On the existence of Even and k-divisible-Matchings

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On the existence of Even and k-divisible-Matchings

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A Dissertation

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Doctor of Philosophy

Auburn, Alabama
May 10, 2008

On the existence of Even and k-divisible-Matchings

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Emilia Anna Moore, daughter of Piotr and Anna Lusnia, was born on October 3, 1982, in Garwolin, Poland. She graduated from Loveless Academic Magnet Program High School in Montgomery, Alabama, in 2000. She then attended Huntingdon College in Montgomery, Alabama, for three years and graduated magna cum laude with Bachelor of Art degrees in Mathematics and Computer Science in May 2003. She entered the PhD program at Auburn University, in June 2003 and was awarded a Master of Science degree in Mathematics in May 2006.

Dissertation Abstract

On the existence of Even and k-divisible-Matchings

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Doctor of Philosophy, May 10, 2008
(M.S., Auburn University, 2006)
(B.A., Huntingdon College, 2003)

98 Typed Pages

Directed by Dean Hoffman

The concept of an an even matching was first introduced by Billington and Hoffman. They were used to find gregarious 4-cycle decompositions of $K_{8t(a),b}$ with a and b odd. Their paper contains even matchings of type (α^8, β) for α, β even and $0 \leq \beta \leq 4\alpha$. This paper considers the necessary and sufficient conditions for the existence of even matchings as well as k-divisible matchings. We present a construction of even matchings and 3-divisible matchings of type $(a_1, a_2, \ldots, a_p)$ provided the necessary conditions are satisfied.

Acknowledgments

The author wishes to express her appreciation for her family members Jakub, Pawel, Anna and Piotr who have given of their love throughout her life. She would like to thank her husband Robert for his abundant support, guidance and love. The author would also like to thank the numerous families who have supported her throughout her life, the Jinrights, the Henrys, the Lindleys and the Stakelys.

The author would also like to thank the professors from her advisory committee for their contribution to this dissertation, with special thanks to Dr. Dean Hoffman for the considerable time, thought, and energy which he used in order to further the author's progress in her studies of design theory.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliography follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

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Chapter 1

Introduction

1.1 Even Matchings

In a recent paper by Dean Hoffman and Elizabeth Billington a definition of an even matching was introduced [1].

Definition 1.1 *Let $a_1, \ldots, a_p$ be non-negative integers. We define the complete multipartite graph $K(a_1, a_2, \ldots, a_p)$ as the graph whose vertex set is partitioned into parts $A_1, \ldots, A_p$ of size $a_1, \ldots, a_p$ respectively. Two vertices are adjacent if and only if they are in two different parts.*

Definition 1.2 *Let $a_1, a_2, \ldots, a_p$ be non-negative integers, and let A_i denote the vertex partite set of size a_i, for $1 \leq i \leq p$. Then for the graph $K(a_1, a_2, \ldots, a_p)$, the ordered set $M = (M_1, M_2, \ldots, M_p)$ is an* **even matching** *of type $(a_1, a_2, \ldots, a_p)$ if*

1. *for each i, $1 \leq i \leq p$, the set M_i is a perfect matching in the graph $K(a_1, a_2, \ldots, a_p) \backslash A_i$, and*
2. *every edge of $K(a_1, a_2, \ldots, a_p)$ lies in an even number of matchings M_i (this number could be zero). We will refer to this as the "evenness" condition.*

In the above mentioned paper, a limited number of even matchings was used to help construct gregarious 4-cycle decompositions of $K(a, a, a, a, a, a, a, a, b)$ for odd size parts and $a \geq 3$. The matchings used were of the form (α^8, β) for all even α and β, with $0 \leq \beta \leq 4\alpha$. We wanted to know the necessary and sufficient conditions for the existence of even matchings. Throughout the paper we will use the notation (a^p) to represent a matching type with p parts of size a.

In Chapter 2 we prove the following theorem.

Theorem 1.3 *Assume even matchings of the types* $(0^{2h}, 2^{2h}, 4h-2)$, $(1^{3h+c}, 3^{h-c}, 6h-3-2c)$, $(1^{h+c}, 3^{3h-c}, 10h-5-2c)$, $(1^{h+c}, 3^{2h-c}, 7^h, 14h-7-2c)$ *for* $h \geq c$, $(1^{4h}, i)$ *for* $3 \leq i \leq 4h-3$ *odd,* $(1^{4h-1}, j, i)$ *for* $3 \leq i \leq 4h-1$ *and* $3 \leq j \leq i$ *odd, and* $(0^{2h+1}, 2^{2h+1}, 4h)$ *exist. Assume* $a_1 \leq a_2 \leq \ldots \leq a_p$ *and define* $n = \sum_{i=1}^{p} a_i$. *An even matching of the type* $(a_1, \ldots, a_p)$ *exists if and only if*

1. p *is odd,*
2. $2a_p + a_{p-1} \leq n$,
3. *either* a_i *are even for all* $1 \leq i \leq p$ *or* a_i *are odd for all* $1 \leq i \leq p$ *and* $p \equiv 1 \pmod 4$,
4. $2(p-2)a_p \leq (p-3)n$.

In Chapter 3 we will generalize this notion to that of k-divisible-matchings and consider the existence problem for them. More precisely, we prove the following theorem.

Theorem 1.4 *Assume 3-divisible-matchings of the types* $(0^{4h}, 2^{2h}, 4h-2)$, $(1^{4h+c}, 3^{2h-c}, 10h-5-2c)$, $(1^{2h+c}, 3^{4h-c}, 7^h, 14h-7-2c)$ *for* $0 \leq c \leq 2h-3$, $(1^{6h}, i)$ *for* $3 \leq i \leq 6h-3$ *odd,* $(1^{6h-1}, j, i)$ *for* $3 \leq i \leq 6h-1$ *and* $3 \leq j \leq i$ *odd, and* $(0^{4h+2}, 2^{2h+1}, 4h)$ *exist. Assume* $a_1 \leq a_2 \leq \ldots \leq a_p$ *and define* $n = \sum_{i=1}^{p} a_i$.
A 3-divisible-matching of the type $(a_1, \ldots, a_p)$ *exists if and only if*

1. $p \equiv 1 \pmod 3$,

2. $2a_p + a_{p-1} \leq n$,

3. *Either all* a_i *are even or all* a_i *are odd and* p *is odd.*

4. $(2p-5)a_p \leq (p-4)n$.

Chapter 2

Existence of Even Matchings

2.1 Necessary conditions

In this Section we present the necessary conditions for the existence of even matchings of type $(a_1, a_2, \ldots, a_p)$. Let us define the set

$S_2 = \{(a_1, \ldots, a_p)|$ an even matching on the graph $K(a_1, \ldots, a_p)$ exists$\}$.

We assume $a_1 \leq a_2 \leq \ldots \leq a_p$ and define $n = \sum_{i=1}^{p} a_i$. The necessary conditions are as follows:

1. p is odd,
2. $2a_p + a_{p-1} \leq n$,
3. either a_i are even for all $1 \leq i \leq p$ or a_i are odd for all $1 \leq i \leq p$ and $p \equiv 1 \ (mod\ 4)$,
4. $2(p-2)a_p \leq (p-3)n$.

Let us confirm the above conditions. We assume that $(M_1, \ldots, M_p)$ is an even matching of the type $(a_1, \ldots, a_p)$.

1. Each vertex (element of A_i) will be used in $p-1$ edges of $\bigcup_{i=1}^{p} M_i$, and the number of edges that vertex is in must be even, equivalently $\frac{p-1}{2}$ must be an integer. Therefore p must be odd.

2. For every $i < p$, $n - a_i - 2a_p \geq 0$, since we must have enough vertices in each $K(a_1, \ldots, a_p)\backslash A_i$ to "match" the vertices of the largest part. Since $a_1 \leq a_2 \leq \ldots \leq a_p$, it is sufficient that $n - a_{p-1} - 2a_p \geq 0 \Rightarrow 2a_p + a_{p-1} \leq n$.

3. Since each M_i is to be a perfect matching, $(\sum_{i=1}^{p} a_i) - a_i$ must be even for all i; hence all a_i have same parity. The "evenness" condition requires $\sum_{i=1}^{p} \frac{n-a_i}{2} = \frac{(p-1)n}{2}$ be even. Therefore either all a_i are even, or all a_i are odd and $\frac{p-1}{2}$ even, hence $p \equiv 1 \ (mod\ 4)$.

4. None of the edges in M_p use vertices in A_p. There must be enough edges in $M_1 \cup M_2 \cup \ldots \cup M_{p-1}$ not intersecting A_p to satisfy the **evenness condition**. So,

$$\frac{n - a_p}{2} \leq \sum_{i=1}^{p-1} \frac{n - a_i - 2a_p}{2} = \frac{(p-2)n - (2p-3)a_p}{2},$$

hence

$$2(p-2)a_p \leq (p-3)n.$$

Notice that, by Property 4, $p > 3$.

2.2 Sufficiency of Conditions

The paper by Billington and Hoffman contains the following Lemmas regarding even matchings.

Lemma 2.1 *If* $M = (M_1, M_2, \ldots, M_p)$ *is an even matching of type* $(a_1, a_2, \ldots, a_p)$ *and* $N = (N_1, N_2, \ldots, N_p)$ *is an even matching of type* $(b_1, b_2, \ldots, b_p)$*, on disjoint vertex sets, then*

$M \cup N = (M_1 \cup N_1, M_2 \cup N_2, \ldots, M_p \cup N_p)$ is an even matching of type $(a_1 + b_1, a_2 + b_2, \ldots, a_p + b_p)$.

Lemma 2.2 *If $M = (M_1, M_2, \ldots, M_p)$ is an even matching of type $(a_1, a_2, \ldots, a_p)$ and $N = (N_1, N_2, \ldots, N_q)$ is an even matching of type $(a_1, b_2, \ldots, b_q)$, then there exists an even matching for $K(a_1, a_2, \ldots, a_p, b_2, b_3, \ldots, b_q)$ of type $(a_1, a_2, \ldots, a_p, b_2, b_3, \ldots, b_q)$.*

Using the above Lemma 2.2 we can inductively construct even matchings of type $(c_1, c_2, \ldots, c_r)$, for any $r \geq 9$, from matchings with five, $(a_1, a_2, a_3, a_4, a_5)$, and seven, $(b_1, b_2, \ldots, b_7)$, parts. This construction is limited as the converse of Lemma 2.2 is not true.

We use the following Lemma in various proofs throughout the paper.

Lemma 2.3 *If the graph $K(a_1, \ldots, a_p)$ satisfies the four necessary conditions and a_i is even for all i, then there exists a perfect matching on $K(a_1, \ldots, a_p)$.*

Proof: By property 2, $a_p < \frac{n}{2}$. Also, the total number of vertices is even. The rest of the proof is trivial. □

The following Lemmas are useful when working with even matchings.

Lemma 2.4 *If $a_1, \ldots, a_p$ are even and $(a_1, \ldots, a_p)$ is in S_2, then $(0^{2n}, a_1, \ldots, a_p)$ is also in S_2 for any integer n.*

Proof: Let $(M_1, \ldots, M_p)$ be an even matching of type $(a_1, \ldots, a_p)$. Let N be any perfect matching on $K(a_1, \ldots, a_p)$. By Lemma 2.3, N exists. We construct an even matching of $K(0^{2n}, a_1, \ldots, a_p)$ as follows.

$M'_i = N$ for $1 \leq i \leq kn$

$M'_{j+kn} = M_j$ for $1 \leq j \leq p$

Clearly this is an even matching. □

Lemma 2.5 *If $a_1, \ldots, a_p, b_2, \ldots, b_q$ are all even and $(a_1, \ldots, a_p)$, $(0, b_2, \ldots, b_q)$ are both in S_2, then $(b_2, \ldots, b_q, a_1, \ldots, a_p)$ is in S_2.*

Proof: Let $(M_1, \ldots, M_p)$ be an even matching of type $(a_1, \ldots, a_p)$ and $(N_1, \ldots, N_q)$ be an even matching of type $(0, b_2, \ldots, b_q)$. Let R be any perfect matching on $K(a_1, \ldots, a_p)$. By Lemma 2.3, R exists. We construct an matching of $K(b_2, \ldots, b_q, a_1, \ldots, a_p)$ as follows.

$M'_i = N_{i+1} \cup R$ for $1 \leq i \leq q-1$

$M'_{j+q-1} = N_1 \cup M_j$ for $1 \leq j \leq p$

Since $p-1$ and $q-1$ are both even this is an even matching. $\square$

In the next Sections we will consider $p = 5$, $p = 7$ and $p = 9$. Then we will generalize the argument for any odd p.

2.2.1 Any number of parts of the same size

Let us consider even matchings of the type (a^5) and (a^7), with a even in the latter case. It is sufficient to consider $(1,1,1,1,1)$, $(2,2,2,2,2)$ and $(2,2,2,2,2,2,2)$. We can then use Lemma 2.1, with an appropriate number of copies of $(2,2,2,2,2)$ or $(2,2,2,2,2,2,2)$, to construct even matchings of types (a^5) and (a^7). Also, using Lemma 2.2 we can construct even matchings of type (a^p) for any odd $p \geq 5$.

Through out the paper we let the parts of $K(a_1, \ldots, a_p)$ be $\{1, 1', 1'', 1''', 1^4, \ldots, 1^{a_1-1}\}, \ldots, \{p, p', p'', p''', p^4, \ldots, p^{a_p-1}\}$. We let $\{1', p\}$ be the edge joining $1'$ and p.

