On the role of 1-LC and semi 1-LC properties in determining the fundamental group of a one point union of spaces

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Vita

Emilia Anna Moore, daughter of Piotr and Anna Lusnia, was born on October 3, 1982, in Garwolin, Poland. She graduated from Loveless Academic Magnet Program High School in Montgomery, Alabama, in 2000. She then attended Huntingdon College in Montgomery, Alabama, for three years and graduated magna cum laude with Bachelor of Art degrees in Mathematics and Computer Science in May 2003. She entered the PhD program at Auburn University, in June 2003.

Thesis Abstract

On the role of 1-LC and semi 1-LC properties in determining the fundamental group of a one point union of spaces

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Concerning the fundamental group of spaces written as a union of two topological spaces, the following result by Seifert and van Kampen is well known and frequently used.

Theorem 0.1 *Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let $\phi_1 : \pi_1(U, x_0) \to H$, and $\phi_2 : \pi_1(V, x_0) \to H$ be homomorphisms. Let $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$, $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$, $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphisms induced by inclusion. If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.*

Over the years there have been various theorems published on the topic of fundamental groups of the unions of spaces. A portion of those theorems deal with spaces whose intersection is a one point set. In 1954 Griffiths' result was published stating the following theorem [6].

Theorem 0.2 *If one of the spaces X_1 or X_2 is 1-LC at x, and both X_1 and X_2 are closed in X and satisfy the first axiom of countability, then*

$$(i_1 \wedge i_2) : \pi_1(X_1, x) * \pi_1(X_2, x) \simeq \pi_1(X, x)$$

where $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{x\}$.

The goal of this paper is to show that the 1-LC property in Griffiths' result cannot be generalized to semi 1-LC. Spanier indicated this problem in one of his homework exercises [2]. This result is not, however, contained in Griffiths' paper or Spanier's book. To provide the necessary proof two spaces are constructed with the following properties:

1. X is semi 1-LC (but not 1-LC)

2. $\pi_1(X, x_0)$ is trivial

3. the fundamental group of the one point union of X with itself is not trivial.

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Chapter 1

Introduction

One of the fundamental problems in the field of topology is determining whether two topological spaces are homeomorphic. In general, there is no method for solving this problem. Showing that two spaces are homeomorphic is equivalent to constructing a continuous bijection between them with a continuous inverse, called a homeomorphism. To show that two spaces are not homeomorphic requires proof that such a mapping does not exist. The topological properties, also known as topological invariants, of spaces provide us with some methods of showing the lack of a homeomorphism. A topological invariant of a space X is a property that depends only on the topology of the space, i.e. it is shared by any topological space homeomorphic to X. If we can show that two topological spaces differ in some topological property (such as compactness, second countability, etc) we know that they cannot be homeomorphic. Throughout this paper we will consider spaces which are Hausdorff. In the following chapters we will discuss a concept that helps us show which topological spaces are not homeomorphic. We will introduce the idea of a fundamental group, and observe that two spaces that are homeomorphic have isomorphic fundamental groups (i.e. fundamental group is a topological invariant). This will help us distinguish between spaces by showing their fundamental groups are not isomorphic. We will discuss theorems that help us calculate the fundamental group of spaces that can be written as unions of topological spaces. Let us state some definitions and theorems that will be useful throughout this paper.

1.1 Fundamental Group

Let us quote Munkres' definitions of several terms that will be used in this paper [1].

Definition 1.1 *A **topology** on a set X is a collection τ of subsets of X having the following properties:*

1. *$\emptyset$ and X are in τ.*
2. *The union of the elements of any subcollection of τ is in τ.*
3. *The intersection of the elements of any finite subcollection of τ is in τ.*

*A **topological space** is a set X together with a collection τ. If X is a topological space with topology τ, we say that a subset U of X is an **open set** of X if U belongs to the collection τ. A set K is **closed** if $X \setminus K$ is open.*

Throughout this paper, unless otherwise specified, X is a topological space, $x_0 \in X$ is a point in X and $I = [0, 1]$.

Definition 1.2 *A function $f : X \to Y$ is **continuous** if for every open set $U \subset Y$ the set $f^{-1}(U)$ is open in X.*

Definition 1.3 *Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.*

Definition 1.4 *A space X is said to be **compact** if every open covering of X contains a finite subcollection that also covers X.*

Definition 1.5 *Let X be a topological space, and $x_0, x_1, x_2 \in X$, then f is called a* ***path*** *in X from x_1 to x_2 if $f : I \to X$ is a continuous function such that $f(0) = x_1$ and $f(1) = x_2$; f is called a* ***loop*** *in X based at x_0 if $f : I \to X$ is a continuous function such that $f(0) = x_0$ and $f(1) = x_0$.*

Definition 1.6 *A space X is said to be* ***path connected*** *if every pair of points of X can be joined by a path in X.*

Definition 1.7 *If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the* ***product*** *$f * g$ of f and g to be the path h given by the equations*

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}], \\ g(2s-1) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

Definition 1.8 *Two paths f and f' in X, are* ***path homotopic*** *if they have the same initial point x_1 and the same final point x_2, and if there is a continuous map $H : I \times I \to X$ such that*

$$H(s, 0) = f(s) \qquad H(s, 1) = f'(s),$$

$$H(0, t) = x_1 \qquad H(1, t) = x_2,$$

for each $s \in I$ and each $t \in I$. We call H a ***path homotopy*** *between f and f'. If f is path homotopic to f', we write $f \simeq f'$.*

Definition 1.9 *An **equivalence relation** on a set A is a relation $\sim$ on A having the following properties:*

1. *(Reflexivity) $x \sim x$ for every x in A.*

2. *(Symmetry) If $x \sim y$, then $y \sim x$.*

3. *(Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.*

Throughout our discussion of homotopy groups we will frequently use the following lemma called the Pasting Lemma[[1], p.108].

Lemma 1.10 *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.*

Proof of Lemma: Let K be a closed subset of Y. We will show that the inverse image of K under h is closed. Notice that $h^{-1}(K) = f^{-1}(K) \cup g^{-1}(K)$. Since both f and g are continuous, $f^{-1}(K)$ is closed in A and $g^{-1}(K)$ is closed in B. Therefore they are both closed in X. A union of two sets closed in X is closed in X, hence $h^{-1}(K)$ is closed in X and h is continuous.□

Lemma 1.11 *The relation $\simeq$ (path homotopy) is an equivalence relation.*

If f is a path, we will denote its path-homotopy equivalence class by $[f]$ [[1], p.324].

Proof of Lemma: First, let us show that $\simeq$ is reflexive. Given f, the map $F(x,t) = f(x)$ is a homotopy between f and itself. Hence $f \simeq f$. Next, let us show that $\simeq$ is symmetric. Let f and g be paths such that $f \simeq g$ and $F(x,t)$ be the homotopy between them. Then

$H(x,t) = F(x, 1-t)$ is a homotopy between g and f, since $H(x,0) = F(x,1) = g(x)$ and $H(x,1) = F(x,0) = f(x)$. Hence $g \simeq f$. Lastly, let us show that $\simeq$ is transitive. Let f, g, and h be paths and $F(x,t)$ be a homotopy between f and g ($f \simeq g$), and $H(x,t)$ a homotopy between g and h ($g \simeq h$). Then consider $G(x,t)$ defined by

$$G(x,t) = \begin{cases} F(x, 2t) & \text{for } t \in [0, \frac{1}{2}], \\ H(x, 2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

By the Pasting Lemma, $G(x,t)$ is well defined and continuous since $F(x, 2(\frac{1}{2})) = F(x,1) = g(x)$ and $H(x, 2(\frac{1}{2}) - 1) = H(x,0) = g(x)$. It is a homotopy between f and h, since $G(x,0) = F(x,0) = f(x)$ and $G(x,1) = H(x,1) = h(x)$. Hence $f \simeq h$.$\square$

Let us define the product operation on equivalence classes of loops based at x_0. Let $[f] * [g] = [f * g]$. We need to show this operation is well defined, i.e. if $f \simeq g$, $h \simeq j$, then $f * h \simeq g * j$. Let $F : I \times I \to X$ be a homotopy of f to g, and $G : I \times I \to X$ be a homotopy of h to j. Consider the map $H : I \times I \to X$ defined by:

$$H(s,t) = \begin{cases} F(2s, t) & \text{for } s \in [0, \frac{1}{2}], \\ G(2s-1, t) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

By the Pasting Lemma, $H(s,t)$ is well defined and continuous since

$$H(\frac{1}{2}, t) = F(1,t) = f(1) = h(0) = G(0,t).$$

It is a homotopy between $f * h$ and $g * j$, since

$$H(s,0) = \begin{cases} F(2s,0) = f(2s) & \text{for } s \in [0, \frac{1}{2}], \\ G(2s-1,0) = h(2s-1) & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

and

$$H(s,1) = \begin{cases} F(2s,1) = g(2s) & \text{for } s \in [0, \frac{1}{2}], \\ G(2s-1,1) = j(2s-1) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

Definition 1.12 *The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the* ***fundamental group*** *of X relative to the* ***base point*** *x_0. It is denoted by $\pi_1(X, x_0)$.*

Definition 1.13 *Let G be a nonempty set together with a binary operation $\circ$. G is a* ***group*** *under this operation if the following properties are satisfied:*

1. *Associativity. The operation is associative; that is, $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$.*

2. *Identity. There is an element e (called the identity) in G, such that $a \circ e = e \circ a = a$ for all $a \in G$.*

3. *Inverses. For each element a in G, there is an element b in G (called an inverse of a) such that $a \circ b = b \circ a = e$.*

Definition 1.14 *Let G and G' be groups with group operation $\circ$ in each. A* ***homomorphism*** *$f : G \to G'$ is a map such that $f(x \circ y) = f(x) \circ f(y)$ for all x, y.*

When the context is clear the group operation symbol is omitted, i.e. we write $x_1 x_2 x_3$ instead of $x_1 \circ x_2 \circ x_3$.

Definition 1.15 *Let G be a group, let $\{G_i\}_{i \in J}$ be a family of subgroups of G such that every element $x \in G$ can be written as a finite product of elements of groups G_i. This means that there is a sequence $(x_1, \ldots, x_n)$ of elements of the groups G_i such that $x = x_1 \ldots x_n$. Such a sequence is called a* ***word*** *(of length n) in the groups G_i; it is said to represent the element x of G. If x_i and x_{i+1} both belong to the same group G_j, we can group them together, obtaining the word $(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n)$ of length $n-1$, which also represents x. If any x_i equals 1, we can delete x_i from the sequence. A word obtained by applying these reductions until no group G_j contains two consecutive elements of the sequence and no $x_i = 1$ is called a* ***reduced word****.*

Definition 1.16 *Let G be a group, let $\{G_i\}_{i \in J}$ be a family of subgroups of G such that every element $x \in G$ can be written as a finite product of elements of groups G_i. We say that G is the* ***free product*** *of the group G_i if for each $x \in G$, there is only one reduced word in the groups G_i that represents x.*

Definition 1.17 *The* ***kernel*** *of a homomorphism $f : G \to G'$ is the set $f^{-1}(e')$, where e' is the identity of G'.*

Definition 1.18 *Let G be a group with group operation $\circ$. H is a* ***normal subgroup*** *of G if $x \circ h \circ x^{-1} \in H$ for each $x \in G$ and each $h \in H$.*

Theorem 1.19 *The fundamental group of X at the point x_0, $\pi_1(X, x_0)$, is a group.*

Proof: First, let us show that $*$ is associative. Let f, g, and h be loops based at x_0. We want to show that $(f * g) * h \simeq f * (g * h)$. Since all three loops are based at the same point, the product operation is defined, and it suffices to find a homotopy between the above loops. Consider a function $H : I \times I \to X$ defined as follows:

$$H(s,t) = \begin{cases} f(\frac{4s}{1+t}) & 0 \leq s \leq \frac{t+1}{4}, \\ g(4s-1-t) & \frac{t+1}{4} \leq s \leq \frac{t+2}{4}, \\ h(1-\frac{4(1-s)}{2-t}) & \frac{t+2}{4} \leq s \leq 1. \end{cases}$$

