Orthogonal bases of certain symmetry classes of tensors associated with Brauer characters

by

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Abstract

The main focus of this dissertation is on the existence of an orthogonal basis consisting of standard symmetrized tensors (o-basis for short) of a symmetry class of tensors associated with a Brauer character of a finite group. Most of the work is done for the dihedral group and some results are given for the symmetric group. The existence of an o-basis of a symmetry class of tensors associated with an (ordinary) character of a finite group have been studied by several authors. My study was motivated by the work done on the existence of such a basis of a symmetry class of tensors associated with an (ordinary) irreducible character of a dihedral group.

In Chapter 1 we introduce the basic definitions in character theory. In this a Brauer characters, character of a projective indecomposable module (PI) and a block of a finite group will be introduced. Also in this chapter a generalised orthogonality relation of blocks of a finite group is established. In chapter 2 we introduce the symmetrizer and related notions. Some general results associated with Brauer characters of a finite group will also be given in this chapter. Chapter 3 consists of the results associated with Brauer characters, PIs and blocks of a dihedral group. Finally Chapter 4 lists some result associated with the Brauer characters of the symmetric group.

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Chapter 1

Character Theory

1.1 Group representations and Group Algebra

Let G be a finite group and V be a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. By $\mathrm{GL}(V)$ we denote the group of invertible linear transformations from V to itself. A *representation* of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. The *degree* of the representation is the dimension of V.

Denote by $\mathbb{C}G$ the vector space over $\mathbb{C}$ with the basis G. $\mathbb{C}G$ is a ring with the multiplication defined by,

$$(\sum_{a\in G} \alpha_a a)(\sum_{b\in G} \beta_b b) = \sum_{a,b\in G} \alpha_a \beta_b ab.$$

A $\mathbb{C}$-*algebra* is a ring A that is also a vector space over $\mathbb{C}$ such that $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in \mathbb{C}$ and $a, b \in A$. Note that $\mathbb{C}G$ is a $\mathbb{C}$-algebra and is called the *group algebra* of G over $\mathbb{C}$. $\mathbb{C}G$ has an identity given by $1e \neq 0$, where e is the identity of G. Define a map from $\mathbb{C}$ to $\mathbb{C}G$ by $\alpha \mapsto \alpha 1$, where $1 \neq 0$ is the identity of $\mathbb{C}G$. This is a well defined ring monomorphism, hence $\mathbb{C}$ is viewed as a subring of $\mathbb{C}G$.

Let V be a finite dimensional vector space over $\mathbb{C}$ and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then V can be viewed as a (left) $\mathbb{C}G$ module by defining $av = \rho(a)(v)$ for $a \in G$, $v \in V$ and extending linearly to $\mathbb{C}G$.

On the other hand, let V be a $\mathbb{C}G$ module. Then V is a vector space over $\mathbb{C}$ by viewing $\mathbb{C}$ as a subring of $\mathbb{C}G$. When we say that V is a $\mathbb{C}G$ module, we always assume that V is finite dimensional when viewed as a vector space over $\mathbb{C}$ in this way. Define a map ρ from G to $\mathrm{GL}(V)$ by $\rho(a)(v) = av$ for $a \in G$, $v \in V$. Then ρ is a well defined group homomorphism and hence a representation of G called the representation afforded by V.

An *irreducible representation* of G is a representation afforded by a simple $\mathbb{C}G$ module.

1.2 Character

Let V be a $\mathbb{C}G$ module and let ρ be the representation of G afforded by V. Let $\sigma \in G$, so that $\rho(\sigma) \in \mathrm{GL}(V)$.

Let $M_{\rho(\sigma)}$ be the matrix representation of $\rho(\sigma)$ corresponding to a fixed basis of V. The trace of $\rho(\sigma)$ is given by $tr(\rho(\sigma)) = tr(M_{\rho(\sigma)})$. Note that the value of $tr(\rho(\sigma))$ does not depend on the choice of the basis since similar matrices have the same trace.

The (*ordinary*) *character* of G afforded by ρ or V is the function $\eta : G \to \mathbb{C}$ defined by $\eta(\sigma) = tr(\rho(\sigma))$ for $\sigma \in G$. We say $\eta(e)$ the *degree* of η where e is the identity element of G.

Theorem 1.1 ([11, Lemma 2.15, page 20]). *Let η be a character of G. Let $\sigma \in G$ and let m be the order of σ. Then*

i) $\eta(\sigma)$ is a sum of mth roots of unity,

ii) $\overline{\eta(\sigma)} = \eta(\sigma^{-1})$.

Two representations ρ and ϱ of the same degree n are said to be *similar* if there exists an invertible matrix P of size $n \times n$ such that $M_{\rho(\sigma)} = P^{-1} M_{\varrho(\sigma)} P$ for all $\sigma \in G$. It is easy to observe that the following result holds using the property that $tr(AB) = tr(BA)$ for all square matrices A, B.

Theorem 1.2 ([11, Lemma 2.3, page 14]).

i) Similar representations of G afford equal characters.

ii) Characters are constant on the conjugacy classes of G.

A *class function* is a function on G that is constant on conjugacy classes. The theorem states that the characters of G are class functions.

A character of G is called an *irreducible character* of G if it is afforded by an irreducible representation of G (or, equivalently, a simple module of $\mathbb{C}G$). Representations of G afforded by isomorphic $\mathbb{C}G$-modules are similar. There is a one-to-one correspondence between isomorphism classes of $\mathbb{C}G$-modules and similarity classes of representations of G (see [11, page 10]). Therefore in light of the theorem above the number of different irreducible characters of a group G is the same as the number of isomorphism classes of simple $\mathbb{C}G$-modules.

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G. Maschke's theorem stated below provides a way to reduce the study of characters of G to the study of irreducible characters of G.

Theorem 1.3 (Maschke)**.** *Let K be a field. If $\mathrm{char} K \nmid |G|$, then every KG-module is a direct sum of simple KG-modules.*

Since char $\mathbb{C} = 0$, Maschke's theorem holds for the field $\mathbb{C}$. Now let V and V' be two $\mathbb{C}G$-modules and let ρ and ρ' be the representations they afford respectively. Let θ be the representation afforded by the direct sum $V \oplus V'$. Then for any $\sigma \in G$ the matrix representation of $\theta(\sigma)$ relative to an ordered basis formed by taking an ordered basis for V and appending an ordered basis for V' is given by a block diagonal matrix with blocks the matrix representations of $\rho(\sigma)$ and $\rho'(\sigma)$ as given by

$$M_{\theta(\sigma)} = \begin{pmatrix} M_{\rho(\sigma)} & 0 \\ 0 & M_{\rho'(\sigma)} \end{pmatrix}.$$

Then the character afforded by the $\mathbb{C}G$-module $V \oplus V'$ is $\eta + \eta'$, since $\mathrm{tr}\, M_{\theta(\sigma)} = \mathrm{tr}\, M_{\rho(\sigma)} + \mathrm{tr}\, M_{\rho'(\sigma)}$.

Because of the above results we see that to study the characters of G it is enough to look at the irreducible characters of G. Once all the irreducible characters of G are known the other characters of G are known as well since they are simply sums of irreducible characters.

One gets the number of irreducible characters of G from the number of conjugacy classes of the group G as stated below.

Theorem 1.4 ([11, Corollary 2.7, page 16])**.** *Let G be a group. The number of irreducible characters of G equals the number of conjugacy classes of G.*

1.3 Brauer character

The Brauer characters are the main focus of this entire thesis. These characters are also known as modular characters. The modular representation theory was founded by Richard Brauer in the 1930's. We begin by setting basic definitions.

Let R be the ring of algebraic integers in $\mathbb{C}$. Fix a prime p, and let M be a maximal ideal of R such that $pR \subseteq M$. Set $K = R/M$. Then K is a field. Considering the natural homomorphism $\pi : R \to K$ we have $p \mapsto 0$, so K has characteristic p. The natural homomorphism is going from characteristic 0 to characteristic p.

Theorem 1.5 ([11, Lemma 15.1, page 263])**.** *Let $U = \{\lambda \in \mathbb{C} \mid \lambda^m = 1$ for some integer m with $p \nmid m\}$ and let R, K be as above. Then*

i) $U \subseteq R$,

ii) the natural homomorphism maps U isomorphically onto $K \setminus \{0\}$,

iii) K is algebraically closed and algebraic over its prime field.

An element of G is called a *p-regular* element if its order is not divisible by p. Denote by $\hat{G}$ the set of all p-regular elements of G.

Let V be a KG-module of finite dimension n and let ρ be the representation of G afforded by V. Let $\sigma \in \hat{G}$ and let $\kappa_1, \ldots, \kappa_n \in K \setminus \{0\}$ be the eigenvalues of $\rho(\sigma)$. Then by the theorem above there exist unique $\lambda_1, \ldots, \lambda_n \in$ U such that $\lambda_i \mapsto \kappa_i$ via the natural homomorphism. Define a function $\varphi : \hat{G} \to \mathbb{C}$ by

$$\varphi(\sigma) = \sum_{i=1}^{n} \lambda_i. \tag{1.1}$$

Then φ is called the *Brauer character* of G afforded by ρ.

Let $\sigma \in \hat{G}$ and suppose $\kappa \in K$ is an eigenvalue of $\rho(\sigma)$. Then κ^{-1} is an eigenvalue of $\rho(\sigma^{-1})$ since

$$\rho(\sigma^{-1})v = \rho(\sigma^{-1})\kappa^{-1}\rho(\sigma)v = \kappa^{-1}\rho(e)v = \kappa^{-1}v.$$

Also if $\pi(\lambda) = \kappa$ $(\lambda \in U)$, then $1 = \pi(\lambda\bar{\lambda}) = \pi(\lambda)\pi(\bar{\lambda}) = \kappa\pi(\bar{\lambda})$, so $\pi(\bar{\lambda}) = \kappa^{-1}$. Therefore $\overline{\varphi(\sigma)} = \varphi(\sigma^{-1})$. If φ is a Brauer character of G, then $\bar{\varphi}$, the complex conjugate of φ is also a Brauer character [11]. A Brauer character corresponding to a simple KG-module is called an *irreducible* Brauer character of the group G. We denote the set of irreducible Brauer characters of G by $\mathrm{IBr}(G)$. The irreducible Brauer characters are linearly independent over $\mathbb{C}$ [11, Theorem 15.5, page 265].

Brauer characters are constant on conjugacy classes. The number of irreducible Brauer characters is equal to the number of conjugacy classes of G containing p-regular elements of G as stated by the following theorem.

Theorem 1.6 ([14, Corollary 3, page 150])**.** *The number of classes of simple KG-modules is equal to the number of p-regular conjugacy classes of G.*

Let $\hat{\chi}$ denote the restriction of an ordinary character χ of G to the set $\hat{G}$ of p-regular elements of G. The following result is a well known relationship between the ordinary characters and the Brauer characters of a group.

Theorem 1.7 ([11, Theorem 15.6, page 265])**.** *Let χ be an ordinary character of G. Then $\hat{\chi}$ is a Brauer character of G.*

The character χ uniquely determines the Brauer character $\hat{\chi}$. We note here that if p does not divide the order of the group G, then $\hat{G} = G$ and the Brauer characters of G coincide with the ordinary characters G.

The set of complex valued class functions on $\hat{G}$ form a vector space over $\mathbb{C}$

Theorem 1.8 (R. Brauer). *The irreducible Brauer characters of a group G form a basis of the vector space of complex valued class functions on $\hat{G}$.*

1.4 PIs

We are also interested in the characters associated with projective indecomposable modules of RG. Note that RG is of finite dimension so satisfies the A.C.C. and D.C.C. Then by [2, Theorem 14.2, page 81] RG can be written as a direct sum of indecomposable RG-modules. A summand of this direct sum is a principle indecomposable module (a PIM) of RG. Similarly KG can be expressed as a direct sum of principle indecomposable KG-modules. As direct summands of free modules, PIMs of RG and KG are projective. [5, Theorem I.13.7, page 44] states that there is a one to one correspondence between the isomorphism classes of PIMs of RG and those of KG.

Theorem 1.9 ([2, Theorem 54.11, page 372]). *Let P be a* PIM *of KG. Then P has a unique maximal submodule N_P. Two* PIM*s P and Q are isomorphic if and only if the irreducible modules P/N_P and Q/N_Q are isomorphic.*

Theorem 1.10 ([2, Corollary 54.14, page 374]). *There is a one-to-one correspondence between the isomorphism classes of* PIM*s and the isomorphism classes of irreducible KG-modules.*

The character afforded by a PIM P of RG (PI for short) is the character afforded by $\mathbb{C} \bigotimes_R P$. By the theorem above we have that there is a one to one correspondence between the irreducible Brauer characters of G and the PIs of G. We will denote by Φ_φ the PI corresponding to $\varphi \in \mathrm{IBr}(G)$. A PI Φ is a complex valued function defined on G with the property that $\Phi(\sigma) = 0$ for $\sigma \in G \setminus \hat{G}$ [5, Corollary IV.2.5, page 144].

1.5 Relationships

Let $\eta \in \mathrm{Irr}(G)$ and let $\hat{\eta}$ be the restriction of η to $\hat{G}$. Recall by Theorem 1.7 $\hat{\eta}$ is a Brauer character of G, so

$$\hat{\eta} = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\eta\varphi}\varphi \tag{1.2}$$

for some uniquely determined nonnegative integers $d_{\eta\varphi}$. The integers $d_{\eta\varphi}$ ($\eta \in \mathrm{Irr}(G), \varphi \in \mathrm{IBr}(G)$) are called the *decomposition numbers* of G for the prime p. The matrix of size $|\,\mathrm{Irr}(G)| \times |\,\mathrm{IBr}(G)|$ with the $d_{\eta\varphi}$'s as entries is called the *decomposition matrix* of G.

Let $\varphi \in \mathrm{IBr}(G)$. By [14, page 151] we have the following relationships

$$\Phi_\varphi = \sum_{\eta \in \mathrm{Irr}(G)} d_{\eta\varphi}\eta, \tag{1.3}$$

$$\Phi_\varphi = \sum_{\psi \in \mathrm{IBr}(G)} c_{\psi\varphi}\psi,$$

where the coefficients $c_{\psi\varphi}$ are the entries of the matrix $C = DD^T$, with D^T the transpose of D. The matrix C is called the *Cartan matrix*.

1.6 Block

Let e be a centrally primitive idempotent of the group algebra $\mathbb{C}G$. The *block* $B = B_e$ corresponding to e is the category of $\mathbb{C}G$-modules V such that $eV = V$. A $\mathbb{C}G$-module V is said to belong to B if it is an object of B, that is, if $eV = V$. By [5, Theorem 7.8, page 23] a finitely generated indecomposable $\mathbb{C}G$-module V belongs to a unique block. If V belongs to B, then every submodule and homomorphic image of V belongs to the same block B.

A character or a Brauer character of G is said to belong to a block B if the associated module belongs to B. If $\eta \in \mathrm{Irr}(G)$ belongs to the block B, then $\varphi \in \mathrm{IBr}(G)$ belongs to B if the decomposition number $d_{\eta\varphi}$ is nonzero. Further each irreducible character and irreducible Brauer character belongs to a unique block. Two irreducible characters η and ϕ are in a same block B of G if there is $\varphi \in \mathrm{IBr}(G)$ such that $d_{\eta\varphi}$ and $d_{\phi\varphi}$ are both nonzero. In this case the block is the unique block that contains the Brauer character φ [11].

If ψ is a character or a Brauer character, we write $\psi \in B$ to mean that ψ belongs to the block B. More generally, if S is a set of characters or Brauer characters, we write $\psi \in B \cap S$ to mean that ψ belongs to the block B and $\psi \in S$.

Let B be a block of G. The *Osima idempotent* of $\mathbb{C}G$ corresponding to B is given by

$$s_B = \sum_{\eta \in B \cap \mathrm{Irr}(G)} s_\eta = \frac{1}{|G|} \sum_{\eta \in B \cap \mathrm{Irr}(G)} \sum_{\sigma \in G} \eta(e)\eta(\sigma^{-1})\sigma. \tag{1.4}$$

Theorem 1.11 ([11, Theorem 15.30, page 277])**.** *For blocks B and B' of G, we have*

$$s_B s_{B'} = \delta_{BB'} s_B.$$

The following theorem holds due to the Equations 1.2 and 1.3.

Theorem 1.12 (Osima)**.**

$$\begin{aligned} s_B &= \frac{1}{|G|} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\sigma \in G} \varphi(e)\Phi_\varphi(\sigma^{-1})\sigma \\ &= \frac{1}{|G|} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\sigma \in \hat{G}} \Phi_\varphi(e)\varphi(\sigma^{-1})\sigma. \end{aligned}$$

1.7 Orthogonality

Let $\mathrm{Fun}(G,\mathbb{C})$ denote the set of all functions from G to $\mathbb{C}$. $\mathrm{Fun}(G,\mathbb{C})$ is a vector space over $\mathbb{C}$. For $f,g \in \mathrm{Fun}(G,\mathbb{C})$ set,

$$(f,g) = \frac{1}{|G|}\sum_{\sigma\in G} f(\sigma)\overline{g(\sigma)}.$$

$(\cdot,\cdot)$ is an inner product on $\mathrm{Fun}(G,\mathbb{C})$. By definition the characters of G are in $\mathrm{Fun}(G,\mathbb{C})$. Now we will state some known orthogonality relations of characters of G. For η a character of G and $\sigma \in G$, we have $\overline{\eta(\sigma)} = \eta(\sigma^{-1})$ by Theorem 1.1. It is a known fact that $\mathrm{Irr}(G)$ forms a basis for the set of class functions from G to $\mathbb{C}$. It is indeed an orthonormal basis due to the following result.

Theorem 1.13 ([11, Corollary 2.14, page 20]). *Let $\eta,\eta' \in \mathrm{Irr}(G)$. Then*

$$(\eta,\eta') = \frac{1}{|G|}\sum_{\sigma\in G}\eta(\sigma)\eta'(\sigma^{-1}) = \delta_{\eta\eta'}.$$

Theorem 1.14 (Generalized Orthogonality Relation). *Let $\eta,\eta' \in \mathrm{Irr}(G)$. For any $\tau \in G$*

$$\sum_{\sigma\in G}\eta(\sigma\tau)\eta'(\sigma^{-1}) = \delta_{\eta\eta'}\frac{|G|\eta(\tau)}{\eta(e)}.$$

For complex-valued functions f and g on $\hat{G}$ define

$$(f,g)\hat{} = \frac{1}{|G|}\sum_{\sigma\in \hat{G}} f(\sigma)\overline{g(\sigma)}.$$

Theorem 1.15 ([5, Lemma 3.3, page 145]). *For $\varphi,\phi \in \mathrm{IBr}(G)$, we have*

$$(\Phi_\varphi,\phi)\hat{} = \frac{1}{|G|}\sum_{\sigma\in\hat{G}}\Phi_\varphi(\sigma)\phi(\sigma^{-1}) = \delta_{\varphi\phi}.$$

We establish an orthogonality relation associated with Osima idempotents of blocks of a group G which we call the *generalized orthogonality relation of blocks*. Below we discuss the formulation of this new result.

Theorem 1.16. *For $\mu \in G$,*

$$\sum_{\sigma \in \hat{G}} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\phi \in B' \cap \mathrm{IBr}(G)} \Phi_\varphi(e)\varphi(\sigma)\phi(e)\Phi_\phi(\sigma^{-1}\mu) = \delta_{BB'}|G| \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\mu).$$

Proof. By using Theorem 1.12 we have

$$\begin{aligned} s_B s_{B'} &= \frac{1}{|G|^2} \sum_{\sigma \in \hat{G}} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \Phi_\varphi(e)\varphi(\sigma)\sigma \sum_{\tau \in G} \sum_{\phi \in B' \cap \mathrm{IBr}(G)} \phi(e)\Phi_\phi(\tau)\tau \\ &= \frac{1}{|G|^2} \sum_{\sigma \in \hat{G}} \sum_{\tau \in G} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\phi \in B' \cap \mathrm{IBr}(G)} \Phi_\varphi(e)\varphi(\sigma)\phi(e)\Phi_\phi(\tau)\sigma\tau \\ &= \frac{1}{|G|^2} \sum_{\mu \in G} \sum_{\sigma \in \hat{G}} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\phi \in B' \cap \mathrm{IBr}(G)} \Phi_\varphi(e)\varphi(\sigma)\phi(e)\Phi_\phi(\sigma^{-1}\mu)\mu \end{aligned}$$

and

$$\delta_{BB'} s_B = \sum_{\mu \in G} \delta_{BB'} \frac{1}{|G|} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\mu)\mu.$$

Now, by Theorem 1.11, $s_B s_{B'} = \delta_{BB'}$, so by comparing the coefficients on both sides for a fixed $\mu \in G$ we get,

$$\sum_{\sigma \in \hat{G}} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\phi \in B' \cap \mathrm{IBr}(G)} \Phi_\varphi(e)\varphi(\sigma)\phi(e)\Phi_\phi(\sigma^{-1}\mu) = \delta_{BB'}|G| \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\mu).$$

□

Chapter 2

Symmetrized Tensors

In this chapter we will state some basic definitions and results of different symmetrizers corresponding to characters discussed in Chapter 1.

