THE OPTIMUM UPPER SCREENING LIMIT AND OPTIMUM MEAN FILL LEVEL TO MAXIMIZE EXPECTED NET PROFIT IN THE CANNING PROBLEM FOR FINITE CONTINUOUS DISTRIBUTIONS

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Victoria Spooner Jordan

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Victoria Spooner Jordan

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THE OPTIMUM UPPER SCREENING LIMIT AND OPTIMUM MEAN FILL

LEVEL TO MAXIMIZE EXPECTED NET PROFIT IN THE CANNING

PROBLEM FOR FINITE CONTINUOUS DISTRIBUTIONS

Victoria Spooner Jordan

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Date of Graduation

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Victoria Spooner Jordan, daughter of Joe and June Spooner, was born January 26, 1963, in Athens, Georgia. She graduated from Auburn High School in 1980. She received a Bachelor of Science degree in Statistics from the University of Kentucky in May, 1983, a Master of Science degree in Industrial Engineering from Auburn University in 1987, and a Master of Business Administration from the Ohio State University in 1989. After working for Ampex Corporation and General Electric Company in Quality Assurance, she was a Senior Staff Associate for Luftig & Warren, a management consulting firm. She entered Graduate School, Auburn University, in 2000. She married T. Frank Jordan, son of Terry and Fayron Jordan, on January 1, 1995. Together they have three children: Taylor Frank Jordan, born April 2, 1997; Eben Glenn Jordan, born October 13, 1998; and Audrey Elise Jordan, born July 15, 2001.

DISSERTATION ABSTRACT

THE OPTIMUM UPPER SCREENING LIMIT AND OPTIMUM MEAN FILL LEVEL TO MAXIMIZE EXPECTED NET PROFIT IN THE CANNING PROBLEM FOR FINITE CONTINUOUS DISTRIBUTIONS

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The "canning problem" occurs when a process has a minimum specification such that any product produced below that minimum incurs a scrap/rework cost and any product over the minimum incurs a "give-away" cost. The objective of the canning problem is to determine the target mean for production that minimizes both of these costs. An upper screening limit can also be determined; above which give-away cost is so high that reworking the product maximizes net profit.

Examples of the canning problem are found in the food industry (filling jars or cans) and in the metal industry (thickness.)

In this dissertation, continuous, finite range space distributions are considered, specifically the Uniform and Triangular distributions. For the Uniform distribution, an optimum upper screening limit and an optimum value for the mean fill level is found using three net profit models. Each model assumes a fixed selling price and a linear cost to produce, but costs differ as follows:

- Model 1 uses fixed rework/scrap and reprocessing costs
- Model 2 has linear rework/scrap and reprocessing costs, and
- Model 3 has fixed rework/scrap and reprocessing costs but adds an additional, higher cost associated with a limited capacity of the container.

A discussion is included relating the selection of an optimum set point for the mean to process capability.

For the Triangular distribution, an optimum upper screening limit and an optimum value for the mean fill level is found for both the symmetrical and skewed cases using a net profit model that has fixed rework/scrap and reprocessing costs.

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X = the random variable, fill level

x = a specific value of X

A = selling price of an acceptable product

C = variable cost per unit to produce

C_o = fixed cost per unit to produce

D = selling price of a non-conforming product

$E\left[P\left(x\right)\right]$ = expected net profit

L = lower specification limit

m = mode of distribution

$P(x)$ = net profit function

R_L = cost to scrap or rework product if $x < L$

R_U = cost to scrap or rework product if $x > U$

R = cost to scrap or rework product when $R_L = R_U$

U = upper specification limit (if given)

U_o = optimum upper screening limit

μ_o = optimum target set point for mean

1.0 Introduction

An interesting problem in process optimization is the "canning problem" which attempts to define the optimum set point for the mean of a manufacturing process to minimize scrap and cost (or maximize net profit). The examples used typically refer to filling jars or cans such as in the food industry. In each case, there is a minimum requirement or specification set by the consumer. Any product produced below that minimum is either scrapped or reworked and therefore incurs an associated cost. On the other hand, any product over the minimum is extra product that is given to the customer beyond the minimum requirement (and has thus been labeled "give-away" cost). The give-away cost is proportional to the distance between the existing fill level and the minimum specification. If a target is set too low, product will be rejected as not meeting the customer requirement (lower specification limit) and reworking or scrap costs will be incurred. If a target is set too high, product will meet customer requirements but at the added expense of "giving away" more material than necessary. The objective of the canning problem is to determine the target mean for production that minimizes both of these costs, given that process variability is known and in a state of statistical control.

In some cases, an upper screening limit, *U*, can also be determined. This limit identifies a level above which give-away cost is so high that net profit is maximized by reworking the product.

Examples of the canning problem can be found in “fill processes” such as the amount of coffee in a jar, paint in a can, etc. Other examples come from the metal industry such as steel, aluminum, and copper where metal thickness (gauge) has a required minimum, but additional gauge just adds to the cost.

In this paper, continuous, finite range distributions, specifically the Uniform and Triangular distributions will be considered. The Uniform distribution is presented as a common, symmetric finite range distribution and the Triangular distribution is presented with varying levels of skewness. An optimum upper screening limit (U_o) and an optimum value for the mean fill level (μ_o) will be determined for two net profit models – one with fixed rework costs and one with linear rework costs. In the case of the net profit model with fixed rework costs, the results for the Uniform distribution will be compared with a generalized optimum mean and upper screening limit developed by Liu and Raghavachari (1997) for any continuous fill distribution.

2.0 Literature Review

Springer (1951) introduced the concept of determining an optimum target for a process given fixed costs associated with product falling outside of specifications (both lower specification limit (LSL) and upper specification limit (USL.)) He identified a general approach for calculating a process mean target that minimizes the total cost of rejection for both a Normal and Pearson Type III (or Gamma) distribution. Bettes (1962) wrote a similar article that identified a method for determining the process target when a set lower specification but an arbitrary upper specification is given. He specifically used a "foodstuff" example in which items below the lower specification and above the upper specification were reprocessed at a fixed cost. Both Springer's and Bette's solutions involved a tabulation of factor W. W is a factor which varies depending on the values of $\frac{(U-L)}{\sigma}$ and $\frac{C_L}{C_U}$, the cost ratio between rejecting a part with fill level < L and rejecting a part with fill level > U. The optimum mean is $\mu_o = L + W\sigma$ when $C_L \le C_U$ and $\mu_o = U - W\sigma$ when $C_L \ge C_U$ where $W = \frac{1}{2}\left(\frac{U-L}{\sigma}\right) + \left(\frac{\sigma}{U-L}\right)\log\left(\frac{C_L}{C_U}\right)$. (Note: Springer uses the notation C_L and C_U to define the costs associated with rejected material. In subsequent articles and in this research, the notation R_L and R_U is used.)

To avoid tabulation, Nelson (1979) presented a simplified approach to Springer's solution using a nomograph.

Hunter and Kartha (1977) identified a similar problem with a lower specification limit, where items produced below the specification were sold at a reduced price and give-away cost was linear. Since their approach does not provide a closed-form solution, Nelson (1978) provides an approximating function for use in a calculator or computer allowing an error of about three-decimal accuracy.

The “canning problem” is not specific to cans. Any process that has a minimum specification and costs associated with “under-fill” (and “over-fill”) can be classified as a “canning problem.” Another application of the canning problem is the production of steel beams. The beams have minimum web and flange widths. Beams produced below the width specification cannot be sent to customers. Those with width levels above the minimum use more steel (and therefore cost significantly more) than those produced at minimum. What makes the steel beam example different from the traditional canning problem is that beams produced below the minimum cannot be reworked as in the typical canning problem (more steel cannot be added). The rejected beams are either sold at a reduced price or scrapped and reprocessed. Beams with thickness levels greater than U are most likely melted down and reprocessed. Carlsson presented this example (1984) with the two classes of rejects previously mentioned and give-away cost is measured on a cost per unit basis.

Bisgaard, Hunter, and Pallesen (1984) pointed out that Hunter and Kartha’s assumption that under-filled items can be sold for a fixed price implied that even empty cans could be sold. They expanded Hunter and Kartha’s model to determine the optimum mean, but instead of using a constant selling price for under-filled items, they used a proportional price. They also addressed the possibility and associated costs of reworking under-filled items. Dodson (1993) provided a similar model with his specific example of rolled aluminum sheet metal. Using the characteristic of footage on a finished coil, he identified two costs: when footage is below the lower specification, the entire coil is scrapped (at a cost proportional to footage) and when product is produced above the upper specification, the extra footage is scrapped. He provided a method using graphs or a spreadsheet to identify the target process mean.

Gohlar (1987) first coined the term “canning problem”. He addressed the specific case where over-filled cans are sold for a fixed price (and thus incur a “give-away” cost) and under-filled cans are emptied and refilled at a fixed reprocessing cost. Gohlar and Pollock (1988) added the determination of an upper specification limit for cases where a manufacturer may choose to empty and refill expensive product when the fill level is too high. Gohlar (1988) provided a Fortran program to calculate these values.

Many variations of the original models have been written to include other constraints. Schmidt and Pfeifer (1989) determined the cost benefit associated with reducing variability based on a percentage reduction in standard deviation. Their model extends Gohlar (1987) in which under-filled product is emptied and reprocessed at a fixed cost and over-filled product is sold at the regular price. In 1991, Schmidt and Pfeifer also extended the analysis to include limited capacity as a constraint. Usher, Alexander, and Duggins (1996) recognized that handling rejects reduces efficiency, so they extended Gohlar and Pollock (1988) to include the effect that the target mean and upper limit have on the efficiency of a production line. Cain and Janssen (1997) identified a target value when there is asymmetry in the cost function in the cases of asymmetric linear, asymmetric quadratic, and combined linear and quadratic costs. Pulak and Al-Sultan (1997) developed Fortran programs for solving nine selected targeting models: Hunter and Kartha ’77, Bisgaard, Hunter, and Pallesen ’84, Carlsson ’84, Gohlar ’87, Schmidt and Pfeifer ’91, Boucher and Jafari ’91 (sampling), Gohlar and Pollock ’88, Arcellus and Rahim ’90 (sampling), and Al-Sultan ’94 (two machines in series with sampling.)

Many articles have been written that extend the target selection problem to processes that are subjected to acceptance sampling rather than 100% inspection. Carlsson (1989) identified a method for determining the target mean when using variable data, a one-sided specification, a known variance, and a sampling plan such as MIL-STD-414B. Lee and Elsayed (2002) calculated optimum process mean and screening limits by maximizing profit when a 2-stage screening process is used with a surrogate variable. Lee, Hong, and Elsayed (2001) calculated the optimum process mean and screening limits for a correlated variable under single- and 2-stage screening. In their article, the single screening was based on the quality characteristic being measured and the 2-stage screening was being done on a correlated variable first, then on the quality characteristic of interest.

Recent articles have explored the objective functions and use of a fixed variance. Pfeifer (1999) identified two competing objectives: expected profit per fill attempt and expected profit per can to be filled. Rather than setting the first derivative of expected profit equal to zero and solving for the optimum target, he evaluated expected profit over a range of values and found one that maximized expected profit using spreadsheets and search routines. Misiorek and Barnett (2000) examined the effect of a change in variance on the solution of optimum mean and expected profit. They also explored the implications to “Weights and Measures” requirements. In this article, over-fill was either recaptured or lost and under-filled containers were emptied and material re-used or they were “topped-up”. Kim, Cho, and Phillips (2000) calculated the optimum mean while keeping the process capability at a predetermined level. They used a cost function that increased as variance decreased which seems counter to Taguchi’s loss function and would be difficult to quantify in practical application.

Liu and Raghavachari (1997) generalized the determination of an optimal process mean for the canning problem and an upper screening limit for any continuous distribution. They used a simple profit model given by Schmidt and Pfeifer (1991) and determined an optimal value of U (upper screening limit) and μ which maximized the expected net profit for any continuous fill distribution. Their work addressed infinite-range distributions that they truncated on the low end.

A flow chart of articles reviewed for this research follows:

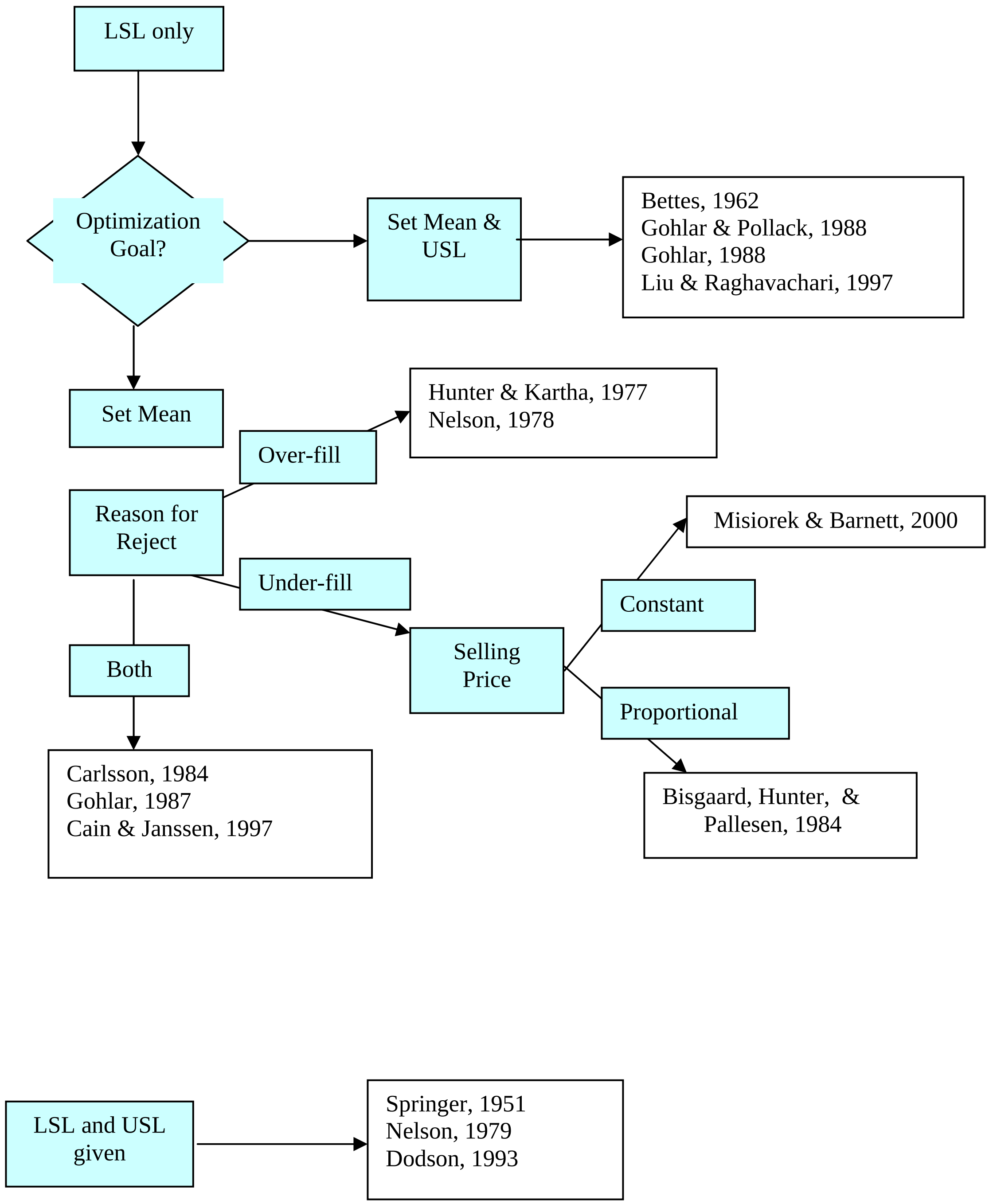


FIGURE 1. References that Address the Canning Problem With 100 % Inspection

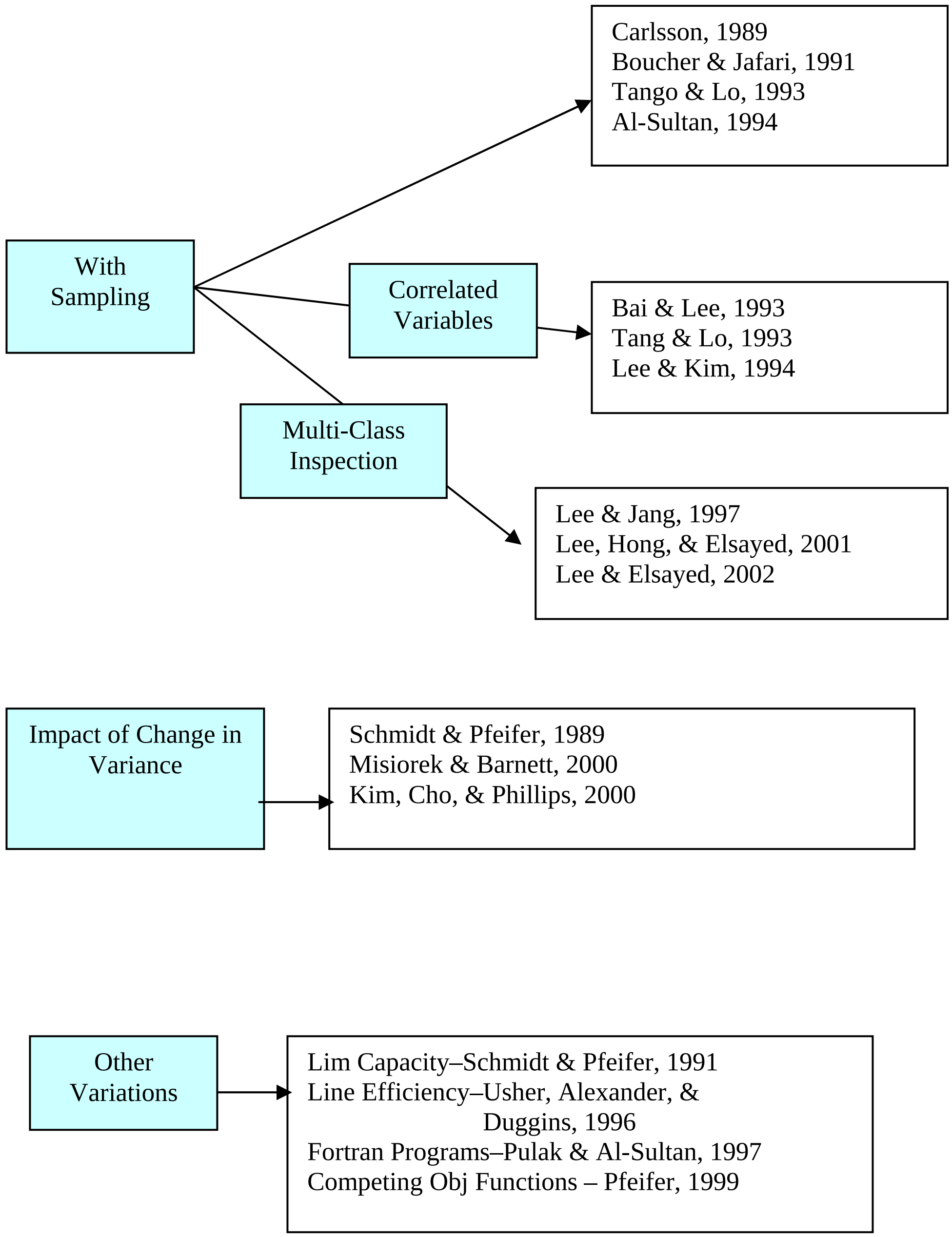


FIGURE 2. Other References that Address the Canning Problem

Since many different models of net profit have been used in literature (with many different forms of notation), they are summarized in Table 1 using the following notation:

X = the random variable, fill level

x = a specific value of X

A = selling price of an acceptable product

D = selling price of a non-conforming product

C = variable cost per unit to produce

C_o = fixed cost per unit to produce

R_L = cost to scrap or rework product if $x < L$

R_U = cost to scrap or rework product if $x > U$

R = cost to scrap or rework product if $R_L = R_U$

L = lower specification limit

U = upper specification limit (if given)

U_o = optimum upper screening limit

μ_o = optimum target mean

m = the modal point

TABLE 1. Net Profit Model Comparison from Literature Review

Author (Date) of Reference	**Net Profit Model**	**Calculate U_o in addition to μ_o**
Springer (1951), Nelson (1979)	Springer and Nelson do not use a net profit model, just the cost of rejection: $-R_L$ if x < L $-R_U$ if x > U	No
Bettes (1962)	First application of give-away cost, no net profit model, costs are as follows: $-R_L$ if x < L $-Cx$ if x > U_o	Yes
Hunter & Kartha (1977), Nelson (1978)	D-Cx if $x < L$ A-Cx if $x \geq L$	No
Carlsson (1984)	(Differs from Hunter & Kartha in that cost to produce is both fixed and variable.) D-(C_o+Cx) if $x < L$ A-(C_o+Cx) if $x \geq L$	No
Bisgaard, Hunter, & Pallesen (1984)	(D-C)x-C_o if $x < L$ A-Cx- C_o if $x \geq L$	No
Gohlar (1987)	A-R-Cx if $x < L$ A-Cx if $x \geq L$	No

TABLE 1 (continued). Net Profit Model Comparison from Literature Review

Author (Date) of Reference	**Net Profit Model**	**Calculate U_o in addition to μ_o**
Gohlar & Pollock (1988)	$\bar{P}(\mu,U)$-*R* if $x < L$, where $\bar{P}(\mu,U)$ is the expected profit at the new level when the can is refilled. *A-Cx* if $L \le x \le U_o$ $\bar{P}(\mu,U)$-*R* if $x > U_o$	Yes
Schmidt & Pfeifer (1991), Liu & Raghavachari (1997)	*A-Cx* if $L \le x \le U$ -*R* otherwise	No Yes
Dodson (1993)	(*A-C*)*x* if $L \le x \le U$ - $R_L\, x$ if $x < L$ - $R_U\, x$ if $x > U$	No

The differences in Table 1 come from several sources. One variation is the selling price. In some models, selling price is fixed, in others it is proportional, and in still other models, there may be more than one selling price if rejected material can be sold in a secondary market. Another variation in the models addresses how material is handled if it is rejected. Some models assume a fixed rejection cost (whether due to scrap or rework) and some models treat rejection cost as proportional to *x* (again this cost may be due to scrap or rework.) In the following chapter, the two net profit models used assume a fixed selling price, no secondary market for selling rejected product, and fixed rejection costs for the first model and proportional rejection costs for the second model.

3.0 Uniform distribution

Most of the work in the literature regarding the canning problem has focused on the Normal distribution with many different net profit functions. All assume a continuous distribution and an infinite range. In actual practice, however, finite distributions may sometimes be more plausible models for a canning process. Few authors use a truncated distribution and those who do (Liu and Raghavachari, 1997) only truncate on the low end of the distribution. This research is preliminarily focused on the application of the canning problem to two standard statistical distributions that have a finite range: the Uniform distribution (with two different net profit models) and the Triangular distribution (with fixed rework/reprocessing costs.)

3.1 Uniform underlying distribution – constant scrap cost

In every case, analysis should begin with a distribution fit (χ^2 Goodness of Fit Test for example) of the data's dimension (fill level, metal thickness, etc.), to determine the appropriate distribution to use in the analysis. In this chapter, it is assumed that such an analysis has been completed and data are found to be most closely approximated by a Uniform distribution.

In this first application, the basic profit function used in the Liu and Raghavachari (1997) article is used, so results for the continuous Uniform distribution can be compared to their's for any continuous distribution for that particular profit model. In the net profit function, first introduced by Schmidt and Pfeifer (1991), the cost of reworking product below L or above U is fixed and the product is only sold if fill level, x, falls within the limits of L and U. A slight modification of Schmidt and Pfeifer's (1991) net profit model is defined below. This net profit model is generalized such that R_L need not be equal to R_U.

The Net Profit function:

$$P(x) = \begin{cases} -R_L, & a \le x < L \\ A - Cx, & L \le x \le U \\ -R_U, & U < x \le b \\ 0, & otherwise \end{cases}$$

where L = lower specification limit, below which the customer will not accept the product. For example, if a jar is to be filled with L = 8 oz. of an ingredient, anything less than L = 8 oz. is not allowed. U = an upper screening limit (to be determined) which is not based on customer preferences but is a value that minimizes giveaway cost such that any fill level above U costs more in giveaway cost than would be received in income. R_L = the rejection cost per container when $x < L$, R_U = the rejection cost per container when $x > U$, A = revenue received for an acceptable container, and C = the production cost per unit of ingredient. The constants A, R_L, R_U, C, and L are known and > 0. In the Liu and Raghavachari (1997) article, $R_L = R_U = R$, but since the cost of rejecting a unit with fill level less than L may be different from the cost of rejecting a unit with fill level greater than U, a slight generalization is made for this model. For example, in the steel industry, if sheets of steel are being produced to a minimum thickness specification, L, then sheets with thickness less than L may be scrapped or sold in a secondary market, while sheets with thickness greater than U, may be reprocessed, melted down and used as raw material. The machine's "fill level" per attempt will be used throughout this research to describe the random variable, X. The unit of measure will depend on the application. For example, "per unit" may be per ounce (in the case of filling a container) or per fraction of an inch (in the case of steel thickness.)