This homotopy is well defined and continuous by the Pasting Lemma, since:

$$f(\frac{4(\frac{t+1}{4})}{1+t}) = f(1) = x_0,$$

$$g(4(\frac{t+1}{4}) - 1 - t) = g(0) = x_0,$$

$$g(4(\frac{t+2}{4}) - 1 - t) = g(1) = x_0,$$

$$h(1 - \frac{4(1-\frac{t+2}{4})}{2-t}) = h(0) = x_0.$$

It is a path homotopy between $(f * g) * h$ and $f * (g * h)$ since:

$$H(0,t) = f(0) = x_0,$$

$$H(1,t) = h(1) = x_0,$$

$$H(s,0) = (f * g) * h,$$

$$H(s,1) = f * (g * h).$$

This concludes the proof of associativity. Second, let us show the existence of an identity. Let $e : I \to X$ be the constant map such that $e(a) = x_0$ for all $a \in I$. Let f be a loop based at x_0. We want to show that $e * f \simeq f$ and $f * e \simeq f$. Let us define the homotopy between f and $e * f$, $H : I \times I \to X$, by

$$H(s,t) = \begin{cases} x_0 & 0 \leq s \leq \frac{1}{2}t, \\ f(\frac{2s-t}{2-t}) & \frac{1}{2}t \leq s \leq 1. \end{cases}$$

This homotopy is well defined and continuous by the Pasting Lemma, since:

$$f(\frac{2(\frac{1}{2}t) - t}{2 - t}) = f(0) = x_0.$$

It is a path homotopy between f and $e * f$ since:

$$H(0,t) = x_0,$$

$$H(1,t) = f(1) = x_0,$$

$$H(s,0) = f(s),$$

$$H(s,1) = \begin{cases} x_0 & \text{for } 0 \leq s \leq \frac{1}{2}, \\ f(2s-1) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases} = (e * f)(s).$$

Now, let us define the homotopy between f and $f * e$, $H : I \times I \to X$, by

$$H(s,t) = \begin{cases} f(\frac{2s}{2-t}) & 0 \leq s \leq 1 - \frac{1}{2}t, \\ x_0 & 1 - \frac{1}{2}t \leq s \leq 1. \end{cases}$$

This homotopy is well defined and continuous by the Pasting Lemma, since:

$$f(\frac{2(1-\frac{1}{2}t)}{2-t}) = f(1) = x_0.$$

It is a path homotopy between f and $f * e$ since:

$$H(0,t) = f(0) = x_0,$$

$$H(1,t) = x_0,$$

$$H(s,0) = f(s),$$

$$H(s,1) = \begin{cases} f(2s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\ x_0 & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases} = (f * e)(s).$$

This concludes the proof of the existence of the identity. Finally, let us show the existence of inverses. Let f be a loop based at x_0 and $\overline{f}$ be the reverse of f defined by $\overline{f}(t) = f(1-t)$. We want to show that $f * \overline{f} \simeq e$, and $\overline{f} * f \simeq e$. Let us define the homotopy between x_0 and $f * \overline{f}$, $H : I \times I \to X$, by

$$H(s,t) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2}t, \\ f(t) & \frac{1}{2}t \leq s \leq 1 - \frac{1}{2}t, \\ f(2-2s) & 1 - \frac{1}{2}t \leq s \leq 1. \end{cases}$$

This homotopy is well defined and continuous by the Pasting Lemma, since:

$$f(2(\frac{1}{2}t)) = f(t),$$

$$f(2-2(1-\frac{1}{2}t)) = f(t).$$

It is a path homotopy between x_0 and $f * \overline{f}$ since:

$$H(0,t) = f(0) = x_0,$$

$$H(1,t) = f(0) = x_0,$$

$$H(s,0) = f(0) = x_0,$$

$$H(s,1) = \begin{cases} f(2s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\ \overline{f}(2s) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Now, let us define the homotopy between x_0 and $\overline{f} * f$, $H : I \times I \to X$, by

$$H(s,t) = \begin{cases} f(1-2s) & 0 \leq s \leq \frac{1}{2}t, \\ f(1-t) & \frac{1}{2}t \leq s \leq 1 - \frac{1}{2}t, \\ f(2s-1) & 1 - \frac{1}{2}t \leq s \leq 1. \end{cases}$$

This homotopy is well defined and continuous by the Pasting Lemma, since:

$$f(1-2(\frac{1}{2}t)) = f(1-t),$$

$$f(2(1-\frac{1}{2}t)-1) = f(1-t).$$

It is a path homotopy between x_0 and $\overline{f} * f$ since:

$$H(0,t) = f(1) = x_0,$$

$$H(1,t) = f(1) = x_0,$$

$$H(s,0) = f(1) = x_0,$$

$$H(s,1) = \begin{cases} \overline{f}(2s) & \text{for } 0 \leq s \leq \frac{1}{2}, \\ f(2s) & \text{for } \frac{1}{2} \leq s \leq 1. \end{cases}$$

This concludes the proof of existence of inverses. Hence, $\pi_1(X, x_0)$ is a group[[5], p.59].

□

Chapter 2

Fundamental Groups of unions of spaces

2.1 Seifert-Van Kampen Theorem

One way to calculate the fundamental group of a space is to express the space as a union of two other spaces whose fundamental groups are already known, or easily computed. There must, of course, be some guidelines for such a process. The Seifert-van Kampen Theorem provides us with a way of determining such fundamental groups. There are many forms of this theorem, below are two versions found in Munkres [[1], p.426,431].

Theorem 2.1 *Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let $\phi_1 : \pi_1(U, x_0) \to H$, and $\phi_2 : \pi_1(V, x_0) \to H$ be homomorphisms. Let $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$, $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$, $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphisms induced by inclusion. If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.*

Theorem 2.2 *Let $X = U \cup V$, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let $\phi_1 : \pi_1(U, x_0) \to H$, and $\phi_2 : \pi_1(V, x_0) \to H$ be homomorphisms. Let $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$, $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$, $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphisms induced by inclusion. Let $j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphism of the free product that extends homomorphisms j_1 and j_2 induced* *by inclusion. If* $\phi_1 \circ i_1 = \phi_2 \circ i_2$*, then* j *is surjective, and its kernel is the least normal subgroup* N *of the free product that contains all elements represented by words of the form*

$$([i_1(g)]^{-1}, i_2(g)),$$

for $g \in \pi_1(U \cap V, x_0)$*.*

Assuming the hypothesis of the Seifert-van Kampen theorem we can state the following:

Corollary 2.3 *If* $U \cap V$ *is simply connected, then there is an isomorphism*

$$k : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0).$$

The following version of the theorem is found in Engelking's book[[8], p.166].

Theorem 2.4 *If a polyhedron* X *is the union of connected polyhedra* X_1 *and* X_2 *whose intersection is simply connected, then the fundamental group* $\pi_1(X, x_0)$*, where* $x_0 \in X_1 \cap X_2$*, is isomorphic to the free product of the groups* $\pi_1(X_1, x_0)$ *and* $\pi_1(X_2, x_0)$*.*

The following result by Van Kampen can be found in a paper by Paul Olum [[10], p.667].

Theorem 2.5 *Let* $X = X_1 \cup X_2$ *be a separable, regular topological space. Let* $X_1 \cap X_2$ *be closed in* X*,* $X_1 - (X_1 \cap X_2)$*,* $X_2 - (X_1 \cap X_2)$ *open in* X*,* X_1 *and* X_2 *locally connected at* $X_1 \cap X_2$*,* $X_1 \cap X_2$ *locally connected and* X_1*,* X_2*,* $X_1 \cap X_2$ *path connected. If* $x_0 \in X_1 \cap X_2$ *then the fundamental group* $\pi_1(X, x_0)$ *is isomorphic to the free product of the groups* $\pi_1(X_1, x_0)$ *and* $\pi_1(X_2, x_0)$*.*

In this paper we are concerned with one point unions of spaces. A one point set is obviously simply connected and closed. Therefore a one point union of two spaces satisfying the conditions of the Seifert-van Kampen Theorem will have a fundamental group equal to the free product of the fundamental groups of the spaces composing it.

2.2 Griffiths' Paper

In 1954 a paper by H.B. Griffiths was published in Quarterly Journal of Mathematics. The paper contained the following result[[6], Theorem 1].

Theorem 2.6 *If one of X_1, X_2 is 1-LC at x, and both X_1 and X_2 are closed in X, Hausdorff, and satisfy the first axiom of countability, and $X_1 \cap X_2 = \{x\}$, then*

$$(i_1 \wedge i_2) : \pi_1(X_1, x) * \pi_1(X_2, x) \simeq \pi_1(X, x)$$

where $X = X_1 \cup X_2$.

Let us first clarify the notation used. If X_1, X_2, and X are groups with injection homomorphisms $j_i : \pi_1(X_i, x) \to \pi_1(X, x)$ for $i = 1, 2$, then $(j_1 \wedge j_2)$ is a homomorphism of the free product $\pi_1(X_1, x) * \pi_1(X_2, x) \to \pi_1(X, x)$ defined by

$$(j_1 \wedge j_2)(a_1 b_1 \ldots a_m b_m) = (j_1 a_1)(j_2 b_1) \ldots (j_1 a_m)(j_2 b_m),$$

where $a_i \in \pi_1(X_1, x)$, $b_i \in \pi_1(X_2, x)$.

Definition 2.7 *A space X is said to have a* ***countable basis at the point x*** *if there is a countable collection $\{U_n\}_{n \leq \infty}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n. A space X that has a countable basis at each of its points is said to satisfy the* ***first countability axiom****.*

Definition 2.8 *A homomorphism is* ***trivial*** *if it maps everything to the identity element.*

Definition 2.9 *A space is* ***1-LC at x*** *(locally simply connected at x) if for every open set* $U \ni x$ *there exists an open set* $V \subseteq U$, $V \ni x$ *such that the homomorphism* $i_* : \pi_1(V,x) \to \pi_1(U,x)$ *is trivial.*

Definition 2.10 *A space is* ***semi 1-LC at x*** *(semilocally simply connected at x) if there exists an open set* $U \ni x$ *such that the homomorphism* $i_* : \pi_1(U,x) \to \pi_1(X,x)$ *is trivial.*

The goal of this paper is to show that the 1-LC property in Griffiths' theorem cannot be replaced with the semi 1-LC property. Griffiths stated that Theorem 2.6 is an immediate consequence of the following two theorems [[6], p.176]:

Theorem 2.11 *If* X_1 *and* X_2 *are Hausdorff and* X_1 *is 1-LC at x, then the homomorphism* $(i_1 \wedge i_2) : \pi_1(X_1,x) * \pi_1(X_2,x) \to \pi_1(X,x)$ *is onto, where X is the one point union of* X_1 *and* X_2.

Theorem 2.12 *If both* X_1 *and* X_2 *are closed in X, Hausdorff, and satisfy the first axiom of countability, and* $X_1 \cap X_2 = \{x\}$, *then the homomorphism* $(i_1 \wedge i_2)$ *has a zero kernel.*

Only the proof of Theorem 2.11 requires the 1-LC property, therefore the proof of Theorem 2.12 will be omitted.

Proof of Theorem 2.11: Let $f : I \to X$ be a loop based at x. Since [0,1] is compact, X is Hausdorff, and f is continuous, then $F = f([0,1])$ is closed in X; since X_1 and X_2 are closed in X, we have $F_1 = F \cap X_1$ and $F_2 = F \cap X_2$ closed in X_1 and X_2 respectively.

Therefore $F_i' = f^{-1}(F_i)$ is closed in [0,1] for $i = 1, 2$.

Claim 1: Every open set in $\mathbb{R}$ can be written as the countable union of disjoint open intervals[[3], p.136].

Proof of Claim 1: Let $U \subseteq \mathbb{R}$ be open. For each $x \in U$, let I_x be the union of open intervals J containing x. $I_x = \bigcup_{J_p \subseteq U, x \in J_p} J_p$ exists for every x since U is open, hence every point $x \in U$ is an interior point. I_x is an open interval, since it is the union of open intervals. Notice that if $x_1 \neq x_2$, $x_1, x_2 \in U$ then either $I_{x_1} = I_{x_2}$ or $I_{x_1} \cap I_{x_2} = 0$. Let $a \in I_{x_1} \cap I_{x_2}$, then $I_{x_1} \cup I_{x_2}$ is an open interval containing both x_1 and x_2. Hence $I_{x_1} \cup I_{x_2} \subseteq I_{x_1}$, and $I_{x_1} \cup I_{x_2} \subseteq I_{x_2}$. Therefore $I_{x_1} = I_{x_2}$. Finally, let us fix a rational number in each I_x. Since the rational numbers are countable, there are only countable many disjoint open intervals. Obviously, each $x \in U$ is in some I, namely I_x. End of Claim 1.