2.1 Background

For fixed positive integers n, m set

$$\Gamma_{n,m} = \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^n \mid 1 \leq \gamma_i \leq m\}.$$

Let G be a subgroup of the symmetric group S_n. Define a right action on $\Gamma_{n,m}$ by G as follows. For $\sigma \in G$ and $\gamma \in \Gamma_{n,m}$

$$\gamma\sigma = (\gamma_{\sigma(e)}, \ldots, \gamma_{\sigma(n)}). \tag{2.1}$$

Consider the relation for $\gamma, \theta \in \Gamma_{n,m}$ given by $\gamma \sim \theta$ if there is an element $\sigma \in G$ such that $\gamma\sigma = \theta$. This is an equivalence relation on $\Gamma_{n,m}$. We fix a set Δ of representatives of the equivalence classes of $\Gamma_{n,m}$ with respect to $\sim$.

Let V be a complex inner product space of dimension m with orthonormal basis $\{e_1, e_2, \ldots, e_m\}$. $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ (n factors) is the nth tensor power of V. For $\gamma \in \Gamma_{n,m}$ let $e_\gamma = e_{\gamma_1} \otimes e_{\gamma_2} \otimes \cdots \otimes e_{\gamma_n}$. The inner product induced on $V^{\otimes n}$ is given by $\langle e_\gamma, e_\theta \rangle = \prod_{i=1}^{n} (e_{\gamma_i}, e_{\theta_i})$ where $(\cdot, \cdot)$ is the inner product of V. Under this inner product $\{e_\gamma \mid \gamma \in \Gamma_{n,m}\}$ is an orthonormal basis for $V^{\otimes n}$. $V^{\otimes n}$ is a $\mathbb{C}G$-module with the action $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ for $\sigma \in G$ extended linearly to $\mathbb{C}G$.

2.2 Symmetrizers associated with ordinary and Brauer characters of G

In the following discussion $*$ stands for either an irreducible ordinary character or an irreducible Brauer character of G. Denote by S a subset of G where $S = \hat{G}$ when $* \in \mathrm{IBr}(G)$ and $S = G$ when $* \in \mathrm{Irr}(G)$.

The *symmetrizer* corresponding to $*$ is defined by,

$$s_* = \frac{*(e)}{|S|} \sum_{\sigma \in S} *(\sigma)\sigma.$$

The theorems in this section pertaining to $* \in \mathrm{Irr}(G)$ are well known. On the other hand, we generalize some well-known results for $* \in \mathrm{Irr}(G)$ to handle the case of $* \in \mathrm{IBr}(G)$.

Theorem 2.1. *The elements s_η for $\eta \in \mathrm{Irr}(G)$ are orthogonal idempotents.*

Proof. Let $\eta, \chi \in \mathrm{Irr}(G)$. Then by Theorem 1.14 we get,

$$\begin{aligned} s_\eta s_\chi &= \frac{\eta(e)\chi(e)}{|G|^2} \sum_{\sigma \in G} \sum_{\tau \in G} \eta(\sigma)\chi(\tau)\sigma\tau = \frac{\eta(e)\chi(e)}{|G|^2} \sum_{\mu \in G} \sum_{\sigma \in G} \eta(\sigma)\chi(\sigma^{-1}\mu)\mu \\ &= \delta_{\eta\chi} \frac{\eta(e)\eta(e)}{|G|^2} \sum_{\mu \in G} \frac{|G|}{\eta(e)} \eta(\mu)\mu = \delta_{\eta\chi} \frac{\eta(e)}{|G|} \sum_{\mu \in G} \eta(\mu)\mu = \delta_{\eta\chi} s_\eta. \end{aligned}$$

$\square$

The *symmetry class of tensors* V_* corresponding to $*$ is the image of $V^{\otimes n}$ under the symmetrizer s_*:

$$V_* = s_* V^{\otimes n}.$$

Corollary 2.2. *If $\eta, \chi \in \mathrm{Irr}(G)$ and $\eta \neq \chi$, then the vector spaces V_η and V_χ are orthogonal.*

Proof. If $s_\eta v = s_\chi w$ for some $v, w \in V^{\otimes n}$, then

$$s_\eta v = s_\eta(s_\eta v) = s_\eta(s_\chi w) = 0.$$

$\square$

Let $\gamma \in \Gamma_{n,m}$. The *standard symmetrized tensor* e^*_γ corresponding to γ is the image of $e_\gamma \in V^{\otimes n}$ under s_*:

$$e^*_\gamma = s_* e_\gamma = \frac{*(e)}{|S|} \sum_{\sigma \in S} *(\sigma)\sigma e_\gamma = \frac{*(e)}{|S|} \sum_{\sigma \in S} *(\sigma) e_{\gamma\sigma^{-1}}. \tag{2.2}$$

By *o-basis* of a subspace W of $V^{\otimes n}$ we mean an orthogonal basis of W that consists of standard symmetrized tensors. An interesting question to ask is "For which W does there exist an o-basis?" In 1991 Wang and Gong gave an example in [16] of such an o-basis for a symmetry class of tensors V_χ with $\chi \in \mathrm{Irr}(G)$ when G is the dihedral group of order eight. Ever since there have been papers [1, 3, 4, 7, 8, 10, 15] answering the question when such an o-basis exists. All these papers however address the problem in the ordinary character case.

This dissertation is devoted to answering the question of when an o-basis exists for a symmetry class of tensors symmetrized by a Brauer symmetrizer for particular choices of G.

Let $V_\gamma = \langle e_{\gamma\sigma} \mid \sigma \in G\rangle$ and let $V^*_\gamma = s_*(V_\gamma)$. Observe that $V^*_\gamma = \langle e^*_{\gamma\sigma} \mid \sigma \in G\rangle$. V^*_γ is called the *orbital subspace* corresponding to γ. Using the orbital subspaces we can write the symmetry class of tensors as an orthogonal direct sum.

Theorem 2.3 ([9, Theorem 1.1])**.** *We have*

$$V_* = \dot{\sum_{\gamma \in \Delta}} V^*_\gamma \quad (\textit{orthogonal direct sum}).$$

In particular, V_* has an o-basis if and only if V^*_γ has an o-basis for each $\gamma \in \Delta$.

Proof. Let $\beta \in \Gamma_{n,m}$. Then $\beta = \gamma\sigma$ for some $\gamma \in \Delta$ and $\sigma \in G$, so that $e^*_\beta = e^*_{\gamma\sigma} \in V^*_\gamma$. This shows that V_* is contained in (and hence equals) the indicated sum.

The sets $E_\gamma = \{e_{\gamma\sigma} \,|\, \sigma \in G\}$, $\gamma \in \Delta$, are pairwise disjoint subsets of the orthogonal set $\{e_\beta \,|\, \beta \in \Gamma_{n,m}\}$ and are therefore pairwise orthogonal. For each $\gamma \in \Delta$ the subspace V^*_γ is contained in the span of E_γ, so the indicated sum is an orthogonal direct sum.

Assume that V_* has an o-basis B. By the first paragraph, B is the union of the sets $B_\gamma = B \cap V_\gamma^*$, $\gamma \in \Delta$, and these sets are pairwise disjoint by the second paragraph, so B_γ is an o-basis for V_γ^* for each $\gamma \in \Delta$.

Finally, if V_γ^* has an o-basis for each $\gamma \in \Delta$, then the union of these bases is an o-basis for V_*. □

For $\gamma \in \Gamma_{n,m}$ let $G_\gamma = \{\sigma \in G \mid \gamma\sigma = \gamma\}$ the stabilizer subgroup of γ in G.

Theorem 2.4. *For $\gamma \in \Gamma_{n,m}$ and a fixed $\sigma \in G$, we have*

$$(e_{\gamma\sigma}^*, e_\gamma^*) = \frac{*(e)^2}{|S|^2} \sum_{\mu \in S} \sum_{\tau \in \sigma\mu^{-1}S \cap G_\gamma} *(\mu) * (\tau^{-1}\sigma\mu^{-1}).$$

Proof. Let $\gamma \in \Gamma_{n,m}$ and $\sigma \in S$ be fixed. Now by using the Equation 2.2 we get,

$$\begin{aligned}
(e_{\gamma\sigma}^*, e_\gamma^*) &= \frac{*(e)^2}{|S|^2} \sum_{\mu \in S} \sum_{\rho \in S} *(\mu)\overline{*(\rho)}(e_{\gamma\sigma\mu^{-1}}, e_{\gamma\rho^{-1}}) \\
&= \frac{*(e)^2}{|S|^2} \sum_{\mu \in S} \sum_{\substack{\rho \in S \\ \sigma\mu^{-1}\rho \in G_\gamma}} *(\mu) * (\rho^{-1}) \\
&= \frac{*(e)^2}{|S|^2} \sum_{\mu \in S} \sum_{\tau \in \sigma\mu^{-1}S \cap G_\gamma} *(\mu) * (\tau^{-1}\sigma\mu^{-1}).
\end{aligned}$$

□

Corollary 2.5. *For $\gamma \in \Gamma_{n,m}$, $\eta \in \mathrm{Irr}(G)$, and a fixed $\sigma \in G$*

$$(e_{\gamma\sigma}^\eta, e_\gamma^\eta) = \frac{\eta(e)}{|G|} \sum_{\rho \in G_\gamma\sigma} \eta(\rho).$$

Proof. From the above Theorem 2.4 we get

$$(e_{\gamma\sigma}^\eta, e_\gamma^\eta) = \frac{\eta(e)^2}{|G|^2} \sum_{\mu \in G} \sum_{\tau \in G_\gamma} \eta(\mu)\eta(\tau^{-1}\sigma\mu^{-1}) = \frac{\eta(e)^2}{|G|^2} \sum_{\tau \in G_\gamma} \sum_{\mu \in G} \eta(\mu)\eta(\tau^{-1}\sigma\mu^{-1}).$$

Then by using the generalized orthogonality relation (Theorem 1.14) we get

$$(e^{\eta}_{\gamma\sigma}, e^{\eta}_{\gamma}) = \frac{\eta(e)^2}{|G|^2} \sum_{\tau \in G_\gamma} \frac{|G|\eta(\tau^{-1}\sigma)}{\eta(e)} = \frac{\eta(e)}{|G|} \sum_{\tau \in G_\gamma} \eta(\tau^{-1}\sigma) = \frac{\eta(e)}{|G|} \sum_{\rho \in G_\gamma \sigma} \eta(\rho).$$

$\square$

For $\eta \in \mathrm{Irr}(G)$ and for $\gamma \in \Delta$ Freese gives the dimension of the orbital subspace V^{η}_{γ} in [6]:

$$\dim V^{\eta}_{\gamma} = \frac{\eta(e)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \eta(\sigma). \tag{2.3}$$

2.2.1 Symmetrizers associated with PIs

Let $\varphi \in \mathrm{IBr}(G)$ and put $\Phi = \Phi_\varphi$.

The symmetrizer associated with Φ is defined by

$$s_\Phi = \frac{\varphi(e)}{|G|} \sum_{\sigma \in G} \Phi(\sigma)\sigma. \tag{2.4}$$

Note that $s_\Phi \in \mathbb{C}G$. For a PI Φ of G the symmetry class of tensors is defined by $V_\Phi = s_\Phi V^{\otimes n}$. For $\gamma \in \Gamma_{n,m}$ the standard symmetrized tensor is defined by $e^{\Phi}_{\gamma} = s_\Phi e_\gamma$ and the orbital subspace is defined by $V^{\Phi}_{\gamma} = \langle e^{\Phi}_{\gamma\sigma} \mid \sigma \in G \rangle$. With a similar argument as in Theorem 2.3 we have

$$V_\Phi = \dot{\sum}_{\gamma \in \Delta} V^{\Phi}_{\gamma} \quad (\textit{orthogonal direct sum}). \tag{2.5}$$

Theorem 2.6. *For $\sigma \in G$ and $\gamma \in \Gamma_{n,m}$*

$$(e^{\Phi}_{\gamma\sigma}, e^{\Phi}_{\gamma}) = \frac{\varphi(e)^2}{|G|^2} \sum_{\tau \in G} \sum_{\alpha \in G_\gamma} \Phi(\sigma^{-1}\alpha\tau)\overline{\Phi(\tau)}.$$

Proof.

$$\begin{aligned}
\left(e^{\Phi}_{\gamma\sigma}, e^{\Phi}_{\gamma}\right) &= \frac{\varphi(e)^2}{|G|^2} \sum_{\mu\in G}\sum_{\tau\in G} \Phi(\mu)\overline{\Phi(\tau)}\,(e_{\gamma\sigma\mu}, e_{\gamma\tau}) \\
&= \frac{\varphi(e)^2}{|G|^2} \sum_{\tau\in G} \sum_{\substack{\mu\in G \\ \sigma\mu\tau^{-1}\in G_\gamma}} \Phi(\mu)\overline{\Phi(\tau)} \\
&= \frac{\varphi(e)^2}{|G|^2} \sum_{\tau\in G} \sum_{\alpha\in\sigma G\tau^{-1}\cap G_\gamma} \Phi(\sigma^{-1}\alpha\tau)\overline{\Phi(\tau)} \\
&= \frac{\varphi(e)^2}{|G|^2} \sum_{\tau\in G} \sum_{\alpha\in G_\gamma} \Phi(\sigma^{-1}\alpha\tau)\overline{\Phi(\tau)}.
\end{aligned}$$

□

2.2.2 Symmetrizers associated with blocks

Let B be a block of G. The symmetrizer corresponding to B is the Osima idempotent s_B of B (Equation 1.4). The symmetry class of tensors is defined by $V_B = s_B V^{\otimes n}$. For $\gamma \in \Gamma_{n,m}$ the standard symmetrized tensor is defined by $e^B_\gamma = s_B e_\gamma$ and the orbital subspace is defined by $V^B_\gamma = \langle e^B_{\gamma\sigma} \mid \sigma \in G\rangle$.

Lemma 2.7. *For $\gamma \in \Gamma_{n,m}$*

$$e^B_\gamma = \sum_{\eta\in B\cap \mathrm{Irr}(G)} e^\eta_\gamma.$$

Proof. By Equation 1.4 we get,

$$e^B_\gamma = s_B(e_\gamma) = \sum_{\eta\in B\cap\mathrm{Irr}(G)} s_\eta(e_\gamma) = \sum_{\eta\in B\cap\mathrm{Irr}(G)} e^\eta_\gamma.$$

□

Theorem 2.8.

$$V_B = \dot{\sum_{\gamma\in\Delta}} V^B_\gamma \quad (\textit{orthogonal direct sum}).$$

Proof. The argument in the proof is similar to that of the proof of Theorem 2.3, and we omit the details. □

Theorem 2.9. *For* $\gamma \in \Delta$

$$\begin{aligned}\dim V_\gamma^B &= \frac{1}{|G_\gamma|} \sum_{\sigma \in \hat{G}_\gamma} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\sigma^{-1}) \\ &= \frac{1}{|G_\gamma|} \sum_{\sigma \in \hat{G}_\gamma} \sum_{\eta \in B \cap \mathrm{Irr}(G)} \eta(e)\eta(\sigma^{-1}).\end{aligned}$$

Proof. Since s_B is an idempotent we have $\mathrm{rank}\,(s_B) = \mathrm{tr}\,(s_B)$. Then by Equation 1.4 we get,

$$\begin{aligned}\dim V_\gamma^B &= \mathrm{rank}\,(s_B) = \mathrm{tr}\,(s_B) = \mathrm{tr}\,\frac{1}{|G|} \sum_{\sigma \in G} \sum_{\eta \in B \cap \mathrm{Irr}(G)} \eta(e)\eta(\sigma^{-1})\sigma \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{\eta \in B \cap \mathrm{Irr}(G)} \eta(e)\eta(\sigma^{-1})\mathrm{tr}\,(\sigma).\end{aligned}$$

Note here that it makes sense to write $\mathrm{tr}\,(\sigma)$ by viewing σ as a linear transformation on $V^{\otimes n}$. In [6, Equation 13] Freese shows that $\frac{1}{|G|} \sum_{\sigma \in G} \eta(e)\eta(\sigma^{-1})\mathrm{tr}\,(\sigma) = \frac{\eta(e)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \eta(\sigma^{-1})$. So we get,

$$\dim V_\gamma^B = \sum_{\eta \in B \cap \mathrm{Irr}(G)} \frac{\eta(e)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \eta(\sigma^{-1}) = \frac{1}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \sum_{\eta \in B \cap \mathrm{Irr}(G)} \eta(e)\eta(\sigma^{-1}).$$

Now by Equations 1.2 and 1.3 we get, for any $\sigma \in G$,

$$\sum_{\eta \in B \cap \mathrm{Irr}(G)} \eta(e)\eta(\sigma^{-1}) = \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \sum_{\eta \in B \cap \mathrm{Irr}(G)} \varphi(e) d_{\eta\varphi}\eta(\sigma^{-1}) = \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\sigma^{-1}).$$

So it gives

$$\begin{aligned}\dim V_\gamma^B &= \frac{1}{|G_\gamma|}\sum_{\sigma\in G_\gamma}\sum_{\varphi\in B\cap \mathrm{IBr}(G)}\varphi(e)\Phi_\varphi(\sigma^{-1}) = \frac{1}{|G_\gamma|}\sum_{\sigma\in \hat{G}_\gamma}\sum_{\varphi\in B\cap \mathrm{IBr}(G)}\varphi(e)\Phi_\varphi(\sigma^{-1})\\ &= \frac{1}{|G_\gamma|}\sum_{\sigma\in \hat{G}_\gamma}\sum_{\eta\in B\cap \mathrm{Irr}(G)}\eta(e)\eta(\sigma^{-1}).\end{aligned}$$

□

Theorem 2.10. *For $\sigma \in G$,*

$$(e_{\gamma\sigma}^B, e_\gamma^B) = \frac{1}{|G|}\sum_{\mu\in\sigma G_\gamma}\sum_{\varphi\in B\cap \mathrm{IBr}(G)}\varphi(e)\Phi_\varphi(\mu). \tag{2.6}$$

Proof. Recall that V_η and V_χ are orthogonal for $\eta, \chi \in \mathrm{Irr}(G)$ with $\eta \neq \chi$ by Corollary 2.2. Then using Lemma 2.7 and Corollary 2.5 we get

$$\begin{aligned}(e_{\gamma\sigma}^B, e_\gamma^B) &= (\sum_{\eta\in B\cap \mathrm{Irr}(G)} e_{\gamma\sigma}^\eta, \sum_{\chi\in B\cap \mathrm{Irr}(G)} e_\gamma^\chi) = \sum_{\eta,\chi\in B\cap \mathrm{Irr}(G)}(e_{\gamma\sigma}^\eta, e_\gamma^\chi)\delta_{\eta\chi} = \sum_{\eta\in B\cap \mathrm{Irr}(G)}(e_{\gamma\sigma}^\eta, e_\gamma^\eta)\\ &= \sum_{\eta\in B\cap \mathrm{Irr}(G)}\frac{\eta(e)}{|G|}\sum_{\mu\in G_\gamma\sigma}\eta(\mu) = \sum_{\eta\in B\cap \mathrm{Irr}(G)}\frac{\eta(e)}{|G|}\sum_{\mu\in\sigma G_\gamma}\eta(\mu)\\ &= \frac{1}{|G|}\sum_{\mu\in\sigma G_\gamma}\sum_{\eta\in B\cap \mathrm{Irr}(G)}\eta(e)\eta(\mu) = \frac{1}{|G|}\sum_{\mu\in\sigma G_\gamma}\sum_{\varphi\in B\cap \mathrm{IBr}(G)}\varphi(e)\Phi_\varphi(\mu).\end{aligned}$$

□

The following is a useful lemma which we call the translation principle of the orthogonality of symmetrized tensors.

Lemma 2.11. *If the standard symmetrized tensors $e_{\gamma\tau}^B$ and $e_{\gamma\tau'}^B$ are orthogonal, then $e_{\gamma\tau\delta}^B$ and $e_{\gamma\tau'\delta}^B$ are orthogonal for every $\delta \in G$.*

Proof. By Theorem 2.10 we get, for each $\delta \in G$,

$$
\begin{aligned}
(e^B_{\gamma\tau'\delta}, e^B_{\gamma\tau\delta}) &= \frac{1}{|G|} \sum_{\sigma \in \tau'\delta(\tau\delta)^{-1} G_\gamma} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\sigma) \\
&= \frac{1}{|G|} \sum_{\sigma \in \tau'\tau^{-1} G_\gamma} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\sigma) = (e^B_{\gamma\tau'}, e^B_{\gamma\tau}) = 0.
\end{aligned}
$$

□

Chapter 3

Dihedral group

In this chapter we focus on the existence of an o-basis of a symmetry class of tensors associated with a Brauer character, a PI and an Osima idempotent of a block of the dihedral group.