In order to determine the optimum value of U, U_o, the equation for maximizing expected profit must be determined and then the first derivative with respect to U is set equal to zero to solve for U_o.

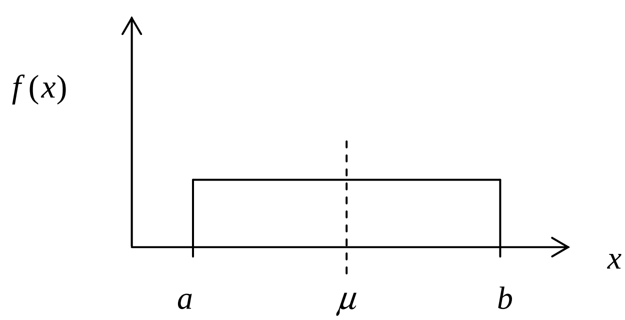
The Uniform distribution has the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \dfrac{1}{2k}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

where $b - a = 2k$.

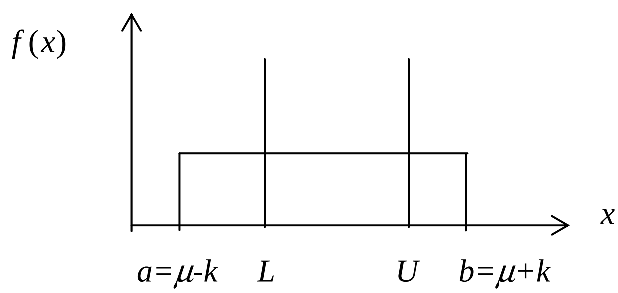
Graphically, the density function appears as:



where $a = \mu - k$, $b = \mu + k$, and $2k$ is a constant value that describes the spread of the Uniform distribution such that $b - a = 2k$, and variance of X is $V(x) = \frac{k^2}{3}$ (Appendix D.)

3.1.1 Optimum upper screening limit

With respect to the canning problem, there is a cost (R_L) when the quantity falls below the lower specification, L ($L \geq a$). There is also an arbitrary upper screening limit, U ($U < b$), such that the profit ($A - Cx$) above U is actually less than the cost to rework or reprocess when fill level is too high, R_U.



Assume that the filling machine variability is such that the process range is $\mu \pm k$ where $a = \mu - k$ and $b = \mu + k$. The objective is to maximize the expected net profit, which is obtained first by combining the equation for the net profit, $P(x)$, and the probability density function of x given by $f(x) = \frac{1}{2k}$, $a \leq x \leq b$.

Assuming that $L \leq \mu \leq U$, $a = \mu - k \leq L$, and $b = \mu + k \geq U$, the expected net profit is given by

$$E\left[P(X)\right] = \int_{\mu-k}^{L} -\frac{R_L}{b-a}dx + \int_{L}^{U}\frac{(A-Cx)}{b-a}dx + \int_{U}^{\mu+k} -\frac{R_U}{b-a}dx$$

$$= -\frac{R_L x}{b-a}\Big|_{\mu-k}^{L} + \frac{1}{b-a}\left(Ax - \frac{Cx^2}{2}\right)\Big|_{L}^{U} - \frac{R_U x}{b-a}\Big|_{U}^{\mu+k}$$

$$= \frac{1}{2k}\left[R_L\left(\mu-k-L\right) + A\left(U-L\right) - \frac{C\left(U^2-L^2\right)}{2} + R_U\left(U-\mu-k\right)\right]$$

$$= \frac{1}{2k}\left[U\left(R_U + A - \frac{CU}{2}\right) + R_L\left(\mu-k\right) - R_U\left(\mu+k\right) + L\left(\frac{CL}{2} - A - R_L\right)\right] \qquad (1)$$

In order to solve for the optimum value of the upper limit, U_U , the first derivative of the expected profit equation with respect to *U* is set equal to zero:

$$\frac{\partial E\left[P(x)\right]}{\partial U} = \frac{1}{2k}\left(R_U + A - CU\right) \qquad (2)$$

The second derivative with respect to *U* is $-\frac{C}{2k}$ which is < 0. Since the second derivative is < 0, the function is strictly concave and setting the first derivative = 0 will yield a maximum expected net profit.

Setting Equation (2) equal to zero,

$$\frac{1}{2k}\left(R_U + A - CU_U\right) = 0 \rightarrow \quad \left(R_U + A - CU_U\right) = 0$$

or,

$$U_U = \frac{R_U + A}{C} \tag{3}$$

In the case where $R = R_U = R_L$, the optimum point obtained in Equation (3) matches the solution found by Liu and Raghavachari (1997) for any distribution.

Substituting U_U into the expected net profit equation, Equation (1), yields the following optimum expected net profit:

$$E_o\left[P(X)\right] =$$

$$\frac{1}{2k}\left[\left(\frac{R_U + A}{C}\right)\left(R_U + A - \frac{R_U + A}{2}\right) + R_L(\mu - k) - R_U(\mu + k) + L\left(\frac{CL}{2} - A - R_L\right)\right]$$

$$= \frac{1}{2k}\left[\frac{(R_U + A)^2}{2C} + R_L(\mu - k) - R_U(\mu + k) + L\left(\frac{CL}{2} - A - R_L\right)\right]$$

$$= \frac{1}{2k}\left[\frac{(R_U + A)^2}{2C} + \mu(R_L - R_U) - k(R_L + R_U) + L\left(\frac{CL}{2} - A - R_L\right)\right].$$

Because the variance of the Uniform density is given by $V(x) = \frac{(b-a)^2}{12} = \frac{k^2}{3}$,

$$E_o\left[P(X)\right] = \frac{1}{2\sigma\sqrt{3}}\left[\frac{(R_U + A)^2}{2C} + \mu(R_L - R_U) - \sigma\sqrt{3}(R_L + R_U) + L\left(\frac{CL}{2} - A - R_L\right)\right] \tag{4}$$

In the case where $R_L = R_U = R$, Equation (4) simplifies further to

$$E_o\left[P(x)\right] = \frac{1}{2\sigma\sqrt{3}}\left[\frac{(R+A)^2}{2C} + L\left(\frac{CL}{2} - A - R\right)\right] - R. \qquad (5)$$

Equations (4) and (5) show that as process variability is reduced, the expected net profit increases. Further, $E_o\left[P(x)\right]$ decreases as costs (*R* and *C*) increase.

If $R_U > R_L$, it is clear based on the net profit model in Figure 3, that when $2k > U_U - L$, fill level should be positioned below *L* rather than above U_U, so in determining the optimum value for μ the probability of $x > U_U$ will be zero. However, if $-R_L > P(U_U)$, a higher profit can be realized if μ is set lower than $U_U - k$. U_U was determined to be the "break-even" point for costs related to R_U, but in the case where $R_U > R_L$, even within the range between *L* and U_U, the cost for a given fill attempt, Cx, may be higher than the selling price, *A*, such that net profit becomes negative, and if *A*-*Cx* < -R_L, then the cost to rework when fill level is less than *L* is actually less than the cost to produce and sell at the given fill level (below U_U.)

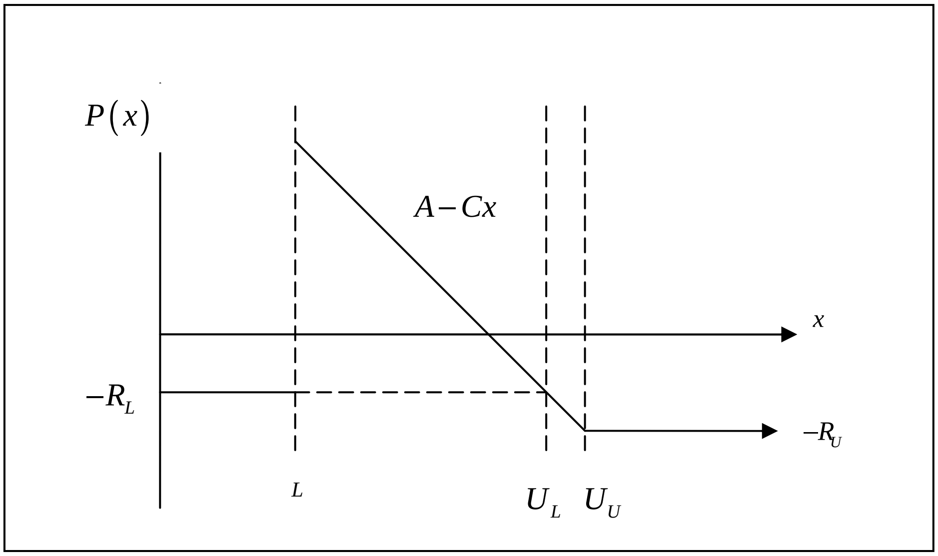


FIGURE 3. Net Profit Model When $R_U > R_L$.

In the case where $R_U > R_L$, an alternative upper limit exists at $U_L = \frac{R_L + A}{C}$ for the purpose of determining μ_o . U_U , in Equation (3), was determined to be the "break-even" point for costs related to R_U , but when $R_U > R_L$, another "break-even" point occurs when $-R_L = P(x)$. This occurs when $A - Cx = -R_L$ and leads to an alternative upper screening limit for determining μ_o :

$$U_L = \frac{R_L + A}{C} \tag{6}$$

$U_L = \frac{R_L + A}{C}$ is the "break-even point" for net profit with the cost of reprocessing material with fill level below L. If it costs less to scrap and reprocess under-fills than to fill and sell product with fill level > U_L , then the production process should be centered such that reject occur below L rather than above U_L .

So, the optimum value for the upper screening limit is

$$U_o = \min\left(U_L, U_U\right) = \min\left(\frac{R_L + A}{C}, \frac{R_U + A}{C}\right) \tag{7}$$

3.1.2 Optimum target set point for the mean

The next step is to determine an optimum machine set point for the mean. Of course, for a Uniform distribution, $\mu = \frac{a+b}{2}$, so the actual mean is defined. In this chapter, the optimum set point for the mean will be determined that maximizes expected net profit per unit. Rather than fixing *a* or *b*, in this analysis, the assumption is that the filling machine has a fixed amount of variability, but the set point can be adjusted such that $b - a = 2k$ is fixed, and the optimum mean, μ_o, is accordingly determined.

The ideal (though technically impossible) optimum situation would occur if the process mean was centered at *L* and there was no variability. The fact that variability exists in all processes, however, makes the canning problem a practical issue.

For a Uniform distribution with *L* and *k* given, the optimum set point for μ, μ_o, is based on the equations for expected net profit. There are two cases to be considered: Case 1: $2k \le U_o - L$ and Case 2: $2k > U_o - L$. Within each case, the equations for expected net profit are different, for different ranges of μ.

Case 1: $2k \le U_o - L$**:**

When $2k \le U_o - L$, there are five different scenarios:

a) $\mu + k \le L$

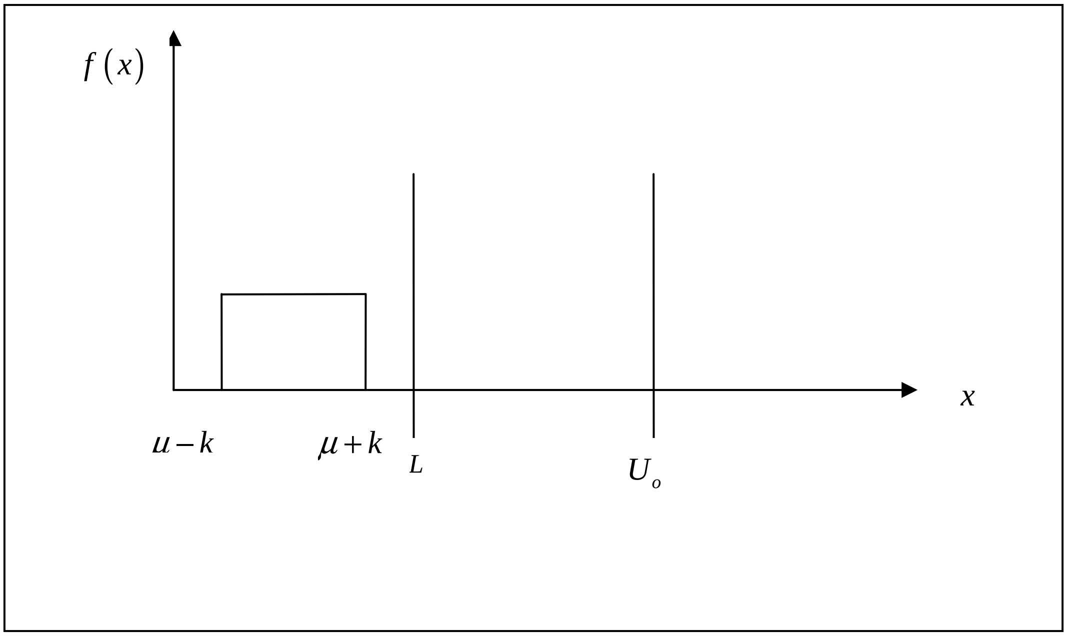


FIGURE 4. Distribution of fill level when $2k \le U_o - L$ and $\mu + k \le L$.

b) $\mu - k \le L$ and $L \le \mu + k \le U_o$

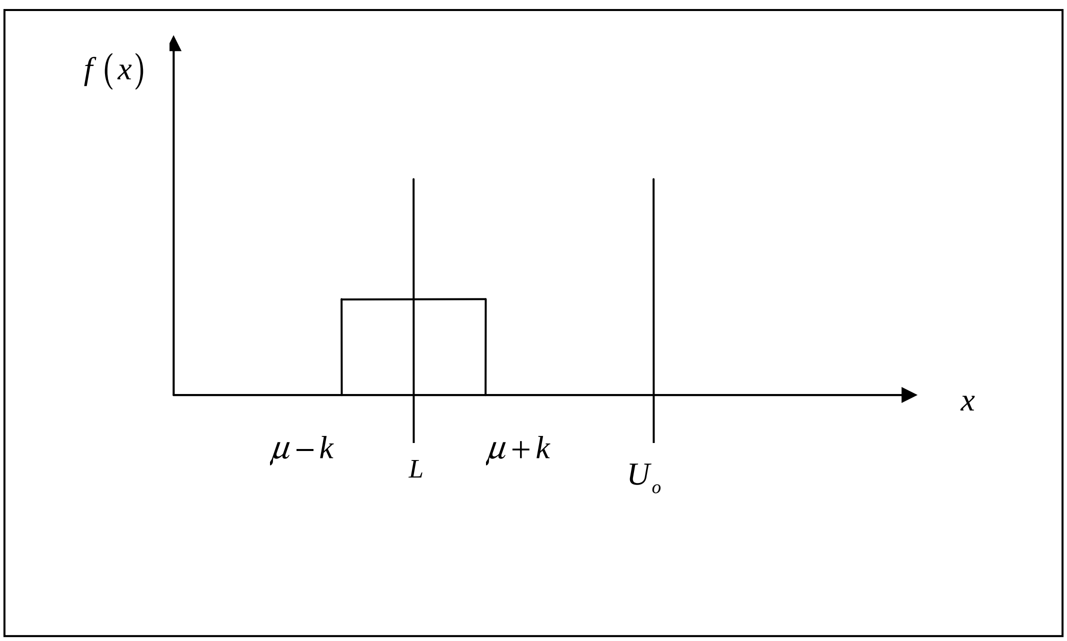


FIGURE 5. Distribution of fill level when $2k \le U_o - L$, $\mu - k \le L$ and $L \le \mu + k \le U_o$.

c) $\mu - k \geq L$ and $\mu + k \leq U_o$

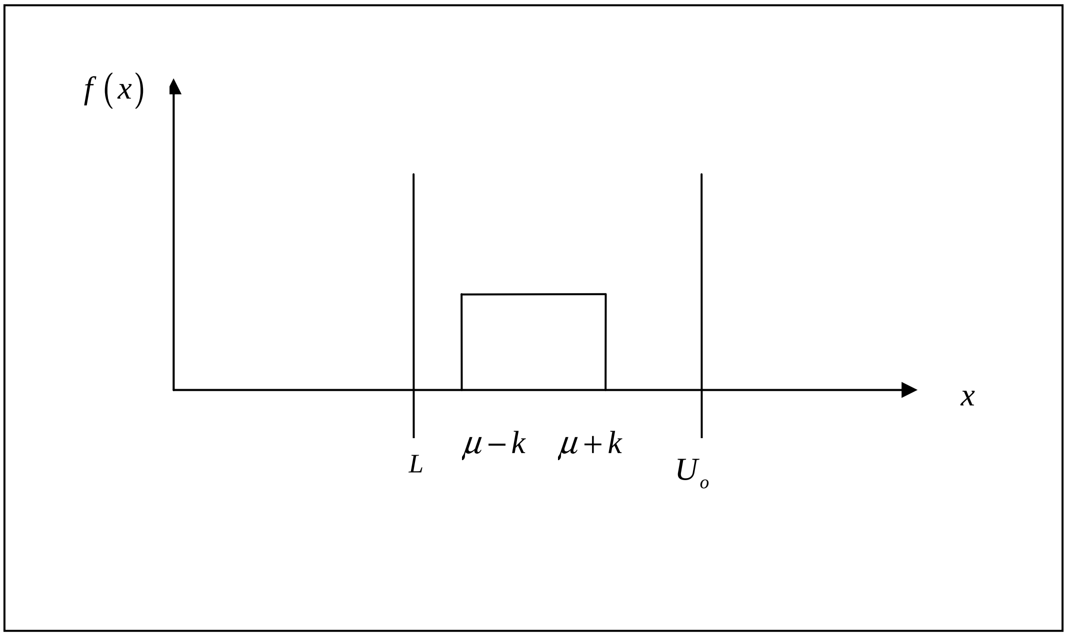


FIGURE 6. Distribution of fill level when $2k \leq U_o - L$, $\mu - k \geq L$ and $\mu + k \leq U_o$.

d) $L \leq \mu - k \leq U_o$ and $\mu + k \geq U_o$

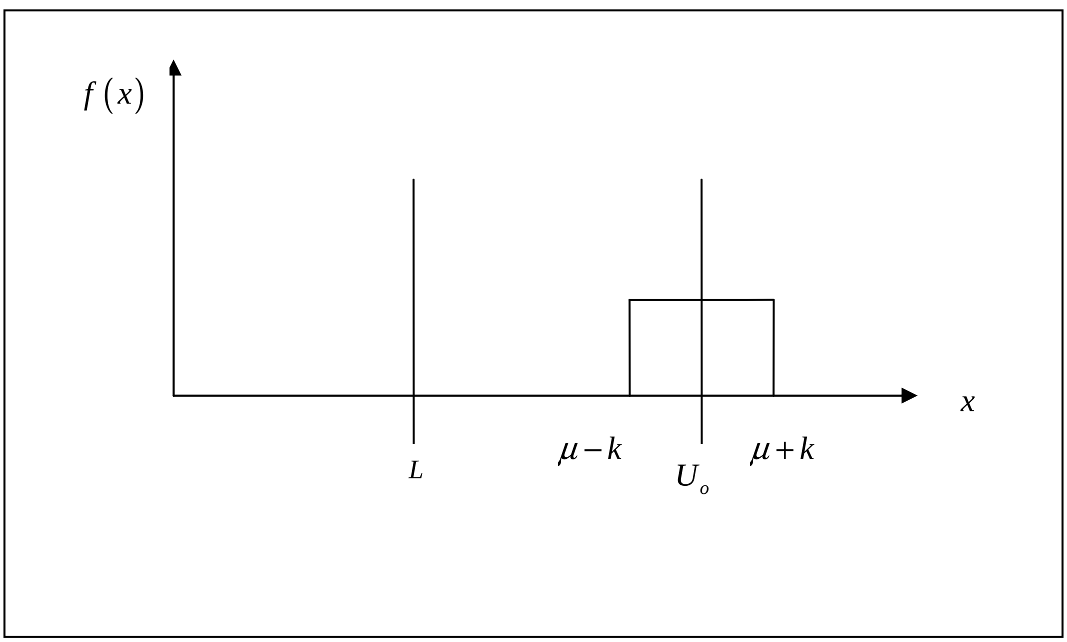


FIGURE 7. Distribution of fill level when $2k \leq U_o - L$, $L \leq \mu - k \leq U_o$, and $\mu + k \geq U_o$.

e) $\mu - k \geq U_o$

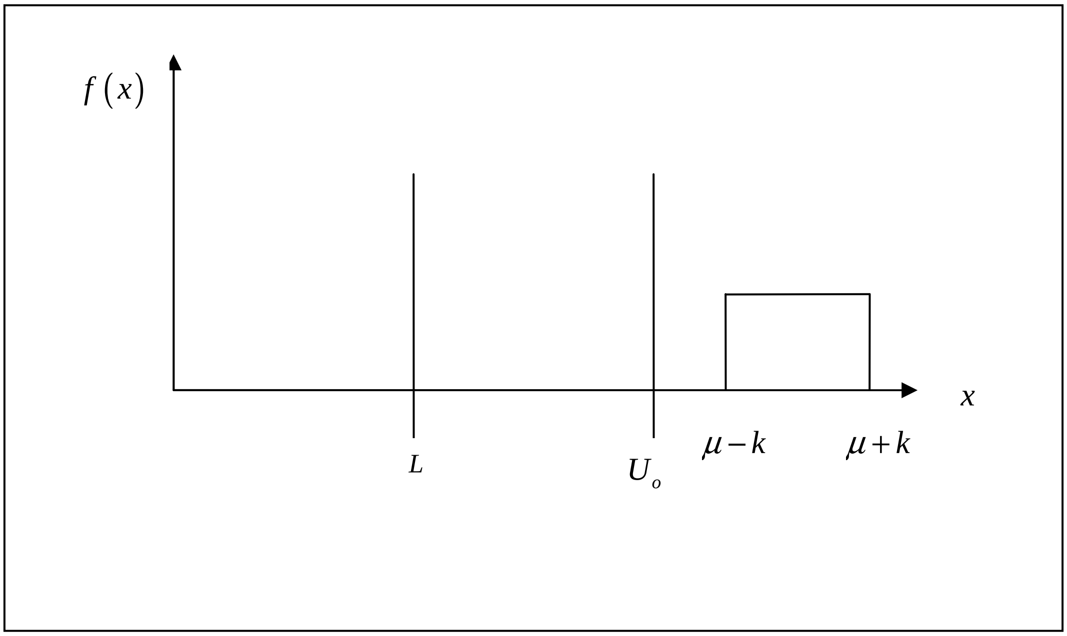


FIGURE 8. Distribution of fill level when $2k \leq U_o - L$ and $\mu - k \geq U_o$.

These scenarios and the formulas for expected net profit are summarized in Table 2:

TABLE 2. Case 1: Expected Net Profit for Ranges of μ when $2k \le U_o - L$.

Scenario	Range of μ	Expected Net Profit, $E\left[P(x)\right]$
a	$k \le \mu \le L-k$	$-R_L$
b	$L-k \le \mu < L+k$	$\frac{1}{2k}\left\{ R_L(\mu-k)+A(\mu+k)-(R_L+A)L-\frac{C}{2}\left[(\mu+k)^2-L^2\right]\right\}$
c	$L+k \le \mu < U_o-k$	$A-C\mu$
d	$U_o-k \le \mu < U_o+k$	$\frac{1}{2k}\left\{(A+R_U)(U_o-\mu)+k(A-R_U)-\frac{C}{2}\left[U_o^2-(\mu-k)^2\right]\right\}$
e	$U_o+k \le \mu$	$-R_U$

Expected Net Profit equations are calculated below and an attempt is made to find through differentiation the optimum set point for μ, μ_o, to maximize expected net profit:

Case 1 (a) $k \le \mu \le L-k$ **:**

$$E\left[P(x)\right]=\int_{\mu-k}^{\mu+k}-R_L f(x)dx=\frac{1}{2k}\left\{-R_L\left[(\mu+k)-(\mu-k)\right]\right\}=-R_L$$

So, there are no critical points in $(k, L-k)$.

Case 1 (b) $L-k \le \mu < L+k$ **:**

$$E\left[P(x)\right]=\int_{\mu-k}^{L}-R_L f(x)dx+\int_{L}^{\mu+k}(A-Cx)f(x)dx$$

$$=\frac{1}{2k}\left\{-R_L\left[L-(\mu-k)\right]+A\left[(\mu+k)-L\right]-\frac{C}{2}\left[(\mu+k)^2-L^2\right]\right\}$$

$$=\frac{1}{2k}\left\{R_L(\mu-k)+A(\mu+k)-(R_L+A)L-\frac{C}{2}\left[(\mu+k)^2-L^2\right]\right\}$$

Now, in order to find the optimum value of μ, the equation for expected net profit is differentiated with respect to μ:

$$\frac{\partial}{\partial\mu}E\left[P(x)\right]=\frac{1}{2k}\left[(R_L+A)-C(\mu+k)\right]$$

The second derivative with respect to μ, $\frac{\partial^2}{\partial\mu^2}E\left[P(x)\right]=-\frac{C}{2k}$, is less than zero, so setting the first derivative equal to zero will result in a maximum:

$$\frac{1}{2k}\left[(R_L+A)-C(\mu+k)\right]=0 \Rightarrow \qquad \mu_o=\frac{R_L+A}{C}-k=U_L-k$$

But, $L-k \le \mu \le L+k$, and $2k \le U_o-L$ is assumed in Case 1. Since $U_o=\min(U_U,U_L)$, $U_L \ge U_o$, so $U_L-k \ge U_o-k > L+k$, and $\mu_o \ne U_L-k$ in this range.

