Maximum and minimum degree in iterated line graphs

by

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Abstract

In this thesis we analyze two papers, both by Dr.Stephen G. Hartke and Dr.Aparna W. Higginson, on maximum [2] and minimum [3] degrees of a graph G under iterated line graph operations. Let Δ_k and δ_k denote the minimum and the maximum degrees, respectively, of the k^{th} iterated line graph $L^k(G)$. It is shown that if G is not a path, then, there exist integers A and B such that for all $k > A$, $\Delta_{k+1} = 2\Delta_k - 2$ and for all $k > B$, $\delta_{k+1} = 2\delta_k - 2$.

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Chapter 1

Introduction

The *line graph* $L(G)$ of a graph G is the graph having edges of G as its vertices, with two vertices being adjacent if and only if the corresponding edges are adjacent in G. Please note that all graphs in this discussion are simple. We restrict our discussion to connected graphs. Refer to [4] for basic definitions of graph theory.

One of the most important resutls in line graphs has been by Beineke, who provides in [1], a new characterization of line graphs in terms of nine excluded subgraphs, also unifying some of the previous characterizations. We provide only the theorem here without the proof.

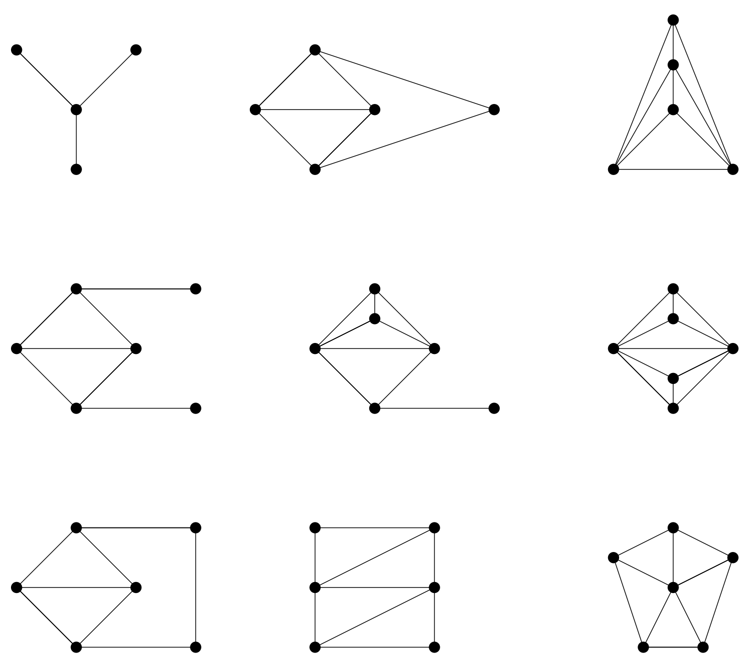

Figure 1.1

Theorem 1.1. *A graph G is a line graph of some graph if and only if none of the nine graphs in Figure 1.1 is an induced subgraph of G.*

The *iterated line graph* is defined recursively as $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) = G$. Let Δ and δ be the maximum and the minimum degree, respectively, of a graph G. We denote the minimum degree of $L^k(G)$ by δ_k and the maximum degree by Δ_k. Hartke and Higgins [2] show that if G is not a path, then, there exists an integer A, such that, $\Delta_{k+1} = 2\Delta_k - 2$ for all $k > A$. Using similar concepts, they show in [3] that there exists an integer B such that $\delta_{k+1} = 2\delta_k - 2$ for all $k > B$. Rather than focusing on the vertices of minimum and maximum degrees, they observe the behavior of particular kinds of regular subgraphs, of which, the vertices of maximum and minimum degrees form a special case. However, this proves only the existence and the question of tight bounds of A and B is still open.

We now define some notation which will be used throughout the proofs. *Neighborhood* of a vertex v, denoted by $N(v)$, is defined as the set of all vertices adjacent to v. Then, if S is a set of vertices of G, we use the following notation:-

1. $N(S) = \bigcup_{v \in S} N(v)$

2. $N[S] = N(S) \cup S$

2. $N\langle S \rangle = N(S) \setminus S$

We would first prove a result in Chapter 2 which was used in [2] and [3] without proof. Then, the result for the maximum degree is proved in Chapter 3 and for the minimum degree is proved in Chapter 4.

Chapter 2

An elementary result

In this chapter we will prove that for most graphs, minimum degree is unbounded under line graph iteration. Notice that, if G is not a path, then δ_k is defined for all k. As mentioned in the introduction, all graphs under consideration are simple and we restrict out discussion to connected graphs.

A *leaf* of a graph is a vertex of degree 1.

Lemma 2.1. *If there exists an integer A such that $\delta_A > 2$, then $\delta_k > 2$ for all $k > A$. Moreover, δ_k is a strictly increasing sequence for all $k \geq A$, and hence $\lim_{k\to\infty} \delta_k = \infty$.*

Proof: Clearly, the minimum possible value of δ_{k+1} is $2\delta_k - 2$. Now,

$$\begin{aligned} \delta_A &> 2 \\ 2\delta_A &> \delta_A + 2 \\ 2\delta_A - 2 &> \delta_A. \end{aligned}$$

But $2\delta_A - 2$ is the minimum possible value of δ_{A+1}, hence, $\delta_{A+1} > \delta_A$ which implies $\delta_{A+1} > 2$. Now, let $\delta_{A+i} > 2$ for some i. Then, following similar set of equations, $\delta_{A+i+1} > \delta_{A+i}$ and $\delta_{A+i+1} > 2$. It follows inductively that $\delta_{k+1} > \delta_k > 2$ for all $k > A$ and therefore δ_k is a strictly increasing sequence. This also implies that the minimum degree is unbounded under line graph operation. □

Lemma 2.2. *Let s_k be the number of vertices of degree 1 in $L^k(G)$. Then, $\{s_k\}$ is non-increasing.*

Proof: Every vertex of degree 1 in a graph $L(G)$ corresponds to an edge in G which is incident with exactly one edge. So, a leaf in $L^k(G)$ corresponds to one leaf in $L^{k-1}(G)$. Also, a leaf in G will give a single leaf under the line graph operation. □

Lemma 2.3. *Let G be a graph which is not a path or a cycle. If $\delta = 2$ then $\lim_{k\to\infty} \delta_k = \infty$.*

Proof: A vertex of degree 2 in $L(G)$ will correspond to an edge in G which is incident with exactly two edges. It can either be a leaf or an edge in a path or cycle as shown in the

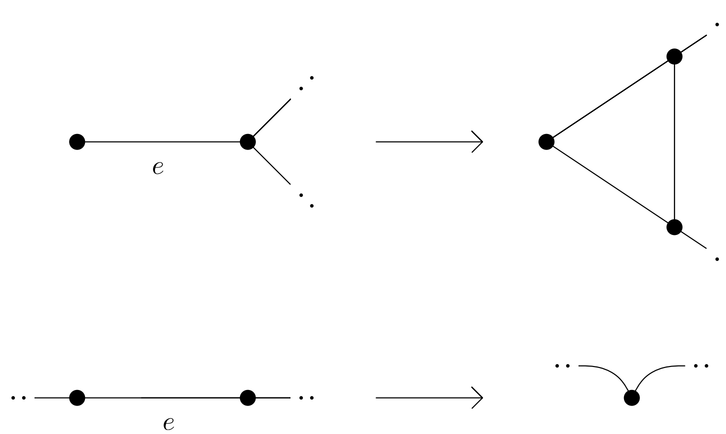

Figure 2.1

Figure 2.1. But as $\delta = 2$, G has no leaf. Hence, we only need to consider vertices of degree 2 in G.

Now, as G is not a path or a cycle, there exists at least one vertex, say v, of degree greater than 2. Also, as $\delta = 2$, G is not a $K_{1,3}$. Let u be a vertex of degree 2 in G. As G is connected, there is a path from u to v, say $P_0 = (u = y_1^0, y_2^0, ..., y_n^0 = v)$, as shown in Figure 2.2. Now, P_0 induces a path $P_1 = (y_1^1, y_2^1, ..., y_{n-1}^1)$ in $L(G)$ where $d_{L(G)}(y_j^1) \geq 2$ for $1 \leq j \leq n-2$ and $d_{L(G)}(y_{n-1}^1) \geq 3$. Now, let $P_i = (y_1^i, y_2^i, ..., y_{n-i}^i)$ with $d_{L^i(G)}(y_j^i) \geq 2$ for $1 \leq j \leq n-i-1$ and $d_{L^i(G)}(y_{n-i}^i) \geq 3$. Then P_i induces P_{i+1} in $L^{i+1}(G)$ such that $P_{i+1} = (y_1^{i+1}, y_2^i + 1, ..., y_{n-i-1}^{i+1})$.

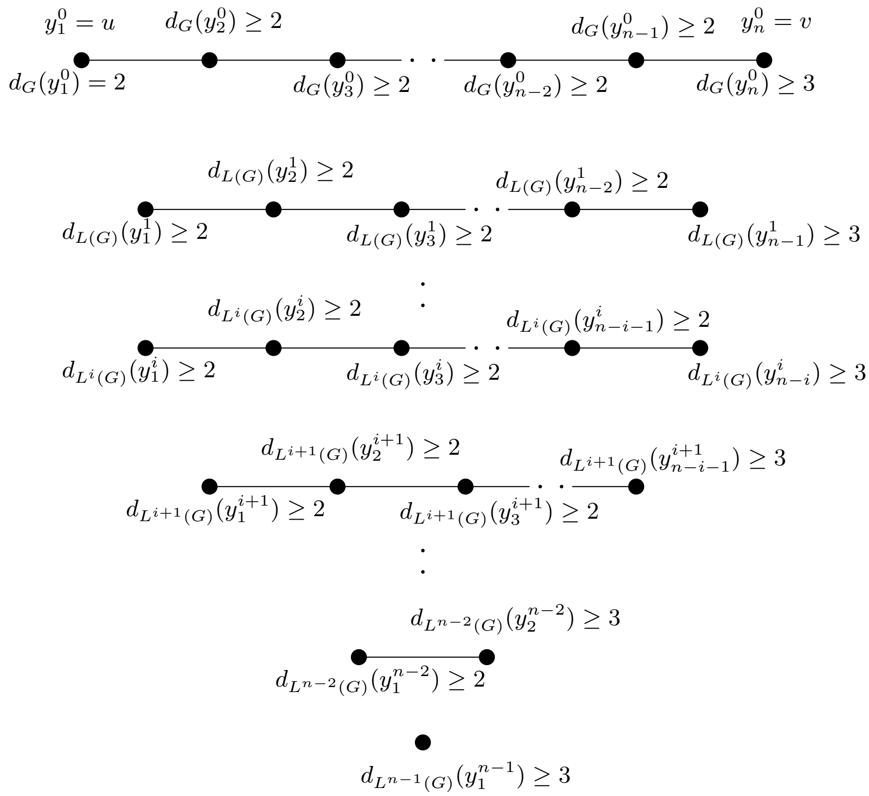

Figure 2.2: Disappearing vertex of degree two

Now, for $1 \leq j \leq k-2$,

$$d_{L^i(G)}(y_2^i) \geq 2$$

$$d_{L^i(G)}(y_2^i) + d_{L^i(G)}(y_1^i) \geq 2 + 2$$

$$d_{L^i(G)}(y_2^i) + d_{L^i(G)}(y_1^i) - 2 \geq 2 + 2 - 2$$

$$d_{L^{i+1}(G)}(y_1^{i+1}) \geq 2.$$

and, for $j = k-1$,

$$d_{L^i(G)}(y^i_{k-1}) \geq 2$$
$$d_{L^i(G)}(y^i_{k-1}) + d_{L^i(G)}(y^i_k) \geq 2+3$$
$$d_{L^i(G)}(y^i_{k-1}) + d_{L^i(G)}(y^i_k) - 2 \geq 2+3-2$$
$$d_{L^{i+1}(G)}(y^{i+1}_{k-1}) \geq 3.$$

Also, $|P_{i+1}| = |P_i| - 1$. Applying inductively, $P_{n-1} = (y^{n-1}_1)$ where $d_{L^{n-1}(G)}(y^{n-1}_1) \geq 3$ as shown in Figure 2.2, and we get that every vertex of degree 2 will definitely 'disappear' after $n-1$ line graph iterations. Doing this for every vertex of degree 2, there exists an integer N such that $L^N(G)$ has no vertex of degree 2, hence $\delta_N \geq 3$ and we are done from Lemma 2.1. □

Lemma 2.4. *If G is neither a path, cycle nor a $K_{1,3}$, then the minimum degree is unbounded under line graph iteration and moreover, there exists an integer A such that $\lim_{k\to\infty} \delta_k = \infty$ for all $k > A$.*

Proof: From Lemma 2.1 it is sufficient to show that for any graph G, as specified, there exists an integer A such that $\delta_A > 2$. As G is neither a path, cycle nor a $K_{1,3}$, there exists an edge, say e, such that, $e = xz$ is incident with at least three edges.

Let $\delta(G) = 1$. From Lemma 2.2, the number of leaves is a non-increasing sequence over line graph iteration. Moreover, a leaf in $L(G)$ corresponds to exactly one leaf in G. So it would suffice to consider line graph operation on leaves of G and show that it disappears at some iteration.

