A Brief Exploration of the Sorgenfrey Line and the Lexicographic Order

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A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
May 11, 2006

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Vita

Regina Marie Greiwe, daughter of Michael Greiwe and Karen Broome, was born February 4, 1982. She attended Tri-County High School in Buena Vista, Georgia where she graduated fourth in her class in June, 1999. She then entered Columbus State University under an Honors scholarship where she graduated Magna cum Laude in June, 2003. She was awarded with a Bachelor of Science degree in Applied Mathematics. While at Columbus State University, she was president of the local chapter of the MAA and was awarded the Mathematics Award in 2003. The following year she entered Auburn Univerty to pursue a Master of Science degree in Mathematics. While at Auburn University, she served as president of the local chapter of SIAM and has been involved in Science Olympiad and mathematics education. She is the mother of Laura Jeanette Amilia Greiwe.

Thesis Abstract

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Master of Science, May 11, 2006
(B.S., Columbus State University, 2003)

45 Typed Pages

Directed by Jo Heath

The author will compare properties of the usual topology on the real line with the properties of the Sorgenfrey line. After that she will look at properties of the Lexicographic order on $\mathbb{R} \times [0, 1]$. The main properties of interest will be the separation properties, along with compact and connected subsets.

Acknowledgments

I would like to thank my family for helping me take care of "life's little bumps" so that I could concentrate on my work. I would also like to thank my topology professors for providing me with inspiration along the way.

I would like to thank my advisor for her patience with every attempt at understanding that I presented to her. Because of her unwavering ability to not give away the answers, I have made an expansive growth in my mathematical maturity. I would also like to thank you, the reader, for taking the time to look at my thoughts and work.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

Table of Contents

Chapter 1

Introduction

In each area of topology, there are interesting spaces that many people are familiar. In this paper, I will be presenting two such spaces that I took a personal interest in.

When I began my research, I was very new to the study of topology in general. After a semester in my first topology course, I began my work by investigating some basic ideas about different topological properties. From here I was given eight different spaces to look into. All that was given to me was a list of eight sets and a base for each. In the beginning I did not know what any of these spaces were called, or even know that they had a name. What you are about to read are the conjectures and proofs that I came up with for two of these spaces.

Many of the theorems that you will see were inspired by work I had done in classes and from colloquia I have attended. During the research portion of my study, I assumed the well known properties of the real line with the usual topology without proof. Other than that, any ideas that I may have needed, I personally proved. For the sake of the reader, many of the proofs of the easier facts have been left out.

Chapter 2 may be used as a reference chapter. In it the reader will find common definitions and statements. Included with these will be proofs of theorems that will be used throughout the other two chapters. Many of these came from the initial fundamental research I did, along with general assignments given in different courses. I believe that some of the ideas used in these proofs are interesting concepts in themselves.

In chapter 3, the reader will encounter a brief investigation of the Sorgenfrey topology on the real line. For this space, I was forced to break away from the preconceived notions that I had had about the real line and intervals in general. Even though many properties are shared by the usual topology and the Sorgenfrey topology, I feel that the interesting part is found when the two are contrasted.

The Lexicographic Order is the subject of chapter 4. I would say that out of the eight spaces given to me, this one is my favorite. I began completely lost on any idea as to what to do with this space. I struggled throughout my research to come to many of the conclusions that are contained in this chapter, yet I feel that everything that I learned has come together to describe this space.

As you read through this paper, I hope that you will find something that you enjoy and maybe something that you find unfamiliar.

Chapter 2

Fundamental Definitions and Theorems

The following are definitions and proofs of statements which will be needed in other parts of the paper. The first two theorems were given as homework problems in my undergraduate level general topology course taught by Dr. Jo Heath.

Definition 2.1. *A space X is said to be* ***regular*** *if for each $x \in X$ and each closed set, $M \subset X$ such that $x \notin M$, there exist disjoint open sets U and V such that $x \in U$ and $M \subseteq V$.*

Theorem 2.1. *A space X is regular if and only if for each $x \in X$ and U open in X such that $x \in U$, there exists V open in X such that $x \in V \subseteq \overline{V} \subseteq U$.*

Proof. Suppose X is a regular topological space. Let $x \in X$ and U be open in X such that $x \in U$. Then U^c is a closed subset of X such that $x \notin U^c$. Thus we can find U_0 and V_0 open subsets of X such that $x \in U_0$, $U^c \subseteq V_0$, and $U_0 \cap V_0 = \emptyset$. Let $V = U_0$. Then $x \in V \subseteq \overline{V} \subseteq V_0^c \subseteq U$.

Now we must prove the reverse implication. So, suppose that for all $x \in X$ and U open in X such that $x \in U$, there exists V open in X such that $x \in V \subseteq \overline{V} \subseteq U$. Let $x \in X$ and M be a closed subset of X such that $x \notin M$. Then $x \in M^c$ and M^c is open, which implies that there exists V open in X such that $x \in V \subseteq \overline{V} \subseteq M^c$. Let $U = \overline{V}^c$. Then U, and V are open sets such that $M \subseteq U$, $x \in V$, and $U \cap V = \overline{V}^c \cap V = \emptyset$. □

Definition 2.2. *Nonempty subsets A and B of a space X* ***separate*** *X if $X = A \cup B$ and $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. A space is* ***connected*** *if there do not exist subsets A and B which separate X.*

Theorem 2.2. *A space is not connected iff it contains a proper clopen subset.*

Proof. Let X be a topological space. If we assume that X contains a proper clopen subset, K, then K and K^c are both nonempty clopen subsets of X. Also, $X = K \cup K^c$. So K and K^c separate X. Hence X is not connected.

Conversely, let's assume that X is not connected. Then there exist A and B, nonempty subsets of X such that $X = A \cup B$ and $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. Hence $X = A \cup B = \overline{A} \cup B$, which implies that $B^c = \overline{A}$. So B is open. Similarly, $X = A \cup \overline{B}$ implies that $B = \overline{B}$. Thus B is closed. Therefore B is a proper clopen subset of X. $\square$

The next six theorems are just a few of the things that were proven during my preliminary research. They give alternate definitions for some topological terms based on basic open sets, and extend certain properties to subspaces. Let us first define a basis for a space.

Definition 2.3. *Let X be a set, and let $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is called a* ***basis****, or* ***base****, for X if $\mathcal{B}$ satisfies the following conditions.*

1. *$\mathcal{B}$ covers X (i.e., $X = \bigcup \mathcal{B}$).*

2. *Whenever $B_1 \cap B_2 \neq \emptyset$ for B_1, $B_2 \in \mathcal{B}$, there exists $B_x \in \mathcal{B}$ for each $x \in B_1 \cap B_2$ such that $x \in B_x \subseteq B_1 \cap B_2$.*

   If X is a topological space, then we need a third condition.

3. *Each open set U can be written as a union of elements of $\mathcal{B}$.*

An element of $\mathcal{B}$ is called a ***basic open set****.*

Definition 2.4. *A space is said to be* ***zero-dimensional*** *if it has a basis consisting of clopen sets.*

Theorem 2.3. *A collection of open sets $\mathcal{B}$ is a base for a space X iff for each $x \in X$ and for each open neighborhood U of x, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$.*

Proof. Let X be a topological space, and let $\mathcal{B}$ be a collection of open sets. Suppose that $\mathcal{B}$ is a base for X. Let $x \in X$, and let U be an open neighborhood of x. Then $U = \bigcup_{i \in I} B_i$ where $B_i \in \mathcal{B}$ for each $i \in I$. So there exists $i \in I$ such that $x \in B_i$, and $B_i \subseteq U$.

Conversely, suppose that for each $x \in X$ and for each open neighborhood U of x, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then obviously $\mathcal{B}$ covers X and for each U open $U = \bigcup_{x \in U} B_x$. Now suppose that $x \in B_1 \cap B_2$ for some B_1 and $B_2 \in \mathcal{B}$. Then $B_1 \cap B_2$ is open which implies that there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq B_1 \cap B_2$. Therefore $\mathcal{B}$ is a base for X. □

Definition 2.5. *A space is called a* ***Hausdorff space*** *if for each pair of distinct points x and y, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.*

Theorem 2.4. *A space X is a Hausdorff space iff for every pair of distinct points, x and $y \in X$, there exist disjoint basic open sets B_x and B_y such that $x \in B_x$ and $y \in B_y$.*

Proof. Let X have a basis $\mathcal{B}$, and let x and $y \in X$ be distinct. Suppose that X is a Hausdorff space. Then there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. We know that $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} B_j$ where B_i and $B_j \in \mathcal{B}$ for each $i \in I$ and $j \in J$. Thus there exists an $i \in I$ and a $j \in J$ such that $x \in B_i$ and $y \in B_j$. Since U and V are disjoint, B_i and B_j are also disjoint.