Case 1 (c) $L+k \le \mu < U_o-k$ **:**

$$E\left[P(x)\right]=\int_{\mu-k}^{\mu+k}(A-Cx)f(x)dx$$

$$=\frac{1}{2k}\left\{A\left[(\mu+k)-(\mu-k)\right]-\frac{C}{2}\left[(\mu+k)^2-(\mu-k)^2\right]\right\}$$

$$= A - C\mu$$

So, there are no critical points in $\left(L + k, U_o - k\right)$.

Case 1 (d) $U_o - k \le \mu < U_o + k$ **:**

$$E\left[P\left(x\right)\right] = \int_{\mu-k}^{U_o} (A - Cx) f\left(x\right) dx + \int_{U_o}^{\mu+k} -R_U f\left(x\right) dx$$

$$= \frac{1}{2k}\left\{A\left[U_o - \left(\mu - k\right)\right] - \frac{C}{2}\left[U_o^{\ 2} - \left(\mu - k\right)^2\right] - R_U\left[\left(\mu + k\right) - U_o\right]\right\}$$

$$= \frac{1}{2k}\left\{\left(A + R_U\right)\left(U_o - \mu\right) + k\left(A - R_U\right) - \frac{C}{2}\left[U_o^{\ 2} - \left(\mu - k\right)^2\right]\right\}$$

Now, differentiating with respect to μ,

$$\frac{\partial}{\partial \mu} E\left[P\left(x\right)\right] = \frac{1}{2k}\left[C\left(\mu - k\right) - \left(R_U + A\right)\right]$$

Here, the second derivative with respect to μ is $\frac{C}{2k} > 0$, so setting the first derivative equal to zero will result in a minimum, not a maximum net profit.

Case 1 (e) $U_o + k \le \mu$ **:**

$$E\left[P\left(x\right)\right] = \int_{\mu-k}^{\mu+k} -R_U f\left(x\right) dx$$

$$= \frac{1}{2k}\left\{-R_U\left[\left(\mu + k\right) - \left(\mu - k\right)\right]\right\} = -R_U$$

So, there are no critical points when $\mu > U_o - k$.

In each case, differentiation does not lead to a solution for μ_o, so the extreme points for each interval are calculated to determine the optimum value of μ, μ_o, to maximize $E\left[P(x)\right]$.

Case 1(a): At $\mu = L - k$, $E\left[P(x)\right] = -R_L$

Case 1(b): At $\mu = L - k$, $E\left[P(x)\right] = -R_L$,

At $\mu = L + k$, $E\left[P(x)\right] = A - C(L + k)$

Case 1(c): At $\mu = L + k$, $E\left[P(x)\right] = A - C(L + k)$,

At $\mu = U_o - k$, $E\left[P(x)\right] = A - C(U_o - k)$

Case 1(d): At $\mu = U_o - k$, $E\left[P(x)\right] = A - C(U_o - k)$,

At $\mu = U_o + k$, $E\left[P(x)\right] = -R_U$

Case 1(e): At $\mu = U_o + k$, $E\left[P(x)\right] = -R_U$

Since $2k \le U_o - L$, $L + k \le U_o - k$, so $A - C(L + k) > A - C(U_o - k)$. Because U_o is calculated using Equation (7), $L + k < U_L$, so $A - C(L + k) > -R_L$ and $L + k < U_U$, so $A - C(L + k) > -R_U$. Therefore, the optimum value of μ when $2k \le U_o - L$, is:

$$\mu_o = L + k \tag{8}$$

and

$$E_o\left[P(x)\right] = A - C(L + k) \tag{9}$$

Example 1 ($2k \leq U_o - L$.)

Let A = \$40, C = \$0.10, R_L =\$5, R_U = \$6, k = 50, and L = 200. U_o can be calculated using Equation (7) to be U_o = 450. Using Equation (8), μ_o = 250. Various values for μ are presented in Figure 9 with the corresponding expected net profit, $E\left[P(x)\right]$, to show that μ_o =250 does, in fact, give the highest expected net profit (at \$15):

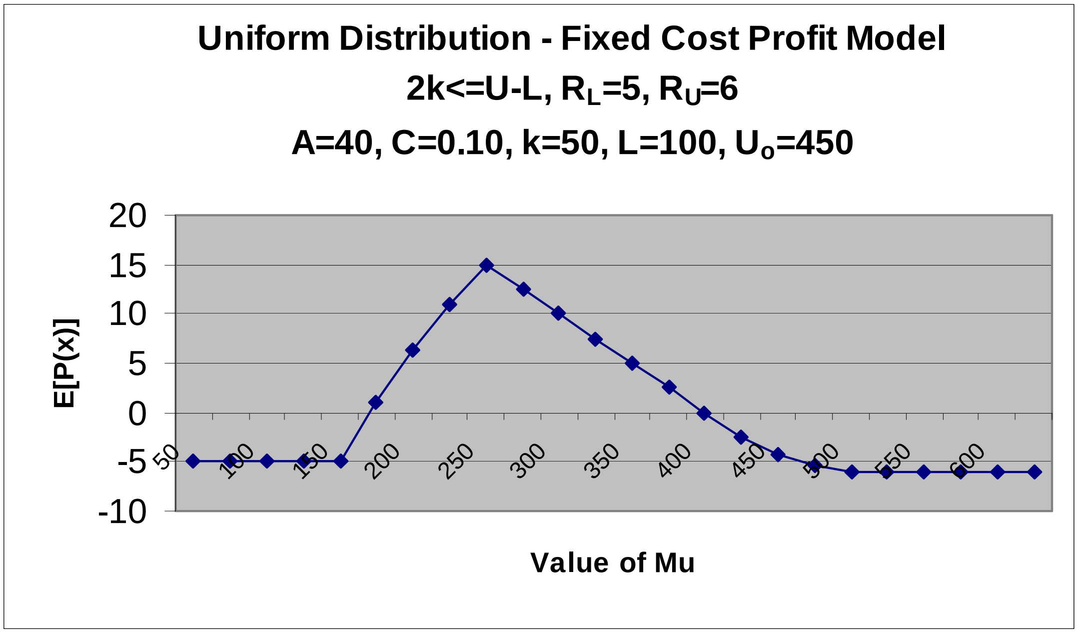


FIGURE 9. Example of Expected Net Profit for Different Values of μ when $2k \leq U_o - L$.

Case 2: $2k > U_o - L$

As in Case 1, for Case 2 where $2k > U_o - L$, there are five different scenarios. The ranges of μ change since $2k > U_o - L$**,** but the expected net profit equations remain the same as in Case 1 with the exception of Case 2(c). Case 2(c) differs from Case 1(c) in that with $2k > U_o - L$**,** fill level will fall below L and/or above U_o as illustrated below:

c) $\mu - k \le L$ and $\mu + k \ge U_o$

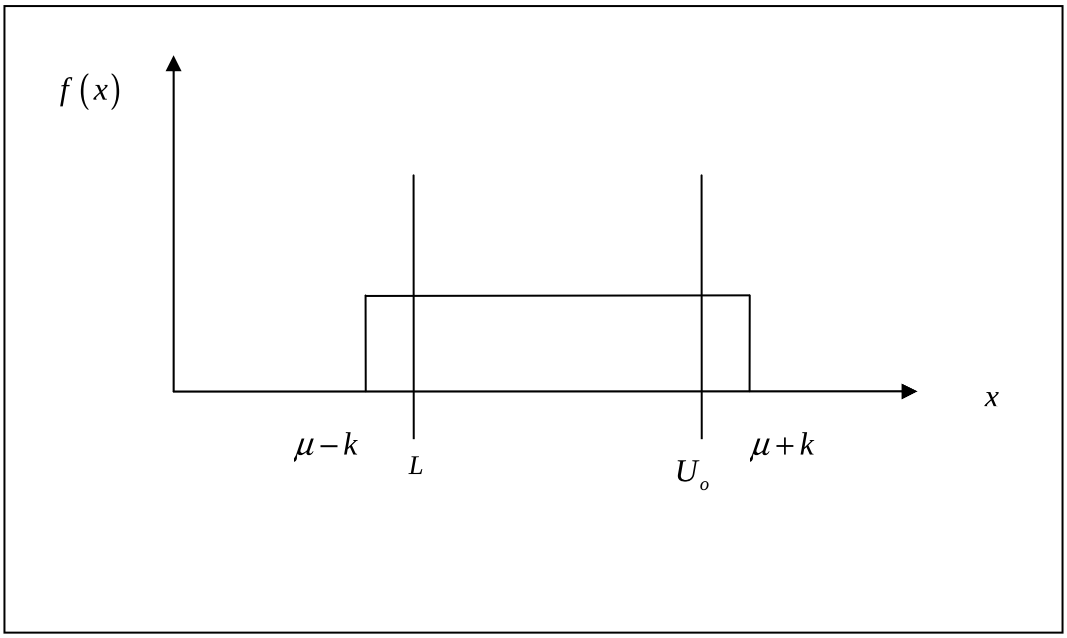


FIGURE 10. Distribution of fill level when $2k > U_o - L$, $\mu - k \le L$ and $\mu + k \ge U_o$.

These scenarios and the formulas for expected net profit are summarized in Table 3:

TABLE 3. Case 2: Expected Net Profit for Ranges of μ when $2k > U_o - L$.

Scenario	**Range of** μ	**Expected Net Profit,** $E\left[P(x)\right]$
a	$k \le \mu \le L-k$	$-R_L$
b	$L-k \le \mu < U_o - k$	$\frac{1}{2k}\left\{\begin{array}{l} R_L(\mu-k)+A(\mu+k)- \\ (R_L+A)L-\frac{C}{2}\left[(\mu+k)^2-L^2\right] \end{array}\right\}$
c	$U_o - k \le \mu < L+k$	$\frac{1}{2k}\left[\begin{array}{l} R_L(\mu-k-L)+A(U_o-L)- \\ \frac{C}{2}(U_o^{\,2}-L^2)-R_U(\mu+k-U_o) \end{array}\right]$
d	$L+k \le \mu < U_o + k$	$\frac{1}{2k}\left\{\begin{array}{l} (A+R_U)(U_o-\mu)+k(A-R_U)- \\ \frac{C}{2}\left[U_o^{\,2}-(\mu-k)^2\right] \end{array}\right\}$
e	$U_o + k \le \mu$	$-R_U$

Expected Net Profit equations were calculated the same as with Case 1 with the exception of 2(b) and 2(c).

Case 2 (b) $L-k \le \mu < U_o - k$**:**

As in Case 1, taking the first derivative with respect to μ of the equation for expected net profit in the range $L-k \le \mu < L+k$, leads to $\mu = \frac{R_L + A}{C} - k = U_L - k$. However, when $2k > U_o - L$, $\mu = \frac{R_L + A}{C} - k = U_L - k$ is a feasible value for μ_o.

Case 2 (c) $U_o - k \le \mu < L + k$ **:**

$$E\left[P(x)\right] = \int_{\mu-k}^{L} -R_L f(x)\,dx + \int_{L}^{U_o} (A - Cx) f(x) dx + \int_{U_o}^{\mu+k} -R_U f(x)\,dx$$

$$= \frac{1}{2k}\left\{-R_L\left[L - (\mu - k)\right] + A[U_o - L] - \frac{C}{2}\left[U_o^{\,2} - L^2\right] - R_U\left[(\mu + k) - U_o\right]\right\}$$

$$= \frac{1}{2k}\left[R_L(\mu - k - L) + A(U_o - L) - \frac{C}{2}\left(U_o^{\,2} - L^2\right) - R_U(\mu + k - U_o)\right]$$

Now, in order to find the optimum value of μ, the equation for expected net profit is differentiated with respect to μ:

$$\frac{\partial}{\partial \mu} E\left[P(x)\right] = \frac{1}{2k}(R_L - R_U)$$

which leads to no closed form solution for μ.

With the exception of Case 2(b), differentiation does not lead to a solution for μ_o, so the extreme points for each interval are calculated to determine the optimum set point for μ, μ_o.

Case 2(a): At $\mu = L - k$, $E\left[P(x)\right] = -R_L$

Case 2(b): At $\mu = L - k$, $E\left[P(x)\right] = -R_L$,

At $\mu = U_o - k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left[R_L(U_o - L - 2k) + A(U_o - L) - \frac{C}{2}\left(U_o^{\,2} - L^2\right)\right]$$

Case 2(c): At $\mu = U_o - k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left[R_L(U_o - L - 2k) + A(U_o - L) - \frac{C}{2}\left(U_o^{\,2} - L^2\right)\right],$$

At $\mu = L + k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left[R_U\left(U_o - L - 2k\right) + A\left(U_o - L\right) - \frac{C}{2}\left(U_o{}^2 - L^2\right)\right]$$

Case 2(d): At $\mu = L + k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left[R_U\left(U_o - L - 2k\right) + A\left(U_o - L\right) - \frac{C}{2}\left(U_o{}^2 - L^2\right)\right],$$

At $\mu = U_o + k$, $E\left[P(x)\right] = -R_U$

Case 2(e): At $\mu = U_o + k$, $E\left[P(x)\right] = -R_U$

Clearly, the maximum expected net profit occurs either at $\mu = U_L - k$, from the differentiation in Case 2(b), at $\mu = L + k$ (from Case 2(c)) where $E[P(x)] = \frac{1}{2k}\left[R_L\left(U_o - L - 2k\right) + A\left(U_o - L\right) - \frac{C}{2}\left(U_o{}^2 - L^2\right)\right]$, or at $\mu = U_o - k$ (from Case 2(c)) where $E\left[P(x)\right] = \frac{1}{2k}\left[R_U\left(U_o - L - 2k\right) + A\left(U_o - L\right) - \frac{C}{2}\left(U_o{}^2 - L^2\right)\right]$. Which value of μ provides a higher expected net profit depends on the relationship between R_L and R_U. If $R_L > R_U$, then the expected net profit when $\mu = L + k$ is $\frac{1}{2k}\left[R_L\left(U_o - L - 2k\right) + A\left(U_o - L\right) - \frac{C}{2}\left(U_o{}^2 - L^2\right)\right]$ which is greater than the expected net profit when $\mu = U_o - k$, or $\frac{1}{2k}\left[R_U\left(U_o - L - 2k\right) + A\left(U_o - L\right) - \frac{C}{2}\left(U_o{}^2 - L^2\right)\right]$. This is consistent with Case 2(b), because when $R_U > R_L$, $U_o = U_L$, so $\mu = U_o - k = U_L - k$.

Therefore, if $R_L > R_U$,

$$\mu_o = L + k \tag{10}$$

and

$$E_o\left[P(x)\right]=\frac{1}{2k}\left[R_L\left(U_o-L-2k\right)+A\left(U_o-L\right)-\frac{C}{2}\left(U_o^{\,2}-L^2\right)\right] \qquad (11)$$

If $R_U > R_L$, then

$$\mu_o = U_o - k \qquad (12)$$

and

$$E_o\left[P(x)\right]=\frac{1}{2k}\left[R_U\left(U_o-L-2k\right)+A\left(U_o-L\right)-\frac{C}{2}\left(U_o^{\,2}-L^2\right)\right] \qquad (13)$$

In the case where $R_U = R_L = R$, Equations (11) and (13) are equal and the optimum set point for μ is any value in the range

$$\mu_o \in \left[U_o - k, L + k\right] \qquad (14)$$

and

$$E_o\left[P(x)\right]=\frac{1}{2k}\left[R\left(U_o-L-2k\right)+A\left(U_o-L\right)-\frac{C}{2}\left(U_o^{\,2}-L^2\right)\right] \qquad (15)$$

If $\mu_o = U_o - k$, then $a = U_o - 2k < L$ and $b = \left(U_o - k\right) + k = U_o$ where there are no rejects on the high side, but the proportion rejected on the low side is $p_L = \frac{\left(L - U_o + 2k\right)}{2k}$. Further, as $U_o - L$ approaches $2k$, $p_L \to 0$. On the other hand, if $\mu_o = L + k$, then $a = L$ and $b = L + 2k > U_o$ where there are no rejects on the low side, but the proportion rejected on the high side is $p_U = \frac{\left(L + 2k - U_o\right)}{2k}$. Again, as $U_o - L$ approaches $2k$, $p_U \to 0$. Note that, as expected, $p_L = p_U$.

Example 2 ($2k > U_o - L$ and $R_L > R_U$ **.)**

Let A = \$40, C = \$0.10, R_L = \$6, R_U = \$5, k = 150, and L = 200. U_o is calculated using Equation (7) to be U_o = 450. Using Equation (10), μ_o = 350. Various values for μ are presented in Figure 11 with the corresponding expected net profit, $E\left[P(x)\right]$, to show that calculating μ_o by Equation (10) does, in fact, give the maximum expected net profit (at \$5.42):

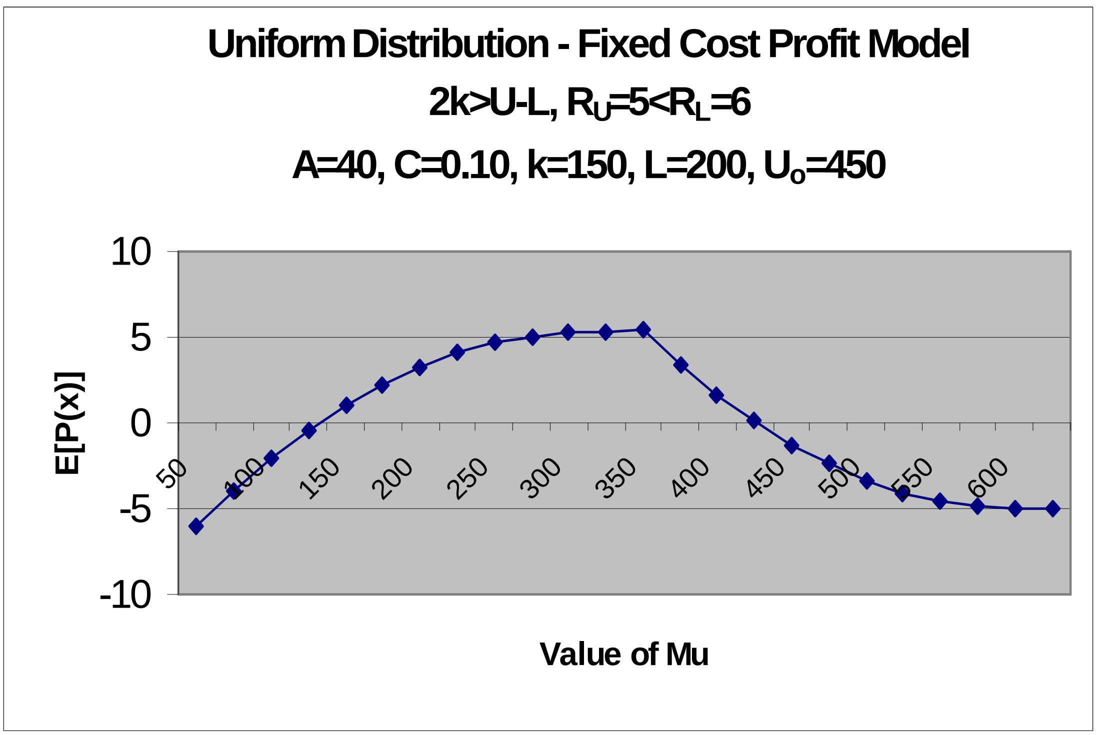


FIGURE 11. Example of Expected Net Profit for Different Values of μ when $2k > U_o - L$ and $R_U < R_L$.

Example 3 ($2k > U_o - L$ **and** $R_U > R_L$ **.)**

Let A = \$40, C = \$0.10, R_L = \$5, R_U = \$6, k = 150, and L = 200. U_o can be calculated using Equation (7) to be $U_o = 450$. Using Equation (12), μ_o = 300. Various values for μ are presented in Figure 11 with the corresponding expected net profit, $E\left[P(x)\right]$, to show that μ_o =300 does, in fact, give the highest expected net profit (at \$5.42.)

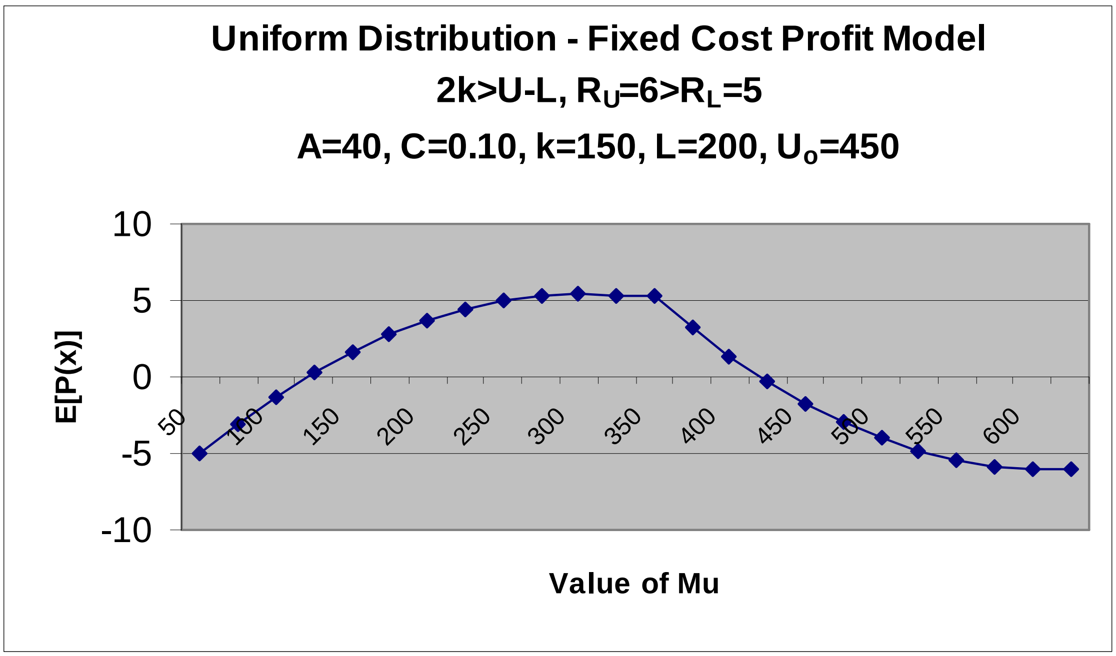


FIGURE 12. Example of Expected Net Profit for Different Values of μ when $2k > U_o - L$ and $R_U > R_L$.

Example 4 ($2k > U_o - L$ **and** $R_L = R_U = R$ **.)**

Let A = \$40, C = \$0.10, R_L = R_U = \$5, k = 150, and L = 200. U_o can be calculated using Equation (7) to be U_o = 450. Using Equation (14), $\mu_o \in [300, 350]$. Various values for μ

are presented in Figure 13 with the corresponding expected net profit, $E\left[P(x)\right]$, to show that $\mu_o \in [300, 350]$ does, in fact, give the highest expected net profit (at $5.42.)

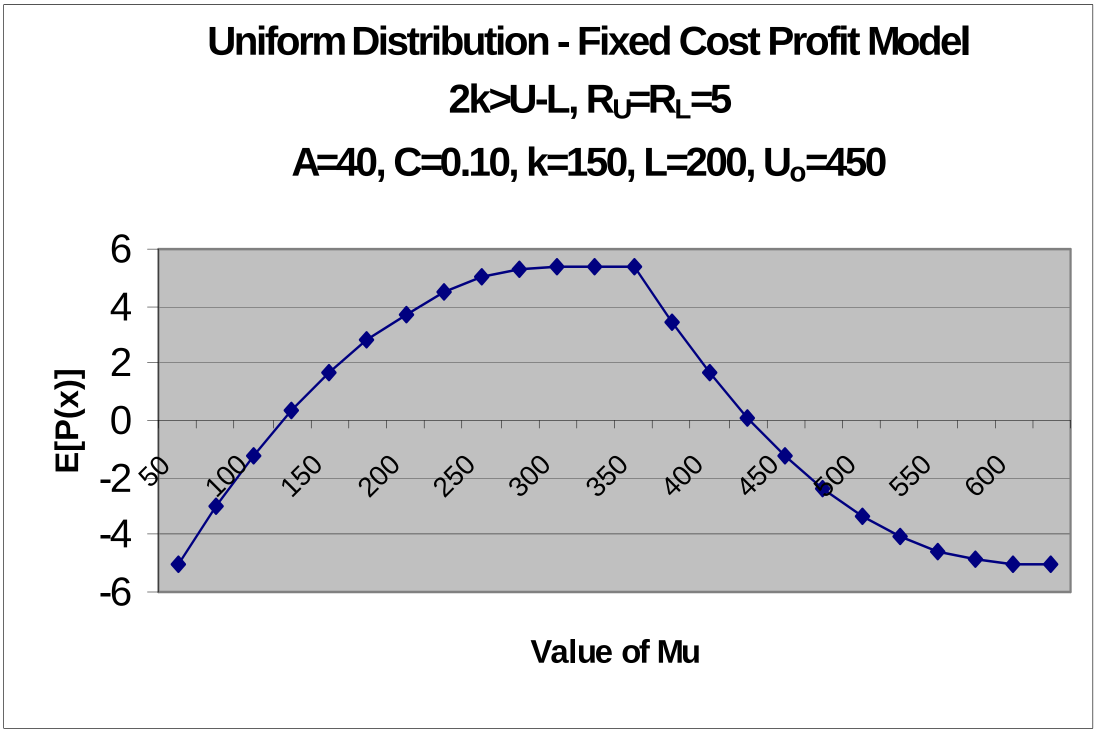


FIGURE 13. Example of Expected Net Profit for Different Values of μ when $2k > U_o - L$ and $R_L = R_U = R$.