Let v be incident on a leaf of G such that $d_G(v) = 1$. Then, as G is connected, there is a path $P_0 = (v = y^0_1, y^0_2, ..., y^0_{n-1} = x, y^0_n = z)$ from v to the edge e such that $d_G(y^0_i) \geq 2$ for $2 \leq i \leq n-2$, as shown in Figure 2.3. Now, P_0 induces a path, say P_1, in $L(G)$ such that $P_1 = (y^1_1, y^1_2, ..., y^1_{n-1})$ where y^1_j corresponds to the edge $y^0_j y^0_{j+1} \in E(G)$ for $2 \leq j \leq n-1$, as shown in Figure 2.3. Now, as xz is incident with at least three edges, $d_G(y^1_{n-1}) \geq 3$. Also,

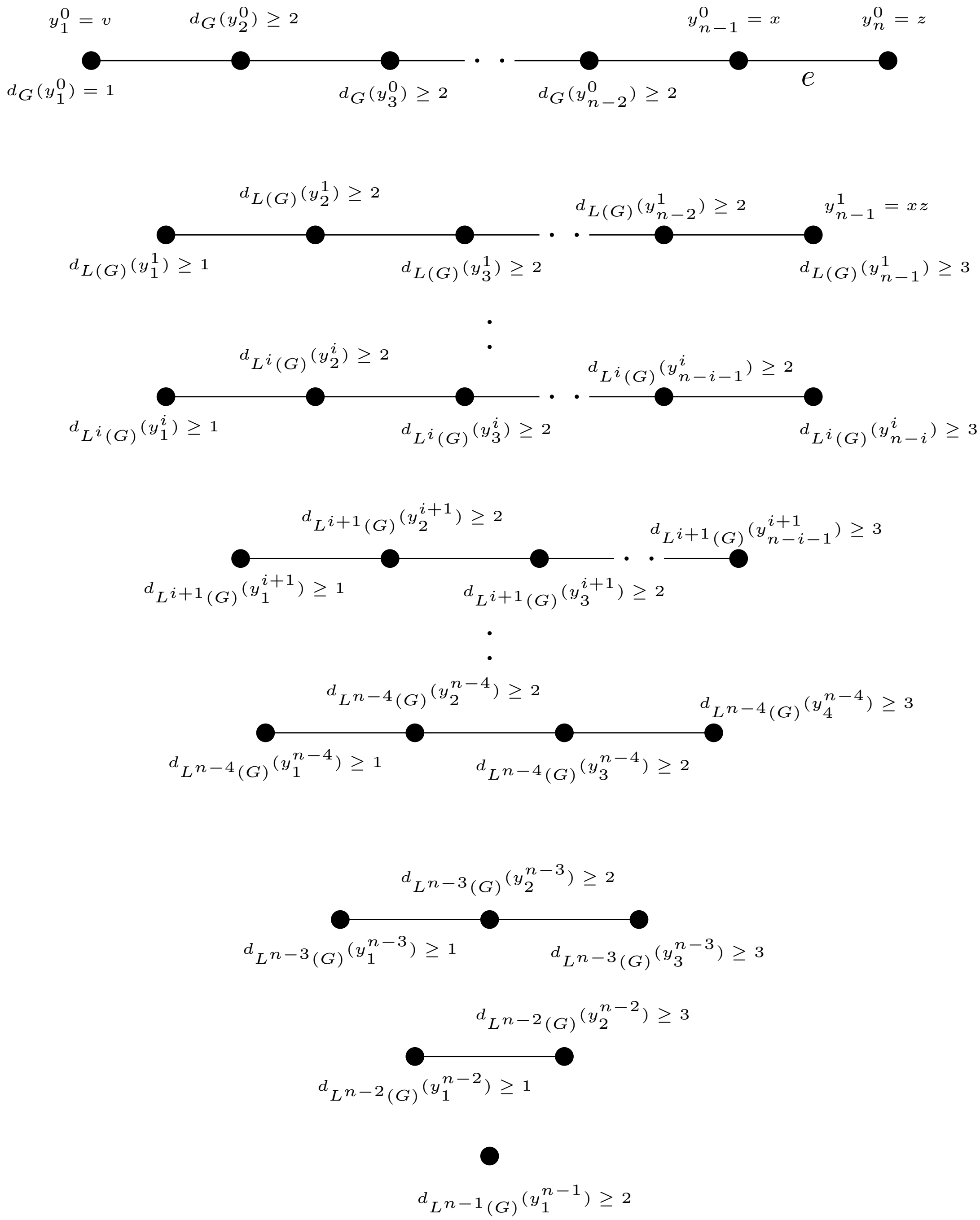

Figure 2.3: Disappearing leaf

$d_{L(G)}(y_1^1) \geq 1$, $d_{L(G)}(y_j^1) \geq 2$ for $2 \leq j \leq n-2$ and $d_{L(G)}(y_{n-1}^1) \geq 3$, as shown in Figure 2.3. Notice that $|P_1| = |P_0| - 1$.

Now, let $P_i = (y_1^i, y_2^i, ..., y_{n-i}^i)$ in $L^i(G)$, such that, $d_{L^i(G)}(y_1^i) \geq 1, d_{L^i(G)}(y_j^i) \geq 2$ for $2 \leq j \leq n-i-1$ and $d_{L^i(G)}(y_{n-i}^i) \geq 3$. Then P_i induces a path P_{i+1} in $L^{i+1}(G)$ such that $P_{i+1} =$

$(y_1^{i+1}, y_2^{i+1}, ..., y_{n-i-1}^{i+1})$ where y_j^{i+1} corresponds to the edge $y_j^i y_{j+1}^i$ in P_i for $1 \leq j \leq n-i-1$ as shown in the Figure 2.3.

Now,

$$\begin{aligned} d_{L^i(G)}(y_2^i) &\geq 2 \\ d_{L^i(G)}(y_2^i) + d_{L^i(G)}(y_1^i) &\geq 2+1 \\ d_{L^i(G)}(y_2^i) + d_{L^i(G)}(y_1^i) - 2 &\geq 2+1-2 d_{L^{i+1}(G)}(y_1^{i+1}) \qquad \geq 1. \end{aligned}$$

Also, for $2 \leq j \leq k-2$,

$$\begin{aligned} d_{L^i(G)}(y_j^i) &\geq 2 \\ d_{L^i(G)}(y_j^i) + d_{L^i(G)}(y_{j+1}^i) &\geq 2+2 \\ d_{L^i(G)}(y_j^i) + d_{L^i(G)}(y_{j+1}^i) - 2 &\geq 2+2-2 \\ d_{L^{i+1}(G)}(y_j^{i+1}) &\geq 2, \end{aligned}$$

and, for $j = k-1$,

$$\begin{aligned} d_{L^i(G)}(y_{k-1}^i) &\geq 2 \\ d_{L^i(G)}(y_{k-1}^i) + d_{L^i(G)}(y_k^i) &\geq 2+3 \\ d_{L^i(G)}(y_{k-1}^i) + d_{L^i(G)}(y_k^i) - 2 &\geq 2+3-2 \\ d_{L^{i+1}(G)}(y_{k-1}^{i+1}) &\geq 3. \end{aligned}$$

So, $d_{L^{i+1}(G)}(y_1^{i+1}) \geq 1, d_{L^{i+1}(G)}(y_j^{i+1}) \geq 2$ for $2 \leq j \leq k-2$ and $d_{L^{i+1}(G)}(y_{k-1}^{i+1}) \geq 3$. Also, $|P_{i+1}| = |P_i| - 1$, then, following inductively starting from P_1 we get that $P_{n-1} = (y_1^{n-1})$ where $d_{L^{n-1}(G)}(y_1^{n-1}) \geq 2$ as shown in the Figure 2.3. Hence, the number of vertices of degree 1 goes down by one.

Let G have N vertices, say $v_1, v_2, ..., v_N$, of degree 1. Then, for every vertex v_j of degree 1 there exists an integer I_j such that there is no vertex of degree 1 in $L^{I_j}(G)$ corresponding to v_j. Then, for the integer $I = max\{I_j \mid 1 \leq j \leq N\}$, there would be no vertex of degree 1 corresponding to any v_j. As there is no other way to get degree 1 vertices under line graph operation, $L^I(G)$ will have no vertices of degree 1. Also, as $L^k(G)$ is connected for all k we conclude that $\delta_I \geq 2$ and we are done from Lemma 2.1 and Lemma 2.3. $\square$

Chapter 3

Maximum degree growth in iterated line graphs

In this chapter it will be shown that for any graph G, which is not a path, there exists an integer D such that $\Delta_{k+1} = 2\Delta_k - 2$ for all $k > D$, where Δ_k is the maximum degree of $L^k(G)$.

If G is a path, then as G is a finite graph, there exists an integer I such that $L^I(G)$ is undefined.

If G is a cycle, then for all $k \in \mathbb{Z}^+$, $\Delta_{k+1} = 2\Delta_k - 2 = 2$.

If G is a $K_{1,3}$, then $L(G)$ is a K_3 and hence, for all $k > 1$, $\Delta_{k+1} = 2\Delta_k - 2 = 2$.

Now we have to prove the theorem for any graph G where it is not a path, a cycle or a $K_{1,3}$.

Definition: A vertex v is a *locally maximum vertex* or a *l.max. vertex* if no vertex in the neighborhood of v has degree greater than that of v.

Definition: The subgraph of G induced by its l.max. vertices is called the *locally maximum subgraph* or *l.max. subgraph* of G and is denoted by $LM(G)$.

Definition: A vertex $v \in L^k(G)$ is *generated by a vertex* $u \in G$ if there is a sequence of vertices $u = v_0, v_1, ..., v_k = v$ such that $v_{i+1} \in L^{i+1}(G)$ corresponds to an edge incident at $v_i \in L^i(G)$. A subgraph J of $L^k(G)$ is *generated by a subgraph* H of G if, for each vertex $v \in J$, v is generated by a vertex in H.

Lemma 3.1. *All vertices in the same component of $LM(G)$ have the same degree in G.*

Proof: Let v and u be two vertices in a component of $LM(G)$. Then v and u are l.max. vertices of the graph G. As $v \in N(u)$, $d(v) \leq d(u)$ from definition. Similarly, as $u \in N(v)$, $d(u) \leq d(v)$. Hence, $d(u) = d(v)$. □

Lemma 3.2. *The vertices of $L(G)$ corresponding to edges of G incident with the same vertex, say v, of G, form a clique in $L(G)$. In particular, all the vertices of $LM(L(G))$ generated by v are in the same component of $LM(L(G))$.*

Proof: It follows from the definition of line graphs that the vertices of $L(G)$, corresponding to the edges of G that share a vertex, will be adjacent to each other. □

Lemma 3.3. *If w is a l.max. vertex of $L(G)$, then w corresponds to an edge e in G such that at least one end of e, say v, is l.max. in G and the other end of e, say u, has the maximum degree among the neighbors of v in G.*

Proof: Assume that neither v nor u is a l.max. vertex. Let $d_G(v) \geq d_G(u)$. Then, as v is not a l.max. vertex, there exists a vertex $y \in N(v)$ such that $d_G(y) > d_G(v)$.

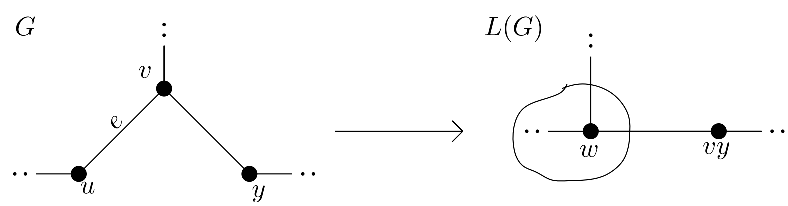

Figure 3.1

Now, the edge vy of G corresponds to a vertex vy of $L(G)$, adjacent to w as shown in the Figure 3.1. Also,

$$d_G(v) \geq d_G(u).$$

But, as $d_G(y) > d_G(v)$,

$$d_G(v) + d_G(y) - 2 > d_G(u) + d_G(v) - 2$$

$$d_{L(G)}(vy) > d_{L(G)}(w),$$

contradicting that w is a l.max. vertex of $L(G)$.

Hence, no such y exists, implying that v is a l.max. vertex of G.

Now, let there exist a vertex $z \in N(v)$ such that $d_G(z) > d_G(u)$.

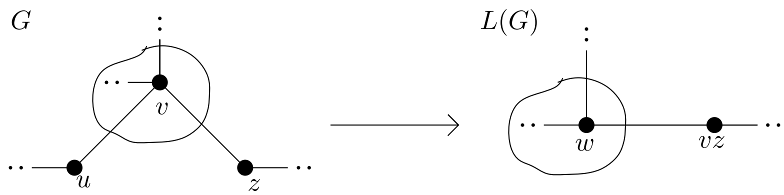

Figure 3.2

Then the edge vz of G corresponds to a vertex vz adjacent to w in $L(G)$ as shown in the Figure 3.2.