Hence the open set $[0, 1] \backslash F_2'$ is the union of a countable set of disjoint intervals I_i, each I_i being open in [0,1]. Because the space X_1 is first countable and 1-LC at x, we can construct the following sequence. Let U_1 be a neighborhood of x in X_1. Let $\{V_i\}$ be the countable collection of open sets in X_1 containing x such that any open set in X_1 containing x contains at least one member of $\{V_i\}$. Since X_1 is 1-LC there is an open set $V \subseteq U_1$ such that $\pi_1(V, x) \to \pi_1(U_1, x)$ is trivial. Let $U_2 = V_i$ such that $V_i \subseteq V$. Then obviously $\pi_1(U_2, x) \to \pi_1(U_1, x)$ is trivial. Continuing in the above manner we obtain a sequence

$$X_1 \supseteq U_1 \supseteq \ldots \supseteq U_m \supseteq \ldots$$

of neighborhoods of $x \in X_1$, such that

$$\bigcap_{m=1}^{\infty} U_m = x$$

and, for each $m > 1$, the injection

$$j_m : \pi_1(U_m, x) \to \pi_1(U_{m-1}, x)$$

is trivial. For each m, let S_m be the collection of all those $I_i \in S$ for which $f(I_i) \subseteq U_m$. Since f is uniformly continuous on [0,1] and $\bigcap U_m = x$, we have that

$$S_m - S_{m+1} = \{I_{m1}, I_{m2}, \dots, I_{mp(m)}\}$$

is a finite set. Also each $\dot{I}_{mn} \subseteq F_2' \cap F_1'$ ($\dot{I}$ represents the frontier of I; i.e. $\dot{I} = \overline{I} \cap \overline{\mathbb{R} \backslash I}$), so that $f(\dot{I}_{mn}) = x$; thus $f_{mn} = (f|I_{mn})$ defines an element of $\pi_1(U_m, x)$. This means that $F_{mn} = f_{mn} \circ \alpha_{mn}$ is an element of $\pi_1(U_m, x)$, where $\alpha_{mn} : [0,1] \to I_{mn}$ is a linear map. Therefore, if $m > 1$, then F_{mn} is path homotopic to the trivial loop in U_{m-1} (with x kept fixed during the homotopy), say by homotopy

$$\psi_{mn} : \overline{I_{mn}} \times I \to U_{m-1}.$$

Every $a \in [0,1]$ is either in F_2' or in some (unique) I_{mn}, and so we can define without ambiguity a homotopy

$$\psi : I \times I \to X$$

by

$$\psi(a,t) = \begin{cases} f(a) & \text{if } a \in F_2' \cup I_{11} \cup \ldots \cup I_{1p(1)}, \\ \psi_{mn}(a,t) & \text{if } a \in I_{mn}(m > 1). \end{cases}$$

We must prove ψ continuous at all points $z \in I \times I$. Consider $z \in (0,1) \times I$. If $z \in I_{mn}$, $m > 1$ then $\psi = \psi_{mn}$ is continuous at z by continuity of ψ_{mn}. If $z = (a,t)$ such that $a \in F_2' \cup I_{11} \cup \ldots \cup I_{1p(1)}$ then $\psi = f$ is unique and hence continuous at z by continuity of f at a. We are dealing with two continuous functions, defined on disjoint sets of points. $\psi_{mn}(a,t)$ is continuous for every $(a,t) \in I_{mn} \times [0,1]$ where $m \neq 1$. Since all I_{mn} are open intervals, we do not need to consider continuity of ψ_{mn} at the endpoints of each I_{mn}. f is continuous for every $a \in [0,1]$, however $f(0) = f(1)$. Therefore, the continuity follows for all z except when $z \in \{0,1\} \times I$, and here by definition of continuity it suffices to show that, given U_q, there exists a neighborhood $V = V(z) \subseteq I \times I$ such that $\psi(V) \subseteq U_q$. If the point 0 is in some I_{mn}, then the continuity of ψ for all $z = (0,t)$ follows from that of ψ_{mn}. Suppose then that 0 is not in the closure of any interval I_{mn}. Now f is continuous at 0, so there is a neighborhood $W = W(0) \in [0,1]$ for which $f(W) \subseteq U_q$. The uniform continuity of f on [0,1] also implies that

$$\lim_{m\to\infty} \{lub_{1\leq n \leq p(m)}(length I_{mn})\} = 0.$$

Claim 2: We can assume W to be such that every I_{mn}, which meets W is contained in W.

Proof of Claim 2: Let $b_{mn} \in I_{mn} \cap W$. Since $I_{mn} \backslash W \neq \emptyset$, let $a_{mn} \in I_{mn} \backslash W$. Clearly, $d(b_{mn}, a_{mn}) > 0$. Since

$$\lim_{m \to \infty} \{lub_{1 \leq n \leq p(m)} (length I_{mn})\} = 0,$$

we have $\lim_{m \to \infty} d(b_{mn}, a_{mn}) = 0$. Therefore for every $\epsilon > 0$, there exists an $n_\epsilon \in \mathbb{N}$ such that $d(b_{mn}, a_{mn}) < \epsilon$ for every $m \geq n_\epsilon$. If $I_{mn} \cap W \neq \emptyset$ and $I_{mn} \backslash W \neq \emptyset$, let $W = W \backslash (I_{mn} \cap W)$. By the above, there are only finitely many I_{mn}'s that intersect but are not contained in W, hence this would be repeated at most finitely many times giving us the desired set W. End of Claim 2.

We can also assume that the I_{mn}'s mentioned above are so small that $\psi_{mn}(I_{mn} \times I) \subseteq U_q$. Hence, if $a \in W$, then either $a \in F_2'$ or a is in some I_{mn}. If $a \in F_2'$, then $\psi(a, t) = f(a)$ for all t, and $f(a) \in f(W) \subseteq U_q$; and if $a \in I_{mn}$, then $a \in W$, hence $\psi(a, t) \in \psi_{mn}(I_{mn} \times I) \subseteq U_q$. Therefore $\psi(W \times I) \subseteq U_q$, i.e. ψ is continuous at all points $(0, t) \in I \times I$. Similar argument shows that ψ is continuous at all points $(1, t)$. Hence ψ is continuous everywhere in $I \times I$, as desired.

Clearly $\psi(a, 0) = f(a)$; define f' by $f'(a) = \psi(a, 1)$, so that $f' \simeq f$ in X (with x kept fixed during the homotopy). We now express [0,1] as

$$[0, 1] = J_1 \cup I_{11} \cup J_2 \cup I_{12} \cup \ldots \cup J_p \cup I_{1p} \cup J_{p+1},$$

where $p = p(1)$, the J's are closed intervals disjoint from the I's and each other, and the numbering is such that, if

$$s \in J_i, \qquad q \in I_{1j}, \qquad r \in J_{i+1},$$

then $s < q < r$ for each appropriate i. By definition $I_{1j} \cap I_{1k} = \emptyset$ for all $j \neq k$. Since each I_{1j} is open, $0 \notin I_1 j$ for all j. Let $I_{11} = (a_1, a_2)$ and define $J_1 = [0, a_1]$. Let $I_{12} = (a_3, a_4)$ and define $J_2 = [a_2, a_3]$. Continue in such a way by defining $J_k = [a_{2(k-1)}, a_{2(k-1)+1}]$ where $I_{1k} = (a_{2(k-1)+1}, a_{2k})$ for $k \leq p$. Define J_{p+1} as $J_{p+1} = [a_{2p}, 1]$. Clearly the collection of I_{1j} and J_j as defined above satisfies the conditions listed. Write

$$f_j = f'|\bar{I}_{1j}, \qquad g_j = f'|J_j.$$

Then $f_j(\bar{I}_{1j}) \subseteq X_1$ and $g_j(J_j) \subseteq X_2$, so that f' is the product mapping

$$f' = g_1 f_1 g_2 f_2 \ldots g_p f_p g_{p+1}.$$

Therefore, if homotopy classes in $\pi_1(X, x)$, $\pi_1(X_i, x)$ are denoted by $R(.)$, $R_i(.)$, respectively, we have

$$\begin{aligned} Rf = Rf' &= R(g_1 f_1 g_2 f_2 \dots g_p f_p g_{p+1}) \\ &= (Rg_1)(Rf_1)\dots(Rg_p)(Rf_p)(Rg_{p+1}) \\ &= (i_2 R_2 g_1)(i_1 R_1 f_1)\dots(i_2 R_2 g_p)(i_1 R_1 f_p)(i_2 R_2 g_{p+1}) \\ &= (i_1 \wedge i_2)\{(R_2 g_1)(R_1 f_1)\dots(R_2 g_p)(R_1 f_p)(R_2 g_{p+1})\} \\ &= (i_1 \wedge i_2)\delta, \end{aligned}$$

where $\delta \in \pi_1(X_1, x) * \pi_1(X_2, x)$. Therefore $(i_1 \wedge i_2)$ is onto, as desired.□

Chapter 3

Expansion of the theorem

3.1 Introduction

It is a common practice in the field of mathematics to generalize already existing theorems. In the previous chapter we introduced Griffiths' result for the fundamental group of one point union of spaces with the 1-LC property. One might ask if this result can be generalized for semi 1-LC spaces. The claim of this paper is that the theorem presented by Griffiths cannot be generalized to semi 1-LC spaces. We will consider two spaces that will contradict the statement: "If one of X_1, X_2 is semi 1-LC at x, and both X_1 and X_2 are closed in X and satisfy the first axiom of countability, then

$$(i_1 \wedge i_2) : \pi_1(X_1, x) * \pi_1(X_2, x) \simeq \pi_1(X, x),$$

where $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{x\}$." In the next section we will construct two spaces with the following properties:

1. X is semi 1-LC (but not 1-LC)

2. $\pi_1(X, x_0)$ is trivial

3. the fundamental group of the one point union of X with itself is not trivial.

Obviously if the above spaces exist they would provide the necessary contradiction. Clearly if $\pi_1(X_1, x_0) \approx 0$, and $\pi_1(X_2, x_0) \approx 0$ then $\pi_1(X_1, x_0) * \pi_1(X_2, x_0) \approx 0 \neq \pi_1(X, x_0)$.

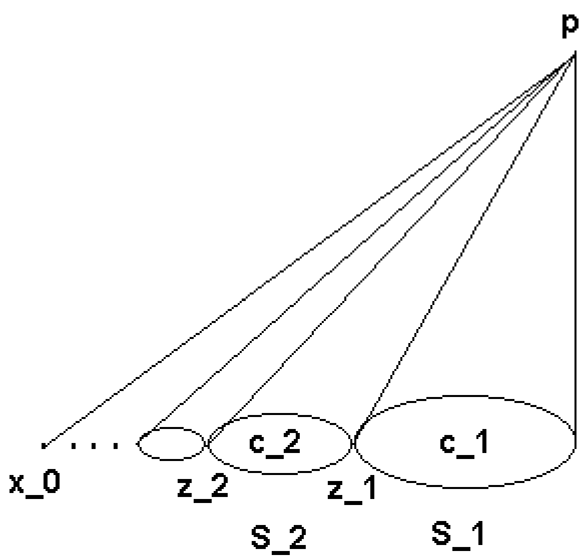

Figure 3.1: Space X

3.2 First counterexample

3.2.1 Description of the space X

In this section we will be dealing with a space that we will refer to as X. Let us first define the space carefully. X is the cone over the space B defined as the union of circles with radius approaching 0. The largest circle, call it S_1, has radius $r_1 = \frac{1}{4}$, and each consecutive circle S_n has radius $r_n = \frac{1}{2^{n+1}}$. The center of each S_n is $c_n = (\frac{3}{2^{n+1}}, 0, 0)$ and the point of intersection of two consecutive circles is $z_n = S_n \cap S_{n-1} = (\frac{1}{2^n}, 0, 0)$. Let us label the "tip" of the cone, $p = (1, 0, 1)$. Let $x_0 = (0, 0, 0)$ and $S = x_0 t + p(1 - t)$ the straight line segment between x_0 and p.