For an integer $n(\geq 3)$, the dihedral group of degree n is the subgroup D_n of the symmetric group S_n, generated by the elements

$$r = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & 1 \end{pmatrix} \quad s = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & n & \dots & 3 & 2 \end{pmatrix},$$

That is $D_n = \{r^k, sr^k \mid 0 \leq k \leq n-1\}$ and $|D_n| = 2n$.

D_n with n even has 4 degree one irreducible characters. Let $\psi_1, \psi_2, \psi_3, \psi_4$ denote these characters. D_n with n odd has only 2 degree one irreducible characters; we denote them with ψ_1, ψ_2. For all n the degree two irreducible characters of D_n are given by χ_h where $1 \leq h < \frac{n}{2}$. For each integer k we get $\chi_h(r^k) = \omega^{hk} + \omega^{-hk} = 2\cos\frac{2\pi hk}{n}$ where $\omega^n = 1$ [14, page 37]. The character table for D_n is given by

	r^k	sr^k	
ψ_1	1	1	
ψ_2	1	-1	
ψ_3	$(-1)^k$	$(-1)^k$	(n even)
ψ_4	$(-1)^k$	$(-1)^{k+1}$	(n even)
χ_h	$2\cos(\frac{2\pi hk}{n})$	0	$1 \leq h < \frac{n}{2}$

We observe from the table that if η a character of degree one or degree two of D_n, then for $\sigma \in D_n$ we have

$$\eta(\sigma) = \eta(\sigma^{-1}). \tag{3.1}$$

Let $G = D_n$. For a fixed prime p write $n = p^q \ell$ with $p \nmid \ell$. The set $\hat{G}$ of p-regular elements of G is given by,

$$\hat{G} = \begin{cases} \{r^{ap^q}, sr^k \mid 0 \le a < \ell, 0 \le k < n\}, & p \neq 2; \\ \{r^{ap^q} \mid 0 \le a < \ell\}, & p = 2. \end{cases}$$

The set of p-regular conjugacy classes of G is,

$$\begin{array}{ll} \{r^{ap^q}, r^{-ap^q}\},\ 0 \le a \le \ell/2,\ \{sr^{2k} \,|\, 0 \le k < n/2\},\ \{sr^{2k+1} \,|\, 0 \le k < n/2\}, & \ell \text{ even, } p \neq 2, \\ \{r^{ap^q}, r^{-ap^q}\},\ 0 \le a \le (\ell-1)/2,\ \{sr^k \,|\, 0 \le k < n\}, & \ell \text{ odd, } p \neq 2, \\ \{r^{ap^q}, r^{-ap^q}\},\ 0 \le a < \ell/2, & p = 2. \end{array}$$

So the number of p-regular conjugacy classes is

$$\varepsilon = \begin{cases} \frac{\ell}{2} + 3, & \text{if } \ell \text{ even, } p \neq 2; \\ \frac{\ell-1}{2} + 2, & \text{if } \ell \text{ odd, } p \neq 2; \\ \frac{\ell-1}{2} + 1, & \text{if } p = 2. \end{cases} \tag{3.2}$$

3.1 Brauer characters of D_n

Our effort in this section is to find conditions for the existence of an o-basis for the symmetry class of tensors corresponding to a Brauer character of the dihedral group $G = D_n$. We begin by listing the distinct irreducible Brauer characters of G.

Recall for an ordinary character η of G the restriction of η to $\hat{G}$ is denoted by $\hat{\eta}$. By 1.7 $\hat{\eta}$ is a Brauer character of G.

For each $1 \leq h \leq \frac{n}{2}$, the Brauer character $\hat{\chi}_h$ is of degree 2. The next lemma gives conditions for when two Brauer characters of degree two are the same.

Lemma 3.1. *For $1 \leq i, j < \frac{n}{2}$, $\hat{\chi}_i = \hat{\chi}_j$ if and only if either $i + j \equiv 0 \mod \ell$ or $i - j \equiv 0 \mod \ell$.*

Proof. Since any degree two character of G is zero on sr^k for all k it is enough to check when two characters are the same on the elements $r^{ap^q} \in G$. Suppose $\hat{\chi}_i = \hat{\chi}_j$. Then for any $0 \leq a < \ell$ we have,

$$0 = \hat{\chi}_i(r^{ap^q}) - \hat{\chi}_j(r^{ap^q}) = 2\cos\frac{2\pi ap^q i}{n} - 2\cos\frac{2\pi ap^q j}{n} = 4\sin\frac{\pi ap^q(i+j)}{n}\sin\frac{\pi ap^q(i-j)}{n}.$$

So $\frac{\pi ap^q(i+j)}{n} = k\pi$ or $\frac{\pi ap^q(i-j)}{n} = k\pi$ for some integer k. This gives $\frac{a(i+j)}{\ell} = k$ or $\frac{a(i-j)}{\ell} = k$, so the result follows since $\ell \nmid a$.

Conversely suppose either $i + j \equiv 0 \mod \ell$ or $i - j \equiv 0 \mod \ell$. With out loss of generality we may assume $i + j = k\ell$ for some integer k. Then

$$\begin{aligned}\hat{\chi}_i(r^{ap^q}) - \hat{\chi}_j(r^{ap^q}) &= 4\sin\frac{\pi ap^q(i+j)}{n}\sin\frac{\pi ap^q(i-j)}{n} = 4\sin\frac{\pi ap^q k\ell}{n}\sin\frac{\pi ap^q(i-j)}{n} \\ &= 4\sin\pi ak\sin\frac{\pi ap^q(i-j)}{n} = 0.\end{aligned}$$

So $\hat{\chi}_i = \hat{\chi}_j$ as desired. □

Some of the restricted degree two characters are not irreducible as given by the following lemma.

Lemma 3.2. *For all n we have $\hat{\chi}_{k\ell} = \hat{\psi}_1 + \hat{\psi}_2$ for $1 \leq k < \frac{p^q}{2}$. When n is even with $p \neq 2$ we also have $\hat{\chi}_{\frac{\ell}{2}+k\ell} = \hat{\psi}_3 + \hat{\psi}_4$ for $0 \leq k < \frac{p^q-1}{2}$.*

Proof. For $0 \leq a < \ell$

$$\hat{\chi}_{k\ell}(r^{ap^q}) = 2\cos(\frac{2\pi k\ell ap^q}{n}) = 2 = \hat{\psi}_1(r^{ap^q}) + \hat{\psi}_2(r^{ap^q}) = (\hat{\psi}_1 + \hat{\psi}_2)(r^{ap^q}),$$

and for $0 \leq b < n$

$$\hat{\chi}_{k\ell}(sr^b) = 0 = \hat{\psi}_1(sr^b) + \hat{\psi}_2(sr^b) = (\hat{\psi}_1 + \hat{\psi}_2)(sr^b).$$

So we have $\hat{\chi}_{k\ell} = \hat{\psi}_1 + \hat{\psi}_2$.

Now assume n is even and $p \neq 2$. Then it makes sense to consider the characters of G given by $\hat{\chi}_{\frac{\ell}{2}+k\ell}$. Now for $0 \leq a < \ell$, we have

$$\hat{\chi}_{\frac{\ell}{2}+k\ell}(r^{ap^q}) = 2\cos\Big(\frac{2\pi\ell(1+2k)ap^q}{2n}\Big) = 2\cos\Big(\pi(1+2k)a\Big),$$

so $\hat{\chi}_{\frac{\ell}{2}+k\ell}(r^{ap^q})$ equals 2 if a is even and -2 if a is odd, and also we have that $\hat{\psi}_3(r^{ap^q})+\hat{\psi}_4(r^{ap^q})$ equals 2 if a is even and -2 if a is odd. For $0 \leq k < n$ we have $\hat{\chi}_{\frac{\ell}{2}+k\ell}(sr^k) = 0 = \hat{\psi}_3(sr^k) + \hat{\psi}_4(sr^k)$. So we get $\hat{\chi}_{\frac{\ell}{2}+k\ell} = \hat{\psi}_3 + \hat{\psi}_4$ as desired. □

Let

$$\epsilon = \begin{cases} 4, & \text{if } \ell \text{ even, } p \neq 2; \\ 2, & \text{if } \ell \text{ odd, } p \neq 2; \\ 1, & \text{if } p = 2. \end{cases} \tag{3.3}$$

For each $1 \leq j \leq \epsilon$, the Brauer character $\varphi_j^1 = \hat{\psi}_j$ is of degree 1.

Theorem 3.3. *Let $G = D_n$. The complete list of distinct irreducible Brauer characters of G is*

$$\varphi_j^1 = \hat{\psi}_j, \textit{ for } 1 \leq j \leq \epsilon, \quad \varphi_i^2 = \hat{\chi}_i \textit{ for } 1 \leq i < \tfrac{\ell}{2}.$$

Proof. For each $1 \leq j \leq \epsilon$, the Brauer character $\hat{\psi}_j$ is of degree one and hence is irreducible. We see the distinctness of these characters by observing the character values for sr^k in the character table above. So we have ϵ distinct irreducible Brauer character of degree one of G.

To see the distinctness of the degree two Brauer characters in the given list observe that for any i, j such that $1 \leq i < j < \frac{\ell}{2}$ we have $j - i < \ell$ and $i + j < \ell$, so $\ell \nmid j - i, i + j$ which gives that $\hat{\chi}_i \neq \hat{\chi}_j$ by Lemma 3.1. Now to show that they are irreducible assume that φ_i^2 is not irreducible for some $1 \le i < \frac{\ell}{2}$. Then φ_i^2 is a sum of two irreducible Brauer characters of degree one. So we write $\varphi_i^2 = \varphi_j^1 + \varphi_k^1$. Now $r^{2p^q} \in \hat{G}$ and $\varphi_i^2(r^{2p^q}) = \varphi_j^1(r^{2p^q}) + \varphi_k^1(r^{2p^q}) = 2$. Now since $1 \le i < \frac{\ell}{2}$ we have $0 < \frac{4\pi p^q i}{n} = \frac{4\pi i}{\ell} < \frac{4\pi \frac{\ell}{2}}{\ell} = 2\pi$. So $\varphi_i^2(r^{2p^q}) = 2\cos\frac{4\pi p^q i}{n} \neq 2$. This gives a contradiction. Therefore φ_i^2 is irreducible. The number of irreducible Brauer characters of G equals the number of p-regular conjugacy classes of G by Theorem 1.6. Using Equation 3.3 we see that the number of all characters in the given list is

$$\frac{\ell}{2} + 3, \text{ if } \ell \text{ even}, p \neq 2; \qquad \frac{\ell - 2}{2} + 2, \text{ if } \ell \text{ even}, p \neq 2; \qquad \frac{\ell - 2}{2} + 1 \text{ if } p = 2.$$

This is same as the number of p-regular conjugacy classes as given by Equation 3.2, so the indicated set is a complete set of irreducible Brauer characters of G. $\square$

Assume $p \neq 2$. For $1 \le i < \ell/2$, put

$$A_i = \{i, k\ell + i, k\ell - i \mid 1 \le k \le \frac{p^q - 1}{2}\}$$

and note that $|A_i| = p^q$.

Lemma 3.4. *Let $1 \le i < \ell/2$. We have $\hat{\chi}_a = \hat{\chi}_i = \varphi_i^2$ for all $a \in A_i$.*

Proof. Let $1 \le i < \ell/2$. No proof is needed when $a = i$. Suppose $a = k\ell + i$ or $a = k\ell - i$. Then $a - i = k\ell$ or $a + i = k\ell$. So by Lemma 3.1 we get $\hat{\chi}_a = \hat{\chi}_i = \varphi_i^2$. $\square$

3.2 PIs of D_n

Let $G = D_n$. Let Φ_j^1, Φ_i^2 be the PIs corresponding to $\varphi_j^1, \varphi_i^2 \in \mathrm{IBr}(G)$ where $j = 1, 2, 3, 4$ and $1 \le i < \frac{\ell}{2}$. Let $\eta \in \mathrm{Irr}(G)$. By Equation 1.2 we have

$$\hat{\eta} = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\eta\varphi}\varphi,$$

where $d_{\eta\varphi}$ are uniquely defined nonnegative integers.

In the following tables we give the values of the decomposition matrix entries $d_{\eta\varphi}$ corresponding to $\eta \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G)$.

For odd n we get

η	φ	$d_{\eta\varphi}$	
ψ_j	φ_j^1	1	$(j = 1, 2)$
$\chi_{k\ell}$	φ_j^1	1	$(1 \leq k \leq \frac{p^q-1}{2},\ j = 1, 2)$
χ_a	φ_i^2	1	$(a \in A_i,\ 1 \leq i < \frac{\ell}{2})$

and for even n we get

η	φ	$d_{\varphi\eta}$	
ψ_j	φ_j^1	1	$(j = 1, 2, 3, 4)$
$\chi_{k\ell}$	φ_j^1	1	$(1 \leq k \leq \frac{p^q-1}{2},\ j = 1, 2)$
$\chi_{\frac{\ell}{2}+k\ell}$	φ_j^1	1	$(0 \leq k \leq \frac{p^q-1}{2} - 1,\ j = 3, 4)$
χ_a	φ_i^2	1	$(a \in A_i,\ 1 \leq i < \frac{\ell}{2})$

Theorem 3.5. *Let $G = D_n$. Then the complete list of* PI*s of G is*

$$\Phi_j^1 = \psi_j + \sum_{k=1}^{\frac{p^q-1}{2}} \chi_{k\ell} \qquad \textit{for } j = 1, 2,$$

$$\Phi_j^1 = \psi_j + \sum_{k=0}^{\frac{p^q-1}{2}-1} \chi_{\frac{\ell}{2}+k\ell} \qquad \textit{for } j = 3, 4,$$

$$\Phi_i^2 = \sum_{a \in A_i} \chi_a, \qquad \textit{for } 1 \leq i < \frac{\ell}{2}.$$

Proof. The proof follows from the tables above and the Equation 1.3. $\square$

3.3 Blocks of D_n

Let $G = D_n$. The relationships of characters belonging to a block of G can be understood by means of the decomposition numbers of G for a considered prime p. Recall that both $\eta, \phi \in \mathrm{Irr}(G)$ belong to the block containing $\varphi \in \mathrm{IBr}(G)$ if both decomposition numbers $d_{\eta\varphi}$ and $d_{\phi\varphi}$ are nonzero. Also recall that each $\eta \in \mathrm{Irr}(G)$ belongs to a unique block. Now by observing the tables above one notes the consistence of characters in blocks of G as given below. For the blocks we give the notation B_b^a, where a gives the lowest degree of the irreducible Brauer characters it contains and b gives the lowest index of the degree a irreducible characters it contains.

- $\varphi_j^1, \psi_j, \chi_{k\ell}$ in a block B_1^1 where $j = 1, 2$ and $1 \leq k \leq \frac{p^q-1}{2}$.
- $\varphi_j^1, \psi_j, \chi_{\frac{\ell}{2}+k\ell}$ in a block B_3^1 where $j = 3, 4$ and $0 \leq k \leq \frac{p^q-1}{2} - 1$.
- for $1 \leq i < \frac{\ell}{2}$, φ_i^2, χ_a in a block B_i^2 where $a \in A_i$.

3.4 Block idempotent symmetrization

In this section we will establish necessary and sufficient conditions for the existence of an o-basis of the symmetry class of tensors V_B corresponding to an Osima idempotent s_B of a block B of $G = D_n$. We will consider the two cases namely B_j^1 the block containing degree one irreducible characters of G and B_i^2 the block consisting only of degree two irreducible characters of G separately when finding the o-basis.

First we state a specialized formula for the inner product given in Theorem 2.10.

Corollary 3.6. *Let $G = D_n$ and let B be a block of G. Then*

$$(e_{\gamma\sigma}^B, e_\gamma^B) = \frac{1}{|G|} \sum_{\mu \in \sigma G_\gamma \cap C_n} \sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\mu).$$

Proof. Fix $0 \leq k \leq n-1$. We observe from the character table for G that $\psi_1(sr^k)+\psi_2(sr^k) = 0$ and $\psi_3(sr^k) + \psi_4(sr^k) = 0$. Therefore by using Theorem 3.5 we get,

$$\sum_{\varphi \in B_1^1 \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(sr^k) = \varphi_1^1(e)\Phi_1^1(sr^k) + \varphi_2^1(e)\Phi_2^1(sr^k)$$

$$= \psi_1(sr^k) + \psi_2(sr^k) + 2\sum_{k=1}^{\frac{p^q-1}{2}} \chi_{k\ell}(sr^k) = 0$$

and

$$\sum_{\varphi \in B_3^1 \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(sr^k) = \varphi_3^1(e)\Phi_3^1(sr^k) + \varphi_4^1(e)\Phi_4^1(sr^k)$$

$$= \psi_3(sr^k) + \psi_4(sr^k) + 2\sum_{k=1}^{\frac{p^q-1}{2}} \chi_{\frac{\ell}{2}+k\ell}(sr^k) = 0.$$

Also for fixed i with $1 \leq i < \frac{\ell}{2}$ we have

$$\sum_{\varphi \in B_i^2 \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(sr^k) = \varphi_i^2(e)\Phi_i^2(sr^k) = \varphi_i^2(e)\sum_{a \in A_i} \chi_a(sr^k) = 0.$$

Now write B for B_1^1, B_3^1, or B_i^2. By Theorem 2.10 we have

$$(e^B_{\gamma\sigma}, e^B_\gamma) = \frac{1}{|G|}\sum_{\mu \in \sigma G_\gamma}\sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\mu) = \frac{1}{|G|}\sum_{\mu \in \sigma G_\gamma \cap C_n}\sum_{\varphi \in B \cap \mathrm{IBr}(G)} \varphi(e)\Phi_\varphi(\mu).$$

□

Lemma 3.7. *Let $G = D_n$. Let B be a block of G. Fix $\gamma \in \Delta$. Let $G_\gamma \cap C_n = \langle r^k \rangle$ with $k|n$ and let t be the largest such that $p^t|k$.*

i) If $H = \{r^{a_0}, r^{a_1}, ..., r^{a_{p^t-1}}\}$ is a list of coset representatives of $\langle r^{p^t}\rangle$ in C_n, then $\{e^B_{\gamma\tau} | \tau \in H\}$ is an orthogonal set. In particular $\{e^B_{\gamma\tau} | \tau \in H'\}$ is an orthogonal set where $H' = \{1, r, ..., r^{p^t-1}\}$.

ii) If $G_\gamma \subseteq C_n$, then $\{e^B_{\gamma\tau}, e^B_{\gamma s\tau} | \tau \in H'\}$ is an orthogonal set.

iii) If e^B_γ and $e^B_{\gamma\rho}$ are orthogonal for some $\rho \in \hat{G} \cap C_n$, then $\{e^B_{\gamma\tau}, e^B_{\gamma\tau\rho} | \tau \in H'\}$ is an orthogonal set.

iv) If $G_\gamma \subseteq C_n$ and if e^B_γ and $e^B_{\gamma\rho}$ are orthogonal for some $\rho \in \hat{G} \cap C_n$, then $\{e^B_{\gamma\tau}, e^B_{\gamma\tau\rho}, e^B_{\gamma s\tau}, e^B_{\gamma s\tau\rho} | \tau \in H'\}$ is an orthogonal set.