But,

$$d_G(z) > d_G(u)$$

$$d_G(z) + d_G(v) - 2 > d_G(u) + d_G(v) - 2$$

$$d_{L(G)}(vz) > d_{L(G)}(w),$$

contradicting that w is a l.max. vertex of $L(G)$. Hence, no such z exists, implying that u has the maximum degree in $N(v)$. □

Lemma 3.4. *Let v be an isolated vertex of $LM(G)$.*

(a) If v has any neighbor of the same degree as that of v, then, v generates no l.max. vertices of $L(G)$.

(b) If all neighbors of v have degree less than that of v, and u is such a neighbor, then the edge uv corresponds to a l.max. vertex of $L(G)$ if and only if u has the maximum degree among the neighbors of v and for all $z \in N(u) \setminus \{v\}$, $d_G(z) \leq d_G(v)$.

Proof:

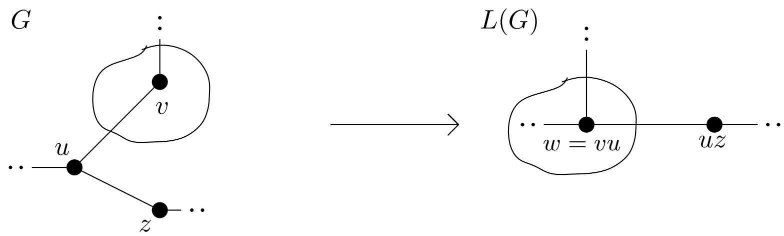

Figure 3.3

(a) As u is not a l.max. vertex of G, there exists a vertex z adjacent to u, such that, $d_G(z) > d_G(u) = d_G(v)$. Then, u and z generate a vertex uz adjacent to w, generated by v and u, as shown in Figure 3.3. Now, $d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 > d_G(u) + d_G(v) - 2 = d_{L(G)}(w)$, therefore, the edge vu does not correspond to a l.max. vertex of $L(G)$, for any u with $d_G(u) = d_G(v)$. Hence, by Lemma 3.3, v does not generate a l.max. vertex of $L(G)$.

(b) Let there exist a vertex $z \in N(u) \setminus \{v\}$ such that $d_G(z) > d_G(v)$. Then the edge uz corresponds to a vertex uz in $L(G)$ adjacent to a vertex w, which corresponds to the edge uv in G, as shown in Figure 3.3. Now, $d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 > d_G(u) + d_G(v) - 2 = d_{L(G)}(w)$, therefore, w will not be a l.max. vertex.

Now, let, for all $z \in N(u) \setminus \{v\}$, $d_G(z) \leq d_G(v)$. Then,

$$d_G(u) + d_G(z) - 2 \leq d_G(u) + d_G(v) - 2,$$

$$d_{L(G)}(uz) \leq d_{L(G)}(w),$$

where w corresponds to the edge uv of G. Therefore, the edge uv corresponds to a l.max. vertex of $L(G)$.

Moreover, if uz is a l.max. vertex, it would be adjacent to w implying that the number of components will not increase. □

Lemma 3.5. *Let C be a component of $LM(G)$ which is not a single vertex.*

a) If v_1 and v_2 are adjacent vertices in C, then the vertex $w \in L(G)$, corresponding to the edge v_1v_2, is a l.max. vertex.

b) If $u \in N\langle C\rangle$, then no edge joining u to a vertex in C corresponds to a l.max. vertex of $L(G)$.

Proof:

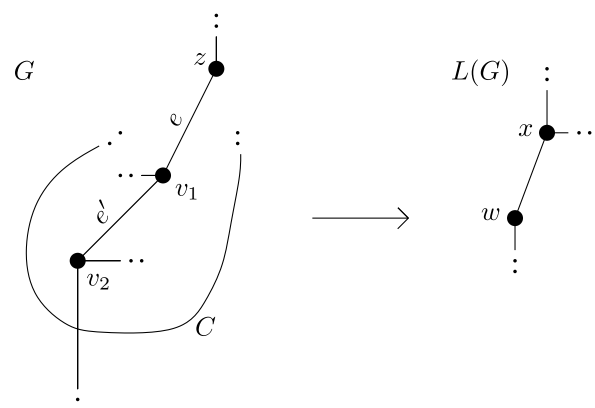

Figure 3.4

a) Let $e' = v_1v_2$ be an edge in C. Let $w \in L(G)$ be the vertex corresponding to e'. Then, any neighbor x of w will correspond to an edge e, in G, incident at either v_1 or v_2. Let e be incident at v_1 and some vertex $z \in N(v_1)$, as shown in the Figure 3.4. Then, as v_1 is a l.max. vertex,

$$d_G(z) \leq d_G(v_1)$$

$$d_G(z) + d_G(v_2) - 2 \leq d_G(v_1) + d_G(v_2) - 2$$

From Lemma 3.1, $d_G(v_1) = d_G(v_2)$,

$$d_G(z) + d_G(v_1) - 2 \leq d_G(v_1) + d_G(v_2) - 2$$
$$d_{L(G)}(x) \leq d_{L(G)}(w),$$

hence, w is a l.max. vertex.

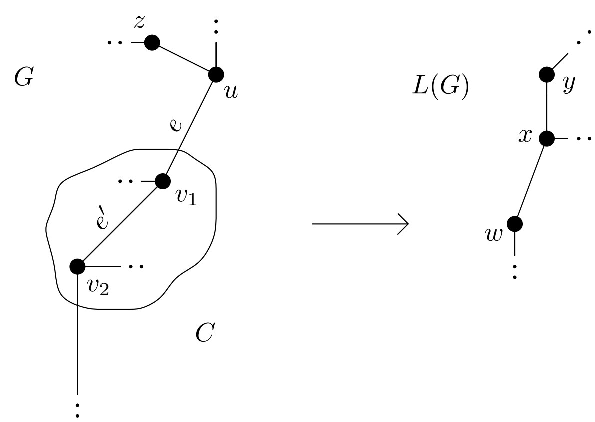

Figure 3.5

b) As $u \in N\langle C\rangle$, it is adjacent to a vertex, say v_1, in C. As C is not a single vertex, there exists a vertex $v_2 \in C$ adjacent to v_1. Let w be the vertex in $L(G)$ corresponding to the edge v_1v_2 and let r be the common degree of vertices in C. Then, $d_{L(G)}(w) = 2r - 2$. Now, the edge uv_1 corresponds to a vertex x adjacent to w in $L(G)$, as shown in the Figure 3.5. Also, $d_{L(G)}(x) = d_G(u) + r - 2$ and as v_1 is a l.max. vertex, we get that $d_G(u) \leq r$.

If $d_G(u) < r$, then,

$$d_G(u) + r - 2 < r + r - 2$$
$$d_{L(G)}(x) < d_{L(G)}(w),$$

hence, x can not be a l.max. vertex.

If $d_G(u) = r$ then as u is not a l.max. vertex, there exists a vertex $z \in N(u) \setminus \{v_1\}$ such that $d_G(z) > d_G(u)$. Then, the edge uz corresponds to a vertex y in $L(G)$, adjacent to x as shown in Figure 3.5. Now,

$$d_G(z) > d_G(u)$$
$$d_G(z) + d_G(u) - 2 > d_G(u) + d_G(u) - 2$$
$$d_G(z) + d_G(u) - 2 > d_G(u) + r - 2$$
$$d_{L(G)}(y) > d_{L(G)}(x),$$

and hence, x can not be a l.max. vertex. □

Corollary 3.1: It follows from Lemma 3.5 that $L(C)$ is a component of $LM(L(G))$.

Corollary 3.2: If C is a single vertex, then from Lemma 3.4 it generates at most one component of $LM(L(G))$. Otherwise, if C is not a single vertex, then every vertex of C generates a l.max. vertex from Lemma 3.5(a). As the line graph operation preserves connectivity, C will generate at most one component of $LM(L(G))$. Hence, in either case, C generates at most one component.

Lemma 3.6. *There exists an integer A such that for all $k > A$, every component of $LM(L^k(G))$ generates exactly one component of $LM(L^{k+1}(G))$.*

Proof: Let c_k be the number of components of $LM(L^k(G))$. From Corollary 3.2, $\{c_k\}$ is a non-increasing sequence. But as c_k is a non-negative number for all k, there exists an integer A, such that c_k is constant for all $k > A$. □

We now define new notation which would be followed in the rest of this chapter. Let C_{A+1} be a component of $LM(L^{A+1}(G))$ where A is the integer from Lemma 3.6. Inductively, for each $k > A$, let C_{k+1} be the component of $L^{k+1}(G)$ generated by C_k. Let r_k be the common degree of vertices in C_k. We can further choose A to be sufficiently large so that $\delta_k > 2$ for all $k > A$ from Lemma 2.1.

Lemma 3.7. *Let $u \in N\langle C_D \rangle$ be adjacent to a vertex $v_D \in C_D$, where D is an integer greater than A. Let $y \in L^{D+1}(G)$ correspond to the edge uv of $L^D(G)$, so $y \in N[C_{D+1}]$.*

(a) If C_D is not a single vertex, so $y \in N\langle C_{D+1} \rangle$, and,

$$r_{D+1} - d_{L^{D+1}(G)}(y) = r_D - d_{L^D(G)}(u).$$

(b) In case C_D is a single vertex, then,

$$r_{D+1} - d_{L^{D+1}(G)}(y) < r_D - d_{L^D(G)}(u)$$

Proof:

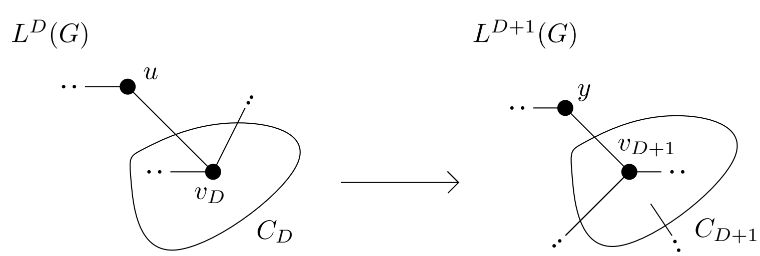

Figure 3.6: When C_D is not a single vertex

(a) From Lemma 3.5(a), if C_D has an edge then it generates C_{D+1}, as shown in Figure 3.6, and $r_{D+1} = 2r_D - 2$.

Also, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. So,

$$r_{D+1} - d_{L^{D+1}(G)}(y) = (2r_D - 2) - (d_{L^D(G)}(u) + r_D - 2)$$
$$r_{D+1} - d_{L^{D+1}(G)}(y) = r_D - d_{L^D(G)}(u).$$

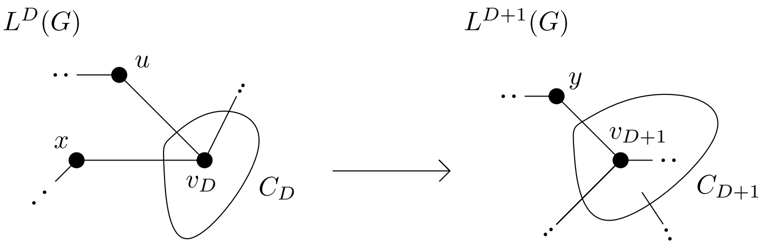

Figure 3.7: When C_D is a single vertex

(b) Suppose $y \in N\langle C_{D+1}\rangle$. Again, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. Let x be a vertex of largest degree in $N(v_D)$ such that the edge xv_D corresponds to a l.max. vertex v_{D+1} in C_{D+1} from Lemma 3.6. Such a vertex x exists from Lemma 3.3 and as C_{D+1} is non-empty, we have,

$$r_{D+1} = d_{L^D(G)}(x) + r_D - 2.$$

As C_D is a single vertex, from Lemma 3.4(a), $d_{L^D(G)}(x) < r_D$ as C_D generates C_{D+1} and hence,

$$d_{L^D(G)}(x) + r_D - 2 < r_D + r_D - 2$$
$$r_{D+1} < 2r_D - 2$$
$$r_{D+1} - d_{L^{D+1}(G)}(y) < (2r_D - 2) - d_{L^{D+1}(G)}(y)$$

But, since $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$, we have,

$$r_{D+1} - d_{L^{D+1}(G)}(y) < (2r_D - 2) - (d_{L^D(G)}(u) + r_D - 2)$$
$$r_{D+1} - d_{L^{D+1}(G)}(y) < r_D - d_{L^D(G)}(u)$$

Now, suppose $y \in C_{D+1}$. Then $r_{D+1} - d_{L^{D+1}(G)}(y) = 0$. Also, as C_D is a single vertex, $r_D - d_{L^D(G)}(u) \neq 0$ as otherwise C_D will not generate a component. Hence, $r_{D+1} - d_{L^{D+1}(G)}(y) < r_D - d_{L^D(G)}(u)$. □

Lemma 3.8. *If $u \in N\langle C_k\rangle$ then u generates a vertex $y \in N[C_{k+1}]$.*

Proof: As $u \in N\langle C_k \rangle$, u is adjacent to a vertex $v \in C_k$. Let the edge uv correspond to the vertex $y \in L^{k+1}(G)$. If C_k has an edge, from Lemma 3.5(a) v generates a vertex in C_{k+1}. Also, if $C_k = \{v\}$, as $k > A$, v generates every vertex in C_{k+1}. So, there exists a vertex $w \in C_{k+1}$ generated by v. Now, the edges in $L^k(G)$ corresponding to y and w, are incident at the vertex v. Hence, y is adjacent to the vertex w in $L^{k+1}(G)$, implying that, if y is a l.max. vertex then $y \in C_{k+1}$ or else $y \in N\langle C_{k+1} \rangle$. □

Let $N\langle C_B \rangle = \{u_1, u_2, \ldots, u_n\}$. Then from Lemma 3.8, for every $1 \leq j \leq n$, u_j generates a vertex, say y_j^1, in $N[C_{B+1}]$.