Another way to describe X is as $X = \bigcup_{n\in N} Con(S_n) \cup S$ where

$$Con(S_n) = \{(x,y,z)|(x,y,z) = a_n t + p(1-t) \text{ for } a_n \in A_n, t \in [0,1]\}$$

and

$$A_n = \{(x,y,0)|y = \pm\sqrt{(\frac{1}{2^{n+1}})^2 - (x - \frac{3}{2^{n+1}})^2},\ \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}}\}.$$

In this notation $B = \bigcup_{n\in N} A_n \cup \{x_0\}$. Let

$$Con(S_n^+) = \{(x,y,z)|(x,y,z) = a_n^+ t + p(1-t) \text{ for } a_n^+ \in A_n^+, t \in [0,1]\}$$

where

$$A_n^+ = \{(x,y,0)|y = \sqrt{(\frac{1}{2^{n+1}})^2 - (x - \frac{3}{2^{n+1}})^2},\ \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}}\}.$$

Let

$$Con(S_n^-) = \{(x,y,z)|(x,y,z) = a_n^- t + p(1-t) \text{ for } a_n^- \in A_n^-, t \in [0,1]\}$$

where

$$A_n^- = \{(x,y,0)|y = -\sqrt{(\frac{1}{2^{n+1}})^2 - (x - \frac{3}{2^{n+1}}))^2},\ \frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}}\}.$$

For each sequence $(\epsilon_1, \epsilon_2, \epsilon_3, \ldots)$ with $\epsilon_i = \pm 1$ the set $X_{\epsilon_1,\epsilon_2,\epsilon_3,\ldots} = \bigcup_{i=1}^{\infty} Con(S_n^{\epsilon_i}) \cup S$ is homeomorphic to the solid triangle T with vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$.

Let us state the Tube Lemma[[1], p. 168] used in the proof of the next Claim.

Lemma 3.1 *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

Claim 1: The projection $h : X_{\epsilon_1,\epsilon_2,\epsilon_3,\ldots} \to T$, $h(x,y,z) = (x,0,z)$ is a homeomorphism.

Proof of Claim 1: Notice that $X_{\epsilon_1,\epsilon_2,\epsilon_3,\ldots} \subseteq T \times [-\frac{1}{2}, \frac{1}{2}]$, so h is a restriction of a projection of $\pi : T \times [-\frac{1}{2}, \frac{1}{2}] \to T$. It is a well known fact that a projection of a product space onto one of its components is continuous, and a restriction of a continuous function is continuous. Hence h is continuous. Let us show h is injective. Assume $(x_1, 0, z_1) = (x_2, 0, z_2)$, then $y_1 = y_2$, since $y = \pm\sqrt{(\frac{1}{2^{n+1}})^2 - (x - \frac{3}{2^{n+1}})^2}$ and the points are limited to $Con(S_n^+)$ or $Con(S_n^-)$. Hence $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ and h is one-to-one. Let us show h is onto. For any $(x, 0, z) \in T$, take $(x, y, z) \in X_{\epsilon_1,\epsilon_2,\epsilon_3,\ldots}$ such that $(x, y, z) \in Con(S_n^+)$ or $(x, y, z) \in Con(S_n^-)$ for some n. Such point exists since $(0, 0, 0)t + (1, 0, 1)(1 - t)$ is a side of the triangle T, $(1, 0, 0)t + (1, 0, 1)(1 - t)$ is a side of the triangle T, and $0 \leq x \leq 1$, $z = 0$ is a side of the triangle T. The rest of the points form a path connected convex space between these edges. Hence there is a y such that $(x, y, z) \in Con(S_n^+)$ or $(x, y, z) \in Con(S_n^-)$ for some n. So h is onto.

Claim 2: A closed subset of a compact space is compact.

Proof of Claim 2: Let A be a closed subset of a compact space X. Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be an open covering of A. Then $\mathcal{U} \cup (X \backslash A)$ is an open covering of X. Since X is compact, there is a finite subcollection of the above sets, say V that covers X. Either $(X \backslash A)$ is in that finite collection, in which case we remove $(X \backslash A)$ from V, or $(X \backslash A)$ is not in the subcollection, in which case we already have a finite subcollection of U covering A. Hence A is compact. End of Claim 2.

Claim 3: A product of two compact spaces is compact.

Proof of Claim 3: Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be an open covering of $X \times Y$, where both X and Y are compact. Let $p \in X$, then $p \times Y$ is homeomorphic to Y, hence it is compact. We can cover $p \times Y$ with finitely many elements of $\mathcal{U}$, say $U_1, U_2, \ldots, U_n$. Let $V = U_1 \cup U_2 \cup \ldots \cup U_n$. Then V is an open set containing $p \times Y$. By the Tube Lemma, V contains an open set $H \times Y \supseteq p \times Y$, where H is an open set in X. Hence $H \times Y$ can be covered by finitely many elements $U_1, U_2, \ldots, U_n$. Take $x \in X$, then we can choose an open set H_x in X such that $H_x \times Y$ can be covered with finitely many elements of $\mathcal{U}$. Since we can choose H_x for every $x \in X$, the collection $\{H_x\}_{x \in X}$ covers X. Since X is compact, we can cover X with finitely many members of $\{H_x\}_{x \in X}$, say $H_{x_1}, H_{x_2}, \ldots, H_{x_n}$. Then $H_{x_1} \times Y, H_{x_2} \times Y, \ldots, H_{x_n} \times Y$ covers $X \times Y$. Since there are finitely many of these sets, and each one of them can be covered with finitely many elements of $\mathcal{U}$, $X \times Y$ can be covered with finitely many members of $\mathcal{U}$. Hence $X \times Y$ is compact. End of Claim 3.

Therefore, we have that $T \times [-\frac{1}{2}, \frac{1}{2}]$ is compact. Take any infinite sequence of points in $X_{\epsilon_1, \epsilon_2, \epsilon_3, \ldots}$. Either there is an n such that all but finitely many points of the sequence lie in $Con(S_n)$ or only a finite number of points lie in each $Con(S_n)$ implying that the points (just as $Con(S_n)$) limit to the segment S. In the first case the sequence limits to a point in the specified $Con(S_n)$ and since each $Con(S_n)$ is closed, the limit point is in $X_{\epsilon_1, \epsilon_2, \epsilon_3, \ldots}$. In the second case, the limit point of the sequence is in $S \subseteq X_{\epsilon_1, \epsilon_2, \epsilon_3, \ldots}$. Therefore $X_{\epsilon_1, \epsilon_2, \epsilon_3, \ldots}$ is a closed subset of $T \times [-\frac{1}{2}, \frac{1}{2}]$, hence it is compact. Since h is a continuous, one-to-one, and onto function on a compact set it is a homeomorphism. End of Claim 1.

3.2.2 Local connectedness and Semi Local Connectedness of X

Definition 3.2 *A space X is said to be* ***contractible*** *if the identity map $i_X : X \to X$ is homotopic to a constant map.*

Let us show that X as described above is semi 1-LC at x_0 but not 1-LC at x_0. By Theorem 3.3, $\pi_1(X, x_0) = 0$. Therefore the homomorphism $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ is trivial for any $x_0 \in X$. Hence, X is semi 1-LC. Now let us show X is not 1-LC. Take any open set $U \ni x_0$. By definition of X, $U \supseteq D_n$ for some n, where $D_n = B \backslash \{S_i\}_{i \le n} \cup \{z_n\}$. Any set $V \subseteq U$, $V \ni x_0$ will again have the property that $V \supseteq D_m$ for some $m \geq n$. Notice that if $p \notin U$, $U \supseteq D_n$ but U does not contain D_{n-1} then $\pi_1(U, x_0) \approx \pi_1(D_n, x_0)$. Since $\pi_1(D_m, x_0) \to \pi_1(D_n, x_0)$ is not trivial for any $m, n \in N$, $m \geq n$, we have that $\pi_1(V, x_0) \to \pi_1(U, x_0)$ is not trivial for any $V \subseteq U$. Hence X is not 1-LC.

3.2.3 The fundamental group of X

Theorem 3.3 *If X is the space defined in this section, and $x_0 = (0, 0, 0) \in X$, then $\pi_1(X, x_0)$ is trivial.*

Two proofs will be provided for the above theorem. The first proof will use the fact that X is a cone, and the second one will involve construction of a homotopy.

First Proof of Theorem 3.3:

Let us introduce the concept of quotient spaces[[9], p.161].

Definition 3.4 *Let X be a space and S an equivalence relation on X. Then S partitions X into a family X/S of equivalence classes. The* ***quotient topology*** *for X/S is defined by the following condition: A set U of equivalence classes in X/S is open if and only if* *the union of the members of U is open in X. The* ***quotient space of X modulo S*** *is the set X/S with the quotient topology.*

Let B be the infinite chain of circles with radius approaching zero union the point $\{x_0\}$ as defined previously. Define the relation on $B \times I$ by $(x,t) \sim (x',t')$ if $t = t' = 1$. Denote the equivalence class of (x,t) by $[x,t]$. Let $X = B \times I/\sim$ be the cone over B.

Step 1: Let us show that X is contractible to $p = [x,1]$. Define a map $F : X \times I \to X$ by $F([x,t],s) = [x,(1-s)t+s]$. Obviously $F([x,t],0) = [x,t]$ and $F([x,t],1) = [x,1] = p$. We have a deformation retraction of X to p, hence X is contractible to p.

Step 2: Show that the fundamental group at x of any space contractible to x is trivial (in this case $\pi_1(X,p) \approx 0$). Take any path $\alpha : I \to X$, use the map defined above as the homotopy of α to p. Hence, any path in X is homotopic to the constant map at p (i.e. $\pi_1(X,p) \approx 0$).

Step 3: Show that if X is path connected and $x, x' \in X$ then $\pi_1(X,x) \approx \pi_1(X,x')$. Let $\gamma : I \to X$, be such that $\gamma(0) = x$ and $\gamma(1) = x'$. We will write $[f] * [g]$ as $[f][g]$ when it is clear we are dealing with products of paths. Define a map $\Psi : \pi_1(X,x) \to \pi_1(X,x')$ by $\Psi([f]) = [\gamma^{-1}][f][\gamma]$ where $f \in \pi_1(X,x)$. This map is a homomorphism since:

$$\begin{aligned}\Psi([f]) * \Psi([g]) = [\gamma^{-1}][f][\gamma][\gamma^{-1}][g][\gamma] = \\ = [\gamma^{-1}][f][g][\gamma] = \Psi([f][g]).\end{aligned}$$

Since γ is fixed $[\gamma^{-1}][f][\gamma] = [\gamma^{-1}][g][\gamma]$ implies $[f] = [g]$, hence Ψ is one-to-one. To show Ψ is onto, let $[g] \in \pi_1(X,x')$. Then $[\gamma][g][\gamma^{-1}] \in \pi_1(X,x)$ and $\Psi([\gamma][g][\gamma^{-1}]) = [\gamma^{-1}][\gamma][g][\gamma^{-1}][\gamma] = [g]$. Hence Ψ is onto. Therefore Ψ is an isomorphism and $\pi_1(X,x) \approx$

$\pi_1(X, x')$.

We conclude that since $\pi_1(X,p) \approx 0$, and $\pi_1(X,p) \approx \pi_1(X,x_0)$, then $\pi_1(X,x_0) \approx 0$.

Second Proof of Theorem 3.3:

Let $f : I \to X$ be such that $f(0) = f(1) = x_0$. Let $\alpha : I \to X$ be the straight line segment between x_0 and p, $\alpha(t) = (1-t)x_0 + tp$, and $\overline{\alpha}$ the reverse of α defined as $\overline{\alpha}(t) = \alpha(1-t)$. Let $g(s,t) = (1-t)f(s) + tp$. Notice that $g(0,t) = \alpha(t)$ and $g(1,t) = \alpha(t)$. Define the homotopy of f to $\alpha * \overline{\alpha}$, $\overline{H} : I \times I \to X$, by:

$$\overline{H}(s,t) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2}t, \\ \begin{cases} g(\frac{2s-t}{2-2t}, t) & \text{if } t \neq 1 \\ g(s,1) = p & \text{if } t = 1 \end{cases} & \frac{1}{2}t \le s \le 1 - \frac{1}{2}t, \\ \alpha(2-2s) & 1 - \frac{1}{2}t \le s \le 1. \end{cases}$$

Let us check that this homotopy is well defined and continuous.