To summarize, for fill level that follows a Uniform distribution when there is a constant scrap cost, the optimum value for the upper screening limit was determined to be

$$U_o = \min(U_L, U_U) = \min\left(\frac{R_L + A}{C}, \frac{R_U + A}{C}\right)$$

The optimum target set point for the process mean was obtained for the various scenarios:

Case 1: $2k \le U_o - L \rightarrow \mu_o = L + k$

Case 2: $2k > U_o - L$:

If $R_U < R_L \rightarrow \mu_o = L + k$

If $R_U > R_L \rightarrow \mu_o = U_o - k$

If $R_U = R_L \rightarrow \mu_o \in [U_o - k, L + k]$

3.2 Uniform underlying distribution – linear scrap cost

In this section, the optimum upper screening limit and optimum target mean are obtained for a different net profit model: one with a linear scrap cost. This net profit model is appropriate when the majority of the scrap/rework/reprocessing cost is from the cost of the material, for example in the gauge of steel beams. Using a model similar to that used by Dodson (1993), the net profit model is:

$$P(x) = \begin{cases} -Rx, & x < L \\ A - Cx, & x \geq L \end{cases}$$

The above model differs from Dodson (1993) in that in his model, price for conforming product was also linear. This model maintains a constant selling price, *A*, for comparison with the previous model.

Modifying the model to match the finite Uniform distribution,

$$P(x) = \begin{cases} -R_L x, & a \leq x < L \\ A - Cx, & L \leq x \leq U \\ -R_U x, & U < x \leq b \\ 0, & otherwise \end{cases}$$

where $a = \mu - k$ and $b = \mu + k$.

3.2.1 Optimum upper screening limit

Using the same approach as in section 3.1.1, the expected net profit equation when $L \le \mu \le U$, $a = \mu - k \le L$, and $b = \mu + k \ge U$, is given by:

$$E\left[P(x)\right] = \int_{\mu-k}^{L} \frac{-R_L x}{2k} + \int_{L}^{U} \frac{(A - Cx)}{2k} + \int_{U}^{\mu+k} \frac{-R_U x}{2k} =$$

$$\frac{1}{2k}\left\{ -\frac{R_L}{2}x^2\Big|_{\mu-k}^{L} + \left[Ax - \frac{C}{2}x^2 \right]_L^U - \frac{R_U}{2}x^2\Big|_{U}^{\mu+k} \right\} =$$

$$\frac{1}{2k}\left[\frac{R_L}{2}\left((\mu-k)^2 - L^2\right) + A(U-L) + \frac{C}{2}\left(L^2 - U^2\right) - \frac{R_U}{2}\left((\mu+k)^2 - U^2\right) \right] =$$

$$\frac{1}{2k}\left\{ \frac{R_L}{2}(\mu-k)^2 + A(U-L) + \frac{L^2}{2}(C - R_L) - \frac{C}{2}U^2 + \frac{R_U}{2}\left[U^2 - (\mu+k)^2\right] \right\} \qquad (16)$$

To determine the optimum upper screening limit, U_o, the derivative of $E\left[P(x)\right]$ with respect to U is calculated and set equal to zero:

$$\frac{\partial}{\partial U} E\left[P(x)\right] = \frac{1}{2k}\left[A - CU + R_U U\right]$$

$$A - CU + R_U U = 0 \Rightarrow$$

$$U_o = \frac{A}{C - R_U} \tag{17}$$

U_o in Equation (17) differs from the optimum value of U_o from Section 3.1.1, Equation (7), which was $U_o = \min\left(\frac{R_U + A}{C}, \frac{R_L + A}{C}\right)$. With the previous profit model, selling price (*A*) and rework cost (*R*) were fixed and only per unit production cost (*C*) was linear. In this model, both material cost (*C*) and rework cost (*R*) are linear.

Note that Equation (17) requires $C > R_U$, which means the per unit cost to produce must be greater than the incremental cost to reprocess in order to calculate an optimum upper screening limit. If $C < R_U$, no upper screening limit is necessary, because net profit will be higher to just produce all product with fill level greater than *L*, sell at *A* and absorb the "give-away" cost. However, $C > R_U$ implies that it costs more to manufacture the part than to manufacture and scrap or rework it. Therefore, an upper screening limit is not appropriate for this model. The remainder of this section will assume a fixed value of *U* that is set by customer requirements or defined by limitations of the container or the production equipment and that *U* is determined such that $A - C(U) \geq 0$.

3.2.2 Optimum target set point for mean

To determine an optimum target mean, μ_o, the two cases (Case 1: $2k \le U - L$ and Case 2: $2k > U - L$) and the same five ranges of μ that were presented in Section 3.1.2 also apply for the linear model as presented in Tables 4 and 5.

Table 4 lists the formulas for expected net profit for different ranges of μ for Case 1: $2k \le U - L$**:**

TABLE 4. Case 1: Expected Net Profit for Ranges of μ when $2k \le U - L$.

Scenario	**Range of** μ	**Expected Net Profit,** $E\left[P(x)\right]$
a	$k \le \mu \le L - k$	$-R_L\mu$
b	$L - k \le \mu < L + k$	$\frac{1}{2k}\left\{\begin{matrix}\frac{-R_L}{2}\left[L^2-(\mu-k)^2\right]+A(\mu+k-L)- \\ \frac{C}{2}\left[(\mu+k)^2-L^2\right]\end{matrix}\right\}$
c	$L + k \le \mu < U - k$	$A - C\mu$
d	$U - k \le \mu < U + k$	$\frac{1}{2k}\left\{\begin{matrix}A\left[U-(\mu-k)\right]-\frac{C}{2}\left[U^2-(\mu-k)^2\right]- \\ \frac{R_U}{2}\left[(\mu+k)^2-U^2\right]\end{matrix}\right\}$
e	$U + k \le \mu$	$-R_U\mu$

Expected Net Profit equations are calculated below and an attempt is made to find through differentiation the optimum set point for μ, μ_o, to maximize expected net profit:

Case 1 (a) $k \le \mu \le L-k$ **:**

$$E\left[P(x)\right] = \int_{\mu-k}^{\mu+k} -R_L x f(x) dx \;= \frac{1}{2k}\left\{\frac{-R_L}{2}\left[(\mu+k)^2 - (\mu-k)^2\right]\right\} = -R_L \mu$$

So, there are no critical points in $(k, L+k)$.

Case 1 (b) $L-k \le \mu < L+k$ **:**

$$E\left[P(x)\right] = \int_{\mu-k}^{L} -R_L x f(x) dx + \int_{L}^{\mu+k} (A - Cx) f(x) dx$$

$$= \frac{1}{2k}\left\{\frac{-R_L}{2}\left[L^2 - (\mu-k)^2\right] + A\left[(\mu+k) - L\right] - \frac{C}{2}\left[(\mu+k)^2 - L^2\right]\right\}$$

Now, in order to find the optimum value of μ, the equation for expected net profit is differentiated with respect to μ:

$$\frac{\partial}{\partial \mu} E\left[P(x)\right] = \frac{1}{2k}\left[R_L(\mu-k) + A - C(\mu+k)\right]$$

The second derivative with respect to μ, $\frac{\partial^2}{\partial \mu^2} E\left[P(x)\right] = \frac{1}{2k}\left(R_L - C\right) < 0$ iff $C > R_L$.

So, setting the first derivative equal to zero will result in a maximum when $C > R_L$:

$\frac{1}{2k}\left[R_L(\mu-k) + A - C(\mu+k)\right] = 0 \Rightarrow \mu = \frac{A}{R_L - C} + k$. But, when $C > R_L$,

$\mu = \frac{A}{R_L - C} + k < k$ and then $\mu - k < 0$ which is not feasible.

Case 1 (c) $L+k \le \mu < U-k$ **:**

$$E\left[P(x)\right] = \int_{\mu-k}^{\mu+k} (A - Cx) f(x) dx$$

$$= \frac{1}{2k}\left\{A\left[(\mu+k)-(\mu-k)\right]-\frac{C}{2}\left[(\mu+k)^2-(\mu-k)^2\right]\right\}$$

$$= A - C\mu$$

So, there are no critical points in $(L+k, U-k)$.

Case 1 (d) $U - k \le \mu < U + k$ **:**

$$E\left[P(x)\right] = \int_{\mu-k}^{U_o}(A-Cx)f(x)dx + \int_{U_o}^{\mu+k} -R_U x f(x)dx$$

$$= \frac{1}{2k}\left\{A\left[U-(\mu-k)\right]-\frac{C}{2}\left[U^2-(\mu-k)^2\right]-\frac{R_U}{2}\left[(\mu+k)^2-U^2\right]\right\}$$

Now, differentiating with respect to μ,

$$\frac{\partial}{\partial\mu}E\left[P(x)\right] = \frac{1}{2k}\left[-A + C(\mu-k) - R_U(\mu+k)\right]$$

Here, the second derivative with respect to μ is $\frac{1}{2k}(C-R_U) < 0$ iff $R_U > C$, so setting the first derivative equal to zero will result in a maximum expected net profit if $R_U > C$:

$\frac{\partial}{\partial\mu}E\left[P(x)\right] = \frac{1}{2k}\left[-A + C(\mu-k) - R_U(\mu+k)\right] = 0 \rightarrow \mu = \frac{A+(C+R_U)k}{C-R_U}$. But this value of μ leads to a maximum expected net profit in this range only if $R_U > C$. And, if $R_U > C$, then $\mu = \frac{A+(C+R_U)k}{C-R_U} < 0$ which is not feasible.

Case 1 (e) $U + k \le \mu$**:**

$$E\left[P(x)\right]=\int_{\mu-k}^{\mu+k}-R_U xf(x)dx \ =\frac{1}{2k}\left\{-\frac{R_U}{2}\left[(\mu+k)^2-(\mu-k)^2\right]\right\}=-R_U\mu$$

So, there are no critical points when $\mu>U-k$.

In each case, differentiation does not lead to a solution for μ_o, so the extreme points for each interval are calculated to determine the optimum value of μ, μ_o.

Case 1(a): At $\mu=L-k$, $E\left[P(x)\right]=-R_L(L-k)<0$

Case 1(b): At $\mu=L-k$, $E\left[P(x)\right]=\frac{1}{2k}\left\{\frac{-R_L}{2}\left[L^2-(L-2k)^2\right]\right\}-R_L(L-k)<0$

At $\mu=L+k$,

$$E\left[P(x)\right]=\frac{1}{2k}\left\{A(2k)-\frac{C}{2}\left[(L+2k)^2-L^2\right]\right\}=A-C(L+k)$$

Case 1(c): At $\mu=L+k$, $E\left[P(x)\right]=A-C(L+k)$

At $\mu=U-k$, $E\left[P(x)\right]=A-C(U-k)$

Case 1(d): At $\mu=U-k$,

$$E\left[P(x)\right]=\frac{1}{2k}\left\{A(2k)-\frac{C}{2}\left[U^2-(U-2k)^2\right]\right\}=A-C(U-k)$$

At $\mu=U+k$,

$$E\left[P(x)\right]=\frac{1}{2k}\left\{-\frac{R_U}{2}\left[(U+2k)^2-U^2\right]\right\}=-R_U(U+k)<0$$

Case 1(e): At $\mu=U+k$, $E\left[P(x)\right]=-R_U(U+k)<0$

Since $2k\le U-L,\ L+k\le U-k$, so $A-C(L+k)>A-C(U-k)$ and clearly, $A-C(L+k)>-R_L(L-k)$ and $A-C(L+k)>-R_U(U+k)$, so the maximum expected net profit occurs at $\mu = L + k$ with $E\left[P(x)\right] = A - C(L+k)$. Therefore, the optimum value of μ when $2k \leq U_o - L$, is:

$$\mu_o = L + k \tag{18}$$

and

$$E\left[P(x)\right] = A - C(L+k) \tag{19}$$

This conclusion is illustrated in Figure 14 below.

Example 5 ($2k \leq U - L$ **.)**

Let A = \$40, C = \$0.10, R_L = \$0.20, R_U = \$0.05, k = 50, L = 200, and $U = A/C = 400$. Using Equation (18), $\mu_o = L + k$ = 250. Various values for μ are presented in Figure 14 with the corresponding expected net profit, $E\left[P(x)\right]$ to show that $\mu_o = L + k$ = 250 does, in fact, give the highest expected net profit (at $E_o\left[P(x)\right]$=\$15.)

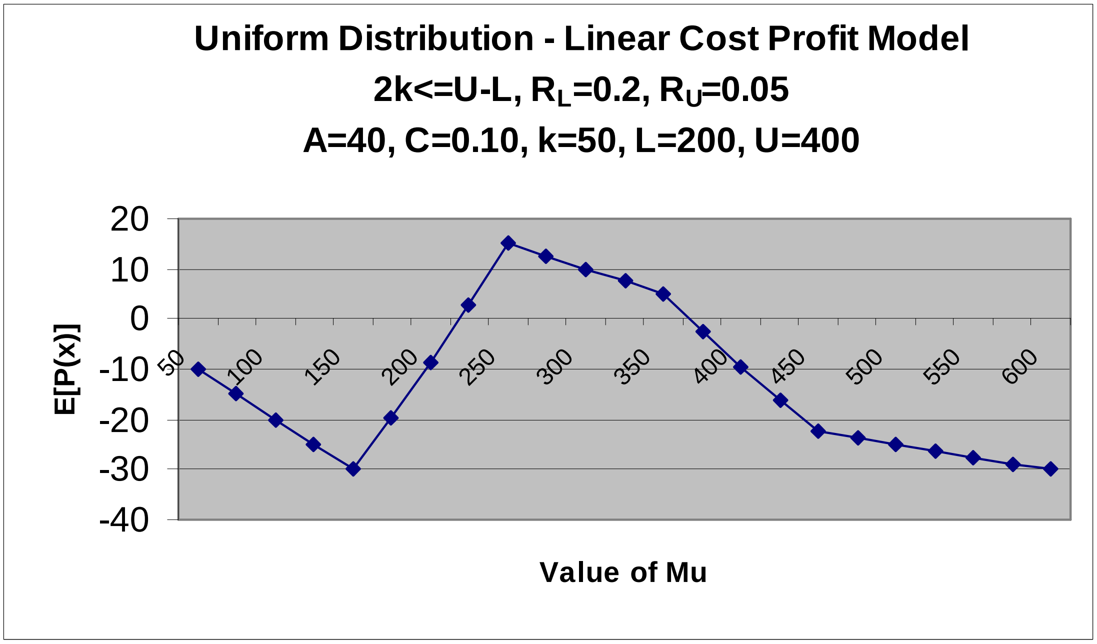


FIGURE 14. Example of Expected Net Profit for Different Values of μ when $2k \leq U - L$.

Case 2: $2k > U - L$

If $2k > U - L$, the fill level distribution is illustrated by Figure 15:

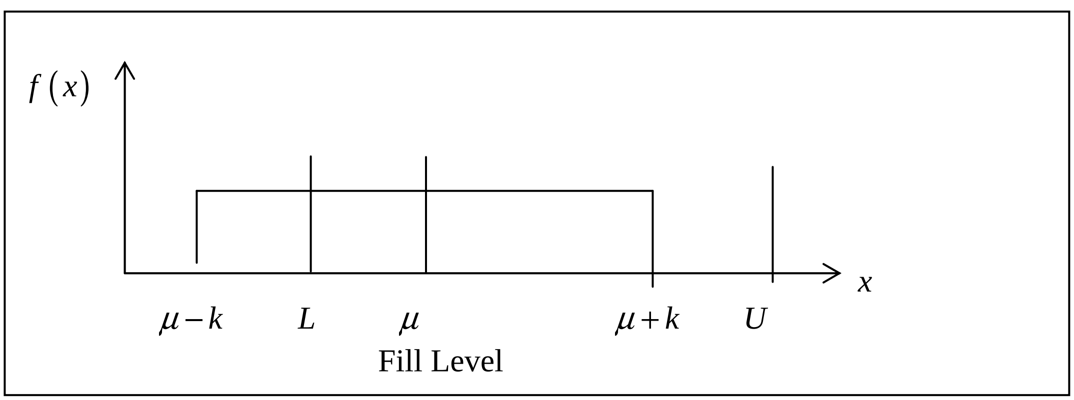


FIGURE 15. Distribution of Fill Level when $2k > U - L$.

The optimum set point for the mean, μ_o, to maximize net profit will depend on the relationship between the values of $R_L, R_U, C,$ and A as illustrated in Figure 16.

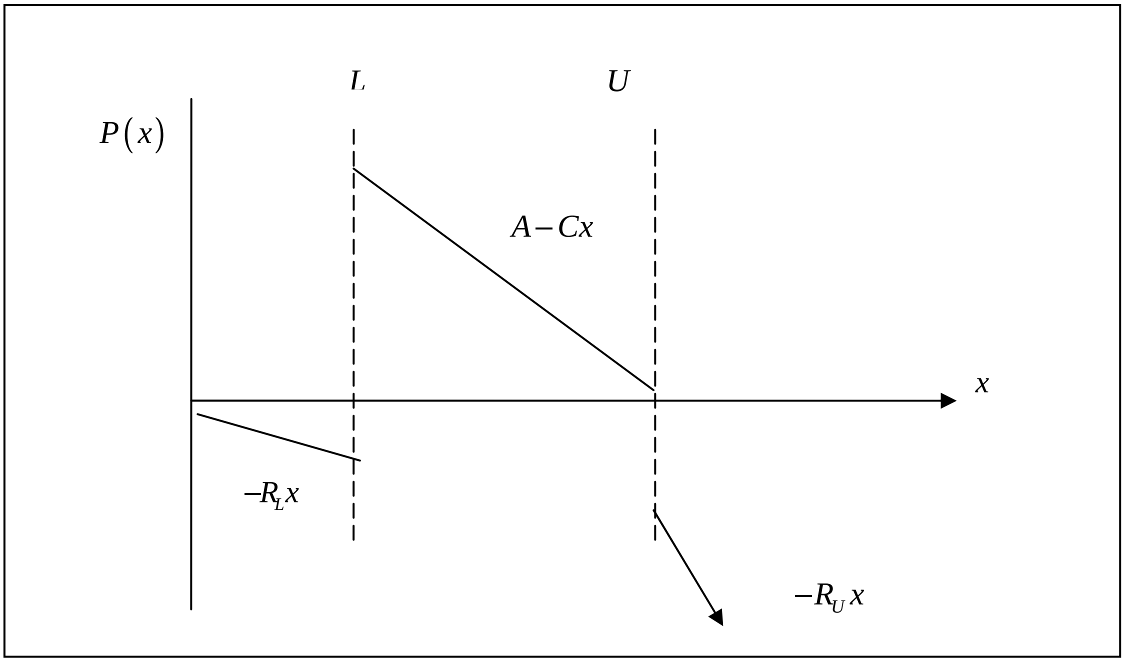


FIGURE 16. Net Profit Equation when $R_U > R_L$

If $R_U > C$, it is clear that there is no optimum value for U_o, as discussed in Section 3.2.1. Even if $R_U < C$, such that the optimum upper screening limit is defined by Equation (17), the fact that the cost is a linear function of fill level, means that it would be unlikely that a business would choose to allow fill level to run that high, since the cost is so great. More likely, a fixed upper limit is given.

Table 5 lists the formulas for expected net profit for different ranges of μ when $2k > U - L$. Note that (as in Case 2 of the fixed cost model in Section 3.1.2) the ranges of μ are different from Case 1 because $2k > U - L$.

TABLE 5. Case 2: Expected Net Profit for Ranges of μ when $2k > U - L$.

Scenario	**Range of μ**	**Expected Net Profit, $E\left[P(x)\right]$**
a	$\mu \le L - k$	$-R_L \mu$
b	$L - k \le \mu < U - k$	$\frac{1}{2k}\left\{\frac{-R_L}{2}\left[L^2 - (\mu - k)^2\right] + A(\mu + k - L) - \frac{C}{2}\left[(\mu + k)^2 - L^2\right]\right\}$
c	$U - k \le \mu < L + k$	$\frac{1}{2k}\left\{\frac{R_L}{2}\left[(\mu - k)^2 - L^2\right] + A(U - L) + \frac{C}{2}\left(L^2 - U^2\right) + \frac{R_U}{2}\left[U^2 - (\mu + k)^2\right]\right\}$
d	$L + k \le \mu < U + k$	$\frac{1}{2k}\left\{A\left[U - (\mu - k)\right] - \frac{C}{2}\left[U^2 - (\mu - k)^2\right] - \frac{R_U}{2}\left[(\mu + k)^2 - U^2\right]\right\}$
e	$U + k \le \mu$	$-R_U \mu$

Expected Net Profit equations are calculated as in Case 1 with the exception of Case 2(c) which is presented below:

Case 2 (c):

$$E\left[P(x)\right] = \int_{\mu-k}^{L} -R_L x f(x) dx + \int_{L}^{U} (A - Cx) f(x) dx + \int_{U}^{\mu+k} -R_U x f(x) dx$$

$$= \frac{1}{2k}\left\{\frac{R_L}{2}\left[(\mu - k)^2 - L^2\right] + A(U - L) + \frac{C}{2}\left(L^2 - U^2\right) + \frac{R_U}{2}\left[U^2 - (\mu + k)^2\right]\right\}$$

Now, differentiating with respect to μ,

$$\frac{\partial}{\partial \mu} E\left[P(x)\right] = \frac{1}{2k}\left[R_L(\mu - k) - R_U(\mu + k)\right]$$

Here, the second derivative with respect to μ is $\frac{1}{2k}(R_L - R_U) < 0$ iff $R_U > R_L$, so setting the first derivative equal to zero will result in a maximum expected net profit if $R_U > R_L$:

$\frac{\partial}{\partial \mu} E\left[P(x)\right] = \frac{1}{2k}\left[R_L(\mu - k) - R_U(\mu + k)\right] = 0 \rightarrow \mu = \frac{k(R_L + R_U)}{R_L - R_U}$. But this value of μ leads to a maximum expected net profit in this range only if $R_U > R_L$. And, if $R_U > R_L$, then $\mu = \frac{k(R_L + R_U)}{R_L - R_U} < 0$ which is not feasible.

In each range of μ, differentiation does not lead to a solution for μ_o, so the extreme points for each interval are calculated to determine the optimum value of μ, μ_o.

