C_B$ is a single vertex. If $d_{LB(G)}(u_1) = d_{LB(G)}(u_2)$ is minimum in $N(C_B)$, then $y_1^1, y_2^1 \in C_{B+1}$ from Lemma 4.11. Else, if $d_{LB(G)}(u_1) = d_{LB(G)}(u_2)$ is not minimum in $N(C_B)$, then $y_1^1, y_2^1 \in N(C_{B+1})$.

Let, for $i = n$, $d_{LB+n(G)}(y_1^n) = d_{LB+n(G)}(y_2^n)$ and either $y_1^n, y_2^n \in C_{B+n}$ or $y_1^n, y_2^n \in N(C_{B+n})$.

Now, if $y_1^n, y_2^n \in C_{B+n}$ then from Lemma 4.4(a), $y_1^{n+1}, y_2^{n+1} \in C_{B+n+1}$ and $d_{LB+n+1(G)}(y_1^{n+1}) = d_{LB+n+1(G)}(y_2^{n+1}) = r_{B+n+1}$.

Otherwise $y_1^n, y_2^n \in N(C_{B+n})$. If $C_{B+n}$ has an edge, then, from Lemma 4.4(b), we have $y_1^{n+1}, y_2^{n+1} \in N(C_{B+n+1})$. Then,

$$
\begin{align*}
  d_{LB+n+1(G)}(y_1^{n+1}) &= d_{LB+n(G)}(y_1^n) + r_{B+n} - 2 \\
  &= d_{LB+n(G)}(y_2^n) + r_{B+n} - 2 \\
  &= d_{LB+n+1(G)}(y_2^{n+1})
\end{align*}
$$

But, if $y_1^n, y_2^n \in N(C_{B+n})$ and $C_{B+n}$ is a single vertex, then, if $d_{LB+n(G)}(y_1^n) = d_{LB+n(G)}(y_2^n)$ is minimum in $N(C_{B+n})$, from Lemma 4.11, $y_1^n$ and $y_2^n$ generate $y_1^{n+1}$ and $y_2^{n+1}$, respectively,
in $C_{B+n+1}$. Else, if $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ is not minimum in $N(C_{B+n})$, then from Lemma 4.2, $y_1^{n+1}$ and $y_2^{n+1}$ are in $N(C_{B+n+1})$ and $d_{L^{B+n+1}(G)}(y_1^{n+1}) = d_{L^{B+n}(G)}(y_1^n) + r_{B+n} - 2 = d_{L^{B+n}(G)}(y_2^n) + r_{B+n} - 2 = d_{L^{B+n+1}(G)}(y_2^{n+1})$.

**Lemma 4.13.** If $u_1, u_2, ..., u_n \in N(C_B)$ with $d_{L^B(G)}(u_i) = d_{L^B(G)}(u_j)$, then there exists an integer $E > B$ such that $u_1, u_2, ..., u_n$ generate vertices $y_1^{E-B}, y_2^{E-B}, ..., y_n^{E-B} \in C_E$ which form a clique.

Proof: From Lemma 4.9 and Lemma 4.12, there exists an integer $D \geq B$ such that $u_j$ generates $y_j^{D-B} \in N(C_D)$, $1 \leq j \leq n$, where $C_D$ is a single vertex, say $v_D$, and $d_{L^D(G)}(y_j^{D-B})$ is minimum in $N(C_D)$. Then, from Lemma 4.2, $y_j^{D-B}$ for $1 \leq j \leq n$, induce a complete subgraph in $C_{D+1}$.

Continuing rest of the proof of Theorem 4.1: Now, $\delta_A > 3$. Hence, $\delta_k > 3$ for all $k > A$. Pick any integer $B > A$. Let $v_B \in C_B$ and $w_B \in L^B(G)$ be a vertex of maximum degree, $\Delta_B$. As $G$ is connected, there exists a path $P_B = (w_B = v_1^B, v_2^B, ..., v_n^B = v_B)$ from $w_B$ to $v_B$.