First, $\lim_{t\to 1}\ g(s,t) = p$, hence that part is continuous. Now let us check if the Pasting Lemma can be applied:

$$\alpha(2(\frac{1}{2}t)) = \alpha(t).$$

For $t \neq 1$ we have:

$$g(\frac{2(\frac{1}{2}t) - t}{2-2t}, t) = g(0,t) = \alpha(t),$$

$$g(\frac{2(1-\frac{1}{2}t) - t}{2-2t}, t) = g(1,t) = \alpha(t).$$

For $t = 1$, we have

$$g(s, 1) = p = \alpha(1),$$

and lastly

$$\alpha(2 - 2(1 - \frac{1}{2}t)) = \alpha(t).$$

Hence $\overline{H}$ is well defined and continuous by the Pasting Lemma. Also,

$$\overline{H}(s, 0) = g(s, 0) = f(s) \qquad \overline{H}(0, t) = \alpha(0) = x_0,$$

$$\overline{H}(s, 1) = \alpha * \overline{\alpha} \qquad \overline{H}(1, t) = \alpha(0) = x_0.$$

Define a homotopy of $\alpha * \overline{\alpha}$ to x_0, $\widetilde{H} : I \times I \to X$, by

$$\widetilde{H}(s, t) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} - \frac{1}{2}t, \\ \alpha(1 - t) & \frac{1}{2} - \frac{1}{2}t \leq s \leq \frac{1}{2}t + \frac{1}{2}, \\ \alpha(2 - 2s) & \frac{1}{2}t + \frac{1}{2} \leq s \leq 1. \end{cases}$$

Let us check that this homotopy is well defined and continuous.

$$\alpha(2(\frac{1}{2} - \frac{1}{2}t)) = \alpha(1 - t),$$

$$\alpha(2 - 2(\frac{1}{2}t + \frac{1}{2})) = \alpha(1 - t).$$

Hence $\widetilde{H}$ is well defined and continuous by the pasting lemma. Also,

$$\widetilde{H}(s,0) = \alpha * \overline{\alpha} \qquad \widetilde{H}(0,t) = \alpha(0) = x_0,$$

$$\widetilde{H}(s,1) = \alpha(0) = x_0 \qquad \widetilde{H}(1,t) = \alpha(0) = x_0.$$

Define a homotopy of f to x_0, $H : I \times I \to X$, by

$$H = \begin{cases} \overline{H}(s,2t) & \text{for } t \in [0, \frac{1}{2}], \\ \widetilde{H}(s,2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Let us check that this homotopy is well defined and continuous:

$$\overline{H}(s, 2 \cdot \frac{1}{2}) = \overline{H}(s,1) = \alpha(2s) * \alpha(2-2s)$$

and

$$\widetilde{H}(s, 2 \cdot \frac{1}{2} - 1) = \widetilde{H}(s,0) = \alpha(2s) * \alpha(2-2s).$$

Hence H is well defined and continuous by the Pasting Lemma.

Let us check that the following are satisfied:

$$H(s,0) = f(s) \qquad H(0,t) = x_0,$$

$$H(s,1) = x_0 \qquad H(1,t) = x_0.$$

$$H(s,0) = \overline{H}(s,0) = g(s,0) = f(s),$$

$$H(s,1) = \widetilde{H}(s,1) = \alpha(1-1) = \alpha(0) = x_0,$$

$$H(0,t) = H(1,t) = \alpha(0) = x_0.$$

Hence we have a homotopy between any path in X originating at x_0 and a constant path at x_0. This concludes the second proof.□

3.2.4 The fundamental group of a one point union of two copies of X

In this section we will consider the one point union of two copies of the space X as described above. We will call the first copy of this cone space X_1 and the second copy X_2. The space X will be defined as $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \{x_0\}$, x_0 is the limiting point as defined in previous section, $x_0 = (0,0,0)$. The points p_1 and p_2 are the "tips" of the spaces X_1 and X_2 respectively (i.e. $p_1 = (1,0,1)$ and $p_2 = (-1,0,1)$). When we described the space in the previous section we had the concept of z_n and c_n. From now on $-z_n = (-\frac{1}{2^n}, 0, 0)$ is the equivalent of z_n in X_2. In other words $z_n \in X_1$ and $-z_n \in X_2$. Also, X_1 is the cone over B_1 and X_2 is the cone over B_2.

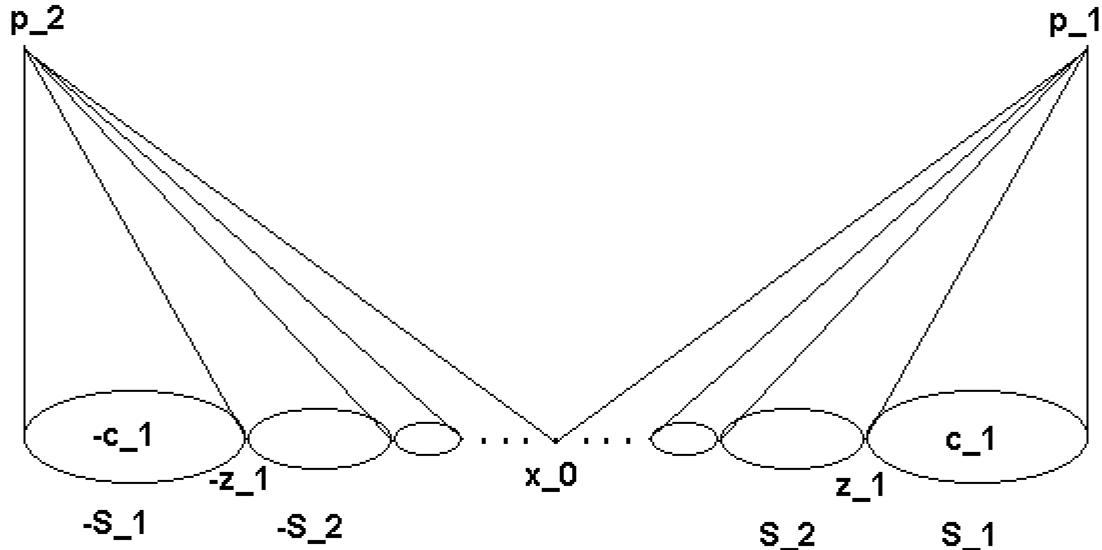

Figure 3.2: Union of two copies of X

Let us define the metric d_X on X as follows:

$$d_X(a,b) = \begin{cases} d(a,x_0) + d(b,x_0) & \text{for } a \in X_1, b \in X_2 \;\; \textit{or} \;\; a \in X_2, b \in X_1 \\ d(a,b) & \text{for } a,b \in X_1 \;\; \textit{or} \;\; a,b \in X_2, \end{cases}$$

where d is the default metric on $\mathbb{R}^2$ and $a, b \in X$. Let us show that d_X is truly a metric on X. First let us recall the definition of a metric [[1], p.119].

Definition 3.5 *A **metric** on a set X is a function*

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. *$d(x,y) \geq 0$ for all $x, y \in X$; $d(x,y) = 0$ if and only if $x = y$.*
2. *$d(x,y) = d(y,x)$ for all $x, y \in X$.*
3. *$d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$.*

Let us verify the above for d_X.

First, $d_X(x,y) \geq 0$ since d is a metric and hence $d(x,x_0) + d(y,x_0) \geq 0$ and $d(x,y) \geq 0$ for all $x, y \in X$. Also, if $x = y$ then $x, y \in X_1$ or $x, y \in X_2$, hence $d_X(x,y) = d(x,y) = 0$.
If $x, y \in X_1$ or $x, y \in X_2$ we have $d_X(x,y) = d(x,y) = d(y,x) = d_X(y,x)$. If $x \in X_1$ and $y \in X_2$, then $d_X(x,y) = d(x,x_0) + d(y,x_0) = d(y,x_0) + d(x,x_0) = d_X(y,x)$. Symmetric argument works when $x \in X_2$ and $y \in X_1$.
If $x, y, z \in X_1$ or $x, y, z \in X_2$ we have $d_X(x,y) + d_X(y,z) = d(x,y) + d(y,z) \geq d(x,z) = d_X(x,z)$. If $x, y \in X_1$ and $z \in X_2$, then $d_X(x,y) + d_X(y,z) = d(x,y) + d(y,x_0) + d(z,x_0) \geq$

$d(x, x_0) + d(z, x_0) = d_X(x, z)$. Similar argument follows for $x \in X_1$ and $y, z \in X_2$. This concludes the proof of d_X being a metric on X.

An open set in the topology on X induced by this metric is of the form $B_\epsilon(a) = \{(x, y) \in X_2 | x^2 + y^2 < (\epsilon - d(a, x_0))^2\} \cup \{(x, y) \in X_1 | d(a, (x, y)) < \epsilon\}$ for $a \in X_1$ and $B_\epsilon(a) = \{(x, y) \in X_1 | x^2 + y^2 < (\epsilon - d(a, x_0))^2\} \cup \{(x, y) \in X_2 | d(a, (x, y)) < \epsilon\}$ for $a \in X_2$. The following Lemma will be used in the proof of the next Theorem.

Lemma 3.6 $\pi_1(B_1, x_0) \simeq \pi_1(X_1 \backslash \{p_1\}, x_0)$*, and* $\pi_1(B_2, x_0) \simeq \pi_1(X_2 \backslash \{p_2\}, x_0)$.

Proof of Lemma 3.6: First let us introduce a definition used in the proof of this Lemma [[7]p.209].

Definition 3.7 *Let A be a subspace of X. Then A is **a strong deformation retract** of X if there is a continuous map $F : X \times I \to X$ such that*

$$F(x, 0) = x \textit{ for all } x \in X,$$

$$F(x, 1) \in A \textit{ for all } x \in X,$$

$$F(a, t) = a \textit{ for all } a \in A \textit{ and } t \in I.$$

Step 1: We will show that if A is a strong deformation retract of X and $x_0 \in A \subseteq X$, then the inclusion map $j : (A, x_0) \to (X, x_0)$ induces an isomorphism of fundamental groups, $j_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$. Let r be the strong deformation retraction of X, $r : X \to A$, $r(x) = F(x, 1)$, where $F : X \times I \to X$ is such that $F(X, 1) = A$, $F(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$, $F(x, 0) = x$ for all $x \in X$. Then $rj = id_A$, hence $r_* j_* = id_{\pi_1(A, x_0)}$, and j_* is injective. Let $[g] \in \pi_1(X, x_0)$, j_* is surjective if there exists $[f] \in \pi_1(A, x_0)$

such that $j_*([f]) = [g]$. Since $F \circ g$ is a homotopy from a loop g in X to a loop in A, $[g] \to [f] \in \pi_1(A, x_0)$, j_* is surjective. Therefore j_* is an isomorphism.