Proof. i) Take $r^{a_x}, r^{a_y} \in H$. Then $r^{a_x}\langle r^{p^t}\rangle \neq r^{a_y}\langle r^{p^t}\rangle$ and $p^t \nmid a_x - a_y$. Now we will show that there are no regular elements in the set $r^{a_x}r^{-a_y}G_\gamma \cap C_n$. Take $r^{a_x - a_y + mk} \in r^{a_x}r^{-a_y}G_\gamma \cap C_n$ for some integer $m = 0, ..., \frac{n}{k} - 1$. Assume $a_x - a_y + mk = m'p^q$. Then $a_x - a_y = m'p^q - mk$, implying $p^t \mid a_x - a_y$ which is a contradiction. Therefore by Corollary 3.6 we get $(e^B_{\gamma r^{a_x} r^{-a_y}}, e^B_\gamma) = 0$. Now for any $r^x, r^y \in H'$ with $x \neq y$, we have $p^t \nmid x - y$, so r^x, r^y are distinct coset representatives of $\langle r^{p^t}\rangle$ in C_n. Therefore this is a special case of the argument above so $\{e^B_{\gamma\tau} | \tau \in H'\}$ is an orthogonal set.

ii) By part i) we know that $(e^B_{\gamma\tau}, e^B_{\gamma\mu}) = 0$ for $\tau, \mu \in H'$. To show that the set $\{e^B_{\gamma\tau} e^B_{\gamma s\tau} | \tau \in H'\}$ is orthogonal it remains to show that $(e^B_{\gamma s\tau}, e^B_{\gamma\mu}) = 0$ and $(e^B_{\gamma s\tau}, e^B_{\gamma s\mu}) = 0$ for $\tau, \mu \in H'$. First for $(e^B_{\gamma s\tau}, e^B_{\gamma\mu})$ we have, $(e^B_{\gamma s\tau}, e^B_{\gamma\mu}) = (e^B_{\gamma s\tau\mu^{-1}}, e^B_\gamma) = 0$ by Corollary 3.6 since $s\tau\mu^{-1}G_\gamma \cap C_n = \emptyset$ when $G_\gamma \subseteq C_n$. Now for the other case

$$(e^B_{\gamma s\tau}, e^B_{\gamma s\mu}) = (e^B_{\gamma s\tau s\mu}, e^B_\gamma) = (e^B_{\gamma\tau^{-1}\mu}, e^B_\gamma) = (e^B_{\gamma\mu\tau^{-1}}, e^B_\gamma) = (e^B_{\gamma\mu}, e^B_{\gamma\tau}) = 0,$$

by part i) and this completes the proof.

iii) Suppose $(e^B_\gamma, e^B_{\gamma\rho}) = 0$ for some $\rho \in \hat{G} \cap C_n$. To show that $\{e^B_{\gamma\tau}, e^B_{\gamma\tau\rho} | \tau \in H'\}$ is an orthogonal set we only need to show that $(e^B_{\gamma\tau\rho}, e^B_{\gamma\mu\rho}) = 0$ and $(e^B_{\gamma\mu\rho}, e^B_{\gamma\tau}) = 0$ since part i) takes care of the case for two elements of the form $e^B_{\gamma\tau}$ where $\tau \in H'$. It is easily seen by the translation principle (Lemma 2.11) and part i) above that $(e^B_{\gamma\tau\rho}, e^B_{\gamma\mu\rho}) = (e^B_{\gamma\tau}, e^B_{\gamma\mu}) = 0$.

Now consider $(e^B_{\gamma\mu\rho}, e^B_{\gamma\tau})$. We have $\tau = r^x, \mu = r^y$ and $\rho = r^{\alpha p^q}$ for some integers x, y, and α with $0 \le x, y < p^t$. Then

$$(e^B_{\gamma\mu\rho}, e^B_{\gamma\tau}) = (e^B_{\gamma r^y r^{\alpha p^q}}, e^B_{\gamma r^x}) = (e^B_{\gamma r^{\alpha p^q + y - x}}, e^B_{\gamma}).$$

Now to show that $(e^B_{\gamma\mu\rho}, e^B_{\gamma\tau}) = 0$ it is enough by Corollary 3.6 to show that $r^{\alpha p^q + y - x} G_\gamma$ has no p-regular elements. Assume to the contrary. Then for some integer m, we have $r^{mk + \alpha p^q + y - x} \in r^{\alpha p^q + y - x} G_\gamma \cap \hat{G}$, implying $r^{mk + \alpha p^q + y - x} = r^{m' p^q}$ for some integer m'. But this implies $mk + \alpha p^q + y - x - m' p^q = jn$ for some integer j, which is equivalent to $y - x = m' p^q - mk - \alpha p^q + jn$ implying $p^t \mid y - x$. This is a contradiction since $0 \le x, y < p^t$.

iv) Suppose $(e^B_\gamma, e^B_{\gamma\rho})$=0. To show $\{e^B_{\gamma\tau}, e^B_{\gamma\tau\rho}, e^B_{\gamma s\tau}, e^B_{\gamma s\tau\rho} | \tau \in H'\}$ is an orthogonal set, we only need to show that $(e^B_{\gamma s\mu\rho}, e^B_{\gamma\tau}) = 0, (e^B_{\gamma s\tau}, e^B_{\gamma\mu\rho}) = 0, (e^B_{\gamma s\tau\rho}, e^B_{\gamma\mu\rho}) = 0, (e^B_{\gamma s\tau}, e^B_{\gamma s\mu\rho}) = 0$ and $(e^B_{\gamma s\tau\rho}, e^B_{\gamma s\mu\rho}) = 0$ since all other combinations of elements in the set are shown to be orthogonal in the three previous parts. Note that we have $(e^B_{\gamma s\mu\rho}, e^B_{\gamma\tau}) = (e^B_{\gamma s\mu\rho\tau^{-1}}, e^B_{\gamma})$. Now since $G_\gamma \subseteq C_n$ we see that $s\mu\rho\tau^{-1} G_\gamma \cap C_n = \emptyset$ whence $(e^B_{\gamma s\mu\rho\tau^{-1}}, e^B_\gamma) = 0$ by Corollary 3.6. The argument for the cases $(e^B_{\gamma s\tau}, e^B_{\gamma\mu\rho})$ and $(e^B_{\gamma s\tau\rho}, e^B_{\gamma\mu\rho})$ is the same. Next due to part iii)

$$\begin{aligned}(e^B_{\gamma s\tau}, e^B_{\gamma s\mu\rho}) &= (e^B_{\gamma s\tau(s\mu\rho)^{-1}}, e^B_\gamma) = (e^B_{\gamma s\tau s\mu\rho}, e^B_\gamma) = (e^B_{\gamma\tau^{-1}\mu\rho}, e^B_\gamma) = (e^B_{\gamma\mu\rho\tau^{-1}}, e^B_\gamma) = (e^B_{\gamma\mu\rho}, e^B_{\gamma\tau})\\ &= 0.\end{aligned}$$

Finally by part ii) we get $(e^B_{\gamma s\tau\rho}, e^B_{\gamma s\mu\rho}) = (e^B_{\gamma s\tau}, e^B_{\gamma s\mu}) = 0$ and it completes the proof. $\square$

Now we look at the block B_1^1 of G.

Lemma 3.8. *Fix $\gamma \in \Delta$. We have $G_\gamma \cap C_n = \langle r^a \rangle$ with $a \mid n$. Let t be the largest such that $p^t | a$. Then*

$$\dim V_\gamma^{B_1^1} = \begin{cases} 2p^t, & \text{if } G_\gamma \subseteq C_n; \\ p^t, & \text{otherwise.} \end{cases}$$

Proof. First observe that for any $0 \leq \alpha < \ell$,

$$\chi_{k\ell}(r^{\alpha p^q}) = 2\cos\frac{2\pi k\ell\alpha p^q}{n} = 2.$$

Write $\varphi = \varphi_j^1$. Then by Theorem 2.9, Theorem 3.5 and the fact that $\psi_1(\sigma) + \psi_2(\sigma) = 0$ for all $\sigma \notin C_n$ and $\psi_1(\sigma) + \psi_2(\sigma) = 2$ for all $\sigma \in C_n$ we get

$$\begin{aligned}
\dim V_\gamma^{B_1^1} &= \frac{1}{|G_\gamma|}\sum_{\sigma\in\hat{G}_\gamma}\sum_{\varphi\in B_1^1\cap \mathrm{IBr}(G)}\varphi(e)\Phi_\varphi(\sigma) = \frac{1}{|G_\gamma|}\sum_{\sigma\in\hat{G}_\gamma\cap C_n}\Big(\psi_1(\sigma)+\psi_2(\sigma)+2\sum_{k=1}^{\frac{p^q-1}{2}}\chi_{k\ell}(\sigma)\Big)\\
&= \frac{1}{|G_\gamma|}\Big(2+(p^q-1)2\Big)|\hat{G}_\gamma\cap C_n| = \frac{2p^q}{|G_\gamma|}\frac{n}{ap^{q-t}} = \frac{2}{|G_\gamma|}\frac{n}{a}p^t.
\end{aligned}$$

Now if $G_\gamma \subseteq C_n$, then $|G_\gamma| = \frac{n}{a}$, in which case we get $\dim V_\gamma^{B_1^1} = 2p^t$ and otherwise $|G_\gamma| = \frac{2n}{a}$ which gives $\dim V_\gamma^{B_1^1} = p^t$. $\square$

Let n be even and consider the block B_3^1 of G containing the characters ψ_3 and ψ_4.

Lemma 3.9. *Let $G = D_n$ with n even and fix $\gamma \in \Delta$. We have $G_\gamma \cap C_n = \langle r^a\rangle$ with $a \mid n$. Let t be the largest such that $p^t|a$. Then*

$$\dim V_\gamma^{B_3^1} = \begin{cases} 0, & \text{if } a \text{ is odd;}\\ 2p^t, & \text{if } G_\gamma \subseteq C_n,\ a \text{ is even;}\\ p^t, & \text{if } G_\gamma \nsubseteq C_n,\ a \text{ is even.}\end{cases}$$

Proof. By Theorem 2.9, Theorem 3.5 and the fact that $\psi_3(\sigma) + \psi_4(\sigma) = 0$ for all $\sigma \notin C_n$ we get,

$$\begin{aligned}
\dim V_\gamma^{B_3^1} &= \frac{1}{|G_\gamma|}\sum_{\sigma\in\hat{G}_\gamma}\sum_{\varphi\in B_3^1\cap \mathrm{IBr}(G)}\varphi(e)\Phi_\varphi(\sigma) = \frac{1}{|G_\gamma|}\sum_{\sigma\in\hat{G}_\gamma}\Big(\psi_3(\sigma)+\psi_4(\sigma)+2\sum_{k=1}^{\frac{p^q-1}{2}}\chi_{\frac{\ell}{2}+k\ell}(\sigma)\Big)\\
&= \frac{1}{|G_\gamma|}\sum_{\sigma\in\hat{G}_\gamma\cap C_n}\Big(\psi_3(\sigma)+\psi_4(\sigma)+2\sum_{k=1}^{\frac{p^q-1}{2}}\chi_{\frac{\ell}{2}+k\ell}(\sigma)\Big).
\end{aligned}$$

Suppose that a is odd. Let $a = a'p^t$ for some odd integer a'. Then $\hat{G}_\gamma \cap C_n = \{r^{\epsilon a p^{q-t}} \mid 0 \leq \epsilon \leq \frac{\ell}{a'} - 1\}$. Note that $\frac{\ell}{a'}$ is even.

$$\chi_{\frac{\ell}{2}+k\ell}(r^{\epsilon a p^{q-t}}) = 2\cos\frac{2\pi(\frac{\ell}{2}+k\ell)\epsilon a p^{q-t}}{n} = 2\cos 2\pi(\frac{1}{2}+k)\epsilon a' = 2\cos\pi\epsilon a',$$

which is equal to 2 or -2 depending upon whether ϵ is even or odd. Also $\psi_3(r^{\epsilon a p^{q-t}}) = \psi_4(r^{\epsilon a p^{q-t}})$ equals 1 or -1 according as ϵ is even or odd respectively. So the sum $\psi_3(r^{\epsilon a p^{q-t}}) + \psi_4(r^{\epsilon a p^{q-t}}) + 2\sum_{k=1}^{\frac{p^q-1}{2}} \chi_{\frac{\ell}{2}+k\ell}(r^{\epsilon a p^{q-t}})$ on the right side of the formula for $\dim V_\gamma^{B_3^1}$ above equals $2p^q$ if ϵ is even and $-2p^q$ if ϵ is odd. Therefore in the case a is odd we get

$$\dim V_\gamma^{B_3^1} = \frac{1}{|G_\gamma|}\Big(2p^q\frac{\ell}{2a'} + -2p^q\frac{\ell}{2a'}\Big) = 0.$$

Now suppose a is even. Then any p-regular element in $\langle r^a\rangle$ can be expressed as $r^{2\epsilon p^q}$ for some integer ϵ. We have

$$\chi_{\frac{\ell}{2}+k\ell}(r^{2\epsilon p^q}) = 2\cos\frac{2\pi(\frac{\ell}{2}+k\ell)2\epsilon p^q}{n} = 2\cos 2\pi(\frac{1}{2}+k)2\epsilon = 2,$$

in which case the proof is similar to the proof of Lemma 3.8 above. So we get $\dim V_{B_3^1} = 2p^t$ if G_γ is contained in C_n and $\dim V_{B_3^1} = p^t$ if G_γ is not contained in C_n. □

Theorem 3.10. *Let $G = D_n$. Write $B = B_1^1$. The symmetry class of tensors V_B has an o-basis.*

Proof. By Theorem 2.8 we have

$$V_B = \dot{\sum_{\gamma\in\Delta}} V_\gamma^B,$$

so it suffices to show that V_γ^B has an o-basis for each $\gamma \in \Delta$. Let $\gamma \in \Delta$. By Lemma 3.8 if $G_\gamma \nsubseteq C_n$, then $\dim V_\gamma^B = p^t$. Therefore by part i) of Lemma 3.7 the set $\{e_{\gamma\tau}^B | \tau \in H'\}$ is an orthogonal basis. If $G_\gamma \subseteq C_n$, then $\dim V_\gamma^B = 2p^t$ and in this case $\{e_{\gamma\tau}^B, e_{\gamma s\tau}^B | \tau \in H'\}$ is an orthogonal basis by part ii) of Lemma 3.7. □

Theorem 3.11. *Let $G = D_n$ with n even. Write $B = B_3^1$. The symmetry class of tensors V_B has an o-basis.*

Proof. Due to Theorem 2.8 it suffices to show that V_γ^B has an o-basis for each $\gamma \in \Delta$. Let $\gamma \in \Delta$. We have $G_\gamma \cap C_n = \langle r^a \rangle$ for some integer a. If a is odd, then by Lemma 3.9 $\dim V_\gamma^B = 0$, so V_γ^B has an o-basis. Suppose a is even. Then $\dim V_\gamma^B = p^t$ if $G_\gamma \nsubseteq C_n$, so by part i) of Lemma 3.7 the set $\{e_{\gamma\tau}^B | \tau \in H'\}$ is an orthogonal basis and $\dim V_\gamma^B = 2p^t$ if $G_\gamma \subseteq C_n$, so $\{e_{\gamma\tau}^B, e_{\gamma s\tau}^B | \tau \in H'\}$ is an orthogonal basis by part ii) of Lemma 3.7. $\square$

Now we will bring our attention to the blocks consisting only of degree two characters of G. For each $1 \le i < \frac{\ell}{2}$ the block B_i^2 contains $\varphi_i^2 \in \mathrm{IBr}(G)$ and this is the only irreducible Brauer character of G it contains. Below is a statement for conditions when the dimension of the orbital subspace $V_\gamma^{B_i^2}$ corresponding to a $\gamma \in \Delta$ is not zero.

Theorem 3.12. *Fix $\gamma \in \Gamma_{n,m}$. Then for $1 \le i < \frac{\ell}{2}$ we have $\dim V_\gamma^{B_i^2} \neq 0$ if and only if $\hat{G}_\gamma \cap C_n \subseteq \langle r^{n'} \rangle$, where $n' = \frac{n}{\gcd(n,i)}$.*

Proof. Suppose $\hat{G}_\gamma \cap C_n \nsubseteq \langle r^{n'} \rangle$. We have $\hat{G}_\gamma \cap C_n = \langle r^b \rangle$ with $b | n$, so $r^b \notin \langle r^{n'} \rangle$. Fix i with $1 \le i < \frac{\ell}{2}$. Then for $\sigma \in \hat{G}$ we have by Theorem 3.5 and Lemma 3.4

$$\Phi_i^2(\sigma) = \sum_{a \in A_i} \chi_a(\sigma) = |A_i| \chi_i(\sigma) = p^q \chi_i(\sigma). \tag{3.4}$$

Now by Theorem 2.9

$$\begin{aligned}
\dim V_\gamma^{B_i^2} &= \frac{1}{|G_\gamma|} \sum_{\sigma \in \hat{G}_\gamma} \varphi_i^2(e) \Phi_i^2(\sigma) = \frac{\varphi_i^2(e)}{|G_\gamma|} \sum_{\sigma \in \hat{G}_\gamma} p^q \chi_i(\sigma) = \frac{p^q \varphi_i^2(e)}{|G_\gamma|} \sum_{\sigma \in \hat{G}_\gamma \cap C_n} \chi_i(\sigma) \\
&= \frac{p^q \varphi_i^2(e)}{|G_\gamma|} \sum_{j=0}^{\frac{n}{b}-1} \chi_i(r^{jb}) = \frac{p^q \varphi_i^2(e)}{|G_\gamma|} \sum_{j=0}^{\frac{n}{b}-1} (\omega^{ibj} + \omega^{-ibj}),
\end{aligned}$$

where ω^{ib} is an $\frac{n}{b}th$ root of unity. Now $\sum_{j=0}^{\frac{n}{b}-1} (\omega^{ibj} + \omega^{-ibj}) \neq 0$ if $\omega^{ib} = 1$. But if $\omega^{ib} = 1$, then there is an integer m such that $ib = mn$, so $i''b = mn'$ where $i'' = \frac{i}{\gcd(n,i)}$. Now since

$\gcd(n', i'') = 1$ we get $n'|b$ which is a contradiction. Therefore $\sum_{j=0}^{\frac{n}{b}-1}(\omega^{ibj} + \omega^{-ibj}) = 0$ and hence $\dim V_\gamma^{B_i^2} = 0$.

Conversely suppose $\hat{G}_\gamma \cap C_n \subseteq \langle r^{n'} \rangle$. Note that for any integer c we have

$$\chi_i(r^{cn'}) = 2\cos\frac{2\pi icn'}{n} = 2\cos 2\pi i''c = 2,$$

and in this case $\dim V_\gamma^{B_i^2} = \frac{\phi_i^2(e)}{|G_\gamma|}\sum_{\sigma \in \hat{G}_\gamma \cap C_n} p^q \chi_i(\sigma) \neq 0$. $\square$

Theorem 3.13. *Let $G = D_n$. Fix i with $1 \leq i < \frac{\ell}{2}$. Let $G_\gamma \cap C_n = \langle r^k \rangle$ with $k \mid n$ and let t be the largest integer such that $p^t \mid k$. If $\hat{G}_\gamma \cap C_n \subseteq \langle r^{n'} \rangle$, then*

$$\dim V_\gamma^{B_i^2} = \begin{cases} 4p^t, & \text{if } G_\gamma \subseteq C_n \\ 2p^t, & \text{otherwise.} \end{cases}$$

Proof. Let $\sigma \in \hat{G}_\gamma \cap C_n \subseteq \langle r^{n'} \rangle$. Then $\sigma = r^{cn'}$ for some integer c and

$$\chi_i(\sigma) = \chi_i(r^{cn'}) = 2\cos\frac{2\pi icn'}{n} = 2\cos 2\pi i''c = 2.$$

Therefore by Theorem 2.9 and Equation 3.4

$$\begin{aligned} \dim V_\gamma^{B_i^2} &= \frac{1}{|G_\gamma|}\sum_{\sigma \in \hat{G}_\gamma} \varphi_i^2(e)\Phi_i^2(\sigma) = \frac{\varphi_i^2(e)}{|G_\gamma|}\sum_{\sigma \in \hat{G}_\gamma} p^q \chi_i(\sigma) = \frac{p^q \varphi_i^2(e)}{|G_\gamma|}\sum_{\sigma \in \hat{G}_\gamma \cap C_n} \chi_i(\sigma) \\ &= \frac{2p^q}{|G_\gamma|}\sum_{\sigma \in \hat{G}_\gamma \cap C_n} 2 = \frac{4p^q}{|G_\gamma|}|\hat{G}_\gamma \cap C_n| = \frac{4p^q}{|G_\gamma|}\frac{n}{ap^{q-t}} = \frac{4}{|G_\gamma|}\frac{n}{a}p^t. \end{aligned}$$

Now if $G_\gamma \subseteq C_n$, then $|G_\gamma| = \frac{n}{a}$, so $\dim V_\gamma^{B_i^2} = 4p^t$, and if $G_\gamma \not\subseteq C_n$, then $|G_\gamma| = \frac{2n}{a}$, so $\dim V_\gamma^{B_i^2} = 2p^t$. $\square$

Theorem 3.14. *Let $G = D_n$ and assume $\dim V \geq 2$. For fixed i with $1 \leq i < \frac{\ell}{2}$ write $B = B_i^2$. The space V_B has an o-basis if and only if $\ell' \equiv 0 \mod 4$, where $\ell' = \frac{\ell}{\gcd(\ell, i)}$.*

Proof. Suppose V_B has an o-basis. Then by Theorem 2.8 it follows that V_γ^B has an o-basis for each $\gamma \in \Delta$. Let $\gamma = (1, 2, 2, ..., 2)$. Then $G_\gamma = \{1, s\}$, so $G_\gamma \cap C_n = \langle r^n \rangle$. Now since q is the largest such that $p^q | n$, by Theorem 3.13 $\dim V_\gamma^B = 2p^q$. The space V_γ^B has an o-basis, that is, an orthogonal basis of the form $E = \{e^B_{\gamma\tau_x} \mid \tau_x \in G, 1 \leq x \leq 2p^q\}$. Consider the subgroup $J = \langle r^{p^q} \rangle G_\gamma$ of G. The index of J in G is p^q, so by the pigeonhole principle there is at least one right coset of J containing τ_x and τ_y for some $1 \leq x, y \leq 2p^q$ with $x \neq y$. Then we have $\tau_x\tau_y^{-1} \in J$, so $\tau_x\tau_y^{-1} = r^{mp^q}\beta$ for some integer m and $\beta \in G_\gamma$. Therefore $\tau_x\tau_y^{-1}G_\gamma = r^{mp^q}\beta G_\gamma = r^{mp^q}G_\gamma = \{r^{mp^q}, sr^{-mp^q}\}$. Then by Corollary 3.6 and Equation 3.4

$$0 \quad = \quad (e^B_{\gamma\tau_x}, e^B_{\gamma\tau_y}) = \frac{1}{|G|} \sum_{\sigma \in \tau_x\tau_y^{-1}G_\gamma \cap C_n} \varphi_i^2(e)\Phi_i^2(\sigma) = \frac{\varphi_i^2(e)}{|G|}\Phi_i^2(r^{mp^q}) = \frac{\varphi_i^2(e)}{|G|}p^q\chi_i(r^{mp^q}).$$

So we get $0 = \chi_i(r^{mp^q}) = 2\cos\frac{2\pi imp^q}{n} = \cos\frac{2\pi im}{\ell}$ which gives $\frac{2\pi im}{\ell} = (2k+1)\frac{\pi}{2}$ for some integer k. Let $i' = \frac{i}{\gcd(\ell,i)}$. Then

$$4i'm = \frac{4im}{\gcd(\ell,i)} = \frac{(2k+1)\ell}{\gcd(\ell,i)} = (2k+1)\ell'.$$

So ℓ' is divisible by 4.