Figure 4.8: Path from $w_B$ to $v_B$
$v_B$ as shown in Figure 4.8. Now,

\[
\begin{align*}
\delta_B &> 3 \\
-\delta_B &< -3 \\
\Delta_B - \delta_B &< \Delta_B - 3 \\
\Delta_B - \delta_B + 1 &< \Delta_B - 2.
\end{align*}
\]

Degree of any neighbor of $w_B$ can be any of $\Delta_B - \delta_B + 1$ possible values. But there are $\Delta_B - 1$ neighbors of $w_B$ apart from $v_B^B$. From Pigeonhole principle, there exist at least two vertices, say $z_1^B, z_2^B \in N(w_B) \setminus \{v_2^B\}$ such that $d_{L(G)}(z_1^B) = d_{L(G)}(z_2^B)$, as shown in Figure 4.9.

Now, $L(P_B)$ will be a path in $L^{B+1}(G)$. Let the edge $z_1^B v_1^B$ correspond to the vertex $z_1^{B+1}$ in $L^{B+1}(G)$. Let the edge $z_2^B v_1^B$ correspond to the vertex $z_2^{B+1}$ in $L^{B+1}(G)$. Let the edge $v_i^B v_{i+1}^B$ correspond to the vertex $v_i^{B+1}$ in $L^{B+1}(G)$ for $1 \leq i \leq n - 2$. From Lemma 4.7, $v_{n-1}^B$ generates a vertex, say $v_{n-1}^{B+1}$, such that either $v_{n-1}^{B+1} \in C_{B+1}$ or $v_{n-1}^{B+1} \in N(C_{B+1})$. When $v_{n-1}^{B+1} \in N(C_{B+1})$, there exists a vertex $v_n^{B+1} \in C_{B+1}$ adjacent to $v_{n-1}^{B+1}$.
Define \( P_{B+1} = (v_1^{B+1}, v_2^{B+1}, \ldots, v_{n-1}^{B+1}) \) if \( v_{n-1}^{B+1} \in N(C_{B+1}) \), as shown in Figure 4.10.

Otherwise, define \( P_{B+1} = (v_1^{B+1}, v_2^{B+1}, \ldots, v_{n-1}^{B+1}) \) if \( v_{n-1}^{B+1} \in C_{B+1} \), as shown in Figure 4.11.

Notice that \( d_{L^{B+1}(G)}(z_1^{B+1}) = d_{L^{B}(G)}(z_1^{B}) + d_{L^{B}(G)}(v_1^{B}) - 2 = d_{L^{B}(G)}(z_2^{B}) + d_{L^{B}(G)}(v_1^{B}) - 2 = d_{L^{B+1}(G)}(z_2^{B+1}) \). Also, if \( v_{n-1}^{B+1} \in N(C_{B+1}) \), then \( |P_{B+1}| = |P_B| \), and if \( v_{n-1}^{B+1} \in C_{B+1} \), then \( |P_{B+1}| = |P_B| - 1 \).

From Lemma 4.8 there exists an integer \( I_{n-1} \) such that \( v_{n-1}^{B} \) generates \( v_{n-1}^{B+I_{n-1}} \in C_{B+I_{n-1}} \).

Let \( I_{n-1} \) be the smallest such integer. Then \( P_{B+I_{n-1}} = (v_1^{B+I_{n-1}}, v_2^{B+I_{n-1}}, \ldots, v_{n-1}^{B+I_{n-1}}) \) and \( |P_{B+I_{n-1}}| = |P_B| - 1 \). 

\[ L^{B+1}(G) \]
Following inductively, there exists an integer \( I = I_{n-1} + I_{n-2} + ... + I_1 \) such that \( P_{B+I} = (v_1^{B+I}) \) and \( d_{L^{B+I}(G)}(z_1^{B+I}) = d_{L^{B+I}(G)}(z_2^{B+I}) \) as shown in Figure 4.12. From Lemma 4.9 and

\[
L^D(G)
\]

\[
\begin{array}{c}
\text{Figure 4.13}
\end{array}
\]