Step 2: Next we will show that B_1 is a strong deformation retraction of $X_1 \backslash \{p_1\}$. Consider the map $F : X_1 \backslash \{p_1\} \times I \to X_1 \backslash \{p_1\}$ defined by $F([x, s], t) = [x, s(1-t)]$. By definition of a cone, points in B_1 are of the form $[x, 0]$ and points in $X_1 \backslash \{p_1\}$ are of the form $[x, t]$ for $t \in [0, 1)$. Clearly $F([x, 0], t) = [x, 0]$ (i.e. $F(a, t) = a$), $F([x, s], 1) = [x, 0]$ (i.e. $F(X, 1) = A$), and $F([x, s], 0) = [x, s]$ (i.e. $F(x, 0) = x$).Therefore B_1 is a strong deformation retract of $X_1 \backslash \{p_1\}$. We conclude that $\pi_1(B_1, x_0) \simeq \pi_1(X_1 \backslash \{p_1\}, x_0)$. Similar argument shows that $\pi_1(B_2, x_0) \simeq \pi_1(X_2 \backslash \{p_2\}, x_0)$. $\square$

Theorem 3.8 *With X the space described above, $\pi_1(X, x_0) \neq 0$.*

To prove the above theorem it is enough to show that there is a loop f in X based at x_0 and there is no homotopy between f and x_0 in X. Consider a loop $f : I \to X$ defined by:

$$f(0) = x_0 = (0, 0, 0),$$

$$f(s) = \begin{cases} (4n\frac{1}{2^n}[(n+1)s - 1], y, 0) \quad y \geq 0 & \text{for } \frac{1}{n+1} \leq s \leq \frac{4n+1}{4n(n+1)}, \\ (-\frac{4n[(n+1)s-1]+3}{2^{n+1}}, y, 0) \quad y \leq 0 & \text{for } \frac{4n+1}{4n(n+1)} \leq s \leq \frac{2n+1}{2n(n+1)}, \\ (\frac{4n[(n+1)s-1]-1}{2^{n+1}}, y, 0) \quad y \geq 0 & \text{for } \frac{2n+1}{2n(n+1)} \leq s \leq \frac{4n+3}{4n(n+1)}, \\ (\frac{-4(n^2 s - n + ns - 1)}{2^n}, y, 0) \quad y \leq 0 & \text{for } \frac{4n+3}{4n(n+1)} \leq s \leq \frac{1}{n}, \end{cases}$$

for n odd, where n is the largest integer smaller than $\frac{1}{s}$, and $y = \pm\sqrt{(\frac{1}{2^{n+1}})^2 - (x - \frac{3}{2^{n+1}})^2}$.

$$f(s) = \begin{cases} (-4n\sum_{i=n+1}^{\infty}\frac{1}{2^i}((n+1)s-1), y, 0) \;\; y \geq 0 & \text{for } \frac{1}{n+1} \leq s \leq \frac{4n+1}{4n(n+1)}, \\ (\frac{4n(n+1)}{2^{n+1}}s - \frac{1}{2^n} - \frac{4n+1}{2^{n+1}}, y, 0) \;\; y \leq 0 & \text{for } \frac{4n+1}{4n(n+1)} \leq s \leq \frac{2n+1}{2n(n+1)}, \\ (-\frac{4n(n+1)}{2^{n+1}}s - \sum_{i=n+2}^{\infty}\frac{1}{2^i} + \frac{2(2n+1)}{2^{n+1}}, y, 0) \;\; y \geq 0 & \text{for } \frac{2n+1}{2n(n+1)} \leq s \leq \frac{4n+3}{4n(n+1)}, \\ (4\sum_{i=n+1}^{\infty}\frac{1}{2^i}(n+1)(ns-1), y, 0) \;\; y \leq 0 & \text{for } \frac{4n+3}{4n(n+1)} \leq s \leq \frac{1}{n}, \end{cases}$$

for n even, where n and y are defined as above. We need to show $f(0) = f(1) = x_0 = (0,0,0)$ and f is continuous. By definition $f(0) = x_0$. If $s = 1$, $n = 1$ so $f(1) = (-\sum_{i=2}^{\infty}\frac{1}{2^i}4(1)(2)(1) + \sum_{i=2}^{\infty}\frac{1}{2^i}4(2), y, 0) = (0, y, 0) = (0,0,0) = x_0$. To show continuity we use the pasting lemma. We need to show the definition of f agrees on all the intersections. If $s = 0$, $\lim_{s\to 0^+}\frac{1}{s} = \infty$.

Consider $f(0) = (\sum_{i=\infty}^{\infty}\frac{1}{2^i}4n(n+1)0 - \sum_{i=\infty}^{\infty}\frac{1}{2^i}4n, y, 0) = (0, y, 0) = (0,0,0) = x_0$.

Now consider:

$$f(\frac{4n+1}{4n(n+1)}) = (\sum_{i=n+1}^{\infty} \frac{4n+1}{2^i} - \sum_{i=n+1}^{\infty} \frac{4n}{2^i}, y, 0) = (\frac{1}{2^n}, y, 0) = z_n,$$

$$f(\frac{4n+1}{4n(n+1)}) = (-\frac{4n+1}{2^{n+1}} + \frac{1}{2^n} + \frac{4n+1}{2^{n+1}}, y, 0) = z_n,$$

$$f(\frac{2n+1}{2n(n+1)}) = (-\frac{4n+2}{2^{n+1}} + \frac{1}{2^n} + \frac{4n+1}{2^{n+1}}, y, 0) =$$

$$= (-\frac{1}{2^{n+1}} + \frac{1}{2^n}, y, 0) = (\sum_{i=n+2}^{\infty} \frac{1}{2^i}, y, 0) = z_{n+1},$$

$$f(\frac{2n+1}{2n(n+1)}) = (\frac{4n+2}{2^{n+1}} + \sum_{i=n+2}^{\infty} \frac{1}{2^i} - \frac{4n+2}{2^{n+1}}, y, 0) = (\sum_{i=n+2}^{\infty} \frac{1}{2^i}, y, 0) = z_{n+1},$$

$$f(\frac{4n+3}{4n(n+1)}) = (\frac{4n+3}{2^{n+1}} + \sum_{i=n+2}^{\infty} \frac{1}{2^i} - \frac{4n+2}{2^{n+1}}, y, 0) =$$

$$= (\frac{1}{2^{n+1}} + \sum_{i=n+2}^{\infty} \frac{1}{2^i}, y, 0) = (\frac{1}{2^n}, y, 0) = z_n,$$

$$f(\frac{4n+3}{4n(n+1)}) = (-\sum_{i=n+1}^{\infty} \frac{4n+3}{2^i} + \sum_{i=n+1}^{\infty} \frac{4n+4}{2^i}, y, 0) = (\frac{1}{2^n}, y, 0) = z_n.$$

So the function is continuous for n odd. Since $f(\frac{1}{n}) = (0, 0, 0)$ for all $n \in \mathbb{Z}$ the function stays continuous during the change from odd to even and vice versa. Lastly, let us show

f is continuous for n even:

$$f(\frac{4n+1}{4n(n+1)}) = (-\sum_{i=n+1}^{\infty}\frac{4n+1}{2^i} + \sum_{i=n+1}^{\infty}\frac{4n}{2^i}, y, 0) = (-\frac{1}{2^n}, y, 0) = -z_n,$$
$$f(\frac{4n+1}{4n(n+1)}) = (\frac{4n+1}{2^{n+1}} - \frac{1}{2^n} - \frac{4n+1}{2^{n+1}}, y, 0) = -z_n,$$
$$f(\frac{2n+1}{2n(n+1)}) = (\frac{4n+2}{2^{n+1}} - \frac{1}{2^n} - \frac{4n+1}{2^{n+1}}, y, 0) =$$
$$= (\frac{1}{2^{n+1}} - \frac{1}{2^n}, y, 0) = (-\sum_{i=n+2}^{\infty}\frac{1}{2^i}, y, 0) = -z_{n+1},$$
$$f(\frac{2n+1}{2n(n+1)}) = (-\frac{4n+2}{2^{n+1}} - \sum_{i=n+2}^{\infty}\frac{1}{2^i} + \frac{4n+2}{2^{n+1}}, y, 0) = (-\sum_{i=n+2}^{\infty}\frac{1}{2^i}, y, 0) = -z_{n+1},$$
$$f(\frac{4n+3}{4n(n+1)}) = (-\frac{4n+3}{2^{n+1}} - \sum_{i=n+2}^{\infty}\frac{1}{2^i} + \frac{4n+2}{2^{n+1}}, y, 0) =$$
$$= (-\frac{1}{2^{n+1}} - \sum_{i=n+2}^{\infty}\frac{1}{2^i}, y, 0) = (-\frac{1}{2^n}, y, 0) = -z_n,$$
$$f(\frac{4n+3}{4n(n+1)}) = (\sum_{i=n+1}^{\infty}\frac{4n+3}{2^i} - \sum_{i=n+1}^{\infty}\frac{4n+4}{2^i}, y, 0) = (-\frac{1}{2^n}, y, 0) = -z_n.$$

This concludes the proof of continuity of f.

Theorem 3.8 is a direct consequence of the following theorem.

Theorem 3.9 *The loop f, as described above, is not homotopic to a constant map at x_0 in X.*

Proof of Theorem 3.9: First, let us try to visualize the loop f. It alternates between "circles" in X_1 and "circles" in X_2. It starts at x_0, runs along the top of the circles to z_n, loops once around the circle S_{n+1} in X_1 and returns to x_0 via the bottom of the circles. It then follows a similar pattern in X_2; it runs on top of the circles to $-z_n$, loops around the circle $-S_{n+1}$ in X_2 once, then returns to x_0 via the bottom of the circles. This repeats for every other circle (i.e. f loops around S_n in X_1 only for odd n, and in X_2 only for even n). Consider any homotopy $H : I \times I \to X$ of f to x_0. We will show that H is not continuous. Since $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{x_0\}$ we have that $A = H^{-1}(x_0)$ separates $H^{-1}(X)$ into components $\{C_i\}$ and $\{D_j\}$ where $C_i \subseteq H^{-1}(X_1)$ and $D_j \subseteq H^{-1}(X_2)$ for every i and every j.

Claim 1: There are infinitely many components C_i such that $C_i \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n odd or there are infinitely many components D_j such that $D_j \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n even.

Proof of Claim 1: Let us assume that there are finitely many components $\{C_i\}$. Then we know that at least one component C_i must contain infinitely many intervals $I_n = [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for n odd. Let $N \in \mathbb{N}$ and assume that $C_k \cap C_l = \emptyset$, where C_k contains infinitely many intervals I_n, $n \geq N$ odd and C_l contains infinitely many intervals I_n, $n \geq N$ odd. Clearly there exist $s, r, p \in \mathbb{N}$ such that $r < p < s$, $I_r, I_s \subseteq C_k$, and $I_p \subseteq C_l$. Since I_r and I_s are contained in C_k then C_k contains an arc, say L, joining the points $p_r \in int(I_r)$ and $p_s \in int(I_s)$. This means that there is a function $g : [0, 1] \to C_k$ such that $g(0) = p_r$ and $g(1) = p_s$. Consider the simple closed curve $J = L \cup (\{0, 1\} \times [-\frac{1}{2}, 1]) \cup (([0, p_s] \cup [p_r, 1]) \times \{-\frac{1}{2}\}) \cup ([0, 1] \times \{1\}) \cup (\{p_s, p_r\} \times [-\frac{1}{2}, 0]$. As a simple closed curve J separates the plane into two components, say Z_1 and Z_2. Without loss of generality let $I_p \subseteq Z_1$, then Z_2 contains the intervals I_n for $n > s$, n odd. Consider the sets $Z_1 \cap C_l$ and $Z_2 \cap C_l$. Since $Z_1 \cup Z_2 = \mathbb{R} \backslash J$ and $J \cap C_l = \emptyset$ we have $C_l \subseteq Z_1 \cup Z_2$. Therefore $(Z_1 \cap C_l) \cup (Z_2 \cap C_l) = C_l$. By connectedness of C_l either $Z_1 \cap C_l = \emptyset$ or $Z_2 \cap C_l = \emptyset$. Since $I_p \subseteq Z_1$ and $I_p \subseteq C_l$, $Z_1 \cap C_l \neq \emptyset$. Hence

$Z_2 \cap C_l = \emptyset$. However, $\bigcup_{n>s} I_n \subseteq Z_2$ for n odd, so C_l does not contain I_n for any $n > s$ odd. Therefore if there are finitely many components C_i, there is an i and $N \in \mathbb{N}$ such that C_i contains all intervals I_n for $n \geq N$, n odd. We know that for n even, $I_n \subseteq D_j$ for some j. Let $\bigcup_{n \geq N} I_n \subseteq C_i$ where n are odd. Let $p_n \in int(I_n)$ and let g_n be an arc from p_n to p_{n+2} in C_i. By assumption C_i contains all g_n for n odd. For a fixed $n \in \mathbb{N}$, $n \geq N$, let $J_n = \bigcup_{n \geq k \geq N} g_k \cup (\{0,1\} \times [-\frac{1}{2}, 1]) \cup ([0,1] \times \{1\}) \cup (([p_N, 1] \cup [0, p_{n+2}]) \times \{-\frac{1}{2}\}) \cup (\{p_N, p_{n+2}\} \times [-\frac{1}{2}, 0])$. J_n separates the plane since it is a simple closed curve. Say J_n separates the plane into components J_{1n} and J_{2n}. Consider a component D_l. Let $s, r \in \mathbb{N}$ such that $r > n > s > N$, r, s even and $I_s \subseteq D_l$. Assume that $I_s \subseteq J_{1n}$. Just as above $J_{1n} \cup J_{2n} = \mathbb{R} \backslash J_n$ and $J_n \cap D_l = \emptyset$ implies $D_l \subseteq J_{1n} \cup J_{2n}$. Therefore $(J_{1n} \cap D_l) \cup (J_{2n} \cap D_l) = D_l$. By connectedness of D_l either $J_{1n} \cap D_l = \emptyset$ or $J_{2n} \cap D_l = \emptyset$. We have that $I_s \subseteq J_{1n}$ and $I_s \subseteq D_l$, hence $J_{2n} \cap D_l = \emptyset$. Since $\bigcup_{k>n} I_k \subseteq J_{2n}$ for n even, D_l does not contain I_k for any $k > n$ even. Therefore each component D_j can contain at most finitely many I_n's for n even. The existence of infinitely many intervals I_n implies that there are infinitely many components D_j. A similar argument follows if we assume there are finitely many D_j's. End of Claim 1.