Case 2(a): At $\mu = L - k$, $E\left[P(x)\right] = -R_L(L - k) < 0$

Case 2(b): At $\mu = L - k$, $E\left[P(x)\right] = \frac{1}{2k}\left\{\frac{-R_L}{2}\left[L^2 - (L - 2k)^2\right]\right\} - R_L(L - k) < 0$

At $\mu = U - k$,

$$\begin{aligned}
E\left[P(x)\right] &= \frac{1}{2k}\left\{-\frac{R_L}{2}\left[L^2 - (U - 2k)^2\right] + A(U - L) - \frac{C}{2}\left(U^2 - L^2\right)\right\} \\
&= \frac{1}{2k}\left[\frac{R_L}{2}\left(U^2 - 4kU + 4k^2 - L^2\right) + A(U - L) - \frac{C}{2}\left(U^2 - L^2\right)\right] \\
&= \frac{1}{2k}\left[\frac{R_L}{2}\left(U^2 - L^2\right) + A(U - L) - \frac{C}{2}\left(U^2 - L^2\right)\right] - R_L(U - k) \\
&= \frac{(U - L)}{2k}\left[\frac{(R_L - C)(U + L)}{2} + A\right] - R_L(U - k)
\end{aligned}$$

Case 2(c): At $\mu = U - k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left\{\frac{R_L}{2}\left[(U-2k)^2 - L^2\right] + A(U-L) + \frac{C}{2}\left(L^2 - U^2\right)\right\}$$

$$= \frac{1}{2k}\left\{\frac{R_L}{2}\left(U^2 - 4kU + 4k^2 - L^2\right) + A(U-L) - \frac{C}{2}\left(U^2 - L^2\right)\right\}$$

$$= \frac{(U-L)}{2k}\left[\frac{(R_L - C)(U+L)}{2} + A\right] - R_L\,(U-k)$$

At $\mu = L + k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left\{A(U-L) + \frac{C}{2}\left(L^2 - U^2\right) + \frac{R_U}{2}\left[U^2 - (L+2k)^2\right]\right\}$$

$$= \frac{1}{2k}\left[A(U-L) - \frac{C}{2}\left(U^2 - L^2\right) + \frac{R_U}{2}\left(U^2 - L^2 - 4kL - 4k^2\right)\right]$$

$$= \frac{(U-L)}{2k}\left[\frac{(R_U - C)(U+L)}{2} + A\right] - R_U\,(L+k)$$

Case 2(d): At $\mu = L + k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left\{A(U-L) - \frac{C}{2}\left(U^2 - L^2\right) - \frac{R_U}{2}\left[(L+2k)^2 - U^2\right]\right\}$$

$$= \frac{1}{2k}\left[A(U-L) - \frac{C}{2}\left(U^2 - L^2\right) - \frac{R_U}{2}\left(L^2 + 4kL + 4k^2 - U^2\right)\right]$$

$$= \frac{(U-L)}{2k}\left[\frac{(R_U - C)(U+L)}{2} + A\right] - R_U\,(L+k)$$

At $\mu = U + k$,

$$E\left[P(x)\right] = \frac{1}{2k}\left\{-\frac{R_U}{2}\left[(U+2k)^2 - U^2\right]\right\} = -R_U\,(U+k) < 0$$

Case 2(e): At $\mu = U + k$, $E\left[P(x)\right] = -R_U\left(U + k\right) < 0$

So, $\mu_o = L + k$ if

$$\frac{(U-L)}{2k}\left[\frac{(R_U - C)(U+L)}{2} + A\right] - R_U\left(L+k\right) > \frac{(U-L)}{2k}\left[\frac{(R_L - C)(U+L)}{2} + A\right] - R_L\left(U-k\right)$$

Otherwise, $\mu_o = U - k$.

Simplifying,

$$\frac{(U-L)}{2k}\left[\frac{(R_U - C)(U+L)}{2} + A\right] - R_U\left(L+k\right) > \frac{(U-L)}{2k}\left[\frac{(R_L - C)(U+L)}{2} + A\right] - R_L\left(U-k\right) \rightarrow$$

$$R_U\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(L+k\right)\right] > R_L\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(U-k\right)\right]$$

Therefore, the optimum value of μ when $2k > U - L$ is:

$$\mu_o = L + k \text{ if } R_U\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(L+k\right)\right] > R_L\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(U-k\right)\right] \tag{20}$$

with
$$E_o\left[P(x)\right] = \frac{(U-L)}{2k}\left[\frac{(R_U - C)(U+L)}{2} + A\right] - R_U\left(L+k\right) \tag{21}$$

and

$$\mu_o = U - k \text{ if } R_U\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(L+k\right)\right] \le R_L\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(U-k\right)\right] \tag{22}$$

with
$$E_o\left[P(x)\right] = \frac{(U-L)}{2k}\left[\frac{(R_L - C)(U+L)}{2} + A\right] - R_L\left(U-k\right) \tag{23}$$

The following examples illustrate these conclusions:

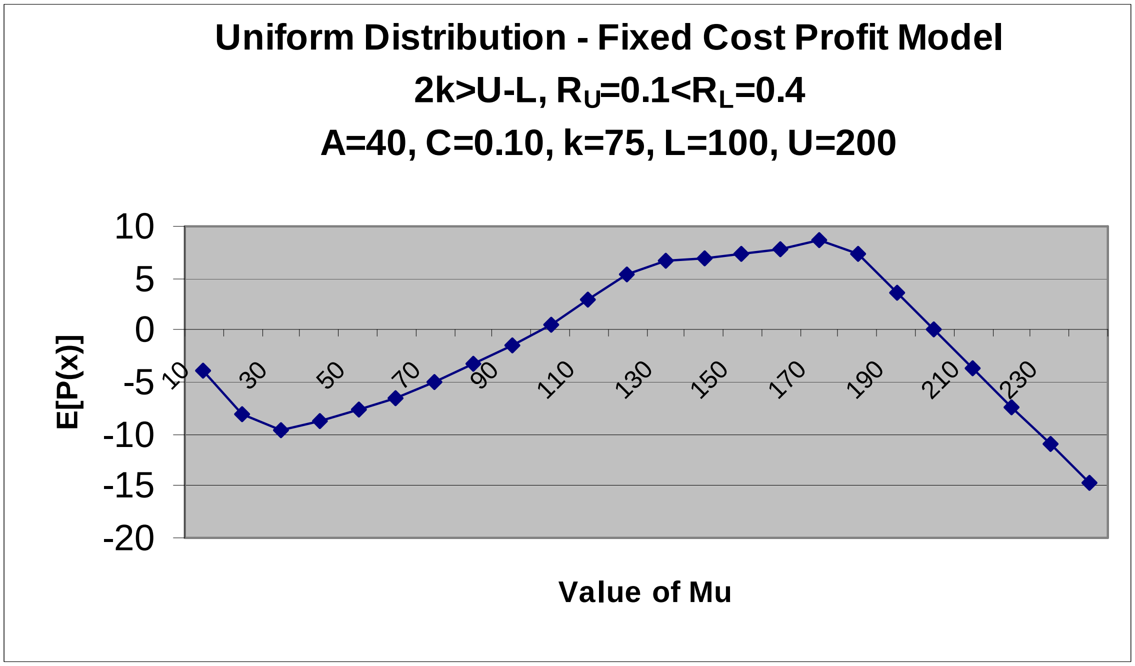
Example 6: $2k > U - L$ and $R_U\left[\frac{(U^2 - L^2)}{4k} - (L+k)\right] > R_L\left[\frac{(U^2 - L^2)}{4k} - (U-k)\right]$

Let A = \$40, C = \$0.10, R_L = \$0.40, R_U = \$0.10, k = 75, and L = 100, U=200. In this case,

$R_U\left[\frac{(U^2 - L^2)}{4k} - (L+k)\right] = -7.5 > R_L\left[\frac{(U^2 - L^2)}{4k} - (U-k)\right] = -10.5$, so based on

Equations (20 and 21), $\mu_o = L + k = 175$ with an expected net profit of \$9.17. Various

values for μ are presented in Figure 17 with the corresponding expected net profit to

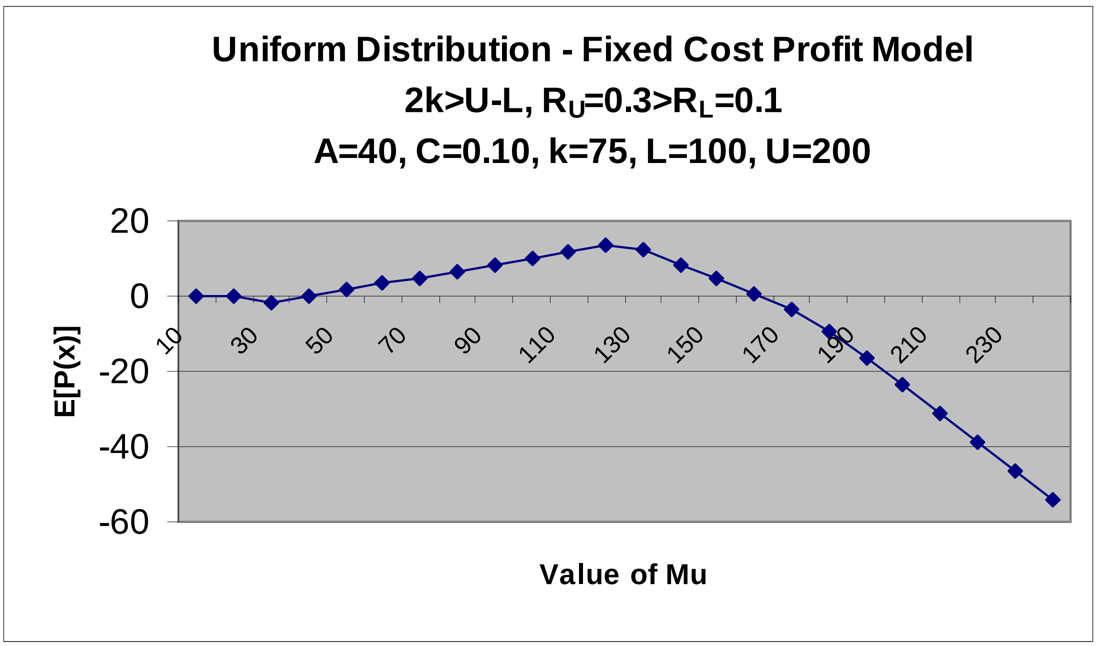


show that $\mu_o = L + k$ does, in fact, provide the maximum expected net profit.

FIGURE 17. Example of Expected Net Profit for Different Values of μ when $2k > U - L$, $\mu_o = L + k$.

Example 7: $2k > U - L$ and $R_U\left[\frac{(U^2-L^2)}{4k}-(L+k)\right] \le R_L\left[\frac{(U^2-L^2)}{4k}-(U-k)\right]$

Let A = \$40, C = \$0.10, R_L = \$0.10, R_U = \$0.30, k = 75, L = 100, and U=200. In this case, $R_U\left[\frac{(U^2-L^2)}{4k}-(L+k)\right] = -22.5 \le R_L\left[\frac{(U^2-L^2)}{4k}-(U-k)\right] = -2.5$, so based on Equations (22 and 23), $\mu_o = U - k = 125$ with an expected net profit of \$14.67. Various values for μ are presented in Figure 18 with the corresponding expected net profit to



show that $\mu_o = U - k$ does, in fact, provide the maximum expected net profit.

FIGURE 18. Example of Expected Net Profit for Different Values of μ when $2k > U - L$ and $\mu_o = U - k$.

To summarize, for fill level that follows a Uniform distribution when there is a linear scrap cost, the optimum target set point for the process mean was obtained for the various scenarios:

Case 1: $2k \le U - L$

$$\mu_o = L + k$$

Case 2: $2k > U - L$

$$\mu_o = L + k \text{ if } R_U\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(L + k\right)\right] > R_L\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(U - k\right)\right]$$

$$\mu_o = U - k \text{ if } R_U\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(L + k\right)\right] \le R_L\left[\frac{\left(U^2 - L^2\right)}{4k} - \left(U - k\right)\right]$$

3.3 Fixed scrap cost with capacity constraint

In practice, there may be some maximum capacity, CAP, such that if fill level exceeds CAP, an additional (usually very high) cost is incurred. This cost may be due to spillage over the maximum capacity of the container, for example, which results in clean up, downtime, restarting the equipment, etc.

The additional cost results in a new net profit equation:

$$P(x) = \begin{cases} -R_L & x < L \\ A - Cx & L \le x \le \min\left(U, CAP\right) \\ -R_U & U < x \le CAP \\ -Q & x > CAP \end{cases}$$

When the fill level is described by a Uniform distribution, the calculation of U_o as shown below is the same as that found in Section 3.1.1 where Equation (7) is

$$U_o = \min\left(\frac{R_U + A}{C}, \frac{R_L + A}{C}\right).$$

If $U_o \le CAP$, $2k > U_o - L$, and $L \le \mu \le U$ the expected net profit can be calculated as follows:

$$E\left[P(x)\right] = \int_{\mu-k}^{L} -R_L f(x)\,dx + \int_{L}^{U} (A - Cx) f(x)\,dx + \int_{U}^{CAP} -R_U f(x)\,dx + \int_{CAP}^{\mu+k} -Q f(x)\,dx$$

$$= \frac{1}{2k}\left[R_L(\mu - k - L) + A(U - L) - \frac{C\left(U^2 - L^2\right)}{2} + R_U(U - CAP) + Q(CAP - \mu - k) \right]$$

$$= \frac{1}{2k}\left[\begin{array}{l} U\left(R_U + A - \dfrac{CU}{2}\right) + R_L(\mu - k) - R_U(CAP) + \\ L\left(\dfrac{CL}{2} - A - R_L\right) + Q(CAP - \mu - k) \end{array} \right] \qquad (24)$$

Since the second derivative with respect to U is $-\frac{C}{2k}$ which is < 0, setting the first derivative of Equation (24) with respect to U equal to zero will provide an optimum value of the upper limit, U_o:

$$\frac{1}{2k}\left(R_U + A - CU\right) = 0 \rightarrow U_o = \frac{R_U + A}{C}$$

Which is the same as Equation (3).

Differentiating Equation (24) with respect to μ, $\frac{\partial}{\partial\mu}\left\{E\left[P\left(x\right)\right]\right\} = \frac{1}{2k}\left[R_L - Q\right]$

which results in no closed form solution. The impact on the optimum set point for the mean, μ_o, of a capacity constraint is to add to all of the scenarios in Section 3.1.2 the condition that if $CAP - k$ is less than the optimum value calculated for that scenario, then $\mu_o = CAP - k$. Specifically, the target set point for the process mean defined for the various scenarios:

Case 1: $2k < \min\left(CAP - L, U_o - L\right) \rightarrow \mu_o = L + k$

Case 2: $2k \geq \min\left(CAP - L, U_o - L\right)$

$$R_U < R_L \rightarrow \mu_o = \min\left(L + k, CAP - k\right)$$

$$R_U > R_L \rightarrow \mu_o = \min\left(U_o - k, CAP - k\right)$$

$$R_U = R_L \rightarrow \mu_o \in \left[U_o - k, \min\left(L + k, CAP - k\right)\right]$$

When $CAP < U_o$, the effect is that CAP replaces U_o in equations for expected net profit with the higher cost, Q, replacing R_U. An example follows:

Example 8: $CAP < U_o$ **and** $2k > CAP - L$

Let A = \$40, C = \$0.10, $R_L = \$6$, $R_U = 5$, CAP=400, Q=\$500, k = 175, and L = 100. In this case, $U_o = \min\left(\frac{R_U + A}{C}, \frac{R_L + A}{C}\right) = 450$, so $CAP < U_o$, $2k > CAP - L$, and $R_L > R_U$, so $\mu_o = \min\left(L + k, CAP - k\right) = \min\left(275, 225\right) = 225$. Various values for μ

are presented in Figure 19 with the corresponding expected net profit to show that $\mu_o = 225$, does in fact lead to the maximum expected net profit at $12.

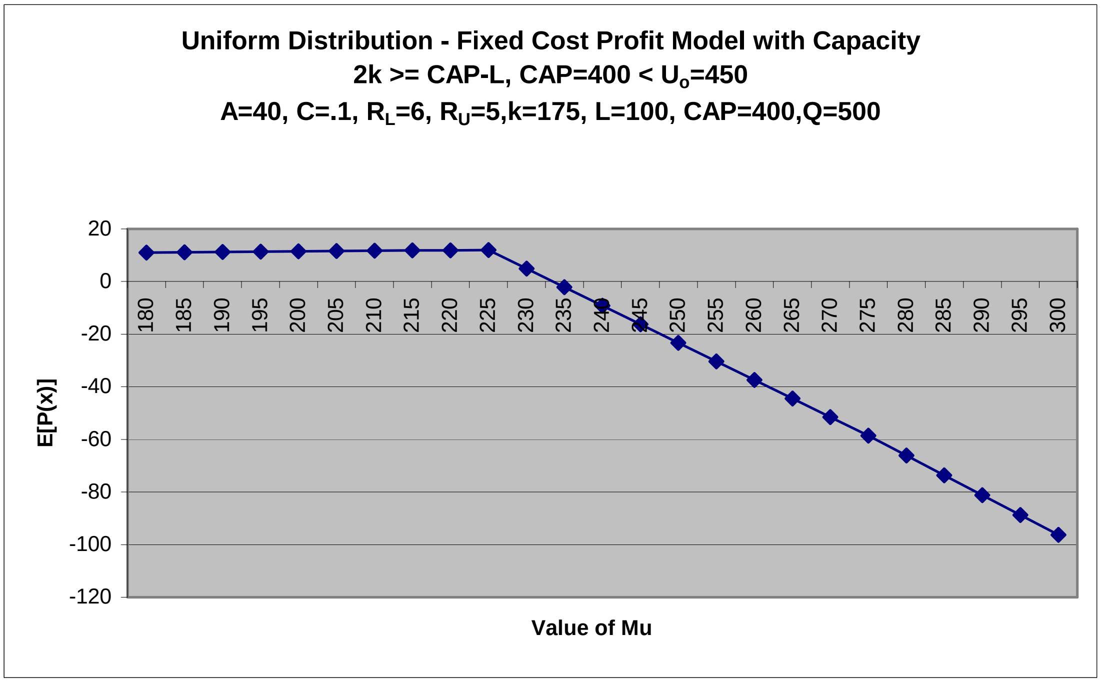


FIGURE 19. Example of Expected Net Profit for Different Values of μ when $CAP < U_o$ and $2k > CAP - L$.

4.0 Relationship between canning problem and process capability

The optimum target mean and upper screening limit for the canning problem is determined to maximize expected net profit, and net profit is increased as the rejection rate is decreased. The selection of μ_o to maximize expected net profit is consistent with the goal of improving process capability. In this chapter, the relationship between the canning problem and process capability is explored.

The term "process capability" refers to the likelihood that a process meets customer requirements or specifications as measured by a ratio of those requirements to process variation. Before process capability can be determined for a process, the process must be shown (e.g. using a control chart) to be stable over time. This means that only common causes of variation are present and the process parameters can accurately be estimated from empirical (or sample) data.

4.1 Relationship between process capability and the selection of U_o:

There is no relationship between process capability and the selection of U_o, since U_o depends only on the values of *A*, R_L, R_U, and *C* and not on process variation. There is, however, a relationship between process capability and the selection of μ_o which is presented in Section 4.2.

4.2 Relationship between process capability and the selection of μ_o:

Specific measures of process capability include C_P, C_{PK}, C_{PL}, C_{PU}, and C_{PM}. Each measure will be defined and discussed in this section as it relates to the canning problem.

4.2.1 Process Capability Measure, C_P

C_P is the most general capability index which compares the width of the specifications to the spread of the distribution (i.e. the natural tolerance of the quality characteristic, fill level.) It is generally calculated as the ratio between the blueprint (*BT*) and natural tolerances (*NT*) of the process: $C_P = \frac{BT}{NT} = \frac{USL - LSL}{UNL - LNL}$ where *USL* = upper specification limit, *LSL* = lower specification limit, *UNL* = upper natural tolerance limit of the distribution and *LNL* = lower natural tolerance limit of the distribution. The natural tolerance is defined as the middle 99.73% of the distribution, so for a normal distribution C_P is calculated as $C_P = \frac{USL - LSL}{6\sigma}$. For a Uniform distribution, $C_P = \frac{USL - LSL}{x_{.99865} - x_{.00135}} = \frac{USL - LSL}{.9973(2k)}$. Since, for a Uniform distribution, $\sigma = \frac{b-a}{\sqrt{12}} = \frac{2k}{\sqrt{12}}$, $2k = 2\sigma\sqrt{3}$, and thus $C_P = \frac{USL - LSL}{.9973\left(2\sigma\sqrt{3}\right)}$. The typically used definition of capability is that a process (that is in-control) is "potentially capable" if the width of the natural tolerance is smaller than the width of the specifications. This corresponds to $C_P \geq 1$.

Since C_P is independent of the position of μ, it is only an effective measure of process capability if the target and the process mean are both centered inside the specification, which is not the case in the canning problem. In cases where the mean is not centered (as in the canning problem), C_{PK} may be a better estimate. C_{PK} will be addressed in Section 4.2.2

Clearly, for any distribution, as variance decreases, the width of the natural tolerances of the distribution decreases, and the value of C_P increases. It has been shown that in the canning problem, as variance decreases, net profit increases. This means that as process capability is improved (and the value of C_P increases), variability decreases, and the mean can be set closer to L, so that $E_o\left[P(x)\right]$ increases.

With the canning problem, a lower specification limit, L, is defined by the customer, but (in the case of the fixed cost model) no upper specification is given, so C_P is not an appropriate measure of process capability. C_P can only be calculated when an upper specification exists (as in the case of the linear cost or capacity constrained models), or when C_P is calculated using U_o as the upper specification limit. But even when an upper specification limit is used, the target for the canning process is not the center of the specifications. In cases of a one-sided specification, C_{PK} is a better measure and is discussed in the next section.

4.2.2 Process Capability Measure, C_{PL}

In the canning problem, only a lower specification is given, so the C_{PL} index can be calculated. For a process that can be approximated with a normal distribution, $C_{PL} = \frac{(\mu - L)}{3\sigma}$. In the case of the Uniform distribution, $C_{PL} = \frac{(\mu - L)}{\mu - x_{.00135}} = \frac{(\mu - L)}{.49865(2k)}$.

$C_{PL} \geq 1$ means that $\leq$ 0.135% of the distribution of the specific quality characteristic is expected to fall below the lower specification.

Since process capability is inversely related to process standard deviation, it relates directly to the canning problem. As the standard deviation decreases, C_{PL} increases, and the process mean can be moved closer to *L*. As the mean is moved closer to *L*, cost decreases and net profit increases.

When a process has an upper specification, C_{PU} can be calculated in a similar fashion. In the case of a process with both an upper and lower specification limit, $C_{PK} = \min(C_{PL}, C_{PU})$. Since the canning problem has only a lower specification, clearly these measures of process capability do not apply.

4.2.3 Process Capability Measure, C_{PM}

C_{PK}, C_{PL}, and C_{PU} all assume the process target is centered inside the specifications. C_{PM} is a measure of process capability that considers deviation from target rather than the center of the specifications. Since the canning problem has a target not centered inside specification, it may be a better measure of process capability. In the special case of the canning problem, the target is actually the lower specification limit if there is no variation. Fill level below *L* is rejected, while fill level above *L* is accepted, but net profit is highest when fill level is equal to *L*. Given that fill level has variation, the target is actually some value above *L*, specifically μ_o. Once μ_o is determined, C_{PM} could be calculated as a measure of process capability. However, $C_{PM} = \frac{C_P}{\sqrt{1+\frac{(\mu - T)^2}{\sigma^2}}}$ where *T* = the target. So, to calculate C_{PM}, one must first calculate C_P which requires both lower and upper specification limits.

4.3 Summary of Relationship between Canning Problem and Process Capability

Any measure of process capability compares the process variation to specifications or requirements. Higher levels of process capability result when process variation decreases. In the canning problem, this means that as process capability is improved, variability decreases, and the mean can be set closer to *L*, so that $E_o\left[P(x)\right]$ increases.

At least one article has addressed the relationship between process capability and the canning problem. Kim, Cho, and Phillips (2000) show an economic model to determine the optimum process mean while maintaining a specific C_P value when fill level can be approximated by a Normal distribution. They assume that achieving a smaller variance leads to an increase in manufacturing costs that they incorporate into the net profit model. They provide a case study and sensitivity analysis.

5.0 Triangular distribution

Another finite distribution that can be a plausible model for a canning process is the Triangular distribution. In this chapter, the Triangular distribution is studied to determine an optimal set point for the mean of the production operation. An optimum upper limit is first calculated which provides a "cut-off" to maximize expected net profit by minimizing "give-away" cost.

Again, the basic profit function of Liu and Raghavachari (1997) article is used, so that Triangular results can also be compared to their's which pertained to continuous distributions.

The Net Profit function:

$$P(x) = \begin{cases} -R_L, & x \le L \\ A - Cx, & L \langle x \langle U \\ -R_U, & x \ge U \end{cases}$$

where L = lower specification limit, below which the customer will not accept the product. For example, if a jar is to be filled with 8 oz. of an ingredient, anything less than L=8 oz. is not allowed. U = an upper limit (U is either set by the producer as an upper specification limit, *USL*, or determined as a value that minimizes giveaway cost such that any fill level above U costs more in giveaway cost than would be received in income.

R_L = the rejection cost per container when fill level is less than *L*, R_U = the rejection cost per container when fill level is greater than *U*, *A* = revenue received for an acceptable container, and *C* = the production cost per unit of ingredient. *A*, R_L, R_U, *C*, and *L* are known and > 0.

A Triangular distribution has the following probability density function:

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(m-a)}, & a \le x \le m \\ \dfrac{2(b-x)}{(b-a)(b-m)}, & m \le x \le b \\ 0, & otherwise \end{cases}$$

where *m* is the mode and graphically the distribution appears below in Figure 20.