Now, if y_j^i is a vertex in $N\langle C_{B+i} \rangle$, then from Lemma 3.8, it generates a vertex, say y_j^{i+1}, in $N[C_{B+i+1}]$. Otherwise, if y_j^i is a vertex in C_{B+i}, then from Lemma 3.5(a), it generates a vertex, say y_j^{i+1}, in C_{B+i+1}. It follows inductively that u_j generates a sequence of vertices $(u_j = y_j^0, y_j^1, y_j^2, y_j^3, ...)$ where $y_j^i \in N[C_{B+i}]$ and, moreover, $y_j^i \in C_{B+i}$ for all $i > I$ if $y_j^I \in C_{B+I}$ for some integer I.

Then we define a function $f(u_j, i) : N\langle C_B \rangle \to \mathbb{R}$ by $f(u_j, i) = r_{B+i} - d_{L^{B+i}(G)}(y_j^i)$ where $i \in \mathbb{Z}^+$. Clearly $f(u_j, i)$ is non-negative and from Lemma 3.7 it is a non-increasing function of i. Also, if C_{B+i} is a single vertex and $y_j^i \in N\langle C_{B+i} \rangle$, then, from Lemma 3.4(a), $f(u_j, i)$ can not equal to zero because otherwise C_{B+i} will not generate a component.

Theorem 3.1. *Let G be a simple connected graph. Let C_A be a component of $LM(L^A(G))$. Then, there are a finite number of integers $k > A$, such that C_k, generated by C_A, is a single vertex.*

Proof: The proof is by contradiction. Let us assume that there are an infinite number of integers $k > A$ such that C_k is a single vertex. Then we prove the following series of lemmas.

Lemma 3.9. *If $u_1 \in N\langle C_B \rangle$ generates $(y_1^0, y_1^1, y_1^2, y_1^3, ...)$, then there exists an integer I such that $y_1^I \in C_{B+I}$.*

Proof: We prove this by contradiction. Let $y_1^i \in N\langle C_{B+i}\rangle$ for all i. The function $f(u_1, i)$ is non-increasing and decreases when C_{B+i} is a single vertex. As there are infinite number of integers $k > A$ such that C_k is a single vertex, there are infinite integers i such that C_{B+i} is a single vertex as $B > A$. Hence, from Lemma 3.7(b) there exists an integer $D > B$ such that $f(u_1, D - B) = 0$.

Now, if C_D is a single vertex, then as $y_1^i \in N\langle C_{B+i}\rangle$ for all i, it follows that $f(u_1, D - B)$ can not be zero and we have a contradiction. Otherwise, if C_D has an edge, then let E be the smallest integer greater than D such that C_E is a single vertex. From Lemma 3.7(a), $f(u_1, E - B) = f(u_1, D - B) = 0$, and we again have a contradiction. □

Lemma 3.10. *If $u_1 \in N\langle C_B\rangle$ then there exists an integer $D \geq B$ such that u_1 generates $y_1^{D-B} \in N\langle C_D\rangle$ where C_D is a single vertex and $d_{L^D(G)}(y_1^{D-B})$ is maximum in $N\langle C_D\rangle$.*

Proof: From Lemma 3.9 there exists an integer I such that u_1 generates $y_1^I \in C_{B+I}$. Let I be the smallest such integer. Hence, $y_1^{I-1} \in N\langle C_{B+I-1}\rangle$. From Lemma 3.5, if C_{B+I-1} has an edge then y_1^{I-1} cannot generate a vertex in C_{B+I}. Hence, C_{B+I-1} is a single vertex. Also, from Lemma 3.3, $d_{L^{B+I-1}(G)}(y_1^{I-1})$ is maximum in $N\langle C_{B+I-1}\rangle$. □

Lemma 3.11. *If $u_1 \in N\langle C_B\rangle$ where C_B is not a single vertex, then, $d_{L^B(G)}(u_1) \neq r_B$.*

Proof: Assume that $d_{L^B(G)}(u_1) = r_B$ and hence, $f(u_1, 0) = 0$. But as $f(u_i, j)$ is non-negative and non-increasing, $f(u_1, j) = 0$ for all j. But, from Lemma 3.10, there exists an integer $D \geq B$ such that u_1 generates $y_1^{D-B} \in N\langle C_D\rangle$ where C_D is a single vertex with $f(u_1, D - B) = 0$, which is a contradiction. □

Corollary 3.3. From Lemma 3.4(a) and Lemma 3.11, if $u \in N\langle C_k\rangle$ then $d_{L^k(G)}(u_1) \neq r_k$.

Lemma 3.12. *Let $C_B = \{v_B\}$ and $u_1, u_2, ..., u_n$ be vertices of equal degree in $N\langle C_B\rangle$ such that $d_{L^B(G)}(u_i)$ is maximum in $N\langle C_B\rangle$. Then, u_i generates a vertex $v_i \in C_{B+1}$ for all $1 \leq i \leq n$. Moreover, $u_1, u_2, ..., u_n$ generate l.max. vertices which induce a complete subgraph in C_{B+1}.*

Proof: As C_B generates C_{B+1}, from Lemma 3.3 there exists an integer $I \in [1, n]$ such that u_I generates a vertex $v_{B+1} \in C_{B+1}$. Let there be some $J \neq I$ such that u_J does not generate any vertex in C_{B+1}. Then, from Lemma 3.8, u_J generates a vertex, say u, in $N\langle C_{B+1}\rangle$. Now, $r_{B+1} = d_{L^{B+1}(G)}(v_{B+1}) = d_{L^B(G)}(u_I) + r_B - 2 = d_{L^B(G)}(u_J) + r_B - 2 = d_{L^{B+1}(G)}(u)$ which is a contradiction from Corollary 3.3 and hence no such J exists.

So, all $u_1, u_2, ..., u_n$ generate l.max. vertices, say $v_1, v_2, ..., v_n$, in C_{B+1} such that v_i corresponds to the edge $u_i v_B$ in $L^B(G)$. As all the corresponding edges share the vertex v_B, the vertices $v_1, v_2, ..., v_n$ induce a complete subgraph. □

Lemma 3.13. *Let $u_1, u_2 \in N\langle C_B\rangle$ with $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$. Furthermore, let u_1 generate the sequence $(u_1 = y_1^0, y_1^1, y_1^2, y_1^3, ...)$ and u_2 generate the sequence $(u_2 = y_2^0, y_2^1, y_2^2, y_2^3, ...)$. Then, $d_{L^{B+i}(G)}(y_1^i) = d_{L^{B+i}(G)}(y_2^i)$ for all $i \in \mathbb{Z}^+$ and either $y_1^i, y_2^i \in C_{B+i}$ or $y_1^i, y_2^i \in N\langle C_{B+i}\rangle$.*

Proof: For $i = 1$,

$$\begin{aligned} d_{L^{B+1}(G)}(y_1^1) &= d_{L^B(G)}(u_1) + r_B - 2 \\ &= d_{L^B(G)}(u_2) + r_B - 2 \\ &= d_{L^{B+1}(G)}(y_2^1). \end{aligned}$$

If C_B has an edge, then $y_1^1, y_2^1 \in N\langle C_{B+1}\rangle$ from Lemma 3.5(b) as $u_1, u_2 \in N\langle C_B\rangle$, otherwise, C_B is a single vertex. If $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$ is maximum in $N\langle C_B\rangle$, then $y_1^1, y_2^1 \in C_{B+1}$ from Lemma 3.12. On the other hand, if $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$ is not maximum in $N\langle C_B\rangle$, then $y_1^1, y_2^1 \in N\langle C_{B+1}\rangle$.

Let, for $i = n$, $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ and either $y_1^n, y_2^n \in C_{B+n}$ or $y_1^n, y_2^n \in N\langle C_{B+n}\rangle$.

Now, if $y_1^n, y_2^n \in C_{B+n}$ then from Lemma 3.5(a), $y_1^{n+1}, y_2^{n+1} \in C_{B+n+1}$ and $d_{L^{B+n+1}(G)}(y_1^{n+1}) = d_{L^{B+n+1}(G)}(y_2^{n+1}) = r_{B+n+1}$.

Otherwise $y_1^n, y_2^n \in N\langle C_{B+n}\rangle$. If C_{B+n} has an edge, from Lemma 3.5(b) we get that

$y_1^{n+1}, y_2^{n+1} \in N\langle C_{B+n+1}\rangle$. Then,

$$\begin{aligned} d_{L^{B+n+1}(G)}(y_1^{n+1}) &= d_{L^{B+n}(G)}(y_1^n) + r_{B+n} - 2 \\ &= d_{L^{B+n}(G)}(y_2^n) + r_{B+n} - 2 \\ &= d_{L^{B+n+1}(G)}(y_2^{n+1}). \end{aligned}$$

But, if $y_1^n, y_2^n \in N\langle C_{B+n}\rangle$ and C_{B+n} is a single vertex, then, if $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ is maximum in $N\langle C_{B+n}\rangle$, and then from Lemma 3.12, y_1^n and y_2^n generate y_1^{n+1} and y_2^{n+1}, respectively, in C_{B+n+1}. Else, if $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ is not maximum in $N\langle C_{B+n}\rangle$, then from Lemma 3.3, y_1^{n+1} and y_2^{n+1} are in $N\langle C_{B+n+1}\rangle$ and $d_{L^{B+n+1}(G)}(y_1^{n+1}) = d_{L^{B+n}(G)}(y_1^n) + r_{B+n} - 2 = d_{L^{B+n}(G)}(y_2^n) + r_{B+n} - 2 = d_{L^{B+n+1}(G)}(y_2^{n+1})$. □

Lemma 3.14. *If $u_1, u_2, ..., u_n \in N\langle C_B\rangle$ with $d_{L^B(G)}(u_i) = d_{L^B(G)}(u_j)$, then there exists an integer $E > B$ such that $u_1, u_2, ..., u_n$ generate vertices $y_1^{E-B}, y_2^{E-B}, \ldots, y_n^{E-B} \in C_E$ which form a clique.*

Proof: From Lemma 3.10 and Lemma 3.13, there exists an integer $D \geq B$ such that u_j generates $y_j^{D-B} \in N\langle C_D\rangle$, $1 \leq j \leq n$, where C_D is a single vertex, say v_D, and $d_{L^D(G)}(y_j^{D-B})$ is maximum in $N\langle C_D\rangle$. Then, from Lemma 3.12, y_j^{D-B} for $1 \leq j \leq n$ induce a complete subgraph in C_{D+1}. □

Lemma 3.15. *There exists an integer $E > A$ such that C_{E-1} has exactly one edge.*

Proof: Pick an integer $B > A$ such that $C_B = \{v_B\}$. Such an integer exists from our assumption that there are infinite integers $k > A$ where C_k is a single vertex. Then, as

$\delta_A > 2$, we have, from Lemma 2.1,

$$\delta_B > 2$$
$$-\delta_B < -2$$
$$r_B - \delta_B + 1 < r_B - 2 + 1$$
$$r_B - \delta_k + 1 < r_B$$

Now, there are r_B neighbors of v_B with $r_B - \delta_B + 1$ possible unique degrees. Hence, by Pigeonhole principle, there exists at least two vertices $u_1, u_2 \in N\langle C_B\rangle$ such that $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$. Then from Lemma 3.14, u_1, u_2 will induce an edge in C_D for some $D > B$. But, as there are infinite integers such that C_k is a single vertex, there exists an integer $E > D$ such that C_E is a single vertex. Let E be the smallest such integer. Then, C_{E-1} will have exactly one edge and the lemma is proved. □

Lemma 3.16. *There exists an integer E such that $C_E = \{v_E\}$ and there are three vertices $u_1, u_2, u_3 \in N\langle C_E\rangle$ with equal degree.*

Proof: Let $\delta_k^{'}$ denote the minimum degree in $N\langle C_k\rangle$. Then,

$$\delta_k^{'} = \delta_{k-1}^{'} + r_{k-1} - 2.$$

Now, pick an integer B such that C_B has exactly one edge. Such an integer exists from Lemma 3.15. Then, $r_{B+1} = 2r_B - 2$ from Lemma 3.5(a). But,

$$\begin{aligned}
\delta'_B &> 2 \\
\delta'_B + r_B - 1 &> 2 + r_B - 1 \\
r_B - 1 &> 2 + r_B - 1 - \delta'_B \\
r_B - 1 &> 2 + 2r_B - 1 - r_B - \delta'_B + 2 - 2 \\
r_B - 1 &> (2r_B - 2) - (r_B + \delta'_B - 2) + 1 \\
r_B - 1 &> r_{B+1} - \delta'_{B+1} + 1 \\
2r_B - 2 &> 2(r_{B+1} - \delta'_{B+1} + 1) \\
r_{B+1} &> 2(r_{B+1} - \delta'_{B+1} + 1)
\end{aligned}$$

Now, r_{B+1} is the number of neighbors of v_{B+1} and as v_{B+1} is a l.max. vertex, we have that $(r_{B+1} - \delta'_{B+1} + 1)$ is the number of possible unique values of degree of a neighbor of v_{B+1}. Also, because C_B has exactly one edge, $C_{B+1} = \{v_{B+1}\}$, and therefore, from Pigeonhole principle, there exist at least three vertices of equal degree in $N\langle C_{B+1}\rangle$. □

Continuing rest of the proof of Theorem 3.1:

Now, from Lemma 3.16 and Lemma 3.14, there exists an integer $F > E$ such that C_F contains a K_3. Hence, from Lemma 3.5(a), C_k contains K_3 all $k > F$ which contradicts that there are infinite number of integers $k > A$ such that C_k is a single vertex. Hence, there are finite values of $k > A$ where C_k is a single vertex and there exists an integer I, such that C_{I+i}, generated by C_k, has at least one edge, for all i. □

So, from Theorem 3.1, for each component C^j_B of $LM(L_B(G))$ where $B > A$, there exists an integer $I_j > B$ such that $C^j_{I_j+i}$ generated by C^j_B has at least one edge for all i. Suppose $LM(L^B(G))$ has N components. Using this reasoning for all components C^j_B, $1 \leq j \leq N$, there exists an integer $D = max\{I_j \mid 1 \leq j \leq N\}$, such that every component of $LM(L^{D+i}(G))$ has at least one edge for all i.