The following is an even matching of type $(1,1,1,1,1)$:

M_1: $\{\{2,3\},\{4,5\}\}$

M_2: $\{\{1,3\},\{4,5\}\}$

M_3: $\{\{1,5\},\{2,4\}\}$

M_4: $\{\{1,5\},\{2,3\}\}$

M_5: $\{\{1,3\},\{2,4\}\}$

Here is an even matching of type $(2,2,2,2,2)$:

M_1: $\{\{2,3\},\{4,5\},\{2',3'\},\{4',5'\}\}$

M_2: $\{\{3,4\},\{5,1'\},\{3',4'\},\{5',1\}\}$

M_3: $\{\{1,2\},\{4,5\},\{1',2'\},\{4',5'\}\}$

M_4: $\{\{2,3\},\{5,1'\},\{2',3'\},\{5',1\}\}$

M_5: $\{\{3,4\},\{1',2'\},\{3',4'\},\{1,2\}\}$

Here is an even matching of type $(2,2,2,2,2,2,2)$:

M_1: $\{\{2,3\},\{2',4\},\{3',4'\},\{5,6\},\{5',7\},\{6',7'\}\}$

M_2: $\{\{3,4\},\{3',5\},\{4',5'\},\{6,7\},\{1,6'\},\{1',7'\}\}$

M_3: $\{\{4,5\},\{4',6\},\{5',6'\},\{1,7\},\{2,7'\},\{1',2'\}\}$

M_4: $\{\{5,6\},\{5',7\},\{6',7'\},\{1,2\},\{1',3\},\{2',3'\}\}$

M_5: $\{\{6,7\},\{1,6'\},\{1',7'\},\{2,3\},\{2',4\},\{3',4'\}\}$

M_6: $\{\{1,7\},\{2,7'\},\{1',2'\},\{3,4\},\{3',5\},\{4',5'\}\}$

M_7: $\{\{1,2\},\{1',3\},\{2',3'\},\{4,5\},\{4',6\},\{5',6'\}\}$

2.2.2 Five parts of any size

In this Section we will give even matchings of type $(a_1, a_2, a_3, a_4, a_5)$. Since $p = 5 \equiv 1$ $(mod\ 4)$ we can have all parts of even size or all parts of odd size. We will describe how to use Lemma 2.1 with $(2,0,0,2,2)$ to construct an even matching $(a_1,\ldots,a_5)$ from $(1,1,1,\_,\_)$ or $(0,2,\_,\_,\_)$.

Here are the "building blocks" we will use.

$(0,0,2,2,2)$

M_1: $\{\{3,5\},\{3',4\},\{4',5'\}\}$

M_2: $\{\{3,4'\},\{4,5\},\{3',5'\}\}$

M_3: $\{\{4,5\},\{4',5'\}\}$

M_4: $\{\{3,5\},\{3',5'\}\}$

M_5: $\{\{3,4'\},\{3',4\}\}$

$(0,2,2,2,2)$

M_1: $\{\{2,4'\},\{3,5'\},\{4,2'\},\{5,3'\}\}$

M_2: $\{\{3,4'\},\{4,5'\},\{5,3'\}\}$

M_3: $\{\{2,4'\},\{4,5'\},\{5,2'\}\}$

M_4: $\{\{2,3'\},\{3,5'\},\{5,2'\}\}$

M_5: $\{\{2,3'\},\{3,4'\},\{4,2'\}\}$

To find an even matching of type $(a_1,\ldots,a_5)$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist and we stop. If $a_1 = 0$ and $a_2 = 0, 2$ or $a_1 = a_2 = a_3 = 1$ look at the listing below to find the even matching. Otherwise continue below.

2. Subtract $(2,0,0,2,2)$ to obtain $(a_1-2, a_2, a_3, a_4-2, a_5-2)$.

3. If necessary, rearrange the terms to ensure that the sequence is nondecreasing.

4. Repeat the above steps until you obtain $(1,\_,\_,\_,\_)$ or $(0,\_,\_,\_,\_)$.

5. Subtract $(0,2,0,2,2)$ and rearrange the terms when necessary until you obtain $(1,1,\_,\_,\_)$ or $(0,2,\_,\_,\_)$.

6. Subtract $(0,0,2,2,2)$ and rearrange the terms when necessary until you obtain $(1,1,1,\_,\_)$.

7. Look up the obtained matching in the list provided below.

Let us consider the four necessary conditions during the "subtracting process."

1. $p = 5$ is not affected.

2. If $(a_1, a_2, a_3, a_4, a_5)$ satisfies $2a_5 + a_4 \leq a_1 + \ldots + a_5$, then $(a_1 - 2, a_2, a_3, a_4 - 2, a_5 - 2)$ satisfies $2(a_5 - 2) + (a_4 - 2) \leq (a_1 - 2) + a_2 + a_3 + (a_4 - 2) + (a_5 - 2)$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_5 - 2$ is not the largest, but the second largest part, then $a_3 = a_5$. So we started with $(a_1, a_2, a_5, a_5, a_5)$ and now have $(a_1 - 2, a_2, a_5 - 2, a_5 - 2, a_5)$. By Properties 2 and 4, $(a_1 - 2, a_2, a_5 - 2, a_5 - 2, a_5)$ is in S_2 as long as $6 \leq a_1 + a_2$. Let us consider the cases when $a_1 + a_2 < 6$. We could have $(1, 1, a_5, a_5, a_5)$, $(1, 3, a_5, a_5, a_5)$, $(0, 0, a_5, a_5, a_5)$, $(0, 2, a_5, a_5, a_5)$, $(0, 4, a_5, a_5, a_5)$ and $(2, 2, a_5, a_5, a_5)$. Each one of those is listed below.

   Case 2: If after rearranging $a_5 - 2$ is not the largest or second largest part, then $a_2 = \ldots = a_5$. So we started with $(a_1, a_5, a_5, a_5, a_5)$ and now have $(a_1 - 2, a_5 - 2, a_5 - 2, a_5, a_5)$. By Properties 2 and 4, $(a_1 - 2, a_5 - 2, a_5 - 2, a_5, a_5)$ is in S_2 as long as $6 \leq a_1 + a_5$. Let us consider the cases when $a_1 + a_5 < 6$. We could have $(1, 1, 1, 1, 1)$, $(1, 3, 3, 3, 3)$, $(0, 2, 2, 2, 2)$, $(0, 4, 4, 4, 4)$ and $(2, 2, 2, 2, 2)$. Each one of those is included in Case 1.

Case 3: If after rearranging $a_5 - 2$ is the largest, but $a_4 - 2$ is not the second largest part, then $a_3 = a_4$. So we started with $(a_1, a_2, a_4, a_4, a_5)$ and now have $(a_1 - 2, a_2, a_4 - 2, a_4, a_5 - 2)$. By Property 2, $(a_1 - 2, a_2, a_4 - 2, a_4, a_5 - 2)$ is in S_2 as long as $a_5 \neq a_1 + a_2 + a_4$. Let us consider the case when $a_5 = a_1 + a_2 + a_4$, then by Property 4, $a_4 = a_5$ and we had $(a_1, a_2, a_5, a_5, a_5)$. But since by assumption $a_5 = a_1 + a_2 + a_4 = a_1 + a_2 + a_5$, we have $0 = a_1 + a_2$, hence we started with $(0, 0, a_5, a_5, a_5)$ which is discussed below and shown to be in S_2.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, a_2, a_3, a_4, a_5)$ satisfies $3a_5 \leq a_1 + a_2 + a_3 + a_4 + a_5$, then $(a_1 - 2, a_2, a_3, a_4 - 2, a_5 - 2)$ satisfies $3(a_5 - 2) \leq (a_1 - 2) + a_2 + a_3 + (a_4 - 2) + (a_5 - 2)$. The problems that arise with rearranging the terms were discussed under Property 2.

Therefore, each time we subtract we obtain a member of S_2. When we reach $(1, 1, 1, \_, \_)$ or $(0, 2, \_, \_, \_)$ all four conditions are satisfied. Hence, an even matching of such a type exists. Following is a list of all even matchings of type $(1, 1, 1, \_, \_)$.

$(1, 1, 1, 1, 1)$ was given in Section 2.2.1

$(1, 1, 1, 3, 3)$:

M_1: $\{\{2, 5'\}, \{3, 4'\}, \{4'', 5''\}, \{4, 5\}\}$

M_2: $\{\{1, 4\}, \{3, 5\}, \{4', 5'\}, \{4'', 5''\}\}$

M_3: $\{\{1, 5''\}, \{2, 4''\}, \{4, 5\}, \{4', 5'\}\}$

M_4: $\{\{1, 5''\}, \{2, 5'\}, \{3, 5\}\}$

M_5: $\{\{1, 4\}, \{2, 4''\}, \{3, 4'\}\}$

$(1,1,a_5,a_5,a_5)$=$(1,1,1,1,1)$+$\frac{a_5-1}{2}$ copies of $(0,0,2,2,2)$

$(1,3,a_5,a_5,a_5)$=$(1,1,1,1,1)$+$(0,2,2,2,2)$+$\frac{a_5-3}{2}$ copies of $(0,0,2,2,2)$

Following are even matchings of types $(0,2,\_,\_,\_)$.

$(0,0,2,2,2)$ given in Section 2.2.2

By Property 2, an even matching of type $(0,0,a_3,a_4,a_5)$ must satisfy $a_5 \leq a_3$. Since $a_3 \leq a_5$ by definition, we must have types $(0,0,a_5,a_5,a_5)$.

$(0,0,a_5,a_5,a_5)$ from $a_5/2$ copies of $(0,0,2,2,2)$

$(0,2,2,2,2)$ given in Section 2.2.2

By Properties 2 and 4 the possible elements of S_2 with $a_1 = 0$ and $a_2 = 2$ are of the form $(0,2,a_5-2,a_5,a_5)$ or $(0,2,a_5,a_5,a_5)$.

$(0,2,a_5-2,a_5,a_5) = (0,2,0,2,2) + \frac{a_5-2}{2}$ copies of $(0,0,2,2,2)$

$(0,2,a_5,a_5,a_5)$= $(0,2,2,2,2) + \frac{a_5-2}{2}$ copies of $(0,0,2,2,2)$

$(0,4,a_5,a_5,a_5)$= 2 copies of $(0,2,2,2,2) + \frac{a_5-4}{2}$ copies of $(0,0,2,2,2)$

$(2,2,a_5,a_5,a_5)$= $(2,2,2,2,2) + \frac{a_5-2}{2}$ copies of $(0,0,2,2,2)$

We now have even matchings of type (a_1,a_2,a_3,a_4,a_5) as long as (a_1,a_2,a_3,a_4,a_5) satisfies the necessary conditions.

2.2.3 Seven parts of any size

Let us now consider even matchings of type $(a_1,\ldots,a_7)$ where the sequence is non-decreasing, i.e. $a_1 \leq \ldots \leq a_7$. Since $p = 7 \not\equiv 1 \ (mod\ 4)$, all parts must be of even size. Similar to the case of five parts, to find an even matching of type $(a_1,\ldots,a_7)$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = a_2 = a_3 = 0$ and $a_4 = 0, 2$ or $K = (0, 0, 2, 2, 2, 2, 2)$ look up the matching in list provided. Otherwise continue below.

2. For $0 \le i \le 3$ repeat the following steps.

3. If $5a_7 = 2n$ skip down to the Special Case 1 section.

   If $a_5 = a_6$ and $a_7 = a_1 + \ldots + a_5$ skip down to Special Case 2 section. Otherwise continue below.

4. Subtract $(0^i, 2, 0^{4-i}, 2, 2)$.

5. If necessary, rearrange the terms to ensure that the sequence is nondecreasing.

6. Repeat steps 3-5 until you obtain $(0, 0, 0, 2, \_, \_, \_)$.

7. The even matchings of type $(0, 0, 0, 2, \_, \_, \_)$ are listed below.

Let us consider the four necessary conditions during steps 3-5 of the algorithm.

1. $p = 7$ is not affected.

2. If $(a_1, \ldots, a_7)$ satisfies $2a_7 + a_6 \le a_1 + \ldots + a_7$, then $(a_1 - 2, a_2, a_3, a_4, a_5, a_6 - 2, a_7 - 2)$ satisfies $2(a_7 - 2) + (a_6 - 2) \le (a_1 - 2) + a_2 + a_3 + a_4 + a_5 + (a_6 - 2) + (a_7 - 2)$. We are, however, rearranging the terms. Let us consider the following cases:

   Case 1: If after such rearranging $a_7 - 2$ is not the largest, but the second largest part, then $a_5 = a_7$. So we started with $(a_1, a_2, a_3, a_4, a_7, a_7, a_7)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_7 - 2, a_7 - 2, a_7)$. By Properties 2 and 4, $(a_1 - 2, a_2, a_3, a_4, a_7 - 2, a_7 - 2, a_7)$ is in S_2 as long as $4 \le a_1 + a_2 + a_3 + a_4$ and $12 - a_7 \le 2a_1 + 2a_2 + 2a_3 + 2a_4$. Let us consider the case when $a_1 + a_2 + a_3 + a_4 < 4$. We could have $(0, 0, 0, 0, a_7, a_7, a_7)$ or $(0, 0, 0, 2, a_7, a_7, a_7)$. Each one of those is discussed and shown to be an element of S_2 below. Let us consider the case when $12 - a_7 > 2a_1 + 2a_2 + 2a_3 + 2a_4$. We could have $(0, 0, 0, 0, 2, 2, 2), (0, 0, 0, 0, 4, 4, 4), \ldots, (0, 0, 0, 0, 10, 10, 10)$, $(0, 0, 0, 2, 2, 2, 2)$, $(0, 0, 0, 2, 4, 4, 4), (0, 0, 0, 2, 6, 6, 6)$ or $(0, 0, 2, 2, 2, 2, 2)$. Each one of those is shown to be in S_2.

Case 2: If after rearranging $a_7 - 2$ is not the largest or second largest part, then $a_4 = \ldots = a_7$. So we started with $(a_1, a_2, a_3, a_7, a_7, a_7, a_7)$ and now have $(a_1 - 2, a_2, a_3, a_7 - 2, a_7 - 2, a_7, a_7)$. By Properties 2 and 4, $(a_1 - 2, a_2, a_3, a_7 - 2, a_7 - 2, a_7, a_7)$ is in S_2 as long as $6 - a_7 \leq a_1 + a_2 + a_3$ and $12 - 3a_7 \leq 2a_1 + 2a_2 + 2a_3$. Let us consider the cases when $6 - a_7 > a_1 + a_2 + a_3$ or $12 - 3a_7 > 2a_1 + 2a_2 + 2a_3$. We could have $(0, 0, 0, 2, 2, 2, 2)$, $(0, 0, 0, 4, 4, 4, 4)$ or $(0, 0, 2, 2, 2, 2, 2)$. Each one of those is in S_2.