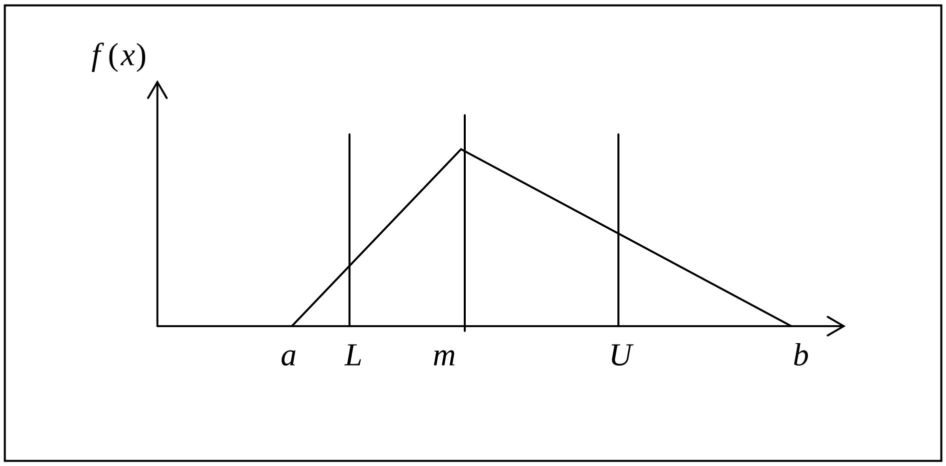


FIGURE 20. The triangular probability density function with lower specification, *L*, and upper screening limit, *U*.

With respect to the canning problem, there is a cost (R_L) when the quantity falls below the lower specification, *L (L* $\geq$ *a),* and a cost (R_U) when the fill level falls above an arbitrary upper specification, *U (U < b)*. Giveaway cost above *U* is greater than the cost of simply scrapping the unit (R_U .)

5.1 Symmetric Triangular underlying distribution

First, a symmetric Triangular distribution will be analyzed using the fixed rework cost model from the previous chapter. Then the impact of skewness will be examined. Assuming a symmetric Triangular distribution, the probability density function is given below:

$$f(x) = \begin{cases} \dfrac{x-m+k}{k^2}, & m-k \le x \le m \\ \dfrac{m+k-x}{k^2}, & m \le x \le m+k \\ 0, & otherwise \end{cases}$$

where *m* is the modal point and 2*k* represents the spread of the distribution and it can be verified (in the Appendix) that the $V(x) = \dfrac{k^2}{6}$. In this chapter, m_o, the optimum set point for *m* is computed for the case where $m - k_1 < L$, $m + k_2 > U$, and $L \le m < U$.

5.1.1. Optimum upper screening limit

If there is no upper specification limit given, an optimum upper limit can be calculated as in previous chapters. The equation for expected net profit for the symmetrical triangular distribution is found below:

$$E\left[P(x)\right]=\begin{cases}\int_{m-k}^{L}-R_L\left[\frac{x-m+k}{k^2}\right]dx+\int_{L}^{m}(A-Cx)\left[\frac{x-m+k}{k^2}\right]dx+\\ \int_{m}^{U}(A-Cx)\left[\frac{m+k-x}{k^2}\right]dx+\int_{U}^{m+k}-R_U\left[\frac{m+k-x}{k^2}\right]dx=\end{cases}$$

$$\frac{1}{k^2}\begin{Bmatrix}-R_L\left[\frac{\left[x-(m-k)\right]^2}{2}\right]_{m-k}^{L}+A\left[\frac{\left[x-(m-k)\right]^2}{2}\right]_{L}^{m}-C\left[\frac{x^3}{3}-\frac{mx^2}{2}+\frac{kx^2}{2}\right]_{L}^{m}-\\ A\left[\frac{\left[(m+k)-x\right]^2}{2}\right]_{m}^{U}-C\left[\frac{mx^2}{2}+\frac{kx^2}{2}-\frac{x^3}{3}\right]_{m}^{U}+R_U\left[\frac{\left[(m+k)-x\right]^2}{2}\right]_{U}^{m+k}\end{Bmatrix}=$$

$$\frac{1}{k^2}\begin{Bmatrix}-\frac{R_L}{2}\left[L-(m-k)\right]^2+A\left[-m^2+m(L+U)+k(U-L)-\frac{1}{2}\left(U^2+L^2\right)\right]-\\ C\left[\frac{m^3}{3}-\frac{m^3}{2}-\frac{L^3}{3}+\frac{mL^2}{2}-\frac{kL^2}{2}+\frac{mU^2}{2}+\frac{kU^2}{2}-\frac{U^3}{3}-\frac{m^3}{2}+\frac{m^3}{3}\right]-\frac{R_U}{2}\left[(m+k)-U\right]^2\end{Bmatrix}=$$

$$E\left[P(x)\right]=\frac{1}{k^2}\begin{Bmatrix}-\frac{R_L}{2}\left[L-(m-k)\right]^2-A\left[\frac{(m-L)^2}{2}+\frac{(m-U)^2}{2}+k(L-U)\right]-\\ C\left[-\frac{\left(m^3+L^3+U^3\right)}{3}+\frac{L^2}{2}(m-k)+\frac{U^2}{2}(m+k)\right]-\frac{R_U}{2}\left[(m+k)-U\right]^2\end{Bmatrix}\qquad(25)$$

Equation (25) assumes that the distribution is such that $m-k<L$ and $m+k>U$ and shows that the expected net profit is inversely proportional to the $V(x)=\frac{k^2}{6}$.

Differentiating Equation (25) with respect to *U*, results in the following:

$$\frac{\partial}{\partial U}E\left[P(x)\right]=\frac{1}{k^2}\left\{A\left[(m+k)-U\right]-CU(m+k)+CU^2+R_U\left[(m+k)-U\right]\right\}=$$

$$\frac{1}{k^2}\left\{(A+R_U)(m+k)-\left[A+R_U+C(m+k)\right]U+CU^2\right\}$$

To determine the optimum value of *U*, U_o, the first derivative with respect to *U* is set equal to zero. (This will be an optimum value for *U* if $U_o<\frac{A+R_U}{2C}+\frac{(m+k)}{2}=U_U$.)

$$\frac{\partial}{\partial U}\left\{E\left[P(x)\right]\right\}=0\Rightarrow$$

$$CU_o^{\ 2}-\left[A+R_U+C(m+k)\right]U_o+(A+R_U)(m+k)=0$$

Let $c_1=\left[A+R_U+C(m+k)\right]$ and $c_2=(A+R_U)(m+k)$, then, $CU_o^{\ 2}-c_1U_o+c_2=0$, which means that $U_o=\frac{c_1\pm\sqrt{c_1^{\ 2}-4Cc_2}}{2C}$

$$U_o=\frac{\left[(A+R_U)+C(m+k)\right]\pm\sqrt{\left[(A+R_U)+C(m+k)\right]^2-4C(A+R_U)(m+k)}}{2C}=$$

$$\frac{\left[\left(A+R_U\right)+C\left(m+k\right)\right]\pm\sqrt{\left(A+R_U\right)^2-2C\left(A+R_U\right)\left(m+k\right)+\left[C\left(m+k\right)\right]^2}}{2C}=$$

$$\frac{\left[\left(A+R_U\right)+C\left(m+k\right)\right]\pm\sqrt{\left[\left(A+R_U\right)-C\left(m+k\right)\right]^2}}{2C}=$$

$$\frac{\left[\left(A+R_U\right)+C\left(m+k\right)\right]\pm\left[\left(A+R_U\right)-C\left(m+k\right)\right]}{2C}$$

Case 1:

$$\frac{\left[\left(A+R_U\right)+C\left(m+k\right)\right]+\left[\left(A+R_U\right)-C\left(m+k\right)\right]}{2C}=$$

$$\frac{2\left[A+R_U\right]}{2C}\Rightarrow U_o=\frac{A+R_U}{C} \tag{26}$$

This is consistent with the value found for U_o in previous sections.

Case 2: $\frac{\left[\left(A+R_U\right)+C\left(m+k\right)\right]-\left[\left(A+R_U\right)-\left[C\left(m+k\right)\right]\right]}{2C}=m+k$ which is not < U

(since $2k>U-L$.)

So, the optimum value for U_o is given in Equation (26).

5.1.2 Optimum target set point for *m*

The objective is to maximize the expected net profit. Since the profit function is defined as:

$$P(x) = \begin{cases} -R_L, & x \le L \\ A - Cx, & L < x < U_o \\ -R_U, & x \ge U_o \end{cases}$$

where L is the customer driven lower specification limit as defined in previous chapters and U_o is the optimum upper screening limit given in Equation (26.)

As in the case of the Uniform distribution, since the Triangular distribution is finite, various cases must be considered. Here, the only cases considered are those where $L \le m \le U$. (In the case of the symmetric Triangular distribution, $k_1 = k_2 = k$.)

Case 1:

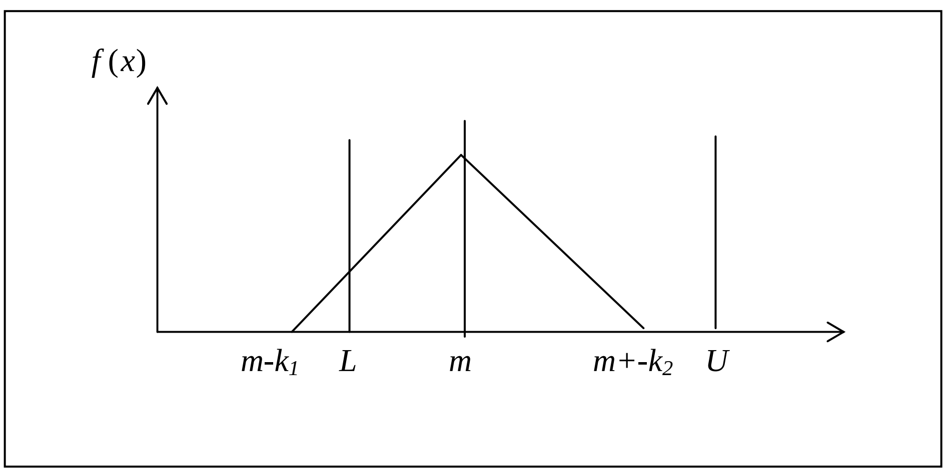


FIGURE 21. The triangular probability density function with lower specification, L, and upper screening limit, U when $m - k_1 < L$ and $L < m + k_2 \le U$.

Case 2:

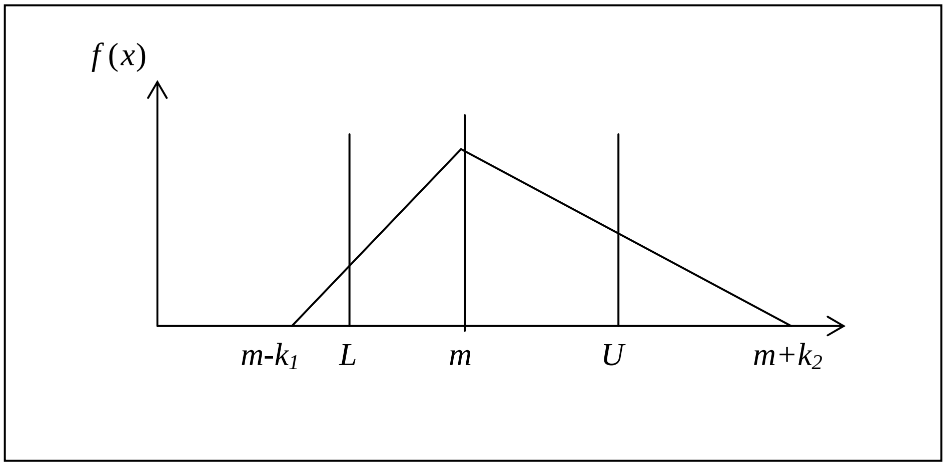


FIGURE 22. The triangular probability density function with lower specification, *L*, and upper screening limit, *U* when $m-k_1 < L$ and $m+k_2 > U$.

Case 3:

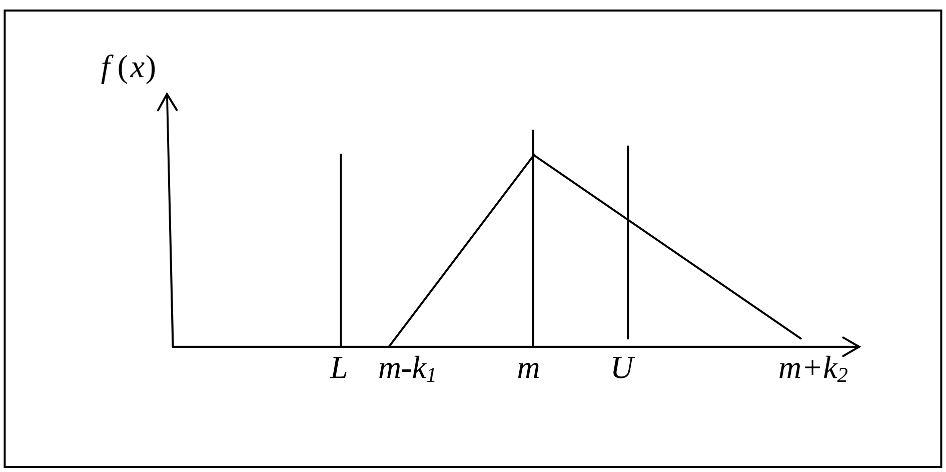


FIGURE 23. The triangular probability density function with lower specification, *L*, and upper screening limit, *U* when $L \le m-k_1 \le U$ and $m+k_2 > U$.

Due to the complexity of the distribution, the only case considered in this work is Case 2, Figure 22, when $L \le m \le U$, $m-k_1 < L$, and $m+k_2 > U$.

To determine the optimum value of *m* to maximize the average net profit, the first derivative of Equation (25) with respect to *m* is calculated below:

$$\frac{\partial}{\partial m}E\left[P(x)\right]=\frac{1}{k^2}\begin{bmatrix}R_L\left[L-(m-k)\right]-A\left[2m-(L+U_o)\right]+\\C\left(m^2-\frac{1}{2}\left(L^2+U_o{}^2\right)\right)-R_U\left[(m+k)-U_o\right]\end{bmatrix} \tag{27}$$

If the second derivative with respect to *m* is less than zero, then Equation (27) can be set to zero to solve for the value of *m* that maximizes the expected net profit:

$$\frac{\partial^2}{\partial m^2}E\left[P(x)\right]=\frac{1}{k^2}\left[-R_L-2A+2Cm-R_U\right]$$

The above second derivative is less than zero only if $m<\dfrac{R_L+R_U+2A}{2C}$ (28)

Setting Equation (27) equal to zero results in a quadratic equation:

$$\frac{1}{k^2}\left[R_L\left[L-(m-k)\right]-A\left[2m-(L+U_o)\right]+C\left(m^2-\frac{1}{2}\left(L^2+U_o{}^2\right)\right)-R_U\left[(m+k)-U_o\right]\right]=0\Rightarrow$$

$$Cm^2-(R_L+R_U+2A)m+\left[R_L(L+k)+A(L+U_o)+R_U(U_o-k)-\frac{C}{2}\left(L^2+U_o{}^2\right)\right]=0$$

Let $c_3=R_L+R_U+2A$

and $c_4=R_L(L+k)+A(L+U_o)+R_U(U_o-k)-\frac{C}{2}\left(L^2+U_o{}^2\right)$,

then, $Cm_o{}^2-c_3m_o+c_4=0$

Thus, the optimum set point for *m* is given by,

$$m_o = \frac{c_3 \pm \sqrt{c_3^{\ 2} - 4Cc_4}}{2C}$$

$$m_o = \frac{\left(R_L + R_U + 2A\right) \pm \sqrt{\begin{matrix}\left(R_L + R_U + 2A\right)^2 - \\ 4C\left[R_L\left(L+k\right) + A\left(L+U_o\right) + R_U\left(U_o - k\right) - C\left(\frac{L^2}{2} + \frac{U_o^{\ 2}}{2}\right)\right]\end{matrix}}}{2C}$$

In the case where $R_L = R_U = R$,

$$m_o = \frac{2\left(R+A\right) \pm \sqrt{\left(2R+2A\right)^2 - 4C\left[R\left(L+U_o\right) + A\left(L+U_o\right) - C\left(\frac{L^2}{2} + \frac{U_o^{\ 2}}{2}\right)\right]}}{2C}$$

$$= \frac{2\left(R+A\right) \pm \sqrt{4\left(R+A\right)^2 - 4C\left[\left(R+A\right)\left(L+U_o\right) - C\left(\frac{L^2}{2} + \frac{U_o^{\ 2}}{2}\right)\right]}}{2C}$$

$$= \frac{R+A}{C} \pm \sqrt{\frac{\left(R+A\right)^2 - C\left[\left(R+A\right)\left(L+U_o\right) - C\left(\frac{L^2}{2} + \frac{U_o^{\ 2}}{2}\right)\right]}{C^2}}$$

$$= \frac{R+A}{C} \pm \sqrt{\left(\frac{R+A}{C}\right)^2 - \frac{\left(R+A\right)\left(L+U_o\right)}{C} + \left(\frac{L^2 + U_o^{\ 2}}{2}\right)} \qquad (29)$$

This requires $\left(\frac{R+A}{C}\right)^2 > \frac{\left(R+A\right)\left(L+U_o\right)}{C} - \left(\frac{L^2 + U_o^{\ 2}}{2}\right)$

$$\left(R+A\right)^2 > C\left(R+A\right)\left(L+U_o\right) - \frac{C^2\left(L^2+U_o{}^2\right)}{2}$$

$$\left(R+A\right)^2 > C\left[\left(R+A\right)\left(L+U_o\right) - \frac{C\left(L^2+U_o{}^2\right)}{2}\right]$$

In the case where $R_L = R_U = R$, the expected net profit Equation (25), can be simplified as follows:

$E\left[P(x)\right] =$

$$\frac{1}{k^2}\left\{\begin{array}{l} -\frac{R}{2}\left\{\left[L-(m-k)\right]^2 + \left[(m+k)-U_o\right]^2\right\} - A\left[m^2 - m\left(L+U_o\right) - k\left(U_o - L\right) + \frac{1}{2}\left(U_o{}^2 + L^2\right)\right] - \\ C\left[-\frac{m^3}{3} + \frac{m}{2}\left(L^2+U_o{}^2\right) + \frac{k}{2}\left(U_o{}^2 - L^2\right) - \left(\frac{U_o{}^3}{3} + \frac{L^3}{3}\right)\right] \end{array}\right\} \quad (30)$$

There are two possible values of *m* from Equation (29),

Case 1: $m_1 = \frac{R+A}{C} + \sqrt{\left(\frac{R+A}{C}\right)^2 - \frac{(R+A)\left(L+U_o\right)}{C} + \left(\frac{L^2+U_o{}^2}{2}\right)}$

Case 2: $m_2 = \frac{R+A}{C} - \sqrt{\left(\frac{R+A}{C}\right)^2 - \frac{(R+A)\left(L+U_o\right)}{C} + \left(\frac{L^2+U_o{}^2}{2}\right)}$

To determine which value of *m* maximizes net profit, substitute each value of *m* into Equation (28) which is the requirement for maximizing net profit.

When $R_L = R_U = R$,

$$m < \frac{R_L + R_U + 2A}{2C} \quad \Rightarrow \quad m < \frac{R + A}{C}$$

Clearly, this is only true for Case 2, so the optimum value of *m* must be:

$$m_o = m_2 = \frac{R+A}{C} - \sqrt{\left(\frac{R+A}{C}\right)^2 - \frac{(R+A)(L+U_o)}{C} + \left(\frac{L^2 + U_o^{\,2}}{2}\right)} \qquad (31)$$

The optimum $E\left[P(x)\right]$ is obtained by inserting m_o from Equation (31) into Equation (25.)

Example 11.

An example similar to that used in the previous chapter follows. Assume that *L* = the lower specification for fill level (*L=100*.) If an upper specification limit, *USL*, is given for fill level, let *U=USL (USL=200*.) The revenue if fill level is between *L* and *U* is *A* = \$20. *C* is the unit cost to produce and *C* = \$0.10. The scrap / reprocessing cost if fill level is less than *L* or greater than *U* is *R* = \$6. The spread of the distribution is characterized by *k* such that the process limits are *m* - *k* and *m* + *k* and in this example, *k* = 100.

TABLE 6. Example of expected net profit calculations for symmetric triangular distribution with $2k > U - L$ with a fixed value of U.

A	*C*	*R*	*k*	*U*	*L*	*m*	*E[P(x)]*
20	0.1	6	100	200	100	139.1695	2.387498
20	0.1	6	100	200	100	10	-5.92167
20	0.1	6	100	200	100	20	-5.69333
20	0.1	6	100	200	100	30	-5.325
20	0.1	6	100	200	100	40	-4.82667
20	0.1	6	100	200	100	50	-4.20833
20	0.1	6	100	200	100	60	-3.48
20	0.1	6	100	200	100	70	-2.65167
20	0.1	6	100	200	100	80	-1.73333
20	0.1	6	100	200	100	90	-0.735
20	0.1	6	100	200	100	100	0.333333
20	0.1	6	100	200	100	110	1.276667
20	0.1	6	100	200	100	120	1.92
20	0.1	6	100	200	100	130	2.283333
20	0.1	6	100	200	100	140	2.386667
20	0.1	6	100	200	100	150	2.25
20	0.1	6	100	200	100	160	1.893333
20	0.1	6	100	200	100	170	1.336667
20	0.1	6	100	200	100	180	0.6
20	0.1	6	100	200	100	190	-0.29667
20	0.1	6	100	200	100	200	-1.33333
20	0.1	6	100	200	100	210	-2.355
20	0.1	6	100	200	100	230	-3.95833
20	0.1	6	100	200	100	240	-4.56
20	0.1	6	100	200	100	250	-5.04167
20	0.1	6	100	200	100	260	-5.41333
20	0.1	6	100	200	100	270	-5.685
20	0.1	6	100	200	100	280	-5.86667
20	0.1	6	100	200	100	290	-5.96833
20	0.1	6	100	200	100	300	-6
20	0.1	6	100	200	100	310	-6

According to Equation (29), the optimum set point for the mean is $m = 139.17$. At that point, the expected net profit is *$2.39.*

A graph of expected net profit for the various values of m is shown in Figure 24. With U set at 200, there is a decrease in net profit as the distribution approaches that point. Clearly, profit would be higher if U was set higher. In the next example, an optimum value of U, U_o, is determined.

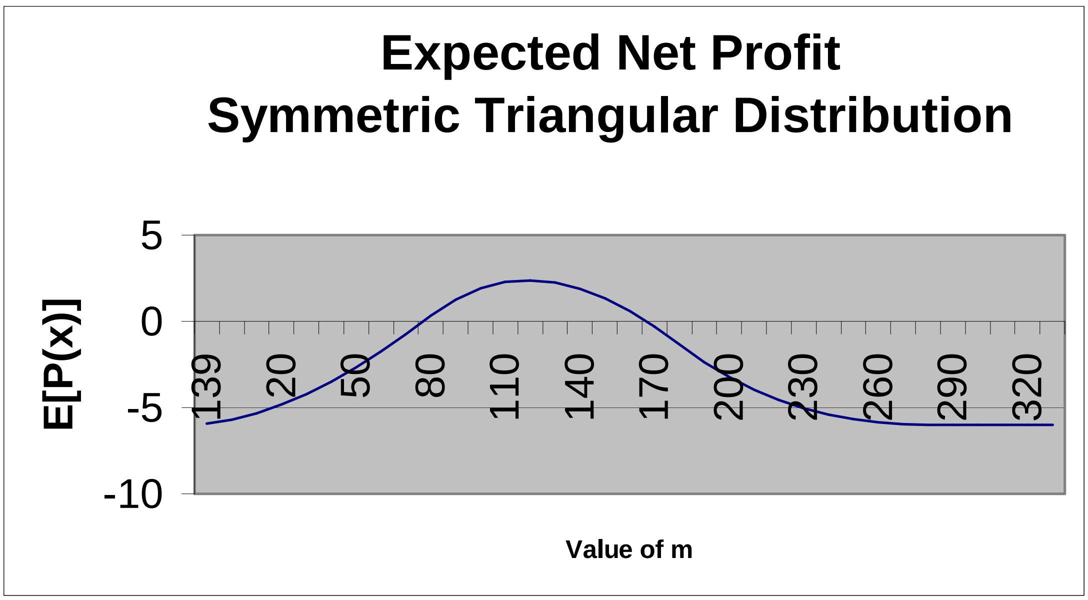


FIGURE 24. Expected net profit model for symmetric triangular distribution with $2k > U - L$ and a fixed value of U.

Example 12.

Now the information from Example 11 is presented, but using the optimum upper screening limit, U_o, instead of a given upper limit,

$$U_o = \frac{A + R_U}{C} \Rightarrow U_o = \frac{20 + 6}{.10} = 260$$

TABLE 7. Expected net profit calculations for symmetric triangular distribution with $2k > U - L$ with a calculated value of *U*.