Claim 2: In every component C_i such that $C_i \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n odd, there is a point $(x, t_x) \in C_i$ such that $H(x, t_x) = p_1$ and in every component D_j such that $D_j \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n even, there is a point $(y, t_y) \in D_j$ such that $H(y, t_y) = p_2$.

Proof of Claim 2: Let us consider the collection $\{C_i\}$ where $C_i \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n odd. Assume there is an element of this collection, say C_k, such that $H(x, t_x) \neq p_1$ for all $(x, t_x) \in C_k$. Then consider the homotopy $F : I \times I \to X_1 \backslash \{p_1\}$, defined by

$$F(x,t) = \begin{cases} H(x,t) & \text{if } (x,t) \in \overline{C_k} \\ \\ x_0 & \text{if } (x,t) \notin C_k. \end{cases}$$

Since $([0,1] \times [0,1]\backslash C_k) \cap \overline{C_k} \subset A$ this function is continuous by the Pasting Lemma. We know that $C_k \supseteq \bigcup_{p \leq n \leq t} [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $p, t \in \mathbb{N}$. Let g be f restricted to $\bigcup_{p \leq n \leq t} [\frac{1}{n+1}, \frac{1}{n}]$. Let $\alpha_n : [0,1] \to [\frac{1}{n+1}, \frac{1}{n}]$ be a linear map and $a_n : [0,1] \to X$ defined as $a_n = f \circ \alpha_n$. We have shown in Chapter 1 that $f * (g * h) \simeq (f * g) * h$. This result can be generalized to any finite product of paths. Let us define $f_1 \overline{*} f_2 \overline{*} \ldots \overline{*} f_n$ as follows: On $[0, \frac{1}{n}]$ it equals the positive linear map of $[0, \frac{1}{n}]$ to $[0,1]$ followed by f_1; on $[\frac{1}{n}, \frac{1}{n-1}]$ it equals the positive linear map of $[\frac{1}{n}, \frac{1}{n-1}]$ to $[0,1]$ followed by f_2; continue the pattern and on $[\frac{1}{2}, 1]$ it equals the positive linear map of $[\frac{1}{2}, 1]$ to $[0,1]$ followed by f_n. By Step 2 of the proof of Theorem 51.2 by Munkres in [1], the path $e \overline{*} a_t \overline{*} e \overline{*} \ldots \overline{*} a_p \overline{*} e$ is homotopic to the product of the same paths with any parenthesis placements. Therefore we will write $h = e * a_t * e * \ldots * a_p * e$ without the parenthesis. Let us show that F is a homotopy of h to a constant map at x_0 in $X_1 \backslash \{p_1\}$. If $(x, 0) \in \overline{C_k}$ then $x \in \bigcup_{p \leq n \leq t} [\frac{1}{n+1}, \frac{1}{n}]$ and $F(x,0) = H(x,0) = f(x)$. If $(x,0) \notin \overline{C_k}$ then $F(x,0) = x_0$. Since $f(\frac{1}{n+1}) = f(\frac{1}{n}) = x_0$, we have that $F(x,0) = h$. If $(x,1) \in \overline{C_k}$ then $F(x,1) = H(x,1) = x_0$. Therefore $F(x,1) = x_0$. Lastly, $F(0,t) = x_0$ and $F(1,t) = x_0$. Since $H(x,t) \neq p_1$ for all $(x,t) \in C_k$, and $F(x,t) = x_0 \neq p_1$ for all other points, $F([0,1] \times [0,1]) \subseteq X_1 \backslash \{p_1\}$. Hence we have a homotopy of h to a constant map at x_0 in $X_1 \backslash \{p_1\}$; h is homotopic to g. However g is not homotopic to a constant map at x_0 in B_1. By Lemma 3.6, this implies that g is not homotopic to the constant map at x_0 in $X_1 \backslash \{p_1\}$. This causes a contradiction.

Therefore the homotopy F as described above cannot exist and hence every component C_i contains a point (x, t_x) such that $H(x, t_x) = p_1$. A similar argument shows the result for components D_j. End of Claim 2.

Claim 3: For every $i > 0$, let B_{ϵ_i} be an open ball of radius ϵ_i centered at (x_i, t_i), where (x_i, t_i) is a point in C_i such that $H(x_i, t_i) = p_1$, and $B_{\epsilon_i} \subseteq C_i$, $B_\epsilon \backslash C_i \neq \emptyset$ for every $\epsilon > \epsilon_i$. If the collection $\{C_i\}$ is infinite then $\lim_{i\to\infty} \epsilon_i = 0$. Similarly, if $B_{\epsilon_j} \subseteq D_j$ and the collection $\{D_j\}$ is infinite then $\lim_{j\to\infty} \epsilon_j = 0$.

Proof of Claim 3: If $\lim_{t\to\infty} \epsilon_i \neq 0$ then there is an $M \in \mathbb{R}$ such that $\epsilon_i \geq M$ for infinitely many i's. Consider the collection of open balls of radius ϵ_i, $\{B_{\epsilon_i}\}$, such that $B_{\epsilon_i} \subseteq C_i$. The area of each B_{ϵ_i}, say A_i, is greater than or equal to $T = \pi(M)^2$. Since there are infinitely many C_i's and $C_i \cap C_j = \emptyset$ for all $i \neq j$, there are infinitely many B_{ϵ_i}'s such that $B_{\epsilon_i} \cap B_{\epsilon_j} = \emptyset$ for all $i \neq j$. Therefore the area S of the square $[0, 1] \times [0, 1]$ is $S \geq \sum_{i=1}^{\infty} A_i \geq T \times \infty = \infty$. However, the area of the unit square is 1. This contradiction shows that the radius of the open balls contained in the sets C_i approaches 0 as i approaches infinity. The same argument works for the collection $\{D_j\}$. End of Claim 3.

Since H is a continuous function on a compact set $[0, 1] \times [0, 1]$ it is uniformly continuous. Therefore we have that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $(x, t), (y, t') \in [0, 1] \times [0, 1]$ are such that $d((x, t), (y, t')) < \delta$, then $d(H(x, t), H(y, t')) < \epsilon$. Let $\epsilon = \frac{1}{2}$, then there exists a $\delta > 0$ satisfying the above condition. If there are infinitely many $C_i \supseteq I_n$ for some $n \in \mathbb{N}$, n odd, then the collection $\{C_i\}$ is infinite. If there are infinitely many $D_j \supseteq I_n$ for some $n \in \mathbb{N}$, n even, then the collection $\{D_j\}$ is infinite. By Claim 1, either $\{C_i\}$ is infinite or $\{D_j\}$ is infinite. Without loss of generality, let $\{C_i\}$ be an infinite collection. Let $\epsilon_i < \delta$, $(x_i, t_i) \in B_{\epsilon_i} \subseteq C_i$ such that $H(x_i, t_i) = p_1$ and $(a, t_a) \in A \cap \overline{B_{\epsilon_i}}$. Then $d((x_i, t_i), (a, t_a)) \leq \epsilon_i < \delta$, and $d(H(x_i, t_i), H(a, t_a)) = d(p_1, x_0) < \frac{1}{2}$. This causes a contradiction, because p_1 and x_0 were defined in such a way that $d(p_1, x_0) > 1$. Therefore H is not continuous. Hence there does not exist a homotopy $H : I \times I \to X$ of f to x_0. This concludes the proof of Theorem 3.9. □

3.3 Second counterexample

3.3.1 Description of the space Y

In this section we will be dealing with a space that we will refer to as Y. This space was suggested in an Exercise in a book by Spanier [[2], p.59]. Let us first define the space carefully. Let W be defined as the union of circles C_n, where C_n has center $c_n = (\frac{1}{n(n+1)}, 0, 0)$ and radius $r_n = \frac{1}{n(n+1)}$ for all positive integers n. Y is the set of points on the closed line segments joining the point $q = (1, 0, 1)$ to W. Let $y_0 = (0, 0, 0)$. Another way to describe Y is as $X = \bigcup_{n \in N} Con(C_n) \cup C$ where

$$Con(C_n) = \{(x, y, z) | (x, y, z) = v_n t + q(1 - t) \text{ for } v_n \in V_n, t \in [0, 1]\},$$

where $V_n = \{(x, y, 0) | y = \pm\sqrt{-x^2 + \frac{2}{n(n+1)}x}\}$. In this notation $W = \bigcup_{n \in N} V_n \cup \{y_0\}$. Let

$$Con(C_n^+) = \{(x, y, z) | (x, y, z) = v_n^+ t + q(1 - t) \text{ for } v_n^+ \in V_n^+, t \in [0, 1]\},$$

where $V_n^+ = \{(x, y, 0) | y = \sqrt{-x^2 + \frac{2}{n(n+1)}x}\}$. Let

$$Con(C_n^-) = \{(x, y, z) | (x, y, z) = v_n^- t + q(1 - t) \text{ for } v_n^- \in V_n^-, t \in [0, 1]\},$$

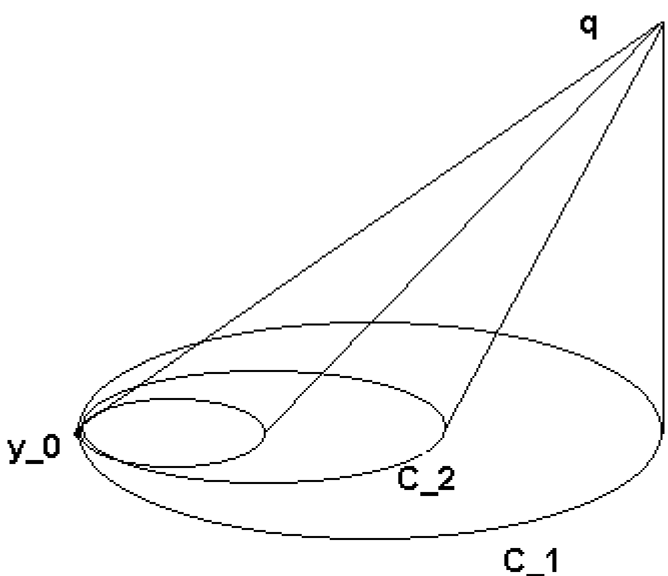

Figure 3.3: Space Y

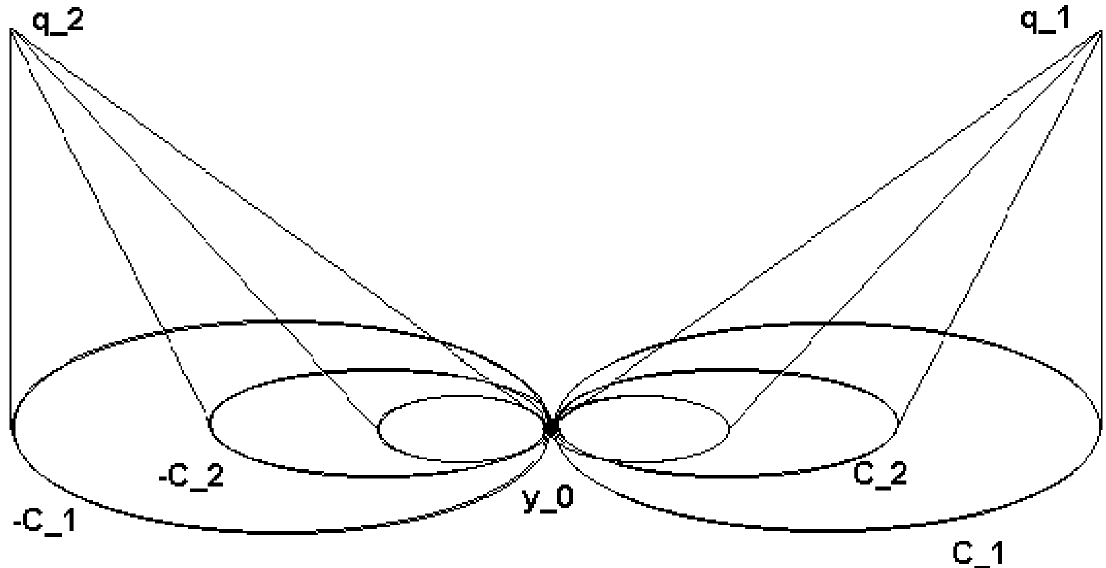

Figure 3.4: Union of two copies of Y

where $V_n^- = \{(x, y, 0)|y = -\sqrt{-x^2 + \frac{2}{n(n+1)}x}\}$.