Clearly, the vertices of maximum degree of any graph G are also l.max. vertices and hence are components of $LM(G)$. But every component of $LM(L^{D+i})$ has edges for all i, hence, every vertex of maximum degree is adjacent to at least one vertex of maximum degree, and so, $\Delta_k = 2\Delta_{k-1} - 2$ for all $k > D$.

Chapter 4

Minimum degree growth in iterated line graphs

In this chapter it will be shown that for any graph G, which is not a path, there exists an integer D such that $\delta_{k+1} = 2\delta_k - 2$ for all $k > D$, where δ_k is the minimum degree of $L^k(G)$.

Note that, most of the lemmas for this proof parallel the lemmas, proved in Chapter 3, with the inequalities reversed. However, a different line of reasoning is used in the second half of the proof to contradict a theorem similar to Theorem 3.1. If G is a path, then as G is a finite graph, there exists an integer I such that $L^I(G)$ is undefined.

If G is a cycle, then for all $k \in \mathbb{Z}^+$, $\delta_{k+1} = 2\delta_k - 2 = 2$.

If G is a $K_{1,3}$, then $L(G)$ is a K_3 and hence, for all $k > 1$, $\delta_{k+1} = 2\delta_k - 2 = 2$.

Now we have to prove the theorem for any graph G where it is not a path, a cycle or a $K_{1,3}$.

Definition: A vertex v is a *locally minimum vertex* or a *l.min. vertex* if no vertex in the neighborhood of v has degree smaller than that of v.

Definition: The subgraph of G induced by its l.min. vertices is called the *locally minimum subgraph* or *l.min. subgraph* of G and is denoted by $lm(G)$.

Lemma 4.1. *All vertices in the same component of $lm(G)$ have the same degree in G.*

Proof: Let v and u be two vertices in a component of $lm(G)$. Then v and u are l.min. vertices of the graph G. As $v \in N(u)$, $d(v) \geq d(u)$ from definition. Similarly, as $u \in N(v)$, $d(u) \geq d(v)$. Hence, $d(u) = d(v)$. □

Lemma 4.2. *If w is a l.min. vertex of $L(G)$, then w corresponds to an edge e in G such that at least one end of e, say v, is l.min. in G and the other end of e, say u, has the smallest degree among the neighbors of v in G.*

Proof: Assume that neither v nor u is a l.min. vertex. Let $d_G(v) \leq d_G(u)$. Then, as v is not a l.min. vertex, there exists a vertex $y \in N(v)$ such that $d_G(y) < d_G(v)$.

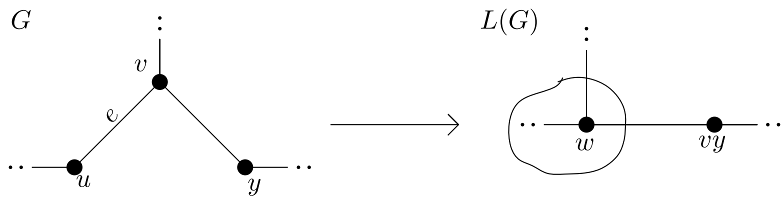

Figure 4.1

Now, the edge vy of G corresponds to the vertex vy of $L(G)$, adjacent to w, as shown in the Figure 4.1. Also,

$$d_G(v) \leq d_G(u).$$

But, as $d_G(y) < d_G(v)$,

$$d_G(v) + d_G(y) - 2 < d_G(u) + d_G(v) - 2,$$

$$d_{L(G)}(vy) < d_{L(G)}(w),$$

contradicting that w is a l.min. vertex of $L(G)$, hence, no such y exists, implying that v is a l.min. vertex of G.

Now, let there exist a vertex $z \in N(v)$ such that $d_G(z) < d_G(u)$.

Then the edge vz of G corresponds to the vertex vz adjacent to w in $L(G)$, as shown

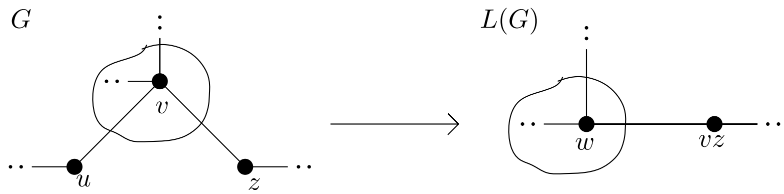

Figure 4.2

in the Figure 4.2.

But,

$$d_G(z) < d_G(u)$$

$$d_G(z) + d_G(v) - 2 < d_G(u) + d_G(v) - 2$$

$$d_{L(G)}(vz) < d_{L(G)}(w),$$

contradicting that w is a l.min. vertex of $L(G)$. Hence, no such z exists, implying that u has the minimum degree in $N(v)$. □

Lemma 4.3. *Let v be an isolated vertex of $LM(G)$.*

(a) If v has any neighbor of the same degree as that of v, then, v generates no l.min. vertices of $L(G)$.

(b) If all neighbors of v have degree greater than that of v, and u is such a neighbor, then the edge uv corresponds to a l.min. vertex of $L(G)$ if and only if u has the minimum degree among the neighbors of v, and for all $z \in N(u) \setminus \{v\}$, $d_G(z) \geq d_G(v)$.

Proof:

(a) Let u be a neighbor of v such that $d_G(u) = d_G(v)$. As u is not a l.min. vertex of G, there exists a vertex z adjacent to u, such that, $d_G(z) < d_G(u) = d_G(v)$. Then,

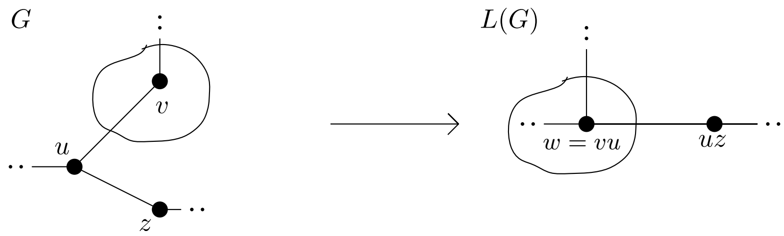

Figure 4.3

the edge uz of G corresponds to a vertex uz in $L(G)$, adjacent to a vertex w, which in turn corresponds to the edge uv of G, as shown in Figure 4.3. Now, $d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 < d_G(u) + d_G(v) - 2 = d_{L(G)}(w)$.

So, the edge uv cannot correspond to a l.min. vertex of $L(G)$ for any u with $d_G(u) = d_G(v)$. Hence, by Lemma 4.2, v does not generate a l.min. vertex in $L(G)$.

(b) Let there exist a vertex $z \in N(u) \setminus \{v\}$ such that $d_G(z) < d_G(v)$. Then, the edge uz corresponds to a vertex uz in $L(G)$ which is adjacent to a vertex w, which in turn corresponds to the edge vu of G, as shown in Figure 4.3. Now, $d_{L(G)}(uz) = d_G(u) + d_G(z) - 2 < d_G(u) + d_G(v) - 2 = d_{L(G)}(w)$.

Therefore, the edge uv will not correspond to a l.min. vertex of $L(G)$. Now, let, for all $z \in N(u) \setminus \{v\}$, $d_G(z) \geq d_G(v)$. Then,

$$d_G(u) + d_G(z) - 2 \geq d_G(u) + d_G(v) - 2$$
$$d_{L(G)}(uz) \geq d_{L(G)}(w),$$

where w corresponds to the edge vu. Therefore, w would be a l.min. vertex.

Moreover, if uz is a l.min. vertex, it would be adjacent to w implying that the number of components will not increase. □

Lemma 4.4. *Let C be a component of $lm(G)$ which is not a single vertex.*

a) If v_1 and v_2 are adjacent vertices in C, then the vertex $w \in L(G)$, corresponding to the edge v_1v_2, is a l.min. vertex.

b) If $u \in N\langle C\rangle$, then no edge joining u to a vertex in C corresponds to a l.min. vertex of $L(G)$.

Proof:

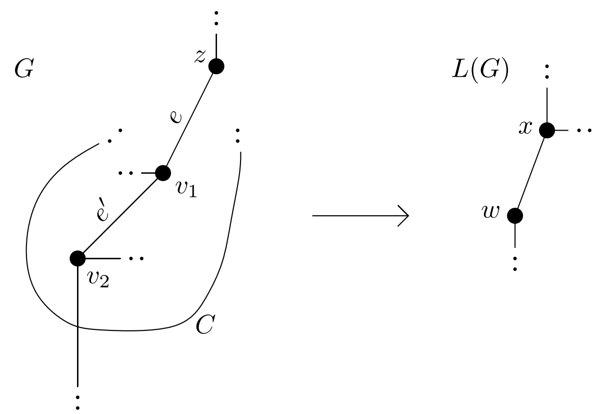

Figure 4.4

a) Let v_1, v_2 be two vertices of G such that $e' = v_1v_2$ is an edge in C. Let $w \in L(G)$ be the vertex corresponding to e'. Then, any neighbor x of w will correspond to an edge e, in G, incident at either v_1 or v_2. Let e be incident at v_1 and some vertex $z \in N(v_1)$ as shown in the Figure 4.4. Then, as v_1 is a l.min. vertex,

$$d_G(z) \geq d_G(v_1),$$

$$d_G(z) + d_G(v_2) - 2 \geq d_G(v_1) + d_G(v_2) - 2.$$

From Lemma 4.1, $d_G(v_1) = d_G(v_2)$,

$$d_G(z) + d_G(v_1) - 2 \geq d_G(v_1) + d_G(v_2) - 2,$$

$$d_{L(G)}(x) \geq d_{L(G)}(w).$$

Hence, w is a l.min. vertex.

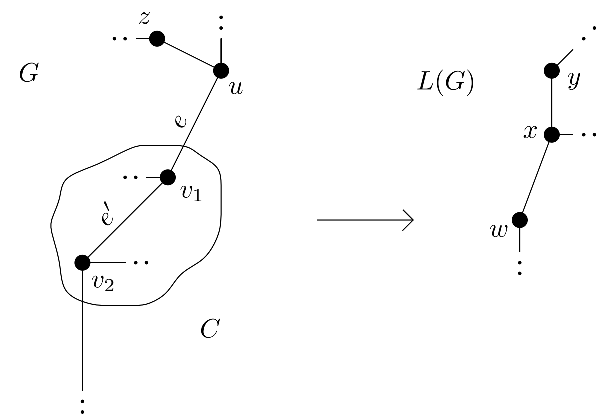

Figure 4.5

b) As $u \in N\langle C\rangle$, it is adjacent to a vertex, say v_1, in C. As C is not a single vertex, there exists a vertex $v_2 \in C$ adjacent to v_1. Let w be the vertex in $L(G)$ corresponding to the edge v_1v_2 and let r be the common degree of vertices in C. Then, $d_{L(G)}(w) = 2r - 2$. Now, the edge uv_1 corresponds to a vertex x adjacent to w, as shown in Figure 4.5. Also, $d_{L(G)}(x) = d_G(u) + r - 2$ and as v_1 is a l.min. vertex, we get that $d_G(u) \geq r$.
If $d_G(u) > r$, then,

$$d_G(u) + r - 2 > r + r - 2,$$

$$d_{L(G)}(x) > d_{L(G)}(w),$$

and x can not be a l.min. vertex.
Otherwise, $d_G(u) = r$. But as u is not a l.min. vertex, there exists a vertex $z \in N(u) \setminus \{v_1\}$ such that $d_G(z) < d_G(u)$. Then, the edge uz of G corresponds to a vertex y adjacent to x, as shown in Figure 4.5. Now,

$$d_G(z) < d_G(u),$$

$$d_G(z) + d_G(u) - 2 < d_G(u) + d_G(u) - 2,$$

$$d_G(z) + d_G(u) - 2 < d_G(u) + r - 2,$$

$$d_{L(G)}(y) < d_{L(G)}(x),$$

and hence, x can not be a l.min. vertex. □

Corollary 4.1: It follows from Lemma 4.4 that $L(C)$ is a component of $lm(L(G))$.