A	***C***	***R***	***k***	***U***	***L***	***m***	***E[P(x)]***
20	0.1	6	100	260	100	146.8629	2.80481
20	0.1	6	100	260	100	0	-6
20	0.1	6	100	260	100	10	-5.92167
20	0.1	6	100	260	100	20	-5.69333
20	0.1	6	100	260	100	30	-5.325
20	0.1	6	100	260	100	40	-4.82667
20	0.1	6	100	260	100	50	-4.20833
20	0.1	6	100	260	100	60	-3.48
20	0.1	6	100	260	100	70	-2.65167
20	0.1	6	100	260	100	80	-1.73333
20	0.1	6	100	260	100	90	-0.735
20	0.1	6	100	260	100	100	0.333333
20	0.1	6	100	260	100	110	1.305
20	0.1	6	100	260	100	120	2.026667
20	0.1	6	100	260	100	130	2.508333
20	0.1	6	100	260	100	140	2.76
20	0.1	6	100	260	100	150	2.791667
20	0.1	6	100	260	100	160	2.613333
20	0.1	6	100	260	100	170	2.236667
20	0.1	6	100	260	100	180	1.68
20	0.1	6	100	260	100	190	0.963333
20	0.1	6	100	260	100	200	0.106667
20	0.1	6	100	260	100	210	-0.79167
20	0.1	6	100	260	100	220	-1.64
20	0.1	6	100	260	100	230	-2.42833
20	0.1	6	100	260	100	240	-3.14667
20	0.1	6	100	260	100	250	-3.785
20	0.1	6	100	260	100	260	-4.33333
20	0.1	6	100	260	100	270	-4.785
20	0.1	6	100	260	100	280	-5.14667
20	0.1	6	100	260	100	290	-5.42833
20	0.1	6	100	260	100	300	-5.64

TABLE 7 (continued). Expected net profit calculations for symmetric triangular distribution with $2k > U - L$ with a calculated value of U.

20	0.1	6	100	260	100	310	-5.79167
20	0.1	6	100	260	100	320	-5.89333
20	0.1	6	100	260	100	330	-5.955
20	0.1	6	100	260	100	340	-5.98667
20	0.1	6	100	260	100	350	-5.99833
20	0.1	6	100	260	100	360	-6
20	0.1	6	100	260	100	370	-6
20	0.1	6	100	260	100	380	-6
20	0.1	6	100	260	100	390	-6
20	0.1	6	100	260	100	400	-6

And, graphically, the expected net profit for various values of *m* (when an optimum value of U, U_o, is determined) is shown in Figure 25.

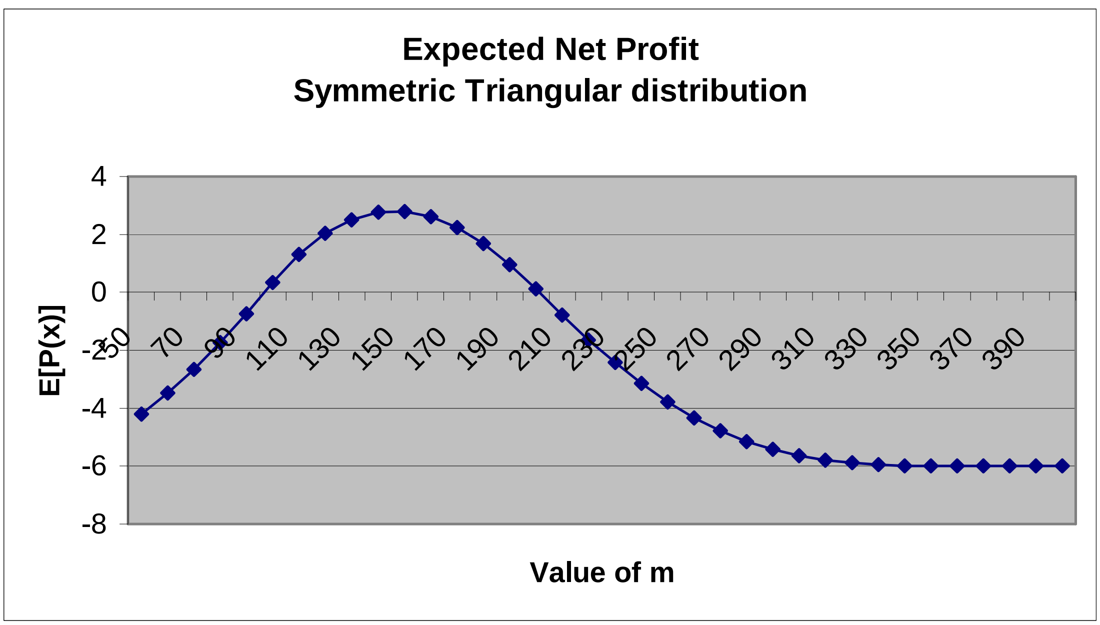


FIGURE 25. Expected net profit model for symmetric triangular distribution with $2k > U - L$ and a calculated value for U.

The optimum set point for *m* is m_o =146.86. At that point, the expected net profit is $2.80 which is higher than it was in the previous example when *U* was set lower (at 200) than U_o.

In general, the value of *U* should be set as the minimum of an upper specification limit or the optimum value of *U*, U_o. $U = \min(USL, U_o)$

5.2 Skewed Triangular underlying distribution

In the case of a skewed triangular distribution, the lower limit of the distribution, *a*, is set equal to $m-k_1$ and the upper limit of the distribution, *b*, is set equal to $m+k_2$ such that the probability density function is:

$$f(x)=\begin{cases}\dfrac{2\left(x-m+k_1\right)}{k_1\left(k_1+k_2\right)}, & m-k_1\le x\le m\\[2ex] \dfrac{2\left(m+k_2-x\right)}{k_2\left(k_1+k_2\right)}, & m\le x\le m+k_2\\[2ex] 0, & otherwise\end{cases}$$

The expected net profit is given by:

$$E\left[P(x)\right] = \int_{m-k_1}^{L} -R_L\left[\frac{2(x-m+k_1)}{k_1(k_1+k_2)}\right]dx + \int_{L}^{m}(A-Cx)\left[\frac{2(x-m+k_1)}{k_1(k_1+k_2)}\right]dx$$
$$+\int_{m}^{U}(A-Cx)\left[\frac{2(m+k_2-x)}{k_2(k_1+k_2)}\right]dx + \int_{U}^{m+k_2} -R_U\left[\frac{2(m+k_2-x)}{k_2(k_1+k_2)}\right]dx =$$

$$\frac{2}{k_1(k_1+k_2)}\left\{-R_L\left[\frac{x^2}{2}-(m-k_1)x\right]_{m-k_1}^{L} + A\left[\frac{x^2}{2}-(m-k_1)x\right]_{L}^{m} - C\left[\frac{x^3}{3}-\frac{(m-k_1)x^2}{2}\right]_{L}^{m}\right\} + \quad =$$
$$\frac{2}{k_2(k_1+k_2)}\left\{A\left[(m+k_2)x-\frac{x^2}{2}\right]_{m}^{U} - C\left[\frac{(m+k_2)x^2}{2}-\frac{x^3}{3}\right]_{m}^{U} - R_U\left[(m+k_2)x-\frac{x^2}{2}\right]_{U}^{m+k_2}\right\}$$

$$\frac{2}{k_1(k_1+k_2)}\left\{\begin{array}{l} -\frac{R_L}{2}\left[L-(m-k_1)\right]^2 - A\left[\frac{(m-L)^2}{2}-k_1(m-L)\right] - \\ C\left[\frac{m^3-L^3}{3}+\frac{(L^2-m^2)(m-k_1)}{2}\right] \end{array}\right\} + \tag{32}$$
$$\frac{2}{k_2(k_1+k_2)}\left\{\begin{array}{l} -A\left[\frac{(m-U)^2}{2}+k_2(m-U)\right] - C\left[\frac{m^3-U^3}{3}+\frac{(U^2-m^2)(m+k_2)}{2}\right] \\ -\frac{R_U}{2}\left[U-(m+k_2)\right]^2 \end{array}\right\}$$

Note that when the distribution is symmetrical, $k = k_1 = k_2$ and Equation (32) reduces to Equation (25). Further, the $E\left[P(x)\right]$ is inversely proportional to the spread of the distribution, $k_1 + k_2$.

5.2.1 Optimum upper screening limit

In order to determine the optimum value of *U*, the first derivative of the Equation (32) with respect to *U* is calculated below:

$$\frac{\partial}{\partial U}E\left[P(x)\right]=\frac{2}{k_2\left(k_1+k_2\right)}\left\{A\left[(m-U)+k_2\right]+C\left[U^2-U\left(m+k_2\right)\right]-R_U\left[U-\left(m+k_2\right)\right]\right\} \tag{33}$$

The second derivative with respect to *U* is less than zero (hence expected net profit is at a maximum) only if

$$U<\frac{A+R_U+C\left(m+k_2\right)}{2C}=\frac{A+R_U}{2C}+\frac{\left(m+k_2\right)}{2} \tag{34}$$

Setting Equation (33) equal to zero to solve for the value of *U*, U_o, that maximizes expected net profit results in the following:

$$\frac{2}{k_2\left(k_1+k_2\right)}\left\{CU^2-\left[A+R_U+C\left(m+k_2\right)\right]U+\left(A+R_U\right)\left(m+k_2\right)\right\}=0$$

$$CU^2-\left[A+R_U+C\left(m+k_2\right)\right]U+\left(A+R_U\right)\left(m+k_2\right)=0$$

Let $c_5=\left[A+R_U+C\left(m+k_2\right)\right]$ and $c_6=\left(A+R_U\right)\left(m+k_2\right)$, then $CU_o^{\,2}-c_5U_o+c_6=0$, which means that

$$U_o=\frac{c_5\pm\sqrt{c_5^{\,2}-4Cc_6}}{2C}$$

$$U_o=\frac{\left[A+R_U+C\left(m+k_2\right)\right]\pm\sqrt{\left[A+R_U+C\left(m+k_2\right)\right]^2-4C\left[\left(A+R_U\right)\left(m+k_2\right)\right]}}{2C}=$$

$$\frac{\left[A+R_U+C\left(m+k_2\right)\right]\pm\sqrt{\left(A+R_U\right)^2+2C\left(A+R_U\right)\left(m+k_2\right)+\left[C\left(m+k_2\right)\right]^2-4C\left(A+R_U\right)\left(m+k_2\right)}}{2C}=$$

$$\frac{\left[A+R_U+C\left(m+k_2\right)\right]\pm\sqrt{\left(A+R_U\right)^2-2C\left(A+R_U\right)\left(m+k_2\right)+C^2\left(m+k_2\right)^2}}{2C}=$$

$$\frac{\left[A+R_U+C\left(m+k_2\right)\right]\pm\sqrt{\left[\left(A+R_U\right)-C\left(m+k_2\right)\right]^2}}{2C}=$$

$$\frac{\left[A+R_U+C\left(m+k_2\right)\right]\pm\left[\left(A+R_U\right)-C\left(m+k_2\right)\right]}{2C}=$$

Case 1:

$$U_1=\frac{\left[A+R_U+C\left(m+k_2\right)\right]+\left[\left(A+R_U\right)-C\left(m+k_2\right)\right]}{2C}=\frac{2\left(A+R_U\right)}{2C}=\frac{A+R_U}{C}$$

Which is consistent with previous results.

Case 2:

$$U_2=\frac{\left[A+R_U+C\left(m+k_2\right)\right]-\left[\left(A+R_U\right)-C\left(m+k_2\right)\right]}{2C}=\frac{2C\left(m+k_2\right)}{2C}=m+k_2$$

However, Equation (34) requires that setting Equation (33) equal to zero will lead to an optimum value of U iff $U<\frac{A+R_U}{2C}+\frac{\left(m+k_2\right)}{2}$, i.e. $U_o=\frac{A+R_U}{C}$ iff $\frac{A+R_U}{C}<\frac{A+R_U}{2C}+\frac{\left(m+k_2\right)}{2}\rightarrow\frac{A+R_U}{C}<m+k_2$. Therefore $m+k_2$ cannot be the optimum value for U, and thus

$$U_o = \frac{A+R_U}{C} \tag{35}$$

5.2.2 Optimum target set point for *m*

Similarly, the optimum value of *m*, m_o, can be found by setting the first derivative of Equation (32) with respect to *m* equal to zero:

$$\frac{\partial}{\partial m}\left[\begin{array}{l} \frac{2}{k_1(k_1+k_2)}\left\{\begin{array}{l} -\frac{R_L}{2}[L-(m-k_1)]^2 - A\left[\frac{(m-L)^2}{2}+k_1(L-m)\right] \\ -C\left[\frac{m^3-L^3}{3}+\frac{(L^2-m^2)(m-k_1)}{2}\right] \end{array}\right\}+ \\ \frac{2}{k_2(k_1+k_2)}\left\{\begin{array}{l} -A\left[\frac{(m-U_o)^2}{2}+k_2(m-U_o)\right]-C\left[\frac{m^3-U_o{}^3}{3}+\frac{(U_o{}^2-m^2)(m+k_2)}{2}\right] \\ -\frac{R_U}{2}\left[U_o-(m+k_2)\right]^2 \end{array}\right\} \end{array}\right]$$

$$\left[\begin{array}{l} \frac{2}{k_1(k_1+k_2)}\left\{R_L[L-(m-k_1)]-A\left[(m-k_1)-L\right]-C\left[\frac{L^2-m^2}{2}+mk_1\right]\right\}+ \\ \frac{2}{k_2(k_1+k_2)}\left\{-A\left[(m+k_2)-U_o\right]-C\left[\frac{U_o{}^2-m^2}{2}-mk_2\right]+R_U\left[U_o-(m+k_2)\right]\right\} \end{array}\right] \tag{36}$$

Note that in the case of the skewed triangular distribution, *m* is not equal to $E(x)=\mu$ (See Appendix D for the first four moments.)

The second derivative with respect to *m* is negative if

$$\frac{2}{(k_1+k_2)}\left\{\left[\frac{-(R_L+A)+C(m-k_1)}{k_1}\right]+\left[\frac{-(A+R_U)+C(m+k_2)}{k_2}\right]\right\}<0$$

$$m < \frac{(R_L + A)k_2 + (A + R_U)k_1}{C(k_1 + k_2)} \tag{37}$$

(Note that when $k = k_1 = k_2$ and $R_L = R_U = R$, Equation (37) simplifies to

$m < \dfrac{R + A}{C} = U_o$ as before.)

Setting Equation (36) equal to zero and solving for m_o results in

$$\frac{1}{k_1}\left\{R_L\left[L-(m-k_1)\right]-A\left[(m-k_1)-L\right]-C\left[\frac{L^2-m^2}{2}+mk_1\right]\right\}+$$
$$\frac{1}{k_2}\left\{-A\left[(m+k_2)-U_o\right]-C\left[\frac{U_o{}^2-m^2}{2}-mk_2\right]+R_U\left[U_o-(m+k_2)\right]\right\}=0$$

$$\frac{C}{2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)m^2+\left[\frac{1}{k_1}(-R_L-A-Ck_1)+\frac{1}{k_2}(-A+Ck_2-R_U)\right]m+$$
$$\frac{1}{k_1}\left[R_L(L+k_1)+A(L+k_1)-\frac{CL^2}{2}\right]+\frac{1}{k_2}\left[A(U_o-k_2)-\frac{CU_o{}^2}{2}+R_U(U_o-k_2)\right]=0$$

$$\frac{C}{2}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)m^2-\left[\frac{(R_L+A)}{k_1}+\frac{(R_U+A)}{k_2}\right]m+\frac{1}{k_1}\left[(R_L+A)(L+k_1)-\frac{CL^2}{2}\right]$$
$$+\frac{1}{k_2}\left[(R_U+A)(U_o-k_2)-\frac{CU_o{}^2}{2}\right]=0$$

$$\frac{C(k_1+k_2)}{2k_1k_2}m^2-\left[\frac{k_2(R_L+A)+k_1(R_U+A)}{k_1k_2}\right]m+\frac{1}{k_1k_2}\left\{\begin{array}{l}k_2\left[(R_L+A)(L+k_1)-\dfrac{CL^2}{2}\right]\\+k_1\left[(R_U+A)(U_o-k_2)-\dfrac{CU_o{}^2}{2}\right]\end{array}\right\}=0$$

$$\frac{C(k_1+k_2)}{2}m^2-\left[k_2(R_L+A)+k_1(R_U+A)\right]m+k_2\left[(R_L+A)(L+k_1)-\frac{CL^2}{2}\right]$$
$$+k_1\left[(R_U+A)(U_o-k_2)-\frac{CU_o^{\,2}}{2}\right]=0$$

Let $c_7=\dfrac{C(k_1+k_2)}{2}$, $c_8=k_2(R_L+A)+k_1(R_U+A)$, and

$c_9=k_2\left[(R_L+A)(L+k_1)-\dfrac{CL^2}{2}\right]+k_1\left[(R_U+A)(U_o-k_2)-\dfrac{CU_o^{\,2}}{2}\right]$, then $c_7m_o^{\,2}-c_8m_o+c_9=0$,

which means that

$$m_o=\frac{c_8\pm\sqrt{c_8^{\,2}-4c_7c_9}}{2c_7}$$

$$m_o=\frac{k_2(R_L+A)+k_1(R_U+A)}{C(k_1+k_2)}\pm$$
$$\frac{\sqrt{\left[k_2(R_L+A)+k_1(R_U+A)\right]^2-2C(k_1+k_2)\begin{Bmatrix}k_2\left[(R_L+A)(L+k_1)-\dfrac{CL^2}{2}\right]+\\ k_1\left[(R_U+A)(U_o-k_2)-\dfrac{CU_o^{\,2}}{2}\right]\end{Bmatrix}}}{C(k_1+k_2)}$$

Given the requirement from Equation (37) that $m_o<\dfrac{(R_L+A)k_2+(A+R_U)k_1}{C(k_1+k_2)}$, it follows

that

$$m_o = \frac{k_2\left(R_L + A\right) + k_1\left(R_U + A\right)}{C\left(k_1 + k_2\right)} - \frac{\sqrt{\begin{array}{l}\left[k_2\left(R_L + A\right) + k_1\left(R_U + A\right)\right]^2 - \\ 2C\left(k_1 + k_2\right)\left\{k_2\left[\left(R_L + A\right)\left(L + k_1\right) - \frac{CL^2}{2}\right] + k_1\left[\left(R_U + A\right)\left(U_o - k_2\right) - \frac{CU_o^{\ 2}}{2}\right]\right\}\end{array}}}{C\left(k_1 + k_2\right)} \tag{38}$$

Which reduces to Equation (31) when $k_1 = k_2$ and $R_L = R_U$.

Example 13.

For example, when *A*=20, *C*=.1, $R_L = 5$, $R_U = 6$, $k_1 = 20$, $k_2 = 180$, and *L*=100, U_o=260 and $m_o = 107.47$ from Equation (38).

It can be shown (in the Appendix), that the mean of a skewed triangular distribution is $\mu = E\left[X\right] = m + \frac{k_2 - k_1}{3}$, so the optimum value of the mean is given below:

$$\mu_o = m_o + \frac{k_2 - k_1}{3} \tag{39}$$

5.2.3 Sensitivity analysis of skewness

Note that the triangular distribution is negatively skewed when $k_1 > k_2$ and positively skewed when $k_2 > k_1$. Variations of Example 13 are presented below to illustrate the impact of skewness on the calculation of m_o.

Example 14.

When A=20, C=0.10, $R_L = 5$, $R_U = 6$, and L=100, U_o =260, values of m_o are shown in Table 8 for various levels of skewness and the expected net profit is calculated by substituting m_o for the value of m in Equation (32):

TABLE 8. m_o and the expected net profit for various levels of skewness

k_1	k_2	m_o	$E\left[P(x)\right]$
20	180	107.47	3.334
80	120	136.06	2.953
100	100	144.43	2.948
120	80	156.32	2.781
180	20	208.00	1.755

As illustrated in this example, as the distribution becomes negatively skewed, the optimum set point must be higher. In the canning problem, where only a lower specification is given, a positively skewed distribution allows for a lower optimum set point and a higher expected net profit.

5.3 Process capability with underlying Triangular distribution

As presented in Chapter 4, a process is generally deemed capable if $C_P \geq 1.0$ which means, if the process is centered inside the specifications, then the middle 99.73% of the distribution will fall inside the specification limits. For a normal distribution, this translates to $C_P = BT / NT = BT / 6\sigma$. However, for a Triangular distribution, the limits defined by $\mu \pm 3\sigma$ do not define the middle 99.73% of the distribution. In fact, the limits defined by $\mu \pm 3\sigma$ actually fall outside the complete range of the distribution defined by $\left(m-k_1, m+k_2\right)$:

$$\mu \pm 3\sigma = m + \frac{k_2 - k_1}{3} \pm \sqrt{\frac{k_1{}^2 + k_1 k_2 + k_2{}^2}{2}}$$

Thus, the 6-sigma spread for a triangular distribution is given by $2\sqrt{\frac{k_1{}^2 + k_1 k_2 + k_2{}^2}{2}}$. As a result, as is the case with the Uniform distribution, the 6-sigma spread $\sqrt{2\left(k_1{}^2 + k_1 k_2 + k_2{}^2\right)} = \sqrt{\left(k_1 + k_2\right)^2 + \left(k_1{}^2 + k_2{}^2\right)} > k_1 + k_2$ so that the 99.73% natural tolerances of the Triangular distribution must be obtained from $x_{.99865}$ - $x_{.00135}$.

It can be shown that the cdf of any Triangular distribution is given by

$$F\left(x\right) = \begin{cases} 0, & x \le m - k_1 \\ \dfrac{\left(x - m + k_1\right)^2}{k_1\left(k_1 + k_2\right)}, & m - k_1 \le x \le m \\ 1 - \dfrac{\left(m + k_2 - x\right)^2}{\left(k_1 + k_2\right)k_2}, & m \le x \le m + k_2 \\ 1, & x \ge m + k_2 \end{cases}$$

Let x_p be the pth quantile (or fractile) of *x*, then upon inverting the above cdf, the following percentile function is obtained:

$$x_p = \begin{cases} \left(m - k_1\right) + \sqrt{\left(k_1 + k_2\right)k_1 p}, & 0 \le p \le \dfrac{k_1}{k_1 + k_2} \\ \left(m + k_2\right) - \sqrt{k_2\left(k_1 + k_2\right)\left(1 - p\right)}, & \dfrac{k_1}{k_1 + k_2} \le p \le 1 \end{cases}$$

The C_P index by definition is $C_P = \frac{USL - LSL}{x_{.99865} - x_{.00135}}$, where the middle 99.73% of the spread of the distribution is given by

$$x_{.99865} - x_{.00135} = \sqrt{k_2 + k_1}\left[\sqrt{k_2 + k_1} - 0.036741\left(\sqrt{k_2} + \sqrt{k_1}\right)\right]$$

6.0 Conclusion

The work presented in this dissertation extends the previous research on the canning problem (which focused on infinite range distributions, specifically the normal distribution, for fill level) to finite distributions. Three finite distributions were analyzed: Uniform, Symmetric Triangular, and Skewed Triangular. In each case, an optimum set point for the mean fill level was determined to maximize expected net profit. When appropriate, an upper screening limit for fill level was also determined.

In the case of the Uniform distribution, three net profit models were studied: fixed rework/reprocessing costs, linear rework/reprocessing costs, and capacity constrained. Few closed form solutions were obtained by differentiation for determining an optimum set point for the mean to maximize the expected net profit. However, the optimum set point was determined to maximize expected net profit by evaluating expected net profits at the extreme points for each range of μ.

6.1 Summary of Results

For fill level that follows a Uniform distribution when there is a constant scrap cost, the optimum value for the upper screening limit was determined to be

$$U_o = \min\left(U_L, U_U\right) = \min\left(\frac{R_L + A}{C}, \frac{R_U + A}{C}\right)$$

The optimum target set point for the process mean was obtained for the various scenarios:

Case 1: $2k \le U_o - L \rightarrow \mu_o = L + k$

Case 2: $2k > U_o - L$:

If $R_U < R_L \rightarrow \mu_o = L + k$

If $R_U > R_L \rightarrow \mu_o = U_o - k$

If $R_U = R_L \rightarrow \mu_o \in [U_o - k, L + k]$

For fill level that follows a Uniform distribution when there is a linear scrap cost, the optimum target set point for the process mean was obtained for the various scenarios:

Case 1: $2k \le U - L$

$$\mu_o = L + k$$

Case 2: $2k > U - L$

$$\mu_o = L + k \text{ if } R_U\left[\frac{(U^2 - L^2)}{4k} - (L + k)\right] > R_L\left[\frac{(U^2 - L^2)}{4k} - (U - k)\right]$$

$$\mu_o = U - k \text{ if } R_U\left[\frac{(U^2 - L^2)}{4k} - (L + k)\right] \le R_L\left[\frac{(U^2 - L^2)}{4k} - (U - k)\right]$$

The target set point for the process mean, μ_o, of a profit model with a capacity constraint was defined for the various scenarios:

Case 1: $2k < \min(CAP - L, U_o - L) \rightarrow \mu_o = L + k$

Case 2: $2k \ge \min(CAP - L, U_o - L)$

$R_U < R_L \rightarrow \mu_o = \min(L + k, CAP - k)$

$R_U > R_L \rightarrow \mu_o = \min(U_o - k, CAP - k)$

$$R_U = R_L \rightarrow \mu_o \in \left[U_o - k, \min\left(L + k, CAP - k\right)\right]$$

For both the symmetric and skewed Triangular distributions, a net profit model with fixed rework/reprocessing costs was assumed. Assuming that $L \le m \le U$, $m - k_1 < L$, and $m + k_2 > U$, the optimum set point for m was determined to be:

$$m_o = \frac{k_2\left(R_L + A\right) + k_1\left(R_U + A\right)}{C\left(k_1 + k_2\right)} - \frac{\sqrt{\left[k_2\left(R_L + A\right) + k_1\left(R_U + A\right)\right]^2 - 2C\left(k_1 + k_2\right)\left\{k_2\left[\left(R_L + A\right)\left(L + k_1\right) - \frac{CL^2}{2}\right] + k_1\left[\left(R_U + A\right)\left(U_o - k_2\right) - \frac{CU_o^{\,2}}{2}\right]\right\}}}{C\left(k_1 + k_2\right)}$$

which reduces to $m_o = \frac{R + A}{C} - \sqrt{\left(\frac{R + A}{C}\right)^2 - \frac{\left(R + A\right)\left(L + U_o\right)}{C} + \left(\frac{L^2 + U_o^{\,2}}{2}\right)}$ when $k_1 = k_2$ and $R_L = R_U$.