Lemma 4.12, there exists an integer \( D \geq B + I \) such that \( z_1^{B+I} \) and \( z_2^{B+I} \) generate \( z_1^D \) and \( z_2^D \), respectively, in \( N(C_D) \), where \( C_D = \{v_D\} \) and \( d_{L^D(G)}(z_1^D) = d_{L^D(G)}(z_2^D) \) is minimum in \( N(C_D) \), as shown in Figure 4.13.

\[
L^D(G)
\]

\[
\begin{array}{c}
\text{Figure 4.14}
\end{array}
\]

But, as \( \delta_k > 3 \) for all \( k > A \), there are at least two more neighbors of \( v_D \), say \( x \) and \( y \) and let \( d_{L^D(G)}(x) \leq d_{L^D(G)}(y) \). As \( C_D \) is a single vertex, from Lemma 4.11 it follows that \( z_1^D \) and \( z_2^D \) generate \( z_1^D v_D \) and \( z_2^D v_D \), respectively, in \( C_{D+1} \), which are adjacent to each other, as is shown in Figure 4.14. In the next iteration, we get four vertices,
Figure 4.15

\[(x_D)(z_D^1v_D), (x_D)(z_D^2v_D), (y_D)(z_D^1v_D)\) and \((y_D)(z_D^2v_D)\), as are shown in the Figure 4.15.

If \(d_{L^D}(x) = d_{L^D}(y)\), then, from Lemma 4.12, we have

\[d_{L^{D+2}}((x_D)(z_D^1v_D)) = d_{L^{D+2}}((x_D)(z_D^2v_D)) = d_{L^{D+2}}((y_D)(z_D^1v_D)) = d_{L^{D+2}}((y_D)(z_D^2v_D))\].

So, from Lemma 4.13, there exists an integer \(F > E\), such that, \(C_F\) contains a \(K_4\).

Otherwise, let \(d_{L^D}(x) < d_{L^D}(y)\). From Lemma 4.9 and Lemma 4.12, there exists an integer \(E > D + 2\) such that \(C_E\) is a single vertex, say \(v_E\), and, \(x_Dz_D^1v_D\) and \(x_Dz_D^2v_D\) generate vertices, say \(x_1\) and \(x_2\), respectively, in \(N(C_E)\), such that they have the same degree which is minimum in \(N(C_E)\). Let \(y_1\) and \(y_2\) be the vertices generated by \((y_D)(z_D^1v_D)\) and \((y_D)(z_D^2v_D)\), respectively, in \(L^E(G)\), as shown in the Figure 4.16. Notice that

\[d_{L^E(G)}(y_1) = d_{L^E(G)}(y_2)\].

Then, we have the line graph iterations as shown in Figure 4.17. Now,

\[d_{L^{E+1}(G)}(y_1v_E) = d_{L^E(G)}(y_1) + d_{L^E(G)}(v_E)\]
\[= d_{L^E(G)}(y_2) + d_{L^E(G)}(v_E)\]
\[= d_{L^{E+1}(G)}(y_2v_E)\].
Also,

\[ d_{LE+2}(G)((y_1v_E)(x_1v_E)) = d_{LE+1}(G)(y_1v_E) + d_{LE+1}(G)(x_1v_E) - 2 \]
\[ = d_{LE+1}(G)(y_1v_E) + r_{E+1} - 2, \]

\[ d_{LE+2}(G)((y_1v_E)(x_2v_E)) = d_{LE+1}(G)(y_1v_E) + d_{LE+1}(G)(x_2v_E) - 2 \]
\[ = d_{LE+1}(G)(y_1v_E) + r_{E+1} - 2, \]

\[ d_{LE+2}(G)((y_2v_E)(x_1v_E)) = d_{LE+1}(G)(y_2v_E) + d_{LE+1}(G)(x_1v_E) - 2 \]
\[ = d_{LE+1}(G)(y_1v_E) + r_{E+1} - 2, \]

and,

\[ d_{LE+2}(G)((y_2v_E)(x_2v_E)) = d_{LE+1}(G)(y_2v_E) + d_{LE+1}(G)(x_2v_E) - 2 \]
\[ = d_{LE+1}(G)(y_1v_E) + r_{E+1} - 2. \]

So, there are four vertices of same degree in \( N(C_{E+2}) \). From Lemma 4.13, there exists an integer \( F > E + 2 \) such that \( C_F \) will contain a \( K_4 \).