Corollary 4.2: If C is a single vertex, then from Lemma 4.3 it generates at most one component of $lm(L(G))$. Otherwise, if C is not a single vertex, every vertex of C generates a l.min. vertex from Lemma 4.4(a). As the line graph operation preserves connectivity, C will generate at most one component of $lm(L(G))$. Hence, in either case, C generates at most one component.

Lemma 4.5. *There exists an integer A such that for all $k > A$, every component of $lm(L^k(G))$ generates exactly one component of $lm(L^{k+1}(G))$.*

Proof: Let c_k be the number of components of $lm(L^k(G))$. From Corollary 4.2, $\{c_k\}$ is a non-increasing sequence. But as c_k is a non-negative number for every k, there exists an integer A, such that c_k is constant for all $k > A$. □

We now define new notation which would be followed in the rest of this chapter. Let C_{A+1} be a component of $lm(L^{A+1}(G))$ where A is the integer from Lemma 4.5. Inductively, for each $k > A$, let C_{k+1} be the component of $L^{k+1}(G)$ generated by C_k. Let r_k be the common degree of vertices in C_k. We can further choose A to be sufficiently large so that $\delta_k > 2$ for all $k > A$, from Lemma 2.1.

Lemma 4.6. *Let $u \in N\langle C_D\rangle$ be adjacent to a vertex $v \in C_D$, where D is an integer greater than A. Further, let $y \in L^{D+1}(G)$ correspond to the edge uv of $L^D(G)$, so $y \in N[C_{D+1}]$. Then the following holds.*

(a) If C_D is not a single vertex, then

$$d_{L^{D+1}(G)}(y) - r_{D+1} = d_{L^D(G)}(u) - r_D$$

(b) Otherwise, if C_D is a single vertex, then,

$$d_{L^{D+1}(G)}(y) - r_{D+1} < d_{L^D(G)}(u) - r_D$$

Proof:

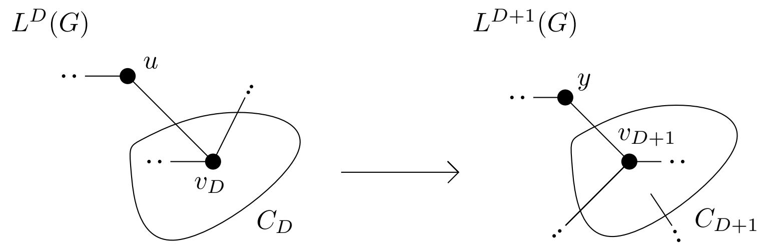

Figure 4.6: When C_D is not a single vertex

(a) From Lemma 4.4(a), if C_D has an edge then it generates C_{D+1}, as shown in Figure 4.6, and $r_{D+1} = 2r_D - 2$.

Also, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. So,

$$d_{L^{D+1}(G)}(y) - r_{D+1} = (d_{L^D(G)}(u) + r_D - 2) - (2r_D - 2)$$
$$d_{L^{D+1}(G)}(y) - r_{D+1} = d_{L^D(G)}(u) - r_D.$$

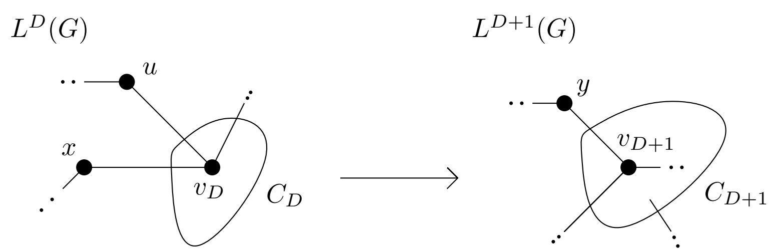

Figure 4.7: When C_D is a single vertex

(b) Suppose $y \in N\langle C_{D+1}\rangle$. Again, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$. Let x be a vertex of smallest degree in $N(v_D)$ such that the edge xv_D corresponds to a l.min. vertex, say

v_{D+1}, in C_{D+1} from Lemma 4.5. Such a vertex x exists from Lemma 4.2 and because C_{D+1} is non-empty. Then,

$$r_{D+1} = d_{L^D(G)}(x) + r_D - 2.$$

Since C_D is a single vertex, hence from Lemma 4.3(a), $d_{L^D(G)}(x) > r_D$ as C_D generates C_{D+1}. So,

$$d_{L^D(G)}(x) + r_D - 2 > r_D + r_D - 2,$$

$$r_{D+1} > 2r_D - 2,$$

$$r_{D+1} - d_{L^{D+1}(G)}(y) > (2r_D - 2) - d_{L^{D+1}(G)}(y).$$

But, $d_{L^{D+1}(G)}(y) = d_{L^D(G)}(u) + r_D - 2$, therefore,

$$r_{D+1} - d_{L^{D+1}(G)}(y) > (2r_D - 2) - (d_{L^D(G)}(u) + r_D - 2),$$

$$r_{D+1} - d_{L^{D+1}(G)}(y) > r_D - d_{L^D(G)}(u),$$

$$d_{L^{D+1}(G)}(y) - r_{D+1} < d_{L^D(G)}(u) - r_D.$$

Now, suppose $y \in C_{D+1}$. Then $d_{L^{D+1}(G)}(y) - r_{D+1} = 0$. But as C_D is a single vertex, $d_{L^D(G)}(u) - r_D \neq 0$, as otherwise C_D will not generate a component. Hence $d_{L^{D+1}(G)}(y) - r_{D+1} < d_{L^D(G)}(u) - r_D$. □

Lemma 4.7. *If $u \in N\langle C_k\rangle$ then u generates a vertex $y \in N[C_{k+1}]$.*

Proof: As $u \in N\langle C_k\rangle$, u is adjacent to a vertex $v \in C_k$. Let the edge uv correspond to the vertex $y \in L^{k+1}(G)$. If C_k has an edge, from Lemma 4.4(a) v generates a vertex in C_{k+1}. Also, if $C_k = \{v\}$, as $k > A$, v generates every vertex in C_{k+1}. Then, there exists a vertex $w \in C_{k+1}$ generated by v. Now, the edges in $L^k(G)$ corresponding to y and w, are incident at the vertex v. Hence, y is adjacent to the vertex w in $L^{k+1}(G)$, implying that, if y is a l.min. vertex then $y \in C_{k+1}$ or else $y \in N\langle C_{k+1}\rangle$. □

Let $N\langle C_B\rangle = \{u_1, u_2, \ldots, u_n\}$. Then, from Lemma 4.7, for every $1 \leq j \leq n$, u_j generates a vertex, say y_j^1, in $N[C_{B+1}]$.

Now, if y_j^i is a vertex in $N\langle C_{B+i}\rangle$, then from Lemma 4.7, it generates a vertex, say y_j^{i+1} in $N[C_{B+i+1}]$. Otherwise, if y_j^i is a vertex in C_{B+i}, then from Lemma 4.4(a), it generates a vertex, say y_j^{i+1} in C_{B+i+1}. It follows inductively that u_j generates a sequence of vertices $(u_j = y_j^0, y_j^1, y_j^2, y_j^3, ...)$ where $y_j^i \in N[C_{B+i}]$ and, moreover, $y_j^i \in C_{B+i}$ for all $i > I$ if $y_j^I \in C_{B+I}$ for some integer I.

Then, we define a function $f(u_j, i) : N\langle C_B\rangle \to \mathbb{R}$ by $f(u_j, i) = d_{L^{B+i}(G)}(y_j^i) - r_{B+i}$ where $i \in \mathbb{Z}^+$. Clearly $f(u_j, i)$ is non-negative and from Lemma 4.6 it is a non-increasing function of i. Also, if C_{B+i} is a single vertex and $y_j^i \in N\langle C_{B+i}\rangle$, then, from Lemma 4.3(a), $f(u_j, i)$ can not equal to zero as otherwise C_{B+i} will not generate a component.

Theorem 4.1. *Let G be a simple and connected graph. Let C_A be a component of $lm(L^A(G))$. Then, there are a finite number of integers $k > A$, such that C_k, generated by C_A, is a single vertex.*

Proof: The proof is by contradiction. Let us assume that there are infinite number of integers $k > A$ such that C_k is a single vertex. Then we prove the following series of lemmas.

Lemma 4.8. *If $u_1 \in N\langle C_B\rangle$ generates $(y_1^0, y_1^1, y_1^2, y_1^3, ...)$, then there exists an integer I such that $y_1^I \in C_{B+I}$.*

Proof: We prove this by contradiction. Let $y_1^i \in N\langle C_{B+i}\rangle$ for all i. The function $f(u_1, i)$ is non-increasing and decreases when C_{B+i} is a single vertex. As there are infinite number of integers $k > A$ such that C_k is a single vertex, there are infinite integers i such that C_{B+i} is a single vertex as $B > A$. Hence, from Lemma 4.6(b) there exists an integer $D > B$ such that $f(u_1, D - B) = 0$.

Now, if C_D is a single vertex, then as $y_1^i \in N\langle C_{B+i}\rangle$ for all i, $f(u_1, D - B)$ can not be zero and we have a contradiction. Otherwise, if C_D has an edge, then let E be the smallest integer greater than D such that C_E is a single vertex. From Lemma 4.6(a), $f(u_1, E-B) = f(u_1, D-B) = 0$, and we again have a contradiction. □

Lemma 4.9. *If $u_1 \in N\langle C_B\rangle$ then there exists an integer $D \geq B$ such that u_1 generates $y_1^{D-B} \in N\langle C_D\rangle$ where C_D is a single vertex and $d_{L^D(G)}(y_1^{D-B})$ is minimum in $N\langle C_D\rangle$.*

Proof: From Lemma 4.8 there exists an integer I such that u_1 generates $y_1^I \in C_{B+I}$. Let I be the smallest such integer. Then, $y_1^{I-1} \in N\langle C_{B+I-1}\rangle$. From Lemma 4.4, if C_{B+I-1} has an edge then y_1^{I-1} cannot generate a vertex in C_{B+I}. Hence, C_{B+I-1} is a single vertex. Also, from Lemma 4.2, $d_{L^{B+I-1}(G)}(y_1^{I-1})$ is minimum in $N\langle C_{B+I-1}\rangle$. □

Lemma 4.10. *If $u_1 \in N\langle C_B\rangle$ where C_B is not a single vertex, then, $d_{L^B(G)}(u_1) \neq r_B$.*

Proof: Assume that $d_{L^B(G)}(u_1) = r_B$ and hence, $f(u_1, 0) = 0$. But as $f(u_i, j)$ is non-negative and non-increasing, $f(u_1, j) = 0$ for all j. But, from Lemma 4.9, there exists an integer $D \geq B$ such that u_1 generates $y_1^{D-B} \in N\langle C_D\rangle$ where C_D is a single vertex with $f(u_1, D-B) = 0$, which is a contradiction. □

Corollary 4.3. From Lemma 4.3(a) and Lemma 4.10, if $u \in N\langle C_k\rangle$ then $d_{L^k(G)}(u) \neq r_k$.

Lemma 4.11. *Let $C_B = \{v_B\}$ and $u_1, u_2, ..., u_n$ be vertices of equal degree in $N\langle C_B\rangle$ such that $d_{L^B(G)}(u_i)$ is minimum in $N\langle C_B\rangle$. Then, u_i generates a vertex $v_i \in C_{B+1}$ for all $1 \leq i \leq n$. Moreover, $u_1, u_2, ..., u_n$ generate l.min. vertices which induce a complete subgraph in C_{B+1}.*

Proof: As C_B generates C_{B+1}, from Lemma 4.2 there exists an integer $I \in [1, n]$ such that u_I generates a vertex in C_{B+1}. Let there be some $J \neq I$ such that u_J does not generate any vertex $v \in C_{B+1}$. Then, from Lemma 4.7 it follows that u_J generates a vertex, say u, in $N\langle C_{B+1}\rangle$. Now, $r_{B+1} = d_{L^{B+1}(G)}(v_{B+1}) = d_{L^B(G)}(u_I) + r_B - 2 = d_{L^B(G)}(u_J) + r_B - 2 = d_{L^{B+1}(G)}(u)$ which is a contradiction from Corollary 4.3 and hence no such J exists.