Conversely suppose $\ell' \equiv 0 \mod 4$. Fix $\gamma \in \Delta$ such that $\dim V_\gamma^B \neq 0$. Then by Theorem 3.12 we have $\hat{G}_\gamma \cap C_n \subseteq \langle r^{n'} \rangle$ where $n' = \frac{n}{\gcd(n,i)}$. Let $\hat{G}_\gamma \cap C_n = \langle r^a \rangle$ where $a \mid n$. Note that $a = a'n'$ for some integer a'. We will first show that there exists $\tau \in \hat{C}_n$ such that $(e^B_{\gamma\tau}, e^B_\gamma) = 0$. Let $\tau = r^{\delta p^q}$ where $\delta = \frac{\ell'}{4} \in \mathbb{Z}$. Note here that the set of p-regular elements of $r^{\delta p^q}G_\gamma \cap C_n$ is the same as $r^{\delta p^q}\hat{G}_\gamma \cap C_n$ since for some $\tau \in \hat{C}_n$, $\tau\mu \in \hat{C}_n$ if and only if $\mu \in \hat{C}_n$. Then by Corollary 3.6

$$\begin{aligned}(e^B_{\gamma r^{\delta p^q}}, e^B_\gamma) \quad &= \quad \frac{1}{|G|} \sum_{\sigma \in r^{\delta p^q}G_\gamma \cap C_n} \varphi_i^2(e)\Phi_i^2(\sigma) = \frac{\varphi_i^2(e)}{|G|} \sum_{\sigma \in r^{\delta p^q}\hat{G}_\gamma \cap C_n} \Phi_i^2(\sigma) \\ &= \quad \frac{\varphi_i^2(e)}{|G|} \sum_{\iota=0}^{\frac{n}{a}-1} \Phi_i^2(r^{\delta p^q + \iota a}) = \frac{\varphi_i^2(e)}{|G|} \sum_{\iota=0}^{\frac{n}{a}-1} p^q\chi_i(r^{\delta p^q + \iota a}).\end{aligned}$$

Note that since $4 \mid \ell'$, the number $i' = \frac{i}{\gcd(\ell,i)}$ is odd. Therefore using $i'' = \frac{i}{\gcd(n,i)}$ and $\delta = \frac{\ell'}{4}$ we get

$$\begin{aligned}\chi_i(r^{\delta p^q + \iota a}) &= 2\cos\frac{2\pi i(\delta p^q + \iota a'n')}{n} = 2\cos\left(\frac{2\pi i' \gcd(\ell,i)\delta}{\ell} + \frac{2\pi i'' \gcd(n,i)\iota a'n'}{n}\right) \\ &= 2\cos\left(\frac{\pi i'}{2} + 2\pi i''\iota a'\right) = 2\cos\frac{\pi i'}{2} = 0,\end{aligned}$$

so $(e^B_{\gamma r^{\delta p^q}}, e^B_{\gamma}) = 0$. Let t be the largest such that $p^t \mid a$. Then by Theorem 3.13, we have $\dim V^B_\gamma = 2p^t$ if $G_\gamma \nsubseteq C_n$, in which case $\{e^B_{\gamma\tau}, e^B_{\gamma\tau\rho} | \tau \in H'\}$ is an o-basis by Lemma 3.7 part iii), and $\dim V^B_\gamma = 4p^t$ if $G_\gamma \subseteq C_n$, in which case $\{e^B_{\gamma\tau}, e^B_{\gamma\tau\rho}, e^B_{\gamma s\tau}, e^B_{\gamma s\tau\rho} | \tau \in H'\}$ is an o-basis by Lemma 3.7 part iv). □

3.5 Irreducible symmetrization

In this section we will give necessary and sufficient conditions for the existence of an o-basis of the symmetry class of tensors corresponding to an irreducible Brauer character of D_n. We will also show the existence of an orthogonal basis for the Brauer symmetry class of tensors that consists of ordinary standard symmetrized tensors in the case of degree two irreducible Brauer characters of D_n.

Let $G = D_n$. Recall that $\varphi_i^2 \in \mathrm{IBr}(G)$ for $1 \le i < \frac{\ell}{2}$ is of degree two and $\chi_i, \chi_{jl-i}, \chi_{jl+i} \in \mathrm{Irr}(G)$ for $1 \le j \le \frac{p^q-1}{2}$ are of degree two. Recall also that Φ_i^2 denotes the PI corresponding to φ_i^2 and that $\hat{\Phi}_i^2$ denotes the restriction of Φ_i^2 to $\hat{G}$.

Lemma 3.15. *For each $1 \le i < \frac{\ell}{2}$ we have*

$$\hat{\Phi}_i^2 = p^q \varphi_i^2.$$

Proof. By using Theorem 3.5 we write

$$\hat{\Phi}_i^2 = \sum_{a \in A_i} \hat{\chi}_a = \sum_{a \in A_i} \varphi_i^2 = |A_i| \varphi_i^2 = p^q \varphi_i^2$$

$\square$

Lemma 3.16. *For each* $1 \le i < \frac{\ell}{2}$,

$$s_{\varphi_i^2} = \frac{|G|}{p^q|\hat{G}|} \sum_{a \in A_i} s_{\chi_a}.$$

Proof. By the definition of a symmetrizer $s_{\varphi_i^2} = \frac{\varphi_i^2(e)}{|\hat{G}|} \sum_{\sigma \in \hat{G}} \varphi_i^2(\sigma)\sigma$. Now by the Lemma 3.15 above

$$s_{\varphi_i^2} = \frac{2}{|\hat{G}|} \sum_{\sigma \in \hat{G}} \frac{1}{p^q} \hat{\Phi}_i^2(\sigma)\sigma = \frac{2}{|\hat{G}|} \sum_{\sigma \in G} \frac{1}{p^q} \Phi_i^2(\sigma)\sigma,$$

where the second equality holds because Φ_i^2 vanishes off of $\hat{G}$. Now by Theorem 3.5 we get

$$\begin{aligned} s_{\varphi_i^2} &= \frac{2}{p^q|\hat{G}|} \sum_{\sigma \in G} \sum_{a \in A_i} \chi_a(\sigma)\sigma = \frac{|G|}{p^q|\hat{G}|} \sum_{a \in A_i} \frac{2}{|G|} \sum_{\sigma \in G} \chi_a(\sigma)\sigma = \frac{|G|}{p^q|\hat{G}|} \sum_{a \in A_i} \frac{\chi_a(e)}{|G|} \sum_{\sigma \in G} \chi_a(\sigma)\sigma \\ &= \frac{|G|}{p^q|\hat{G}|} \sum_{a \in A_i} s_{\chi_a}, \end{aligned}$$

which completes the proof. $\square$

Lemma 3.17. *For each* $1 \le i < \frac{\ell}{2}$ *we have*

$$V_{\varphi_i^2} = \dot{\sum_{a \in A_i}} V_{\chi_a} \quad (\textit{orthogonal direct sum}).$$

Proof. Let $v \in V^{\otimes n}$. Then $s_{\varphi_i^2}(v) \in V_{\varphi_i^2}$. Let $g = \frac{|G|}{p^q|\hat{G}|}$. By using Theorem 3.16 we get

$$s_{\varphi_i^2}(v) = \sum_{a \in A_i} s_{\chi_a}(gv),$$

so $V_{\varphi_i^2} \subseteq \sum_{a \in A_i} V_{\chi_a}$. Now to show the other inclusion, note that for some arbitrary $v_a \in V^{\otimes n}$, $\sum_{a \in A_i} s_{\chi_a}(v_a)$ is in $\sum_{a \in A_i} V_{\chi_a}$. Therefore there exists an element $\sum_{a \in A_i} s_{\chi_a}(\frac{1}{g} v_a)$ in $V^{\otimes n}$ such that

$$\sum_{a\in A_i} s_{\chi_a}(v_a) = g\sum_{a\in A_i} s_{\chi_a}\Big(\sum_{a\in A_i} s_{\chi_a}(\frac{1}{g}v_a)\Big) = s_{\varphi_i^2}\Big(\sum_{a\in A_i} s_{\chi_a}(\frac{1}{g}v_a)\Big) \in V_{\varphi_i^2},$$

so $\sum_{a\in A_i} V_{\chi_a} \subseteq V_{\varphi_i^2}$ as desired. In the above computation we have used the Lemma 3.16 and the fact that the symmetrizers corresponding to ordinary irreducible characters are orthogonal projections (Theorem 2.1). Since ordinary symmetrized spaces are orthogonal by Corollary 2.2 we have the result. □

Recall that for $\chi \in \mathrm{Irr}(G)$ a standard decomposable symmetrized tensor corresponding to χ is given by e^{χ}_{γ} where $\gamma \in \Gamma_{n,m}$.

Theorem 3.18. *For $1 \leq i < \frac{\ell}{2}$, $V_{\varphi_i^2}$ has an orthogonal basis consisting of decomposable tensors of the form e^{χ}_{γ}, $\chi \in \mathrm{Irr}(G)$, if and only if $\ell' \equiv 0 \mod 4$, where $\ell' = \frac{\ell}{\gcd(\ell,i)}$.*

Proof. Suppose $V_{\varphi_i^2}$ has an orthogonal basis of the stated form. Then in particular V_{χ_i} has an orthogonal basis due to Lemma 3.17. Therefore by [10, Theorem 3.1] we get $n \equiv 0 \mod 4i_2$ where i_2 is the power of 2 such that $\frac{i}{i_2}$ is odd. This means that i_2 is a factor of ℓ and further $\frac{\gcd(\ell,i)}{i_2} = \gcd(\frac{\ell}{i_2}, \frac{i}{i_2})$ is odd. Then since $\ell' = \frac{\ell}{\gcd(\ell,i)}$, for some integer m we get

$$4m = \frac{n}{i_2} = \frac{p^q\ell}{i_2} = \frac{p^q\ell'\gcd(\ell,i)}{i_2} = p^q\ell'\frac{\gcd(\ell,i)}{i_2},$$

so $4 \mid \ell'$ as desired.

Conversely suppose $\ell' \equiv 0 \mod 4$. Let ℓ_2 be the largest factor of ℓ that is expressed as a power of 2. Now since $4 \mid \ell' = \frac{\ell}{\gcd(\ell,i)}$ it is the case that $4 \mid \frac{\ell_2}{\gcd(\ell_2,i_2)}$, which gives $\gcd(\ell_2,i_2) = i_2$, so $4i_2 \mid \ell_2$ and hence $4i_2 \mid \ell$. So we have $n \equiv 0 \mod 4i_2$ and therefore V_{χ_i} has an o-basis by [10, Theorem 3.1]. Now, since $4i_2 \mid \ell$, we have $\ell = a4i_2$ for some integer a, so for a fixed k where $1 \leq k \leq \frac{p^q-1}{2}$ we have

$$k\ell \pm i = k(a4i_2) \pm i_2 i_{2'} = i_2(ka4 \pm i_{2'}),$$

where $i_{2'} = \frac{i}{i_2}$ is odd. Then if $(k\ell \pm i)_2$ is the largest factor of $k\ell \pm i$ as a power of 2, then $(k\ell \pm i)_2 = i_2$, so $n \equiv 0 \mod 4(k\ell \pm i)_2$. Therefore $V_{\chi_{k\ell \pm i}}$ has an orthogonal basis by [10, Theorem 3.1]. This means V_{χ_a} has an o-basis for each $a \in A_i$ and hence by Lemma 3.17 we get the result. $\square$

Recall for a character $*$ of G, $V_\gamma^* = \langle e_{\gamma\sigma}^* \mid \sigma \in G \rangle$.

Lemma 3.19. *For $1 \leq i < \frac{\ell}{2}$ and $\gamma \in \Gamma_{n,m}$*

$$V_\gamma^{\varphi_i^2} = \dot{\sum_{a \in A_i}} V_\gamma^{\chi_a} \quad (\textit{orthogonal direct sum}).$$

Proof. Fix $1 \leq i < \frac{\ell}{2}$ and $\gamma \in \Gamma_{n,m}$. Take $s_{\varphi_i^2}(w) \in V_\gamma^{\varphi_i^2}$ for some $w \in V_\gamma$. Let $g = \frac{|G|}{p^q|\hat{G}|}$. Then by Lemma 3.16,

$$s_{\varphi_i^2}(w) = g \sum_{a \in A_i} s_{\chi_a}(w) = \sum_{a \in A_i} s_{\chi_a}(gw),$$

and this gives the inclusion $V_\gamma^{\varphi_i^2} \subseteq \sum_{a \in A_i} V_\gamma^{\chi_a}$. For the other inclusion consider arbitrary $w_a \in V_\gamma$ and set

$$v := \sum_{a \in A_i} s_{\chi_a}(w_a) \in \sum_{a \in A_i} V_\gamma^{\chi_a}.$$

Note that v is also in V_γ, so $\frac{1}{g}v \in V_\gamma$. Then

$$v = g \sum_{a \in A_i} s_{\chi_a}(\frac{1}{g}v) = s_{\varphi_i^2}(\frac{1}{g}v) \in V_\gamma^{\varphi_i^2},$$

so we have $\sum_{a \in A_i} V_\gamma^{\chi_a} \subseteq V_\gamma^{\varphi_i^2}$.

Since orbital subspaces are orthogonal by Theorem 2.3 we have an orthogonal direct sum as desired. $\square$

For $\gamma \in \Gamma_{n,m}$ recall that G_γ is the stabilizer subgroup of γ and $\overline{\Delta}_\eta = \{\gamma \in \Delta \mid \sum_{\sigma \in G_\gamma} \eta(\sigma) \neq 0\}$. Using Theorem 2.3 and Equation 2.3 we have

$$V_\eta = \dot{\sum_{\gamma\in\Delta}} V_\gamma^\eta = \dot{\sum_{\gamma\in\overline{\Delta}_\eta}} V_\gamma^\eta. \tag{3.5}$$

For $1 \le i < \frac{\ell}{2}$ put $\Lambda_i = \bigcup_{a\in A_i} \overline{\Delta}_{\chi_a}$.

Theorem 3.20. *For $1 \le i < \frac{\ell}{2}$ we have*

$$V_{\varphi_i^2} = \dot{\sum_{\gamma\in\Lambda_i}} V_\gamma^{\varphi_i^2}.$$

Proof. By using Lemma 3.17 and Equation 3.5 we get,

$$V_{\varphi_i^2} = \dot{\sum_{a\in A_i}} V_{\chi_a} = \dot{\sum_{a\in A_i}} \dot{\sum_{\gamma\in\overline{\Delta}_{\chi_a}}} V_\gamma^{\chi_a} = \dot{\sum_{\gamma\in\Lambda_i}} \dot{\sum_{a\in A_i}} V_\gamma^{\chi_a} = \dot{\sum_{\gamma\in\Lambda_i}} V_\gamma^{\varphi_i^2},$$

where the last equality is due to Lemma 3.19. □

Lemma 3.21. *For each $1 \le i < \frac{\ell}{2}$, $\gamma \in \Gamma_{m,n}$, and $\sigma \in G$, we have*

$$(e_{\gamma\sigma}^{\varphi_i^2}, e_\gamma^{\varphi_i^2}) = g^2 \sum_{a\in A_i} (e_{\gamma\sigma}^{\chi_a}, e_\gamma^{\chi_a}),$$

where $g = |G|/(p^q|\hat{G}|)$.

Proof. Fix $1 \le i < \frac{\ell}{2}$, $\gamma \in \Gamma_{m,n}$, and $\sigma \in G$. By Lemma 3.16 we get,

$$e_\gamma^{\varphi_i^2} = s_{\varphi_i^2} e_\gamma = g \sum_{a\in A_i} s_{\chi_a} e_\gamma = g \sum_{a\in A_i} e_\gamma^{\chi_a},$$

so

$$(e_{\gamma\sigma}^{\varphi_i^2}, e_\gamma^{\varphi_i^2}) = g^2 (\sum_{a\in A_i} e_{\gamma\sigma}^{\chi_a}, \sum_{a\in A_i} e_\gamma^{\chi_a}) = g^2 \sum_{a\in A_i} (e_{\gamma\sigma}^{\chi_a}, e_\gamma^{\chi_a}).$$

$\square$

Lemma 3.22. *For a fixed integer a consider the list of numbers of the form $k\ell + a$ where $k = 0, \ldots, p^q - 1$. Then for any integer $0 \leq \delta \leq q$, there are exactly $p^{q-\delta}$ numbers in the list that are divisible by p^δ.*

Proof. Let $0 \leq \delta \leq q$. We first show that for any list of p^δ numbers of the form $k\ell + a$ where $k = 0, \ldots, p^\delta - 1$ there is exactly one number in the list divisible by p^δ. We will show that the remainders when divided by p^δ of the numbers $k\ell + a$ where $k = 0, \ldots, p^\delta - 1$ are distinct.

Let $0 \leq k_1 < k_2 \leq p^\delta - 1$ and write $k_1\ell + a = Q_1 p^\delta + R_1$ and $k_2\ell + a = Q_2 p^\delta + R_2$ with $Q_i, R_i \in \mathbb{Z}$. Assume $R_1 = R_2$. Then $Q_1 < Q_2$ since if $Q_1 \geq Q_2$, then $k_1 \geq k_2$. Now we have

$$k_1\ell + a - Q_1 p^\delta = k_2\ell + a - Q_2 p^\delta \Rightarrow (Q_2 - Q_1)p^\delta = (k_2 - k_1)\ell.$$

This is a contradiction since $p^\delta \nmid (k_2 - k_1)\ell$. So all the remainders of the numbers in the list when divided by p^δ are distinct and hence are $0, 1, \ldots, p^\delta - 1$. Therefore there is exactly one number of the form $k\ell + a$ where $k = 0, \ldots, p^\delta - 1$ that is divisible by p^δ.

Now we consider the list of p^q numbers of the form $k\ell + a$ where $k = 0, \ldots, p^q - 1$ and a is any integer. By the above result we have that there is exactly one number in the list of the first p^δ numbers that is divisible by p^δ. Say $p^\delta \mid k'\ell + a$ where $0 \leq k' \leq p^\delta - 1$. Then to be divisible by p^δ, a number in the list $\{k\ell + a | k = 0, \ldots, p^q - 1\}$ must have the form $(k' + bp^\delta)\ell + a$ where $b = 0, \ldots, p^{q-\delta-1}$. Therefore there are exactly $p^{q-\delta}$ numbers in the list divisible by p^δ. $\square$

Theorem 3.23. *Fix i where $1 \leq i < \frac{\ell}{2}$. There are exactly $p^{q-\delta}$ elements of the set A_i that are divisible by p^δ.*

Proof. For $a = k\ell - i \in A_i$ where $1 \leq k \leq \frac{p^q - 1}{2}$ note that $p^\delta \mid k\ell - i$ if and only if $p^\delta \mid p^q\ell - (k\ell - i) = (p^q - k)\ell + i$. Then by writing $i = 0\ell + i$ we see that the number of elements of A_i that are divisible by p^δ is the same as the number of integers $k\ell + i$, $0 \leq k \leq p^q - 1$, that are divisible by p^δ, which number is $p^{q-\delta}$ by Lemma 3.22. $\square$

Let $n'_a = \frac{n}{\gcd(n,a)}$ for $a \in A_i$.