Case 3: If after rearranging $a_7 - 2$ is the largest, but $a_6 - 2$ is not the second largest part, then $a_5 = a_6$. So we started with $(a_1, a_2, a_3, a_4, a_6, a_6, a_7)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_6 - 2, a_6, a_7 - 2)$. By Property 2, $(a_1 - 2, a_2, a_3, a_4, a_6 - 2, a_6, a_7 - 2)$ is in S_2 as long as $a_7 \neq a_1 + \ldots + a_5$. The case when $a_7 = a_1 + \ldots + a_5$ and $a_5 = a_6$ is discussed below as Special Case 2.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, \ldots, a_7)$ satisfies $5a_7 < 2(a_1 + \ldots + a_7)$, then $(a_1 - 2, a_2, a_3, a_4, a_5, a_6 - 2, a_7 - 2)$ satisfies $5(a_7 - 2) \leq 2((a_7 - 2) + a_2 + a_3 + a_4 + a_5 + (a_6 - 2) + (a_7 - 2))$. If $5a_7 = 2n$ we

refer to Special Case 1. Problems arising from rearranging the terms are discussed under item 2 above.

Therefore, each time we repeat steps 2-5 we obtain an element of S_2 and if we reach $(0,0,0,2,\_,\_,\_)$ all four conditions are satisfied. Notice that the above holds for the remaining steps of the algorithm. So it is sufficient to give even matchings of type $(0,0,0,2,\_,\_,\_)$ and consider the special cases mentioned above.

Let us consider the **Special Case 1**.

We define the set $\overline{SP1}$ as the set of all matching types satisfying the four necessary conditions and $3a_7 = 2(a_1 + \ldots + a_6)$. We construct an even matching of type $K = (a_1, \ldots, a_7) \in \overline{SP1}$ by induction. We start by subtracting $(0,0,0,2,2,2,4)$ to obtain $K' = (a_1, a_2, a_3, a_4 - 2, a_5 - 2, a_6 - 2, a_7 - 4)$. As long as no rearranging is necessary K' is in $\overline{SP1}$. If rearranging is necessary, we started with $K = (a_1, a_2, a_6, a_6, a_6, a_6, a_7)$ and now have $K'' = (a_1, a_2, a_6 - 2, a_6 - 2, a_6 - 2, a_6, a_7 - 4)$. This is always in $\overline{SP1}$. This concludes the inductive argument, since each time we repeat this process we obtain a smaller element of $\overline{SP1}$. Thus Special Case 1 is solved.

Now let us consider **Special Case 2**.

If $a_5 = a_6$ and $a_7 = a_1 + \ldots + a_5$ we have types of the form $(a_1, a_2, a_3, a_4, a_7 - (a_1 + \ldots + a_4), a_7 - (a_1 + \ldots + a_4), a_7)$. We will construct an even matching of such type as follows: $a_1/2$ copies of $(2,0,0,0,2,2,4)$, $a_2/2$ copies of $(0,2,0,0,2,2,4)$, $a_3/2$ copies of $(0,0,2,0,2,2,4)$, $a_4/2$ copies of $(0,0,0,2,2,2,4)$ and $(a_7 - 2(a_1 + \ldots + a_4))/2$ copies of $(0,0,0,0,2,2,2)$. Notice that by Property 4, $a_7 \geq 2(a_1 + \ldots + a_4)$. This concludes Special Case 2.

The following are even matchings of type $(0,0,0,2,\_,\_,\_)$.

$(0,0,0,0,2,2,2)$:

M_1: $\{\{5,7\},\{6,7'\},\{5',6'\}\}$

M_2: $\{\{5,6'\},\{6,7'\},\{5',7\}\}$

M_3: $\{\{5,6\},\{6',7'\},\{5',7\}\}$

M_4: $\{\{5,6'\},\{6,7\},\{5',7'\}\}$

M_5: $\{\{6,7\},\{6',7'\}\}$

M_6: $\{\{5,7\},\{5',7'\}\}$

M_7: $\{\{5,6\},\{5',6'\}\}$

By Property 2, an even matching of type $(0,0,0,0,a_5,a_6,a_7)$ exists if $a_7 \leq a_5$. Hence we must have $(0,0,0,0,a_7,a_7,a_7)$.

$(0,0,0,0,a_7,a_7,a_7)$ from $a_7/2$ copies of $(0,0,0,0,2,2,2)$

$(0,0,0,2,2,2,2)$:

M_1: $\{\{4,5\},\{6,7\},\{4',5'\},\{6',7'\}\}$

M_2: $\{\{4,5\},\{6,7\},\{4',7'\},\{5',6'\}\}$

M_3: $\{\{4,5\},\{6,7\},\{4',7'\},\{5',6'\}\}$

M_4: $\{\{5,7'\},\{5',6'\},\{6,7\}\}$

M_5: $\{\{4,7\},\{4',6\},\{6',7'\}\}$

M_6: $\{\{4,7\},\{4',5'\},\{5,7'\}\}$

M_7: $\{\{4,5\},\{4',6\},\{5',6'\}\}$

$(0,0,0,2,2,2,4)$:

M_1: $\{\{4,5\},\{6,7'\},\{6',7\},\{4',7''\},\{5',7'''\}\}$

M_2: $\{\{4,7''\},\{4',6\},\{5,7'''\},\{5',7'\},\{6',7\}\}$

M_3: $\{\{4,7\},\{4',7''\},\{5,7'''\},\{5',6'\},\{6,7'\}\}$

M_4: $\{\{5,7''\},\{5',7'''\},\{6,7\},\{6',7'\}\}$

M_5: $\{\{4,7''\},\{4',7'''\},\{6,7\},\{6',7'\}\}$

M_6: $\{\{4,7\},\{4',7'''\},\{5,7''\},\{5',7'\}\}$

M_7: $\{\{4,5\},\{4',6\},\{5',6'\}\}$

In general, by Property 2, an even matching of type $(0,0,0,2,a_5,a_6,a_7)$ exists if $a_5 \geq a_7 - 2$.

$(0,0,0,2,a_7,a_7,a_7)$ from $(0,0,0,2,2,2,2)$ and $(a_7-2)/2$ copies of $(0,0,0,0,2,2,2)$

$(0,0,0,2,a_7-2,a_7,a_7)$ from $(0,0,0,2,0,2,2)$ and $(a_7-2)/2$ copies of $(0,0,0,0,2,2,2)$

Since $a_7 \geq 2$, $a_7 - 2 \geq 0$.

$(0,0,0,2,a_7-2,a_7-2,a_7)$ from $(0,0,0,2,2,2,4)$ and $(a_7-4)/2$ copies of $(0,0,0,0,2,2,2)$

Since $a_7 - 2 \geq 2$, $a_7 - 4 \geq 0$.

$(0,0,0,4,4,4,4)$= 2 copies of $(0,0,0,2,2,2,2)$

Hence we have even matchings of type $(a_1,\ldots,a_7)$.

2.2.4 Nine parts of any size

Notice that when $p = 9$ condition 4 becomes $7a_9 \leq 3n$ which is equivalent to $4a_9 \leq 3(a_1 + \ldots + a_8)$. Condition 2 is $a_9 \leq a_1 + \ldots + a_7$. With the use of Lemma 2.4 finding an even matching of type $(0,0,a_3,\ldots,a_9)$ simply requires the even matching $(a_3,\ldots,a_9)$, which we constructed in the previous Section.

Let us now consider even matchings of type $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9$. Since $p = 9 \equiv 1(mod\ 4)$, we can have all parts of odd or all parts of even size. Similar to the case of five or seven parts, to find an even matching of type $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, $a_6 = 2$ or $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 1$ or $K = (0, 0, 0, 0, 2, 2, 2, 2, 2)$ or $K = (0, 0, 0, 0, 2, 2, 2, 2, 6)$ look up the matching in list provided. Otherwise continue below.

2. For $0 \le i \le 5$ repeat the following steps.

3. If $7a_9 = 3n$ skip down to the Special Case 1 section.

   If $7a_9 = 3n - 2$ skip down to the Special Case 2 section.

   If $a_7 = a_8$ and $a_9 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$ skip down to Special Case 3 section. Otherwise continue below.

4. Subtract $(0^i, 2, 0^{6-i}, 2, 2)$.

5. If necessary, rearrange the terms to ensure that the parts are in a nondecreasing sequence.

6. Repeat steps 3-5 until you obtain $(0, 0, 0, 0, 0, 2, \_, \_, \_)$ or $(1, 1, 1, 1, 1, 1, 1, \_, \_)$. If at any point $K = (0, 0, 0, 0, 2, 2, 2, 2, 2)$ or $K = (0, 0, 0, 0, 2, 2, 2, 2, 6)$ look it up.

7. If $7a_9 = 3n$ skip down to the Special Case 1 section.

   If $7a_9 = 3n - 2$ skip down to the Special Case 2 section.

   If $a_7 = a_8$ and $a_9 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$ skip down to Special Case 3 section. Otherwise continue below.

8. Subtract $(0^6, 2, 2, 2)$.

9. If necessary, rearrange the terms to ensure that the parts are in a nondecreasing sequence.

10. Repeat until you obtain $(1, 1, 1, 1, 1, 1, 1, \_, \_)$.

11. The even matchings of type $(0, 0, 0, 0, 0, 2, \_, \_, \_)$ and $(1, 1, 1, 1, 1, 1, 1, \_, \_)$ are listed below.

Let us consider the four necessary conditions during steps 2-5 of the algorithm.

1. $p = 9$ is not affected.

2. If $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ satisfies $a_9 \leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$, then $(a_1 - 2, a_2, a_3, a_4, a_5, a_6, a_7, a_8 - 2, a_9 - 2)$ satisfies $a_9 - 2 \leq a_1 - 2 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_9 - 2$ is not the largest, but the second largest part, then $a_7 = a_9$. So we started with $(a_1, a_2, a_3, a_4, a_5, a_6, a_9, a_9, a_9)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_5, a_6, a_9 - 2, a_9 - 2, a_9)$. By Properties 2 and 4 $(a_1 - 2, a_2, a_3, a_4, a_5, a_6, a_9 - 2, a_9 - 2, a_9)$ is in S_2 as long as $4 \leq a_1 + \ldots + a_6$ and $18 - 2a_9 \leq 3(a_1 + \ldots + a_6)$. Let us consider the cases when $4 > a_1 + \ldots + a_6$ or $18 - 2a_9 > 3(a_1 + \ldots + a_6)$. We could have $(0, 0, 0, 0, 0, 0, a_9, a_9, a_9)$ or $(0, 0, 0, 0, 0, 2, a_9, a_9, a_9)$. Each one of those is discussed and shown to be in S_2.

   Case 2: If after rearranging $a_9 - 2$ is not the largest or second largest part, then $a_6 = a_9$. So we started with $(a_1, a_2, a_3, a_4, a_5, a_9, a_9, a_9, a_9)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_5, a_9 - 2, a_9 - 2, a_9, a_9)$. By Properties 2 and 4 $(a_1, a_2, a_3, a_4, a_5, a_9 - 2, a_9 - 2, a_9, a_9)$ is in S_2 as long as $6 - a_9 \leq a_1 + \ldots + a_5$ and $18 - 5a_9 \leq 3(a_1 + \ldots + a_5)$.

Let us consider the cases when $6 - a_9 > a_1 + \ldots + a_5$ or $18 - 5a_9 > 3(a_1 + \ldots + a_5)$. We could have $(0,0,0,0,0,2,2,2,2)$, $(0,0,0,0,0,4,4,4,4)$ or $(0,0,0,0,2,2,2,2,2)$. Each one of those is in S_2.

Case 3: If after rearranging $a_9 - 2$ is the largest, but $a_8 - 2$ is not the second largest part, then $a_7 = a_8$. So we started with $(a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_8, a_9)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_5, a_6, a_8 - 2, a_8, a_9 - 2)$. By Property 2, $(a_1 - 2, a_2, a_3, a_4, a_5, a_6, a_8 - 2, a_8, a_9 - 2)$ is in S_2 as long as $a_9 \neq a_1 + \ldots + a_7$. The case when $a_9 = a_1 + \ldots + a_7$ and $a_7 = a_8$ is discussed below as Special Case 3.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ satisfies $4a_9 < 3(a_1 + \ldots + a_8) - 2$, then $(a_1 - 2, a_2, a_3, a_4, a_5, a_6, a_7, a_8 - 2, a_9 - 2)$ satisfies $4(a_9 - 2) \leq 3(a_1 + \ldots + a_8 - 4)$. If $7a_9 = 3n$ we refer to Special Case 1. If $7a_9 = 3n - 2$ we refer to Special Case 2. Problems arising from rearranging the terms are discussed under Property 2.

Therefore, each time we repeat steps 2-5 we obtain an element of S_2. Notice that the above holds for the remaining steps of the algorithm. So it is sufficient to give even matchings of type $(0,0,0,0,0,2,\_,\_,\_)$ and $(1,1,1,1,1,1,1,\_,\_)$ and consider the special cases mentioned above.

Let us consider the **Special Case 1**.

We need to find even matchings for $(a_1, \ldots, a_9)$ satisfying the four necessary conditions and $4a_9 = 3(a_1 + \ldots + a_8)$. Let us refer to these as matching type $SP1 \subseteq S_2$. Say we need $K = (a_1, a_2, \ldots, a_8, \frac{3(a_1+\ldots+a_8)}{4}) \in SP1$. We will build it by induction.