Throughout the research, examples were provided and proofs, where necessary, were outlined.

6.2 Practical Applications

This research shows that when fill level is not normally distributed, the optimum set points for a canning problem can still be determined if the distribution can be modeled, even if the distribution range is not infinite. In at least one case (steel thickness), the Uniform distribution has semmed to be an appropriate fit. However, in most cases, practical application of this work may be somewhat limited, since fill level is best estimated by an infinite range distribution that is bounded rather than an actual finite distribution.

6.3 Recommendations for Future Work

Given the similarities between the Uniform distribution, symmetric Triangular distribution, and the Normal distribution (symmetric, continuous within the range, and mean centered,) it may be interesting to compare the results presented here with results when a Normal distribution is incorrectly assumed for fill level which is best modeled by a Uniform distribution.. Fill level data generated from a Uniform distribution could be used to calculate the optimum upper screening limit and the optimum mean using the formulas presented here. Those results could then be compared with results from the formulas from research using the same net profit model, but assuming a Normal distribution of fill level. It would be interesting to see how the results differ in terms of magnitude of μ_o and in terms of expected net profit.

Another extension of this work would be to extend the analysis of the Triangular distribution to the other cases presented in Figures 21 and 23. and to more thoroughly complete a sensitivity analysis of skewness using the formula for skewness in Appendix D.

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APPENDICES

Appendix A: Formulas for Excel spreadsheet for calculating expected net profit for various levels of μ with a Uniform distribution and fixed scrap cost.

H2=IF(2*E2<=F2-G2,G2+E2,IF(C2>D2,G2+E2,F2-E2))

I2==IF(H2<=G2-E2,-C2,IF(AND(H2<G2+E2,H2>=G2-E2),L2,IF(AND(H2<F2-E2,H2>=G2+E2),M2,IF(AND(H2>=F2-E2,H2<F2+E2),N2,-D2))))

J2==IF(H2<=G2-E2,-C2,IF(AND(H2<F2-E2,H2>=G2-E2),L2,IF(AND(H2<G2+E2,H2>=F2-E2),O2,IF(AND(H2>=G2+E2,H2<F2+E2),N2,-D2))))

K2==IF(2*E2<=F2-G2,I2,J2)

L2=(1/(2*E2))*(C2*(H2-E2)+A2*(H2+E2)-(C2+A2)*G2-(B2/2)*((H2+E2)^2-G2^2))

M2=A2-B2*H2

N2==(1/(2*E2))*((A2+D2)*(F2-H2)+E2*(A2-D2)-(B2/2)*(F2^2-(H2-E2)^2))

O2 =(1/(2*E2))*(C2*(H2-E2-G2)+A2*(F2-G2)-(B2/2)*(F2^2-G2^2)-D2*(H2+E2-F2))

A	B	C	D	E	F	G	H
A	C	RL	RU	k	U	L	MU
40	0.1	5	6	50	450	100	100

I	J	K	L	M	N	O
<U-L Case	>U-L Case	E[P(x)]				
11.25	11.25	11.25	11.25	30	78	59.25

Appendix B: Formulas for Excel spreadsheet for calculating expected net profit for various levels of μ with a Uniform distribution and linear scrap cost.

H2 =IF(2*E2<=F2-G2,G2+E2,IF(D2*((F2^2-G2^2)/(4*E2)-(G2+E2))>C2*((F2^2-G2^2)/(4*E2)-(F2-E2)),G2+E2,F2-E2))

I2 =IF(H2<=G2-E2,-C2*H2,IF(AND(H2<G2+E2,H2>=G2-E2),L2,IF(AND(H2<F2-E2,H2>=G2+E2),M2,IF(AND(H2>=F2-E2,H2<F2+E2),N2,-D2*H2))))

J2 =IF(H2<=G2-E2,-C2*H2,IF(AND(H2<F2-E2,H2>=G2-E2),L2,IF(AND(H2<G2+E2,H2>=F2-E2),O2,IF(AND(H2>=G2+E2,H2<F2+E2),N2,-D2*H2))))

K2 =IF(2*E2<=F2-G2,I2,J2)

L2 =(1/(2*E2))*(-C2/2*(G2^2-(H2-E2)^2)+A2*(H2+E2-G2)-(B2/2)*((H2+E2)^2-G2^2))

M2 =A2-B2*H2

N2 =(1/(2*E2))*((A2*(F2-(H2-E2))-(B2/2)*(F2^2-(H2-E2)^2)-(D2/2)*((H2+E2)^2-F2^2)))

O2 =(1/(2*E2))*((C2/2)*((H2-E2)^2-G2^2)+A2*(F2-G2)-(B2/2)*(F2^2-G2^2)+(D2/2)*(F2^2-(H2+E2)^2))

A	B	C	D	E	F	G	H
A	C	RL	RU	k	U	L	MU
40	0.1	0.2	0.05	50	800	200	250

I	J	K	L	M	N	O
<U-L Case	>U-L Case	E[P(x)]				
15	15	15	15	15	77.5	77.5

Appendix C: Formulas for Excel spreadsheet for calculating expected net profit for various levels of m with a symmetric Triangular distribution.

```
I2=IF(H2<=G2-E2,-C2,IF(AND(H2<=G2+E2,H2<=F2-E2,H2>G2-
E2),L2,IF(AND(H2<=G2+E2,H2>F2-E2),M2,IF(AND(H2>F2-
E2,H2>G2+E2,H2<F2+E2),O2,-C2))))
```

```
L2= =IF(H2>G2,(1/(E2^2))*(-(C2+A2)/2*(G2-H2+E2)^2-B2*(-
H2^3/3+(H2+E2)^3/6+G2^2/2*(H2-E2)-G2^3/3))+A2,(1/(E2^2))*((A2+C2)/2*(H2+E2-
G2)^2-B2*((H2+E2)^3/6-G2^2/2*(H2+E2)+G2^3/3))-C2)
```

```
M2= =(1/E2^2)*(-C2/2*(G2-H2+E2)^2-A2*(H2^2-H2*(G2+F2)-E2*(F2-
G2)+0.5*(F2^2+G2^2))-B2*(-H2^3/3+H2/2*(G2^2+F2^2)+E2/2*(F2^2-G2^2)-
(F2^3/3+G2^3/3))-C2/2*(H2+E2-F2)^2)
```

```
N2= =A2-B2/(E2^2)*(-H2^3/3+(H2-E2)^3/6+(H2+E2)^3/6)
```

```
O2= =IF(H2<F2,A2+(1/E2^2)*(-(A2+C2)/2*(H2+E2-F2)^2-B2*(-H2^3/3+(H2-
E2)^3/6+F2^2*(H2+E2)/2-F2^3/3)),(1/E2^2)*((A2+C2)/2*(F2-H2+E2)^2-B2*(F2^3/3-
(H2-E2)*F2^2/2+(H2-E2)^3/6))-C2)
```

A	B	C	D	E	F	G	H
A	C	R		k	U	L	m
20	0.1	6		100	260	100	146.8629

I	J	K	L	M	N	O
E[P(x)]						
2.80481			2.804809982	2.801031	5.313708	5.30993

Appendix D: Derivation of the first four moments of the skewed Triangular distribution

First Moment:

$$\mu_1 = E[X] = \frac{2}{k_1+k_2}\left\{\frac{1}{k_1}\int_{m-k_1}^{m}\left[x^2-(m-k_1)x\right]dx+\frac{1}{k_2}\int_{m}^{m+k_2}\left[(m+k_2)x-x^2\right]dx\right\}$$

$$=\frac{2}{k_1+k_2}\left\{\begin{array}{l}\frac{1}{k_1}\left[\frac{m^3-m^3-3k_1{}^2m+3k_1m^2+k_1{}^3}{3}+\frac{(-m^3)+m^2k_1+m^3+3k_1{}^2m-3k_1m^2-k_1{}^3}{2}\right]+\\ \frac{1}{k_2}\left[\frac{m^3+3k_2m^2+3k_2{}^2m+k_2{}^3-m^3-k_2m^2}{2}-\frac{m^3+3k_2m^2+3k_2{}^2m+k_2{}^3-m^3}{3}\right]\end{array}\right\}$$

$$=\frac{2}{k_1+k_2}\left\{\frac{1}{k_1}\left[\frac{k_1{}^2m}{2}-\frac{k_1{}^3}{6}\right]+\frac{1}{k_2}\left[\frac{k_2{}^2m}{2}+\frac{k_2{}^3}{6}\right]\right\}$$

$$=\frac{1}{k_1+k_2}\left[(k_1+k_2)m-\frac{(k_1{}^2-k_2{}^2)}{3}\right]$$

$$E[X]=m+\frac{k_2-k_1}{3}$$

Second Moment:

$$\mu_2 = E\left[\left(X-\mu\right)^2\right];\ \ \mu = m + \frac{k_2 - k_1}{3}$$

$$V\left(X\right) = \mu_2 = \begin{cases} \dfrac{2}{k_1\left(k_1+k_2\right)}\displaystyle\int_{m-k_1}^{m}\left(x-m-\frac{k_2-k_1}{3}\right)^2\left(x-m+k_1\right)dx + \\ \dfrac{2}{k_2\left(k_1+k_2\right)}\displaystyle\int_{m}^{m+k_2}\left(x-m-\frac{k_2-k_1}{3}\right)^2\left(m+k_2-x\right)dx \end{cases} \tag{40}$$

In the first integral on the right hand side of Equation (40), let $y = x - m + k_1$ and in the second integral, let $y = m + k_2 - x$, $-y = x - m - k_2$ so that $V\left(x\right)$ in Equation (40) beomes

$$V\left[X\right] = \frac{2}{k_1\left(k_1+k_2\right)}\int_0^{k_1}\left(y-k_1-\frac{k_2-k_1}{3}\right)^2 ydy + \frac{2}{k_2\left(k_1+k_2\right)}\int_{k_2}^{0}\left(k_2-y-\frac{k_2-k_1}{3}\right)^2 y\left(-dy\right)$$

$$= \frac{2}{k_1\left(k_1+k_2\right)}\int_0^{k_1}\left(y-\frac{2k_1}{3}-\frac{k_2}{3}\right)^2 ydy + \frac{2}{k_2\left(k_1+k_2\right)}\int_0^{k_2}\left(\frac{2k_2}{3}-y+\frac{k_1}{3}\right)^2 ydy$$

$$= \begin{cases} \dfrac{2}{k_1\left(k_1+k_2\right)}\left[\dfrac{y^4}{4}+\dfrac{4k_1^{\,2}}{9}\dfrac{y^2}{2}+\dfrac{k_2^{\,2}}{9}\dfrac{y^2}{2}-\dfrac{4k_1}{3}\dfrac{y^3}{3}-\dfrac{2k_2}{3}\dfrac{y^3}{3}+\dfrac{4k_1k_2}{9}\dfrac{y^2}{2}\right]_0^{k_1} + \\ \dfrac{2}{k_2\left(k_1+k_2\right)}\left[\dfrac{4k_2^{\,2}}{9}\dfrac{y^2}{2}+\dfrac{y^4}{4}+\dfrac{k_1^{\,2}}{9}\dfrac{y^2}{2}-\dfrac{4k_2}{3}\dfrac{y^3}{3}+\dfrac{4k_1k_2}{9}\dfrac{y^2}{2}-\dfrac{2k_1}{3}\dfrac{y^3}{3}\right]_0^{k_2} \end{cases}$$

$$=\begin{cases}\dfrac{2}{k_1+k_2}\left[\dfrac{k_1^{\,3}}{4}+\dfrac{2k_1^{\,3}}{9}+\dfrac{k_2^{\,2}}{18}k_1-\dfrac{4k_1^{\,3}}{9}-\dfrac{2k_2}{9}k_1^{\,2}+\dfrac{2k_2}{9}k_1^{\,2}\right]+\\ \dfrac{2}{k_1+k_2}\left[\dfrac{2k_2^{\,2}}{9}k_2+\dfrac{k_2^{\,3}}{4}+\dfrac{k_1^{\,2}}{18}k_2-\dfrac{4}{9}k_2^{\,3}+\dfrac{2k_1}{9}k_2^{\,2}-\dfrac{2k_1}{9}k_2^{\,2}\right]\end{cases}$$

$$=\frac{2}{k_1+k_2}\left[\frac{k_1^{\,3}}{36}+\frac{k_1k_2^{\,2}}{18}+\frac{k_2^{\,3}}{36}+\frac{k_2k_1^{\,2}}{18}\right]=\frac{1}{18\left(k_1+k_2\right)}\left[k_1^{\,3}+k_2^{\,3}+2k_1k_2^{\,2}+2k_1^{\,2}k_2\right]$$

$$=\frac{\left(k_1+k_2\right)\left(k_1^{\,2}+k_2^{\,2}+k_1k_2\right)}{18\left(k_1+k_2\right)}$$

$$=\frac{k_1^{\,2}+k_2^{\,2}+k_1k_2}{18}$$

Third Moment:

Skewness: $\alpha_3 = \sqrt{\frac{{\mu_3}^2}{{\mu_2}^3}} = \frac{\mu_3}{\sigma^3}$

$$\mu_3 = \begin{cases} \frac{2}{k_1\left(k_1+k_2\right)}\int_{m-k_1}^{m}\left(x-m-\frac{k_2-k_1}{3}\right)^3\left(x-m+k_1\right)dx+ \\ \frac{2}{k_2\left(k_1+k_2\right)}\int_{m}^{m+k_2}\left(x-m-\frac{k_2-k_1}{3}\right)^3\left(m+k_2-x\right)dx \end{cases}$$

In the first integral on the right hand side, set $y = x-m+k_1$ and in the second integral, set $y = m+k_2-x$, so $-y = x-m-k_2 \rightarrow -dy = dx \rightarrow dx = -dy$

$$\mu_3 = \frac{2}{k_1\left(k_1+k_2\right)}\int_0^{k_1}\left(y-\frac{2k_1}{3}-\frac{k_2}{3}\right)^3 ydy + \frac{2}{k_2\left(k_1+k_2\right)}\int_{k_2}^{0}\left(\frac{2k_2}{3}-y+\frac{k_1}{3}\right)^3\left(-ydy\right)$$

Let $c_{10} = \frac{1}{3}\left(2k_1+k_2\right)$ and $c_{11} = \frac{1}{3}\left(2k_2+k_1\right)$, then

$$\mu_3 = \frac{2}{k_1\left(k_1+k_2\right)}\int_0^{k_1}\left(y-c_{10}\right)^3 ydy + \frac{2}{k_2\left(k_1+k_2\right)}\int_0^{k_2}\left(c_{11}-y\right)^3 ydy$$

$$= \begin{bmatrix} \frac{2}{k_1\left(k_1+k_2\right)}\left(\frac{{k_1}^5}{5}-3c_{10}\frac{{k_1}^4}{4}+3{c_{10}}^2\frac{{k_1}^3}{3}-{c_{10}}^3\frac{{k_1}^2}{2}\right)+ \\ \frac{2}{k_2\left(k_1+k_2\right)}\left({c_{11}}^3\frac{{k_2}^2}{2}-3{c_{11}}^2\frac{{k_2}^3}{3}+3c_{11}\frac{{k_2}^4}{4}-\frac{{k_2}^5}{5}\right) \end{bmatrix}$$

$$=\frac{2}{k_1+k_2}\left(\frac{k_1{}^4}{5}-3c_{10}\frac{k_1{}^3}{4}+c_{10}{}^2k_1{}^2-c_{10}{}^3\frac{k_1}{2}+c_{11}{}^3\frac{k_2}{2}-c_{11}{}^2k_2{}^2+3c_{11}\frac{k_2{}^3}{4}-\frac{k_2{}^4}{5}\right)$$

$$=\frac{2}{k_1+k_2}\left[\frac{k_1{}^4-k_2{}^4}{5}+\frac{3}{4}\left(c_{11}k_2{}^3-c_{10}k_1{}^3\right)+c_{10}{}^2k_1{}^2-c_{11}{}^2k_2{}^2+\frac{1}{2}\left(c_{11}{}^3k_2-c_{10}{}^3k_1\right)\right]$$

$$=\frac{2}{k_1+k_2}\left\{\begin{array}{l}\frac{k_1{}^4-k_2{}^4}{5}+\frac{1}{4}\left[\left(2k_2+k_1\right)k_2{}^3-\left(2k_1+k_2\right)k_1{}^3\right]+\frac{1}{9}\left[\left(2k_1+k_2\right)^2k_1{}^2-\left(2k_2+k_1\right)^2k_2{}^2\right]+\\ \frac{1}{54}\left[\left(2k_2+k_1\right)^3k_2-\left(2k_1+k_2\right)^3k_1\right]\end{array}\right\}$$

$$=\frac{2}{k_1+k_2}\left[\begin{array}{l}\frac{\left(k_1{}^4-k_2{}^4\right)}{5}+\frac{1}{4}\left(2k_2{}^4+k_1k_2{}^3-2k_1{}^4-k_2k_1{}^3\right)+\frac{1}{9}\left(4k_1{}^4+4k_1{}^3k_2-4k_2{}^4-4k_1k_2{}^3\right)+\\ \frac{1}{54}\left(8k_2{}^4+12k_1k_2{}^3+6k_1{}^2k_2{}^2+k_1{}^3k_2-8k_1{}^4-12k_1{}^3k_2-6k_1{}^2k_2{}^2-k_1k_2{}^3\right)\end{array}\right]$$

$$=\frac{1}{k_1+k_2}\left(\begin{array}{l}\frac{2k_1{}^4}{5}-\frac{2k_2{}^4}{5}+k_2{}^4+\frac{k_1k_2{}^3}{2}-k_1{}^4-\frac{k_2k_1{}^3}{2}+\frac{8k_1{}^4}{9}+\frac{8k_1{}^3k_2}{9}-\\ \frac{8k_2{}^4}{9}-\frac{8k_1k_2{}^3}{9}+\frac{8k_2{}^4}{27}+\frac{11k_1k_2{}^3}{27}-\frac{8k_1{}^4}{27}-\frac{11k_1{}^3k_2}{27}\end{array}\right)$$

$$=\frac{1}{k_1+k_2}\left[\left(\frac{2}{5}-1+\frac{8}{9}-\frac{8}{27}\right)k_1{}^4+\left(1-\frac{2}{5}-\frac{8}{9}+\frac{8}{27}\right)k_2{}^4+\left(\frac{1}{2}-\frac{8}{9}+\frac{11}{27}\right)k_1k_2{}^3+\left(\frac{8}{9}-\frac{1}{2}-\frac{11}{27}\right)k_1{}^3k_2\right]$$

$$=\frac{1}{27\left(k_1+k_2\right)}\left(\frac{k_2{}^4-k_1{}^4}{5}+\frac{k_1k_2{}^3-k_1{}^3k_2}{2}\right)=\frac{1}{27\left(k_1+k_2\right)}\left(\frac{2k_2{}^4-2k_1{}^4+5k_1k_2{}^3-5k_1{}^3k_2}{10}\right)$$

$$= \frac{2\left(k_2^3 - k_1^3\right) + 3k_1k_2^2 - k_1^2k_2}{270} = \frac{\left(k_2 - k_1\right)}{270}\left(2k_2^2 + 2k_1^2 + 5k_1k_2\right)$$

$$\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{\left(k_2 - k_1\right)\left(2k_2^2 + 2k_1^2 + 5k_1k_2\right)/270}{\left[\left(k_2^2 + k_1k_2 + k_1^2\right)/18\right]^{1.5}}$$

$$= \frac{\left(\sqrt{18}\right)^3}{270} \frac{\left(k_2 - k_1\right)\left(2k_2^2 + 2k_1^2 + 5k_1k_2\right)}{\left(k_2^2 + k_1k_2 + k_1^2\right)^{1.5}}$$

$$= \frac{\left(\sqrt{18}/3\right)^3}{10} \frac{\left(k_2 - k_1\right)\left(2k_2^2 + 2k_1^2 + 5k_1k_2\right)}{\left(k_2^2 + k_1k_2 + k_1^2\right)^{1.5}}$$

$$= \frac{\left(\sqrt{2}\right)^3}{\left(\sqrt{2}\right)^2(5)} \frac{\left(k_2 - k_1\right)\left(2k_2^2 + 2k_1^2 + 5k_1k_2\right)}{\left(k_2^2 + k_1k_2 + k_1^2\right)^{1.5}}$$

$$= \frac{\sqrt{2}}{5} \frac{\left(k_2 - k_1\right)\left(2k_2^2 + 2k_1^2 + 5k_1k_2\right)}{\left(k_2^2 + k_1k_2 + k_1^2\right)^{1.5}} \tag{41}$$

Equation (41) clearly shows that the triangular distribution is positively skewed iff $k_2 > k_1$.

If $k_2 > k_1$, then $k_1 = rk_2$, where $0 < r < 1$ and $r = {k_1}/{k_2}$. Substituting for $k_1 = rk_2$ in Equaion (41) results in

$$\alpha_3 = \sqrt{0.08}\,\frac{\left(k_2 - rk_2\right)\left(2k_2^{\;2} + 2r^2k_2^{\;2} + 5rk_2^{\;2}\right)}{\left(k_2^{\;2} + rk_2^{\;2} + r^2k_2^{\;2}\right)^{1.5}}$$

$$= \sqrt{0.08}\,\frac{\left(1-r\right)\left(2+2r^2+5r\right)}{\left(1+r+r^2\right)^{1.5}}$$

Taking the first derivative of skewness,

$$\frac{d\alpha_3}{dr} = \sqrt{0.08}\,\frac{d}{dr}\left[\left(2+3r-3r^2-2r^3\right)\left(1+r+r^2\right)^{-1.5}\right]$$

$$= \sqrt{0.08}\left[\left(3-6r-6r^2\right)\left(1+r+r^2\right)^{-1.5} - 1.5\left(1+r+r^2\right)^{-2.5}\left(1+2r\right)\left(2+3r-3r^2-2r^3\right)\right]$$

$$= \sqrt{0.08}\left(1+r+r^2\right)^{-2.5}\left[\left(3-6r-6r^2\right)\left(1+r+r^2\right) - 1.5\left(1+2r\right)\left(2+3r-3r^2-2r^3\right)\right]$$

$$= \sqrt{0.08}\left(1+r+r^2\right)^{-2.5}\left(-13.5r-13.5r^2\right)$$

$= -\sqrt{14.58}\,\frac{r(1+r)}{\left(1+r+r^2\right)^{2.5}} < 0$ for all *r* within $0 < r < 1$. Hence, the maximum of α_3 occurs at *r = 0* and all triangular distributions have skewness values in the interval $-\sqrt{0.32} \le \alpha_3 \le \sqrt{0.32}$. Further, all right-triangular distributions have $k_1 = 0$ and $r = 0$, so that $\alpha_3 = -\sqrt{0.32}$, and all left-triangular distributions have $r = 1$, so that $\alpha_3 = \sqrt{0.32}$. When $k_1 = k_2 = 0$, $\alpha_3 = 0$.

Similarly, it can be shown that the kurtosis of all triangular distributions is given by $\beta_4 = \alpha_4 - 3 = E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] - 3 = 2.4 - 3 = -0.60$.

So, to summarize, for the Triangular distribution

$$E[X] = m + \frac{k_2 - k_1}{3}$$

$$V[X] = \frac{k_1^{\,2} + k_2^{\,2} + k_1 k_2}{18}$$

$$\alpha_3 = \sqrt{0.08}\,\frac{(k_2 - k_1)\left(2k_2^{\,2} + 2k_1^{\,2} + 5k_1 k_2\right)}{\left(k_2^{\,2} + k_1 k_2 + k_1^{\,2}\right)^{1.5}}$$

and $\beta_4 = \alpha_4 - 3 = -0.60$

Note that for a symmetric Triangular distribution, the above equations simplify to

$E[X] = m$, $V[X] = \frac{k^2}{6}$, $\alpha_3 = 0$, and $\beta_4 = \alpha_4 - 3 = -0.60$.