Lemma 3.24. *Fix $a \in A_i$. Then $n'_a = p^{q-\delta_a}\ell'$, where δ_a is the largest such that $p^{\delta_a} \mid a$.*

Proof. Let δ_a with $0 \leq \delta_a \leq q$, be the largest such that $p^{\delta_a} \mid a$. Observe that δ_a is also the largest such that $p^{\delta_a} \mid \frac{a}{\gcd(\ell,i)}$ since $p \nmid \gcd(\ell, i)$. Then $\gcd(p^q\ell', \frac{a}{\gcd(\ell,i)}) = \gcd(\frac{p^q\ell}{\gcd(\ell,i)}, \frac{a}{\gcd(\ell,i)}) = tp^{\delta_a}$ for some integer t with $p \nmid t$. Now since $t \mid p^q\ell'$ we get $t \mid \ell'$. If $a = i$ we get $\frac{a}{\gcd(\ell,i)} = \frac{i}{\gcd(\ell,i)} = i'$, so $t \mid i'$. On the other hand, if $a = k\ell \pm i$ we get $\frac{a}{\gcd(\ell,i)} = \frac{k\ell \pm i}{\gcd(\ell,i)} = k\ell' \pm i'$ and since $t \mid k\ell' \pm i'$ and $t \mid \ell'$ we get $t \mid i'$. Then since $\gcd(\ell', i') = 1$ it should be that $t = 1$ and this gives $\gcd(p^q\ell', \frac{a}{\gcd(\ell,i)}) = p^{\delta_a}$. Now

$$n'_a = \frac{p^q\ell}{\gcd(p^q\ell, a)} = \frac{p^q\ell}{\gcd(\ell, i) \cdot \gcd(p^q\ell', \frac{a}{\gcd(\ell,i)})} = \frac{p^q\ell}{\gcd(\ell, i) \cdot p^{\delta_a}} = p^{q-\delta_a}\ell'.$$

$\square$

Recall for $\eta \in \mathrm{Irr}(G)$, $\overline{\Delta}_\eta = \{\gamma \in \Delta \mid \sum_{\sigma \in G_\gamma} \eta(\sigma) \neq 0\}$.

Lemma 3.25. *Let $1 \leq h < \frac{n}{2}$. We have $\gamma \in \overline{\Delta}_{\chi_h}$ if and only if G_γ is of the form H or HT where $H \leqslant \langle r^{n'_h} \rangle$ and $T = \langle s \rangle$*

Proof. Suppose $\gamma \in \overline{\Delta}_{\chi_h}$. In [10] it is shown that if $\gamma \in \overline{\Delta}_{\chi_h}$, then G_γ is of the form H or HT, where $H \leqslant \langle r^{n'_h} \rangle$ and $T = \langle t \rangle$ with $t^2 = 1$ and $t \notin C_n$. So the desired result follows since $s \notin C_n$ and $s^2 = 1$. To prove the other direction suppose G_γ is of the form H or HT. First assume G_γ is of the form H. We have $G_\gamma = \langle r^{mn'_h} \rangle$ for some integer m such that $mn'_h \mid n$. Then for any $r^{\epsilon mn'_h} \in G_\gamma$ with $\epsilon \in \mathbb{Z}$ we have

$$\chi_h(r^{\epsilon mn'_h}) = \omega^{h\epsilon mn'_h} + \omega^{-h\epsilon mn'_h}.$$

Now since $h'' = \frac{h}{\gcd(n,h)}$ and $n'_h = \frac{n}{\gcd(n,h)}$ we get,

$$\chi_h(r^{\epsilon mn'_h}) = \omega^{h''\epsilon mn} + \omega^{-h''\epsilon mn} = 2.$$

So $\sum_{\sigma\in G_\gamma}\chi_h(\sigma)\neq 0$, and hence $\gamma\in\overline{\Delta}_{\chi_h}$.

Now suppose G_γ is of the form HT, so that $G_\gamma=\{r^{\epsilon mn'_h}, sr^{\epsilon mn'_h}\mid 0\leq\epsilon\leq\frac{n}{mn'_h}-1\}$ for some m. For each ϵ we have $\chi_h(sr^{\epsilon mn'_h})=0$. So $\sum_{\sigma\in G_\gamma}\chi_h(\sigma)=\sum_{\sigma\in G_\gamma\cap C_n}\chi_h(\sigma)$, which is the same as the sum in the case of $G_\gamma=H$ and hence is nonzero. So $\gamma\in\overline{\Delta}_{\chi_h}$. $\square$

Theorem 3.26. *Let $1\leq i<\frac{\ell}{2}$. The space $V_{\varphi_i^2}$ has an o-basis if and only if either* $\dim V=1$ *or $\ell'\equiv 0\mod 4$, where $\ell'=\frac{\ell}{\gcd(\ell,i)}$.*

Proof. Put $\varphi=\varphi_i^2$. First suppose V_φ has an o-basis and assume $\dim V\neq 1$. Now by Theorem 3.20 the space V_γ^φ has an o-basis for all $\gamma\in\Lambda_i=\bigcup_{a\in A_i}\overline{\Delta}_{\chi_a}$. Let $\gamma=(1,2,\ldots,2)$. Then $G_\gamma=\{1,s\}$. So for all $a\in A_i$, $\gamma\in\overline{\Delta}_{\chi_a}$ since $\chi_a(e)=2$ and $\chi_a(s)=0$ and hence $\gamma\in\Lambda_i$. By Equation 2.3 we have $\dim V_\gamma^{\chi_a}=2$ for each $a\in A_i$. Then by Lemma 3.19 we have $\dim V_\gamma^\varphi=2p^q$, so V_γ^φ has a nonempty o-basis B. We may assume B contains e_γ^φ.

By Lemma 3.21 and Corollary 2.5,

$$(e_{\gamma\sigma}^\varphi,e_\gamma^\varphi)=\sum_{a\in A_i}(e_{\gamma\sigma}^{\chi_a},e_\gamma^{\chi_a})=\sum_{a\in A_i}\frac{\chi_a(e)}{|G_\gamma|}\sum_{\rho\in G_\gamma\sigma}\chi_a(\rho)=\frac{\chi_a(e)}{|G_\gamma|}\sum_{\rho\in G_\gamma\sigma\cap C_n}\sum_{a\in A_i}\chi_a(\rho).$$

First note that $(e_{\gamma s}^\varphi,e_\gamma^\varphi)\neq 0$ because with $\sigma=s$ we get $G_\gamma\sigma\cap C_n=\{1,s\}s\cap C_n=\{1\}$ whence the above sum is not zero. Now we will show that $(e_{\gamma r^{mp^q}}^\varphi,e_\gamma^\varphi)=0$ or $(e_{\gamma sr^{mp^q}}^\varphi,e_\gamma^\varphi)=0$ for some $1\leq m\leq\ell-1$. Assume to the contrary, that is for all $1\leq m\leq\ell-1$, we have $(e_{\gamma r^{mp^q}}^\varphi,e_\gamma^\varphi)\neq 0$ and $(e_{\gamma sr^{mp^q}}^\varphi,e_\gamma^\varphi)\neq 0$. Then for $0\leq x\leq p^q-1$ we have $|\{e_{\gamma\sigma}^\varphi\mid\sigma\in r^x\langle r^{p^q}\rangle\}\cap B|\leq 1$, because for $e_{\gamma r^x r^{m_1p^q}}^\varphi, e_{\gamma r^x r^{m_2p^q}}^\varphi\in\{e_{\gamma\sigma}^\varphi\mid\sigma\in r^x\langle r^{p^q}\rangle\}$ with $1\leq m_1<m_2\leq\ell-1$ we get

$$(e_{\gamma r^x r^{m_2p^q}}^\varphi, e_{\gamma r^x r^{m_1p^q}}^\varphi)=(e_{\gamma r^{(m_2-m_1)p^q}}^\varphi,e_\gamma^\varphi)\neq 0.$$

So $|\{e_{\gamma\sigma}^\varphi\mid\sigma\in C_n\}\cap B|\leq p^q$. Now by our observation $(e_{\gamma s}^\varphi,e_\gamma^\varphi)\neq 0$ and by our assumption we get $\{e_{\gamma\sigma}^\varphi\mid\sigma\in s\langle r^{p^q}\rangle\}\cap B=\emptyset$. For $1\leq x\leq p^q-1$ we have $|\{e_{\gamma\sigma}^\varphi\mid\sigma\in sr^x\langle r^{p^q}\rangle\}\cap B|\leq 1$. So $|\{e_{\gamma\sigma}^\varphi\mid\sigma\in G\setminus C_n\}\cap B|<p^q$. Therefore we get $\dim V_\gamma^\varphi<2p^q$, which contradicts with the observation above. So $(e_{\gamma r^{mp^q}}^\varphi,e_\gamma^\varphi)=0$ or $(e_{\gamma sr^{mp^q}}^\varphi,e_\gamma^\varphi)=0$ for some $1\leq m\leq\ell-1$.

Note that since $G_\gamma = \{1, s\}$ we have $G_\gamma r^{mp^q} \cap C_n = \{r^{mp^q}\}$ and $G_\gamma s r^{mp^q} \cap C_n = \{r^{mp^q}\}$. Recall since r^{mp^q} is p-regular we have $\chi_{k\ell \pm i}(r^{mp^q}) = \chi_i(r^{mp^q})$, so $\sum_{a \in A_i} \chi_a(r^{mp^q}) = p^q \chi_i(r^{mp^q})$. So for $\sigma \in \{r^{mp^q}, sr^{mp^q} \mid 1 \leq m \leq \ell - 1\}$ we have

$$\begin{aligned} 0 = (e^{\varphi}_{\gamma\sigma}, e^{\varphi}_{\gamma}) &= \frac{2}{|G_\gamma|} \sum_{\rho \in G_\gamma \sigma \cap C_n} \sum_{a \in A_i} \chi_a(\rho) = \sum_{a \in A_i} \chi_a(r^{mp^q}) = p^q \chi_i(r^{mp^q}) \\ &= 2p^q \cos \frac{2\pi m p^q i}{n}. \end{aligned}$$

Therefore, $\frac{2\pi m p^q i}{n} = (2c+1)\frac{\pi}{2}$ for some integer c, so

$$4i'm = \frac{4im}{\gcd(\ell, i)} = \frac{(2c+1)\ell}{\gcd(\ell, i)} = (2c+1)\ell'.$$

So ℓ' is divisible by 4.

Conversely suppose that $\dim V = 1$. Then following the same argument in [9, Theorem 2.2], $V_\varphi = \langle e^{\varphi}_{\gamma} \rangle$ with $\gamma = (1, \ldots, 1)$, so V_φ has o-basis $\{e^{\varphi}_{\gamma}\}$ or $\emptyset$ accordingly as $\dim V_\varphi$ is 1 or 0.

Now suppose $\ell' \equiv 0 \mod 4$. To show that there is an o-basis for V_φ, it is enough by Theorem 3.20 to show that there is an o-basis for V^{φ}_{γ} for each $\gamma \in \Lambda_i$. Fix $\gamma \in \Lambda_i$. Then there is $a \in A_i$ such that $\gamma \in \overline{\Delta}_{\chi_a}$ and by Lemma 3.25, G_γ is of the form H or HT where $H \leqslant \langle r^{n'_a} \rangle$ and $T = \langle s \rangle$. Now let $\bar{a} \in A_i$ be such that $\langle r^{n'_{\bar{a}}} \rangle$ is the smallest for which $H \leqslant \langle r^{n'_{\bar{a}}} \rangle$. Let $\delta_{\bar{a}}$ be the largest such that $p^{\delta_{\bar{a}}} \mid \bar{a}$. Then by Lemma 3.24 we have $n'_{\bar{a}} = p^{q-\delta_{\bar{a}}}\ell'$.

We claim that if $p^{\delta_{\bar{a}}} \mid a$ for some $a \in A_i$, then $\gamma \in \overline{\Delta}_{\chi_a}$. Fix $a \in A_i$ and let δ_a be the largest such that $p^{\delta_a} \mid a$. Then if $p^{\delta_{\bar{a}}} \mid a$ it follows that $p^{\delta_{\bar{a}}} \leq p^{\delta_a}$, so $n'_a = p^{q-\delta_a}\ell' \leq p^{q-\delta_{\bar{a}}}\ell' = n'_{\bar{a}}$. Therefore $H \leqslant \langle r^{n'_{\bar{a}}} \rangle \leqslant \langle r^{n'_a} \rangle$. Then it follows from Lemma 3.25 that, $\gamma \in \overline{\Delta}_{\chi_a}$ as claimed.

Also we claim that $\gamma \in \overline{\Delta}_{\chi_a}$ only when $p^{\delta_{\bar{a}}} \mid a$. To see this assume there is $a \in A_i$ such that $\gamma \in \overline{\Delta}_{\chi_a}$, but $p^{\delta_{\bar{a}}} \nmid a$. Note that $H \leq \langle r^{n'_a} \rangle$ by Lemma 3.25. Letting δ_a be the largest such that $p^{\delta_a} \mid a$ we get $p^{\delta_a} < p^{\delta_{\bar{a}}}$, so $n'_{\bar{a}} = p^{q-\delta_{\bar{a}}}\ell' < p^{q-\delta_a}\ell' = n'_a$. This gives $\langle r^{n'_a} \rangle \leqslant \langle r^{n'_{\bar{a}}} \rangle$ which is a contradiction since $\langle r^{n'_{\bar{a}}} \rangle$ is the smallest subgroup to contain H. Therefore the claim holds and then by Lemma 3.19 we can write

$$V_\gamma^\varphi = \dot{\sum_{a\in A_i}} V_\gamma^{\chi_a} = \dot{\sum}_{\substack{a\in A_i \\ p^{\delta_{\bar{a}}}|a}} V_\gamma^{\chi_a}.$$

By Lemma 3.22 there are $p^{q-\delta_{\bar{a}}}$ summands in the direct sum above. In the proof of [10, Theorem 3.1] it is shown that for $\chi \in \mathrm{Irr}(G)$ of degree two, $\dim V_\gamma^\chi = 4$ if $G_\gamma = H$ and $\dim V_\gamma^\chi = 2$ if $G_\gamma = HT$. Then we have $\dim V_\gamma^\varphi = 4p^{q-\delta_{\bar{a}}}$ if $G_\gamma = H$ and $\dim V_\gamma^\varphi = 2p^{q-\delta_{\bar{a}}}$ if $G_\gamma = HT$.

Suppose $G_\gamma = H$. Write $G_\gamma = \langle r^{mn'_{\bar{a}}} \rangle$ for some integer m with $mn'_{\bar{a}} \mid n$. Now we will show that $\{e^\varphi_{\gamma\sigma}, e^\varphi_{\gamma s\sigma} | \sigma \in X\}$ where $X = \{r^{\frac{x\ell'}{4}} | 0 \le x \le 2p^{q-\delta_{\bar{a}}} - 1\}$ is an orthogonal basis for V_γ^φ. To show this we will compare all possible combinations of elements for orthogonality. Let $\sigma, \tau \in X$. Consider the elements $e^\varphi_{\gamma s\tau}$ and $e^\varphi_{\gamma\sigma}$. Then since $G_\gamma s\tau\sigma^{-1} \cap C_n = \emptyset$ we have

$$(e^\varphi_{\gamma s\tau}, e^\varphi_{\gamma\sigma}) = (e^\varphi_{\gamma s\tau\sigma^{-1}}, e^\varphi_\gamma) = \frac{2}{|G_\gamma|} \sum_{\rho\in G_\gamma s\tau\sigma^{-1}\cap C_n} \sum_{a\in A_i} \chi_a(\rho) = 0.$$

For elements $e^\varphi_{\gamma s\tau}$ and $e^\varphi_{\gamma s\sigma}$ we see that $(e^\varphi_{\gamma s\tau}, e^\varphi_{\gamma s\sigma}) = (e^\varphi_{\gamma\tau^{-1}\sigma}, e^\varphi_\gamma) = (e^\varphi_{\gamma\sigma}, e^\varphi_{\gamma\tau})$, so it is enough to check the orthogonality of elements of the form $e^\varphi_{\gamma\sigma}$. In this case it is sufficient to check that $(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma) = 0$ for each $\sigma \in X$ with $\sigma \ne 1$. Fix $\sigma = r^{\frac{x\ell'}{4}} \in X$ with $\sigma \ne 1$. Then $G_\gamma\sigma \cap C_n = \{r^{\epsilon mn'_{\bar{a}} + \frac{x\ell'}{4}} \mid 0 \le \epsilon \le \frac{n}{mn'_{\bar{a}}} - 1\}$, so we get

$$\begin{aligned}(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma) &= \frac{2}{|G_\gamma|} \sum_{\rho\in G_\gamma\sigma\cap C_n} \sum_{a\in A_i} \chi_a(\rho) = \frac{2}{|G_\gamma|} \sum_{\epsilon=0}^{\frac{n}{mn'_{\bar{a}}}-1} \sum_{a\in A_i} \chi_a(r^{\epsilon mn'_{\bar{a}} + \frac{x\ell'}{4}}) \\ &= \frac{2}{|G_\gamma|} \sum_{\epsilon=0}^{\frac{n}{mn'_{\bar{a}}}-1} \sum_{a\in A_i} \chi_a(r^{\epsilon mp^{q-\delta_{\bar{a}}}\ell' + \frac{x\ell'}{4}}) = \frac{2}{|G_\gamma|} \sum_{\epsilon=0}^{\frac{n}{mn'_{\bar{a}}}-1} \sum_{a\in A_i} \chi_a(r^{\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)}).\end{aligned}$$

We observe here that for any $r^t \in C_n$

$$\begin{aligned}
\sum_{a\in A_i} \chi_a(r^t) &= \chi_i(r^t) + \sum_{k=1}^{\frac{p^q-1}{2}} \chi_{k\ell+i}(r^t) + \sum_{k=1}^{\frac{p^q-1}{2}} \chi_{k\ell-i}(r^t) \\
&= \chi_i(r^t) + \sum_{k=1}^{\frac{p^q-1}{2}} \chi_{k\ell+i}(r^t) + \sum_{k=\frac{p^q+1}{2}}^{p^q-1} \chi_{p^q\ell-(k\ell+i)}(r^t) \\
&= \omega^{ti} + \omega^{-ti} + \sum_{k=1}^{\frac{p^q-1}{2}} (\omega^{t(k\ell+i)} + \omega^{-t(k\ell+i)}) + \sum_{k=\frac{p^q+1}{2}}^{p^q-1} (\omega^{t(p^q\ell-(k\ell+i))} + \omega^{-t(p^q\ell-(k\ell+i))}) \\
&= \sum_{k=0}^{p^q-1} (\omega^{t(k\ell+i)} + \omega^{-t(k\ell+i)}).
\end{aligned}$$

So with $t = \frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}} + x)$ we get,

$$\begin{aligned}
(e^{\varphi}_{\gamma\sigma}, e^{\varphi}_{\gamma}) &= \frac{2}{|G_\gamma|} \sum_{\epsilon=0}^{\frac{n}{mn'_{\bar{a}}}-1} \sum_{k=0}^{p^q-1} (\omega^{\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)(k\ell+i)} + \omega^{-\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)(k\ell+i)}) \\
&= \frac{2}{|G_\gamma|} \sum_{\epsilon=0}^{\frac{n}{mn'_{\bar{a}}}-1} \Big(\omega^{\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)i} \sum_{k=0}^{p^q-1} \omega^{\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)k\ell} \\
&\qquad + \omega^{-\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)i} \sum_{k=0}^{p^q-1} \omega^{-\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)k\ell} \Big) \\
&= \frac{2}{|G_\gamma|} \sum_{\epsilon=0}^{\frac{n}{mn'_{\bar{a}}}-1} \Big(\omega^{\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)i} \sum_{k=0}^{p^q-1} (\omega^{\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)\ell})^k \\
&\qquad + \omega^{-\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)i} \sum_{k=0}^{p^q-1} (\omega^{-\frac{\ell'}{4}(4\epsilon m p^{q-\delta_{\bar{a}}}+x)\ell})^k \Big).
\end{aligned}$$

In order to proceed, we need the fact that if j is a positive integer and $\rho \in \mathbb{C}$ is a jth root of unity with $\rho \neq 1$, then

$$\sum_{k=0}^{j-1} \rho^k = \frac{\rho^j - 1}{\rho - 1} = 0.$$

Note that $\omega^{\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)\ell}$ is a p^qth root of unity for all ϵ. If for some fixed ϵ the expression $\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)$ is not a multiple of p^q, then we have $\sum_{k=0}^{p^q-1}(\omega^{\pm\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)\ell})^k = 0$ by the preceding observation.