Returning to the proof of Theorem 4.1: Therefore, for a component \( C_B^j \) of \( lm(L_B(G)) \) where \( B > A \) there exists an integer \( I_j > B \) such that \( C_{I_j}^j \) generated by \( C_B^j \) has a \( K_4 \) and
hence, from Lemma 4.4(a), $C_{j+1}^i$ contains $K_4$ for all $i$, which is a contradiction to the assumption that there are infinite integers $k > A$ such that $C_k$ is a single vertex. Hence, there exists an integer $I$ such that $C_{I+1}$ has at least one edge for all $i$.

Suppose $lm(L_B(G))$ has $N$ components. Then, from Theorem 4.1, for every component $C_B^j$, $1 \leq j \leq N$, as there are finite number of integers $k$ such that $C_k$ is a single vertex, there exists an integer $I_j > B$ such that $C_{I_j+1}^j$ generated by $C_B^j$, has at least one edge for all $i$. Hence, there exists $D = \max\{I_j \mid 1 \leq j \leq N\}$, such that every component of $lm(L_D+i(G))$ has at least one edge for all $i$.

Clearly, the vertices of minimum degree of any graph $G$ are also $l\text{.min.}$ vertices and, hence, are components of $lm(G)$. But every component of $lm(L_D+i(G))$ has at least one edge for all $i$. Hence, every vertex of minimum degree is adjacent to at least one vertex of minimum degree, so, $\delta_k = 2\delta_{k-1} - 2$ for all $k > D$. 

![Figure 4.17](image.png)
Dr. Hoffman assigned me an interesting puzzle. If $G$ is a connected graph and $L(G)$ is regular, then show that $G$ is either regular or bipartite.

Proof: For any graph $G$, its line graph, $L(G)$, is regular if and only if every edge of $G$ is incident with the same number of edges. Hence, for any two edges $uv$ and $wy$,

\[
d(u) + d(v) - 2 = d(w) + d(y) - 2, \\
 d(u) + d(v) = d(w) + d(y)
\]

Let $uv$ be an edge of $G$ and $w$ be a vertex. As $G$ is connected, then without loss of generality, there exists a path $P = (u, v, v_1, v_2, \ldots, v_n, w)$. Now, $d(v_1) = d(u)$ and hence $d(v_1) = d(u)$. If $d(v_i) = d(u)$, then, $d(v_{i+1}) = d(v)$, otherwise, if $d(v_i) = d(v)$, then, $d(v_{i+1}) = d(u)$. It follows from induction that for any vertex $w$ of $G$, we have that, $d(w) = d(u)$ or $d(w) = d(v)$. Moreover, for any edge $wy$, either $d(w) = d(u)$ and $d(y) = d(v)$ or the other way round. Also, from induction, the degree of the vertices alternates along the path, hence, if $|P|$ is even, then, $d(w) = d(u)$, otherwise, if $|P|$ is odd then $d(w) = d(v)$.

Now, if $G$ has an odd cycle, say, $P = (u, v_1, v_2, \ldots, v_n, u)$, then from above discussion $d(u) = d(v_1)$. But as for any vertex $w$ of $G$, $d(w) = d(u)$ or $d(w) = d(v_1)$, therefore $G$ is regular. It follows that for any connected graph $G$ with $L(G)$ regular, either $G$ is regular and, if it is not regular, then it has no odd cycles, i.e., it is bipartite. \qed
Bibliography