Now suppose that for every pair of distinct points, x and $y \in X$, there exist disjoint basic open sets B_x and B_y such that $x \in B_x$ and $y \in B_y$. Since B_x and B_y are open, we clearly have that X is a Hausdorff space. □

Definition 2.6. *A space is **compact** provided that every open cover of the space has a finite subcover. A space is **Lindelöf** provided that every open cover of the space has a countable subcover.*

Theorem 2.5. *A space X is compact (respectively Lindelöf) iff for every open cover, $\mathcal{G}$, consisting of basic open sets, there is a finite (respectively countable) subcover of $\mathcal{G}$.*

Proof. Let X have a basis $\mathcal{B}$. Suppose X is compact, and let $\mathcal{G}$ be an open cover consisting of basic open sets. Then by the definition of compact, there exists a finite subcover of $\mathcal{G}$.

Now suppose that for every open cover, $\mathcal{G}$, consisting of basic open sets, there is a finite subcover of $\mathcal{G}$. Let $\mathcal{H}$ be an arbitrary open cover of X. Then for each $x \in X$, choose $U_x \in \mathcal{H}$ such that $x \in U_x$. Since $\mathcal{B}$ is a base for X, there exists $B_x \in \mathcal{B}$ such that for each $x \in X$, $x \in B_x \subseteq U_x$. Let $\mathcal{G}_B = \{B_x : x \in X\}$. Then there exists $N \in \mathbb{N}$ such that $\widehat{\mathcal{G}}_B = \{B_{x_n} : n \leq N\}$ is a subcover of $\mathcal{G}_B$. Let $\widehat{\mathcal{G}} = \{U_{x_n} : n \leq N\}$. Then $\widehat{\mathcal{G}}$ is a subcover of $\mathcal{G}$. Therefore X is compact.

This same proof can be used to show that a space is Lindelöf with a few alterations.

□

When showing compactness, the following property is very useful.

Definition 2.7. *A collection of sets, $\mathcal{C}$, is said to have the **finite intersection property** provided that every finite subcollection has nonempty intersection.*

Theorem 2.6. *A space X is compact iff every collection of closed subsets with the finite intersection property has nonempty intersection.*

Proof. Suppose that X is compact. Let $\mathcal{C}$ be a collection of closed subsets of X with the finite intersection property. Suppose that $\bigcap \mathcal{C}\mathcal{G} = \emptyset$. Define a collection $\mathcal{G} = \{K^c :$

$K \in \mathcal{C}\}$. Notice that $\mathcal{G}$ is an open cover of X. Hence there exists $n \in \mathbb{N}$ and some K_1^c, ..., $K_n^c \in \mathcal{G}$ such that $\bigcup_{i=1}^n K_i^c = X$. Thus $\bigcap_{i=1}^n K_i = (\bigcup i = 1^n K_i^c)^c = \emptyset$. This is a contradiction because $\mathcal{C}$ has the finite intersection property.

Now conversely assume that every collection of closed subsets of X with the finite intersection property has nonempty intersection. Let $\mathcal{G}$ be an open cover of X. Suppose that it does not have a finite subcover. Define the collection $\mathcal{C} = \{U^c : U \in \mathcal{G}\}$. Then $\mathcal{C}$ is a collection of closed subsets with the finite intersection property. So we have that $\bigcap \mathcal{C} \neq \emptyset$. Thus $\bigcup \mathcal{G} = (\bigcap \mathcal{C})^c \neq X$, a contradiction since $\mathcal{G}$ covers X. □

Theorem 2.7. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point x in a space X iff for each basic open set B_x containing x, B_x also contains x_n for all but finitely many $n \in \mathbb{N}$.*

Proof. Let X be a space with a basis $\mathcal{B}$. Suppose that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in X$. Then for every open set U such that $x \in U$, $x_n \in U$ for all but finitely many $n \in N$. Thus obviously for each $B_x \in \mathcal{B}$ such that $x \in B_x$, $x_n \in B_x$ for all but finitely many $n \in N$.

Conversely, suppose that for each basic open set B_x containing x, B_x contains x_n for all but finitely many $n \in \mathbb{N}$. Let U be an open neighborhood of x. Then there exist $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. So by the hypothesis, $x_n \in B_x \subseteq U$ for all but finitely many $n \in \mathbb{N}$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ converges to x. □

Definition 2.8. *A function $f : X \to Y$ from a space X to a space Y is* ***continuous*** *if $f^{-1}(V)$ is open in X for every V open in Y.*

Theorem 2.8. *Let $f : X \to Y$ be a function from a space X to a space Y. Then f is continuous iff for each basic open set $B_Y \subseteq Y$, $f^{-1}(B_Y)$ is open in X.*

Proof. Let $f : X \to Y$ be a function from a space X to a space Y. Suppose that f is continuous. Then obviously for each basic open set $B_Y \subseteq Y$, $f^{-1}(B_Y)$ is open in X.

Conversely, suppose that for each basic open set $B_Y \subseteq Y$, $f^{-1}(B_Y)$ is open in X. Let V be open in Y. Then $V = \bigcup_{i \in I} B_i$ where B_i is a basic open set in Y for each $i \in I$. Then $f^{-1}(V) = f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$ which is open in X. Therefore f is continuous. □

Definition 2.9. *A* ***subspace*** *of a space X is a subset A of X with the subspace topology, $T|_A$ defined by $T|_A = \{U \cap A : U \text{ open in } X\}$.*

Definition 2.10. *A property C is said to be* ***hereditary*** *provided that every subspace has property C whenever the whole space has property C. A space is said to have a property* ***hereditarily,*** *if every subspace has the property.*

Theorem 2.9. *The properties of regularity and Hausdorff are hereditary.*

Proof. Assume that X is a Hausdorff space and that A is a subspace of X. Let a and $b \in A$ be distinct. Since $A \subset X$, there exist U and V disjoint open sets in X such that $a \in U$ and $b \in V$. Let $\widehat{U} = U \cap A$ and $\widehat{V} = V \cap A$. Then $\widehat{U}$ and $\widehat{V}$ are disjoint open subsets of A such that $a \in \widehat{U}$ and $b \in \widehat{V}$. Therefore A is a Hausdorff space.

Now let X be regular, and let M be a nonempty closed subset of A. Let $a \in A$ such that $a \notin M$. Then notice that $A \setminus M$ is open. So there exists U open in X such that $A \setminus M = U \cap A$. Then $M = A \setminus (A \setminus M) = A \cap (U \cap A)^c = A \cap U^c$. Let $K = U^c$. Now notice $M = K \cap A$, K is closed in X, and $a \notin K$. This implies that there exists U and V disjoint open subsets of X such that $a \in U$ and $K \subseteq V$. Let $\widehat{U} = U \cap A$ and $\widehat{V} = V \cap A$. Then $\widehat{U}$ and $\widehat{V}$ are disjoint open subsets of A such that $a \in \widehat{U}$ and $M = K \cap A \subseteq V \cap A = \widehat{V}$. Therefore A is regular. □

The next two theorems were very beneficial in looking into normality for the following two chapters. They were introduced as homework in Dr. Gary Gruenhage's course, Set Theoretic Topology.

Lemma 2.1. *Let K and M be disjoint subsets of a space X. Suppose there exist collections $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ of open sets satisfying the following:*

1. *$M \subseteq \bigcup_{n\in\mathbb{N}} U_n$ and $K \subseteq \bigcup_{n\in\mathbb{N}} V_n$;*
2. *for each $n \in \mathbb{N}$, $\overline{U}_n \cap K$ is empty; and,*
3. *for each $n \in \mathbb{N}$, $\overline{V}_n \cap M$ is empty.*

Then there exist disjoint open sets U and V such that $M \subseteq U$ and $K \subseteq V$.

Proof. Let K and M be disjoint subsets of a space X. Suppose there exist collections $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ of open sets satisfying the following:

1. $M \subseteq \bigcup_{n\in\mathbb{N}} U_n$ and $K \subseteq \bigcup_{n\in\mathbb{N}} V_n$;
2. for each $n \in \mathbb{N}$, $\overline{U}_n \cap K$ is empty; and,
3. for each $n \in \mathbb{N}$, $\overline{V}_n \cap M$ is empty.

Let $\widehat{U}_n = U_n \setminus (\bigcup_{k\leq n} \overline{V}_k)$ and $\widehat{V}_n = V_n \setminus (\bigcup_{k\leq n} \overline{U}_k)$. Notice that for each $n \in \mathbb{N}$, $\widehat{U}_n$ and $\widehat{V}_n$ are open.

Now we claim that for each i and $j \in \mathbb{N}$, $\widehat{U}_i$ and $\widehat{V}_j$ are disjoint. Suppose that $i = j$. Notice that $\widehat{U}_i \cap V_i$ is empty. Thus $\widehat{U}_i \cap \widehat{V}_i$ is empty. Now suppose that $i < j$ and that $\widehat{U}_i \cap \widehat{V}_j$ is nonempty. Then there exists a point $x \in \widehat{U}_i \cap \widehat{V}_j$ which implies that $x \in \widehat{U}_i$ and $x \in \widehat{V}_j$. Thus $x \in U_i$ and $x \notin \overline{U}_i$, a contradiction. Hence $\widehat{U}_i \cap \widehat{V}_j$ is empty. Similarly if $i > j$, we have that $\widehat{U}_i \cap \widehat{V}_j$ is empty.

Let $U = \bigcup_{n \in \mathbb{N}} \widehat{U}_n$ and $V = \bigcup_{n \in \mathbb{N}} \widehat{V}_n$. By the preceding claim, U and V are disjoint open sets. Since $M \subseteq \bigcup_{n \in \mathbb{N}} U_n$, we have that $M \subseteq U$. Similarly, $K \subseteq V$. □

Definition 2.11. *A space X is **normal** provided that for each pair of disjoint closed sets H and K, there exist disjoint open sets U and V such that $H \subseteq U$ and $K \subseteq V$.*

Theorem 2.10. *Every regular Lindelöf space is normal.*

Proof. Assume that X is a regular Lindelöf space. Let H and K be nonempty disjoint closed subsets of X. Then for each $h \in H$, there exists U_h and V_h disjoint open subsets of X such that $h \in U_h$ and $K \subseteq V_h$. Notice that this implies that $\overline{U}_h \cap K$ is empty. Now $\{U_h : h \in H\}$ is an open cover of H. So there exist $h_n \in H$ for each $n \in \mathbb{N}$ such that $H \subseteq \bigcup_{n \in \mathbb{N}} U_{h_n}$. Recall that for each $n \in \mathbb{N}$, $\overline{U}_{h_n} \cap K$ is empty.

Similarly, we can find open sets V_{k_n} for some point $k_n \in K$ for each $n \in \mathbb{N}$ such that $K \subseteq \bigcup_{n \in \mathbb{N}} V_{k_n}$ and $\overline{V}_{k_n} \cap H$ is empty. By Lemma 2.1, there exist U and V disjoint open subsets of X such that $H \subseteq U$ and $K \subseteq V$. Therefore X is normal. □

We have finished proving the general statements that will be used throughout this paper. Let us define a few more terms with which the reader may not be familiar.

Definition 2.12. *A set is called **degenerate** if it contains only a single element.*

Definition 2.13. *A subset D of a space X is **dense** in X provided that $\overline{D} = X$. In other words, D is dense in X if for each open set U, $U \cap X$ is nonempty.*

Definition 2.14. *A space is **separable** if it contains a countable dense subset.*

Definition 2.15. *Let X be a set. A function $d : X \times X \to [0, \infty)$ is a **metric** provided that for each x, y, and $z \in X$*

1. $d(x, y) = 0$ *iff* $x = y$,

2. $d(x, y) = d(y, x)$, *and*

3. $d(x, z) \leq d(x, y) + d(y, z)$.

Define an ***open ball*** *centered at* x *with radius* e, *denoted* $B(x, e)$, *as the collection of all elements,* $y \in X$, *such that* $d(x, y) < e$. *A space* (X, τ) *is said to be* ***metrizable*** *if there exists a metric on* X *which can be used to build a basis consisting of open balls which generates* τ.

Definition 2.16. *Let* $f : X \to Y$ *be a continuous bijection from a space* X *to a space* Y. *Then* f *is called a* ***homeomorphism*** *if* f^{-1} *is also continuous. We say that a space* X *is* ***homeomorphic*** *to a space* Y *if there exists a homeomorphism* $f : X \to Y$.

Now that we have gotten familiar with a few terms and general theorems, let us move on to our first space.

Chapter 3

The Sorgenfrey Line

What would happen to the real line if we gave it a finer topology than the usual topology? Would it lose any of its properties? Would it gain any? These are the questions we will explore in this chapter.

Let S be the real line. Define a collection $\mathcal{B}$ by $\mathcal{B} = \{(a, b] \subseteq S : a < b\}$. Notice that this collection of intervals covers S. Also if we note that the intersection of two intervals of the form $(a, b]$ is another interval of that form or the empty set, then it is easy to see that $\mathcal{B}$ is a basis for S. Let T be the topology generated by $\mathcal{B}$. In other words, let T be the collection of all unions of elements of $\mathcal{B}$. We call the topological space (S, T) the Sorgenfrey line. In particular, since the intervals are closed on the right, T is known as the right Sorgenfrey topology. If instead we use intervals closed on the left, then it would be called the left Sorgenfrey topology.

We will discover that (S, T) satisfies most of the separation properties. Let us first show that this is a finer topology than the usual topology. For one topology, τ_1, to be finer than another topology, τ_2, we must show that τ_2 is a subset of τ_1.

Theorem 3.1. *Any open interval in the usual topology on S is open in the Sorgenfrey topology.*

Proof. Let (a, b) be a nonempty subset of S. Then $(a, b) = \bigcup_{n \in \mathbb{N}} \left(a, b - \frac{1}{n}\right]$ is an element of T since it is a union of basic open sets. □

One of the bases for the usual topology is the collection of all open intervals in S. By Theorem 3.1, each open interval is open in the Sorgenfrey topology, T. So T is a finer than the usual topology. What does this tell us? The properties of regularity and of being a Hausdorff space are preserved by finer topologies. Thus (S, T) is a regular Hausdorff space. Notice that a half closed interval such as $(0, 1]$ cannot be written as the union of open intervals. So the Sorgenfrey topology on S is strictly finer than the usual topology.

We will now show that any dense subset of S with the usual topology, is also dense in S with the Sorgenfrey topology. In particular, the rationals would be dense in (S, T). So this would imply that (S, T) is separable.

Theorem 3.2. *Let D be a dense subset of S with the usual topology. Then D is dense in S with the Sorgenfrey topology.*

Proof. Let τ denote the usual topology on S, and let D be dense in (S, τ). Choose an arbitrary interval, $(a, b] \in \mathcal{B}$. Now take note that every basic open set $(a, b] \in \mathcal{B}$ contains the open interval $(a, \frac{a+b}{2})$. So since D is dense in (S, τ), there exists an element of D which we will call $d \in D \cap (a, \frac{a+b}{2})$. Hence $d \in (a, b]$ which implies that $D \cap (a, b]$ is nonempty. Therefore D is dense in (S, T). □

Next we will consider the countability properties. Obviously the two spaces share the same cardinality, since their underlying sets are the same. What about the first and second countability properties? We know that the usual topology has both properties. How about the Sorgenfrey topology?

Theorem 3.3. *(S, T) is first countable.*

Proof. To show that the Sorgenfrey line is first countable, we must show that each point has a countable local base. Suppose $x \in S$. Let $G_x = \{(p, x] : p < x,\, p \in \mathbb{Q}\}$. Then G_x is countable. Now we will show that G_x is a local base at x. Let U be an open neighborhood of x. Then there is a basic open set, $(a,b] \in \mathcal{B}$ such that $x \in (a,b] \subseteq U$. Thus $a < x \leq b$ which gives us a $p \in \mathbb{Q}$ such that $a < p < x \leq b$. Hence $x \in (p,x] \subset (a,b] \subseteq U$ and $(p,x] \in G_x$. □

Theorem 3.4. *The Sorgenfrey line is not second countable.*

Proof. Suppose that (S,T) is second countable. Then (S,T) has a countable basis which we will call $\mathcal{A}$. We will use the usual basic open sets with irrational endpoints to contradict the fact that $\mathcal{A}$ is countable. Let $V = \{(x-1,x] : x \in \mathbb{Q}^c\} \subseteq T$. Since $\mathcal{A}$ is a basis, for each $x \in \mathbb{Q}^c$ we can choose $A_x \in \mathcal{A}$ such that $x \in A_x \subseteq (x-1,x]$. Let $\widehat{\mathcal{A}} = \{A_x : x \in \mathbb{Q}^c\} \subseteq \mathcal{A}$.

Now we will show that there exists an injective function from the irrationals to $\widehat{\mathcal{A}}$. Define a function $f : \mathbb{Q}^c \to \widehat{\mathcal{A}}$ by $f(x) = A_x$. Suppose that f is not injective. Then there are points x, $y \in \mathbb{Q}^c$ such that $x \neq y$ and $f(x) = f(y)$. We may assume that $x < y$. Notice that $f(x) = f(y)$ implies that $A_x = A_y$, but $x < y$ implies that $y \notin A_x$; a contradiction since $y \in A_y$. Therefore f is injective.

Since there is an injective function from $\mathbb{Q}^c$ to $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}} \subseteq \mathcal{A}$, we have that $|\mathbb{Q}^c| \leq |\widehat{\mathcal{A}}| \leq |\mathcal{A}|$. Thus $\mathcal{A}$ is uncountable, a contradiction. □

Interestingly enough, the Sorgenfrey line does not share the second countability property with the usual topology. This will give rise to a very important difference between the two spaces. The reals with the usual topology is a metric space. We will see that the Sorgenfrey Line is not metrizable, using the fact that it is separable, but not second countable. First we will show that a separable metric space is second countable.

Lemma 3.1. *Every separable metric space is second countable.*

Proof. Assume X is a separable metric space. Let d be a metric on X, and D be a countable dense subset of X. Let $\mathcal{B} = \{B(x, e) : x \in X,\ e > 0\}$ which is a basis for a metric space, and let $\mathcal{B}_0 = \{B(x, e) : x \in D,\ e \in \mathbb{Q},\ e > 0\}$. Then $\mathcal{B}_0$ is countable, and $\mathcal{B}_0 \subseteq \mathcal{B}$. Now we will show that $\mathcal{B}_0$ is a basis for X. Let U be an open neighborhood of $x \in X$. Then there is a $B(x_0, e_0) \in \mathcal{B}$ such that $x \in B(x_0, e_0) \subseteq U$. Thus there exists $e_1 > 0$ such that $B(x, e_1) \subseteq B(x_0, e_0)$. Since D is dense in X, there is a $p \in D$ such that $p \in B\left(x, \frac{e_1}{4}\right)$. We know there is an $e \in \mathbb{Q}$ such that $\frac{e_1}{4} < e < \frac{3e_1}{4}$. Hence $B(p, e) \in B_0$. To show that $\mathcal{B}_0$ is a basis, we will show that $x \in B(p, e)$ and that $B(p, e) \subseteq U$.

Claim 1: $x \in B(p, e)$

$d(p, x) = d(x, p) < \frac{e_1}{4} < e \Rightarrow x \in B(p, e)$.

Claim 2: $B(p, e) \subseteq B(x, e_1)$

Let $y \in B(p, e)$. Then $d(p, y) < e$, and

$d(x, y) \leq d(x, p) + d(p, y) < \frac{e_1}{4} + e < \frac{e_1}{4} + \frac{3e_1}{4} = e_1$.

Thus $y \in B(x, e_1) \Rightarrow B(p, e) \subseteq B(x, e_1)$.

So by claims 1 and 2, $x \in B(p, e) \subseteq B(x, e_1) \subseteq B(x_0, e_0) \subseteq U$. Hence $\mathcal{B}_0$ is a countable basis for X, and this implies X is second countable. □

Corollary 3.1. *(S, T) is not metrizable.*

Since (S, T) is not second countable, either it is not separable or it is not metrizable. We have shown that it is separable, and thus it is nonmetrizable.

Now we will show an interesting property of the Sorgenfrey line, not shared with the reals with the usual topology.

Lemma 3.2. *(S, T) is zero-dimensional. In other words, (S, T) has a basis of clopen sets.*

Proof. We will show that the basis $\mathcal{B}$ is made up of clopen sets. Let $(a, b] \in \mathcal{B}$. Then clearly $(a, b]$ is open. We now will show that its complement is open. Notice that $(a, b]^c = (-\infty, a] \cup (b, \infty) = (\bigcup_{n \in \mathbb{N}}(a - n, a]) \cup (\bigcup_{n \in \mathbb{N}}(b, b + n])$ which is open. □

This can be a very useful property, because it would automatically give us regularity.

The natural thing to do now, would be to prove normality. Instead we will consider uncountable subsets of the reals. This property and the resulting theorems were introduced to me by Dr. Gary Gruenhage in a class on Set-Theoretic Topology. This property uses the fact that the usual topology on the reals is second countable, and will be advantageous to prove that (S, T) is hereditarily normal.

Lemma 3.3. *If H is an uncountable subset of the reals, then there exists $x \in H$ such that for each $\varepsilon > 0$, the intersection $(x - \varepsilon, x) \cap H$ is nonempty.*

Proof. Let H be an uncountable subset of $\mathbb{R}$. Suppose that for each $x \in H$, there exists $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x) \cap H$ is empty. Then for each $x \in H$ we can choose a rational number, $p_x \in (x - \varepsilon_x, x)$. We will use this set of rational numbers to contradict the fact that H is uncountable.

Let $f : H \to \mathbb{Q}$ be defined by $f(x) = p_x$. Suppose that $f(x) = f(y)$ for some x and $y \in H$. Then $p_x = p_y$. So $p_x \in (x - \varepsilon_x, x) \cap (y - \varepsilon_y, y)$. Since neither interval meets H, $x \notin (y - \varepsilon_y, y)$ and $y \notin (x - \varepsilon_x, x)$. This gives us $x = y$. Thus f is injective, which implies that $|H| \leq |\mathbb{Q}|$, a contradiction. □

Notice that by altering the proof slightly, we can also show that there exists a point, $y \in H$ such that for each $\varepsilon > 0$, the intersection $(y, y + \varepsilon) \cap H$ is also nonempty. This will come in handy when we begin to discuss compactness.

Now Lemma 3.3 will give us the property of hereditarily Lindelöf. It tells us that even though we can't find a countable basis for (S, T), we can at least find countable subcovers for open covers anywhere in the space.

Theorem 3.5. *The Sorgenfrey line is hereditarily Lindelöf.*

Proof. Note that whenever we have a point $x \in S$, and an open neighborhood, U, of x, we can find a basic open set of the form $(b, x] \in \mathcal{B}$ such that $(b, x] \subseteq U$.

Let X be a subset of S. Now consider an open cover of X of the form, $\mathcal{G} = \{(b_x, x] : x \in X\}$. By the preceding remark, we can always find a collection of this form where each element is contained in an element of our cover for any open cover. Let $M = \{(b_x, x) : (b_x, x) \cap X \neq \emptyset\}$, and let K be the set of all elements of X that are not covered by M.

We would like to show that K is countable. To do this let us first suppose that K is not countable. Then we know that there exists $y \in K$ such that for each $\varepsilon > 0$, $(y - \varepsilon, y) \cap K \neq \emptyset$ implying that $(b_y, y) \cap K \neq \emptyset$. Hence there exists $z \in (b_y, y) \cap K$, which means that $z \in (b_y, y) \cap X$ since $K \subseteq X$. Then z is covered by M, which tells us that it is not in K, a contradiction. Therefore K must be countable.

Notice that M covers $X \setminus K$ in the usual topology on $\mathbb{R}$. If we recall that the usual topology is second countable and thus hereditarily Lindelöf, then we see that M has a countable subcover, $\widehat{M}$. Let $\widehat{G} = \{(b_x, x] : (b_x, x) \in \widehat{M}\} \cup \{(b_y, y] : y \in K\}$. Then $\widehat{G}$ is the union of two countable subcollections of $\mathcal{G}$ which together covers all of X. Thus X is Lindelöf. $\square$

We have shown that each subspace of (S, T) is both regular and Lindelöf which implies that each is normal by theorem 2.10.

Corollary 3.2. (S, T) *is hereditarily normal.*

In the usual topology, every interval is connected and all closed bounded subsets are compact. Since the Sorgenfrey topology is a finer topology on S, we have added more open sets. This gives more opportunity for the subspaces to be disconnected. It also lets us add more open sets to covers which gives the subspaces less of a chance to be compact.

We will start by looking for non-trivial connected subspaces for (S, T). The best place to begin is naturally with the whole space itself.

Theorem 3.6. (S, T) *is not connected.*

This can easily be seen by considering any element of $\mathcal{B}$. Each element is a proper clopen subset of S.

The next lemma will extend our use of the clopen basis to subspaces of (S, T).

Lemma 3.4. *Let K be clopen in a space X and $Y \subseteq X$, then $K \cap Y$ is clopen in the subspace topology on Y.*

Proof. Let X be a topological space, and let $Y \subseteq X$. Let K be a clopen subset of X. Then K and K^c are both open in X. Hence by the definition of subspace, $K \cap Y$ and $Y \setminus K = K^c \cap Y$ are both open in Y. Therefore K is clopen in Y. $\square$

Now that we know that the space (S,T) is not connected one may wonder if there is any nontrivial connected subspaces of (S,T). The next theorem answers this question.

Theorem 3.7. *The only nonempty connected subspaces of the Sorgenfrey line are the degenerate sets.*

Proof. Let X be a nonempty subspace of (S,T). If we suppose that X is degenerate, then clearly X is connected. Suppose that X is nondegenerate. Then there exist distinct points a and $b \in X$. We may assume that $a < b$. By the preceding lemma, $(a,b] \cap X$ is a proper clopen subset of X. Thus X is not connected. $\square$

Earlier we discussed briefly how a finer topology could affect the compact subsets of a space. The next theorems deal with this issue. First we will consider the subsets of S containing intervals.

Theorem 3.8. *If X, a subset of S, contains an interval, then X is not compact.*

Proof. Suppose that X is a subset of S which contains an interval. Then there exist a and $b \in X$ such that $[a,b] \subseteq X$. Let $C = \{(a, \frac{(2^n-1)a+b}{2^n}] : n \in \mathbb{N}\}$. Then C is a collection of closed sets in the subspace X with the finite intersection property. Notice that $\bigcap C = \emptyset$. Hence X is not compact. $\square$

Already with this one theorem we have shown that many of the compact subspaces of S with the usual topology are not compact with the Sorgenfrey topology. This even excludes the space itself from being compact since it contains the interval $(0,1)$.

Corollary 3.3. *(S,T) is not compact.*

Obviously any finite subset of (S,T) is compact, but are there others? We will look into unbounded subsets, and then we will take a look at uncountable subsets.

Theorem 3.9. *Unbounded subsets of S are not compact.*

Proof. Let X be an unbounded subset of (S, T). Suppose that X is compact. Let $\widehat{G} = \{(-n, n] : n \in \mathbb{N}\}$. Notice that $\widehat{G}$ is an open cover of S, and thus it is an open cover of X. Then $\widehat{G}$ has a finite subcover of X. So there exists an $N \in \mathbb{N}$ such that $\{(-n_i, n_i] : i \leq N\}$ covers X.

Let $M = \max\{n_i : i \leq N\}$. Then $X \subset \bigcup_{i \leq N}(-n_i, n_i] = (-M, M]$. So this shows that X is bounded by M and $-M$, a contradiction. Therefore X cannot be compact. $\square$

To deal with uncountable subsets, we will use some earlier theorems. In particular, we will use the fact that a space X is compact if and only if for each collection of closed subsets of X with the finite intersection property, C, the intersection $\bigcap C$ is nonempty. The reader can find the proof of this statement in Cchapter 2.

Theorem 3.10. *Uncountable subsets of S are not compact.*

Proof. Let X be an uncountable subset of S. We may assume that X does not contain an interval by Theorem 3.9. By Lemma 3.3, there exists a point in X such that for each $\varepsilon > 0$, the intersection $X \cap (x, x + \varepsilon)$, is nonempty. Let $\mathcal{C} = \{(x, x + \varepsilon] \cap X : \varepsilon > 0\}$. Notice that since the subsets in $\mathcal{C}$ are nested and nonempty, this collection has the finite intersection property. By Lemmas 3.2 and 3.4 , $\mathcal{C}$ is a collection of closed sets in X. If we look at $\bigcap \mathcal{C}$, we realize that this intersection is empty. Hence X is not compact. $\square$

So far we only have the trivially compact subsets. By now one may be thinking that this is all there is. Surprisingly enough, it's not. Let us digress to look at convergent sequences. Once we show the existence of a nontrivial convergent sequence, we will use it to find an infinite compact subset of the Sorgenfrey line.

Theorem 3.11. *If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence converging to some $x \in \mathbb{R}$ with the usual topology, and for all but finitely many n, $x_n \leq x$, then $\{x_n\}_{n\in\mathbb{N}}$ converges to x in (S, T).*

Proof. Let τ be the usual topology on S. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in S which is convergent to some point x in (S, τ). Suppose this sequence also satisfies the condition that for all but finitely many n, $x_n \leq x$. Now let us show that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in (S, T). Let U be an open neighborhood of x in (S, T). Then there exists a basic open set $(a, b] \subseteq U$ such that $x \in (a, b]$. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x in (S, τ), there exists $N_1 \in \mathbb{N}$ such that $x_n \in (a, b) \subset (a, b]$ for all $n \geq N_1$. Now notice that there exists $N_2 \in \mathbb{N}$ such that $x_n \leq x$ for all $n \geq N_2$. So we have that $x_n \in (a, x] \subseteq (a, b] \subseteq U$ for all $n \geq N_1 + N_2$. Thus $\{x_n\}_{n\in\mathbb{N}}$ converges to x in (S, T). □

Now we know that sequences such as $\frac{n}{n+1}{}_{n\in\mathbb{N}}$ converge. Now that we have convergent sequences, let's see how we can use it.

Theorem 3.12. *If $\{x_n\}_{n\in\mathbb{N}}$ converges to x in (S, T) from the left then $\{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is a compact subspace of (S, T).*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ converge to a point x in (S, T) from the left. Let $X = \{x_n\}_{n\in\mathbb{N}} \cup \{x\}$. Suppose that $\mathcal{G}$ is an open cover of X. Then there exists $U \in \mathcal{G}$ which is an open neighborhood of x. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x, we have that for some $N \in \mathbb{N}$, the points, $x_n \in U$ for all $n \geq N$. Since $\mathcal{G}$ covers X, there exists U_n an open neighborhood of x_n for each $n \leq N$. Let $\widehat{G} = \{U_n : n \leq N\} \cup \{U\}$. Then $\widehat{G}$ is an open cover of X. Therefore X is compact. □

That will conclude our study of the Sorgenfrey line. In summary, by taking a basis which adds an endpoint to the open intervals in $\mathbb{R}$, we have altered the usual topology on the reals into a much finer space, the Sorgenfrey line. Even though many of the nicer qualities of the usual topology have been lost such as metrizability, the Sorgenfrey topology does share some properties of the usual topology, while gaining a few more.

Chapter 4

The Lexicographic Order on $\mathbb{R} \times [0,1]$

To organize our papers and files, we normally alphabetize them. In other words, we order them by the first letter in the name, and then by the consecutive letters. This is a very common order that can be extended to coordinate systems. Let us define a set $L = \mathbb{R} \times [0,1]$. Elements of this set are expressed as ordered pairs. It is now natural to order the first coordinates and then the second coordinates. So define an order " $<$ " on L by $(x,x') < (y,y')$ provided that either $x < y$ or, $x = y$ and $x' < y'$. This is called the *lexicographic order* on L. Clearly this is a linear order on L.

We can now use this ordering to construct "open intervals." Define $((a,a'),(b,b'))$ for (a,a'), $(b,b') \in L$ as the set of points, $\{(x,x') \in L : (a,a') < (x,x') < (b,b')\}$ where $(a,a') < (b,b')$. Notice that the intersection of two open intervals is either another open interval or the empty set. Thus if we let $\mathcal{B}$ be the collection of all open intervals on L, then it is easy to see that $\mathcal{B}$ is a basis for L. Let R be the topology generated by $\mathcal{B}$.

We start by showing that L is a Hausdorff space.

Theorem 4.1. *(L,R) is a Hausdorff space.*

Proof. Let (a,a'), and (b,b') be two distinct points in L. We may assume that $(a,a') < (b,b')$. Then either $a < b$, or $a = b$ and $a' < b'$.

Case 1: Suppose $a < b$. Since a, $b \in \mathbb{R}$ there exists $c \in \mathbb{R}$ such that $a < c < b$. Let $U = ((a-1,a'),(c,a'))$ and $V = ((c,a'),(b+1,b'))$. Then U and V are open, $(a,a') \in U$, $(b,b') \in V$, and $U \cap V = \emptyset$.

Case 2: Suppose $a = b$. Then $a' < b'$. Since $a', b' \in [0,1]$ and $[0,1] \subseteq \mathbb{R}$, there exists $c' \in [0,1]$ such that $a' < c' < b'$. Let $U = ((a-1, a'), (a, c'))$ and $V = ((b, c'), (b+1, b'))$. Then, again, U and V are open, $(a, a') \in U$, $(b, b') \in V$, and $U \cap V = \emptyset$.

Therefore (L, R) is a Hausdorff space. □

We prove regularity using the definition of a regular space. Let H° denotes the interior of H which is an open set.

Theorem 4.2. *(L, R) is a regular space.*

Proof. Let M be a closed subset of L, and let $(x, x') \in L$ such that $(x, x') \notin M$. Then M^c is open and $(x, x') \in M^c$ implies that there exists a basic open set $((a, a'), (b, b'))$ such that $(x, x') \in ((a, a'), (b, b')) \subseteq M^c$. Then there exists (c, c'), $(d, d') \in L$ such that $(a, a') < (c, c') < (x, x') < (d, d') < (b, b')$. Let $U = ((c, c'), (d, d'))$ and $V = (U^c)^\circ$. Then U and V are open, $(x, x') \in U$, $M \subseteq ((a, a'), (b, b'))^c \subset (U^c)^\circ = V$, and $U \cap V = \emptyset$. Hence (L, R) is regular. □

Now it is easy to see that L is uncountable, but is it separable, first countable, or second countable? We will first show that the space L is not separable. From this it will easily follow that the space is not second countable, since if it were, we could construct a countable dense set by choosing a point from each basic open set in the countable basis.

Theorem 4.3. *(L, R) is not separable.*

Proof. Suppose (L, R) is separable. Then there is a dense countable subset of L. Let $D = \{(x_n, y_n) : n \in \mathbb{N}\}$ be this subset. Now we will construct an open set which does not intersect D. Since $\mathbb{R}$ is uncountable, we may choose $x \in L$ such that $x \notin \{x_n :$

$n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, $(x_n, y_n) \notin ((x, 0), (x, 1))$. Hence D is not dense in L, a contradiction. Therefore (L, R) is not separable. $\square$

Corollary 4.1. *(L, R) is not second countable.*

Even though we cannot find a countable basis for the whole space, we will prove that we can find a countable local basis at each point. This proves that (L, R) is first countable.

Theorem 4.4. *(L, R) is first countable.*

Proof. Let $(a, a') \in L$. Then either $a' \in (0, 1)$, $a' = 1$, or $a' = 0$. For each point in each case we will construct a countable local base.

Case 1: Suppose $a' \in (0, 1)$. Let $G_{(a,a')} = \{((a, p), (a, q)) : p, q \in \mathbb{Q} \cap [0, 1]$ and $p < a' < q\}$. Let U be an open neighborhood of (a, a'). Then there is a basic open set, $((b, b'), (c, c'))$ such that $(a, a') \in ((b, b'), (c, c')) \subseteq U$. Notice that there exists p and $q \in \mathbb{Q} \cap [0, 1]$ such that $(b, b') < (a, p) < (a, a') < (a, q) < (c, c')$. Then $((a, p), (a, q))$ is an element of $G_{(a,a')}$, a subset of U, and it contains (a, a'). Hence $G_{(a,a')}$ is a local base of (a, a').

Case 2: Suppose $a' = 0$. Let $G_{(a,0)} = \{((p, 0), (a, q)) : p \in \mathbb{Q}$ and $q \in \mathbb{Q} \cap (0, 1]$ and $p < a\}$ Again, let $U \in T$ such that $(a, 0) \in U$. Then there is a basic open set, $((b, b'), (c, c'))$ such that $(a, 0) \in ((b, b'), (c, c')) \subseteq U$. Notice that there exists p and $q \in \mathbb{Q} \cap [0, 1]$ such that $(b, b') < (p, 0) < (a, a') < (a, q) < (c, c')$. Then $((p, 0), (a, q))$ is an element of $G_{(a,a')}$, is a subset of U, and it contains $(a, 0)$. Hence $G_{(a,a')}$ is a local base of $(a, 0)$.

Case 3: Now we are left with the case when $a' = 1$. We will construct a countable local base for (a, a') in much the same way as in Case 2. Let $G_{(a,1)} = \{((a, p), (q, 1)) : p \in \mathbb{Q} \cap [0, 1) \text{ and } q \in \mathbb{Q} \text{ and } q > a\}$. Again, let $U \in T$ such that $(a, 1) \in U$. Then there is a basic open set $((b, b'), (c, c'))$ such that $(a, 1) \in ((b, b'), (c, c')) \subseteq U$. In much the same way as before, there is an element of $G_{(a,1)}$ which contains $(a, 1)$ and is a subset of $((b, b'), (c, c'))$. Thus $G_{(a,1)}$ is a local base for $(a, 1)$.

Therefore there is a local countable base at each point, which implies that (L, R) is first countable. □

The following theorems deal with specific subspaces and convergent sequences of (L, R). We wish to categorize the subspaces we mention. To do this we will find a homeomorphism between the subspace and a familiar space.

Theorem 4.5. *Let $x \in \mathbb{R}$. Then $[(x, 0), (x, 1)]$ with the subspace topology is homeomorphic to $[0, 1]$ with the usual topology.*

Proof. Let $X = [(x, 0), (x, 1)]$ for some $x \in \mathbb{R}$, and let $T|_X$ be the subspace topology on X. Let $[0, 1]$ have the usual topology. Define a function $f : X \to [0, 1]$ by $f((x, x')) = x'$ for each $(x, x') \in X$. Then it is easy to see that f is a well defined function which is also a bijection. Now we need to show that f is a homeomorphism.

First we will prove that f is continuous by using Theorem 5 in the Chapter 2. This means that we need to show that the preimage of a basic open set is open. Choose A to be a basic open set in [0,1]. Then $A = (a, b) \cap [0, 1]$ for some a, $b \in \mathbb{R}$. Notice that $f^{-1}(A) = ((x, a), (x, b)) \cap X$ which is open in X. Hence f is continuous.

Now we need to show that f^{-1} is continuous. Let $V = ((a,a'),(b,b')) \cap X$ for some (a,a'), $(b,b') \in L$. Then we have one of four situations; either $V = ((x,a'),(x,b'))$, or $V = [(x,0),(x,b'))$, or $V = ((x,a'),(x,1)]$, or $V = X$.

Case 1: Assume $V = ((x,a'),(x,b'))$. Then $f(V) = (a',b')$ which is open in $[0,1]$.

Case 2: Assume $V = ([(x,0),(x,b'))$. Then $f(V) = [0,b') = (-b',b') \cap Y$ which is open in $[0,1]$.

Case 3: Assume $V = ((x,a'),(x,1)]$. Then $f(V) = (a',1] = (a',a'+1) \cap [0,1]$ which is open in $[0,1]$.

Case 4: Assume $V = X$. Since f is a bijection, $f(V) = f(X)$ is [0,1] which is definitely open.

Hence f^{-1} is continuous. Therefore X is homeomorphic to [0,1] with the usual topology.

□

Now we have seen what a projection onto the second coordinate looks like, for any fixed first coordinate. What does a projection onto the first coordinate look like? We will see that it depends on which second coordinate we fix.

Theorem 4.6. *For any $y \in [0,1]$ let $K_y = \mathbb{R} \times \{y\}$, with the subspace topology.*

1. *If $y \in (0,1)$, then K_y is homeomorphic to the reals with the discrete topology.*
2. *If $y = 0$, then K_y is homeomorphic to the reals with the Right Sorgenfrey topology.*
3. *If $y = 1$, then K_y is homeomorphic to the reals with the Left Sorgenfrey topology.*

Proof. Let $y \in [0,1]$, and let $K_y = \mathbb{R} \times \{y\}$. Define $f : K_y \to \mathbb{R}$ by $f((x,y)) = x$ for each $(x,y) \in K_y$. Then it is obvious that f is a well-defined bijection. We will go on to show that when we change the value of y and put the corresponding topology on the reals, f is a homeomorphism.

Lets begin by assuming that $y \in (0,1)$ and that $\mathbb{R}$ has the discrete topology. Then obviously f^{-1} is continuous since $\mathbb{R}$ is discrete. Thus for this case, we only need to show that f is continuous.

Let U be a basic open set in $\mathbb{R}$. Then $U = \{x\}$ for some $x \in \mathbb{R}$. So $f^{-1}(U) = \{(x,y)\} = ((x,0),(x,1)) \cap K_y$ which is open. Hence f is continuous. Therefore K_y is homeomorphic to the reals with the discrete topology.

Now suppose that $y = 0$. Let T_R denote the Right Sorgenfrey topology on $\mathbb{R}$. So we need to show that f and its inverse are both continuous. Let $(a,b] \subset \mathbb{R}$ be a basic open set in $(\mathbb{R}, T_R)$. Then $f^{-1}((a,b]) = (a,b] \times \{0\} = ((a,1),(b,1)) \cap K_0$ which is open in K_0. Hence f is continuous.

Next let A be a basic open set in K_0. Then $A = ((a,a'),(b,b')) \cap K_0$ for some $((a,a'),(b,b')) \in \mathcal{B}$ which impies that either $a = (a,b) \times \{0\}$ or, in the case that $b' = 0$, $A = (a,b) \times \{0\}$. Then $f(A) = (a,b] \in T_R$ or $f(A) = (a,b) \in T_R$ since T_R is finer than the usual topology on the reals. Hence f^{-1} is continuous, and so f is a homeomorphism between K_0 and $\mathbb{R}$ with the Right Sorgenfrey topology.

If we let $y = 1$ and give $\mathbb{R}$ the Left Sorgenfrey topology, then it is easy to see that f is a homeomorphism between K_1 and $\mathbb{R}$ with the Left Sorgenfrey topology by using an argument similar to the $y = 0$ case. $\square$

We will now discuss how a few interesting sequences converge to some very unexpected points.

Theorem 4.7. *Let $x \in \mathbb{R}$, $y \in [0,1]$, and let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in L.*

1. *If $\{y_n\}_{n\in\mathbb{N}}$ converges to y in the usual topology on $[0,1]$ and $x_n = x$ for all but finitely many $n \in \mathbb{N}$, then $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to (x, y) in L.*

2. *If $\{x_n\}_{n\in\mathbb{N}}$ converges to x in the usual topology and $x_n < x$ for all but finitely many $n \in \mathbb{N}$, then $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to $(x, 0)$ in L.*

3. *If $\{x_n\}_{n\in\mathbb{N}}$ converges to x in the usual topology and $x_n > x$ for all but finitely many $n \in \mathbb{N}$, then $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to $(x, 1)$ in L.*

Proof. Let $x \in \mathbb{R}$, $y \in [0,1]$, and let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in L.

Case 1: Suppose $y_n \to y$ in the usual topology on $[0,1]$ and $x_n = x$ for all but finitely many $n \in \mathbb{N}$. Let U be an open neighborhood of (x, y), and let $X = [(x, 0), (x, 1)]$. Then $U \cap X$ is a nonempty open neighborhood of (x, y) in X. Now, by the hypothesis, $(x_n, y_n) \in X$ for all but finitely many $n \in \mathbb{N}$. Since $y_n \to y$ in the usual topology on $[0,1]$ and X is homeomorphic to $[0,1]$, $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to (x, y) in X. Then $(x_n, y_n) \in U \cap X$ for all but finitely many $n \in \mathbb{N}$ which implies that $(x_n, y_n) \in U$ for all but finitely many $n \in \mathbb{N}$. Hence the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to (x, y) in L.

Case 2: Assume that $x \in \mathbb{R}$, and $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq L$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in the usual topology on $\mathbb{R}$ and $x_n < x$ for all but finitely many $n \in \mathbb{N}$. Let $((a, a'), (b, b'))$ be a basic open set in (L, R) such that $(x, 0) \in ((a, a'), (b, b'))$. Then $(a, a') < (x, 0) < (b, b')$. Thus either $a < x < b$ or $a < x = b$.

Suppose that $a < x < b$. Then $x \in (a, b)$, and this implies that $x_n \in (a, b)$ for all but finitely many $n \in \mathbb{N}$. Thus $(x_n, y_n) \in ((a, a'), (b, b'))$ for all but finitely many $n \in \mathbb{N}$.

Now suppose that $a < x = b$. Then $x \in (a, b]$ which implies that $x \in (a, b + 1)$. Hence there exist $N_1 \in \mathbb{N}$ such that $x_n \in (a, b + 1)$ for all $n \geq N_1$. Since $x_n < x$ for all $n \geq N_2$ for some $N_2 \in \mathbb{N}$, $x_n \in (a, b)$ for all $n \geq N_1 + N_2$. Hence $(x_n, y_n) \in ((a, a'), (b, b'))$ for all $n \geq N_1 + N_2$. Therefore $(x_n, y_n) \to (x, 0)$.

Case 3: Suppose that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in the usual topology and $x_n > x$ for all but finitely many $n \in \mathbb{N}$. Then similarly to Case 2, $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to $(x, 1)$.

□

This idea of using the subspaces to prove properties about L is not isolated to showing sequence convergence. We will use this idea to show that L is a Lindelöf space. Then using Theorem 2.10, we have that L is normal as a corollary.

Theorem 4.8. *(L, R) is Lindelöf.*

Proof. Let $\mathcal{G}$ be an open cover of L consisting of basic open sets. Then $\mathcal{G}$ is an open cover of $K_0 = \mathbb{R} \times \{0\}$ with the subspace topology. Since K_0 is homeomorphic to the right Sorgenfrey topology on the reals, by Theorem 3.6, $\mathcal{G}$ has a countable subcover of K_0. Let $G_0 = \{((a_n, a'_n), (b_n, b'_n)) : n \in \mathbb{N}\}$ be such a countable subcover of K_0. Let A_0 be the collection of first coordinates of both endpoints of each interval in G_0. In other words, let $A_0 = \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}$. Notice that G_0 covers $A_0^c \times [0, 1]$, and that A_0 is countable.

We can use a similar argument to find a countable subcover G_1 of $K_1 = \mathbb{R} \times \{1\}$. Let A_1 be the collecton of first coordinates of both endpoints of each interval in G_1. Again notice that G_1 covers $A_1^c \times [0,1]$. Also $G_0 \cup G_1$ covers $(A_0 \cap A_1)^c \times [0,1]$. Note that so far we have a countable subcover of all but those points with first coordinates in the countable set $A_1 \cap A_0$.

Now let $a \in A_0 \cap A_1$. Then $X_a = [(a,0),(a,1)]$ is homeomorphic to the closed interval $[0,1]$ with the usual topology. Hence for each $a \in A_0 \cap A_1$, $\mathcal{G}$ has a countable subcover, G_a, of X_a.

Let $\widehat{G} = G_0 \cup G_1 \cup (\bigcup_{a \in A_0 \cap A_1} G_a)$. Then obviously $\widehat{G} \subseteq \mathcal{G}$. Notice that G_0 and G_1 are countable. Also $A_0 \cap A_1$ is countable and, for each $a \in A_0 \cap A_1$, G_a is countable. Thus we have that $\widehat{G}$ is a countable union of countable collections. Now to show that $\widehat{G}$ is a cover of L, consider the fact that $L = ((A_0 \cap A_1) \times [0,1]) \cup ((A_0 \cap A_1)^c \times [0,1]) = (\bigcup_{a \in A_0 \cap A_1} (\bigcup G_a)) \cup (\bigcup G_0) \cup (\bigcup G_1) = \bigcup \widehat{G}$. Hence $\widehat{G}$ is a countable subcover of $\mathcal{G}$. □

Corollary 4.2. *The space L is normal.*

Let us now look a little closer at our underlying set, L. To show that our space is connected and that certain subsets are compact, it would help to have the least upper bound property for L.

Lemma 4.1. *Bounded subsets of L have the least upper bound property.*

Proof. Let X be a bounded subset of L. Then there exist points (a,a'), $(b,b') \in L$ such that for each point $(x,x') \in X$, $(a,a') < (x,x') < (b,b')$. So if we let $E = \{x : (x,x') \in X\}$, or in other words let E be the set of all first coordinates of points in X, then E is bounded on the real line by a and b. Thus E has a least upper bound which we will call c.

Suppose that $c \in E$. Then $X \cap [(c,0),(c,1)]$ has a least upper bound, (c,c'), since $[(c,0),(c,1)]$ is homeomorphic to the closed interval $[0,1]$. Now we claim that in this case, (c,c') is the least upper bound of X. Let $(x,x') \in X$. Then either $x < c$ or $x = c$ and $x' < c'$. Either way $(x,x') \leq (c,c')$. Now suppose that (d,d') is an upper bound of X. Then either $c < d$ or $d = c$ and $c' \leq d'$. Thus $(c,c') \leq (d,d')$. Hence (c,c') is the least upper bound of X.

Now suppose that $c \notin E$. Then for each $(x,x') \in X$, $x < c$. We claim that in this case $(c,0)$ is the least upper bound of X. First of all, for each $(x,x') \in X$, $x < c$ implies that $(x,x') < (c,0)$. Suppose (d,d') is an upper bound of X. Then either $c < d$ or $c = d$. If $c < d$, then obviously $(c,0) < (d,d')$. If $d = c$, then $0 \leq d'$ and $(c,0) \leq (d,d')$. So $(c,0)$ is the least upper bound of X. □

Earlier we described an open interval by the subset $((a,a'),(b,b')) = \{(c,c') \in L : (a,a') < (c,c') < (b,b')\}$. This is very similar to the definition of an open interval on the real line. The open interval, (a,b), on the real line is the set, $\{c \in \mathbb{R} : a < c < b\}$. In any linearly ordered space, this is the form of an open interval. When we talk about any interval, we are talking about subsets of a linearly ordered set with the same form as intervals on the real line.

So how does the least upper bound property help us? It is an important element in showing that intervals are connected.

Theorem 4.9. *Every interval in L is connected.*

Proof. Let I be an interval in L. Suppose that I is not connected. Then there exist nonempty disjoint open sets, U and V such that $I = U \cup V$. Hence there exist elements (a,a'), $(b,b') \in I$ such that $(a,a') \in U$ and $(b,b') \in V$. Without loss of generality we will assume that $(a, a') < (b, b')$. Let $\widehat{U} = U \cap [(a, a'), (b, b')]$ and $\widehat{V} = V \cap [(a, a'), (b, b')]$. Notice that $\widehat{U}$ and $\widehat{V}$ are open subsets of $[(a, a'), (b, b')]$ that separate $[(a, a'), (b, b')]$.

Now we will find a point of $[(a, a'), (b, b')]$ that is in neither $\widehat{U}$ nor $\widehat{V}$. Let (c, c') be the least upper bound of $\widehat{U}$. Then $(c, c') \geq (x, x')$ for each $(x, x') \in \widehat{U}$ and for any upper bound, (y, y') of $\widehat{U}$, $(c, c') \leq (y, y')$. Since $(a, a') \in \widehat{U}$ and (b, b') is an upper bound on $\widehat{U}$, we have that $(c, c') \in [(a, a'), (b, b')]$.

Suppose that $(c, c') \in \widehat{U}$. Then there is some basic open set B_U such that $(c, c') \in B_U \cap [(a, a'), (b, b')]$. Since $(b, b') \notin \widehat{U}$, we know that $B_U \cap [(a, a'), (b, b')]$ is either of the form $((d, d'), (e, e'))$ for some (d, d'), $(e, e') \in [(a, a'), (b, b')]$ or of the form $[(a, a'), (e, e'))$ for some $(e, e') \in ((a, a'), (b, b')]$. Either way $((c, c'), (e, e')) \subset B_U \cap [(a, a'), (b, b')]$. However we know that there is a point (x, x') such that $(c, c') < (x, x') < (e, e')$ which implies that $(x, x') \in \widehat{U}$, a contradiction since (c, c') is the least upper bound. Hence $(c, c') \notin \widehat{U}$.

So (c, c') must be in $\widehat{V}$. Again there is a basic open set B_V such that $(c, c') \in B_V \cap [(a, a'), (b, b')]$, but this time $B_V \cap [(a, a'), (b, b')]$ is either of the form $((d, d'), (e, e'))$ for some (d, d'), $(e, e') \in [(a, a'), (b, b')]$ or of the form $((d, d'), (b, b')]$ for some $(d, d') \in [(a, a'), (b, b'))$. Thus $((d, d'), (c, c')) \subset B_V \cap [(a, a'), (b, b')]$, yet there exists a point $(x, x') \in ((d, d'), (c, c'))$ which is not in $\widehat{U}$. This makes (x, x') an upper bound on $\widehat{U}$ which is less than (c, c'), a contradiction.

So we have that $(c, c') \notin \widehat{U} \cup \widehat{V}$ meaning that $[(a, a'), (b, b')]$ is not separated by $\widehat{U}$ and $\widehat{V}$. Therefore I must be connected. $\square$

If we recall that any union of connected sets with nonempty intersection is also connected, then we get the following corollary.

Corollary 4.3. (L, R) *is connected.*

Proof. Let $\mathcal{G} = \{((-n,0),(n,1)) : n \in \mathbb{N}\}$. Then $\mathcal{G}$ is a collection of connected sets, and $\bigcap G = ((-1,0),(1,1))$ which is obviously nonempty. So $\bigcup G = L$ is connected. □

Are there any more connected subsets of L? We will next show that there are not.

Theorem 4.10. *If X is a proper nondegenerate subset of L which is not an interval, then X is not connected.*

Proof. Let X be a proper subset of L. Suppose that X is not an interval. Then there are points (a,a'), $(b,b') \in X$ and $(c,c') \notin X$ such that $(a,a') < (c,c') < (b,b')$. Let $U = (\bigcup_{n\in\mathbb{N}}((a-n,a'),(c,c'))) \cap X = (-\infty,(c,c')) \cap X$, and let $V = (\bigcup_{n\in\mathbb{N}}((c,c'),(b+n,b'))) \cap X = ((c,c'),\infty) \cap X$. Then U and V are nonempty disjoint open subsets of X such that $X = U \cap V$. □

It may have been apparent that the arguments for connectedness worked in much the same way as they work for the real line with the usual topology. We will now see that the same holds for compact subsets of L.

Theorem 4.11. *Unbounded subsets of L are not compact.*

Proof. Let X be an unbounded subset of L. Suppose that X is compact. Let $\mathcal{G} = \{((-n,0),(n,1)) : n \in \mathbb{N}\}$. Then $\mathcal{G}$ is an open cover of X. Since we assume that X is compact, there exists $N \in \mathbb{N}$ such that $G_N = \{((-n_i,0),(n_i,1)) : i \leq N\}$ is a subcover of L. Then $X \subseteq \bigcup G_N$. Let $m = \max\{n_i : i \leq N\}$. Then X is bounded by $(-m,0)$ and $(m,1)$, a contradiction. Therefore X is not compact. □

We know that L itself is unbounded, so it is clear to see that L is not compact.

What type of subsets are compact? If we keep with the idea that the compact subsets of L resemble the compact subsets of $\mathbb{R}$ with the usual topology, then why don't we look into closed bounded subsets of L? Lets first show that closed intervals are compact.

The following proof was inspired by a proof that the interval $[0,1]$ is compact in the usual topology on $\mathbb{R}$ by Dr. John O'Connor [2].

Theorem 4.12. *If X is a closed interval in L, then X is compact.*

Proof. Let $X = [(a,a'),(b,b)']$ for some (a,a'), $(b,b') \in L$. Let $\mathcal{G}$ be an open cover of X consisting of basic open sets. Let

$\widehat{X} = \{(x,x') \in X : [(a,a'),(x,x')]$ can be covered by finitely many elements of $G\}$. Let (c,c') be the least upper bound of $\widehat{X}$.

We claim that $(c,c') = (b,b')$. Suppose that $(c,c') < (b,b')$. Then there exists $((d,d'),(e,e')) \in \mathcal{G}$ such that $(c,c') \in ((d,d'),(e,e'))$. Choose $(y,y') \in ((d,d'),(c,c'))$. Then (y,y') must be a point in $\widehat{X}$. So there exists a finite subcover $\widehat{G}$ of $[(a,a'),(y,y')]$. Now choose $(z,z') \in ((c,c'),(e,e'))$. Then $\widehat{G} \cup \{((d,d'),(e,e'))\}$ is a finite subcover of $[(a,a'),(z,z')]$ which implies that $(z,z') \in \widehat{X}$. However since (c,c') is an upper bound on $\widehat{X}$ and $(c,c') < (z,z')$, we have that $(z,z') \notin \widehat{X}$, a contradiction. Also, $(c,c') \not> (b,b')$ because (b,b') is an upper bound on $\widehat{X}$. Hence $(c,c') = (b,b')$.

Now choose $((d,d'),(e,e')) \in \mathcal{G}$ such that $(b,b') \in ((d,d'),(e,e'))$. Then we know that $(d,d') \in \widehat{X}$ by the prior claim. So there exists a finite subcover, G_d, of $[(a,a'),(d,d')]$. Notice that $G_d \cup ((d,d'),(e,e'))$ is a finite subcover of $[(a,a'),(b,b')]$. Therefore $[(a,a'),(b,b')]$ is compact. □

We can now expand this idea to bounded closed subsets of L.

Corollary 4.4. *Every bounded closed subset of L is compact.*

Proof. Let X be a closed subset of L, and let X be bounded by (a, a'), $(b, b') \in L$. By Theorem 4.12,$[(a, a'), (b, b')]$ is compact. Since X is closed in L, X is closed in $[(a, a'), (b, b')]$. Now recall that closed subsets of a compact space are compact. □

Since we have shown that L is a Hausdorff space, every compact subset of L must be closed, and by Theorem 4.11, it must be bounded. So these are the only compact subsets of L.

This will conclude our exploration of the Lexicographic order on $\mathbb{R} \times [0, 1]$, and in essence this paper.

Bibliography

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