We consider parts of odd size first. We start by subtracting $P = (1^6, 3, 3, 9)$ to obtain $K' = (a_1 - 1, \ldots, a_6 - 1, a_7 - 3, a_8 - 3, \frac{3(a_1+\ldots+a_8)}{4} - 9)$. Since $\frac{3(a_1+\ldots+a_8)}{4} \leq a_1 + \ldots + a_7$ in K we have $\frac{3(a_1+\ldots+a_8)}{4} - 9 \leq a_1 + \ldots + a_7 - 9$ in K'. Also, $4(\frac{3(a_1+\ldots+a_8)}{4} - 9) = 3(a_1 + \ldots + a_8 - 12)$. Therefore K' is in S_2. However, if rearranging is necessary and we obtain $K'' = (a_1 - 1, \ldots, a_5 - 1, a_7 - 3, a_8 - 3, a_6 - 1, \frac{3(a_1+\ldots+a_8)}{4} - 9)$ instead of K', we must have $a_6 - 1 = a_8 - 3 + 2$ or $a_6 = a_8$. Hence we started with $(a_1, \ldots, a_5, a_8, a_8, a_8, \frac{3(a_1+\ldots+a_8)}{4})$. In this case instead of subtracting $P = (1^6, 3, 3, 9)$, we subtract $Q = (1^5, 5, 5, 5, 15)$ to obtain $K' = (a_1 - 1, \ldots, a_5 - 1, a_8 - 5, a_8 - 5, a_8 - 5, \frac{3(a_1+\ldots+a_8)}{4} - 15)$. Notice that $4(\frac{3(a_1+\ldots+a_8)}{4} - 15) = 3(a_1 + \ldots + a_8 - 20)$ and $\frac{3(a_1+\ldots+a_8)}{4} - 15 \leq a_1 + \ldots + a_7 - 15$ in K' and so $K' \in S_2$. However, if rearranging is necessary and results in $K'' = (a_1 - 1, \ldots, a_4 - 1, a_8 - 5, a_8 - 5, a_8 - 5, a_5 - 1, \frac{3(a_1+\ldots+a_8)}{4} - 15)$ we must have $a_5 - 1 = a_8 - 5 + 2$ or $a_5 = a_8 - 2$. So we started with $K = (a_1, \ldots, a_4, a_8 - 2, a_8, a_8, a_8, \frac{3(a_1+\ldots+a_8)}{4})$. Notice that by Property 2, we have $\frac{3(a_1+\ldots+a_4+4a_8-2)}{4} \leq a_1 + \ldots + a_4 + 3a_8 - 2$, which implies $2 \leq a_1 + \ldots + a_4$. However, we know that since $a_i \geq 1$ for all i, $4 \leq a_1 + \ldots + a_4$. Therefore, $\frac{3(a_1+\ldots+a_4+4a_8-2)}{4} \leq a_1 + \ldots + a_4 + 3a_8 - 4$. In this case we go back to subtracting $P = (1^6, 3, 3, 9)$ to obtain $K' = (a_1 - 1, \ldots, a_4 - 1, a_8 - 3, a_8 - 1, a_8 - 3, a_8 - 3, \frac{3(a_1+\ldots+a_8)}{4} - 9)$ which would be rearranged to $K'' = (a_1 - 1, \ldots, a_4 - 1, a_8 - 3, a_8 - 3, a_8 - 3, a_8 - 1, \frac{3(a_1+\ldots+a_8)}{4} - 9)$. Notice that $\frac{3(a_1+\ldots+a_8)}{4} - 9 \leq a_1 + \ldots + a_4 + 3a_8 - 13$ and $4(\frac{3(a_1+\ldots+a_8)}{4} - 9) = 3(a_1 + \ldots + a_8 - 12)$ and $K'' \in S_2$. Because $a_4 \leq a_8$ it is not possible for the rearranging to result in $(a_1 - 1, a_2 - 1, a_3 - 2, a_8 - 3, a_8 - 3, a_8 - 3, a_8 - 1, a_4 - 1, \frac{3(a_1+\ldots+a_8)}{4} - 9)$. One more rearranging issue needs to be addressed. Is it possible for $a_8 - 3 \geq \frac{3(a_1+\ldots+a_8)}{4} - 9 + 2$ or $a_8 - 5 \geq \frac{3(a_1+\ldots+a_8)}{4} - 15 + 2$? It is sufficient to show that $a_8 < \frac{3(a_1+\ldots+a_8)}{4} - 8$. If $a_8 = \frac{3(a_1+\ldots+a_8)}{4} - 8$ we have $K = (a_1, \ldots, a_7, 3(a_1 + \ldots + a_7) - 32, 3(a_1 + \ldots + a_7) - 24)$

and by Property 2, this implies $3(a_1+\ldots+a_7)-24 \leq a_1+\ldots+a_7$. So $a_1+\ldots+a_7 \leq 12$, and this is only possible if $K=(1^6,3,3,9)$ which is given below. This shows that when we complete this step, we obtain an element of $SP1$.

For parts of even size we start by subtracting $(0^4,2^4,6)$ to obtain $K'=(a_1,\ldots,a_4,a_5-2,\ldots,a_8-2,\frac{3(a_1+\ldots+a_8)}{4}-6)$. As long as no rearranging is necessary K' is in $SP1$. If rearranging is necessary, we started with $K=(a_1,a_2,a_3,a_8,\ldots,a_8,\frac{3(a_1+\ldots+a_8)}{4})$ and now have $K''=(a_1,a_2,a_3,a_8-2,\ldots,a_8-2,a_8,\frac{3(a_1+\ldots+a_8)}{4}-6)$. This is in $SP1$ as long as $-16 \leq a_1+a_2+a_3+a_8$ which is always true for elements of S_2.

This concludes the induction argument, since each time we perform the above we obtain a smaller element of $SP1$. This concludes Special Case 1.

Let us consider the **Special Case 2**.

We need to find even matchings for $(a_1,\ldots,a_9)$ satisfying the four necessary conditions and $4a_9=3(a_1+\ldots+a_8)-2$. Let us refer to these as matching type $SP2 \subseteq S_2$. Say we need $K=(a_1,a_2,\ldots,a_8,\frac{3(a_1+\ldots+a_8)-2}{4}) \in SP2$. We will build it by reducing the matching to an element of $SP1$ which is considered above.

We start with parts of odd size. First, subtract $P=(1^7,3,7)$ to obtain $K'=(a_1-1,\ldots,a_6-1,a_7-1,a_8-3,\frac{3(a_1+\ldots+a_8)-2}{4}-7)$. Since $\frac{3(a_1+\ldots+a_8)-2}{4} \leq a_1+\ldots+a_7$ in K we have $\frac{3(a_1+\ldots+a_8)-2}{4}-7 \leq a_1+\ldots+a_7-7$ in K'. Also, $4(\frac{3(a_1+\ldots+a_8)-2}{4}-7)=3(a_1+\ldots+a_8-10)$ which takes us back to Special Case 1. However, if rearranging is necessary and we obtain $K''=(a_1-1,a_2-2,\ldots,a_5-1,a_6-1,a_8-3,a_7-1,\frac{3(a_1+\ldots+a_8)-2}{4}-7)$ instead of K', we must have $a_7-1=a_8-3+1$ or $a_7=a_8$. Hence we started with $(a_1,\ldots,a_5,a_6,a_8,a_8,\frac{3(a_1+\ldots+a_8)-2}{4})$. In this case instead of subtracting $P=(1^7,3,7)$, we subtract $Q=(1^5,3,5,5,13)$ to obtain $K'=(a_1-1,\ldots,a_5-1,a_6-3,a_8-5,a_8-$

$5, \frac{3(a_1+...+a_8)-2}{4} - 13)$. Notice that $4(\frac{3(a_1+...+a_8)-2}{4} - 13) = 3(a_1 + \ldots + a_8 - 18)$ and $\frac{3(a_1+...+a_8)-2}{4} - 13 \leq a_1 + \ldots + a_7 - 13$ in K' taking us to Special Case 1. However, if rearranging is necessary and results in $K'' = (a_1 - 1, \ldots, a_4 - 1, a_5 - 1, a_8 - 5, a_8 - 5, a_6 - 3, \frac{3(a_1+...+a_8)-2}{4} - 13)$ we must have $a_6 - 3 = a_8 - 5 + 2$ or $a_6 = a_8$. So we started with $K = (a_1, \ldots, a_4, a_5, a_8, a_8, a_8, \frac{3(a_1+...+a_8)-2}{4})$. In this case we subtract $M = (1^5, 7, 7, 7, 19)$ instead of P or Q. This gives $K' = (a_1 - 1, \ldots, a_5 - 1, a_8 - 7, a_8 - 7, a_8 - 7, \frac{3(a_1+...+a_8)-2}{4} - 19)$ which is back in Special Case 1. If rearranging is necessary, we have $a_5 = a_8 - 4$ and $K'' = (a_1 - 1, \ldots, a_8 - 7, a_8 - 7, a_8 - 7, a_8 - 5, \frac{3(a_1+...+a_8)-2}{4} - 19)$ which came from $K = (a_1, \ldots, a_4, a_8 - 4, a_8, a_8, a_8, \frac{3(a_1+...+a_8)-2}{4})$. Notice that by Property 2, we have $\frac{3(a_1+...+a_4+4a_8-4)}{4} \leq a_1 + \ldots + a_4 + 3a_8 - 4$, which implies $2 \leq a_1 + \ldots + a_4$. However, we know that since $a_i \geq 1$ for all i, $4 \leq a_1 + \ldots + a_4$. Therefore, $\frac{3(a_1+...+a_4+4a_8-4)}{4} \leq a_1 + \ldots + a_4 + 3a_8 - 6$. In this case we go back to subtracting $P = (1^7, 3, 7)$ to obtain $K' = (a_1 - 1, \ldots, a_4 - 1, a_8 - 5, a_8 - 1, a_8 - 1, a_8 - 3, \frac{3(a_1+...+a_8)-2}{4} - 7)$ which would be rearranged to $K'' = (a_1 - 1, \ldots, a_4 - 1, a_8 - 5, a_8 - 3, a_8 - 1, a_8 - 1, \frac{3(a_1+...+a_8)-2}{4} - 7)$. Notice that $\frac{3(a_1+...+a_8)}{4} - 7 \leq a_1 + \ldots + a_4 + 3a_8 - 11$ and $4(\frac{3(a_1+...+a_8)-2}{4} - 7) = 3(a_1 + \ldots + a_8 - 10)$ and $K'' \in S_2$ and under Special Case 1. Because $a_4 \leq a_8$ it is not possible for the rearranging to result in $(a_1 - 1, a_2 - 1, a_3 - 2, a_8 - 5, a_8 - 3, a_8 - 1, a_8 - 1, a_4 - 1, \frac{3(a_1+...+a_8)-2}{4} - 7)$. One more rearranging issue needs to be addressed. Is it possible for $a_8 - 3 \geq \frac{3(a_1+...+a_8)-2}{4} - 7 + 2$, $a_8 - 5 \geq \frac{3(a_1+...+a_8)-2}{4} - 13 + 2$ or $a_8 - 7 \geq \frac{3(a_1+...+a_8)-2}{4} - 17 + 2$? It is sufficient to show that $a_8 < \frac{3(a_1+...+a_8)-2}{4} - 8$. If $a_8 = \frac{3(a_1+...+a_8)-2}{4} - 8$ we have $K = (a_1, \ldots, a_7, 3(a_1 + \ldots + a_7) - 34, 3(a_1 + \ldots + a_7) - 26)$ and by Property 2, this implies $3(a_1 + \ldots + a_7) - 26 \leq a_1 + \ldots + a_7$. So $a_1 + \ldots + a_7 \leq 13$, and this is only possible if $K = (1^7, 3, 7)$ which is given below or $K = (1^4, 3^3, 5, 13) = (1^7, 3, 7) + (0^4, 2^4, 6)$ or

$K = (1^3, 3^5, 13) = (1^7, 3, 7) + (0^3, 2^4, 0, 6)$. This shows that when we complete this step, we obtain an element of $SP1$. And we continue with the second step in Special Case 1.

For parts of even size we start by subtracting $(0^5, 2^3, 4)$ to obtain $K' = (a_1, \ldots, a_5, a_6 - 2, a_7 - 2, a_8 - 2, \frac{3(a_1+\ldots+a_8)-2}{4} - 4)$. As long as no rearranging is necessary K' is in $SP1$. If rearranging is necessary, we started with $K = (a_1, a_2, a_3, a_4, a_8, \ldots, a_8, \frac{3(a_1+\ldots+a_8)-2}{4})$ and now have $K'' = (a_1, a_2, a_3, a_4, a_8 - 2, a_8 - 2, a_8 - 2, a_8, \frac{3(a_1+\ldots+a_8)-2}{4} - 4)$. This K'' is in $SP1$ as long as $-12 \leq a_1 + \ldots + a_4$ which is always true. This concludes Special Case 2.

Now let us consider **Special Case 3**.

If $a_7 = a_8$ and $a_9 = a_1 + \ldots + a_7$ we have types of the form $(a_1, a_2, a_3, a_4, a_5, a_6, a_9 - (a_1 + \ldots + a_6), a_9 - (a_1 + \ldots + a_6), a_9)$. We will construct an even matching of such type as follows: $a_1/2$ copies of $(2, 0, 0, 0, 0, 0, 2, 2, 4)$, $a_2/2$ copies of $(0, 2, 0, 0, 0, 0, 2, 2, 4)$, $\ldots$, $a_6/2$ copies of $(0, 0, 0, 0, 0, 2, 2, 2, 4)$ and $a_9 - 2(a_1 + \ldots + a_6)/2$ copies of $(0, 0, 0, 0, 0, 0, 2, 2, 2)$. Notice that by Property 4 $a_9 \geq 2(a_1 + \ldots + a_6)$. This concludes Special Case 3.

The following are even matchings of type $(0, 0, 0, 0, 0, 2, \_, \_, \_)$ and $(1, 1, 1, 1, 1, 1, 1, \_, \_)$.

$(0, 0, 0, 0, 0, 0, 2, 2, 2)$: from Lemma 2.4

By Property 2, an even matching of type $(0, 0, 0, 0, 0, 0, a_7, a_8, a_9)$ exists if $a_9 \leq a_7$. Hence we must have $(0, 0, 0, 0, 0, 0, a_9, a_9, a_9)$.

$(0, 0, 0, 0, 0, 0, a_9, a_9, a_9)$ from $a_9/2$ copies of $(0, 0, 0, 0, 0, 0, 2, 2, 2)$

$(0, 0, 0, 0, 0, 2, 2, 2, 2)$: from Lemma 2.4

$(0, 0, 0, 0, 0, 2, 2, 2, 4)$: from Lemma 2.4

$(0, 0, 0, 0, 2, 2, 2, 2, 2)$: from Lemma 2.4

By Property 2, an even matching of type $(0, 0, 0, 0, 0, 2, a_7, a_8, a_9)$ exists if $a_7 \geq a_9 - 2$.