So, all $u_1, u_2, ..., u_n$ generate l.min. vertices, say $v_1, v_2, ..., v_n$, in C_{B+1} such that v_i corresponds to the edge u_iv_B in $L^B(G)$. As all the corresponding edges share the vertex v_B, the vertices $v_1, v_2, ..., v_n$ induce a complete subgraph. □

Lemma 4.12. *Let $u_1, u_2 \in N\langle C_B\rangle$ with $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$. Let u_1 generate the sequence $(u_1 = y_1^0, y_1^1, y_1^2, y_1^3, ...)$ and u_2 generate the sequence $(u_2 = y_2^0, y_2^1, y_2^2, y_2^3, ...)$. Then, $d_{L^{B+i}(G)}(y_1^i) = d_{L^{B+i}(G)}(y_2^i)$ for all $i \in \mathbb{Z}^+$ and either $y_1^i, y_2^i \in C_{B+i}$ or $y_1^i, y_2^i \in N\langle C_{B+i}\rangle$.*

Proof: For $i = 1$,

$$\begin{aligned} d_{L^{B+1}(G)}(y_1^1) &= d_{L^B(G)}(u_1) + r_B - 2 \\ &= d_{L^B(G)}(u_2) + r_B - 2 \\ &= d_{L^{B+1}(G)}(y_2^1) \end{aligned}$$

If C_B has an edge, then $y_1^1, y_2^1 \in N\langle C_{B+1}\rangle$ from Lemma 4.4(b) as $u_1, u_2 \in N\langle C_B\rangle$.
Otherwise, C_B is a single vertex. If $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$ is minimum in $N\langle C_B\rangle$, then $y_1^1, y_2^1 \in C_{B+1}$ from Lemma 4.11. Else, if $d_{L^B(G)}(u_1) = d_{L^B(G)}(u_2)$ is not minimum in $N\langle C_B\rangle$, then $y_1^1, y_2^1 \in N\langle C_{B+1}\rangle$.
Let, for $i = n$, $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ and either $y_1^n, y_2^n \in C_{B+n}$ or $y_1^n, y_2^n \in N\langle C_{B+n}\rangle$.
Now, if $y_1^n, y_2^n \in C_{B+n}$ then from Lemma 4.4(a), $y_1^{n+1}, y_2^{n+1} \in C_{B+n+1}$ and $d_{L^{B+n+1}(G)}(y_1^{n+1}) = d_{L^{B+n+1}(G)}(y_2^{n+1}) = r_{B+n+1}$.
Otherwise $y_1^n, y_2^n \in N\langle C_{B+n}\rangle$. If C_{B+n} has an edge, then, from Lemma 4.4(b), we have $y_1^{n+1}, y_2^{n+1} \in N\langle C_{B+n+1}\rangle$. Then,

$$\begin{aligned} d_{L^{B+n+1}(G)}(y_1^{n+1}) &= d_{L^{B+n}(G)}(y_1^n) + r_{B+n} - 2 \\ &= d_{L^{B+n}(G)}(y_2^n) + r_{B+n} - 2 \\ &= d_{L^{B+n+1}(G)}(y_2^{n+1}). \end{aligned}$$

But, if $y_1^n, y_2^n \in N\langle C_{B+n}\rangle$ and C_{B+n} is a single vertex, then, if $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ is minimum in $N\langle C_{B+n}\rangle$, from Lemma 4.11, y_1^n and y_2^n generate y_1^{n+1} and y_2^{n+1}, respectively, in C_{B+n+1}. Else, if $d_{L^{B+n}(G)}(y_1^n) = d_{L^{B+n}(G)}(y_2^n)$ is not minimum in $N\langle C_{B+n}\rangle$, then from Lemma 4.2, y_1^{n+1} and y_2^{n+1} are in $N\langle C_{B+n+1}\rangle$ and $d_{L^{B+n+1}(G)}(y_1^{n+1}) = d_{L^{B+n}(G)}(y_1^n) + r_{B+n} - 2 = d_{L^{B+n}(G)}(y_2^n) + r_{B+n} - 2 = d_{L^{B+n+1}(G)}(y_2^{n+1})$. □

Lemma 4.13. *If $u_1, u_2, ..., u_n \in N\langle C_B\rangle$ with $d_{L^B(G)}(u_i) = d_{L^B(G)}(u_j)$, then there exists an integer $E > B$ such that $u_1, u_2, ..., u_n$ generate vertices $y_1^{E-B}, y_2^{E-B}, \ldots, y_n^{E-B} \in C_E$ which form a clique.*

Proof: From Lemma 4.9 and Lemma 4.12, there exists an integer $D \geq B$ such that u_j generates $y_j^{D-B} \in N\langle C_D\rangle$, $1 \leq j \leq n$, where C_D is a single vertex, say v_D, and $d_{L^D(G)}(y_j^{D-B})$ is minimum in $N\langle C_D\rangle$. Then, from Lemma 4.2, y_j^{D-B} for $1 \leq j \leq n$, induce a complete subgraph in C_{D+1}. □

Continuing rest of the proof of Theorem 4.1: Now, $\delta_A > 3$. Hence, $\delta_k > 3$ for all $k > A$. Pick any integer $B > A$. Let $v_B \in C_B$ and $w_B \in L^B(G)$ be a vertex of maximum degree, Δ_B. As G is connected, there exists a path $P_B = (w_B = v_1^B, v_2^B, ..., v_n^B = v_B)$ from w_B to

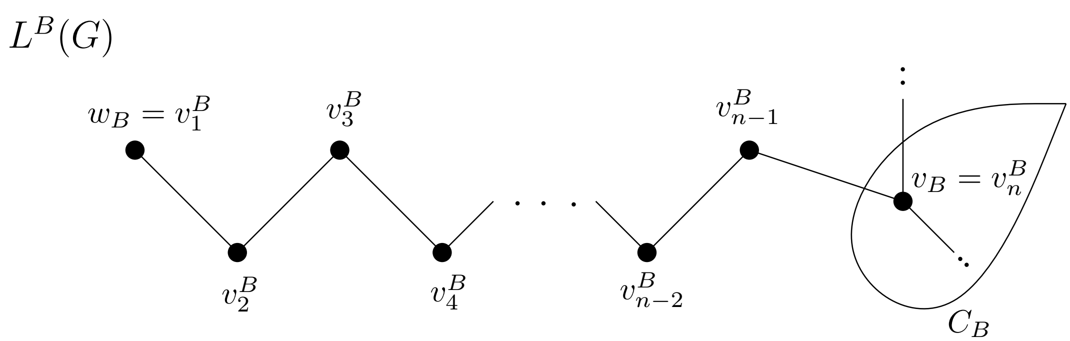

Figure 4.8: Path from w_B to v_B

v_B as shown in Figure 4.8. Now,

$$\delta_B > 3$$

$$-\delta_B < -3$$

$$\Delta_B - \delta_B < \Delta_B - 3$$

$$\Delta_B - \delta_B + 1 < \Delta_B - 2.$$

Degree of any neighbor of w_B can be any of $\Delta_B - \delta_B + 1$ possible values. But there are

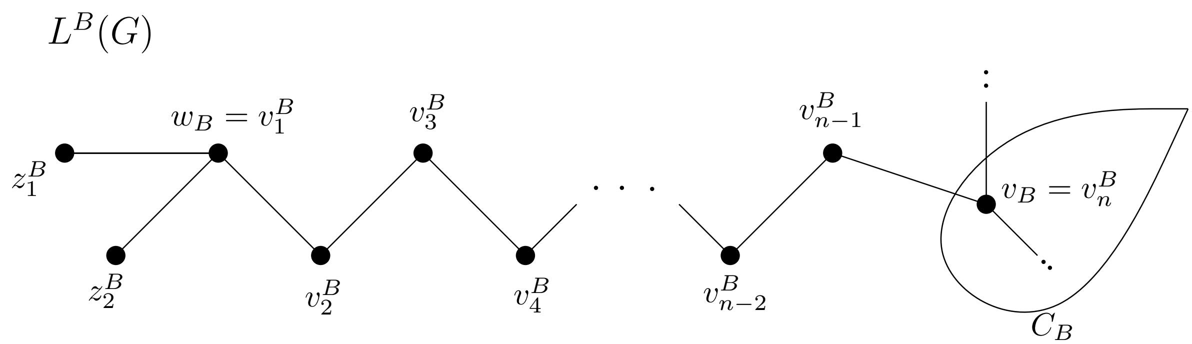

Figure 4.9: Path from w_B to v_B

$\Delta_B - 1$ neighbors of w_B apart from v_2^B. From Pigeonhole principle, there exist at least two vertices, say $z_1^B, z_2^B \in N(w_B) \setminus \{v_2^B\}$ such that $d_{L^B(G)}(z_1^B) = d_{L^B(G)}(z_2^B)$, as shown in Figure 4.9.

Now, $L(P_B)$ will be a path in $L^{B+1}(G)$. Let the edge $z_1^B v_1^B$ correspond to the vertex z_1^{B+1} in $L^{B+1}(G)$. Let the edge $z_2^B v_1^B$ correspond to the vertex z_2^{B+1} in $L^{B+1}(G)$. Let the edge $v_i^B v_{i+1}^B$ correspond to the vertex v_i^{B+1} in $L^{B+1}(G)$ for $1 \leq i \leq n-2$. From Lemma 4.7, v_{n-1}^B generates a vertex, say v_{n-1}^{B+1}, such that either $v_{n-1}^{B+1} \in C_{B+1}$ or $v_{n-1}^{B+1} \in N\langle C_{B+1}\rangle$. When $v_{n-1}^{B+1} \in N\langle C_{B+1}\rangle$, there exists a vertex $v_n^{B+1} \in C_{B+1}$ adjacent to v_{n-1}^{B+1}.

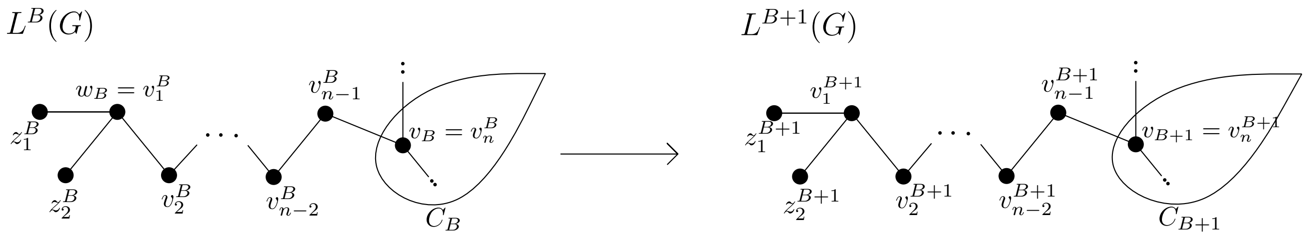

Figure 4.10: $v_{n-1}^{B+1} \in N\langle C_{B+1}\rangle$

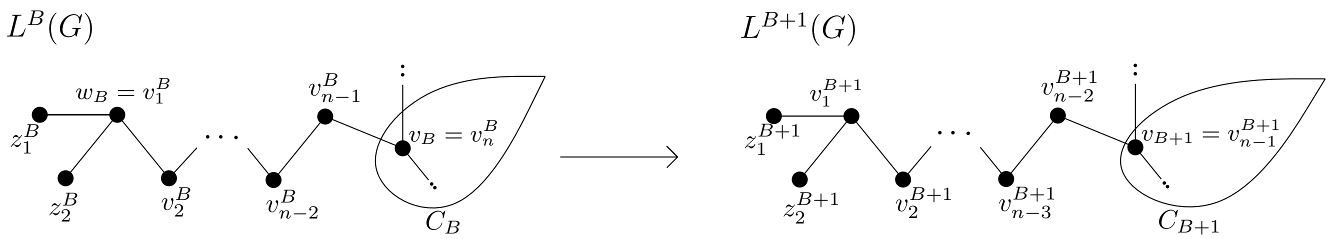

Figure 4.11: $v_{n-1}^{B+1} \in C_{B+1}$

Define $P_{B+1} = (v_1^{B+1}, v_2^{B+1}, ..., v_n^{B+1})$ if $v_{n-1}^{B+1} \in N\langle C_{B+1}\rangle$, as shown in Figure 4.10. Otherwise, define $P_{B+1} = (v_1^{B+1}, v_2^{B+1}, ..., v_{n-1}^{B+1})$ if $v_{n-1}^{B+1} \in C_{B+1}$, as shown in Figure 4.11. Notice that $d_{L^{B+1}(G)}(z_1^{B+1}) = d_{L^B(G)}(z_1^B) + d_{L^B(G)}(v_1^B) - 2 = d_{L^B(G)}(z_2^B) + d_{L^B(G)}(v_1^B) - 2 = d_{L^{B+1}(G)}(z_2^{B+1})$. Also, if $v_{n-1}^{B+1} \in N\langle C_{B+1}\rangle$, then $|P_{B+1}| = |P_B|$, and if $v_{n-1}^{B+1} \in C_{B+1}$, then $|P_{B+1}| = |P_B| - 1$.

From Lemma 4.8 there exists an integer I_{n-1} such that v_{n-1}^B generates $v_{n-1}^{B+I_{n-1}} \in C_{B+I_{n-1}}$. Let I_{n-1} be the smallest such integer. Then $P_{B+I_{n-1}} = (v_1^{B+I_{n-1}}, v_2^{B+I_{n-1}}, ..., v_{n-1}^{B+I_{n-1}})$ and $|P_{B+I_{n-1}}| = |P_B| - 1$.