On the other hand if a fixed ϵ is such that $\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)$ is a multiple of p^q, then since $p \nmid \ell'$ it should be that $4\epsilon mp^{q-\delta_{\bar{a}}}+x = z_\epsilon p^q$ for some integer z_ϵ. Then for such ϵ, on the right hand side of the above equation we get,

$$
\begin{aligned}
\omega^{\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)i} & \sum_{k=0}^{p^q-1}(\omega^{\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)\ell})^k + \omega^{-\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)i}\sum_{k=0}^{p^q-1}(\omega^{-\frac{\ell'}{4}(4\epsilon mp^{q-\delta_{\bar{a}}}+x)\ell})^k \\
&= \omega^{\frac{\ell'}{4}z_\epsilon p^q i}\sum_{k=0}^{p^q-1}\omega^{\frac{\ell'}{4}z_\epsilon p^q \ell k} + \omega^{-\frac{\ell'}{4}z_\epsilon p^q i}\sum_{k=0}^{p^q-1}\omega^{-\frac{\ell'}{4}z_\epsilon p^q \ell k} = p^q(\omega^{\frac{\ell'}{4}z_\epsilon p^q i} + \omega^{-\frac{\ell'}{4}z_\epsilon p^q i}) \\
&= p^q(\omega^{\frac{\ell}{4}z_\epsilon p^q i'} + \omega^{-\frac{\ell}{4}z_\epsilon p^q i'}) = p^q(\omega^{\frac{n}{4}z_\epsilon i'} + \omega^{-\frac{n}{4}z_\epsilon i'}) = p^q\omega^{-\frac{n}{4}z_\epsilon i'}(\omega^{\frac{n}{2}z_\epsilon i'}+1) \\
&= p^q\omega^{-\frac{n}{4}z_\epsilon i'}((-1)^{z_\epsilon i'}+1).
\end{aligned}
$$

Now clearly $i' = \frac{i}{\gcd(\ell,i)}$ is *odd* because when $4 \mid \ell' = \frac{\ell}{\gcd(\ell,i)}$ we have $4 \mid \ell' = \frac{\ell_2}{\gcd(\ell_2,i_2)}$, so $i_2 = \gcd(\ell_2, i_2) = \gcd(\ell,i)_2$, that is, the largest powers of 2 dividing i and $\gcd(\ell,i)$, respectively, are equal. Here we claim that z_ϵ is also an *odd* number. Observe that $x = z_\epsilon p^q - 4\epsilon mp^{q-\delta_{\bar{a}}} = p^{q-\delta_{\bar{a}}}(z_\epsilon p^{\delta_{\bar{a}}} - 4\epsilon m)$, but since x is an integer such that $0 \le x \le 2p^{q-\delta}-1$ we get $z_\epsilon p^{\delta_{\bar{a}}} - 4\epsilon m = 1$, so $z_\epsilon p^{\delta_{\bar{a}}} = 4\epsilon m + 1$ and therefore z_ϵ is odd as claimed. Then the last expression of the above equation $p^q\omega^{-\frac{n}{4}z_\epsilon i'}((-1)^{z_\epsilon i'}+1) = 0$ and this gives us the desired result that $(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma) = 0$.

Now suppose $G_\gamma = HT$. Write $H = \langle r^{mn'_{\bar{a}}}\rangle$ with m an integer satisfying $mn'_{\bar{a}} \mid n$. In this case we will show that $\{e^\varphi_{\gamma\sigma} | \sigma \in X\}$ where $X = \{\sigma = r^{\frac{x\ell'}{4}} | 0 \le x \le 2p^{q-\delta_{\bar{a}}}-1\}$ is an orthogonal basis for V^φ_γ. Let $\sigma \in X$ not be the identity element. It is sufficient to check that $(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma) = 0$. We have $HT\sigma \cap C_n = H\sigma \cap C_n$, so the computation is the same as in the case of $G_\gamma = H$, whence we have shown the desired result.

$\square$

We will state the result for the existence of an o-basis in the case of a degree one irreducible Brauer character as it appears in [9].

Theorem 3.27 ([9, Theorem 2.2])**.** *Let $0 \leq j < \varepsilon$, and put $\varphi = \hat{\psi}_j$. The space V_φ has an o-basis if and only if at least one of the following holds:*

i) $\dim V = 1$,

ii) $p = 2$,

iii) *m is not divisible by p.*

3.6 Projective symmetrization

In this section we will discuss the existence of an o-basis associated with a PI of $G = D_n$. To prevent the redundancy of some computations to follow we introduce some notation below.

$$\varepsilon = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 4, & \text{if } n \text{ is even.} \end{cases}$$

$$T_j = \begin{cases} \{k\ell \mid 1 \leq k \leq \frac{p^q-1}{2}\}, & j = 1, 2; \\ \{\frac{\ell}{2} + k\ell \mid 0 \leq k \leq \frac{p^q-1}{2} - 1\}, & j = 3, 4. \end{cases} \tag{3.6}$$

Then using Theorem 3.5 we can write

$$\Phi_j^1 = \psi_j + \sum_{t \in T_j} \chi_t, \quad \text{for } 1 \leq j \leq \varepsilon, \tag{3.7}$$

$$\Phi_i^2 = \chi_i + \sum_{k=1}^{\frac{p^q-1}{2}} (\chi_{k\ell+i} + \chi_{k\ell-i}), \quad \text{for } 1 \leq i < \frac{\ell}{2}. \tag{3.8}$$

First considering the PIs corresponding to degree one Brauer characters and using the Equation 2.4 the symmetrizer is given by

$$s_{\Phi_i^1} = \frac{\varphi_i^1(e)}{|G|} \sum_{\sigma \in G} \Phi_i^1(\sigma)\sigma.$$

Proposition 3.28. *Fix j with $1 \leq j \leq \varepsilon$. Then*

$$s_{\Phi_j^1} = s_{\psi_j} + \frac{1}{2} \sum_{t \in T_j} s_{\chi_t}.$$

Proof. By Equations 2.4 and 3.7 we get

$$\begin{aligned}
s_{\Phi_j^1} &= \frac{\varphi_j^1(e)}{|G|} \sum_{\sigma \in G} \Phi_j^1(\sigma)\sigma = \frac{\varphi_j^1(e)}{|G|} \sum_{\sigma \in G} (\psi_j(\sigma) + \sum_{t \in T_j} \chi_t(\sigma))\sigma \\
&= \frac{1}{|G|} \sum_{\sigma \in G} \psi_j(\sigma)\sigma + \sum_{t \in T_j} \frac{1}{|G|} \chi_t(\sigma)\sigma \\
&= \frac{\psi_j^1(e)}{|G|} \sum_{\sigma \in G} \psi_j(\sigma)\sigma + \sum_{t \in T_j} \frac{\chi_t(e)}{2|G|} \chi_t(\sigma)\sigma \\
&= s_{\psi_j} + \frac{1}{2} \sum_{t \in T_j} s_{\chi_t}.
\end{aligned}$$

□

Recall that Δ is a set of representatives of the orbits of $\Gamma_{n,m}$ under the action given in Equation 2.1. Then for $1 \leq j \leq \varepsilon$ by Equation 2.5 we have

$$V_{\Phi_j^1} = \dot{\sum_{\gamma \in \Delta}} V_\gamma^{\Phi_j^1}.$$

Theorem 3.29. *Fix j with $1 \leq j \leq \varepsilon$ and fix $\gamma \in \Delta$. Then*

$$V_\gamma^{\Phi_j^1} = V_\gamma^{\psi_j} \dot{+} \dot{\sum_{t \in T_j}} V_\gamma^{\chi_t} \quad \textit{(orthogonal direct sum)}$$

Proof. Let $s_{\Phi_j^1}(v) \in V_\gamma^{\Phi_j^1}$. Then it is clear from Proposition 3.28 above that $s_{\Phi_j^1}(v) \in V_\gamma^{\psi_j} + \sum_{t\in T_j} V_\gamma^{\chi_t}$. To show the other inclusion consider $s_{\psi_j}(w) + \sum_{t\in T_j} s_{\chi_t}(w_t) \in V_\gamma^{\psi_j} + \sum_{t\in T_j} V_\gamma^{\chi_t}$. Note that $s_{\psi_j}(w) + \sum_{t\in T_j} s_{\chi_t}(2w_t) \in V_\gamma$. Then

$$\begin{aligned}
s_{\psi_j}(w) + \sum_{t\in T_j} s_{\chi_t}(w_t) &= s_{\psi_j}(w) + \frac{1}{2}\sum_{t\in T_j} s_{\chi_t}(2w_t) = s_{\psi_j}(w) + \frac{1}{2}\sum_{t\in T_j}\sum_{b} s_{\chi_b}s_{\chi_t}(2w_t) \\
&= \Big(s_{\psi_j} + \frac{1}{2}\sum_b s_{\chi_b}\Big)\Big(s_{\psi_j}(w) + \sum_{t\in T_j} s_{\chi_t}(2w_t)\Big) \\
&= \Big(s_{\psi_j} + \frac{1}{2}\sum_b s_{\chi_b}\Big)\Big(s_{\psi_j}(w) + \sum_{t\in T_j} s_{\chi_t}(2w_t)\Big) \\
&= s_{\Phi_j^1}\Big(s_{\psi_j}(w) + \sum_{t\in T_j} s_{\chi_t}(2w_t)\Big) \in V_\gamma^{\Phi_j^1},
\end{aligned}$$

where we have used that $s_\chi s_\psi = \delta_{\chi\psi} s_\chi$ for all $\chi, \psi \in \mathrm{Irr}(G)$. This shows that $V_\gamma^{\Phi_j^1} = V_\gamma^{\psi_j} + \sum_{t\in T_j} V_\gamma^{\chi_t}$. The orthogonality follows from the argument in the proof of Theorem 2.3. $\square$

Fix j with $1 \le j \le \varepsilon$ and $\gamma \in \Gamma_{n,m}$. We have by Theorem 2.6

$$(e_{\gamma\sigma}^{\Phi_j^1}, e_\gamma^{\Phi_j^1}) = \frac{\varphi_j^1(e)^2}{|G|^2} \sum_{\tau\in G} \sum_{\alpha\in G_\gamma} \Phi_j^1(\sigma^{-1}\alpha\tau)\overline{\Phi_j^1(\tau)}.$$

Below we state as a corollary a useful form of this inner product.

Corollary 3.30.

$$(e_{\gamma\sigma}^{\Phi_j^1}, e_\gamma^{\Phi_j^1}) = \frac{1}{2|G|} \sum_{\alpha\in G_\gamma} \Big(2\psi_j(\sigma^{-1}\alpha) + \sum_{t\in T_j} \chi_t(\sigma^{-1}\alpha)\Big) = \frac{1}{2|G|} \sum_{\alpha\in G_\gamma} \Big(\psi_j(\sigma^{-1}\alpha) + \Phi_j^1(\sigma^{-1}\alpha)\Big).$$

Proof. Using Equation 3.7, Theorem 1.13 and Theorem 1.14 we get,

$$
\begin{aligned}
(e_{\gamma\sigma}^{\Phi_j^1}, e_{\gamma}^{\Phi_j^1}) &= \frac{\varphi_j^1(e)^2}{|G|^2} \sum_{\tau \in G} \sum_{\alpha \in G_\gamma} \Phi_j^1(\sigma^{-1}\alpha\tau)\overline{\Phi_j^1(\tau)} \\
&= \frac{\varphi_j^1(e)^2}{|G|^2} \sum_{\tau \in G} \sum_{\alpha \in G_\gamma} \Big(\psi_j(\sigma^{-1}\alpha\tau) + \sum_{t \in T_j} \chi_t(\sigma^{-1}\alpha\tau)\Big)\Big(\overline{\psi_j(\tau)} + \sum_{u \in T_j} \overline{\chi_u(\tau)}\Big) \\
&= \frac{\varphi_j^1(e)^2}{|G|^2} \sum_{\tau \in G} \sum_{\alpha \in G_\gamma} \Big(\psi_j(\sigma^{-1}\alpha\tau)\psi_j(\tau^{-1}) + \sum_{t \in T_j} \chi_t(\sigma^{-1}\alpha\tau)\chi_t(\tau^{-1})\Big) \\
&= \frac{\varphi_j^1(e)^2}{|G|^2} \sum_{\alpha \in G_\gamma} \Big(\sum_{\tau \in G} \psi_j(\sigma^{-1}\alpha\tau)\psi_j(\tau^{-1}) + \sum_{t \in T_j} \sum_{\tau \in G} \chi_t(\sigma^{-1}\alpha\tau)\chi_t(\tau^{-1})\Big) \\
&= \frac{\varphi_j^1(e)^2}{|G|^2} \sum_{\alpha \in G_\gamma} \Big(\frac{|G|}{\psi_j(e)}\psi_j(\sigma^{-1}\alpha) + \sum_{t \in T_j} \frac{|G|}{\chi_t(e)}\chi_t(\sigma^{-1}\alpha)\Big).
\end{aligned}
$$

Now since $\varphi_j^1(e) = \psi_j^1(e) = 1$ and $\chi_t(e) = 2$ for each $t \in T_j$ we get,

$$
(e_{\gamma\sigma}^{\Phi_j^1}, e_{\gamma}^{\Phi_j^1}) = \frac{1}{2|G|} \sum_{\alpha \in G_\gamma} \Big(2\psi_j(\sigma^{-1}\alpha) + \sum_{t \in T_j} \chi_t(\sigma^{-1}\alpha)\Big) = \frac{1}{2|G|} \sum_{\alpha \in G_\gamma} \Big(\psi_j(\sigma^{-1}\alpha) + \Phi_j^1(\sigma^{-1}\alpha)\Big).
$$

□

Theorem 3.31. *For $i = 1, 2, 3, 4$ the space $V_{\Phi_j^1}$ has an o-basis if and only if at least one of the following holds.*

i) $\dim V = 1$,

ii) $p = 2$,

iii) n is not divisible by p.

Proof. If $\dim V = 1$, then $V_{\Phi_j^1} = \langle e_{\gamma}^{\Phi_j^1} \rangle$, where $\gamma = (1, 1, \ldots, 1)$, so $V_{\Phi_j^1}$ has o-basis $\{e_{\gamma}^{\Phi_j^1}\}$ or $\emptyset$ according as $\dim V_{\Phi_j^1}$ is 1 or 0.

Assume $p = 2$. Then Φ_j^1 is an ordinary irreducible character of the group $\hat{G} = \langle r^{p^q} \rangle \leq C_n$. Then using Freese's result for the dimension of an orbital subspace [6], $\dim V_{\gamma}^{\Phi_j^1}$ is at most one and hence has an o-basis. So $V_{\Phi_j^1}$ has an o-basis by Equation 2.5.

Assume n is not divisible by p. Then $\hat{G} = G$ and hence $\Phi_j^1 = \psi_j$. Since ψ_j is of degree one, each orbital subspace has dimension at most one and hence $V_{\Phi_j^1}$ has an o-basis by the same argument as in the above paragraph.

Now assume that none of the three conditions stated in the theorem holds. Let $\gamma = (1, 2, \ldots, 2)$. which is in $\Gamma_{n,m}$ since $\dim V \geq 2$. We show that $(e_{\gamma\sigma}^{\Phi_j^1}, e_{\gamma}^{\Phi_j^1}) \neq 0$ for every $\sigma \in G$. Note that $G_\gamma = \{1, s\}$.

First let $\sigma \in G \setminus \hat{G}$. Since $p \neq 2$ we have $\sigma \in C_n$. Using Corollary 3.30 we get,

$$2|G|(e_{\gamma\sigma}^{\Phi_j^1}, e_{\gamma}^{\Phi_j^1}) = (\psi_j + \Phi_j^1)(\sigma^{-1}) + (\psi_j + \Phi_j^1)(\sigma^{-1}s) = \psi_j(\sigma) + 2\psi_j(s\sigma) \neq 0.$$

Now let $\sigma \in \hat{G}$. Then $\sigma \in \{r^{ap^q}, sr^b \mid 0 \leq a < \ell, 0 \leq b < n\}$. Assume $\sigma = r^{ap^q}$ for some $0 \leq a < \ell$. We have

$$\chi_{k\ell}(\sigma) = \chi_{k\ell}(r^{kp^q}) = 2\cos\frac{2\pi j\ell kp^q}{n} = 2,$$

and

$$\chi_{\frac{\ell}{2}+k\ell}(\sigma) = \chi_{\frac{\ell}{2}+k\ell}(r^{ap^q}) = 2\cos\frac{2\pi(\frac{\ell}{2}+k\ell)r^{ap^q}}{n} = 2\cos\pi ap^q = (-1)^a 2.$$

We also note that for a fixed j we have that $\psi_j(r^{ap^q})$ and $\chi_t(r^{ap^q})$ $(t \in T_j)$ are all positive or all negative at the same time for a given a. Recall by Equation 3.1 we have $\psi_j(r^{-ap^q}) = \psi_j(r^{ap^q})$ and $\chi_t(r^{-ap^q}) = \chi_t(r^{ap^q})$. So

$$\begin{aligned} 2|G|(e_{\gamma\sigma}^{\Phi_j^1}, e_{\gamma}^{\Phi_j^1}) &= (\psi_j + \Phi_j^1)(r^{-ap^q}) + (\psi_j + \Phi_j^1)(r^{-ap^q}s) \\ &= 2\psi_j(r^{ap^q}) + 2\psi_j(sr^{ap^q}) + \sum_{t\in T_j}\chi_t(r^{ap^q}) \neq 0 \end{aligned}$$

using that $p \neq 2$ and that n is divisible by p so that T_j is nonempty. Now assume $\sigma = sr^b$ for some $0 \leq b < n$. By Corollary 3.30 we get

$$\begin{aligned}(e_{\gamma\sigma}^{\Phi_j^1}, e_\gamma^{\Phi_j^1}) &= \frac{1}{2|G|}\Big((\psi_j + \Phi_j^1)(sr^b) + (\psi_j + \Phi_j^1)(sr^b s)\Big)\\ &= \frac{1}{2|G|}\Big((\psi_j + \Phi_j^1)(sr^b) + (\psi_j + \Phi_j^1)(r^{-b})\Big) = (e_{\gamma r^b}^{\Phi_j^1}, e_\gamma^{\Phi_j^1}).\end{aligned}$$

Now we get $(e_{\gamma\sigma}^{\Phi_j^1}, e_\gamma^{\Phi_j^1}) \neq 0$ since $(e_{\gamma r^b}^{\Phi_j^1}, e_\gamma^{\Phi_j^1}) \neq 0$ by the two previous cases. By Equation 2.3 we have $\dim V_\gamma^{\chi_t} = \frac{\chi_t(e)}{|G_\gamma|}\sum_{\sigma \in G_\gamma} \chi_t(\sigma) = 2$ for each $t \in T_j$. As observed earlier, T_j is nonempty, so $\dim V_\gamma^{\Phi_j^1} > 1$. So we conclude that $V_{\Phi_j^1}$ does not have an o-basis and the proof is complete. $\square$

Now we will consider the PIs corresponding to degree two Brauer characters of $G = D_n$.

Corollary 3.32. *For $1 \leq i < \frac{\ell}{2}$*

$$s_{\Phi_i^2} = \frac{p^q|\hat{G}|}{|G|} s_{\varphi_i^2}.$$

Proof. Let $1 \leq i < \frac{\ell}{2}$. By Equation 2.4 and Lemma 3.15,

$$s_{\Phi_i^2} = \frac{\varphi_i^2(e)}{|G|}\sum_{\sigma \in G} \Phi_i^2(\sigma)\sigma = \frac{\varphi_i^2(e)}{|G|}\sum_{\sigma \in \hat{G}} \hat{\Phi}_i^2(\sigma)\sigma = \frac{p^q\varphi_i^2(e)}{|G|}\sum_{\sigma \in \hat{G}} \varphi_i^2(\sigma)\sigma = \frac{p^q|\hat{G}|}{|G|} s_{\varphi_i^2}.$$

$\square$

Recall that $V_{\Phi_i^2} = s_{\Phi_i^2}(V^{\bigotimes n})$ and $V_{\varphi_i^2} = s_{\varphi_i^2}(V^{\bigotimes n})$.