$(0,0,0,0,0,2,a_9,a_9,a_9)$ from $(0,0,0,0,0,2,2,2,2)$ and

$(a_9-2)/2$ copies of $(0,0,0,0,0,0,2,2,2)$

$(0,0,0,0,0,2,a_9-2,a_9,a_9)$ from $(0,0,0,0,0,2,0,2,2)$ and

$(a_9-2)/2$ copies of $(0,0,0,0,0,0,2,2,2)$

Since $a_9 \geq 2$, $a_9-2 \geq 0$.

$(0,0,0,0,0,2,a_9-2,a_9-2,a_9)$ from $(0,0,0,0,0,2,2,2,4)$ and

$(a_9-4)/2$ copies of $(0,0,0,0,0,0,2,2,2)$

Since $a_9-2 \geq 2$, $a_9-4 \geq 0$.

$(0,0,0,0,2,2,2,2,6)$:

M_1: $\{\{5,9\},\{5',6'\},\{6,9'\},\{7,9'''\},\{7',9''\},\{8,9^4\},\{8',9^5\}\}$

M_2: $\{\{5,9^4\},\{5',9^5\},\{6,9'''\},\{6',9''\},\{7,9'\},\{7',8'\},\{8,9\}\}$

M_3: $\{\{5,9\},\{5',9'''\},\{6,9'\},\{6',9''\},\{7,8\},\{7',9^5\},\{8',9^4\}\}$

M_4: $\{\{5,6\},\{5',9'''\},\{6',9''\},\{7,9'\},\{7',9^5\},\{8,9\},\{8',9^4\}\}$

M_5: $\{\{6,9^4\},\{6',9^5\},\{7,9''\},\{7',9'''\},\{8,9\},\{8',9'\}\}$

M_6: $\{\{5,9\},\{5',9'\},\{7,9'''\},\{7',9''\},\{8,9^4\},\{8',9^5\}\}$

M_7: $\{\{5,9^4\},\{5',9^5\},\{6,9'''\},\{6',9''\},\{8,9\},\{8',9'\}\}$

M_8: $\{\{5,9\},\{5',9'\},\{6,9^4\},\{6',9^5\},\{7,9''\},\{7',9'''\}\}$

M_9: $\{\{5,6\},\{5',6'\},\{7,8\},\{7',8'\}\}$

$(0,0,0,0,0,4,4,4,4)$= 2 copies of $(0,0,0,0,0,2,2,2,2)$

$(1,1,1,1,1,1,1,1,1)$: given in Section 2.2.1

$(1,1,1,1,1,1,1,1,3)$:

M_1: $\{\{2,9''\},\{3,9'\},\{4,5\},\{6,8\},\{7,9\}\}$

M_2: $\{\{1,9''\},\{3,7\},\{4,5\},\{6,9'\},\{8,9\}\}$

M_3: $\{\{1,9''\},\{2,8\},\{4,6\},\{5,9\},\{7,9'\}\}$

M_4: $\{\{1,6\},\{2,9''\},\{3,9'\},\{5,9\},\{7,8\}\}$

M_5: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,6\},\{7,8\}\}$

M_6: $\{\{1,2\},\{3,4\},\{5,9''\},\{7,9'\},\{8,9\}\}$

M_7: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,5\},\{6,8\}\}$

M_8: $\{\{1,2\},\{3,4\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_9: $\{\{1,6\},\{2,8\},\{3,7\},\{4,5\}\}$

$(1,1,1,1,1,1,1,1,5)$:

M_1: $\{\{2,9'''\},\{3,4\},\{5,9'\},\{6,9^4\},\{7,9''\},\{8,9\}\}$

M_2: $\{\{1,9^4\},\{3,9'\},\{4,9'''\},\{5,9''\},\{6,8\},\{7,9\}\}$

M_3: $\{\{1,2\},\{4,9'''\},\{5,9'\},\{6,9^4\},\{7,9\},\{8,9''\}\}$

M_4: $\{\{1,9^4\},\{2,9'''\},\{3,9'\},\{5,6\},\{7,9''\},\{8,9\}\}$

M_5: $\{\{1,2\},\{3,9^4\},\{4,9'''\},\{6,9'\},\{7,9\},\{8,9''\}\}$

M_6: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,9'''\},\{5,9^4\},\{7,8\}\}$

M_7: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,9'''\},\{5,9^4\},\{6,8\}\}$

M_8: $\{\{1,2\},\{3,9^4\},\{4,9'''\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_9: $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$

$(1,1,1,1,1,1,1,3,3)$:

M_1: $\{\{2,8'\},\{3,9''\},\{4,9'\},\{5,8\},\{6,8''\},\{7,9\}\}$

M_2: $\{\{1,8''\},\{3,8\},\{4,8'\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_3: $\{\{1,8''\},\{2,8'\},\{4,9''\},\{5,8\},\{6,9'\},\{7,9\}\}$

M_4: $\{\{1,9\},\{2,9'\},\{3,9''\},\{5,8'\},\{6,8''\},\{7,8\}\}$

M_5: $\{\{1,8''\},\{2,8'\},\{3,8\},\{4,9''\},\{6,9'\},\{7,9\}\}$

M_6: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,8''\},\{5,8'\},\{7,8\}\}$

M_7: $\{\{1,9\},\{2,8'\},\{3,9''\},\{4,8''\},\{5,8\},\{6,9'\}\}$

M_8: $\{\{1,9\},\{2,3\},\{4,9'\},\{5,9''\},\{6,7\}\}$

M_9: $\{\{1,8''\},\{2,3\},\{4,8'\},\{5,8\},\{6,7\}\}$

$(1,1,1,1,1,1,1,3,5)$:

M_1: $\{\{2,9^4\},\{3,9'\},\{4,8''\},\{5,9''\},\{6,8\},\{7,9\},\{8',9'''\}\}$

M_2: $\{\{1,8''\},\{3,9^4\},\{4,9'''\},\{5,9'\},\{6,9''\},\{7,8'\},\{8,9\}\}$

M_3: $\{\{1,9\},\{2,8\},\{4,8'\},\{5,9'\},\{6,9''\},\{7,9'''\},\{8'',9^4\}\}$

M_4: $\{\{1,9^4\},\{2,9'''\},\{3,8''\},\{5,8'\},\{6,9''\},\{7,9'\},\{8,9\}\}$

M_5: $\{\{1,9\},\{2,8\},\{3,9'\},\{4,8'\},\{6,9''\},\{7,9'''\},\{8'',9^4\}\}$

M_6: $\{\{1,9^4\},\{2,9'''\},\{3,8''\},\{4,9''\},\{5,8'\},\{7,9'\},\{8,9\}\}$

M_7: $\{\{1,8''\},\{2,9^4\},\{3,5\},\{4,9''\},\{6,9'\},\{8,9\},\{8',9'''\}\}$

M_8: $\{\{1,2\},\{3,9^4\},\{4,9'''\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_9: $\{\{1,2\},\{3,5\},\{4,8''\},\{6,8\},\{7,8'\}\}$

$(1,1,1,1,1,1,1,3,7)$:

M_1: $\{\{2,9^5\},\{3,9^6\},\{4,8\},\{5,9''\},\{6,9'\},\{7,9\},\{8',9'''\},\{8'',9^4\}\}$

M_2: $\{\{1,9^6\},\{3,7\},\{4,9^5\},\{5,9^4\},\{6,9'''\},\{8,9\},\{8',9'\},\{8'',9''\}\}$

M_3: $\{\{1,9^6\},\{2,9^5\},\{4,9'''\},\{5,6\},\{7,9\},\{8,9'\},\{8',9''\},\{8'',9^4\}\}$

M_4: $\{\{1,9^6\},\{2,9^5\},\{3,9^4\},\{5,9''\},\{6,9'\},\{7,8'\},\{8,9\},\{8'',9'''\}\}$

M_5: $\{\{1,9^6\},\{2,9^5\},\{3,7\},\{4,9^4\},\{6,9'\},\{8,9\},\{8',9''\},\{8'',9'''\}\}$

M_6: $\{\{1,9^6\},\{2,9^5\},\{3,8''\},\{4,9^4\},\{5,9''\},\{7,9\},\{8,9'\},\{8',9'''\}\}$

M_7: $\{\{1,2\},\{3,9^6\},\{4,9^5\},\{5,9^4\},\{6,9'''\},\{8,9\},\{8',9'\},\{8'',9''\}\}$

M_8: $\{\{1,9^6\},\{2,9^5\},\{3,9^4\},\{4,9'''\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_9: $\{\{1,2\},\{3,8''\},\{4,8\},\{5,6\},\{7,8'\}\}$

$(1,1,1,1,1,1,1,5,5)$:

M_1: $\{\{2,9\},\{3,8'\},\{4,8'''\},\{5,9''\},\{6,7\},\{8,9'\},\{8'',9'''\},\{8^4,9^4\}\}$

M_2: $\{\{1,9\},\{3,8''\},\{4,8'''\},\{5,9''\},\{6,9'''\},\{7,8'\},\{8,9'\},\{8^4,9^4\}\}$

M_3: $\{\{1,8\},\{2,9\},\{4,9'''\},\{5,8''\},\{6,9'\},\{7,8'\},\{8''',9''\},\{8^4,9^4\}\}$

M_4: $\{\{1,2\},\{3,8'\},\{5,8''\},\{6,9'''\},\{7,9\},\{8,9'\},\{8''',9''\},\{8^4,9^4\}\}$

M_5: $\{\{1,2\},\{3,4\},\{6,7\},\{8,9\},\{8',9'\},\{8'',9''\},\{8''',9'''\},\{8^4,9^4\}\}$

M_6: $\{\{1,9\},\{2,8'\},\{3,9^4\},\{4,8^4\},\{5,9''\},\{7,8'''\},\{8,9'\},\{8'',9'''\}\}$

M_7: $\{\{1,2\},\{3,4\},\{5,6\},\{8,9\},\{8',9'\},\{8'',9''\},\{8''',9'''\},\{8^4,9^4\}\}$

M_8: $\{\{1,2\},\{3,9^4\},\{4,9'''\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_9: $\{\{1,8\},\{2,8'\},\{3,8''\},\{4,8^4\},\{5,6\},\{7,8'''\}\}$

$(1,1,1,1,1,1,1,5,7)$:

M_1: $\{\{2,9\},\{3,8'\},\{4,8'''\},\{5,9''\},\{6,9'''\},\{7,9^6\},\{8,9'\},\{8'',9^5\},\{8^4,9^4\}\}$

M_2: $\{\{1,9^6\},\{3,9^5\},\{4,8'''\},\{5,9''\},\{6,9'''\},\{7,8'\},\{8,9'\},\{8'',9\},\{8^4,9^4\}\}$

M_3: $\{\{1,8\},\{2,9\},\{4,9'''\},\{5,9^6\},\{6,9'\},\{7,8'\},\{8'',9^5\},\{8''',9''\},\{8^4,9^4\}\}$

M_4: $\{\{1,9'''\},\{2,6\},\{3,9^5\},\{5,9''\},\{7,8'''\},\{8,9'\},\{8',9^6\},\{8'',9\},\{8^4,9^4\}\}$

M_5: $\{\{1,2\},\{3,4\},\{6,9^5\},\{7,9^6\},\{8,9\},\{8',9'\},\{8'',9''\},\{8''',9'''\},\{8^4,9^4\}\}$

M_6: $\{\{1,9'''\},\{2,9^5\},\{3,9^4\},\{4,8^4\},\{5,8''\},\{7,9\},\{8,9'\},\{8',9^6\},\{8''',9''\}\}$

M_7: $\{\{1,2\},\{3,4\},\{5,9^6\},\{6,9^5\},\{8,9\},\{8',9'\},\{8'',9''\},\{8''',9'''\},\{8^4,9^4\}\}$

M_8: $\{\{1,9^6\},\{2,9^5\},\{3,9^4\},\{4,9'''\},\{5,9''\},\{6,9'\},\{7,9\}\}$

M_9: $\{\{1,8\},\{2,6\},\{3,8'\},\{4,8^4\},\{5,8''\},\{7,8'''\}\}$

$(1,1,1,1,1,1,1,7,7)$:

M_1: $\{\{2,9^6\},\{3,4\},\{5,6\},\{7,8^6\},\{8,9\},\{8',9'\},\{8'',9''\},\{8''',9'''\},\{8^4,9^4\},\{8^5,9^5\}\}$

M_2: $\{\{1,9\},\{3,8^4\},\{4,9'''\},\{5,8''\},\{6,9^5\},\{7,8\},\{8',9'\},\{8''',9''\},\{8^5,9^4\},\{8^6,9^6\}\}$

M_3: $\{\{1,2\},\{4,9^6\},\{5,6\},\{7,8^6\},\{8,9\},\{8',9'\},\{8'',9'''\},\{8''',9''\},\{8^4,9^4\},\{8^5,9^5\}\}$

M_4: $\{\{1,9\},\{2,9^6\},\{3,8^6\},\{5,6\},\{7,8\},\{8',9'\},\{8'',9'''\},\{8''',9''\},\{8^4,9^5\},\{8^5,9^4\}\}$

M_5: $\{\{1,8^6\},\{2,9'\},\{3,9''\},\{4,8'''\},\{6,8'\},\{7,9^6\},\{8,9\},\{8'',9'''\},\{8^4,9^4\},\{8^5,9^5\}\}$

M_6: $\{\{1,9\},\{2,8^5\},\{3,8^6\},\{4,9^6\},\{5,9^4\},\{7,8\},\{8',9'\},\{8'',9'''\},\{8''',9''\},\{8^4,9^5\}\}$

M_7: $\{\{1,2\},\{3,4\},\{5,6\},\{8,9\},\{8',9'\},\{8'',9''\},\{8''',9'''\},\{8^4,9^4\},\{8^5,9^5\},\{8^6,9^6\}\}$

M_8: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,9'''\},\{5,9^4\},\{6,9^5\},\{7,9^6\}\}$

M_9: $\{\{1,8^6\},\{2,8^5\},\{3,8^4\},\{4,8'''\},\{5,8''\},\{6,8'\},\{7,8\}\}$

$(1,1,1,1,1,1,3,3,9)$:

M_1: $\{\{2,9^8\},\{3,4\},\{5,9^6\},\{6,9^5\},\{7,9^4\},\{7',9''\},\{7'',9^7\},\{8,9\},\{8',9'\},\{8'',9'''\}\}$

M_2: $\{\{1,9'\},\{3,9'''\},\{4,9''\},\{5,6\},\{7,9^5\},\{7',9^4\},\{7'',9^7\},\{8,9\},\{8',9^6\},\{8'',9^8\}\}$

M_3: $\{\{1,9'\},\{2,9^8\},\{4,9''\},\{5,9^6\},\{6,9^5\},\{7,9^4\},\{7',8'\},\{7'',9^7\},\{8,9\},\{8'',9'''\}\}$

M_4: $\{\{1,9^8\},\{2,9^7\},\{3,9^6\},\{5,9^4\},\{6,9'''\},\{7,9^5\},\{7',9''\},\{7'',8''\},\{8,9\},\{8',9'\}\}$

M_5: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,9'''\},\{6,9^5\},\{7,8\},\{7',9^4\},\{7'',9^7\},\{8',9^6\},\{8'',9^8\}\}$

M_6: $\{\{1,2\},\{3,9'''\},\{4,9^5\},\{5,9^4\},\{7,9\},\{7',9'\},\{7'',9''\},\{8,9^8\},\{8',9^7\},\{8'',9^6\}\}$

M_7: $\{\{1,9\},\{2,9'\},\{3,9''\},\{4,9'''\},\{5,9^4\},\{6,9^5\},\{8,9^8\},\{8',9^7\},\{8'',9^6\}\}$

M_8: $\{\{1,9^8\},\{2,9^7\},\{3,9^6\},\{4,9^5\},\{5,9^4\},\{6,9'''\},\{7,9\},\{7',9'\},\{7'',9''\}\}$

M_9: $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{7',8'\},\{7'',8''\}\}$

$(1,1,1,1,1,3,5,5,13)$:

M_1: $\{\{2,9^{12}\},\{3,6''\},\{4,6'\},\{5,9^4\},\{6,9\},\{7,9^6\},\{7',9''\},\{7'',9'''\},\{7''',9^{10}\},\{7^4,9^{11}\},$
$\{8,9'\},\{8',9^7\},\{8'',9^8\},\{8''',9^9\},\{8^4,9^5\}\}$

M_2: $\{\{1,9^8\},\{3,9^7\},\{4,9^6\},\{5,6\},\{6',8'\},\{6'',9^4\},\{7,9^9\},\{7',9''\},\{7'',9^{10}\},\{7''',9^{11}\},$
$\{7^4,9^{12}\},\{8,9\},\{8'',9'\},\{8''',9'''\},\{8^4,9^5\}\}$

M_3: $\{\{1, 9^8\}, \{2, 9^{12}\}, \{4, 9^9\}, \{5, 9^5\}, \{6, 9\}, \{6', 8'\}, \{6'', 9^4\}, \{7, 9^6\}, \{7', 9''\}, \{7'', 8''\},$
$\{7''', 9^{10}\}, \{7^4, 9^{11}\}, \{8, 9'\}, \{8''', 9'''\}, \{8^4, 9^7\}\}$

M_4: $\{\{1, 9^{12}\}, \{2, 9^{11}\}, \{3, 9^{10}\}, \{5, 9^8\}, \{6, 9^4\}, \{6', 9^5\}, \{6'', 9^6\}, \{7, 9^9\}, \{7', 9''\}, \{7'', 9'''\},$
$\{7''', 8'''\}, \{7^4, 8^4\}, \{8, 9\}, \{8', 9^7\}, \{8'', 9'\}\}$

M_5: $\{\{1, 9\}, \{2, 9'\}, \{3, 9''\}, \{4, 9'''\}, \{6, 9^4\}, \{6', 9^5\}, \{6'', 9^6\}, \{7, 8\}, \{7', 8'\}, \{7'', 9^{10}\},$
$\{7''', 9^{11}\}, \{7^4, 9^{12}\}, \{8'', 9^8\}, \{8''', 9^9\}, \{8^4, 9^7\}\}$

M_6: $\{\{1, 2\}, \{3, 9^7\}, \{4, 9^6\}, \{5, 9^5\}, \{7, 9\}, \{7', 9'\}, \{7'', 9''\}, \{7''', 9'''\}, \{7^4, 9^4\}, \{8, 9^8\},$
$\{8', 9^9\}, \{8'', 9^{10}\}, \{8''', 9^{11}\}, \{8^4, 9^{12}\}\}$

M_7: $\{\{1, 9\}, \{2, 9'\}, \{3, 9''\}, \{4, 9'''\}, \{5, 9^4\}, \{6, 9^5\}, \{6', 9^6\}, \{6'', 9^7\}, \{8, 9^8\}, \{8', 9^9\},$
$\{8'', 9^{10}\}, \{8''', 9^{11}\}, \{8^4, 9^{12}\}\}$

M_8: $\{\{1, 9^{12}\}, \{2, 9^{11}\}, \{3, 9^{10}\}, \{4, 9^9\}, \{5, 9^8\}, \{6, 9^5\}, \{6', 9^6\}, \{6'', 9^7\}, \{7, 9\}, \{7', 9'\},$
$\{7'', 9''\}, \{7''', 9'''\}, \{7^4, 9^4\}\}$

M_9: $\{\{1, 2\}, \{3, 6''\}, \{4, 6'\}, \{5, 6\}, \{7, 8\}, \{7', 8'\}, \{7'', 8''\}, \{7''', 8'''\}, \{7^4, 8^4\}\}$

$(1, 1, 1, 1, 1, 5, 5, 5, 15)$:

M_1: $\{\{2, 6'''\}, \{3, 9^7\}, \{4, 9^6\}, \{5, 9^4\}, \{6, 9^{14}\}, \{6', 9'\}, \{6'', 9''\}, \{6^4, 9^{12}\}, \{7, 8\}, \{7', 9^{10}\},$
$\{7'', 9^{13}\}, \{7''', 9^5\}, \{7^4, 9^{11}\}, \{8', 9\}, \{8'', 9^8\}, \{8''', 9'''\}, \{8^4, 9^9\}\}$

M_2: $\{\{1, 9^9\}, \{3, 6''\}, \{4, 6'\}, \{5, 9^5\}, \{6, 9\}, \{6''', 9^6\}, \{6^4, 9^7\}, \{7, 9''\}, \{7', 9'''\}, \{7'', 9^{12}\},$
$\{7''', 9^{13}\}, \{7^4, 9^{14}\}, \{8, 9'\}, \{8', 9^{11}\}, \{8'', 9^8\}, \{8''', 9^{10}\}, \{8^4, 9^4\}\}$

M_3: $\{\{1, 6^4\}, \{2, 9^8\}, \{4, 9^{11}\}, \{5, 6\}, \{6', 9^7\}, \{6'', 9^6\}, \{6''', 9^5\}, \{7, 9''\}, \{7', 9'''\}, \{7'', 9^{12}\},$
$\{7''', 9^{13}\}, \{7^4, 9^{14}\}, \{8, 9'\}, \{8', 9\}, \{8'', 9^4\}, \{8''', 9^{10}\}, \{8^4, 9^9\}\}$

M_4: $\{\{1, 9^{14}\}, \{2, 9^{13}\}, \{3, 9^{12}\}, \{5, 9^{10}\}, \{6, 9\}, \{6', 9'\}, \{6'', 9''\}, \{6''', 9^6\}, \{6^4, 9^7\}, \{7, 9^9\},$
$\{7', 8'\}, \{7'', 8''\}, \{7''', 9^5\}, \{7^4, 9^{11}\}, \{8, 9^8\}, \{8''', 9'''\}, \{8^4, 9^4\}\}$

M_5: $\{\{1, 9\}, \{2, 9'\}, \{3, 9''\}, \{4, 9'''\}, \{6, 9^{14}\}, \{6', 9^7\}, \{6'', 9^6\}, \{6''', 9^5\}, \{6^4, 9^{12}\}, \{7, 9^9\},$

$\{7', 9^{10}\}, \{7'', 9^{13}\}, \{7''', 8'''\}, \{7^4, 8^4\}, \{8, 9^8\}, \{8', 9^{11}\}, \{8'', 9^4\}\}$

M_6: $\{\{1, 9^9\}, \{2, 9^8\}, \{3, 9^7\}, \{4, 9^6\}, \{5, 9^5\}, \{7, 9\}, \{7', 9'\}, \{7'', 9''\}, \{7''', 9'''\}, \{7^4, 9^4\},$

$\{8, 9^{10}\}, \{8', 9^{11}\}, \{8'', 9^{12}\}, \{8''', 9^{13}\}, \{8^4, 9^{14}\}\}$

M_7: $\{\{1, 9\}, \{2, 9'\}, \{3, 9''\}, \{4, 9'''\}, \{5, 9^4\}, \{6, 9^5\}, \{6', 9^6\}, \{6'', 9^7\}, \{6''', 9^8\}, \{6^4, 9^9\},$

$\{8, 9^{10}\}, \{8', 9^{11}\}, \{8'', 9^{12}\}, \{8''', 9^{13}\}, \{8^4, 9^{14}\}\}$

M_8: $\{\{1, 9^{14}\}, \{2, 9^{13}\}, \{3, 9^{12}\}, \{4, 9^{11}\}, \{5, 9^{10}\}, \{6, 9^5\}, \{6', 9^6\}, \{6'', 9^7\}, \{6''', 9^8\},$

$\{6^4, 9^9\}, \{7, 9\}, \{7', 9'\}, \{7'', 9''\}, \{7''', 9'''\}, \{7^4, 9^4\}\}$

M_9: $\{\{1, 6^4\}, \{2, 6'''\}, \{3, 6''\}, \{4, 6'\}, \{5, 6\}, \{7, 8\}, \{7', 8'\}, \{7'', 8''\}, \{7''', 8'''\}, \{7^4, 8^4\}\}$

$(1, 1, 1, 1, 1, 7, 7, 7, 19):$

M_1: $\{\{2, 6^6\}, \{3, 7\}, \{4, 9^8\}, \{5, 9^7\}, \{6, 9'\}, \{6', 9^{15}\}, \{6'', 9''\}, \{6''', 8^4\}, \{6^4, 9^{10}\}, \{6^5, 9^{11}\},$

$\{7', 9\}, \{7'', 9^{14}\}, \{7''', 9^{17}\}, \{7^4, 9^4\}, \{7^5, 9^5\}, \{7^6, 9^6\}, \{8, 9^{12}\}, \{8', 9^{13}\}, \{8'', 9^9\}, \{8''', 9^{16}\},$

$\{8^5, 9'''\}, \{8^6, 9^{18}\}\}$

M_2: $\{\{1, 8^6\}, \{3, 9^9\}, \{4, 7''\}, \{5, 8'''\}, \{6, 9^7\}, \{6', 9^{15}\}, \{6'', 9''\}, \{6''', 9^{16}\}, \{6^4, 9^{10}\}, \{6^5, 9^{11}\},$

$\{6^6, 9^{14}\}, \{7, 9^{12}\}, \{7', 9\}, \{7''', 9^8\}, \{7^4, 9^4\}, \{7^5, 9^5\}, \{7^6, 9^6\}, \{8, 9'\}, \{8', 9^{13}\}, \{8'', 9^{18}\},$

$\{8^4, 9^{17}\}, \{8^5, 9'''\}\}$

M_3: $\{\{1, 9^{11}\}, \{2, 9^{10}\}, \{4, 9^{15}\}, \{5, 9^4\}, \{6, 9'\}, \{6', 9^8\}, \{6'', 7^4\}, \{6''', 9^{16}\}, \{6^4, 7'''\}, \{6^5, 7^5\},$

$\{6^6, 9^{14}\}, \{7, 9\}, \{7', 9^7\}, \{7'', 9''\}, \{7^6, 9^6\}, \{8, 9^{12}\}, \{8', 9^{13}\}, \{8'', 9^9\}, \{8''', 9'''\}, \{8^4, 9^{17}\},$

$\{8^5, 9^5\}, \{8^6, 9^{18}\}\}$

M_4: $\{\{1, 9^{18}\}, \{2, 9^{17}\}, \{3, 9^{16}\}, \{5, 9^{14}\}, \{6, 8'\}, \{6', 8''\}, \{6'', 9^9\}, \{6''', 9^{10}\}, \{6^4, 9^{11}\},$

$\{6^5, 9^{12}\}, \{6^6, 9^{13}\}, \{7, 9\}, \{7', 9^7\}, \{7'', 9''\}, \{7''', 9^8\}, \{7^4, 9^4\}, \{7^5, 8^4\}, \{7^6, 9^6\}, \{8, 9'\},$

$\{8''', 9'''\}, \{8^5, 9^5\}, \{8^6, 9^{15}\}\}$

M_5: $\{\{1, 9\}, \{2, 9'\}, \{3, 9''\}, \{4, 9'''\}, \{6, 9^5\}, \{6', 9^6\}, \{6'', 9^7\}, \{6''', 9^8\}, \{6^4, 9^9\}, \{6^5, 9^{10}\},$

$\{6^6, 9^{11}\}, \{7, 9^{12}\}, \{7', 8\}, \{7'', 9^{14}\}, \{7''', 9^{17}\}, \{7^4, 9^4\}, \{7^5, 8^4\}, \{7^6, 8^5\}, \{8', 9^{13}\}, \{8'', 9^{18}\},$

$\{8''', 9^{16}\}, \{8^6, 9^{15}\}\}$

M_6: $\{\{1, 9^{11}\}, \{2, 9^{10}\}, \{3, 9^9\}, \{4, 9^8\}, \{5, 9^7\}, \{7, 9\}, \{7', 9'\}, \{7'', 9''\}, \{7''', 9'''\}, \{7^4, 9^4\},$
$\{7^5, 9^5\}, \{7^6, 9^6\}, \{8, 9^{12}\}, \{8', 9^{13}\}, \{8'', 9^{14}\}, \{8''', 9^{15}\}, \{8^4, 9^{16}\}, \{8^5, 9^{17}\}, \{8^6, 9^{18}\}\}$

M_7: $\{\{1, 9\}, \{2, 9'\}, \{3, 9''\}, \{4, 9'''\}, \{5, 9^4\}, \{6, 9^5\}, \{6', 9^6\}, \{6'', 9^7\}, \{6''', 9^8\}, \{6^4, 9^9\},$
$\{6^5, 9^{10}\}, \{6^6, 9^{11}\}, \{8, 9^{12}\}, \{8', 9^{13}\}, \{8'', 9^{14}\}, \{8''', 9^{15}\}, \{8^4, 9^{16}\}, \{8^5, 9^{17}\}, \{8^6, 9^{18}\}\}$

M_8: $\{\{1, 9^{18}\}, \{2, 9^{17}\}, \{3, 9^{16}\}, \{4, 9^{15}\}, \{5, 9^{14}\}, \{6, 9^7\}, \{6', 9^8\}, \{6'', 9^9\}, \{6''', 9^{10}\},$
$\{6^4, 9^{11}\}, \{6^5, 9^{12}\}, \{6^6, 9^{13}\}, \{7, 9\}, \{7', 9'\}, \{7'', 9''\}, \{7''', 9'''\}, \{7^4, 9^4\}, \{7^5, 9^5\}, \{7^6, 9^6\}\}$

M_9: $\{\{1, 8^6\}, \{2, 6^6\}, \{3, 7\}, \{4, 7''\}, \{5, 8'''\}, \{6, 8'\}, \{6', 8''\}, \{6'', 7^4\}, \{6''', 8^4\}, \{6^4, 7'''\},$
$\{6^5, 7^5\}, \{7'', 8\}, \{7^6, 8^5\}\}$

Hence we have even matchings of type $(a_1, \ldots, a_9)$.