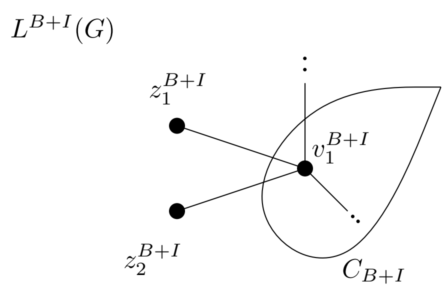

Figure 4.12

Following inductively, there exists an integer $I = I_{n-1} + I_{n-2} + ... + I_1$ such that $P_{B+I} = (v_1^{B+I})$ and $d_{L^{B+I}(G)}(z_1^{B+I}) = d_{L^{B+I}(G)}(z_2^{B+I})$ as shown in Figure 4.12. From Lemma 4.9 and

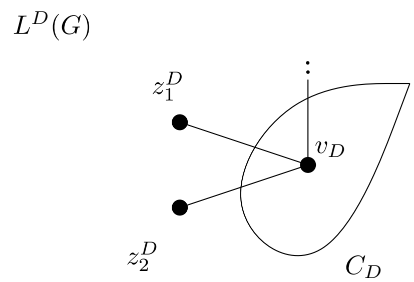

Figure 4.13

Lemma 4.12, there exists an integer $D \geq B + I$ such that z_1^{B+I} and z_2^{B+I} generate z_1^D and z_2^D, respectively, in $N\langle C_D \rangle$, where $C_D = \{v_D\}$ and $d_{L^D(G)}(z_1^D) = d_{L^D(G)}(z_2^D)$ is minimum in $N\langle C_D \rangle$, as shown in Figure 4.13.

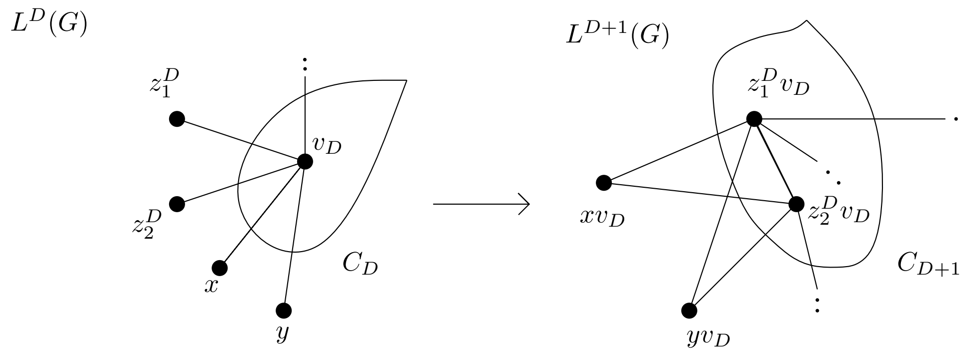

Figure 4.14

But, as $\delta_k > 3$ for all $k > A$, there are at least two more neighbors of v_D, say x and y and let $d_{L^D(G)}(x) \leq d_{L^D(G)}(y)$. As C_D is a single vertex, from Lemma 4.11 it follows that z_1^D and z_2^D generate $z_1^D v_D$ and $z_2^D v_D$, respectively, in C_{D+1}, which are adjacent to each other, as is shown in Figure 4.14. In the next iteration, we get four vertices,

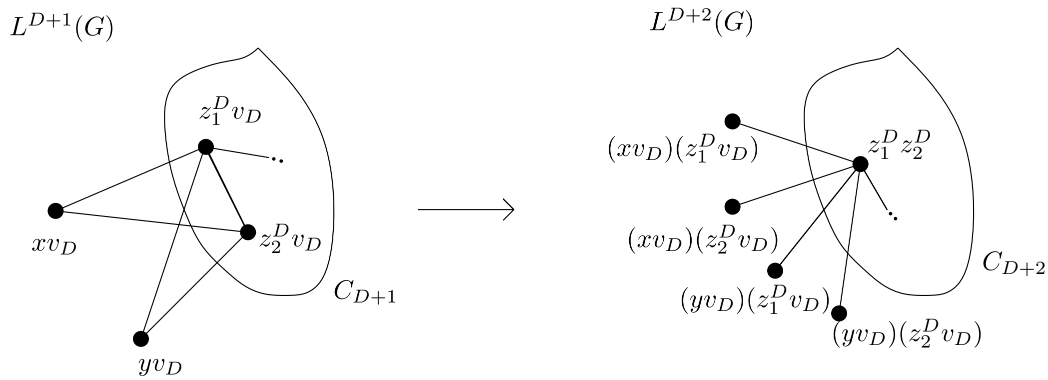

Figure 4.15

$(xv_D)(z_1^D v_D), (xv_D)(z_2^D v_D), (yv_D)(z_1^D v_D)$ and $(yv_D)(z_2^D v_D)$, as are shown in the Figure 4.15. If $d_{L^D}(x) = d_{L^D}(y)$, then, from Lemma 4.12, we have $d_{L^{D+2}(G)}((xv_D)(z_1^D v_D)) = d_{L^{D+2}(G)}((xv_D)(z_2^D v_D)) = d_{L^{D+2}(G)}((yv_D)(z_1^D v_D)) = d_{L^{D+2}(G)}((yv_D)(z_2^D v_D))$. So, from Lemma 4.13, there exists an integer $F > E$, such that, C_F contains a K_4.

Otherwise, let $d_{L^D}(x) < d_{L^D}(y)$. From Lemma 4.9 and Lemma 4.12, there exists an integer $E > D + 2$ such that C_E is a single vertex, say v_E, and, $xv_D z_1^D v_D$ and $xv_D z_2^D v_D$ generate vertices, say x_1 and x_2, respectively, in $N\langle C_E \rangle$, such that they have the same degree which is minimum in $N\langle C_E \rangle$. Let y_1 and y_2 be the vertices generated by $(yv_D)(z_1^D v_D)$ and $(yv_D)(z_2^D v_D)$, respectively, in $L^E(G)$, as shown in the Figure 4.16. Notice that $d_{L^E(G)}(y_1) = d_{L^E(G)}(y_2)$.

Then, we have the line graph iterations as shown in Figure 4.17. Now,

$$\begin{aligned} d_{L^{E+1}(G)}(y_1 v_E) &= d_{L^E(G)}(y_1) + d_{L^E(G)}(v_E) \\ &= d_{L^E(G)}(y_2) + d_{L^E(G)}(v_E) \\ &= d_{L^{E+1}(G)}(y_2 v_E). \end{aligned}$$

Also,

$$
\begin{aligned}
d_{L^{E+2}(G)}((y_1v_E)(x_1v_E)) &= d_{L^{E+1}(G)}(y_1v_E) + d_{L^{E+1}(G)}(x_1v_E) - 2 \\
&= d_{L^{E+1}(G)}(y_1v_E) + r_{E+1} - 2, \\
d_{L^{E+2}(G)}((y_1v_E)(x_2v_E)) &= d_{L^{E+1}(G)}(y_1v_E) + d_{L^{E+1}(G)}(x_2v_E) - 2 \\
&= d_{L^{E+1}(G)}(y_1v_E) + r_{E+1} - 2, \\
d_{L^{E+2}(G)}((y_2v_E)(x_1v_E)) &= d_{L^{E+1}(G)}(y_2v_E) + d_{L^{E+1}(G)}(x_1v_E) - 2 \\
&= d_{L^{E+1(G)}}(y_1v_E) + r_{E+1} - 2,
\end{aligned}
$$

and,

$$
\begin{aligned}
d_{L^{E+2}(G)}((y_2v_E)(x_2v_E)) &= d_{L^{E+1}(G)}(y_2v_E) + d_{L^{E+1}(G)}(x_2v_E) - 2 \\
&= d_{L^{E+1}(G)}(y_1v_E) + r_{E+1} - 2.
\end{aligned}
$$

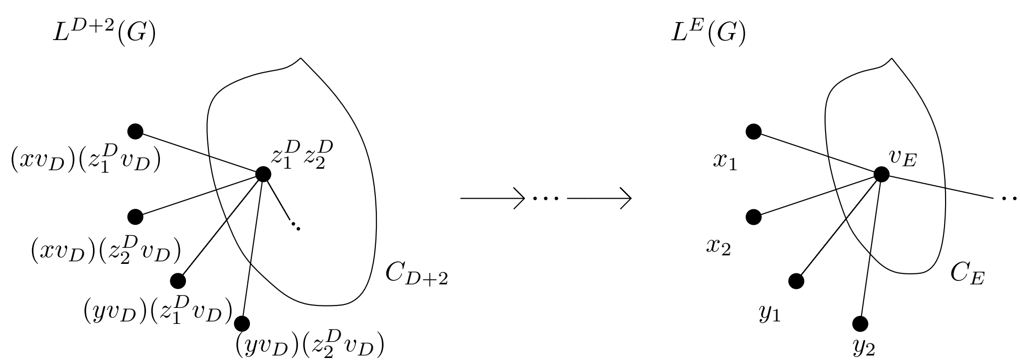

Figure 4.16

So, there are four vertices of same degree in $N\langle C_{E+2}\rangle$. From Lemma 4.13, there exists an integer $F > E + 2$ such that C_F will contain a K_4.

Returning to the proof of Theorem 4.1: Therefore, for a component C_B^j of $lm(L_B(G))$ where $B > A$ there exists an integer $I_j > B$ such that $C_{I_j}^j$ generated by C_B^j has a K_4 and hence, from Lemma 4.4(a), $C^j_{I_j+i}$ contains K_4 for all i, which is a contradiction to the assumption that there are inifinite integers $k > A$ such that C_k is a single vertex. Hence, there exists an integer I such that C_{I+i} has at least one edge for all i. □

Suppose $lm(L_B(G))$ has N components. Then, from Theorem 4.1, for every component C^j_B, $1 \leq j \leq N$, as there are finite number of integers k such that C_k is a single vertex, there exists an integer $I_j > B$ such that $C^j_{I_j+i}$, generated by C^j_B, has at least one edge for all i. Hence, there exists $D = max\{I^j \mid 1 \leq j \leq N\}$, such that every component of $lm(L^{D+i}(G))$ has at least one edge for all i.

Clearly, the vertices of minimum degree of any graph G are also l.min. vertices and, hence, are components of $lm(G)$. But every component of $lm(L^{D+i}(G))$ has at least one edge for all i. Hence, every vertex of minimum degree is adjacent to at least one vertex of minimum degree, so, $\delta_k = 2\delta_{k-1} - 2$ for all $k > D$.

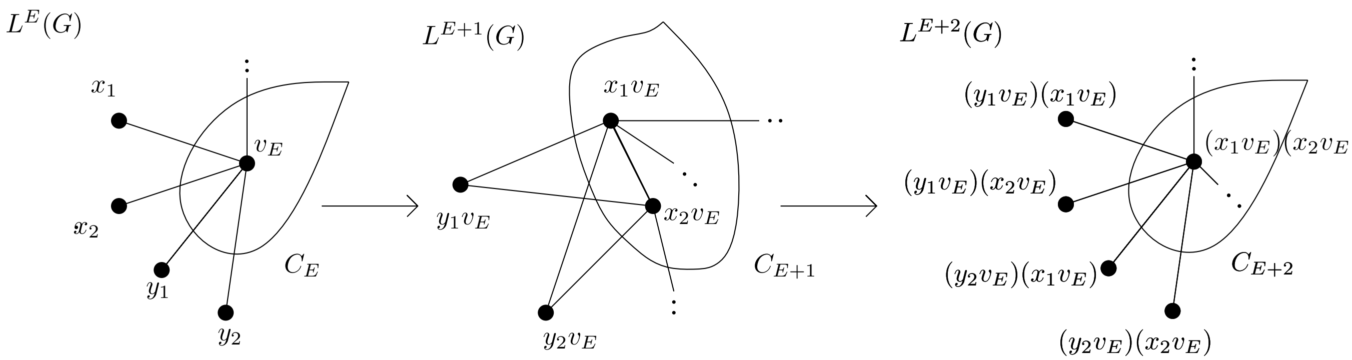

Figure 4.17

Chapter 5

A puzzle

Dr.Hoffman assigned me an interesting puzzle. If G is a connected graph and $L(G)$ is regular, then show that G is either regular or bipartite.

Proof: For any graph G, its line graph, $L(G)$, is regular if and only if every edge of G is incident with the same number of edges. Hence, for any two edges uv and wy,

$$d(u) + d(v) - 2 = d(w) + d(y) - 2,$$

$$d(u) + d(v) = d(w) + d(y)$$

Let uv be an edge of G and w be a vertex. As G is connected, then without loss of generality, there exists a path $P = (u, v, v_1, v_2, \ldots, v_n, w)$. Now, $d(v_1) + d(v) = d(v) + d(u)$ and hence $d(v_1) = d(u)$. If $d(v_i) = d(u)$, then, $d(v_{i+1}) = d(v)$, otherwise, if $d(v_i) = d(v)$, then, $d(v_{i+1}) = d(u)$. It follows from induction that for any vertex w of G, we have that, $d(w) = d(u)$ or $d(w) = d(v)$. Moreover, for any edge wy, either $d(w) = d(u)$ and $d(y) = d(v)$ or the other way round. Also, from induction, the degree of the vertices alternates along the path, hence, if $|P|$ is even, then, $d(w) = d(u)$, otherwise, if $|P|$ is odd then $d(w) = d(v)$.

Now, if G has an odd cycle, say, $P = (u, v_1, v_2, \ldots, v_n, u)$, then from above discussion $d(u) = d(v_1)$. But as for any vertex w of G, $d(w) = d(u)$ or $d(w) = d(v_1)$, therefore G is regular. It follows that for any connected graph G with $L(G)$ regular, either G is regular and, if it is not regular, then it has no odd cycles, i.e., it is bipartite. $\square$

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