Theorem 3.33. *For $1 \leq i < \frac{\ell}{2}$ we have*

$$V_{\Phi_i^2} = V_{\varphi_i^2}.$$

Proof. Take $s_{\Phi_i^2}(w) \in V_{\Phi_i^2}$. Then using Corollary 3.32 we get

$$s_{\Phi_i^2}(w) = \frac{p^q|\hat{G}|}{|G|} s_{\varphi_i^2}(w) = s_{\varphi_i^2}(\frac{p^q|\hat{G}|}{|G|} w) \in V_{\varphi_i^2},$$

which shows the inclusion $V_{\Phi_i^2} \subseteq V_{\varphi_i^2}$. Then take $s_{\varphi_i^2}(u) \in V_{\varphi_i^2}$. Again using Corollary 3.32 we see that

$$s_{\varphi_i^2}(u) = \frac{|G|}{p^q|\hat{G}|} s_{\Phi_i^2}(u) = s_{\Phi_i^2}(\frac{|G|}{p^q|\hat{G}|} u) \in V_{\Phi_i^2},$$

which gives the other inclusion. So we get $V_{\Phi_i^2} = V_{\varphi_i^2}$. $\square$

Theorem 3.34. *Fix i where $1 \leq i < \frac{\ell}{2}$ and let $\gamma \in \Gamma_{n,m}$. Then*

$$e_\gamma^{\Phi_i^2} = \frac{p^q|\hat{G}|}{|G|} e_\gamma^{\varphi_i^2}.$$

Proof. By Corollary 3.32 we get that,

$$e_\gamma^{\Phi_i^2} = s_{\Phi_i^2}(e_\gamma) = \frac{p^q|\hat{G}|}{|G|} s_{\varphi_i^2}(e_\gamma) = \frac{p^q|\hat{G}|}{|G|} e_\gamma^{\varphi_i^2}.$$

$\square$

Lemma 3.35. *Fix i with $1 \leq i < \frac{\ell}{2}$. Then $V_{\Phi_i^2}$ has an o-basis if and only if $V_{\varphi_i^2}$ has an o-basis.*

Proof. We observe that,

$$(e_\gamma^{\Phi_i^2}, e_\tau^{\Phi_i^2}) = (\frac{p^q|\hat{G}|}{|G|} e_\gamma^{\varphi_i^2}, \frac{p^q|\hat{G}|}{|G|} e_\tau^{\varphi_i^2}) = (\frac{p^q|\hat{G}|}{|G|})^2 (e_\gamma^{\varphi_i^2}, e_\tau^{\varphi_i^2}).$$

So $(e_\gamma^{\Phi_i^2}, e_\tau^{\Phi_i^2}) = 0$ if and only if $(e_\gamma^{\varphi_i^2}, e_\tau^{\varphi_i^2}) = 0$. $\square$

Theorem 3.36. *Fix i with $1 \leq i < \frac{\ell}{2}$. Then $V_{\Phi_i^2}$ has an o-basis if and only if either $\dim V = 1$ or ℓ' is divisible by 4, where $\ell' = \ell/\gcd(\ell, i)$.*

Proof. The result follows from Lemma 3.35 and Theorem 3.26. $\square$

Theorem 3.37. *Fix i with $1 \leq i < \frac{\ell}{2}$. Then $V_{\Phi_i^2}$ has an orthogonal basis consisting of decomposable tensors of the form e_γ^χ if and only if $\ell' \equiv 0 \mod 4$. Where $\ell' = \frac{\ell}{\gcd(\ell,i)}$.*

Proof. The result follows from Theorem 3.33 and Theorem 3.18. $\square$

Chapter 4

Symmetric group

In this chapter we will discuss some results associated with Brauer characters of a symmetric group.

For some positive integer n, the symmetric group S_n of degree n is the group of permutations of the set $\{1, 2, \ldots, n\}$ with the binary operation defined by function composition. The number of elements of S_n is $n!$. A permutation σ of S_n is given by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(e) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

or

$$\sigma = (i_{11}, \ldots, i_{1r_1})(i_{21}, \ldots, i_{2r_2}) \cdots (i_{s1}, \ldots, i_{sr_s}),$$

where $1 \leq i_{ab} \leq n$, $i_{ab} = i_{cd}$ implies $a = c$ and $b = d$, and $\sigma(i_{ab}) = i_{a(b+1)}$ $(b < r_a)$, $\sigma(i_{ar_a}) = i_{a1}$. The latter is in a factored form, where the factors are *disjoint cycles*. The length of a cycle is the number of numbers that appear in the cycle. The lengths of the cycles $r_1, \ldots, r_s$ of σ, when arranged in non-increasing order is called the *cycle type* of σ. Two permutations are conjugate in S_n if and only if they have the same cycle type ([13, Theorem 2.4, page 292]). In particular, the number of conjugacy classes of S_n is equal to the number of different cycle types of the elements of S_n. So the number of irreducible characters of S_n is the same as the number of different cycle types of the elements of S_n.

The order of a cycle equals the length of the cycle. Let σ be a permutation with the cycle type $(r_1, r_2, \ldots, r_k)$. Then it can easily be observed that the order of σ equals the least common multiple of the numbers $r_1, r_2, \ldots, r_k$. In light of this a p-regular element of S_n is a permutation with cycle type consisting of numbers not divisible by p.

In the following study we will just consider the principal Braur character of $G = S_n$, which is the character ψ given by $\psi(\sigma) = 1$ for each $\sigma \in G$. As the Brauer character afforded by the trivial KG-module ψ is irreducible.

Theorem 4.1. *Let $G = S_n$ with $n \geq 3$. Assume that $\dim V \geq 2$ and $p \neq 2$. Then $(e^{\psi}_{\gamma\sigma}, e^{\psi}_{\gamma}) \neq 0$ for all $\sigma \in G$, where $\gamma = (1, \ldots, 1, 2)$. In particular, if $\dim V^{\psi}_{\gamma} > 1$, then V_ψ does not have an o-basis.*

Proof. Let $\gamma = (1, \ldots, 1, 2)$. We can assume γ to be the representative of the orbit containing it, so $\gamma \in \Delta$. Observe that $G_\gamma = \{\sigma \in S_n \,|\, \sigma(n) = n\} \cong S_{n-1}$.

By Theorem 2.4, for any $\sigma \in G$ we have

$$(e^{\psi}_{\gamma\sigma}, e^{\psi}_{\gamma}) = \frac{\psi(e)^2}{|\hat{G}|^2} \sum_{\mu \in \hat{G}} \sum_{\tau \in \sigma\mu^{-1}\hat{G} \cap G_\gamma} \psi(\mu)\psi(\tau^{-1}\sigma\mu^{-1}) = \frac{\psi(e)^2}{|\hat{G}|^2} \sum_{\mu \in \hat{G}} \sum_{\rho \in \mu\sigma^{-1}G_\gamma \cap \hat{G}} \psi(\mu)\psi(\rho^{-1})$$
$$= \frac{1}{|\hat{G}|^2} \sum_{\mu \in \hat{G}} |\mu\sigma^{-1}G_\gamma \cap \hat{G}|.$$

So $(e^{\psi}_{\gamma\sigma}, e^{\psi}_{\gamma}) = 0$ only when $\mu\sigma^{-1}G_\gamma \cap \hat{G} = \emptyset$ for some $\mu \in \hat{G}$. We will show that for all $\sigma \in G$ there is some $\mu \in \hat{G}$ such that $\mu\sigma^{-1}G_\gamma \cap \hat{G} \neq \emptyset$, which implies that $(e^{\psi}_{\gamma\sigma}, e^{\psi}_{\gamma}) \neq 0$.

We claim here that the cyclic group $H = \langle(1, 2, \ldots, n)\rangle$ is a set of right coset representatives of G_γ in G. For $h_1, h_2 \in H$ with $h_1 \neq h_2$ we have $h_1 h_2^{-1}(n) \neq n$, so $G_\gamma h_1 \neq G_\gamma h_2$. Also $|G : G_\gamma| = n = |H|$. Now since $G_\gamma H = G$ we have $\{e^{\psi}_{\gamma\sigma} | \sigma \in G\} = \{e^{\psi}_{\gamma h} | h \in H\}$. So it is enough to show $(e^{\psi}_{\gamma h}, e^{\psi}_{\gamma}) \neq 0$ for all $h \in H$. Let $h \in H$. If $h = e$, then letting $\mu = e \in \hat{G}$ we get $\mu h^{-1} G_\gamma \cap \hat{G} = G_\gamma \cap \hat{G}$, and this latter set contains the transposition $(1, 2)$, since $n \geq 3$ and $p \neq 2$, so it is nonempty as desired. Now assume that $h \neq e$. Then $h(n) \neq n$ and there is $1 \leq m \leq n-1$ such that $h(m) = n$. Now let $\mu = (m, n)$ and observe that we get $\mu h^{-1}(n) = n$, so $\mu h^{-1} \in S_{n-1} = G_\gamma$. Therefore, $\mu \in \hat{G}$, since $p \neq 2$, and $\mu h^{-1} G_\gamma \cap \hat{G} = G_\gamma \cap \hat{G} \neq \emptyset$ as desired.

If $\dim V_\gamma^\psi > 1$, then V_γ^ψ does not have an o-basis, so by Theorem 2.3 the space V_ψ does not have an o-basis. $\square$

The following example is a case, where we do not have an o-basis for V_ψ.

Example 4.2. *Let $G = S_3$. Assume $\dim V \geq 2$ and $p = 3$. Let ψ be the principal Brauer character of G and let $\gamma = (1, 1, 2)$. Then $\dim V_\gamma^\psi > 1$.*

Proof. Note since $p = 3$ we have $\hat{G} = \{1, (a, b), (a, c), (b, c)\}$. Write $e_\gamma^\psi = e_{(112)}^\psi$. Then for $\sigma = (a, b, c) \in G$ we get $e_{\gamma\sigma}^\psi = e_{(211)}^\psi$. Now by Equation 2.2 we get

$$
\begin{aligned}
e_{(112)}^\psi &= \frac{1}{4}(2e_{(112)} + e_{(211)} + e_{(121)}), \\
e_{(211)}^\psi &= \frac{1}{4}(2e_{(211)} + e_{(112)} + e_{(121)}).
\end{aligned}
$$

By inspection we see that $e_{(112)}^\psi$ and $e_{(211)}^\psi$ are linearly independent, so $\dim V_\gamma^\psi > 1$. $\square$

The alternating group $G = A_n$ is the subgroup of S_n consisting of all the even permutations of S_n. Let ψ be the irreducible Brauer character of G with $\psi(\sigma) = 1$ for all $\sigma \in G$.

Theorem 4.3. *Let $G = A_n$. Assume that $\dim V \geq 2$, $n(\geq 3)$ is odd, and $p \neq 2$. Then $(e_{\gamma\sigma}^\psi, e_\gamma^\psi) \neq 0$ for all $\sigma \in G$, where $\gamma = (1, \ldots, 1, 2)$. In particular, if $\dim V_\gamma^\psi > 1$, then V_ψ does not have an o-basis.*

Proof. Let $\gamma = (1, \ldots, 1, 2)$. We can assume γ to be the representative of the orbit containing it, so $\gamma \in \Delta$. Observe that $G_\gamma = \{\sigma \in A_n \,|\, \sigma(n) = n\} \cong A_{n-1}$. Since n is odd $H = \langle(1, 2, \ldots, n)\rangle \subseteq G$. Following the same argument as in the proof of Theorem 4.1 we can show that H is a set of right coset representatives of G_γ in G. The argument to show $(e_{\gamma\sigma}^\psi, e_\gamma^\psi) \neq 0$ for all $\sigma \in G$ is the same as in the proof of Theorem 4.1. $\square$

4.1 Special case S_4

The symmetric group $G = S_4$ of degree 4 is the group of permutations of a set $\{a, b, c, d\}$. Let $p = 2$. Then there are two, 2-regular conjugacy classes and

$$\hat{G} = \{1, (abc), (acb), (abd), (adb), (acd), (adc), (bcd), (bdc)\}.$$

The Brauer character table of G in this case is (see [12, page 431])

	$(\cdot)$	$(\cdots)$
φ_1	1	1
φ_2	2	-1

Theorem 4.4. *Assume that* $\dim V > 1$. *The space* V_{φ_i} *does not have an o-basis for* $i \in \{1, 2\}$.

Proof. Fix $i \in \{1, 2\}$ and put $\varphi = \varphi_i$. To show V_φ does not have an o-basis it is enough by Theorem 2.3 to show that V_γ^φ does not have an o-basis for some $\gamma \in \Delta$.

Let $\gamma = (1, 1, 1, 2)$. Then $G_\gamma = \{1, (ab), (ac), (bc), (abc), (acb)\}$.

Let $H = \{1, (ab)(cd), (ac)(bd), (ad)(bc)\}$ and observe that $G = G_\gamma H$. So $\{e_{\gamma\sigma}^\varphi \mid \sigma \in G\} = \{e_{\gamma\sigma}^\varphi \mid \sigma \in H\}$. We will compute the value of $(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi)$ using the formula

$$(e_{\gamma\sigma}^\varphi, e_\gamma^\varphi) = \frac{\varphi(e)^2}{|\hat{G}|^2} \sum_{\mu \in \hat{G}} \sum_{\rho \in \mu\sigma^{-1} G_\gamma \cap \hat{G}} \varphi(\mu)\varphi(\rho^{-1}) \tag{4.1}$$

(see proof of Theorem 4.1). The following table lists the products $\mu\sigma^{-1}$, with $\mu \in \hat{G}$ and $\sigma \in H$.

$\sigma \setminus \mu$	1	(abc)	(acb)	(abd)	(adb)	(acd)	(adc)	(bcd)	(bdc)
1	1	(abc)	(acb)	(abd)	(adb)	(acd)	(adc)	(bcd)	(bdc)
$(ab)(cd)$	$(ab)(cd)$	(acd)	(bcd)	(adc)	(bdc)	(abc)	(abd)	(acb)	(adb)
$(ac)(bd)$	$(ac)(bd)$	(bdc)	(abd)	(acb)	(acd)	(adb)	(bcd)	(adc)	(abc)
$(ad)(bc)$	$(ad)(bc)$	(adb)	(adc)	(bcd)	(abc)	(bdc)	(acb)	(abd)	(acd)

Now we will look at the part $\sum_{\mu\in\hat{G}}\sum_{\rho\in\mu\sigma^{-1}G_\gamma\cap\hat{G}}\varphi(\mu)\varphi(\rho^{-1})$ on the right side of the Equation 4.1. Note from the table above that $\mu\sigma^{-1}$ is even for all μ and σ. Now since $\hat{G}$ does not contain odd cycles we can neglect the products of $\mu\sigma^{-1}$ with the elements of the form $(\cdot\cdot)$ in G_γ when considering $\mu\sigma^{-1}G_\gamma \cap \hat{G}$.

When $\mu = 1$, for all cases of $\sigma \neq 1$ we get:

$$\mu\sigma^{-1}(\cdot) = (\cdot\cdot)(\cdot\cdot)(\cdot) = (\cdot\cdot)(\cdot\cdot) \quad \text{and} \quad \mu\sigma^{-1}(\cdot\cdot\cdot) = (\cdot\cdot)(\cdot\cdot)(\cdot\cdot\cdot) = (\cdot\cdot\cdot) \;\; \text{(two times)}.$$

From the table we see if $\mu = (\cdot\cdot\cdot)$, then for each $\sigma \in H$ we have $\mu\sigma^{-1} = (\cdot\cdot\cdot)$. The table below lists the products $\mu\sigma^{-1}$ (columns) of the form $(\cdot\cdot\cdot)$ with the even permutations in G_γ (rows).

	(abc)	(acb)	(abd)	(adb)	(acd)	(adc)	(bcd)	(bdc)
1	(abc)	(acb)	(abd)	(adb)	(acd)	(adc)	(bcd)	(bdc)
(abc)	(acb)	1	$(ad)(bc)$	(bcd)	(abd)	$(ab)(cd)$	$(ac)(bd)$	(adc)
(acb)	1	(abc)	(acd)	$(ac)(bd)$	$(ad)(bc)$	(bdc)	(adb)	$(ab)(cd)$

Assume that $\varphi = \varphi_1$. Since $\varphi(\theta) = 1$ for all $\theta \in \hat{G}$ and $\mu\sigma^{-1}G_\gamma \cap \hat{G} \neq \emptyset$ we get $(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma) \neq 0$ for all $\sigma \in H$. Now write $e^\varphi_\gamma = e^\varphi_{(1112)}$. Then $e^\varphi_{\gamma(ad)} = e^\varphi_{(2111)}$. Now by Equation 2.2 we get

$$\begin{aligned} e^\varphi_{(1112)} &= \frac{1}{9}(3e_{(1112)} + 2e_{(2111)} + 2e_{(1211)} + 2e_{(1121)}), \\ e^\varphi_{(2111)} &= \frac{1}{9}(3e_{(2111)} + 2e_{(1112)} + 2e_{(1211)} + 2e_{(1121)}). \end{aligned}$$

By inspection we note that $e^\varphi_{(1112)}$ and $e^\varphi_{(2111)}$ are linearly independent, so $\dim V^\varphi_\gamma \geq 2$. So we conclude that V^φ_γ does not have an o-basis and hence V_φ does not have an o-basis.

Now assume that $\varphi = \varphi_2$. Then to evaluate $(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma)$ for each $\sigma \neq 1$ in H we observe from the computations above in the cases of $\mu = 1 = (\cdot)$ and μ of the form $(\cdot\cdot\cdot)$ that

$$\sum_{\mu\in\hat{G}} \sum_{\rho\in\mu\sigma^{-1}G_\gamma\cap\hat{G}} \varphi(\mu)\varphi(\rho^{-1}) = 4\varphi\Big((\cdot)\Big)\varphi\Big((\cdot\cdot\cdot)\Big) + 16\varphi\Big((\cdot\cdot\cdot)\Big)\varphi\Big((\cdot\cdot\cdot)\Big).$$

So by the Brauer character table given above and Equation 4.1 we get

$$(e^\varphi_{\gamma\sigma}, e^\varphi_\gamma) = \frac{\psi(e)^2}{|\hat{G}|^2}\Big(4(2)(-1) + 16(-1)(-1)\Big) \neq 0.$$

Now note that by Equation 2.2 we get

$$\begin{aligned} e^\varphi_{(1112)} &= \frac{2}{9}(-2e_{(2111)} - 2e_{(1211)} - 2e_{(1121)}), \\ e^\varphi_{(2111)} &= \frac{2}{9}(-2e_{(1112)} - 2e_{(1211)} - 2e_{(1121)}). \end{aligned}$$

By inspection $e^\varphi_{(1112)}$ and $e^\varphi_{(2111)}$ are linearly independent implying $\dim V^\varphi_\gamma \geq 2$. So we conclude that V^φ_γ has no o-basis and therefore V_φ does not have an o-basis. $\square$

Bibliography

[1] C. Bessenrodt, M. R. Pournaki, and A. Reifegerste, *A note on the orthogonal basis of a certain full symmetry class of tensors*, Linear Algebra and its Applications **370** (2003), 369-374.

[2] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.

[3] M. R. Darafsheh and N. S. Poursalavati, *Orthogonal basis of the symmetry classes of tensors associated with the direct product of permutation groups*, Pure Mathematics and Applications **10** (3) (1999), 241-248.

[4] J. A. Dias da Silva and M. M. Torres, *On the orthogonal dimension of orbital sets*, Linear Algebra and its Applications **401** (2005), 77-107.

[5] W. Feit, *The Representation Theory of Finite Groups*, North-Holland, New York, 1982.

[6] R. Freese, *Inequalities for generalized matrix functions based on arbitrary characters*, Linear Algebra and its Applications **7** (1973), 337–345.

[7] R. R. Holmes, *Orthogonal bases of symmetrized tensor spaces*, Linear Multilinear Algebra **39** (3) (1995), 241–243.

[8] R. R. Holmes, *Orthogonality of cosets relative to irreducible characters of finite groups*, Linear Multilinear Algebra **52** (2) (2004), 133–143.

[9] R. R. Holmes and A. Kodithuwakku (2012), *Orthogonal bases of Brauer symmetry classes of tensors for the dihedral group*, Linear Multilinear Algebra, DOI:10.1080/03081087.2012.729583.

[10] R. R. Holmes and T.-Y. Tam, *Symmetry classes of tensors associated with certain groups*, Linear Multilinear Algebra **32** (1) (1992), 21–31.

[11] I. M. Isaacs, *Character theory of finite groups*, Dover, New York, 1976.

[12] G. D. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, 1981.

[13] Michio Suzuki, *Group theory I*, Springer, New York, 1982.

[14] J.-P. Serre, *Linear Representations of Finite Groups*, Springer, New York, 1977.

[15] M. A. Shahabi, K. Azizi, and M. H. Jafari, *On the orthogonal basis of symmetry classes of tensors*, Journal of Algebra **237** (2) (2001), 637-646.

[16] B. Y. Wang and M. P. Gong, *A higher symmetry class of tensors with an orthogonal basis of decomposable symmetrized tensors*, Linear and Multilinear Algebra **30** (1-2) (1991), 61–64.