Theorem 3.10 *If Y is the space defined in this section, and $y_0 = (0, 0, 0) \in Y$, then $\pi_1(Y, y_0)$ is trivial.*

The proof of Theorem 3.10 is exactly the same as the proof of Theorem 3.3.

By the argument in section 3.2.2, Y is semi 1-LC but not 1-LC.

3.3.2 The fundamental group of a one point union of two copies of Y

In this section we will consider the one point union of two copies of the space Y described above. We will call the first copy of this cone space Y_1 and the second copy Y_2. The space Y will be defined as $Y = Y_1 \cup Y_2$ where $Y_1 \cap Y_2 = \{y_0\}$, $y_0 = (0, 0, 0)$

is the limiting point as defined previously. The space Y_1 is the cone over W_1 and the space Y_2 is the cone over W_2. The points q_1 and q_2 are the "tips" of the spaces Y_1 and Y_2 respectively (i.e. $q_1 = (1, 0, 1)$ and $q_2 = (-1, 0, 1)$). Let $-c_n = (-\frac{1}{n(n+1)}, 0, 0)$ be the center of $-C_n$, where $-C_n$ is the n^{th} circle in Y_2.

Lemma 3.11 *The fundamental group of W_1 is isomorphic to the fundamental group of $Y_1 \backslash \{q_1\}$ and the fundamental group of W_2 is isomorphic to the fundamental group of $Y_2 \backslash \{q_2\}$.*

The proof of this Lemma is the same as for the space X.

Theorem 3.12 *Let Y be the space described above, then $\pi_1(Y, y_0) \neq 0$.*

Similarly to showing that $\pi_1(X, x_0) \neq 0$ in the previous subsection, it is enough to show that there is a loop f in Y based at y_0 and there is no homotopy between f and y_0 in Y. Consider a loop $f : I \to Y$ defined by:

$$f(0) = y_0 = (0, 0, 0)$$

$$f(s) = \begin{cases} (8s - \frac{8}{n+1}, y, 0) \;\; y \geq 0 & \text{for } \frac{1}{n+1} \leq s \leq \frac{4n+1}{4n(n+1)}, \\ (-8s + \frac{4(2n+1)}{n(n+1)}, y, 0) \;\; y \leq 0 & \text{for } \frac{4n+1}{4n(n+1)} \leq s \leq \frac{2n+1}{2n(n+1)}, \\ (8s - \frac{4(2n+1)}{n(n+1)}, y, 0) \;\; y \geq 0 & \text{for } \frac{2n+1}{2n(n+1)} \leq s \leq \frac{4n+3}{4n(n+1)}, \\ (-8s + \frac{8}{n}, y, 0) \;\; y \leq 0 & \text{for } \frac{4n+3}{4n(n+1)} \leq s \leq \frac{1}{n}, \end{cases}$$

for n odd, where $n = \ulcorner \frac{1}{s} \urcorner$, and $y = \pm\sqrt{-x^2 + \frac{2}{n(n+1)}x}$.

$$f(s) = \begin{cases} (-8s + \frac{8}{n+1}, y, 0) \;\; y \geq 0 & \text{for } \frac{1}{n+1} \leq s \leq \frac{4n+1}{4n(n+1)}, \\ (8s - \frac{4(2n+1)}{n(n+1)}, y, 0) \;\; y \leq 0 & \text{for } \frac{4n+1}{4n(n+1)} \leq s \leq \frac{2n+1}{2n(n+1)}, \\ (-8s + \frac{4(2n+1)}{n(n+1)}, y, 0) \;\; y \geq 0 & \text{for } \frac{2n+1}{2n(n+1)} \leq s \leq \frac{4n+3}{4n(n+1)}, \\ (8s - \frac{8}{n}, y, 0) \;\; y \leq 0 & \text{for } \frac{4n+3}{4n(n+1)} \leq s \leq \frac{1}{n}, \end{cases}$$

for n even, where n and y are defined as above. We need to show $f(0) = f(1) = y_0 = (0,0,0)$ and f is continuous. By definition $f(0) = y_0$. If $s = 1$, $n = 1$ so $f(1) = (8(1) - \frac{8}{1}, y, 0) = (0, y, 0) = (0,0,0) = y_0$. To show continuity we use the pasting lemma. We need to show the definition of f agrees on all the intersections. If $s = 0$, $\lim_{s \to 0^+} \frac{1}{s} = \infty$.

Consider $f(0) = (8(0) - \lim_{n\to\infty} \frac{8}{n}, y, 0) = (0, y, 0) = (0,0,0) = y_0$.

Now consider:

$$
\begin{aligned}
f(\frac{4n+1}{4n(n+1)}) &= (8(\frac{4n+1}{4n(n+1)}) - \frac{8}{n+1}, y, 0) = (\frac{2}{n(n+1)}, y, 0),\\
f(\frac{4n+1}{4n(n+1)}) &= (-8(\frac{4n+1}{4n(n+1)}) + \frac{4(2n+1)}{n(n+1)}, y, 0) = (\frac{2}{n(n+1)}, y, 0),\\
f(\frac{2n+1}{2n(n+1)}) &= (-8(\frac{2n+1}{2n(n+1)}) + \frac{4(2n+1)}{n(n+1)}, y, 0) = (0, y, 0) = y_0,\\
f(\frac{2n+1}{2n(n+1)}) &= (8(\frac{2n+1}{2n(n+1)}) - \frac{4(2n+1)}{n(n+1)}, y, 0) = (0, y, 0) = y_0,\\
f(\frac{4n+3}{4n(n+1)}) &= (8(\frac{4n+3}{4n(n+1)}) - \frac{4(2n+1)}{n(n+1)}, y, 0) = (\frac{2}{n(n+1)}, y, 0),\\
f(\frac{4n+3}{4n(n+1)}) &= (-8(\frac{4n+3}{4n(n+1)}) + \frac{8}{n}, y, 0) = (\frac{2}{n(n+1)}, y, 0).
\end{aligned}
$$

So the function is continuous for n odd. Since $f(\frac{1}{n}) = (0,0,0)$ for all $n \in \mathbb{Z}$ the function stays continuous during the change from odd to even and vice versa. Lastly, let us show f is continuous for n even:

$$
\begin{aligned}
f(\frac{4n+1}{4n(n+1)}) &= (-8(\frac{4n+1}{4n(n+1)}) + \frac{8}{n+1}, y, 0) = (-\frac{2}{n(n+1)}, y, 0),\\
f(\frac{4n+1}{4n(n+1)}) &= (8(\frac{4n+1}{4n(n+1)}) - \frac{4(2n+1)}{n(n+1)}, y, 0) = (-\frac{2}{n(n+1)}, y, 0),\\
f(\frac{2n+1}{2n(n+1)}) &= (8(\frac{2n+1}{2n(n+1)}) - \frac{4(2n+1)}{n(n+1)}, y, 0) = (0, y, 0) = y_0,\\
f(\frac{2n+1}{2n(n+1)}) &= (-8(\frac{2n+1}{2n(n+1)}) + \frac{4(2n+1)}{n(n+1)}, y, 0) = (0, y, 0) = y_0,\\
f(\frac{4n+3}{4n(n+1)}) &= (-8(\frac{4n+3}{4n(n+1)}) + \frac{4(2n+1)}{n(n+1)}, y, 0) = (-\frac{2}{n(n+1)}, y, 0),\\
f(\frac{4n+3}{4n(n+1)}) &= (8(\frac{4n+3}{4n(n+1)}) - \frac{8}{n}, y, 0) = (-\frac{2}{n(n+1)}, y, 0).
\end{aligned}
$$

This concludes the proof of continuity of f.

I claim that the loop f as described above is not homotopic to y_0 in Y. First, let us try to visualize the loop f. It alternates between "circles" in Y_1 and "circles" in Y_2. It starts at y_0, loops around a circle in Y_1 twice in clockwise direction and returns to y_0. It then follows a similar pattern in Y_2, it loops around a circle in X_2 twice in counter-clockwise direction, then returns to y_0. This repeats for every other circle (i.e. f loops around C_n in Y_1 only for odd n, and in Y_2 only for even n). Consider any homotopy $H : I \times I \to Y$ of f to y_0. We will show that H is not continuous. Since $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \{y_0\}$ we have that $A = H^{-1}(y_0)$ separates $H^{-1}(Y)$ into components $\{Q_i\}$ and $\{D_j\}$ where $Q_i \subseteq H^{-1}(Y_1)$ and $D_j \subseteq H^{-1}(Y_2)$ for every i and every j.

It was shown for the first counterexample that there are either infinitely many components Q_i such that $Q_i \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n odd or infinitely many components D_j such that $D_j \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n even. The same argument is true in the space Y.

It was also shown that in every component Q_i such that $Q_i \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n odd, there is a point $(x, t_x) \in Q_i$ such that $H(x, t_x) = q_1$ and in every component D_j such that $D_j \supseteq [\frac{1}{n+1}, \frac{1}{n}] \times \{0\}$ for some $n \in \mathbb{N}$, n even, there is a point $(y, t_y) \in D_j$ such that $H(y, t_y) = q_2$. The argument is the same as for the space X. The following shows that $\pi_1(W_1, y_0) \neq 0$. We know that $\pi_1(C_n, y_0) \neq 0$ for every $n \in N$. Fix $n = n_x$. Let us show that C_n is a strong deformation retract of W_1, then by Step 1 of the previous claim $\pi_1(W_1, y_0) \neq 0$. First, let us define maps $r_m(x, t)$ and $r'_m(x, t)$ as follows. Let $x \in C_m$ for some $m \leq n$, let $x' \in C_n$ be the point on the straight line segment $x(1-s)+c_n s$ for $s \in [0, 1]$. Define $r_m : C_m \times I \to C_n$ by $r_m(x, t) = x't+x(1-t)$.

Let $x \in C_m$ for some $m \geq n$, let $x' \in C_n$ be the point on the straight line constructed by extending the segment $xs + c_m(1-s)$ for $s \in [0,1]$. Define $r'_m : C_m \times I \to C_n$ by $r'_m(x,t) = x't + x(1-t)$. Clearly $r_n(x,t) = r'_n(x,t) = x$ for all $t \in [0,1]$. Both r_m and r'_m are continuous maps for each $m \in N$, since they are a sum of continuous functions. Also,

$$r_m(x,0) = x \in C_m,$$

$$r'_m(x,0) = x \in C_m,$$

$$r_m(x,1) = x' \in C_n,$$

$$r'_m(x,1) = x' \in C_n.$$

Consider the map $F : W_1 \times I \to W_1$ defined by:

$$F(x,t) = \begin{cases} x & \text{if } x \in C_n, \\ r_m(x,t) & \text{if } x \in C_m \quad for \quad m \leq n, \\ r'_m(x,t) & \text{if } x \in C_m \quad for \quad m \geq n. \end{cases}$$

Since $r_n(x,t) = r'_n(x,t) = x$ for all $t \in [0,1]$ and $F(x,t) = x$, $F(x,t) = r_m(x,t)$, $F(x,t) = r'_m(x,t)$ are continuous, by the Pasting Lemma this is a continuous map. Let us verify it is a strong deformation retraction.

$$F(a,t) = a \text{ for } a \in C_n,$$

$$F(W_1, 1) = C_n,$$

$$F(x, 0) = x \text{ for } x \in W_1.$$

Since C_n is a strong deformation retract of W_1, we have that $\pi_1(W_1, y_0) \approx \pi_1(C_n, y_0) \neq 0$. Just as in the case of the space X, the above results imply that f is not homotopic to the constant map at x_0 in X. This concludes the proof of Theorem 3.12.

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