2.2.5 $p \equiv 1\ (mod\ \ 4)$ parts of any size

In this Section we will give a general construction of even matchings of type $(a_1, \ldots, a_{4h+1})$ for any positive integer $h \geq 3$. Since $p \equiv 1\ (mod\ \ 4)$ we can have all parts of even size or all parts of odd size. We will need "building blocks" similar to the ones used before.

$(0^{4h-2}, 2, 2, 2)$: from Lemma 2.4

$(0^{4h-3}, 2, 2, 2, 2)$: from Lemma 2.4

Notice that when $p = 4h + 1$ condition 4 becomes $2ha_p \leq (2h-1)(a_1 + \ldots + a_{p-1})$ and condition 2 remains $a_p \leq a_1 + \ldots + a_{p-2}$. We can construct an even matching of type $(0, 0, a_3, \ldots, a_p)$ inductively. We start with the even matching $(a_3, \ldots, a_p)$ (as long as it exists) and apply Lemma 2.4.

To find an even matching of type $(a_1, \ldots, a_{4h+1})$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = \ldots = a_{p-4} = 0$, $a_{p-3} = 2$ or $a_1 = \ldots = a_{p-2} = 1$ or $K = (0^{4h-4}, 2, 2, 2, 2, 2)$ or $K = (0^{4h-4}, 2, 2, 2, 2, 6)$ look up the matching in list provided. Otherwise continue below.

2. If $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-3$ skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h-1)$ section.

   Otherwise continue below.

3. Subtract $(2, 0^{4h-2}, 2, 2)$ to obtain $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$.

4. If necessary, rearrange the terms to ensure a nondecreasing sequence.

5. Repeat steps 2-4 until you obtain $(0, \_^{4h})$ or $(1, \_^{4h})$.

6. If $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-3$ skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h-1)$ section.

   Otherwise continue below.

7. For $4h - 3 \geq j \geq 1$ repeat the following steps.

8. Subtract $(0^{4h-2-j}, 2, 0^j, 2, 2)$.

9. If necessary, rearrange the terms to ensure the sequence is nondecreasing.

10. Repeat until you obtain $(0^{4h-1-j}, \_^{j+2})$ or $(1^{4h-1-j}, \_^{j+2})$.

11. Stop when you obtain $(0^{4h-3}, 2, \_, \_, \_)$ or $(1^{4h-1}, \_, \_)$.

12. The even matchings of type $(0^{4h-3}, 2, \_, \_, \_)$ and $(1^{4h-1}, \_, \_)$ will need to be given.

Let us consider the four necessary conditions during steps 2-5 of the algorithm.

1. $p = 4h + 1$ is not affected.

2. If $(a_1, \ldots, a_p)$ satisfies $a_p \leq a_1 + \ldots + a_{p-2}$, then $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$ satisfies $a_p - 2 \leq a_1 - 2 + a_2 + \ldots + a_{p-2}$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_p - 2$ is not the largest, but the second largest part, then $a_{p-2} = a_p$. So we started with $(a_1, \ldots, a_{p-3}, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_p - 2, a_p - 2, a_p)$. By Properties 2 and 4 this is in S_2 as long as $4 \leq a_1 + \ldots + a_{p-3}$ and $12h - 6 - (2h-2)a_p \leq (2h-1)(a_1 + \ldots + a_{p-3})$, which is equivalent to $a_1 + \ldots + a_{p-3} \geq 6 - \frac{2h-2}{2h-1}a_p$. Let us consider the cases when $4 > a_1 + \ldots + a_{p-3}$ or $a_1 + \ldots + a_{p-3} < 6 - \frac{2h-2}{2h-1}a_p$. We could have $(0^{4h-2}, a_p, a_p, a_p)$, $(0^{4h-3}, 2, a_p, a_p, a_p)$ or $(0^{4h-4}, 2, 2, 2, 2, 2)$. Each one of those is in S_2.

   Case 2: If after rearranging $a_p - 2$ is not the largest or second largest part, then $a_{p-3} = a_p$. So we started with $(a_1, \ldots, a_{p-4}, a_p, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-4}, a_p - 2, a_p - 2, a_p, a_p)$. By Properties 2 and 4 this is in S_2 as long as $6 - a_p \leq a_1 + \ldots + a_{p-4}$ and $6 - \frac{4h-3}{2h-1}a_p \leq a_1 + \ldots + a_{p-4}$. Let us consider the cases when $6 - a_p > a_1 + \ldots + a_{p-4}$ or $6 - \frac{4h-3}{2h-1}a_p > a_1 + \ldots + a_{p-4}$. We could have $(0^{4h-3}, 2, 2, 2, 2)$, $(0^{4h-3}, 4, 4, 4, 4)$ or $(0^{4h-4}, 2, 2, 2, 2, 2)$. Each one of those is in S_2.

   Case 3: If after rearranging $a_p - 2$ is the largest, but $a_{p-1} - 2$ is not the second largest part, then $a_{p-2} = a_{p-1}$. So we started with $(a_1, \ldots, a_{p-3}, a_{p-1}, a_{p-1}, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_{p-1} - 2, a_{p-1}, a_p - 2)$. By Property 2, this is in S_2 as long as $a_p \neq a_1 + \ldots + a_{p-2}$. The case when $a_p = a_1 + \ldots + a_{p-2}$ and $a_{p-2} = a_{p-1}$ is discussed below as Special Case $(2h-1)$.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, \ldots, a_p)$ satisfies $2ha_p \leq (2h-1)(a_1 + \ldots + a_{p-1}) - 4(h-1)$, then $(a_1 - 2, a_2, \ldots, a_{p-2}, a_{p-1} - 2, a_p - 2)$ satisfies $2h(a_p - 2) \leq (2h-1)(a_1 + \ldots + a_{p-1} - 4)$. Each Special Case $(c+1)$ section covers the instances when $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-3$. Problems arising from rearranging the terms are discussed under condition 2.

Therefore, each time we repeat steps 2-5 we obtain an element of S_2. Notice that the above holds for the remaining steps of the algorithm. So it is sufficient to give even matchings of type $(0^{4h-3}, 2, \_, \_, \_)$ and $(1^{4h-1}, \_, \_)$ and consider the special cases mentioned above.

Let us consider the **Special Case** $(c+1)$ for $0 \leq c \leq 2h-3$.

We need to find even matchings for $(a_1, \ldots, a_p)$ satisfying the four necessary conditions and $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$. Let us refer to these as matching type $SP(c+1) \subseteq S_2$. Say we need $K = (a_1, a_2, \ldots, a_{p-1}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h}) \in SP(c+1)$. We will build it by induction.

We consider parts of odd size first. If $h - c \geq 0$, we start by subtracting $P^1 = (1^{3h+c}, 3^{h-c}, 6h-3-2c)$ to obtain $K' = (a_1 - 1, \ldots, a_{3h+c} - 1, a_{3h+c+1} - 3, \ldots, a_{4h} - 3, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (6h-3-2c))$. Since necessary conditions 2 and 4 were satisfied in K, they are still satisfied in K'. Also, $K' \in SP1$. However, if rearranging is necessary and we obtain $K'' = (a_1 - 1, \ldots, a_{3h+c+1} - 3, a_{3h+c+2} - 3, \ldots, a_{4h} - 3, a_{3h+c} -$

$1, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (6h-3-2c))$ instead of K', we must have $a_{3h+c}-1 = a_{4h}-3+2$ or $a_{3h+c} = a_{4h}$. Hence we started with $K = (a_1, \ldots, a_{3h+c-1}, a_{4h}, \ldots, a_{4h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$. In this case instead of subtracting $P^1 = (1^{3h+c}, 3^{h-c}, 6h-3-2c)$, we subtract $P^2 = (1^{h+c}, 3^{3h-c}, 10h-5-2c)$ to obtain $K' = (a_1-1, \ldots, a_{h+c}-1, a_{h+c+1}-3, \ldots, a_{4h}-3, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (10h-5-2c))$. Notice that as long as no rearranging is necessary, $K' \in SP1$. We will also subtract P^2 if $h-c < 0$, which means we start with $a_p > 6h-3-2c$. However, if rearranging is necessary and results in $K'' = (a_1-1, \ldots, a_{h+c-1}-1, a_{h+c+1}-3, \ldots, a_{4h}-3, a_{h+c}-1, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (10h-5-2c))$ we must have $a_{h+c} = a_{4h}$. So we started with $K = (a_1, \ldots, a_{h+c-1}, a_{4h}, \ldots, a_{4h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$. In this case we subtract $P^3 = (1^{h+c}, 3^{2h-c}, 7^h, 14h-7-2c)$ instead of P^1 or P^2. We then get $K' = (a_1-1, \ldots, a_{h+c-1}-1, a_{4h}-1, a_{4h}-3, \ldots, a_{4h}-3, a_{4h}-7, \ldots, a_{4h}-7, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (14h-7-2c))$ which is rearranged to $K'' = (a_1-1, \ldots, a_{h+c-1}-1, a_{4h}-3, \ldots, a_{4h}-3, a_{4h}-7, \ldots, a_{4h}-7, a_{4h}-1, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (14h-7-2c))$. Satisfying the conditions 2 and 4, this K'' is always in $SP1$. This shows that when we complete this step, we obtain an element of $SP1$.

For parts of even size we start by subtracting $(0^{2h+c}, 2^{2h-c}, 4h-2-2c)$ to obtain $K' = (a_1, \ldots, a_{2h+c}, a_{2h+c+1}-2, \ldots, a_{4h}-2, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (4h-2-2c))$. As long as no rearranging is necessary K' is in $SP1$. If rearranging is necessary, we started with $K = (a_1, \ldots, a_{2h+c-1}, a_{4h}, \ldots, a_{4h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$ and now have $K'' = (a_1, \ldots, a_{2h+c-1}, a_{4h}-2, \ldots, a_{4h}-2, a_{4h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (4h-2-2c))$. This is in $SP1$ as long as $(c-1)a_{4h} - 8h^2 + 8h + 4ch \leq a_1 + \ldots + a_{2h+c-1}$ which is equivalent to $a_{4h} - 2 \leq \frac{a_4 h+1}{2h-1}$ when $c = 2h-3$ (worst case scenario). By Property 2, this inequality is always true for elements of S_2. Notice that no matter what $SP(c+1)$ we start with, after the first subtraction, we will continue the induction process with $SP1$ or $c = 0$.

By induction, this concludes Special Case $(c+1)$. We take any element of $SP(c+1)$ and subtract to get a smaller element of $SP1$.

Now let us consider **Special Case** $(2h-1)$.

If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ we have types of the form $(a_1, a_2, \ldots, a_{p-3}, a_p - (a_1 + \ldots + a_{p-3}), a_p - (a_1 + \ldots + a_{p-3}), a_p)$. We will construct an even matching of such type as follows: $a_j/2$ copies of $(0^{j-1}, 2, 0^{4h-2-j}, 2, 2, 4)$ for each $1 \leq j \leq 4h-2$, and $a_p - 2(a_1 + \ldots + a_{p-3})/2$ copies of $(0^{4h-2}, 2, 2, 2)$. Notice that by Property 4 $a_p \geq 2(a_1 + \ldots + a_{p-3})$. This concludes Special Case $(2h-1)$.

The following are even matchings of type $(0^{4h-3}, 2, \_, \_, \_)$ and $(1^{4h-1}, \_, \_)$ and other matchings used in the above construction.

$(0^{4h-2}, 2, 2, 2)$: from Lemma 2.4

By Property 2, an even matching of type $(0^{4h-2}, a_{p-2}, a_{p-1}, a_p)$ exists if $a_p \leq a_{p-2}$. Hence we must have $(0^{4h-2}, a_p, a_p, a_p)$.

$(0^{4h-2}, a_p, a_p, a_p)$ from $a_p/2$ copies of $(0^{4h-2}, 2, 2, 2)$

$(0^{4h-3}, 2, 2, 2, 2)$: from Lemma 2.4

$(0^{4h-3}, 2, 2, 2, 4)$: from Lemma 2.4

By Property 2, an even matching of type $(0^{4h-3}, 2, a_{p-2}, a_{p-1}, a_p)$ exists if $a_{p-2} \geq a_p - 2$.

$(0^{4h-3}, 2, a_p, a_p, a_p)$ from $(0^{4h-3}, 2, 2, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{4h-2}, 2, 2, 2)$

$(0^{4h-3}, 2, a_p - 2, a_p, a_p)$ from $(0^{4h-3}, 2, 0, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{4h-2}, 2, 2, 2)$

Since $a_p \geq 2$, $a_p - 2 \geq 0$.

$(0^{4h-3}, 2, a_p-2, a_p-2, a_p)$ from $(0^{4h-3}, 2, 2, 2, 4)$ and $(a_p-4)/2$ copies of $(0^{4h-2}, 2, 2, 2)$

Since $a_p - 2 \geq 2$, $a_p - 4 \geq 0$.

$(0^{4h-4}, 2, 2, 2, 2, 2)$: from Lemma 2.4

$(0^{4h-4}, 2, 2, 2, 2, 6)$: from Lemma 2.4

$(0^{2h}, 2^{2h}, 4h-2)$: Needed; not yet found.

The remaining $(0^{2h+c}, 2^{2h-c}, 4h-2-2c)$ for $0 < c \leq 2h-3$ are obtained from Lemma 2.4

(1^{4h+1}): given in Section 2.2.1

For $h \geq c$, $(1^{3h+c}, 3^{h-c}, 6h-3-2c)$: Needed; not yet found.

$(1^{h+c}, 3^{3h-c}, 10h-5-2c)$: Needed; not yet found.

$(1^{h+c}, 3^{2h-c}, 7^h, 14h-7-2c)$: Needed; not yet found.

In the following families of matchings we use i, j odd.

For $3 \leq i \leq 4h-3$, $(1^{4h}, i)$: Needed; not yet found.

For $3 \leq i \leq 4h-1$ and $3 \leq j \leq i$, $(1^{4h-1}, j, i)$: Needed; not yet found.

Hence we have even matchings of type $(a_1, \ldots, a_{4h+1})$.

2.2.6 $p \equiv 3 \ (mod \ \ 4)$ parts of any size

In this Section we will give a general construction of even matchings of type $(a_1, \ldots, a_{4h+3})$ for any positive integer $h \geq 2$. Since $p \equiv 3 \ (mod \ \ 4)$ we can only have all parts of even size. We will use "building blocks" similar to the ones above.

$(0^{4h}, 2, 2, 2)$: from Lemma 2.4

$(0^{4h-1}, 2, 2, 2, 2)$: from Lemma 2.4

Notice that when $p = 4h + 3$ condition 4 becomes $(2h+1)a_p \leq 2h(a_1 + \ldots + a_{p-1})$ and condition 2 remains $a_p \leq a_1 + \ldots + a_{p-2}$. We can still construct an even matching of type $(0, 0, a_3, \ldots, a_p)$ inductively. We start with the even matching $(a_3, \ldots, a_p)$ (as long as it exists) and apply Lemma 2.4.

To find an even matching of type $(a_1, \ldots, a_{4h+3})$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = \ldots = a_{p-4} = 0$, $a_{p-3} = 2$ or $K = (0^{4h-2}, 2, 2, 2, 2, 2)$ or $K = (0^{4h-2}, 2, 2, 2, 2, 6)$ look up the matching in list provided. Otherwise continue below.

2. If $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$, for $0 \leq c \leq 2h-2$, skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h)$ section. Otherwise continue below.

3. Subtract $(2, 0^{4h}, 2, 2)$ to obtain $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$.

4. If necessary, rearrange the terms to ensure a nondecreasing sequence.

5. Repeat steps 2-4 until you obtain $(0, \_^{4h+2})$.

6. If $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-2$ skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h)$ section. Otherwise continue below.

7. For $4h - 1 \geq j \geq 1$ repeat the following steps.

8. Subtract $(0^{4h-j}, 2, 0^j, 2, 2)$.

9. If necessary, rearrange the terms to ensure the sequence is nondecreasing.

10. Repeat until you obtain $(0^{4h+1-j}, \_^{j+2})$.

11. Stop when you obtain $(0^{4h-1}, 2, \_, \_, \_)$.

12. The even matchings of type $(0^{4h-1}, 2, \_, \_, \_)$ will need to be given.

Let us consider the four necessary conditions during steps 2-5 of the algorithm.

1. $p = 4h + 3$ is not affected.

2. If $(a_1, \ldots, a_p)$ satisfies $a_p \leq a_1 + \ldots + a_{p-2}$, then $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$ satisfies $a_p - 2 \leq a_1 - 2 + a_2 + \ldots + a_{p-2}$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_p - 2$ is not the largest, but the second largest part, then $a_{p-2} = a_p$. So we started with $(a_1, \ldots, a_{p-3}, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_p - 2, a_p - 2, a_p)$. By Properties 2 and 4 this is in S_2 as long as $4 \leq a_1 + \ldots + a_{p-3}$ and $a_1 + \ldots + a_{p-3} \geq 6 - \frac{2h-1}{2h} a_p$. Let us consider the cases when $4 > a_1 + \ldots + a_{p-3}$ or $a_1 + \ldots + a_{p-3} < 6 - \frac{2h-1}{2h} a_p$. We could have $(0^{4h}, a_p, a_p, a_p)$, $(0^{4h-1}, 2, a_p, a_p, a_p)$ or $(0^{4h-2}, 2, 2, 2, 2, 2)$. Each one of those is in S_2.

   Case 2: If after rearranging $a_p - 2$ is not the largest or second largest part, then $a_{p-3} = a_p$. So we started with $(a_1, \ldots, a_{p-4}, a_p, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-4}, a_p - 2, a_p - 2, a_p, a_p)$. By Properties 2 and 4 this is in S_2 as long as $6 - a_p \leq a_1 + \ldots + a_{p-4}$ and $6 - \frac{4h-1}{2h} a_p \leq a_1 + \ldots + a_{p-4}$. Let us consider the cases when $6 - a_p > a_1 + \ldots + a_{p-4}$ or $6 - \frac{4h-1}{2h} a_p > a_1 + \ldots + a_{p-4}$. We could have $(0^{4h-1}, 2, 2, 2, 2)$, $(0^{4h-1}, 4, 4, 4, 4)$ or $(0^{4h-2}, 2, 2, 2, 2, 2)$. Each one of those is in S_2.

Case 3: If after rearranging $a_p - 2$ is the largest, but $a_{p-1} - 2$ is not the second largest part, then $a_{p-2} = a_{p-1}$. So we started with $(a_1, \ldots, a_{p-3}, a_{p-1}, a_{p-1}, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_{p-1} - 2, a_{p-1}, a_p - 2)$. By Property 2, this is in S_2 as long as $a_p \neq a_1 + \ldots + a_{p-2}$. The case when $a_p = a_1 + \ldots + a_{p-2}$ and $a_{p-2} = a_{p-1}$ is discussed below as Special Case $(2h)$.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, \ldots, a_p)$ satisfies $(2h+1)a_p \leq 2h(a_1 + \ldots + a_{p-1}) - 2(2h-1)$, then $(a_1 - 2, a_2, \ldots, a_{p-2}, a_{p-1} - 2, a_p - 2)$ satisfies $(2h+1)(a_p - 2) \leq 2h(a_1 + \ldots + a_{p-1} - 4)$. Each Special Case $(c+1)$ section covers the instances when $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-2$. Problems arising from rearranging the terms are discussed under condition 2.

Therefore, each time we repeat steps 2-5 we obtain an element of S_2. Notice that the above holds for the remaining steps of the algorithm. So it is sufficient to give even matchings of type $(0^{4h-1}, 2, \_, \_, \_)$ and consider the special cases mentioned above.

Let us consider the **Special Case** $(c+1)$.

As with $p = 1$ $(mod\ \ 4)$ we define the set $\overline{SP(c+1)}$ as the set of all matching types satisfying the four necessary conditions and $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$. We construct an even matching of type $K = (a_1, \ldots, a_{4h+3}) \in \overline{SP(c+1)}$ by induction. We start by subtracting $(0^{2h+c+1}, 2^{2h-c+1}, 4h-2c)$ to obtain $K' = (a_1, \ldots, a_{2h+c+1}, a_{2h+c+2} - 2, \ldots, a_{4h+2} - 2, \frac{(2h)(a_1 + \ldots + a_{p-1}) - 2c}{2h+1} - (4h - 2c))$. As long as no rearranging is necessary K' is in $\overline{SP1}$. If rearranging is necessary, we started with

$K = (a_1, \ldots, a_{2h+c}, a_{4h+2}, \ldots, a_{4h+2}, \frac{(2h)(a_1+\ldots+a_{p-1})-2c}{2h+1})$ and now have $K'' = (a_1, \ldots, a_{2h+c}, a_{4h+2}-2, \ldots, a_{4h+2}-2, a_{4h+2}, \frac{(2h)(a_1+\ldots+a_{p-1})-2c}{2h+1} - (4h-2c))$. This is always in $\overline{SP1}$. Notice that no matter what $\overline{SP(c+1)}$ we start with, after the first subtraction, we will continue the induction process with $\overline{SP1}$ (i.e. $c = 0$). This concludes the inductive argument, since each time we repeat this process we obtain a smaller element of $\overline{SP1}$. Thus Special Case $(c+1)$ is solved.

Now let us consider **Special Case** $(2h)$.

If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ we have types of the form $(a_1, a_2, \ldots, a_{p-3}, a_p - (a_1 + \ldots + a_{p-3}), a_p - (a_1 + \ldots + a_{p-3}), a_p)$. We will construct an even matching of such type as follows: $a_j/2$ copies of $(0^{j-1}, 2, 0^{4h-j}, 2, 2, 4)$ for each $1 \leq j \leq 4h$, and $a_p - 2(a_1 + \ldots + a_{p-3})/2$ copies of $(0^{4h}, 2, 2, 2)$. This concludes Special Case $(2h)$.

The following are even matchings of type $(0^{4h-1}, 2, \_, \_, \_)$ and other matchings used above.

$(0^{4h}, 2, 2, 2)$: from Lemma 2.4

By Property 2, an even matching of type $(0^{4h}, a_{p-2}, a_{p-1}, a_p)$ exists if $a_p \leq a_{p-2}$. Hence we must have $(0^{4h}, a_p, a_p, a_p)$.

$(0^{4h}, a_p, a_p, a_p)$ from $a_p/2$ copies of $(0^{4h}, 2, 2, 2)$

$(0^{4h-1}, 2, 2, 2, 2)$: from Lemma 2.4

$(0^{4h-1}, 2, 2, 2, 4)$: from Lemma 2.4

By Property 2, an even matching of type $(0^{4h-1}, 2, a_{p-2}, a_{p-1}, a_p)$ exists if $a_{p-2} \geq a_p - 2$.

$(0^{4h-1}, 2, a_p, a_p, a_p)$ from $(0^{4h-1}, 2, 2, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{4h}, 2, 2, 2)$

$(0^{4h-1}, 2, a_p - 2, a_p, a_p)$ from $(0^{4h-1}, 2, 0, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{4h}, 2, 2, 2)$

Since $a_p \geq 2$, $a_p - 2 \geq 0$.

$(0^{4h-1}, 2, a_p - 2, a_p - 2, a_p)$ from $(0^{4h-1}, 2, 2, 2, 4)$ and $(a_p - 4)/2$ copies of $(0^{4h}, 2, 2, 2)$

Since $a_p - 2 \geq 2$, $a_p - 4 \geq 0$.

$(0^{4h-2}, 2, 2, 2, 2, 2)$: from Lemma 2.4

$(0^{4h-2}, 2, 2, 2, 2, 6)$: from Lemma 2.4 For $c \geq 1$ we have $(0^{2h+c+1}, 2^{2h-c+1}, 4h - 2c)$ from Lemma 2.4

$(0^{2h+1}, 2^{2h+1}, 4h)$: Needed; not yet found.

Therefore we have even matchings for $p = 3 \ (mod \ \ 4)$. Once the missing matchings are found, this proves the sufficiency of the necessary conditions listed in Section 2.1.

Chapter 3

Existence of k-divisible-Matchings

The results in Chapter 2 can be extended by considering the following definition.

Definition 3.1 *Let $a_1, a_2, \ldots, a_p$ be non-negative integers, and let A_i denote the vertex partite set of size a_i, for $1 \leq i \leq p$. Then for the graph $K(a_1, a_2, \ldots, a_p)$, the ordered set $M = (M_1, M_2, \ldots, M_p)$ is a k-**divisible-matching** of type $(a_1, a_2, \ldots, a_p)$ if*

1. *for each i, $1 \leq i \leq p$, the set M_i is a perfect matching in the graph $K(a_1, a_2, \ldots, a_p) \backslash A_i$, and*
2. *for each edge e of $K(a_1, a_2, \ldots, a_p)$, the number of matchings M_i containing e is divisible by k.*

   We will refer to this as the **divisibility condition**.

Previously we considered each edge appearing an even number of times. Notice that even matchings are 2-divisible-matchings. Let us define the set
$S_k = \{(a_1, \ldots, a_p) |$ a k-divisible-matching on the graph $K(a_1, \ldots, a_p)$ exists$\}$.

3.1 Necessary conditions

In this Section we present necessary conditions for the existence of k-divisible-matchings of type $(a_1, a_2, \ldots, a_p)$. We assume $a_1 \leq a_2 \leq \ldots \leq a_p$ and define $n = \sum_{i=1}^{p} a_i$. These necessary conditions are as follows:

1. $p \equiv 1 \ (mod \ k)$,

2. $2a_p + a_{p-1} \leq n$,

3. If k is even: either all a_i are even or all a_i are odd and $p \equiv 1 \ (mod\ 2k)$.

   If k is odd: either all a_i are even or all a_i are odd and p is odd.

4. $(2p - k - 2)a_p \leq (p - k - 1)n$.

Notice that condition 4 implies $p > k + 1$.

Let us verify the above conditions.

1. Each vertex (element of A_i) will be used in $p-1$ edges, and the number of edges that vertex is in must be divisible by k, equivalently $\frac{p-1}{k}$ must be an integer. