Structure Theory and a Generalization of the Isomorphism Theorems

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
August 3, 2013

Keywords: Structure, Universal Algebra, Topology, Graph Theory

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Abstract

A general format in which the mathematical structure of topological spaces, algebraic structures, and graphs can be expressed is described. A generalization of the fundamental homomorphism theorem and the isomorphism theorems of algebra is proved.
Acknowledgments

Firstly, special thanks to my wife Yesenia for all her love and support, for bearing with me during the evenings we spent in our studio apartment while I worked out the proofs on my laptop.

Special thanks to my mentor Dr. Gordon Johnson for instilling in me a love for mathematics, and patiently picking apart my arguments while I was learning to think carefully.

Thanks to Dr. Michel Smith for allowing me to run with the idea I had, and his patience in examining my arguments.

Thanks to Dr. Randall Holmes for all his feedback, through which I’ve been able to express my ideas much more clearly, and for his interest early on and willingness to hear my scheme.

Finally thanks to my colleagues who showed an interest in my work, and were willing to wade through my definitions back when I wasn’t sure I had anything: John Asplund, Daniel Brice, Glenn Hughes, Zachary Sarver, Alexander Byaly.
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Chapter 1
Definitions

**Definition** The statement that \( r \) is a relation means \( r \) is a set of ordered pairs. If \( S \) is a set then \( r(S) \) denotes the set to which an element \( y \) belongs if and only if there is an element \( x \in S \) such that \((x, y) \in r\).

**Theorem 1.1.** Suppose each of \( r \) and \( s \) is a relation, \( r \subseteq s \), each of \( U \) and \( V \) is a set, and \( U \subseteq V \). Then \( r(U) \subseteq s(V) \).

**Proof:**

\[
y \in r(U) \\
\implies \exists x \in U \text{ such that } (x, y) \in r \\
\implies x \in V \text{ and } (x, y) \in s \\
\implies y \in s(V)
\]

So \( r(U) \subseteq s(V) \). \( \square \)

**Definition** Suppose each of \( r \) and \( s \) is a relation. The *composition of \( r \) and \( s \)* is the relation to which a pair \((x, z)\) belongs if and only if there is an element \( y \) such that \((x, y) \in s \) and \((y, z) \in r \). Denote the composition of \( r \) and \( s \) by \( rs \).

**Theorem 1.2.** Suppose each of \( r, s, \) and \( t \) is a relation. Then \((rs)t = r(st)\).
Proof:

\[(a, d) \in (rs)t\]
\[\iff \exists c \text{ such that } (a, c) \in rs \text{ and } (c, d) \in t\]
\[\iff \exists b \text{ and } a c \text{ such that } (a, b) \in r, (b, c) \in s, \text{ and } (c, d) \in t\]
\[\iff \exists b \text{ such that } (a, b) \in r \text{ and } (b, d) \in st\]
\[\iff (a, d) \in r(st)\]

So \((rs)t = r(st)\). □

**Theorem 1.3.** Suppose each of \(f, g, r, \) and \(s\) is a relation, \(f \subseteq g\), and \(r \subseteq s\). Then \(rf \subseteq sg\).

**Proof:**

\[(x, z) \in rf\]
\[\implies \exists y \text{ such that } (x, y) \in f \text{ and } (y, z) \in r\]
\[\implies (x, y) \in g \text{ and } (y, z) \in s\]
\[\implies (x, z) \in sg\]

So \(rf \subseteq sg\). □

**Definition** Suppose \(r\) is a relation. The inverse of \(r\) is the relation to which a pair \((x, y)\) belongs if and only if \((y, x)\) is in \(r\). Denote the inverse of \(r\) by \(r^{-1}\).

**Theorem 1.4.** Suppose \(r\) is a relation. Then \((r^{-1})^{-1} = r\).
Proof:

\[(x, y) \in (r^{-1})^{-1} \iff (y, x) \in r^{-1} \iff (x, y) \in r\]

So \((r^{-1})^{-1} = r\). \qed

**Theorem 1.5.** Suppose each of \(r\) and \(s\) is a relation. Then \((rs)^{-1} = s^{-1}r^{-1}\).

**Proof:**

\[(x, z) \in (rs)^{-1} \iff (z, x) \in rs \iff \exists y \text{ such that } (y, x) \in r \text{ and } (z, y) \in s \iff \exists y \text{ such that } (x, y) \in r^{-1} \text{ and } (y, z) \in s^{-1} \iff (x, z) \in s^{-1}r^{-1}\]

So \((rs)^{-1} = s^{-1}r^{-1}\). \qed

**Definition** Suppose \(r\) is a relation. The statement that \(D\) is the domain of \(r\) means \(D\) is the set to which an element \(x\) belongs if and only if \(x\) is the first element of a pair in \(r\). Denote the domain of \(r\) by \(\text{dom}(r)\).

**Definition** Suppose \(r\) is a relation. The statement that \(R\) is the image of \(r\) means \(R\) is the set to which an element \(x\) belongs if and only if \(x\) is the second element of a pair in \(r\). Denote the image of \(r\) by \(\text{im}(r)\).

**Theorem 1.6.** Suppose \(r\) is a relation. Then \(\text{im}(r) = r(\text{dom}(r))\).
Proof:

\[ y \in \text{im}(r) \iff \exists(x, y) \in r \iff \exists x \in \text{dom}(r) \text{ such that } (x, y) \in r \iff y \in r(\text{dom}(r)) \]

So \( \text{im}(r) = r(\text{dom}(r)) \).

**Theorem 1.7.** Suppose each of \( r \) and \( s \) is a relation. Then \( \text{im}(rs) = r(\text{im}(s)) \).

Proof:

\[ z \in \text{im}(rs) \iff \exists(x, z) \in rs, \exists y \text{ such that } (y, z) \in r \text{ and } \exists(x, y) \in s \iff \exists y \in \text{im}(s) \text{ such that } (y, z) \in r \iff z \in r(\text{im}(s)) \]

So \( \text{im}(rs) = r(\text{im}(s)) \).

**Definition** The statement that \( f \) is a function means \( f \) is a relation such that no two pairs in \( f \) share the same first element. If \((x, y) \in f\), then denote \( y \) by \( f(x) \).

**Theorem 1.8.** Suppose each of \( f \) and \( g \) is a function. Then \( fg \) is a function.
Proof:

\[(x, z_1) \in fg \text{ and } (x, z_2) \in fg\]

\[\implies \exists y_1 \text{ such that } (x, y_1) \in g \text{ and } (y_1, z_1) \in f\]

and \(\exists y_2 \text{ such that } (x, y_2) \in g \text{ and } (y_2, z_2) \in f\)

\[\implies y_1 = y_2 \text{ and } (y_1, z_1) \in f \text{ and } (y_1, z_2) = (y_2, z_2) \in f \text{ (since } g \text{ is a function)}\]

\[\implies z_1 = z_2 \text{ (since } f \text{ is a function)}\]

So no two pairs of \(fg\) contain the first element. So \(fg\) is a function. \qed

Definition The statement that \(f\) is an injection means \(f\) is a function and \(f^{-1}\) is a function.

Theorem 1.9. Suppose each of \(f\) and \(g\) is an injection. Then \(fg\) is a injection.

Proof:

\(f\) is an injection and \(g\) is an injection

\[\implies f\) is a function, \(g\) is a function, \(f^{-1}\) is a function, and \(g^{-1}\) is a function\]

\[\implies fg\) is a function and \((fg)^{-1} = g^{-1}f^{-1}\) is a function\]

\[\implies fg\) is an injection \qed\]

Definition The statement that \(f\) is a surjection with respect to \(Y\) means \(f\) is a function with image \(Y\).

Theorem 1.10. Suppose \(S\) is a set, and \(f\) is a surjection with respect to \(S\), and \(g\) is a surjection with respect to \(\text{dom}(f)\). Then \(fg\) is a surjection with respect to \(S\).
Proof:

\[ z \in S \]
\[ \iff z \in \text{im}(f) \]
\[ \iff \exists y \in \text{dom}(f) = \text{im}(g) \text{ such that } (y, z) \in f \]
\[ \iff \exists x \in \text{dom}(g) \text{ and } \exists y \in \text{dom}(f) \text{ such that } (x, y) \in g \text{ and } (y, z) \in f \]
\[ \iff \exists x \in \text{dom}(g) \text{ such that } (x, z) \in fg \]
\[ \iff z \in \text{im}(fg) \]

So \( S = \text{im}(fg) \) and thus \( fg \) is a surjection with respect to \( S \).

Definition The statement that \( f \) is a bijection with respect to \( Y \) means \( f \) is an injection and a surjection with respect to \( Y \).

Definition The notation \( r : X \to Y \) means \( r \) is a relation and \( X \) is the domain of \( r \) and the image of \( r \) is a subset of \( Y \), and henceforth if the terms surjection or bijection are used to describe \( r \) they will be understood to be with respect to \( Y \).

Definition Suppose \( A \) is a set. Denote by \( 1_A \) the relation \( \{(a, a) \mid a \in A\} \).

Theorem 1.11. Suppose \( A \) is a set. Then \( 1_A = 1_A^{-1} \).

Proof:

\[ (a, a) \in 1_A \]
\[ \iff (a, a) \in 1_A^{-1} \]

So \( 1_A = 1_A^{-1} \).

Definition Suppose \( A \) is a set and \( r \) is a relation. Denote by \( r|_A \) the relation to which a pair \( (x, y) \) belongs if and only if \( (x, y) \in r \) and \( x \in A \).
Theorem 1.12. Suppose $A$ is a set and $r$ is a relation. Then $r = r|_A$ if and only if $\text{dom}(r) \subseteq A$.

Proof: Suppose $r = r|_A$.

\[
\begin{align*}
a & \in \text{dom}(r) \\
\implies & \exists b \text{ such that } (a, b) \in r \\
\implies & \exists b \text{ such that } (a, b) \in r|_A \\
\implies & a \in A
\end{align*}
\]

So $\text{dom}(r) \subseteq A$.

Suppose $\text{dom}(r) \subseteq A$.

\[
\begin{align*}
(a, b) & \in r \\
\iff & a \in \text{dom}(r) \subseteq A \text{ and } (a, b) \in r \\
\iff & (a, b) \in r|_A
\end{align*}
\]

So $r = r|_A$. \hfill \square

Theorem 1.13. Suppose $A$ is a set, and $r$ is a relation. Then $r1_A = r|_A$.

Proof:

\[
\begin{align*}
(x, z) & \in r1_A \\
\iff & \exists y \text{ such that } (x, y) \in 1_A \text{ and } (y, z) \in r \\
\iff & \exists y \text{ such that } x \in A, x = y, \text{ and } (y, z) \in r \\
\iff & x \in A \text{ and } (x, z) \in r \\
\iff & (x, z) \in r|_A
\end{align*}
\]
So \( r1_A = r|_A \).

**Definition** Suppose \( A \) is a set and \( r \) is a relation. Denote by \( r|_A \) the relation to which a pair \((x, y)\) belongs if and only if \((x, y) \in r \) and \( y \in A \).

**Theorem 1.14.** Suppose \( A \) is a set and \( r \) is a relation. Then \( r = r|_A \) if and only if \( \text{im}(r) \subseteq A \).

**Proof:** Suppose \( r = r|_A \).

\[
\begin{align*}
a & \in \text{im}(r) \\
\implies & \exists b \text{ such that } (b, a) \in r \\
\implies & \exists b \text{ such that } (b, a) \in r|_A \\
\implies & a \in A
\end{align*}
\]

So \( \text{im}(r) \subseteq A \).

Suppose \( \text{im}(r) \subseteq A \).

\[
\begin{align*}
(b, a) & \in r \\
\iff & a \in \text{im}(r) \subseteq A \text{ and } (b, a) \in r \\
\iff & (b, a) \in r|_A
\end{align*}
\]

So \( r = r|_A \). \qed

**Theorem 1.15.** Suppose \( A \) is a set, and \( r \) is a relation. Then \( 1_A r = r|_A \).
Proof:

\[(x, z) \in 1_A r\]
\[\iff \exists y \text{ such that } (x, y) \in r \text{ and } (y, z) \in 1_A \]
\[\iff \exists y \text{ such that } (x, y) \in r, \ y \in A, \text{ and } y = z \]
\[\iff (x, z) \in r \text{ and } z \in A \]
\[\iff (x, z) \in r^{|A|} \]

So \(1_A r = r^{|A|}\).

\[\square\]

**Theorem 1.16.** Suppose each of \(A\) and \(B\) is a set, and \(f : A \to B\) is a function. Then \(1_A \subseteq f^{-1}f\), and \(1_A = f^{-1}f\) if and only if \(f\) is an injection.

**Proof:**

\[(a, a) \in 1_A\]
\[\implies a \in A\]
\[\implies (a, f(a)) \in f \text{ and } (f(a), a) \in f^{-1}\]
\[\implies (a, a) \in f^{-1}f\]

So \(1_A \subseteq f^{-1}f\).

Suppose \(1_A = f^{-1}f\).

\[(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}\]
\[\implies (a_1, b) \in f \text{ and } (b, a_2) \in f^{-1}\]
\[\implies (a_1, a_2) \in f^{-1}f = 1_A\]
\[\implies a_1 = a_2\]
So $f^{-1}$ is a function and thus $f$ is an injection.

Suppose $f$ is an injection.

$$(a_1, a_2) \in f^{-1}f$$
$$\implies \exists b \text{ such that } (a_1, b) \in f \text{ and } (b, a_2) \in f^{-1}$$
$$\implies (b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}$$
$$\implies a_1 = a_2 \text{ (since } f^{-1} \text{ is a function)}$$
$$\implies (a_1, a_2) \in 1_A$$

So $f^{-1}f \subseteq 1_A$, and thus $1_A = f^{-1}f$. 

\[ \square \]

**Theorem 1.17.** Suppose each of $A$ and $B$ is a set, and $f : A \to B$ is a function. Then $f f^{-1} \subseteq 1_B$, and $f f^{-1} = 1_B$ if and only if $f$ is a surjection.

**Proof:** Suppose each of $b_1$ and $b_2$ is in $B$.

$$(b_1, b_2) \in f f^{-1}$$
$$\implies \exists a \in A \text{ such that } (b_1, a) \in f^{-1} \text{ and } (a, b_2) \in f$$
$$\implies (a, b_1) \in f \text{ and } (a, b_2) \in f$$
$$\implies b_1 = b_2 \text{ (since } f \text{ is a function)}$$
$$\implies (b_1, b_2) \in 1_B$$

So $f f^{-1} \subseteq 1_B$.

Suppose $f f^{-1} = 1_B$. 

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Suppose $b \in B$.

$$(b, b) \in 1_B = ff^{-1}$$

$$\implies \exists a \in A \text{ such that } (b, a) \in f^{-1} \text{ and } (a, b) \in f$$

$$\implies a \in A \text{ such that } b = f(a) \subseteq \text{im}(f)$$

So $f$ is a surjection.

Suppose $f$ is a surjection.

Suppose $(b, b) \in 1_B$ (so $b \in B$). There is an $a \in A$ such that $f(a) = b$.

$$(a, b) \in f \text{ and } (b, a) \in f^{-1}$$

$$\implies (b, b) \in ff^{-1}$$

So $1_B \subseteq ff^{-1}$, and thus $ff^{-1} = 1_B$. \hfill \Box

**Theorem 1.18.** Suppose $A$ is a set. Then $1_A$ is a bijection with respect to $A$.

**Proof:**

1. $1_A$ is an injection:

   $$1_A = 1_A 1_A = 1_A^{-1} 1_A$$

   So $1_A$ is an injection.

2. $1_A$ is a surjection with respect to $A$:

   $$1_A = 1_A 1_A = 1_A 1_A^{-1}$$
So \( A = \text{im}(1_A) \) and thus \( 1_A \) is a surjection with respect to \( A \).

So \( 1_A \) is a bijection with respect to \( A \).

**Definition** Suppose each of \( A \) and \( B \) is a set. Denote by \( A \times B \) the relation to which an ordered pair \((a, b)\) belongs if and only if \( a \in A \) and \( b \in B \).

**Definition** Let \( I \) be a set. The statement that \((A, \mathcal{R})\) is an \( I \)-structure means \( A \) is a set, and \( \mathcal{R} \) is a set of relations each of which is a subset of \( I \times A \). \( A \) is called the base set of \((A, \mathcal{R})\) and \( I \) is called the index set of \((A, \mathcal{R})\).

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that a function \( \alpha : A \to B \) is **preservative** means for each \( r \in \mathcal{R} \), \( \alpha r \in \mathcal{S} \).

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that a function \( \alpha : A \to B \) is **saturating** means for each \( s \in \mathcal{S} \) such that \( \text{im}(s) \subseteq \text{im}(\alpha) \), there is an \( r \in \mathcal{R} \) such that \( \alpha r = s \).

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that a function \( \alpha : A \to B \) is **continuous** means for each \( s \in \mathcal{S} \), \( \alpha^{-1} s \in \mathcal{R} \).

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that a function \( \alpha : A \to B \) is **conservative** means for each \( r \in \mathcal{R} \), there is an \( s \in \mathcal{S} \) such that \( \text{im}(s) \subseteq \text{im}(\alpha) \) and \( \alpha^{-1} s = r \).

**Definition** Suppose \( \varphi : A \to B \) is a function and each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that \( \varphi \) is an \( I \)-structure homomorphism from \((A, \mathcal{R})\) to \((B, \mathcal{S})\) means \( \varphi \) is preservative and saturating.

**Definition** Suppose \( \varphi : A \to B \) is a function and each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that \( \varphi \) is an \( I \)-structure cohomomorphism from \((A, \mathcal{R})\) to \((B, \mathcal{S})\) means \( \varphi \) is continuous and conservative.
**Definition** Suppose \( \varphi : A \to B \) is a function and each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \(I\)-structure. The statement that \( \varphi \) is an \(I\)-structure isomorphism from \((A, \mathcal{R})\) to \((B, \mathcal{S})\) means \( \varphi \) is a continuous, preservative bijection.

**Definition** The statement that an \(I\)-structure \((A, \mathcal{R})\) and an \(I\)-structure \((B, \mathcal{S})\) are isomorphic means there is an isomorphism \( \varphi : A \to B \) from \((A, \mathcal{R})\) to \((B, \mathcal{S})\). In this case \((A, \mathcal{R}) \cong (B, \mathcal{S})\) denotes \("(A, \mathcal{R}) \text{ and } (B, \mathcal{S}) \text{ are isomorphic}"\).

**Lemma 1.19.1.** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \(I\)-structure and \( \varphi : A \to B \) is a continuous function. Then \( \varphi \) is saturating.

**Proof:** Suppose \( s \in \mathcal{S} \) such that \( \text{im}(s) \subseteq \text{im}(\varphi) \).

\( \varphi \) is continuous, so \( \varphi^{-1}s \in \mathcal{R} \).

So \( \varphi^{-1}s \in \mathcal{R} \) and \( s = 1_{\text{im}(\varphi)}s = \varphi\varphi^{-1}s \). So \( \varphi \) is saturating. \( \square \)

**Theorem 1.19.** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \(I\)-structure and \( \alpha : A \to B \) is a function. Then \( \alpha \) is a bijective \(I\)-structure homomorphism if and only if \( \alpha \) is an \(I\)-structure isomorphism.

**Proof:** Suppose \( \alpha \) is a bijective \(I\)-structure homomorphism.

\( \alpha \) is bijective and preservative, so it remains only to show that \( \alpha \) is continuous.

Suppose \( s \in \mathcal{S} \). \( \alpha \) is surjective, so \( \text{im}(s) \subseteq B = \text{im}(\alpha) \). \( \alpha \) is saturating, so there is an \( r \in \mathcal{R} \) such that \( \alpha r = s \).

\[ \alpha^{-1}s = \alpha^{-1}\alpha r = 1_A r = r \in \mathcal{R} \]

So \( \alpha \) is continuous and is thus an isomorphism.
Suppose \( \alpha \) is an \( I \)-structure isomorphism.

\( \alpha \) is bijective and preservative. \( \alpha \) is continuous, so by Lemma 1.19.1, \( \alpha \) is saturating.

So \( \alpha \) is a bijective homomorphism. \( \square \)

**Lemma 1.20.1.** Suppose each of \((A, R)\) and \((B, S)\) is an \( I \)-structure and \( \varphi : A \to B \) is a conservative function. Then \( \varphi \) is preservative.

**Proof:** Suppose \( r \in R \).

\( \varphi \) is conservative, so there is an \( s \in S \) such that \( \text{im}(s) \subseteq \text{im}(\varphi) \) and \( \varphi^{-1}s = r \).

So \( \varphi r = \varphi \varphi^{-1}s = 1_{\text{im}(\varphi)}s = s \in S \).

So \( \varphi \) is preservative. \( \square \)

**Theorem 1.20.** Suppose each of \((A, R)\) and \((B, S)\) is an \( I \)-structure and \( \alpha : A \to B \) is a cohomomorphism. Then \( \alpha \) is a homomorphism.

**Proof:** \( \alpha \) is conservative, so by Lemma 1.20.1, \( \alpha \) is preservative.

\( \alpha \) is continuous, so by Lemma 1.19.1, \( \alpha \) is saturating.

\( \alpha \) is both preservative and saturating, so \( \alpha \) is a homomorphism. \( \square \)

**Theorem 1.21.** Suppose each of \((A, R)\) and \((B, S)\) is an \( I \)-structure and \( \alpha : A \to B \) is a function. Then \( \alpha \) is a bijective cohomomorphism if and only if \( \alpha \) is an isomorphism.

**Proof:** Suppose \( \alpha \) is a bijective cohomomorphism.
α is bijective and continuous. α is conservative, so by Lemma 1.20.1, α is preservative.

So α is an isomorphism.

Suppose α is an isomorphism.

α is bijective and continuous, so it remains only to show that α is conservative.

Suppose \( r \in \mathcal{R} \). α is preservative, so \( \alpha r \in \mathcal{S} \). \( \text{im}(\alpha r) \subseteq \text{im}(\alpha) \).

\[
\alpha^{-1}\alpha r = 1_A r = r
\]

So α is conservative and is thus a bijective cohomomorphism. \( \square \)

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \(I\)-structure. The statement that a function \( \varphi : A \to B \) is a *structure monomorphism* means \( \varphi \) is an injective \(I\)-structure homomorphism.

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \(I\)-structure. The statement that a function \( \varphi : A \to B \) is a *structure epimorphism* means \( \varphi \) is a surjective \(I\)-structure homomorphism.

**Theorem 1.22.** Suppose each of \( M = (A, \mathcal{R}) \), \( N = (B, \mathcal{S}) \), \( L = (C, \mathcal{T}) \) is an \(I\)-structure, \( \alpha : A \to B \) is an epimorphism from \( M \) to \( N \), and \( \beta : B \to C \) is a homomorphism from \( N \) to \( L \). Then \( \beta \alpha \) is a homomorphism from \( M \) to \( L \).

**Proof:** Suppose \( r \in \mathcal{R} \). α is preservative, so \( \alpha r \in \mathcal{S} \). β is preservative, so \( \beta \alpha r \in \mathcal{T} \). So \( \beta \alpha \) is preservative.

Suppose \( t \in \mathcal{T} \) such that \( \text{im}(t) \subseteq \text{im}(\beta \alpha) \). β is saturating, and \( \text{im}(t) \subseteq \text{im}(\beta \alpha) \subseteq \text{im}(\beta) \), so
there is an \( s \in S \) such that \( \beta s = t \).

\[ \text{im}(s) \subseteq B = \text{im}(\alpha), \] so there is an \( r \in R \) such that \( \alpha r = s \).

So \( r \in R \) such that \( \beta \alpha r = \beta s = t \). So \( \beta \alpha \) is saturating.

\( \beta \alpha \) is both preservative and saturating, and is thus a homomorphism. \( \square \)

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that a function \( \varphi : A \to B \) is a *structure comonomorphism* means \( \varphi \) is an injective \( I \)-structure cohomomorphism.

**Definition** Suppose each of \((A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \( I \)-structure. The statement that a function \( \varphi : A \to B \) is a *structure coepimorphism* means \( \varphi \) is a surjective \( I \)-structure cohomomorphism.

**Theorem 1.23.** Suppose each of \( M = (A, \mathcal{R}) \), \( N = (B, \mathcal{S}) \), \( L = (C, \mathcal{T}) \) is an \( I \)-structure, \( \alpha : A \to B \) is an coepimorphism from \( M \) to \( N \), and \( \beta : B \to C \) is a cohomomorphism from \( N \) to \( L \). Then \( \beta \alpha \) is a cohomomorphism from \( M \) to \( L \).

**Proof:** Suppose \( t \in \mathcal{T} \). \( \beta \) is continuous, so \( \beta^{-1} t \in \mathcal{S} \). \( \alpha \) is continuous, so \( (\beta \alpha)^{-1} t = \alpha^{-1} \beta^{-1} t \in \mathcal{R} \). So \( \beta \alpha \) is continuous.

Suppose \( r \in \mathcal{R} \). \( \alpha \) is conservative, so there is an \( s \in \mathcal{S} \) such that \( \text{im}(s) \subseteq \text{im}(\alpha) \) and \( \alpha^{-1} s = r \).

\[ s \in \mathcal{S}, \text{ so there is an } t \in \mathcal{T} \text{ such that } \text{im}(t) \subseteq \text{im}(\beta) \text{ and } \beta^{-1} t = s. \]

So \( t \in \mathcal{T} \) such that \( (\beta \alpha)^{-1} t = \alpha^{-1} \beta^{-1} t = \alpha^{-1} s = r \). So \( \beta \alpha \) is conservative.
$\beta \alpha$ is both continuous and conservative, and is thus a cohomomorphism.
Chapter 2
Equivalence Relations

**Definition** Suppose \( r \) is a relation. The statement that \( r \) is *symmetric* means \( r^{-1} \subseteq r \).

**Lemma 2.1.1.** Let \( r \) be a symmetric relation. Then \( r^{-1} = r \).

**Proof:**

\[
(x, y) \in r \\
\implies (y, x) \in r^{-1} \subseteq r \\
\implies (x, y) \in r^{-1}
\]

So \( r \subseteq r^{-1} \) and since \( r^{-1} \subseteq r \), \( r^{-1} = r \). \( \square \)

**Definition** Suppose \( r \) is a relation. The statement that \( r \) is *transitive* means \( rr \subseteq r \).

**Definition** Suppose \( r \) is a relation. The statement that \( r \) is an *equivalence relation* means \( r \) is symmetric and transitive.

**Definition** Suppose \( r \) is a relation. The statement that \( r \) is reflexive with respect to \( A \) means \( 1_A \subseteq r \).

**Lemma 2.1.2.** Suppose \( r \) is an equivalence relation. Then \( r \) is reflexive with respect to \( \text{dom}(r) \).
Proof:

\[(a, a) \in 1_{\text{dom}(r)} \]
\[\iff a \in \text{dom}(r) \]
\[\iff \exists b \text{ such that } (a, b) \in r \]
\[\iff \exists b \text{ such that } (a, b) \in r \text{ and } (b, a) \in r^{-1} = r \]
\[\iff (a, a) \in rr \]

So \(1_{\text{dom}(r)} \subseteq rr \subseteq r\). □

Remark Suppose \(A\) is a set. Then \(1_A\) is an equivalence relation.

Lemma 2.1.3. Suppose \(r\) is a reflexive relation. Then for each \(a \in \text{dom}(r)\), \(a \in r(\{a\})\).

Proof: Suppose \(a \in \text{dom}(r)\).

\(1_{\text{dom}(r)} \subseteq r\), so \(a \in \{a\} = 1_{\text{dom}(r)}(\{a\}) \subseteq r(\{a\})\). □

Lemma 2.1.4. Suppose \(r\) is an equivalence relation. Then \(rr = r\).

Proof:

\[r = 1_{\text{im}(r)}r = 1_{\text{dom}(r^{-1})}r = 1_{\text{dom}(r)}r \subseteq rr \]

So \(r \subseteq rr\). Then since \(rr \subseteq r\), \(rr = r\). □

Definition Suppose \(A\) is a set. The statement that \(P\) is a partition of \(A\) means if \(P \in \mathcal{P}\) then \(P \subseteq A\), and if \(a \in A\) then \(a\) belongs to exactly one element of \(P\).

Theorem 2.1. Suppose \(r\) is an equivalence relation. Then \(r\) induces a partition \(\mathcal{P}\) of \(\text{dom}(r)\) by \(\mathcal{P} = \{r(\{a\}) \mid a \in \text{dom}(r)\}\), each member of which is nonempty.
Proof:

\[ P \in \mathcal{P} \]
\[ \implies P = r(\{a\}) \text{ for some } a \in \text{dom}(r) \]
\[ \implies r(\{a\}) = r^{-1}(\{a\}) \subseteq \text{dom}(r) \]

So each member of \( \mathcal{P} \) is a subset of \( \text{dom}(r) \).

Suppose \( a \in \text{dom}(r) \).

\[ a \in \{a\} = 1_{\text{dom}(r)}(\{a\}) \subseteq r(\{a\}) \]

So \( a \) belongs to one member of \( \mathcal{P} \).

Suppose \( b \in \text{dom}(r) \) and \( a \in r(\{b\}) \). Then \( (b, a) \in r \) and \( r = r^{-1} \) so \( (a, b) \in r \).

\[ p \in r(\{a\}) \]
\[ (a, p) \in r \]
\[ \implies (b, p) \in rr \]
\[ (b, p) \in r \]
\[ p \in r(\{b\}) \]

So \( r(\{a\}) = r(\{b\}) \).

So \( a \) belongs to no more than one member of \( \mathcal{P} \).

So \( \mathcal{P} \) is a partition.
If $P \in \mathcal{P}$, then $P = r\{a\}$ for some $a \in \text{dom}(r)$ so by Lemma 2.1.3 $a \in r\{a\} = P$, so $P$ is nonempty.

So each member of $\mathcal{P}$ is nonempty.

**Theorem 2.2.** Suppose $A$ is a set and $f$ is a function with domain $A$. Then $f^{-1}f$ is an equivalence relation on $A$.

**Proof:**

1. $f^{-1}f$ is symmetric: $(f^{-1}f)^{-1} = f^{-1}(f^{-1})^{-1} = f^{-1}f$

2. $f^{-1}f$ is transitive: $f^{-1}ff^{-1}f = f^{-1}1_{\text{im}(f)}f = f^{-1}f$

So $f^{-1}f$ is an equivalence relation.

**Definition** Suppose $A$ is a set and $f$ is a function with domain $A$. Then denote by $A/f$ the partition of $A \{f^{-1}(\{a\}) \mid a \in A\}$.

**Theorem 2.3.** Suppose $A$ is a set and $r$ is an equivalence relation with domain $A$. Suppose $\mathcal{P}$ is the partition of $A$ induced by $r$, and $\pi : A \rightarrow \mathcal{P}$ is the function which assigns each member of $A$ to its part in $\mathcal{P}$. Then $r = \pi^{-1}\pi$.

**Proof:**

\[(a_1, a_2) \in r \iff \exists P \in \mathcal{P} \text{ such that } a_1 \in P \text{ and } a_2 \in P \]
\[\iff \exists P \in \mathcal{P} \text{ such that } (a_1, P) \in \pi \text{ and } (a_2, P) \in \pi \]
\[\iff \exists P \in \mathcal{P} \text{ such that } (a_1, P) \in \pi \text{ and } (P, a_2) \in \pi^{-1} \]
\[\iff (a_1, a_2) \in \pi^{-1}\pi \]

So $r = \pi^{-1}\pi$.  

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**Definition** Suppose $f$ is a function with domain $A$. Denote by $\pi_f : A \to A/f$ the function such that for each $a \in A$, $\pi_f(a)$ is the part in $A/f$ to which $a$ belongs.

**Theorem 2.4.** Suppose $A$ is a set, and $f$ is a function with domain $A$. Then $\pi_f$ is a surjection with respect to $A/f$.

**Proof:** Suppose $P \in A/f$. $P$ is nonempty, so there is an $a \in P$, and by the definition of $\pi_f$, $\pi_f(a) = P$. So $\pi_f$ is a surjection with respect to $A/f$. □

**Theorem 2.5.** Suppose $A$ is a set, and $f$ is a function with domain $A$. Then for each $a \in A$, $\pi_f(a) = f^{-1}f(\{a\})$.

**Proof:** For each $a \in A$, $\pi_f(a)$ is the part in $A/f$ to which $a$ belongs.

By Lemma 2.1.3 $a$ belongs to $f^{-1}f(\{a\}) \in A/f$, so $\pi_f(a) = f^{-1}f(\{a\})$. □

**Theorem 2.6.** Suppose $A$ is a set, and $f$ is a function with domain $A$. Then $\pi_f^{-1} \pi_f = f^{-1}f$.

**Proof:**

\[
(a_1, a_2) \in \pi_f^{-1} \pi_f \\
\implies \exists P \in A/f \text{ such that } (a_1, P) \in \pi_f \text{ and } (P, a_2) \in \pi_f^{-1} \\
\implies (a_1, P) \in \pi_f \text{ and } (a_2, P) \in \pi_f \\
\implies f^{-1}(\{f(a_1)\}) = f^{-1}f(\{a_1\}) = \pi_f(a_1) = P = \pi_f(a_2) = f^{-1}f(\{a_2\}) = f^{-1}(\{f(a_2)\}) \\
\implies f(a_1) = 1_{im(f)}f(a_1) = ff^{-1}f(a_1) = ff^{-1}f(a_2) = 1_{im(f)}f(a_2) = f(a_2) \\
\implies (a_1, f(a_1)) \in f \text{ and } (f(a_1), a_2) \in f^{-1} \\
\implies (a_1, a_2) \in f^{-1}f
\]
So $\pi_f^{-1}\pi_f \subseteq f^{-1}f$.

\[(a_1, a_2) \in f^{-1}f \Rightarrow \pi_f(a_1) = f^{-1}(f(\{a_1\})) = f^{-1}(f(\{a_2\})) = \pi_f(a_2) \Rightarrow (a_1, \pi_f(a_1)) \in \pi_f \text{ and } (a_2, \pi_f(a_1)) \in \pi_f \Rightarrow (a_1, \pi_f(a_1)) \in \pi_f \text{ and } (\pi_f(a_1), a_2) \in \pi_f^{-1} \Rightarrow (a_1, a_2) \in \pi_f^{-1}\pi_f \]

So $f^{-1}f \subseteq \pi_f^{-1}\pi_f$.

So $\pi_f^{-1}\pi_f = f^{-1}f$. \qed

**Theorem 2.7.** Suppose $A$ is a set, and $f$ is a function with domain $A$. Then $A/\pi_f = A/f$.

**Proof:**

\[A/f = \{f^{-1}(\{a\}) \mid a \in A\} = \{\pi_f^{-1}\pi_f(\{a\}) \mid a \in A\} = A/\pi_f \quad \square\]

**Theorem 2.8.** Suppose $A$ is a set, and $f$ is a function with domain $A$. Then $\pi_{\pi_f} = \pi_f$.

**Proof:** Suppose $a \in A$.

\[\pi_f(a) = f^{-1}f(\{a\}) = \pi_f^{-1}\pi_f(\{a\}) = \pi_{\pi_f}(a)\]

Since this is true for each $a \in A$, $\pi_{\pi_f} = \pi_f$. \qed

**Theorem 2.9.** Suppose $A$ is a set, and $f$ is a function with domain $A$. Then if $P \in A/f$ then $P = \pi_f^{-1}(\{P\})$ and $\pi_f(P) = \{P\}$.
Proof: Suppose $P \in A/f$ and $a \in P$.

\[ P = \pi_f(a) = f^{-1}f({a}) = \pi_f^{-1}f({a}) = \pi_f^{-1}(\pi_f({a})) = \pi_f^{-1}(\pi_f(a)) = \pi_f^{-1}(\{P\}) \]

So $P = \pi_f^{-1}(\{P\})$.

\[ P = \pi_f^{-1}(\{P\}) \Rightarrow \pi_f(P) = \pi_f(\pi_f^{-1}(\{P\})) = \pi_f\pi_f^{-1}(\{P\}) = 1_{A/f}(\{P\}) = \{P\} \]

So $\pi_f(P) = \{P\}$. \qed

Theorem 2.10. Suppose $A$ is a set, $f$ is a function with domain $A$, and each of $a_1$ and $a_2$ is in $A$. Then the following are equivalent:

1. There is a $P \in A/f$ such that $a_1$ and $a_2$ belong to $P$.

2. $\pi_f(a_1) = \pi_f(a_2)$

3. $a_2 \in \pi_f^{-1}(\pi_f(\{a_1\}))$

4. $(a_1, a_2) \in \pi_f^{-1}\pi_f$

5. $(a_1, a_2) \in f^{-1}f$

6. $a_2 \in f^{-1}(f(\{a_1\}))$

7. $f(a_1) = f(a_2)$

Proof: 1 $\Rightarrow$ 2:

Suppose there is a $P \in A/f$ such that $a_1$ and $a_2$ belong to $P$.

$a_1 \in P$ so $\pi_f(a_1) = P$ and $a_2 \in P$ so $\pi_f(a_2) = P$. 

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So $\pi_f(a_1) = P = \pi_f(a_2)$. \qed

2 $\implies$ 3:

Suppose $\pi_f(a_1) = \pi_f(a_2)$.

$$1_A \subseteq \pi_f^{-1} \pi_f$$

$$\implies \{a_2\} = 1_A(\{a_2\}) \subseteq \pi_f^{-1} \pi_f(\{a_2\}) = \pi_f^{-1}(\{\pi_f(a_2)\}) = \pi_f^{-1}(\{\pi_f(a_1)\}) = \pi_f^{-1}(\pi_f(\{a_1\}))$$

$$\implies a_2 \in \pi_f^{-1}(\pi_f(\{a_1\})) \quad \Box$$

3 $\implies$ 4:

Suppose $a_2 \in \pi_f^{-1}(\pi_f(\{a_1\})) = \pi_f^{-1} \pi_f(\{a_1\})$.

Then there is a pair $(x, a_2) \in \pi_f^{-1} \pi_f$ such that $x \in \{a_1\}$. So $x = a_1$ and $(a_1, a_2) \in \pi_f^{-1} \pi_f$. \qed

4 $\implies$ 5:

Suppose $(a_1, a_2) \in \pi_f^{-1} \pi_f$.

$$\pi_f^{-1} \pi_f = f^{-1} f$$

so $(a_1, a_2) \in \pi_f^{-1} \pi_f = f^{-1} f$. \qed

5 $\implies$ 6:

Suppose $(a_1, a_2) \in f^{-1} f$. 

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Then $a_1 \in \{a_1\}$ such that $(a_1, a_2) \in f^{-1}f$, so $a_2 \in f^{-1}f(\{a_1\}) = f^{-1}(f(\{a_1\}))$. 

6 $\Rightarrow$ 7:

Suppose $a_2 \in f^{-1}(f(\{a_1\}))$.

\[ a_2 \in f^{-1}(f(\{a_1\})) \]

\[ \Rightarrow \{a_2\} \subseteq f^{-1}(\{a_1\}) \]

\[ \Rightarrow \{f(a_2)\} = f(\{a_2\}) \subseteq f(f^{-1}(\{a_1\})) = f^{-1}(f(\{a_1\})) = 1_{\text{im}(f)} f(\{a_1\}) = f(\{a_1\}) = \{f(a_1)\} \]

\[ \Rightarrow f(a_1) = f(a_2) \]  \(\square\)

7 $\Rightarrow$ 1:

Suppose $f(a_1) = f(a_2)$.

Consider $f^{-1}f(\{a_1\}) \in A/f$.

By Lemma 2.1.3

\[ a_1 \in f^{-1}f(\{a_1\}) \]

and $a_2 \in f^{-1}f(\{a_2\}) = f^{-1}(\{f(a_2)\}) = f^{-1}(\{f(a_1)\}) = f^{-1}(f(\{a_1\})) = f^{-1}(\{a_1\})$

So $f^{-1}f(\{a_1\}) \in A/f$ such that $a_1$ and $a_2$ belong to $f^{-1}f(\{a_1\})$. \(\square\)
Remark In this chapter 1 is used set theoretically, i.e., $1 = \{0\}$.

Definition Suppose $(X, \tau)$ is a topological space. Then the structurization of $(X, \tau)$ is the 1-structure $(X, \hat{\tau})$, where $\hat{\tau} = \{1 \times S \mid S \in \tau\}$.

Example Consider the set $\mathbb{R}$ with the standard topology $\tau_{\mathbb{R}}$. Then the structurization of $(\mathbb{R}, \tau_{\mathbb{R}})$ is $(\mathbb{R}, R)$, where $R = \{1 \times S \mid S \in \tau_{\mathbb{R}}\}$. E.g., $1 \times (-3, \infty) \in R$.

Theorem 3.1. Suppose each of $(X, \tau_X)$ and $(Y, \tau_Y)$ is a topological space, and $(X, \hat{\tau}_X)$ is the structurization of $(X, \tau_X)$, and $(Y, \hat{\tau}_Y)$ is the structurization of $(Y, \tau_Y)$. Then $\alpha : X \to Y$ is preservative if and only if $\alpha$ is an open function with respect to $(X, \tau_X)$ and $(Y, \tau_Y)$.

Proof: Suppose $\alpha : X \to Y$ is preservative.

Suppose $S \in \tau_X$. $1 \times S \in \hat{\tau}_X$. Define $r = 1 \times S$. $\alpha r \in \hat{\tau}_Y$ so $\alpha r = 1 \times T$ for some $T$ in $\tau_Y$. $\alpha[S] = \alpha[r[1]] = \alpha r[1] = T \in \tau_Y$.

So $\alpha$ is an open function with respect to $(X, \tau_X)$ and $(Y, \tau_Y)$.

Suppose $\alpha : X \to Y$ is an open function.

Suppose $r \in \hat{\tau}_X$. Then $r = 1 \times S$ for some $S$ in $\tau_X$. Since $\alpha$ is open, $\alpha[S] \in \tau_Y$, so $\alpha r = \alpha(1 \times S) = 1 \times \alpha[S] \in \hat{\tau}_Y$.

So $\alpha$ is preservative with respect to the 1-structures $(X, \hat{\tau}_X)$ and $(Y, \hat{\tau}_Y)$. \qed
Theorem 3.2. Suppose each of \((X,\tau_X)\) and \((Y,\tau_Y)\) is a topological space, and \((X,\hat{\tau}_X)\) is the structurization of \((X,\tau_X)\), and \((Y,\hat{\tau}_Y)\) is the structurization of \((Y,\tau_Y)\). Then \(\alpha : X \to Y\) is continuous if and only if \(\alpha\) is a continuous function with respect to \((X,\tau_X)\) and \((Y,\tau_Y)\).

Proof: Suppose \(\alpha : X \to Y\) is (structurally) continuous.

Suppose \(T \in \tau_Y\). Then \(1 \times T \in \hat{\tau}_Y\). Define \(s = 1 \times T\). \(\alpha^{-1}s \in \hat{\tau}_X\) so \(\alpha^{-1}(T) = \alpha^{-1}[s[1]] = \alpha^{-1}s[1] \in \tau_X\).

So \(\alpha\) is a (topologically) continuous function with respect to \((X,\tau_X)\) and \((Y,\tau_Y)\).

Suppose \(\alpha : X \to Y\) is a (topologically) continuous function.

Suppose \(s \in \hat{\tau}_Y\). Then \(s = 1 \times T\) for some \(T \in \tau_Y\). Since \(\alpha\) is continuous, \(\alpha^{-1}[T] \in \tau_X\), so \(\alpha^{-1}s = \alpha^{-1}(1 \times T) = 1 \times \alpha^{-1}[T] \in \hat{\tau}_X\).

So \(\alpha\) is (structurally) continuous with respect to the 1-structures \((X,\hat{\tau}_X)\) and \((Y,\hat{\tau}_Y)\). \(\square\)

Theorem 3.3. Suppose each of \((X,\tau_X)\) and \((Y,\tau_Y)\) is a topological space, and the 1-structure \((X,\hat{\tau}_X)\) is the structurization of \((X,\tau_X)\), and the 1-structure \((Y,\hat{\tau}_Y)\) is the structurization of \((Y,\tau_Y)\). Then \((X,\tau_X)\) and \((Y,\tau_Y)\) are homeomorphic if and only if \((X,\hat{\tau}_X)\) and \((Y,\hat{\tau}_Y)\) are isomorphic.

Proof: Let \(\varphi : A \to B\) be a function.

\[\varphi\text{ is a homeomorphism}\]
\[\iff \varphi\text{ is bijective, open, and (topologically) continuous}\]
\[\iff \varphi\text{ is bijective, preservative, and (structurally) continuous (Thm 3.1, Thm 3.2)}\]
\[\iff \varphi\text{ is a 1-structure isomorphism}\] \(\square\)
Lemma 3.4.1. Suppose $J$ is a nonempty set, and for each $j \in J$, $r_j$ is a relation, and $\varphi$ is a function. Then $\varphi^{-1}(\bigcap_{j \in J} r_j) = \bigcap_{j \in J}(\varphi^{-1}r_j)$.

Proof:

$$(x, y) \in \varphi^{-1}(\bigcap_{j \in J} r_j)$$

$\iff \exists z \text{ such that } (z, y) \in \varphi^{-1} \text{ and } (x, z) \in \bigcap_{j \in J} r_j$

$\iff \exists z \text{ such that } (z, y) \in \varphi^{-1} \text{ and } (x, z) \in r_j \text{ for each } j \in J$

$\iff (x, y) \in \varphi^{-1}r_j \text{ for each } j \in J$

$\iff (x, y) \in \bigcap_{j \in J}(\varphi^{-1}r_j)$

So $\varphi^{-1}(\bigcap_{j \in J} r_j) = \bigcap_{j \in J}(\varphi^{-1}r_j)$.

Lemma 3.4.2. Suppose $J$ is a set, and for each $j \in J$, $r_j$ is a relation, and $\varphi$ is a function. Then $\varphi(\bigcup_{j \in J} r_j) = \bigcup_{j \in J}(\varphi r_j)$.

Proof:

$$(x, y) \in \varphi(\bigcup_{j \in J} r_j)$$

$\iff \exists z \text{ such that } (z, y) \in \varphi \text{ and } (x, z) \in \bigcup_{j \in J} r_j$

$\iff \exists z \text{ such that } (z, y) \in \varphi \text{ and } (x, z) \in r_j \text{ for some } j \in J$

$\iff (x, y) \in \varphi r_j \text{ for some } j \in J$

$\iff (x, y) \in \bigcup_{j \in J}(\varphi r_j)$

So $\varphi(\bigcup_{j \in J} r_j) = \bigcup_{j \in J}(\varphi r_j)$. 

\[ \square \]
**Theorem 3.4.** Suppose each of \((A, R)\) and \((B, S)\) is a 1-structure, \(\varphi: A \to B\) is a co-epimorphism, and \((A, R)\) is the structurization of a topological space. Then \((B, S)\) is the structurization of a topological space.

**Proof:** \(\varphi\) is a cohomomorphism, so by Theorem 1.20, \(\varphi\) is a homomorphism.

1. \(\emptyset \in R\), and \(\varphi\) is preservative, so \(\emptyset = \varphi\emptyset \in S\).

2. \(1 \times A \in R\), so suppose \(r = 1 \times A\). \(\varphi\) is surjective, and \(\varphi\) is preservative, so \(1 \times B = 1 \times \varphi[A] = \varphi(1 \times A) = \varphi r \in S\).

3. Suppose \(J\) is a set, and for each \(j \in J\), \(s_j \in S\).

   \(\varphi\) is continuous, so for each \(j \in J\), \(\varphi^{-1}s_j \in R\). \((A, R)\) is the structurization of a topological space, so \(\bigcup_{j \in J} \varphi^{-1}s_j \in R\).

   \(\varphi\) is preservative and surjective, so by Lemma 3.4.2

   \[
   \bigcup_{j \in J} s_j = \bigcup_{j \in J} \varphi \varphi^{-1}s_j = \varphi \bigcup_{j \in J} \varphi^{-1}s_j \in S
   \]

4. Suppose each of \(s_0\) and \(s_1\) is in \(S\). \(\varphi\) is continuous, so each of \(\varphi^{-1}s_0\) and \(\varphi^{-1}s_1\) is in \(R\). \((A, R)\) is the structurization of a topological space, so by Lemma 3.4.1

   \(\varphi^{-1}(s_0 \cap s_1) = (\varphi^{-1}s_0) \cap (\varphi^{-1}s_1) \in R\).

   \(\varphi\) is preservative and surjective, so \(s_0 \cap s_1 = \varphi \varphi^{-1}(s_0 \cap s_1) \in S\).

So by the above, \((B, S)\) is the structurization of a topological space.

**Theorem 3.5.** Suppose each of \((A, R)\) and \((B, S)\) is a 1-structure, \(\varphi: A \to B\) is a comonomorphism, and \((B, S)\) is the structurization of a topological space. Then \((A, R)\) is the structurization of a topological space.
**Proof:** Suppose \((B, \mathcal{S})\) is the structurization of a topological space.

1. \(\emptyset \in \mathcal{S}\), and \(\varphi\) is continuous, so \(\emptyset = \varphi^{-1}\emptyset \in \mathcal{R}\).

2. \(1 \times B \in \mathcal{S}\), so suppose \(s = 1 \times B\). \(\varphi\) is continuous, so \(1 \times A = 1 \times \varphi^{-1}[B] = \varphi^{-1}(1 \times B) = \varphi^{-1}s \in \mathcal{R}\).

3. Suppose \(J\) is a set, and for each \(j \in J\), \(r_j \in \mathcal{R}\).

   \(\varphi\) is preservative, so for each \(j \in J\), \(\varphi r_j \in \mathcal{S}\). \((B, \mathcal{S})\) is the structurization of a topological space, so \(\bigcup_{j \in J} \varphi r_j \in \mathcal{S}\).

   \(\varphi\) is continuous and injective, so by Lemma 3.4.2,

   \[
   \bigcup_{j \in J} r_j = \varphi^{-1}(\bigcup_{j \in J} r_j) = \varphi^{-1}(\bigcup_{j \in J} \varphi r_j) \in \mathcal{R}
   \]

4. Suppose each of \(r_0\) and \(r_1\) is in \(\mathcal{R}\). \(\varphi\) is preservative, so each of \(\varphi r_0\) and \(\varphi r_1\) is in \(\mathcal{S}\). \((B, \mathcal{S})\) is the structurization of a topological space, so \((\varphi r_0) \cap (\varphi r_1) \in \mathcal{S}\).

   \(\varphi\) is continuous and injective, so by Lemma 3.4.1, \(r_0 \cap r_1 = (\varphi^{-1}\varphi r_0) \cap (\varphi^{-1}\varphi r_1) = \varphi^{-1}((\varphi r_0) \cap (\varphi r_1)) \in \mathcal{R}\).

So by the above, \((A, \mathcal{R})\) is the structurization of a topological space. \(\square\)

**Definition** The statement that \(\mathcal{F}\) is a type means \(\mathcal{F}\) is a function with domain a set of symbols and image a subset of the cardinal numbers[3].

**Definition** Let \(\mathcal{F}\) be a type. The statement that \(A = (A, F)\) is an algebra of type \(\mathcal{F}\) means \(A\) is a set, \(F\) is a set of functions each having image a subset of \(A\), and there is a bijection \(g: \text{dom}(\mathcal{F}) \to F\) such that for each \(f \in \text{dom}(\mathcal{F})\), \(\text{dom}(g(f)) = A^{F(f)}\). For each \(f \in \text{dom}(\mathcal{F})\), denote \(g(f)\) by \(f^A\).
Definition Let $A = (A,F)$ be an algebra of type $F$. Define $I$ to be the set of symbols $\bigcup_{f \in \text{dom}(F)} \left( \{p_f\} \cup \bigcup_{q \in F(f)} \{q_f\} \right)$. The structurization of $(A,F)$ is the $I$-structure $(A,R)$ where $R$ is the set of functions to which a function $r$ belongs if and only if there is an $f$ in $\text{dom}(F)$ and an element $a$ in $A^F(f)$ such that the domain of $r$ is $\{p_f\} \cup \bigcup_{q \in F(f)} \{q_f\}$ and for each element $q$ in $F(f)$, $r(q_f) = a(q)$, and $r(p_f) = f^A(a)$.

Example Suppose $F = \{(e,0), (-1,1), (\cdot,2)\}$ is the type associated with groups. $e$ is the symbol corresponding with the 0-ary function that for each group, picks out the identity element of the group, $-1$ is the symbol corresponding with the unary function that associates each element of the group with its inverse, and $\cdot$ is the symbol corresponding with the binary function of the group.

Consider the dihedral group $D_6 = \{\epsilon, \sigma, \sigma^2, \tau, \tau \sigma, \tau \sigma^2\}$.

Then the structurization of $D_6$ is the $I$-structure $(D_6,R)$, with $I = \{p_e, p_{-1}, 0_{-1}, p_0, 0_1\}$.
and

\[ \mathcal{R} = \{ \]
\[ \{(p_e, \epsilon)\}, \]
\[ \{(p_1, \epsilon), (0_1, \epsilon)\}, \{(p_1, \sigma), (0_1, \sigma^2)\}, \{(p_1, \sigma^2), (0_1, \sigma)\}, \]
\[ \{(p_1, \tau), (0_1, \tau)\}, \{(p_1, \tau \sigma), (0_1, \tau \sigma)\}, \{(p_1, \tau \sigma^2), (0_1, \tau \sigma^2)\}, \]
\[ \{(p, \epsilon), (0, \epsilon), (1, \epsilon)\}, \{(p, \sigma), (0, \sigma), (1, \epsilon)\}, \{(p, \sigma^2), (0, \sigma^2), (1, \epsilon)\}, \]
\[ \{(p, \tau), (0, \tau), (1, \epsilon)\}, \{(p, \tau \sigma), (0, \tau \sigma), (1, \epsilon)\}, \{(p, \tau \sigma^2), (0, \tau \sigma^2), (1, \epsilon)\}, \]
\[ \{(p, \sigma), (0, \epsilon), (1, \sigma)\}, \{(p, \sigma^2), (0, \sigma), (1, \sigma)\}, \{(p, \epsilon), (0, \sigma^2), (1, \sigma)\}, \]
\[ \{(p, \tau \sigma), (0, \tau), (1, \sigma)\}, \{(p, \tau \sigma^2), (0, \tau \sigma), (1, \sigma)\}, \{(p, \tau), (0, \tau \sigma^2), (1, \sigma)\}, \]
\[ \{(p, \sigma^2), (0, \epsilon), (1, \sigma^2)\}, \{(p, \epsilon), (0, \sigma), (1, \sigma^2)\}, \{(p, \sigma), (0, \sigma^2), (1, \sigma^2)\}, \]
\[ \{(p, \tau \sigma^2), (0, \tau), (1, \sigma^2)\}, \{(p, \tau), (0, \tau \sigma), (1, \sigma^2)\}, \{(p, \tau \sigma), (0, \tau \sigma^2), (1, \sigma^2)\}, \]
\[ \{(p, \tau), (0, \epsilon), (1, \tau)\}, \{(p, \tau \sigma^2), (0, \sigma), (1, \tau)\}, \{(p, \tau \sigma), (0, \sigma^2), (1, \tau)\}, \]
\[ \{(p, \epsilon), (0, \tau), (1, \tau)\}, \{(p, \sigma), (0, \tau \sigma), (1, \tau)\}, \{(p, \sigma^2), (0, \tau \sigma^2), (1, \tau)\}, \]
\[ \{(p, \tau \sigma), (0, \epsilon), (1, \tau \sigma)\}, \{(p, \tau), (0, \sigma), (1, \tau \sigma)\}, \{(p, \tau \sigma^2), (0, \sigma^2), (1, \tau \sigma)\}, \]
\[ \{(p, \sigma), (0, \tau), (1, \tau \sigma)\}, \{(p, \epsilon), (0, \tau \sigma), (1, \tau \sigma)\}, \{(p, \sigma^2), (0, \tau \sigma^2), (1, \tau \sigma)\}, \]
\[ \{(p, \tau \sigma^2), (0, \epsilon), (1, \tau \sigma^2)\}, \{(p, \tau \sigma), (0, \sigma), (1, \tau \sigma^2)\}, \{(p, \tau), (0, \sigma^2), (1, \tau \sigma^2)\}, \]
\[ \{(p, \sigma^2), (0, \tau), (1, \tau \sigma^2)\}, \{(p, \sigma), (0, \tau \sigma), (1, \tau \sigma^2)\}, \{(p, \epsilon), (0, \tau \sigma^2), (1, \tau \sigma^2)\} \} \]
Theorem 3.6. Suppose each of \( A = (A, F) \) and \( B = (B, G) \) is an algebra of type \( F \), and \( (A, R) \) is the structurization of \( (A, F) \) and \( (B, S) \) is the structurization of \( (B, G) \). Then a function \( \varphi : A \to B \) is an algebraic homomorphism if and only if it is a \( \bigcup_{f \in \text{dom}(F)} \left( \{ p_f \} \cup \bigcup_{q \in F(f)} \{ q_f \} \right) \)-structure homomorphism.

Proof: Suppose \( \varphi \) is an algebraic homomorphism.

Suppose \( r \in R \). Then there is an \( f \) in \( \text{dom}(F) \) and an element \( a \) in \( A^{F(f)} \) such that the domain of \( r \) is \( \{ p_f \} \cup \bigcup_{q \in F(f)} \{ q_f \} \) and for each element \( q \) in \( F(f) \), \( r(q_f) = a(q) \), and \( r(p_f) = f^A(a) \).

\( \varphi a \) is an element in \( B^{F(f)} \), so there is an \( s \in S \) such that the domain of \( s \) is \( \{ p_f \} \cup \bigcup_{q \in F(f)} \{ q_f \} \) and for each element \( q \) in \( F(f) \), \( s(q_f) = \varphi a(q) = \varphi(a(q)) = \varphi(r(q_f)) \), \( s(p_f) = f^B(\varphi a) = \varphi(f^A(a)) = \varphi(r(p_f)) = \varphi r(p_f) \).

So \( \varphi r = s \in S \) and \( \varphi \) is preservative.

Suppose \( s \in S \) such that \( \text{im}(s) \subseteq \text{im}(\varphi) \).

There is an \( f \) in \( \text{dom}(F) \) and an element \( b \) in \( B^{F(f)} \) such that the domain of \( s \) is \( \{ p_f \} \cup \bigcup_{q \in F(f)} \{ q_f \} \), and for each element \( q \) in \( F(f) \), \( s(q_f) = b(q) \), and \( s(p_f) = f^B(b) \).

Moreover, for each \( q \in F(f) \), since \( b(q) = s(q_f) \in \text{im}(s) \subseteq \text{im}(\varphi) \), there is an \( a_q \in A \) such that \( b(q) = \varphi(a_q) \).

Define \( a : F(f) \to A \) such that for each \( q \in F(f) \), \( a(q) = a_q \). \( a \) is an element in \( A^{F(f)} \), so there is an \( r \in R \) such that the domain of \( r \) is \( \{ p_f \} \cup \bigcup_{q \in F(f)} \{ q_f \} \) and for each element \( q \) in \( F(f) \), \( r(q_f) = a(q) \), and \( r(p_f) = f^A(a) \). Note for each \( q \in F(f) \), \( b(q) = \varphi(a_q) = \varphi(a(q)) = \varphi a(q) \).
so $b = \varphi a$.

For each $q$ in $\mathcal{F}(f)$, $s(q_f) = b(q) = \varphi a(q) = \varphi(a(q)) = \varphi(r(q_f)) = \varphi(r(q_f))$.

$s(p_f) = f^B(b) = f^B(\varphi a) = \varphi(f^A(a)) = \varphi(r(p_f)) = \varphi(r(p_f))$.

So $r$ is a relation in $\mathcal{R}$ such that $\varphi r = s$, and $\varphi$ is saturating.

Thus $\varphi$ is an $\bigcup_{f \in \text{dom}(\mathcal{F})} \left( \{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\} \right)$-structure homomorphism.

Suppose $\varphi$ is an $\bigcup_{f \in \text{dom}(\mathcal{F})} \left( \{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\} \right)$-structure homomorphism.

Suppose $f$ is in $\text{dom}(\mathcal{F})$ and $a$ is an element in $A^{F(f)}$.

Then there is an $r \in \mathcal{R}$ such that such that the domain of $r$ is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$ and for each element $q$ in $\mathcal{F}(f)$, $r(q_f) = a(q)$, and $r(p_f) = f^A(a)$.

$\varphi$ is an $I$-structure homomorphism, so $\varphi r \in \mathcal{S}$.

Since $\varphi r$ is in $\mathcal{S}$, and has domain $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$, and for each $q \in \mathcal{F}(f)$, $\varphi r(q_f) = \varphi(r(q_f)) = \varphi(a(q)) = \varphi a(q)$, $f$ is the member of $\text{dom}(\mathcal{F})$ and $\varphi a$ is the element in $B^{F(f)}$ such that $\varphi r(p_f) = f^B(\varphi a)$.

$\varphi(f^A(a)) = \varphi(r(p_f)) = \varphi r(p_f) = f^B(\varphi a)$.

So $\varphi$ is an algebraic homomorphism. \qed
Theorem 3.7. Suppose each of $A = (A,F)$ and $B = (B,G)$ is an algebra of type $\mathcal{F}$, and $(A,\mathcal{R})$ is the structurization of $(A,F)$ and $(B,\mathcal{S})$ is the structurization of $(B,G)$. Then a function $\varphi : A \rightarrow B$ is an algebraic isomorphism if and only if it is an $\bigcup_{f \in \text{dom}(\mathcal{F})} \left( \{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\} \right)$-structure isomorphism.

Proof: Suppose $\varphi : A \rightarrow B$ is a function.

$\varphi$ is an algebraic isomorphism

$\iff$ $\varphi$ is a bijective algebraic homomorphism

$\iff$ $\varphi$ is a bijective structure homomorphism

$\iff$ $\varphi$ is an $I$-structure isomorphism \qed

Remark Graph will hereafter be used to refer to graphs which may contain loops and multiple edges\cite{4}.

Definition Let $G$ be a graph. Define $V(G)$ to be the vertex set of $G$.

Definition Let $G$ be a graph. Define $E(G)$ to be the edge set of $G$.

Definition Let $G$ be a graph. The structurization of $G$ is the $\mathbb{N}\setminus\{0\}$-structure $(A,\mathcal{R})$ where $\mathcal{R}$ is the set of relations to which a relation $f : \mathbb{N}\setminus\{0\} \rightarrow V(G)$ belongs if and only if $f$ contains either one or two pairs each having the same first element $n$, and there are at least $n$ edges joining the vertices in $f[\{n\}]$. (If there is exactly one pair $(n,v)$ in $f$, then the $n$ edges correspond to $n$ loops at $v$).

Theorem 3.8. Suppose each of $G$ and $H$ is a graph. Suppose the $\mathbb{N}\setminus\{0\}$-structure $(V(G),\mathcal{R})$ is the structurization of $G$ and the $\mathbb{N}\setminus\{0\}$-structure $(V(H),\mathcal{S})$ is the structurization of $H$. Then the graphs $G$ and $H$ are isomorphic if and only if $(V(G),\mathcal{R})$ and $(V(H),\mathcal{S})$ are isomorphic.
**Proof:** Suppose $G$ and $H$ are isomorphic, and $\varphi : V(G) \to V(H)$ is a graph isomorphism. $\varphi$ is a bijection.

Suppose $r \in \mathcal{R}$. Suppose $r$ contains exactly one element $(n, v)$. Then there are at least $n$ loops at vertex $v$ in $G$, so there are at least $n$ loops at vertex $\varphi(v)$ in $H$. Thus there is a $s \in \mathcal{S}$ such that $s$ contains exactly one element $(n, \varphi(v))$, so $s[\{n\}] = \varphi[r[\{n\}]] = \varphi[r]\{n\}]$. So $s = \varphi r$.

Suppose $r$ contains exactly two elements $(n, v)$ and $(n, w)$. Then there are at least $n$ edges connecting vertices $v$ and $w$ in $G$, so there are at least $n$ edges connecting vertices $\varphi(v)$ and $\varphi(w)$ in $H$. Thus there is a $s \in \mathcal{S}$ such that $s$ contains exactly two elements $(n, \varphi(v))$ and $(n, \varphi(w))$, so $s[\{n\}] = \varphi[r[\{n\}]] = \varphi[r]\{n\}]$. So $s = \varphi r$.

So in both cases there is a $s$ in $\mathcal{S}$ such that $s = \varphi r$, so $\varphi$ is preservative.

Suppose $s \in \mathcal{S}$. Suppose $s$ contains exactly one element $(n, v)$. Then there are at least $n$ loops at vertex $w$ in $H$, so there are at least $n$ loops at vertex $\varphi^{-1}(v)$ in $G$. Thus there is a $r \in \mathcal{R}$ such that $r$ contains exactly one element $(n, \varphi^{-1}(v))$, so $r[\{n\}] = \varphi^{-1}[s[\{n\}]] = \varphi^{-1}s[\{n\}]$. So $r = \varphi^{-1}s$.

Suppose $s$ contains exactly two elements $(n, v)$ and $(n, w)$. Then there are at least $n$ edges connecting vertices $v$ and $w$ in $H$, so there are at least $n$ edges connecting vertices $\varphi^{-1}(v)$ and $\varphi^{-1}(w)$ in $G$. Thus there is a $r \in \mathcal{R}$ such that $r$ contains exactly two elements $(n, \varphi^{-1}(v))$ and $(n, \varphi^{-1}(w))$, so $r[\{n\}] = \varphi^{-1}[s[\{n\}]] = \varphi^{-1}s[\{n\}]$. So $r = \varphi^{-1}s$.

So $\varphi$ is continuous and thus $\varphi$ is a $\mathbb{N}\{0\}$-structure isomorphism. \hfill \Box
Suppose \((V(G), \mathcal{R})\) and \((V(H), \mathcal{S})\) are isomorphic and \(\alpha : V(G) \to V(H)\) is a \(\mathbb{N}\backslash\{0\}\)-structure isomorphism and hence a bijection.

Suppose \(n \in \mathbb{N}\), and there are exactly \(n\) loops at vertex \(v\) in \(G\). If \(n \neq 0\) then there is a relation \(r \in \mathcal{R}\) such that \(r = \{(n, v)\}\), and \(\{(n, \alpha(v))\} = \alpha r \in \mathcal{S}\), so there are at least \(n\) loops in \(H\) at vertex \(\alpha(v)\).

Suppose \(s = \{(n+1, \alpha(v))\}\). If \(s \in \mathcal{S}\), then \(\{(n+1, v)\} = \{(n+1, \alpha^{-1}(\alpha(v)))\} = \alpha^{-1} s \in \mathcal{R}\), and thus there are \(n+1\) loops at \(v\), contradicting the assumption that there are exactly \(n\) loops at \(v\) in \(G\). So \(s \notin \mathcal{S}\), and thus there are not \(n+1\) loops at \(\alpha(v)\). So there are exactly \(n\) loops at \(\alpha(v)\) in \(H\).

Suppose \(n \in \mathbb{N}\), and there are exactly \(n\) edges connecting vertices \(v\) and \(w\) in \(G\). If \(n \neq 0\) then there is a relation \(r \in \mathcal{R}\) such that \(r = \{(n, v), (n, w)\}\), and \(\{(n, \alpha(v)), (n, \alpha(w))\} = \alpha r \in \mathcal{S}\), so there are at least \(n\) edges in \(H\) connecting vertices \(\alpha(v)\) and \(\alpha(w)\).

Suppose \(s = \{(n+1, \alpha(v)), (n+1, \alpha(w))\}\). If \(s \in \mathcal{S}\), then \(\{(n+1, v), (n+1, w)\} = \{(n+1, \alpha^{-1}(\alpha(v))), (n+1, \alpha^{-1}(\alpha(w)))\} = \alpha^{-1} s \in \mathcal{R}\), and thus there are \(n+1\) edges connecting \(v\) and \(w\), contradicting the assumption that there are exactly \(n\) edges connecting \(v\) and \(w\) in \(G\). So \(s \notin \mathcal{S}\), and thus there are not \(n+1\) edges connecting \(\alpha(v)\) and \(\alpha(w)\). So there are exactly \(n\) edges connecting \(\alpha(v)\) and \(\alpha(w)\) in \(H\).

So \(\alpha\) is a graph isomorphism and \(G\) and \(H\) are isomorphic. \(\square\)
Lemma 4.1.1. Suppose each of $A$, $B$, and $C$ is a set, $\alpha : A \to B$ is a surjection, $\beta : A \to C$ is a function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is a unique function with domain $B$ such that $(\beta\alpha^{-1})\alpha = \beta$. Moreover, $\text{im}(\beta\alpha^{-1}) = \text{im}(\beta)$.

Proof:

\[(b, c_1) \in \beta\alpha^{-1} \text{ and } (b, c_2) \in \beta\alpha^{-1} \]

\[\Rightarrow \exists a_1 \in A \text{ such that } (b, a_1) \in \alpha^{-1} \text{ and } (a_1, c_1) \in \beta \]

and $\exists a_2 \in A$ such that $\exists (b, a_2) \in \alpha^{-1}$ and $(a_2, c_2) \in \beta$

\[\Rightarrow (a_1, b) \in \alpha \text{ and } (b, a_2) \in \alpha^{-1}, c_1 = \beta(a_1), c_2 = \beta(a_2) \]

\[\Rightarrow (a_1, a_2) \in \alpha^{-1}\alpha \subseteq \beta^{-1}\beta, c_1 = \beta(a_1), c_2 = \beta(a_2) \]

\[\Rightarrow \exists c \in C \text{ such that } (a_1, c) \in \beta \text{ and } (c, a_2) \in \beta^{-1}, c_1 = \beta(a_1), c_2 = \beta(a_2) \]

\[\Rightarrow (a_1, c) \in \beta \text{ and } (a_2, c) \in \beta, c_1 = \beta(a_1), c_2 = \beta(a_2) \]

\[\Rightarrow c_1 = \beta(a_1) = c = \beta(a_2) = c_2 \]

So $\beta\alpha^{-1}$ is a function.
Note $1_A \subseteq \alpha^{-1} \alpha$ and $\beta^{-1} \beta \subseteq 1_C$.

$$(\beta^{-1}) \alpha = \beta(\alpha^{-1} \alpha) \subseteq \beta(\alpha^{-1} \beta) = (\beta^{-1}) \beta \subseteq 1_C \beta = \beta$$

$$\beta = \beta 1_A \subseteq \beta(\alpha^{-1} \alpha) = (\beta^{-1}) \alpha$$

So $(\beta^{-1}) \alpha = \beta$.

Suppose $\gamma$ is a function with domain $B$ such that $\gamma \alpha = \beta$. $\alpha$ is a surjection, so $\alpha^{-1} = 1_B$

$$\gamma \alpha = \beta$$

$$\implies \gamma = \gamma 1_B = \gamma \alpha \alpha^{-1} = \beta \alpha^{-1}$$

So $\beta \alpha^{-1}$ is the only function having domain $B$ with the property that $(\beta^{-1}) \alpha = \beta$.

$\text{im}(\beta^{-1}) = \beta(\text{im}(\alpha^{-1})) = \beta(\text{dom}(\alpha)) = \beta(A) = \text{im}(\alpha)$

So $\text{im}(\beta^{-1}) = \text{im}(\beta)$. \hfill \square

**Corollary 4.1.1.** Suppose each of $A$, $B$, and $C$ is a set, $\alpha : A \to B$ is a surjection, $\beta : A \to C$ is a function, and $\alpha^{-1} \alpha \subseteq \beta^{-1} \beta$. Then $\beta$ is a surjection if and only if $\beta \alpha^{-1}$ is a surjection with respect to $C$.

**Proof:** Suppose $\beta$ is a surjection.

By Lemma 4.1.1, $\beta \alpha^{-1}$ is a function.

$C = \text{im}(\beta) = \text{im}(\beta^{-1})$, so $\beta \alpha^{-1}$ is a surjection with respect to $C$.

Suppose $\beta \alpha^{-1}$ is a surjection with respect to $C$.
\[ C = \text{im}(\beta \alpha^{-1}) = \text{im}(\beta), \text{ so } \beta \text{ is a surjection.} \]

**Corollary 4.1.2.** Suppose each of \( A, B, \) and \( C \) is a set, \( \alpha : A \to B \) is a surjection, \( \beta : A \to C \) is a function, and \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \). Then \( \alpha^{-1}\alpha = \beta^{-1}\beta \) if and only if \( \beta\alpha^{-1} \) is an injection.

**Proof:** Suppose \( \alpha^{-1}\alpha = \beta^{-1}\beta \).

\[
(\beta\alpha^{-1})^{-1}\beta\alpha^{-1} = \alpha\beta^{-1}\beta\alpha^{-1} = \alpha\alpha^{-1}\alpha\alpha^{-1} = 1_B 1_B = 1_B
\]

So \( \beta\alpha^{-1} \) is an injection.

Suppose \( \beta\alpha^{-1} \) is an injection. Note \( \beta\alpha^{-1}\alpha = \beta \)

\[
\beta^{-1}\beta = 1_A\beta^{-1}\beta \subseteq \alpha^{-1}\alpha\beta^{-1}\beta = \alpha^{-1}\alpha\beta^{-1}(\beta\alpha^{-1}\alpha) = \alpha^{-1}(\beta\alpha^{-1})^{-1}\beta\alpha^{-1}\alpha = \alpha^{-1}1_B\alpha = \alpha^{-1}\alpha
\]

So \( \beta^{-1}\beta \subseteq \alpha^{-1}\alpha \) and thus \( \alpha^{-1}\alpha = \beta^{-1}\beta \).

**Lemma 4.1.2.** Suppose each of \((A, \mathcal{R})\), \((B, \mathcal{S})\), and \((C, \mathcal{T})\) is an \( I \)-structure, \( \alpha : A \to B \) is a saturating surjection, \( \beta : A \to C \) is a preservative function, and \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \). Then \( \beta\alpha^{-1} \) is preservative.

**Proof:** Suppose \( s \in \mathcal{S} \). \( \text{im}(s) \subseteq B = \text{im}(\alpha) \), so since \( \alpha \) is saturating, there is an \( r \in \mathcal{R} \) such that \( \alpha r = s \). \( \beta \) is preservative, so \( \beta r \in \mathcal{T} \). By Lemma 4.1.1, \( \beta\alpha^{-1}\alpha = \beta \), so \( \beta\alpha^{-1}s = \beta\alpha^{-1}\alpha r = \beta r \).

Thus \( \beta\alpha^{-1}s = \beta r \in \mathcal{T} \). So \( \beta\alpha^{-1} \) is preservative.

**Lemma 4.1.3.** Suppose each of \((A, \mathcal{R})\), \((B, \mathcal{S})\), and \((C, \mathcal{T})\) is an \( I \)-structure, \( \alpha : A \to B \) is a preservative surjection, \( \beta : A \to C \) is a saturating function, and \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \). Then \( \beta\alpha^{-1} \) is saturating.
Proof: Suppose $t \in \mathcal{T}$ and $\text{im}(t) \subseteq \text{im}(\beta\alpha^{-1}) = \text{im}(\beta)$. $\beta$ is saturating, so there is an $r \in \mathcal{R}$ such that $\beta r = t$. $\alpha$ is preservative, so $\alpha r \in \mathcal{S}$.

By Lemma 4.1.1 $\beta\alpha^{-1}\alpha = \beta$, so $(\beta\alpha^{-1})(\alpha r) = (\beta\alpha^{-1}\alpha)r = \beta r = t$.

So $\alpha r$ is a relation in $\mathcal{S}$ such that $(\beta\alpha^{-1})\alpha r = t$. So $\beta\alpha^{-1}$ is saturating.

\[ \text{Theorem 4.1.} \quad \text{Suppose each of } (A, \mathcal{R}), (B, \mathcal{S}), \text{ and } (C, \mathcal{T}) \text{ is an I-structure, } \alpha : A \to B \text{ is an epimorphism, } \beta : A \to C \text{ is a homomorphism, and } \alpha^{-1}\alpha \subseteq \beta^{-1}\beta. \text{ Then } \beta\alpha^{-1} \text{ is a unique homomorphism with domain } B \text{ such that } \beta\alpha^{-1}\alpha = \beta. \]

Proof: By Lemma 4.1.1 $\beta\alpha^{-1}$ is a unique function with domain $B$ such that $\beta\alpha^{-1}\alpha = \beta$.

$\alpha$ is a homomorphism and thus is saturating, and $\beta$ is a homomorphism and thus is preservative, so by Lemma 4.1.2 $\beta\alpha^{-1}$ is preservative.

$\alpha$ is a homomorphism and thus is preservative, and $\beta$ is a homomorphism and thus is saturating, so by Lemma 4.1.3 $\beta\alpha^{-1}$ is saturating.

Since $\beta\alpha^{-1}$ is both preservative and saturating, $\beta\alpha^{-1}$ is an I-structure homomorphism. 

\[ \text{Lemma 4.2.1.} \quad \text{Suppose each of } (A, \mathcal{R}), (B, \mathcal{S}), \text{ and } (C, \mathcal{T}) \text{ is an I-structure, } \alpha : A \to B \text{ is a conservative surjection, } \beta : A \to C \text{ is a continuous function, and } \alpha^{-1}\alpha \subseteq \beta^{-1}\beta. \text{ Then } \beta\alpha^{-1} \text{ is continuous.} \]

Proof: Suppose $t \in \mathcal{T}$. $\beta$ is continuous, so, $\beta^{-1}t \in \mathcal{R}$. $\alpha$ is conservative, so there is an $s \in \mathcal{S}$ such that $\text{im}(s) \subseteq \text{im}(\alpha)$ and $\alpha^{-1}s = \beta^{-1}t$. $\alpha$ is surjective, so $\alpha\alpha^{-1} = 1_B$.

\[(\beta\alpha^{-1})^{-1}t = \alpha\beta^{-1}t = \alpha\alpha^{-1}s = 1Bs = s\]
Thus $(\beta \alpha^{-1})^{-1}t = s \in S$. So $\beta \alpha^{-1}$ is continuous.

**Lemma 4.2.2.** Suppose each of $(A, R)$, $(B, S)$, and $(C, T)$ is an $I$-structure, $\alpha : A \to B$ is a continuous surjection, $\beta : A \to C$ is a conservative function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta \alpha^{-1}$ is conservative.

**Proof:** Suppose $s \in S$. $\alpha$ is continuous, so $\alpha^{-1}s \in R$. $\beta$ is conservative, so there is a $t \in T$ such that $\text{im}(t) \subseteq \text{im}(\beta)$ and $\beta^{-1}t = \alpha^{-1}s$. $\alpha$ is surjective, so $\alpha\alpha^{-1} = 1_B$.

$$((\beta \alpha^{-1})^{-1}t = \alpha \beta^{-1}t = \alpha \alpha^{-1}s = 1_B s) = s$$

Thus $(\beta \alpha^{-1})^{-1}t = s$, and $t$ is a relation in $T$ such that $\text{im}(t) \subseteq \text{im}(\beta) = \text{im}(\beta \alpha^{-1})$ and $(\beta \alpha^{-1})^{-1}t = s$. So $\beta \alpha^{-1}$ is conservative.

**Theorem 4.2.** Suppose each of $(A, R)$, $(B, S)$, and $(C, T)$ is an $I$-structure, $\alpha : A \to B$ is a coepimorphism, $\beta : A \to C$ is a cohomomorphism, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta \alpha^{-1}$ is a unique cohomomorphism with domain $B$ such that $\beta \alpha^{-1} \alpha = \beta$.

**Proof:** By Lemma 4.1.1, $\beta \alpha^{-1}$ is a unique function with domain $B$ such that $\beta \alpha^{-1} \alpha = \beta$.

$\alpha$ is a cohomomorphism and thus is conservative, and $\beta$ is a cohomomorphism and thus is continuous, so by Lemma 4.2.1, $\beta \alpha^{-1}$ is continuous.

$\alpha$ is a cohomomorphism and thus is continuous, and $\beta$ is a cohomomorphism and thus is conservative, so by Lemma 4.2.2, $\beta \alpha^{-1}$ is conservative.

Since $\beta \alpha^{-1}$ is both continuous and conservative, $\beta \alpha^{-1}$ is an $I$-structure cohomomorphism.

**Lemma 4.3.1.** Suppose each of $\alpha$ and $\beta$ is a function. Then $\alpha \alpha^{-1} \subseteq \beta \beta^{-1}$ if and only if $\text{im}(\alpha) \subseteq \text{im}(\beta)$.
**Proof:** Suppose $\alpha^{-1} \subseteq \beta^{-1}$.

$$\text{im}(\alpha) = \text{im}(\text{id}_\text{im}(\alpha)) = \text{im}(\alpha^{-1}) \subseteq \text{im}(\beta^{-1}) = \text{im}(\text{id}_\text{im}(\beta)) = \text{im}(\beta)$$

Suppose $\text{im}(\alpha) \subseteq \text{im}(\beta)$.

$$\alpha^{-1} = 1_{\text{im}(\alpha)} \subseteq 1_{\text{im}(\beta)} = \beta^{-1} \quad \square$$

**Lemma 4.3.2.** Suppose each of $A$, $B$, and $C$ is a set, $\alpha : B \to A$ is a function, $\beta : C \to A$ is an injection, and $\alpha^{-1} \subseteq \beta^{-1}$. Then $\beta^{-1}\alpha$ is a unique function with image a subset of $C$ such that $\beta(\beta^{-1}\alpha) = \alpha$. Moreover, $\text{dom}(\beta^{-1}\alpha) = \text{dom}(\alpha)$.

![Diagram](image)

**Proof:**

$$(b,c_1) \in \beta^{-1}\alpha \text{ and } (b,c_2) \in \beta^{-1}\alpha$$

$$\implies \exists a_1 \in A \text{ such that } (b,a_1) \in \alpha \text{ and } (a_1,c_1) \in \beta^{-1}$$

and $\exists a_2 \in A \text{ such that } (b,a_2) \in \alpha \text{ and } (a_2,c_2) \in \beta^{-1}$

$$\implies a_1 = a_2 \text{ since } \alpha \text{ is a function}$$

$$\implies (c_1,a_1) \in \beta \text{ and } (c_2,a_1) \in \beta$$

$$\implies c_1 = c_2 \text{ since } \beta \text{ is an injection}$$

So $\beta^{-1}\alpha$ is a function.
Note $\beta \beta^{-1} \subseteq 1_A$ and $\alpha \alpha^{-1} = 1_{\text{im}(\alpha)}$.

$$
\beta(\beta^{-1} \alpha) = (\beta \beta^{-1}) \alpha \subseteq 1_A \alpha = \alpha
$$

$$
\alpha = 1_{\text{im}(\alpha)} \alpha = (\alpha \alpha^{-1}) \alpha \subseteq (\beta \beta^{-1}) \alpha = \beta(\beta^{-1} \alpha)
$$

So $\beta(\beta^{-1} \alpha) = \alpha$.

Suppose $\gamma$ is a function with image a subset of $C$ such that $\beta \gamma = \alpha$. Note since $\beta$ is an injection, $\beta^{-1} \beta = 1_C$.

$$
\beta \gamma = \alpha
\implies \gamma = 1_C \gamma = \beta^{-1} \beta \gamma = \beta^{-1} \alpha
$$

So $\beta^{-1} \alpha$ is the only function with image a subset of $C$ with the property that $\beta(\beta^{-1} \alpha) = \alpha$.

$$
b \in \text{dom}(\beta^{-1} \alpha)
\implies \exists c \in C \text{ such that } (b,c) \in \beta^{-1} \alpha
\implies \exists a \in A \text{ such that } (b,a) \in \alpha \text{ and } (a,c) \in \beta^{-1}
\implies b \in \text{dom}(\alpha)
$$

So $\text{dom}(\beta^{-1} \alpha) \subseteq \text{dom}(\alpha)$.

$$
b \in \text{dom}(\alpha)
\implies \exists a \in A \text{ such that } (b,a) \in \alpha = \beta(\beta^{-1} \alpha)
\implies \exists c \in C \text{ such that } (c,a) \in \beta \text{ and } (b,c) \in \beta^{-1} \alpha
\implies b \in \text{dom}(\beta^{-1} \alpha)
$$
So $\text{dom}(\alpha) \subseteq \text{dom}(\beta^{-1}\alpha)$, and thus $\text{dom}(\beta^{-1}\alpha) = \text{dom}(\alpha)$. □

**Corollary 4.3.1.** Suppose each of $A$, $B$, and $C$ is a set, $\alpha : B \to A$ is a function, $\beta : C \to A$ is an injection, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $\alpha$ is an injection if and only if $\beta^{-1}\alpha$ is an injection.

**Proof:** Suppose $\alpha$ is an injection. $\text{im}(\alpha) \subseteq \text{im}(\beta)$.

\[
(\beta^{-1}\alpha)^{-1}\beta^{-1}\alpha = \alpha^{-1}\beta\beta^{-1}\alpha = \alpha^{-1}1_{\text{im}(\beta)}\alpha = \alpha^{-1}\alpha = 1_B
\]

So $\beta^{-1}\alpha$ is an injection.

Suppose $\beta^{-1}\alpha$ is an injection.

\[
\alpha^{-1}\alpha = \alpha^{-1}1_{\text{im}(\beta)}\alpha = \alpha^{-1}\beta\beta^{-1}\alpha = (\beta^{-1}\alpha)^{-1}\beta^{-1}\alpha = 1_B
\]

So $\alpha$ is an injection. □

**Corollary 4.3.2.** Suppose each of $A$, $B$, and $C$ is a set, $\alpha : B \to A$ is a function, $\beta : C \to A$ is an injection, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $\alpha\alpha^{-1} = \beta\beta^{-1}$ if and only if $\beta^{-1}\alpha$ is a surjection with respect to $C$.

**Proof:** Suppose $\alpha\alpha^{-1} = \beta\beta^{-1}$.

\[
\beta^{-1}\alpha(\beta^{-1}\alpha)^{-1} = \beta^{-1}\alpha\alpha^{-1}\beta = \beta^{-1}\beta\beta^{-1}\beta = 1_C1_C = 1_C
\]

So $\beta^{-1}\alpha$ is a surjection with respect to $C$.

Suppose $\beta^{-1}\alpha$ is a surjection with respect to $C$.

\[
\beta\beta^{-1} = \beta 1_C\beta^{-1} = \beta\beta^{-1}\alpha(\beta^{-1}\alpha)^{-1}\beta^{-1} = \beta\beta^{-1}\alpha\alpha^{-1}\beta\beta^{-1} = 1_{\text{im}(\beta)}\alpha\alpha^{-1}1_{\text{im}(\beta)} = \alpha\alpha^{-1}
\]

□
Lemma 4.3.3. Suppose each of \((A, R), (B, S),\) and \((C, T)\) is an \(I\)-structure, \(\alpha : B \to A\) is a preservative function, \(\beta : C \to A\) is a saturating injection, and \(\alpha \alpha^{-1} \subseteq \beta \beta^{-1}\). Then \(\beta^{-1} \alpha\) is preservative.

Proof: Suppose \(s \in S\). \(\alpha\) is preservative, so \(\alpha s \in R\). \(\text{im}(\alpha s) \subseteq \text{im}(\alpha) \subseteq \text{im}(\beta)\), so since \(\beta\) is saturating, there is a \(t \in T\) such that \(\beta t = \alpha s\).

Thus \(\beta^{-1} \alpha s = \beta^{-1} \beta t = 1_A t = t \in T\). So \(\beta^{-1} \alpha\) is preservative. \(\square\)

Lemma 4.3.4. Suppose each of \((A, R), (B, S),\) and \((C, T)\) is an \(I\)-structure, \(\alpha : B \to A\) is a saturating function, \(\beta : C \to A\) is a preservative injection, and \(\alpha \alpha^{-1} \subseteq \beta \beta^{-1}\). Then \(\beta^{-1} \alpha\) is saturating.

Proof: Suppose \(t \in T\) such that \(\text{im}(t) \subseteq \text{im}(\beta^{-1} \alpha)\). \(\beta\) is preservative, so \(\beta t \in R\). \(\text{im}(\beta t) = \beta(\text{im}(t)) \subseteq \beta(\text{im}(\beta^{-1} \alpha)) = \text{im}(\beta \beta^{-1} \alpha) \subseteq \text{im}(1_A \alpha) = \text{im}(\alpha)\), so since \(\alpha\) is saturating, there is an \(s \in S\) such that \(\alpha s = \beta t\).

So \(s\) is a relation in \(S\) such that \(\beta^{-1} \alpha s = \beta^{-1} \beta t = 1_A t = t\). So \(\beta^{-1} \alpha\) is saturating. \(\square\)

Theorem 4.3. Suppose each of \((A, R), (B, S),\) and \((C, T)\) is an \(I\)-structure, \(\alpha : B \to A\) is a homomorphism, \(\beta : C \to A\) is a monomorphism, and \(\alpha \alpha^{-1} \subseteq \beta \beta^{-1}\). Then \(\beta^{-1} \alpha\) is a unique homomorphism with image a subset of \(C\) such that \(\beta \beta^{-1} \alpha = \alpha\).

Proof: By Lemma 4.3.2 \(\beta^{-1} \alpha\) is a unique function with image a subset of \(C\) such that \(\beta \beta^{-1} \alpha = \alpha\).

\(\alpha\) is a homomorphism and thus is preservative, and \(\beta\) is a homomorphism and thus is saturating, so by Lemma 4.3.3 \(\beta^{-1} \alpha\) is preservative.
\(\alpha\) is a homomorphism and thus is saturating, and \(\beta\) is a homomorphism and thus is preservative, so by Lemma 4.3.4, \(\beta^{-1}\alpha\) is saturating.

Since \(\beta^{-1}\alpha\) is both preservative and saturating, \(\beta^{-1}\alpha\) is an \(I\)-structure homomorphism.

**Lemma 4.4.1.** Suppose each of \((A, R), (B, S), \) and \((C, T)\) is an \(I\)-structure, \(\alpha : B \to A\) is a continuous function, \(\beta : C \to A\) is a conservative injection, and \(\alpha\alpha^{-1} \subseteq \beta\beta^{-1}\). Then \(\beta^{-1}\alpha\) is continuous.

**Proof:** Suppose \(t \in T\). \(\beta\) is conservative, so there is an \(r \in R\) such that \(\text{im}(r) \subseteq \text{im}(\beta)\) and \(\beta^{-1}r = t\). \(\alpha\) is continuous, so \(\alpha^{-1}r \in S\).

\[
(\beta^{-1}\alpha)^{-1}t = \alpha^{-1}\beta t = \alpha^{-1}\beta\beta^{-1}r = \alpha^{-1}1_{\text{im}(\beta)}r = \alpha^{-1}r
\]

Thus \((\beta^{-1}\alpha)^{-1}t = \alpha^{-1}r \in S\). So \(\beta^{-1}\alpha\) is continuous.

**Lemma 4.4.2.** Suppose each of \((A, R), (B, S), \) and \((C, T)\) is an \(I\)-structure, \(\alpha : B \to A\) is a conservative function, \(\beta : C \to A\) is a continuous injection, and \(\text{im}(\alpha) \subseteq \text{im}(\beta)\). Then \(\beta^{-1}\alpha\) is conservative.

**Proof:** Suppose \(s \in S\). \(\alpha\) is conservative, so there is an \(r \in R\) such that \(\text{im}(r) \subseteq \text{im}(\alpha) \subseteq \text{im}(\beta)\) and \(s = \alpha^{-1}r\). \(\beta\) is continuous, so \(\beta^{-1}r \in T\).

\[
(\beta^{-1}\alpha)^{-1}\beta^{-1}r = \alpha^{-1}\beta\beta^{-1}r = \alpha^{-1}1_{\text{im}(\beta)}r = \alpha^{-1}r = s
\]

Thus \((\beta^{-1}\alpha)^{-1}\beta^{-1}r = s\), and \(\beta^{-1}r\) is a relation in \(T\) such that \(\text{im}(\beta^{-1}r) = \beta^{-1}(\text{im}(r)) \subseteq \beta^{-1}(\text{im}(\alpha)) = \text{im}(\beta^{-1}\alpha)\) and \((\beta^{-1}\alpha)^{-1}\beta^{-1}r = s\). So \(\beta^{-1}\alpha\) is conservative.

**Theorem 4.4.** Suppose each of \((A, R), (B, S), \) and \((C, T)\) is an \(I\)-structure, \(\alpha : B \to A\) is a cohomomorphism, \(\beta : C \to A\) is a comonomorphism, and \(\alpha\alpha^{-1} \subseteq \beta\beta^{-1}\). Then \(\beta^{-1}\alpha\) is a unique cohomomorphism with image a subset of \(C\) such that \(\beta\beta^{-1}\alpha = \alpha\).
Proof: By Lemma 4.3.2, $\beta^{-1}\alpha$ is a unique function with image a subset of $C$ such that $\beta\beta^{-1}\alpha = \alpha$.

$\alpha$ is a cohomomorphism and thus is continuous, and $\beta$ is a cohomomorphism and thus is conservative, so by Lemma 4.4.1, $\beta^{-1}\alpha$ is continuous.

$\alpha$ is a cohomomorphism and thus is conservative, and $\beta$ is a cohomomorphism and thus is continuous, so by Lemma 4.4.2, $\beta^{-1}\alpha$ is conservative.

Since $\beta^{-1}\alpha$ is both continuous and conservative, $\beta^{-1}\alpha$ is an $I$-structure cohomomorphism. \qed
Definition Suppose $M = (A, \mathcal{R})$ is an $I$-structure and $B \subseteq A$. The $I$-substructure of $M$ induced by $B$ is the $I$-structure $(B, \hat{\mathcal{R}})$ where $\hat{\mathcal{R}}$ is the set of relations to which a relation $\hat{r}$ belongs if and only if $\hat{r} \in \mathcal{R}$ and $\text{im}(\hat{r}) \subseteq B$. Denote the $I$-structure $(B, \hat{\mathcal{R}})$ by $M|B$. The statement that $(C, \mathcal{T})$ is an $I$-substructure of $M$ means $C \subseteq A$ and $(C, \mathcal{T})$ is the $I$-substructure of $M$ induced by $C$.

Definition Suppose $M = (A, \mathcal{R})$ is an $I$-structure and $\varphi$ is a function with domain $A$. Suppose $\hat{\mathcal{R}}$ is the set of relations to which a relation $\bar{r}$ belongs if and only if there is a relation $r \in \mathcal{R}$ such that $\bar{r} = \pi_\varphi r$. Denote the $I$-structure $(A/\varphi, \hat{\mathcal{R}})$ by $M/\varphi$.

Lemma 5.1.1. Suppose each of $A$ and $B$ is a set, and $\varphi : A \to B$ is a function. Then $\varphi \pi_\varphi^{-1}$ is a bijection with respect to $\text{im}(\varphi)$.

Proof: By Theorem 2.4, $\pi_\varphi$ is a surjection with respect to $A/\varphi$, and by Theorem 2.6, $\pi_\varphi^{-1} \pi_\varphi \subseteq \varphi^{-1} \varphi$, so by Lemma 4.1.1, $\varphi \pi_\varphi^{-1}$ is a function.

$\varphi$ is a surjection with respect to $\text{im}(\varphi)$, so by Corollary 4.1.1, $\varphi \pi_\varphi^{-1}$ is a surjection with respect to $\text{im}(\varphi)$.

By Theorem 2.6, $\varphi^{-1} \varphi = \pi_\varphi^{-1} \pi_\varphi$, so by Corollary 4.1.2, $\varphi \pi_\varphi^{-1}$ is an injection.

Thus $\varphi \pi_\varphi^{-1}$ is a function which is both an injection and a surjection with respect to $\text{im}(\varphi)$, so $\varphi \pi_\varphi^{-1}$ is a bijection with respect to $\text{im}(\varphi)$. $\square$
Lemma 5.1.2. Suppose each of $M = (A, \mathcal{R})$ and $(B, \mathcal{S})$ is an $I$-structure, and $\varphi : A \to B$ is a function. Then $\pi_\varphi$ is a $I$-structure epimorphism with respect to the $I$-structure $M/\varphi = (A/\varphi, \mathcal{R})$.

Proof:

1. $\pi_\varphi$ is preservative: Suppose $r \in \mathcal{R}$, then $\pi_\varphi r \in \mathcal{R}$ by definition of $M/\varphi$.

2. $\pi_\varphi$ is saturating: Suppose $\bar{r} \in \mathcal{R}$ such that $\text{im}(\bar{r}) \subseteq \text{im}(\pi_\varphi)$, then by the definition of $M/\varphi$ there is an $r \in \mathcal{R}$ such that $\pi_\varphi r = \bar{r}$.

3. $\pi_\varphi$ is a surjection: By Theorem 2.4.

So $\pi_\varphi$ is an epimorphism.

Lemma 5.1.3. Suppose each of $M = (A, \mathcal{R})$ and $N = (B, \mathcal{S})$ is an $I$-structure, and $\varphi : A \to B$ is a function. Then $\varphi$ is an $I$-structure homomorphism from $M$ to $N$ if and only if $\varphi$ is an $I$-structure homomorphism from $A$ to the $I$-substructure of $M$ induced by $\text{im}(\varphi)$, $(\text{im}(\varphi), \hat{\mathcal{S}})$.

Proof: Suppose $\varphi$ is an $I$-structure homomorphism from $M$ to $N$.

Suppose $r \in \mathcal{R}$. $\varphi$ is preservative, so $\varphi r \in \mathcal{S}$.

Suppose $b \in \text{im}(\varphi r)$. Then there is an $i \in I$ such that $(i, b) \in \varphi r$. There is an $a \in A$ such that $\varphi(a) = b$ and $(i, a) \in r$. $b = \varphi(a) \in \text{im}(\varphi)$. So $\text{im}(\varphi r) \subseteq \text{im}(\varphi)$, and thus $\varphi r \in \hat{\mathcal{S}}$.

So $\varphi$ is preservative between $(A, \mathcal{R})$ and $(\text{im}(\varphi), \hat{\mathcal{S}})$.

Suppose $\hat{s} \in \hat{\mathcal{S}}$ such that $\text{im}(\hat{s}) \subseteq \text{im}(\varphi)$. $\varphi$ is saturating and $\hat{s} \in \mathcal{S}$, so there is an $r \in \mathcal{R}$ such that $\varphi r = \hat{s}$.
So \( \varphi \) is saturating between \((A, \mathcal{R})\) and \((\text{im}(\varphi), \hat{\mathcal{S}})\).

\( \varphi \) is both preservative and saturating between \((A, \mathcal{R})\) and \((\text{im}(\varphi), \hat{\mathcal{S}})\), so \( \varphi \) is an \( I \)-structure homomorphism between \((A, \mathcal{R})\) and \((\text{im}(\varphi), \hat{\mathcal{S}})\). \( \square \)

Suppose \( \varphi \) is an \( I \)-structure homomorphism from \( A \) to the \( I \)-substructure of \( M \) induced by \( \text{im}(\varphi) \).

Suppose \( r \in \mathcal{R} \). \( \varphi \) is preservative with respect to \((\text{im}(\varphi), \hat{\mathcal{S}})\), so \( \varphi r \in \hat{\mathcal{S}} \), so \( \varphi r \in \mathcal{S} \).

So \( \varphi \) is preservative between \( M \) and \( N \).

Suppose \( s \in \mathcal{S} \) such that \( \text{im}(s) \subseteq \text{im}(\varphi) \). Then \( s \in \hat{\mathcal{S}} \), and since \( \varphi \) is saturating with respect to \((\text{im}(\varphi), \hat{\mathcal{S}})\), there is an \( r \in \mathcal{R} \) such that \( \varphi r = s \).

So \( \varphi \) is saturating between \( M \) and \( N \).

\( \varphi \) is both preservative and saturating between \( M \) and \( N \), so \( \varphi \) is an \( I \)-structure homomorphism between \( M \) and \( N \). \( \square \)

**Theorem 5.1.** Suppose each of \( M = (A, \mathcal{R}) \) and \( N = (B, \mathcal{S}) \) is an \( I \)-structure and \( \varphi : A \rightarrow B \) is a function. Then \( \varphi \) is an \( I \)-structure homomorphism if and only if \( \varphi \pi_{\varphi}^{-1} \) is an isomorphism from \( M/\varphi \) to the \( I \)-substructure of \( N \) induced by \( \text{im}(\varphi) \).

**Proof:** Suppose \( \varphi \) is an \( I \)-structure homomorphism.

\( M/\varphi = (A/\varphi, \bar{\mathcal{R}}) \) where \( \bar{\mathcal{R}} \) is the set of relations to which a relation \( \bar{r} \) belongs if and only if there is a relation \( r \in \mathcal{R} \) such that \( \pi_{\varphi} r = \bar{r} \).
The $I$-substructure of $N$ induced by $\text{im}(\varphi)$ is $(\text{im}(\varphi), \hat{S})$ where $\hat{S}$ is the set of relations to which a relation $\hat{s}$ belongs if and only if $\hat{s} \in S$ and $\text{im}(\hat{s}) \subseteq \text{im}(\varphi)$.

By Lemma $5.1.2$, $\pi_{\varphi}$ is an $I$-structure epimorphism, by Lemma $5.1.3$ $\varphi$ is an $I$-structure homomorphism between $(A, \mathcal{R})$ and $(\text{im}(\varphi), \hat{S})$, and by Lemma $2.6\quad \pi_{\varphi}^{-1}\pi_{\varphi} \subseteq \varphi^{-1}\varphi$. So by Theorem $4.1$, $\varphi\pi_{\varphi}^{-1}$ is a homomorphism.

By Lemma $5.1.1\quad \varphi\pi_{\varphi}^{-1}$ is a bijection, so by Theorem $1.19\quad \varphi\pi_{\varphi}^{-1}$ is an isomorphism.

So $M/\varphi$ is isomorphic to the $I$-substructure of $N$ induced by $\text{im}(\varphi)$. \hfill \Box

Suppose $\varphi_{\varphi}^{-1}$ is an isomorphism from $M/\varphi$ to the $I$-substructure of $N$ induced by $\text{im}(\varphi)$.

By Lemma $5.1.2\quad \pi_{\varphi}$ is an epimorphism, and by assumption, $\varphi\pi_{\varphi}^{-1}$ is a homomorphism from $M/\varphi$ to the $I$-substructure of $N$ induced by $\text{im}(\varphi)$. So by Theorem $1.22\quad (\varphi\pi_{\varphi}^{-1})\pi_{\varphi}$ is a homomorphism from $M$ to the $I$-substructure of $N$ induced by $\text{im}(\varphi)$.

By Lemma $4.1.1\quad \varphi = (\varphi\pi_{\varphi}^{-1})\pi_{\varphi}$, so $\varphi$ is a homomorphism from $M$ to the $I$-substructure of $N$ induced by $\text{im}(\varphi)$, and thus by Lemma $5.1.3\quad \varphi$ is a homomorphism from $M$ to $N$. \hfill \Box

**Corollary 5.1.1.** Suppose each of $M = (A, \mathcal{R})$ and $(B, \mathcal{S})$ is a 1-structure, and $\varphi : A \to B$ is a homomorphism. Then if each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is the structurization of a topological space, then $(A/\varphi, \mathcal{R})$ is the structurization of a topological space.

**Proof:**

1. $1 \times A/\varphi \in \mathcal{R}$:
$(A, \mathcal{R})$ is the structurization of a topological space, so $1 \times A \in \mathcal{R}$.

$\pi_\varphi$ is an epimorphism, so $1 \times A/\varphi = 1 \times \pi_\varphi[A] = \pi_\varphi(1 \times A) \in \bar{\mathcal{R}}$.

2. $\emptyset \in \mathcal{R}$:

$(A, \mathcal{R})$ is the structurization of a topological space, so $\emptyset \in \mathcal{R}$.

$\pi_\varphi$ is a homomorphism, so $\emptyset = \pi_\varphi \emptyset \in \bar{\mathcal{R}}$.

3. Suppose $J$ is a set and for each $j \in J$, $\bar{r}_j \in \bar{\mathcal{R}}$, then there is an $\bar{r} \in \bar{\mathcal{R}}$ such that $\bar{r}[1] = \bigcup_{j \in J} \bar{r}_j[1]$:

For each $j \in J$, there is an $r_j \in \mathcal{R}$ such that $\pi_\varphi r_j = \bar{r}_j$. Since $(A, \mathcal{R})$ is the structurization of a topological space, there is an $r \in \mathcal{R}$ such that $r[1] = \bigcup_{j \in J} r_j[1]$. $\pi_\varphi r \in \bar{\mathcal{R}}$.

$$P \in \pi_\varphi r[1]$$
$$\iff P \in \pi_\varphi[r[1]]$$
$$\iff P \in \pi_\varphi\left(\bigcup_{j \in J} r_j[1]\right) = \bigcup_{j \in J} \pi_\varphi[r_j[1]] \text{ by Lemma 3.4.2}$$
$$\iff P \in \bigcup_{j \in J} \pi_\varphi r_j[1]$$
$$\iff P \in \bigcup_{j \in J} \bar{r}_j[1]$$

So $\pi_\varphi r \in \bar{\mathcal{R}}$ such that $\pi_\varphi r[1] = \bigcup_{j \in J} \bar{r}_j[1]$.

4. Suppose each of $\bar{r}_1$ and $\bar{r}_2$ is in $\bar{\mathcal{R}}$, then there is an $\bar{r} \in \bar{\mathcal{R}}$ such that $\bar{r}[1] = \bar{r}_1[1] \cap \bar{r}_2[1]$:

By the definition of $\bar{\mathcal{R}}$, there is an $r_1 \in \mathcal{R}$ and an $r_2 \in \mathcal{R}$ such that $\pi_\varphi r_1 = \bar{r}_1$ and $\pi_\varphi r_2 = \bar{r}_2$. 54
Since $\varphi$ is a homomorphism, $\varphi r_1 \in S$ and $\varphi r_2 \in S$, and since $(B, S)$ is the structurization of a topological space, there is an $s \in S$ such that $s[1] = \varphi r_1[1] \cap \varphi r_2[1] \subseteq \text{im}(\varphi)$. So there is an $r \in R$ such that $\varphi r = s.$

$$\pi_{\varphi r}[1] = \pi_{\varphi}[r[1]] = \{\pi_{\varphi}(a) \mid a \in r[1]\}$$

$$= \{\varphi^{-1}[\varphi\{a\}] \mid a \in r[1]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r[1]]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[s\{0\}]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r_1[1] \cap \varphi r_2[1]]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r_1[1]] \cap \varphi r_2[1]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r_1[1] \cap \varphi r_2[1]]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r_1[1]] \cap \varphi^{-1}[\varphi r_2[1]]\}$$

$$= \{\varphi^{-1}[\varphi\{a\}] \mid a \in r_1[1]\} \cap \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r_2[1]]\}$$

$$= \{\pi_{\varphi}(a) \mid a \in r_1[1]\} \cap \{\pi_{\varphi}(a) \mid a \in \varphi^{-1}[\varphi r_2[1]]\}$$

$$= \pi_{\varphi r_1[1]} \cap \pi_{\varphi r_2[1]}$$

$$= \pi_{\varphi r_1[1]} \cap \pi_{\varphi r_2[1]}$$

$$= \pi_{\varphi r_1[1]} \cap \pi_{\varphi r_2[1]}$$

$$= \bar{r}_1[1] \cap \bar{r}_2[1]$$

So $\pi_{\varphi r} \in \bar{R}$ such that $\pi_{\varphi r}[1] = \bar{r}_1[1] \cap \bar{r}_2[1]$.

So by the above properties, $(A/\varphi, \bar{R})$ is the structurization of a topological space. 

\textbf{Corollary 5.1.2.} Suppose $(G, *)$ is a group with identity $e$, $I = \{p_e, p_{-1}, 0_{-1}, p_*, 0_*, 1_*\}$, $M = (G, \mathcal{R})$ is the structurization of $(G, *)$, and $\varphi$ is a function with domain $G$. Then $M/\varphi$ is the structurization of a group under the operation induced by $*$ if and only if for all $(a_1, a_2) \in \varphi^{-1}\varphi$, $(a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$, and for all $(a_1, b_1), (a_2, b_2)$ in $\varphi^{-1}\varphi$, $(a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi$.

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Proof: Suppose for all \((a_1, a_2) \in \varphi^{-1}\varphi, (a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi\), and for all \((a_1, b_1), (a_2, b_2)\) in \(\varphi^{-1}\varphi, (a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi\).

Suppose each of \(P = \varphi^{-1}\varphi(\{x\})\) and \(Q = \varphi^{-1}\varphi(\{y\})\) is in \(G/\varphi\).

\(P \ast Q = \{p \ast q \mid p \in P \text{ and } q \in Q\}\). Consider the element of \(G/\varphi, \varphi^{-1}\varphi(\{x \ast y\})\).

\[g \in P \ast Q\]
\[\implies \exists p \in P, q \in Q \text{ such that } g = p \ast q\]
\[\implies \exists p \in \varphi^{-1}\varphi(\{x\}), q \in \varphi^{-1}\varphi(\{y\}) \text{ such that } g = p \ast q\]
\[\implies \exists p, q \text{ such that } (p, x) \in \varphi^{-1}\varphi \text{ and } (q, y) \in \varphi^{-1}\varphi \text{ and } g = p \ast q\]
\[\implies \exists p, q \text{ such that } (p \ast q, x \ast y) \in \varphi^{-1}\varphi \text{ and } g = p \ast q\]
\[\implies g \in \varphi^{-1}\varphi(\{x \ast y\})\]

So \(P \ast Q \subseteq \varphi^{-1}\varphi(\{x \ast y\})\).

\[g \in \varphi^{-1}\varphi(\{x \ast y\})\]
\[\implies (g, x \ast y) \in \varphi^{-1}\varphi \text{ and } (y^{-1}, y^{-1}) \in \varphi^{-1}\varphi\]
\[\implies (g \ast y^{-1}, x) = (g \ast y^{-1}, x \ast e) = (g \ast y^{-1}, x \ast (y \ast y^{-1})) = (g \ast y^{-1}, (x \ast y) \ast y^{-1}) \in \varphi^{-1}\varphi\]
\[\implies g \ast y^{-1} \in \varphi^{-1}\varphi(\{x\}) = P \text{ and } y \in \varphi^{-1}\varphi(\{y\}) = Q\]
\[\implies g = g \ast e = g \ast (y^{-1} \ast y) = (g \ast y^{-1}) \ast y \in P \ast Q\]

So \(\varphi^{-1}\varphi(\{x \ast y\}) \subseteq P \ast Q\).

Thus \(\varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{y\}) = P \ast Q = \varphi^{-1}\varphi(\{x \ast y\}) \in G/\varphi\).

So \(G/\varphi\) is closed under the operation \(\ast\).
Consider the element \( \varphi^{-1}\varphi(\{e\}) \).

By the above, if \( P = \varphi^{-1}\varphi(\{x\}) \in G/\varphi \), then:

\[
P \ast \varphi^{-1}\varphi(\{e\}) = \varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{e\}) = \varphi^{-1}\varphi(\{x \ast e\}) = \varphi^{-1}\varphi(\{x\}) = P
\]

And

\[
\varphi^{-1}\varphi(\{e\}) \ast P = \varphi^{-1}\varphi(\{e\}) \ast \varphi^{-1}\varphi(\{x\}) = \varphi^{-1}\varphi(\{e \ast x\}) = \varphi^{-1}\varphi(\{x\}) = P
\]

So \( \varphi^{-1}\varphi(\{e\}) \) is an identity in \( G/\varphi \) with respect to the operation \( \ast \).

Suppose \( P = \varphi^{-1}\varphi(\{x\}) \in G/\varphi \). Consider \( \varphi^{-1}\varphi(\{x^{-1}\}) \).

\[
P \ast \varphi^{-1}\varphi(\{x^{-1}\}) = \varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{x^{-1}\}) = \varphi^{-1}\varphi(\{x \ast x^{-1}\}) = \varphi^{-1}\varphi(\{e\})
\]

And

\[
\varphi^{-1}\varphi(\{x^{-1}\}) \ast P = \varphi^{-1}\varphi(\{x^{-1}\}) \ast \varphi^{-1}\varphi(\{x\}) = \varphi^{-1}\varphi(\{x^{-1} \ast x\}) = \varphi^{-1}\varphi(\{e\})
\]

So \( \varphi^{-1}\varphi(\{x^{-1}\}) \) is an inverse for \( P \) with respect to the identity \( \varphi^{-1}\varphi(\{e\}) \).

Suppose each of \( P = \varphi^{-1}\varphi(\{x\}) \), \( Q = \varphi^{-1}\varphi(\{y\}) \), and \( R = \varphi^{-1}\varphi(\{z\}) \) are members of
\[ G/\varphi. \]

\[(P \ast Q) \ast R = (\varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{y\})) \ast \varphi^{-1}\varphi(\{z\}) = \varphi^{-1}\varphi(\{x \ast y\}) \ast \varphi^{-1}\varphi(\{z\})\]

\[= \varphi^{-1}\varphi(\{(x \ast y) \ast z\}) = \varphi^{-1}\varphi(\{x \ast (y \ast z)\}) = \varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{y \ast z\})\]

\[= \varphi^{-1}\varphi(\{x\}) \ast (\varphi^{-1}\varphi(\{y\}) \ast \varphi^{-1}\varphi(\{z\})) = P \ast (Q \ast R)\]

So \(G/\varphi\) is associative with respect to the operation \(\ast\).

So \((G/\varphi, \ast)\) is a group. (Note that saying for all \((a_1, a_2) \in \varphi^{-1}\varphi, (a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi\), and for all \((a_1, b_1), (a_2, b_2)\) in \(\varphi^{-1}\varphi, (a_1 \ast a_2, b_1 \ast b_2) \in \varphi^{-1}\varphi\) implies that \(\varphi^{-1}\varphi(\{e\})\) is a normal subgroup of \(G\).

Now I intend to prove that \(M/\varphi\) is the structurization of said group.

\[ M/\varphi = (G/\varphi, \mathring{R}) \text{ where } I = \{p_e, p_{-1}, 0_{-1}, p_s, 0_s, 1_s\} \text{ and } \mathring{R} = \{\pi r \mid r \in R\}. \]

The structurization of \((G/\varphi, \ast) = (G/\varphi, S)\) where \(S\) is the set of relations to which a relation \(s\) belongs if and only if either \(s = \{(p_e, \varphi^{-1}\varphi(e))\}\), there is a \(P \in G/\varphi\) such that \(s = \{(p_{-1}, P^{-1}), (0_{-1}, P)\}\), or there is a \(P\) and a \(Q\) each of which is in \(G/\varphi\) such that \(s = \{(p_s, P \ast Q), (0_s, P), (1_s, Q)\}\).

Since \(R\) is the relation set for the structurization of a group, for each \(r \in R\) either \(r = \{(p_e, e)\}\), there is an \(x \in G\) such that \(r = \{(p_{-1}, x^{-1}), (0_{-1}, x)\}\), or there is an \(x\)
and a $y$ in $G$ such that $r = \{(p_*, x*y), (0_*, x), (1_*, y)\}$.

$t \in \mathcal{R}$

$\iff \exists r \in \mathcal{R}$ such that $t = \pi r$

$\iff t = \pi r$ where

$r = \{(p_e, e)\}$

or $r = \{(p_{-1}, x^{-1}), (0_{-1}, x)\}$ for some $x \in G$

or $r = \{(p_*, x*y), (0_*, x), (1_*, y)\}$ for some $x, y \in G$

$\iff t = \{(p_e, \pi(e))\}$

or $t = \{(p_{-1}, \pi(x^{-1})), (0_{-1}, \pi(x))\}$ for some $x \in G$

or $t = \{(p_*, \pi(x*y)), (0_*, \pi(x)), (1_*, \pi(y))\}$ for some $x, y \in G$

$\iff t = \{(p_e, \varphi^{-1}\varphi(\{e\}))\}$

or $t = \{(p_{-1}, \varphi^{-1}\varphi(\{x^{-1}\})), (0_{-1}, \varphi^{-1}\varphi(\{x\}))\}$ for some $x \in G$

or $t = \{(p_*, \varphi^{-1}\varphi(\{x*y\})), (0_*, \varphi^{-1}\varphi(\{x\})), (1_*, \varphi^{-1}\varphi(\{y\}))\}$ for some $x, y \in G$

$\iff t = \{(p_e, \varphi^{-1}\varphi(\{e\}))\}$

or $t = \{(p_{-1}, (\varphi^{-1}\varphi(\{x\}))^{-1}), (0_{-1}, \varphi^{-1}\varphi(\{x\}))\}$ for some $x \in G$

or $t = \{(p_*, \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{y\})), (0_*, \varphi^{-1}\varphi(\{x\})), (1_*, \varphi^{-1}\varphi(\{y\}))\}$ for some $x, y \in G$

$\iff t = \{(p_e, \varphi^{-1}\varphi(\{e\}))\}$

or $t = \{(p_{-1}, P^{-1}), (0_{-1}, P)\}$ for some $P \in G/\varphi$

or $t = \{(p_*, P*Q), (0_*, P), (1_*, Q)\}$ for some $P, Q \in G/\varphi$

$\iff t \in \mathcal{S}$

So $\mathcal{R} = \mathcal{S}$, and thus $M/\varphi$ is the structurization of $(G/\varphi,*).$

Suppose $M/\varphi$ is the structurization of a group under the operation induced by $*$. 
Consider $\varphi^{-1}\varphi(\{e\})$. Suppose $P \in G/\varphi$. There is an $x \in G$ such that $P = \varphi^{-1}\varphi(x)$.

$$x \in \varphi^{-1}\varphi(x) \text{ and } x = e \ast x \in \varphi^{-1}\varphi(\{e\}) \ast \varphi^{-1}\varphi(\{x\}) = \varphi^{-1}\varphi(\{e\}) \ast P \in G/\varphi$$

$$\implies \varphi^{-1}\varphi(\{e\}) \ast P = \varphi^{-1}\varphi(\{x\}) = P$$

$$\implies \varphi^{-1}\varphi(\{e\}) \text{ is the identity element for } M/\varphi$$

Suppose $P \in G/\varphi$. There is an $x \in G$ such that $P = \varphi^{-1}\varphi(\{x\})$. Consider $\varphi^{-1}\varphi(\{x^{-1}\})$

$$e \in \varphi^{-1}\varphi(\{e\}) \text{ and } e = x \ast x^{-1} \in \varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{x^{-1}\}) = P \ast \varphi^{-1}\varphi(\{x^{-1}\}) \in G/\varphi$$

$$\implies P \ast \varphi^{-1}\varphi(\{x^{-1}\}) = \varphi^{-1}\varphi(\{e\}) \text{ (elements of a partition intersect if and only if they are equal)}$$

$$\implies P^{-1} = \varphi^{-1}\varphi(\{x^{-1}\})$$

Suppose each of $P$ and $Q$ is in $G/\varphi$. There is an $x$ and a $y$ in $G$ such that $P = \varphi^{-1}\varphi(\{x\})$ and $Q = \varphi^{-1}\varphi(\{y\})$. Consider $\varphi^{-1}\varphi(\{x \ast y\})$.

$$x \ast y \in \varphi^{-1}\varphi(\{x \ast y\}) \text{ and } x \ast y \in \varphi^{-1}\varphi(\{x\}) \ast \varphi^{-1}\varphi(\{y\}) = P \ast Q$$

$$\implies P \ast Q = \varphi^{-1}\varphi(\{x \ast y\})$$

Now the main result follows.

$$(a_1, a_2) \in \varphi^{-1}\varphi$$

$$\iff \varphi^{-1}\varphi(\{a_1\}) = \varphi^{-1}\varphi(\{a_2\})$$

$$\iff (\varphi^{-1}\varphi(\{a_1\}))^{-1} = (\varphi^{-1}\varphi(\{a_2\}))^{-1}$$

$$\iff \varphi^{-1}\varphi(\{a_1^{-1}\}) = \varphi^{-1}\varphi(\{a_2^{-1}\})$$

$$\iff (a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$$


\[(a_1, b_1) \in \varphi^{-1}\varphi \text{ and } (a_2, b_2) \in \varphi^{-1}\varphi\]

\[\implies \varphi^{-1}\varphi(\{a_1\}) = \varphi^{-1}\varphi(\{b_1\}) \text{ and } \varphi^{-1}\varphi(\{a_2\}) = \varphi^{-1}\varphi(\{b_2\})\]

\[\implies \varphi^{-1}\varphi(\{a_1\}) \ast \varphi^{-1}\varphi(\{a_2\}) = \varphi^{-1}\varphi(\{b_1\}) \ast \varphi^{-1}\varphi(\{b_2\})\]

\[\implies \varphi^{-1}\varphi(\{a_1 \ast a_2\}) = \varphi^{-1}\varphi(\{b_1 \ast b_2\})\]

\[\implies (a_1 \ast a_2, b_1 \ast b_2) \in \varphi^{-1}\varphi\]

Thus for all \((a_1, a_2) \in \varphi^{-1}\varphi, (a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi,\) and for all \((a_1, b_1), (a_2, b_2) \in \varphi^{-1}\varphi, (a_1 \ast a_2, b_1 \ast b_2) \in \varphi^{-1}\varphi.\]

**Definition** Suppose \(M = (A, \mathcal{R})\) is an \(I\)-structure and \(B \subseteq A\). The \(I\)-understructure of \(M\) induced by \(B\) is the \(I\)-structure \((B, \hat{\mathcal{R}})\) where \(\hat{\mathcal{R}}\) is the set of relations to which a relation \(\hat{r}\) belongs if and only if there is a relation \(r \in \mathcal{R}\) such that \(\hat{r} = r \cap (I \times B)\). Denote the structure \((B, \hat{\mathcal{R}})\) by \(M \| B\). The statement that \((C, \mathcal{T})\) is an \(I\)-understructure of \(M\) means \(C \subseteq A\) and \((C, \mathcal{T})\) is the \(I\)-understructure of \(M\) induced by \(C\).

**Definition** Suppose \(M = (A, \mathcal{R})\) is an \(I\)-structure. Suppose \(\overline{\mathcal{R}}\) is the set of relations to which a relation \(\overline{r}\) belongs if and only if \(\overline{r} \subseteq I \times A/\varphi\) and \(\pi_\varphi^{-1}\overline{r} \in \mathcal{R}\). Denote the \(I\)-structure \((A/\varphi, \overline{\mathcal{R}})\) by \(M/\varphi\).

**Lemma 5.2.1.** Suppose each of \(M = (A, \mathcal{R})\) and \((B, \mathcal{S})\) is an \(I\)-structure, and \(\varphi : A \to B\) is a cohomomorphism. Then \(\pi_\varphi\) is an \(I\)-structure coepimorphism with respect to the \(I\)-structure \(M/\varphi = (A/\varphi, \overline{\mathcal{R}})\).

**Proof:**

1. \(\pi_\varphi\) is continuous: Suppose \(\overline{r} \in \overline{\mathcal{R}}\), then \(\pi_\varphi^{-1}\overline{r} \in \mathcal{R}\) by definition of \(M/\varphi\). So \(\pi_\varphi\) is continuous.

2. \(\pi_\varphi\) is conservative:
Suppose \( r \in R \). \( \varphi \) is conservative, so there is an \( s \in S \) such that \( \text{im}(s) \subseteq \text{im}(\varphi) \) and \( \varphi^{-1}s = r \). So \( s = \varphi r \) and \( \pi_\varphi^{-1} \pi_\varphi r = \varphi^{-1} \varphi r = \varphi^{-1}s = r \).

Thus \( \pi_\varphi r \in R \), \( \text{im}(\pi_\varphi r) \subseteq \text{im}(\pi_\varphi) \), and \( \pi_\varphi^{-1}(\pi_\varphi r) = r \).

So \( \pi_\varphi \) is conservative.

3. \( \pi_\varphi \) is a surjection with respect to \( A/\varphi \): By Theorem 2.4

So \( \pi_\varphi \) is an coepimorphism with respect to \( M/\varphi \). \( \square \)

**Lemma 5.2.2.** Suppose each of \( M = (A, R) \) and \( N = (B, S) \) is an I-structure, and \( \varphi : A \rightarrow B \) is an I-structure cohomomorphism. Then \( \varphi \) is a cohomomorphism from \( A \) to the understructure of \( M \) induced by \( \text{im}(\varphi) \), \( (\text{im}(\varphi), \hat{S}) \).

**Proof:** Suppose \( \hat{s} \in \hat{S} \). Then there is an \( s \in S \) such that \( \hat{s} = s \cap (I \times \text{im}(\varphi)) \). \( \varphi \) is continuous, so \( \varphi^{-1}s \in R \). I intend to show that \( \varphi^{-1}\hat{s} = \varphi^{-1}s \).

Suppose \( (i, a) \in \varphi^{-1}\hat{s} \). Then there is a \( b \in \text{im}(\varphi) \) such that \( \varphi(a) = b \) and \( (i, b) \in \hat{s} \). \( \hat{s} \subseteq s \), so \( (i, b) \in s \), and thus \( (i, a) \in \varphi^{-1}s \).

So \( \varphi^{-1}\hat{s} \subseteq \varphi^{-1}s \).

Suppose \( (i, a) \in \varphi^{-1}s \). Then there is an \( b \in B \) such that \( b = \varphi(a) \in \text{im}(\varphi) \) and \( (i, b) \in s \). \( (i, b) \in I \times \text{im}(\varphi) \), so \( (i, b) \in \hat{s} \), and thus \( (i, a) \in \varphi^{-1}\hat{s} \).

So \( \varphi^{-1}s \subseteq \varphi^{-1}\hat{s} \).

So \( \varphi^{-1}\hat{s} = \varphi^{-1}s \in R \).
So $\varphi$ is continuous between $(A, R)$ and $(\text{im}(\varphi), \hat{S})$.

Suppose $r \in R$. $\varphi$ is conservative, so there is an $s \in S$ such that $\text{im}(s) \subseteq \text{im}(\varphi)$ and $\varphi^{-1}s = r$. Since $\text{im}(s) \subseteq \text{im}(\varphi)$, $s \cap (I \times \text{im}(\varphi)) = s$, so $s \in \hat{S}$.

So $s \in \hat{S}$, $\text{im}(s) \subseteq \text{im}(\varphi)$, and $\varphi^{-1}s = r$.

So $\varphi$ is conservative between $(A, R)$ and $(\text{im}(\varphi), \hat{S})$.

$\varphi$ is both continuous and conservative between $(A, R)$ and $(\text{im}(\varphi), \hat{S})$, so $\varphi$ is an $I$-structure cohomomorphism between $(A, R)$ and $(\text{im}(\varphi), \hat{S})$. \hfill \Box

**Theorem 5.2.** Suppose each of $M = (A, R)$ and $N = (B, S)$ is an $I$-structure, and $\varphi : A \rightarrow B$ is a cohomomorphism. Then $M/\!/\varphi$ is isomorphic to the understructure of $N$ induced by $\text{im}(\varphi)$.

**Proof:** $M/\!/\varphi = (A/\varphi, R)$ where $R$ is the set of relations to which a relation $\bar{r}$ belongs if and only if $\bar{r} \subseteq I \times A/\varphi$ and $\pi_{\varphi}^{-1}\bar{r} \in R$.

The understructure of $N$ induced by $\text{im}(\varphi)$ is $(\text{im}(\varphi), \hat{S})$ where $\hat{S}$ is the set of relations to which a relation $\hat{s}$ belongs if and only if there is a relation $s \in S$ such that $\hat{s} = s \cap (I \times \text{im}(\varphi))$.

By Lemma 5.2.1 $\pi_{\varphi}$ is an $I$-structure coepimorphism, by Lemma 5.2.2 $\varphi$ is an $I$-structure cohomomorphism between $(A, R)$ and $(\text{im}(\varphi), \hat{S})$, and by Lemma 2.6 $\pi_{\varphi}^{-1}\varphi \subseteq \varphi^{-1}\varphi$. So by Theorem 4.2 $\varphi\pi_{\varphi}^{-1}$ is a cohomomorphism.

By Lemma 5.1.1 $\varphi\pi_{\varphi}^{-1}$ is a bijection, so by Theorem 1.21 $\varphi\pi_{\varphi}^{-1}$ is an isomorphism.

So $M/\!/\varphi$ is isomorphic to the understructure of $N$ induced by $\text{im}(\varphi)$. \hfill \Box
Corollary 5.2.1. Suppose $M = (A, \mathcal{R})$ is an $I$-structure and $\varphi$ is a cohomomorphism between $M$ and another structure. Then $M/\parallel \varphi$ is isomorphic to $M/\varphi$.

Proof: By Theorem 1.20 $\varphi$ is a homomorphism.

By Lemma 5.2.1 $\pi_\varphi$ is a coepimorphism between $M$ and $M/\parallel \varphi$ and (by Lemma 5.1.2) an epimorphism between $M$ and $M/\varphi$. Moreover, $\pi_\varphi^{-1} \pi_\varphi \subseteq \pi_\varphi^{-1} \pi_\varphi$.

So by Theorem 4.1, $\pi_\varphi \pi_\varphi^{-1}$ is a homomorphism between $M/\parallel \varphi$ and $M/\varphi$.

$\pi_\varphi$ is a surjection, so $\pi_\varphi \pi_\varphi^{-1} = 1_{A/\varphi}$, which is a bijection. So by Theorem 1.19 $1_{A/\varphi}$ is an isomorphism.

So $M/\parallel \varphi \cong M/\varphi$.  \qed
Chapter 6
Second Isomorphism Theorem

**Lemma 6.1.1.** Suppose $A$ is a set, $B \subseteq A$, $\varphi$ is a function with domain $A$, $\tilde{B} = \varphi^{-1}\varphi(B)$, and $P \in \tilde{B}/\varphi|_{\tilde{B}}$. Then $P \cap B \in B/\varphi|_{B}$.

**Proof:** Suppose $b \in P \cap B$.

Since $P \in \tilde{B}/\varphi|_{\tilde{B}}$ and $b \in P$, $P = (\varphi|_{B})^{-1}((\varphi|_{B})(\{b\}))$.

$b \in B$ so $b \in (\varphi|_{B})^{-1}((\varphi|_{B})(\{b\})) \in B/\varphi|_{B}$. So $P \cap B \subseteq (\varphi|_{B})^{-1}((\varphi|_{B})(\{b\})) \in B/\varphi|_{B}$.

Suppose $b' \in (\varphi|_{B})^{-1}((\varphi|_{B})(\{b\}))$. Then there is a $c$ such that $(c, b') \in (\varphi|_{B})^{-1}$ and $(b, c) \in \varphi|_{B}$, so $(b', c) \in \varphi|_{B}$, so $(b, c) \in \varphi$, $(b', c) \in \varphi$ and $b' \in B \subseteq \tilde{B}$.

$(b, c) \in \varphi$ and $b \in \tilde{B}$, so $(b, c) \in \varphi|_{\tilde{B}}$. $(b', c) \in \varphi$ and $b' \in \tilde{B}$, so $(b', c) \in \varphi|_{\tilde{B}}$. So $b' \in (\varphi|_{B})^{-1}((\varphi|_{B})(\{b\})) = P$.

So $b' \in P \cap B$ and $P \cap B = (\varphi|_{B})^{-1}((\varphi|_{B})(\{b\})) \in B/\varphi|_{B}$.

**Lemma 6.1.2.** Suppose $A$ is a set, $B \subseteq A$, $\varphi$ is a function with domain $A$, $\tilde{B} = \varphi^{-1}\varphi(B)$. Then the function $\psi : \tilde{B}/\varphi|_{\tilde{B}} \rightarrow B/\varphi|_{B}$ such that for each $P \in \tilde{B}/\varphi|_{\tilde{B}}$, $\psi(P) = P \cap B$, is a bijection.

**Proof:** Suppose each of $P$ and $Q$ is in $\tilde{B}/\varphi|_{\tilde{B}}$, and $\psi(P) = \psi(Q)$.

$P \cap B = \psi(P) = \psi(Q) = Q \cap B$. 

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Since each of $P \cap B$ and $Q \cap B$ is in $B/\varphi_B$, each is nonempty. So there is a $b \in P \cap B = Q \cap B$. $b \in P \cap B \subseteq P$ and $b \in Q \cap B \subseteq Q$. $P$ and $Q$ intersect, and $\bar{B}/\varphi_B$ is a partition, so $P = Q$.

So $\psi$ is an injection.

Suppose $H \in B/\varphi_B$. $H$ is nonempty, so there is a $b \in H$ and $H = (\varphi_B)^{-1}((\varphi_B)(\{b\}))$.

$(b, \varphi(b)) \in \varphi$ and $b \in B \subseteq \bar{B}$, so $b \in (\varphi_B)^{-1}((\varphi_B)(\{b\})) \in B/\varphi_B$.

$\psi((\varphi_B)^{-1}((\varphi_B)(\{b\}))) = (\varphi_B)^{-1}((\varphi_B)(\{b\})) \cap B$.

$b \in H = (\varphi_B)^{-1}((\varphi_B)(\{b\})) \subseteq B$ and $b \in (\varphi_B)^{-1}((\varphi_B)(\{b\}))$, so $b \in (\varphi_B)^{-1}((\varphi_B)(\{b\})) \cap B = \psi((\varphi_B)^{-1}((\varphi_B)(\{b\})))$

Since $B/\varphi_B$ is a partition, and $p \in H \in B/\varphi_B$ and $b \in \psi((\varphi_B)^{-1}((\varphi_B)(\{b\}))) \in B/\varphi_B$, it must be the case that $H = \psi((\varphi_B)^{-1}((\varphi_B)(\{b\})))$.

So $(\varphi_B)^{-1}((\varphi_B)(\{b\}))$ is an element of $\bar{B}/\varphi_B$ such that $\psi((\varphi_B)^{-1}((\varphi_B)(\{b\}))) = H$.

So $\psi$ is a surjection.

So $\psi$ is a bijection. \qed

**Theorem 6.1.** Suppose $M = (A, \mathcal{R})$ is an $I$-structure, $B \subseteq A$, $\varphi$ is a function with domain $A$, $\bar{B} = \varphi^{-1}\varphi(B)$, $M|\bar{B}/\varphi_B = (\bar{B}/\varphi_B, \mathcal{S})$, $\psi : \bar{B}/\varphi_B \to B/\varphi_B$ is the function such that for each $P \in \bar{B}/\varphi_B$, $\psi(P) = P \cap B$, and $N = (B/\varphi_B, \mathcal{T})$ where $\mathcal{T}$ is the set of relations to
which a relation $t$ belongs if and only if there is a relation $s \in S$ such that $\psi s = t$. Then $M|_{B/\varphi|_B} \cong N$.

Proof:

1. $\psi$ is preservative: By definition of $N$, if $s \in S$, then $\psi s \in T$.

2. $\psi$ is saturating: By definition of $N$, if $t \in T$, then there is a relation $s \in S$ such that $\psi s = t$.

3. $\psi$ is a bijection: By Lemma 6.1.2, $\psi$ is a bijection.

So by Theorem 1.19, $\psi$ is an isomorphism.

So $M|_{B/\varphi|_B} \cong N$.  

$\square$
Chapter 7
Third Isomorphism Theorem

Definition Suppose \( A \) is a set, and each of \( \alpha \) and \( \beta \) is a function with domain \( A \) such that \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \). Then define \( \beta/\alpha : A/\alpha \rightarrow A/\beta \) such that if \( P \in A/\alpha \), then \( \beta/\alpha(P) = \pi_\beta \pi_\alpha^{-1} \).

Lemma 7.1.1. Suppose \( M = (A, R) \) is an \( I \)-structure, and each of \( \alpha \) and \( \beta \) is a homomorphism with domain \( A \) such that \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \). Then \( \beta/\alpha \) is an epimorphism between \( M/\alpha \) and \( M/\beta \).

Proof: By Lemma 5.1.2, \( \pi_\alpha \) is an epimorphism.

By Lemma 5.1.2 \( \pi_\beta \) is an epimorphism.

Suppose \( P \in A/\alpha \).

\[
\pi_\beta \pi_\alpha^{-1}(P) = \pi_\beta \pi_\alpha^{-1}(\pi_\alpha(P)) = \pi_\beta \pi_\alpha^{-1}(\pi_\alpha(P)) = \pi_\beta(P) = \beta^{-1}(\beta(P)) = \beta/\alpha(P)
\]

So \( \beta/\alpha = \pi_\beta \pi_\alpha^{-1} \).

So by Theorem 4.1 and Lemma 4.1.1, \( \beta/\alpha = \pi_\beta \pi_\alpha^{-1} \) is an epimorphism between \( M/\alpha \) and \( M/\beta \).

Lemma 7.1.2. Suppose \( A \) is a set, and each of \( \alpha \) and \( \beta \) is a function with domain \( A \) such that \( \alpha^{-1}\alpha = \beta^{-1}\beta \). Then \( A/\alpha = A/\beta \) and \( \beta/\alpha = 1_{A/\alpha} \).
Proof:

\[ A/\alpha = \{ \alpha^{-1}(\{a\}) \mid a \in A \} = \{ \alpha^{-1}\alpha(a) \mid a \in A \} = \{ \beta^{-1}\beta(\{a\}) \mid a \in A \} = A/\beta \]

So \( A/\alpha = A/\beta \).

Suppose \( P \in A/\alpha \).

\[ \beta/\alpha(P) = \beta^{-1}(\beta(P)) = \beta^{-1}\beta(P) = \alpha^{-1}\alpha(P) = \alpha^{-1}(\alpha(P)) = P \]

So \( \beta/\alpha = 1_{A/\alpha} \). □

**Lemma 7.1.3.** Suppose each of \( A \) and \( B \) is a set, and each of \( \alpha \) and \( \beta : A \to B \) is a function with domain \( A \) such that \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \) and \( \gamma \) is a function with domain \( B \). Then \( \alpha^{-1}\alpha \subseteq (\gamma\beta)^{-1}\gamma\beta \).

**Proof:**

\[ \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \subseteq \beta^{-1}1_B\beta \subseteq \beta^{-1}\gamma^{-1}\gamma\beta = (\gamma\beta)^{-1}\gamma\beta \]

So \( \alpha^{-1}\alpha \subseteq (\gamma\beta)^{-1}\gamma\beta \). □

**Lemma 7.1.4.** Suppose \( A \) is a set, and each of \( \mathcal{P} \) and \( \mathcal{Q} \) is a partition, and \( \gamma : \mathcal{P} \to \mathcal{Q} \) is a surjection such that for each \( P \in \mathcal{P}, P \subseteq \gamma(P) \). Suppose \( \pi_1 : A \to \mathcal{P} \) is the function such that for each \( a \in A, \pi_1(a) \) is the part in \( \mathcal{P} \) to which \( a \) belongs, and \( \pi_2 : A \to \mathcal{Q} \) is the function such that for each \( a \in A, \pi_2(a) \) is the part in \( \mathcal{Q} \) to which \( a \) belongs. Then \( \pi_2 = \gamma\pi_1 \).

**Proof:** Suppose \( a \in A \).

\[ a \in \pi_1(a) \subseteq \gamma(\pi_1(a)) = \gamma\pi_1(a) \]
So $\gamma \pi_1(a)$ is the part in $Q$ to which $a$ belongs. So $\gamma \pi_1(a) = \pi_2(a)$.

Since this is true for all $a \in A$, $\pi_2 = \gamma \pi_1$.

**Lemma 7.1.5.** Suppose each of $M = (A, R)$, $N = (B, S)$, and $L = (P, T)$ is an $I$-structure, where $P$ is a partition of $A$, and $\alpha : A \to B$ is a homomorphism, and $\gamma : A/\alpha \to P$ is an epimorphism between $M/\alpha$ and $L$ such that for all $P \in A/\alpha$, $P \subseteq \gamma(P)$. Then there is a homomorphism $\beta$ from $M$ such that $\alpha^{-1} \alpha \subseteq \beta^{-1} \beta$ and $\gamma = \beta/\alpha$.

**Proof:** Define $\beta : A \to P$ such that $\beta = \gamma \pi_\alpha$.

Since $\alpha$ is a homomorphism, $\pi_\alpha$ is an epimorphism. $\gamma$ is a homomorphism. $\beta = \gamma \pi_\alpha$ is the composition of a homomorphism with an epimorphism, so $\beta$ is a homomorphism.

Suppose $(a_1, a_2) \in \alpha^{-1} \alpha$. By Lemma 2.6, $\alpha^{-1} \alpha = \pi_\alpha^{-1} \pi_\alpha$.

$$
(a_1, a_2) \in \alpha^{-1} \alpha = \pi_\alpha^{-1} \pi_\alpha = \pi_\alpha^{-1} A/\alpha \pi_\alpha \subseteq \pi_\alpha^{-1} \gamma \pi_\alpha = (\gamma \pi_\alpha)^{-1} \gamma \pi_\alpha = \beta^{-1} \beta
$$

So $\alpha^{-1} \alpha \subseteq \beta^{-1} \beta$.

Define $\pi : A \to P$ is the function such that for each $a \in A$, $\pi(a)$ is the part in $P$ to which $a$ belongs.

By Lemma 7.1.4, $\beta = \gamma \pi_\alpha = \pi$.

By Theorem 2.8, $\pi = \pi_\beta$.

So $P = A/\beta$. 

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\[ \pi_\alpha^{-1}\pi_\alpha = \alpha^{-1}\alpha \subseteq \beta^{-1}\beta = \pi_\beta^{-1}\pi_\beta. \]

So \( \beta/\alpha = \pi_\beta\pi_\alpha^{-1} \) is the unique homomorphism such that \( (\pi_\beta\pi_\alpha^{-1})\pi_\alpha = \pi_\beta = \beta. \)

\[ \gamma\pi_\alpha = \beta, \text{ so } \gamma = \pi_\beta\pi_\alpha^{-1} = \beta/\alpha. \]

So \( \beta \) is a homomorphism from \( M \) such that \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \) and \( \gamma = \beta/\alpha. \)

**Theorem 7.1.** Suppose \( M = (A, \mathcal{R}) \) is an I-structure, and each of \( \alpha \) and \( \beta \) is a homomorphism with domain \( A \) such that \( \alpha^{-1}\alpha \subseteq \beta^{-1}\beta \). Then \( M/\alpha \big/ \beta/\alpha \cong M/\beta. \)

**Proof:** By Lemma 7.1.1 \( \beta/\alpha \) is an epimorphism between \( M/\alpha \) and \( M/\beta \).

So by Theorem 5.1 \( M/\alpha \big/ \beta/\alpha \cong M/\beta. \)
Chapter 8
Correspondence Theorem

**Definition** Suppose $A$ is a set, $B$ is a subset of $A$, and $\varphi$ is a function with domain $A$. The statement that $B$ is $\varphi$ exact means $B = \varphi^{-1}\varphi(B)$.

**Definition** Suppose $M = (A, \mathcal{R})$ is an $I$-structure, $N = (B, \mathcal{S})$ is an $I$-substructure of $M$, and $\varphi$ is a function with domain $A$. The statement that $N$ is $\varphi$ exact means $B = \varphi^{-1}\varphi(B)$.

**Lemma 8.1.1.** Suppose $A$ is a set, $f$ is a function with domain $A$, $B \subseteq A$ such that $B = f^{-1}f(B)$, and $b \in B$. Then $f|_B^{-1}f|_B(\{b\}) = f^{-1}f(\{b\})$.

**Proof:**

\[
p \in f|_B^{-1}f|_B(\{b\})
\]
\[\iff (b, p) \in f|_B^{-1}f|_B
\]
\[\iff p \in B = f^{-1}f(B) \text{ and } (b, p) \in f^{-1}f
\]
\[\iff p \in f^{-1}f(\{b\})
\]

**Lemma 8.1.2.** Suppose $A$ is a set, $f$ is a function with domain $A$, $B \subseteq A$ such that $B = f^{-1}f(B)$, and $b \in B$. Then $\pi_{f|_B}(b) = \pi_f(b)$.

**Proof:**

\[
\pi_{f|_B}(b) = f|_B^{-1}f|_B(\{b\}) = f^{-1}f(\{b\}) = \pi_f(b)
\]

**Lemma 8.1.3.** Suppose $A$ is a set, $f$ is a function with domain $A$, $B \subseteq A$ such that $B = f^{-1}f(B)$, $r$ is a relation such that $\operatorname{im}(r) \subseteq B/(\varphi|_B)$. Then $\pi_{f|_B}^{-1}r = \pi_f^{-1}r$.
Proof:

\[(i, b) \in \pi_{f|B}^{-1} r\]
\[\iff \exists P \text{ such that } (i, P) \in r \text{ and } (P, b) \in \pi_{f|B}^{-1} \]
\[\iff (i, \pi_{f|B}(b)) \in r\]
\[\iff (i, \pi_f(b)) \in r \text{ and } b \in B\]
\[\iff \exists P \text{ such that } (i, P) \in r \text{ and } (P, b) \in \pi_f^{-1} \text{ (so } P \in B/\varphi|_B)\]
\[\iff (i, b) \in \pi_f^{-1} r\]

So \(\pi_{f|B}^{-1} r = \pi_f^{-1} r\). \(\square\)

**Lemma 8.1.4.** Suppose \(M = (A, R)\) is an I-structure, \(\varphi\) is a function with domain \(A\), and \(N = (B, \hat{R})\) is a \(\varphi\) exact I-substructure of \(M\). Then \(N/(\varphi|_B) = (B/(\varphi|_B), T)\) is the I-substructure of \(M/\varphi = (A/\varphi, S)\) induced by \(B/(\varphi|_B)\).

**Proof:** Suppose \(P \in B/(\varphi|_B)\). Then \(P = \varphi|_B^{-1}\varphi|_B(\{b\})\) for some \(b\) in \(B\).

\[P = \varphi|_B^{-1}\varphi|_B(\{b\}) = \varphi^{-1}\varphi(\{b\}) \in A/\varphi\]

So \(B/(\varphi|_B) \subseteq A/\varphi\).

Note \(T\) is the relation set to which a relation \(t\) belongs if and only if there is an \(\hat{r} \in \hat{R}\) such that \(t = \pi_{p|B}\hat{r}\).

Suppose \((B/(\varphi|_B), \hat{S})\) is the I-substructure of \(M/\varphi\) induced by \(B/(\varphi|_B)\). Note \(\hat{S}\) is the
relation set to which a relation \( \hat{s} \) belongs if and only if \( \hat{s} \in S \) and \( \text{im}(\hat{s}) \subseteq B/\varphi_B \).

\[ k \in T \iff \exists r \in \hat{R} \text{ (so im}(r) \subseteq B \text{) such that } k = \pi_{\varphi} r = \pi_{\varphi|B} r \]
\[ \iff \exists r \in R \text{ such that } \pi_{\varphi} r = k \text{ and } \text{im}(r) \subseteq B \]
\[ \iff \exists r \in R \text{ such that } \pi_{\varphi} r = k \text{ and } \text{im}(k) \subseteq B/(\varphi|B) \]
\[ ( \iff : \text{im}(k) = \pi_{\varphi|B}(\text{im}(r)) \subseteq \pi_{\varphi|B}(B) = B/(\varphi|B)) \]
\[ ( \iff : \text{im}(r) \subseteq \pi_{\varphi^{-1}} \pi_{\varphi} r = \text{im}(\pi_{\varphi^{-1}} k) = \pi_{\varphi^{-1}}(\text{im}(k)) \subseteq \pi_{\varphi^{-1}}(\varphi(B)) = \varphi^{-1}(B) = B) \]
\[ \iff k \in S \text{ and } \text{im}(k) \subseteq B/(\varphi|B) \]
\[ \iff k \in \hat{S} \]

So \( T = \hat{S} \), and \( N/(\varphi|B) = (B/(\varphi|B), T) = (B/(\varphi|B), \hat{S}) \), the I-substructure of \( M/\varphi \) induced by \( B/(\varphi|B) \).

\[ \square \]

**Theorem 8.1.** Suppose \( M = (A, \mathcal{R}) \) is an I-structure, and \( \varphi \) is a function with domain \( A \). Then there is a bijection between the set of \( \varphi \) exact I-substructures of \( M \), and the set of I-substructures of \( M/\varphi \).

**Proof:** Suppose \( S \) is the set of \( \varphi \) exact I-substructures of \( M \).

Suppose \( T \) is the set of I-substructures of \( M/\varphi \).

Define \( f : S \rightarrow T \) such that for each \( N = (B, \hat{R}) \) in \( S \), \( f(N) = N/(\varphi|B) \).

By Lemma 8.1.4, \( f(N) \in T \).

1. \( f \) is an injection: Suppose each of \( N_1 = (B_1, \mathcal{R}_1) \) and \( N_2 = (B_2, \mathcal{R}_2) \) is in \( S \), and \( f(N_1) = f(N_2) \). Note each of \( N_1 \) and \( N_2 \) is \( \varphi \) exact, so \( B_1 = \varphi^{-1}(\varphi(B_1)) \) and \( B_2 = \varphi^{-1}(\varphi(B_2)) \)
\[ \varphi^{-1}\varphi(B_2). \]

\[
(B_1/(\varphi|_{B_1}), \hat{R}_1) = N_1/(\varphi|_{B_1}) = f(N_1) = f(N_2) = N_2/(\varphi|_{B_2}) = (B_2/(\varphi|_{B_2}), \hat{R}_2)
\]

So \( B_1/(\varphi|_{B_1}) = B_2/(\varphi|_{B_2}). \)

\[
\pi_\varphi(B_1) = \pi_{\varphi|_{B_1}}(B_1) = B_1/(\varphi|_{B_1}) = B_2/(\varphi|_{B_2}) = \pi_{\varphi|_{B_2}}(B_2) = \pi_\varphi(B_2)
\]

\[
\implies B_1 = \varphi^{-1}\varphi(B_1) = \pi_{\varphi}^{-1}\pi_\varphi(B_1) = \pi_{\varphi}^{-1}\varphi(B_2) = \varphi^{-1}\varphi(B_2) = B_2
\]

\[ \mathcal{R}_1 = \{ r \mid r \in \mathcal{R} \text{ and } \text{im}(r) \subseteq B_1 \} = \{ r \mid r \in \mathcal{R} \text{ and } \text{im}(r) \subseteq B_2 \} = \mathcal{R}_2 \]

So \( N_1 = (B_1, \mathcal{R}_1) = (B_2, \mathcal{R}_2) = N_2. \)

So \( f \) is an injection.

2. \( f \) is a surjection: Suppose \( (\mathcal{P}, \mathcal{T}) \) is an \( I \)-substructure of \( M/\varphi = (A/\varphi, \bar{\mathcal{R}}) \) (namely the \( I \)-substructure of \( M/\varphi \) induced by \( \mathcal{P} \)).

Then \( \mathcal{P} \subseteq A/\varphi = \pi_\varphi(A) \) so \( \pi_\varphi^{-1}(\mathcal{P}) \subseteq \pi_\varphi^{-1}(\pi_\varphi(A)) = A. \)

Define \( B = \pi_\varphi^{-1}(\mathcal{P}). \)

Consider \( L = (B, \mathcal{S}) \) where \( \mathcal{S} \) is the relation set to which a relation \( s \) belongs if and only if \( s \in \mathcal{R} \) and \( \text{im}(s) \subseteq B. \) So \( L \) is an \( I \)-substructure of \( M. \)

\[
B = \pi_\varphi^{-1}(\mathcal{P}) = \pi_\varphi^{-1}\pi_\varphi\pi_\varphi^{-1}(\mathcal{P}) = \pi_\varphi^{-1}\pi_\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \varphi^{-1}\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \varphi^{-1}\varphi(B)
\]
So $L$ is $\varphi$ exact.

So $L \in S$.

\begin{align*}
B/(\varphi|_B) &= \pi_{\varphi|_B}(B) = \pi_{\varphi}(B) = \pi_{\varphi}(\pi_{\varphi}^{-1}(P)) = P
\end{align*}

By Lemma 8.1.4, $L/(\varphi|_B)$ is the $I$-substructure of $M/\varphi$ induced by $B/(\varphi|_B) = P$.

So $f(L) = L/(\varphi|_B) = (P, T)$.

So $f$ is a surjection.

Thus $f$ is a bijection.

**Definition** Suppose $M = (A, \mathcal{R})$ is an $I$-structure, $N = (B, \mathcal{S})$ is an understructure of $M$, and $\varphi$ is a function with domain $A$. The statement that $N$ is $\varphi$ exact means $B = \varphi^{-1}(\varphi(B))$.

**Lemma 8.2.1.** Suppose each of $A$ and $B$ is a set, $r$ is a relation, and $B \subseteq \text{dom}(r)$. Then $r(A \times B) = A \times r(B)$.

**Proof:**

\begin{align*}
(a, c) &\in r(A \times B) \\
\iff &\exists b \text{ such that } (a, b) \in A \times B \text{ and } (b, c) \in r \\
\iff &a \in A \text{ and } \exists b \in B \subseteq \text{dom}(r) \text{ such that } c \in r(\{b\}) \\
\iff & (a, c) \in A \times r(B)
\end{align*}

So $r(A \times B) = A \times r(B)$.

**Lemma 8.2.2.** Suppose $r$ is a relation, $I$ is a set such that $\text{dom}(r) \subseteq I$, $\varphi$ is a function, and $B$ is a set such that $B \subseteq \text{dom}(\varphi)$ and $B$ is $\varphi$ exact. Then $\pi_{\varphi}(r \cap (I \times B)) = (\pi_{\varphi}r) \cap (\pi_{\varphi}(I \times B))$. 

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Proof:

\[(i, P) \in \pi_\varphi(r \cap (I \times B))\]
\[\iff \exists b \text{ such that } (b, P) \in \pi_\varphi \text{ and } (i, b) \in r \cap (I \times B)\]
\[\iff \exists b \text{ such that } (b, P) \in \pi_\varphi \text{ and } (i, b) \in r \text{ and } (i, b) \in I \times B\]
\[\iff \exists b_1, b_2 \text{ such that } (b_1, P) \in \pi_\varphi, (b_2, P) \in \pi_\varphi, (i, b_1) \in r, \text{ and } (i, b_2) \in I \times B\]
\[\iff (i, P) \in \pi_\varphi r \text{ and } (i, P) \in \pi_\varphi(I \times B)\]
\[\iff (i, P) \in (\pi_\varphi r) \cap (\pi_\varphi(I \times B))\]

So \(\pi_\varphi(r \cap (I \times B)) = (\pi_\varphi r) \cap (\pi_\varphi(I \times B))\). \hfill \Box

Lemma 8.2.3. Suppose each of \(f\) and \(g\) is a relation, \(I\) is a set such that \(\text{dom}(f) \subseteq I\) and \(\text{dom}(g) \subseteq I\), \(\varphi\) is a function such that \(\text{im}(f) \subseteq \text{dom}(\varphi)/\varphi\) and \(\text{im}(g) \subseteq \text{dom}(\varphi)/\varphi\), and \(B\) is a set such that \(B \subseteq \text{dom}(\varphi)\) and \(B\) is \(\varphi\) exact. Then \(f = g \cap (I \times B/(\varphi|_B))\) if and only if \(\pi_\varphi^{-1} f = (\pi_\varphi^{-1} g) \cap (I \times B)\).

Proof: Suppose \(f = g \cap (I \times B/(\varphi|_B))\).

\[\pi_\varphi^{-1} f = \pi_\varphi^{-1}(g \cap (I \times B/(\varphi|_B))) = (\pi_\varphi^{-1} g) \cap (\pi_\varphi^{-1}(I \times \pi_\varphi|_B(B))) = (\pi_\varphi^{-1} g) \cap (\pi_\varphi^{-1}(I \times \pi_\varphi(B)))\]
\[= (\pi_\varphi^{-1} g) \cap (I \times \pi_\varphi^{-1} \pi_\varphi(B)) = (\pi_\varphi^{-1} g) \cap (I \times \varphi^{-1} \varphi(B)) = (\pi_\varphi^{-1} g) \cap (I \times B)\]

So \(\pi_\varphi^{-1} f = (\pi_\varphi^{-1} g) \cap (I \times B)\). \hfill \Box

Suppose \(\pi_\varphi^{-1} f = (\pi_\varphi^{-1} g) \cap (I \times B)\).

\[f = \pi_\varphi \pi_\varphi^{-1} f = \pi_\varphi((\pi_\varphi^{-1} g) \cap (I \times B)) = (\pi_\varphi \pi_\varphi^{-1} g) \cap (\pi_\varphi(I \times B)) = g \cap (I \times \pi_\varphi(B))\]
\[= g \cap (I \times \pi_\varphi|_B(B)) = g \cap (I \times B/(\varphi|_B))\]
So $f = g \cap (I \times B/(\varphi|_B))$. \hfill \Box

**Lemma 8.2.4.** Suppose $M = (A, \mathcal{R})$ is an $I$-structure, $\varphi$ is a cohomomorphism from $M$, and $N = (B, \hat{\mathcal{R}})$ is a $\varphi$ exact understructure of $M$. Then $N/\varphi|_B = (B/(\varphi|_B), \mathcal{T})$ is the understructure of $M/\varphi = (A/\varphi, \mathcal{S})$ induced by $B/(\varphi|_B)$.

**Proof:** Suppose $P \in B/(\varphi|_B)$. Then $P = \varphi|_B^{-1}\varphi|_B(\{b\})$ for some $b$ in $B$.

$$P = \varphi|_B^{-1}\varphi|_B(\{b\}) = \varphi^{-1}\varphi(\{b\}) \in A/\varphi$$

So $B/(\varphi|_B) \subseteq A/\varphi$.

Note $\mathcal{T}$ is the relation set to which a relation $t$ belongs if and only if $t \subseteq I \times B/(\varphi|_B)$ and $\pi_{\varphi|_B}^{-1}t \in \hat{\mathcal{R}}$.

Suppose $(B/(\varphi|_B), \hat{\mathcal{S}})$ is the understructure of $M/\varphi$ induced by $B/(\varphi|_B)$. Note $\hat{\mathcal{S}}$ is the relation set to which a relation $\hat{s}$ belongs if and only if there is an $s \in \mathcal{S}$ such that $\hat{s} = s \cap (I \times B/\varphi|_B)$.

$$k \in \hat{\mathcal{S}}$$

$$\iff \exists s \in \mathcal{S} \text{ such that } k = s \cap (I \times B/(\varphi|_B))$$

$$\iff \exists s \in \mathcal{S} \text{ such that } \pi_{\varphi|_B}^{-1}s \cap (I \times B) = \pi_{\varphi|_B}^{-1}k$$

$$\iff \exists r \in \mathcal{R} \text{ such that } r \cap (I \times B) = \pi_{\varphi|_B}^{-1}k$$

$$\iff \pi_{\varphi|_B}^{-1}k \in \hat{\mathcal{R}} \text{ and } \pi_{\varphi|_B}^{-1}k \subseteq I \times B$$

$$\iff \pi_{\varphi|_B}^{-1}k \in \hat{\mathcal{R}} \text{ and } k \subseteq I \times B/(\varphi|_B)$$

$$\iff k \in \mathcal{T}$$

So $\hat{\mathcal{S}} = \mathcal{T}$, and $N/\varphi|_B = (B/(\varphi|_B), \mathcal{T}) = (B/(\varphi|_B), \hat{\mathcal{S}})$, the understructure of $M/\varphi$ induced by $B/(\varphi|_B)$. \hfill \Box
Example Suppose $M = (A, \mathcal{R})$ is an $I$-structure, $\varphi$ is a function with domain $A$, and $N = (B, \hat{\mathcal{R}})$ is a $\varphi$ exact understructure of $M$. Then $N/\varphi|_B = (B/(\varphi|_B), \mathcal{T})$ is not necessarily the understructure of $M/\varphi = (A/\varphi, \mathcal{S})$ induced by $B/(\varphi|_B)$.

Proof: Define the following:

$M = (\{a_1, a_2, b\}, \{0\}, \{(0, a_1), (0, b)\})$

$B = \{b\}$

$\varphi = \{(a_1, x), (a_2, x), (b, y)\}$

$N = (\{b\}, \{0\}, \{(0, b)\})$, a $\varphi$ exact understructure of $M$

Then $\pi_\varphi = \{(a_1, \{a_1, a_2\}), (a_2, \{a_1, a_2\}), (b, \{b\})\}$

$M/\varphi = (\{a_1, a_2\}, \{a_2\}, \{0\})$

$\varphi|_B = \{(0, b)\}$

$B/(\varphi|_B) = \{b\}$

$N/(\varphi|_B) = (\{b\}, \{0\}, \{(0, b)\})$

$\{(b), \{0\}, \emptyset\}$ is the understructure of $M/\varphi$ induced by $B/(\varphi|_B)$.

$N/\varphi|_B \neq (\{b\}, \{0\}, \emptyset)$

Theorem 8.2. Suppose $M = (A, \mathcal{R})$ is an $I$-structure, and $\varphi$ is a cohomomorphism from $M$. Then there is a bijection between the set of $\varphi$ exact understructures of $M$, and the set of understructures of $M/\varphi$.

Proof: Suppose $S$ is the set of $\varphi$ exact understructures of $M$.

Suppose $T$ is the set of understructures of $M/\varphi$.

Define $f : S \to T$ such that for each $N = (B, \hat{\mathcal{R}})$ in $S$, $f(N) = N/(\varphi|_B)$. 

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By Lemma 8.2.4, $f(N) \in T$.

1. $f$ is an injection: Suppose each of $N_1 = (B_1, \mathcal{R}_1)$ and $N_2 = (B_2, \mathcal{R}_2)$ is in $S$, and $f(N_1) = f(N_2)$. Note each of $N_1$ and $N_2$ is $\phi$ exact, so $B_1 = \phi^{-1}(\phi(B_1))$ and $B_2 = \phi^{-1}(\phi(B_2))$.

$$(B_1/((\phi|_{B_1}), \tilde{\mathcal{R}}_1) = N_1/((\phi|_{B_1}) = f(N_1) = f(N_2) = N_2/((\varphi|_{B_2}) = (B_2/((\phi|_{B_2}), \tilde{\mathcal{R}}_2))$$

So $B_1/((\varphi|_{B_1}) = B_2/((\varphi|_{B_2})$.

$$\pi_\phi(B_1) = \pi_{\phi|_{B_1}}(B_1) = B_1/((\varphi|_{B_1}) = B_2/((\varphi|_{B_2}) = \pi_{\phi|_{B_2}}(B_2) = \pi_\phi(B_2)$$

$$\implies B_1 = \phi^{-1}(\phi(B_1)) = \pi_\phi(B_1) = \pi_{\phi|_{B_2}}(B_2) = \phi^{-1}(\phi(B_2)) = B_2$$

$\mathcal{R}_1 = \{\tilde{r} \mid \exists r \in \mathcal{R} \text{ such that } \tilde{r} = r \cap (I \times B_1)\} = \{\tilde{r} \mid \exists r \in \mathcal{R} \text{ such that } \tilde{r} = r \cap (I \times B_2)\} = \mathcal{R}_2$

So $N_1 = (B_1, \mathcal{R}_1) = (B_2, \mathcal{R}_2) = N_2$.

So $f$ is an injection.

2. $f$ is a surjection: Suppose $(\mathcal{P}, \mathcal{T})$ is an understructure of $M/\phi = (A/\phi, \mathcal{R})$ (namely, the understructure of $M/\phi$ induced by $\mathcal{P}$).

Then $\mathcal{P} \subseteq A/\phi = \pi_\phi(A)$ so $\pi^{-1}_\phi(\mathcal{P}) \subseteq \pi^{-1}_\phi(\pi_\phi(A)) = A$.

Define $B = \pi^{-1}_\phi(\mathcal{P})$.

Consider $L = (B, \mathcal{S})$ where $\mathcal{S}$ is the relation set to which a relation $s$ belongs if and only if there is an $r \in \mathcal{R}$ such that $s = r \cap (I \times B)$. So $L$ is an understructure of
Set $M$.  

\[ B = \pi_{\varphi}^{-1}(P) = \pi_{\varphi}^{-1}(\pi_{\varphi}^{-1}(P)) = \pi_{\varphi}^{-1}(\varphi^{-1}(P)) = \varphi^{-1}(\varphi^{-1}(P)) = \varphi^{-1}(B) \]

So $L$ is $\varphi$ exact.

So $L \in S$.  

\[ B/\varphi|_B = \pi_{\varphi|_B}(B) = \pi_{\varphi}(B) = \pi_{\varphi}(\pi_{\varphi}(P)) = \varphi(P) \]

By Lemma 8.2.4, $L/\varphi|_B$ is the understructure of $M/\varphi$ induced by $B/\varphi|_B = \varphi$.

So $f(L) = L/\varphi|_B = (P, T)$.

So $f$ is a surjection.

Thus $f$ is a bijection.  \(\square\)
Bibliography


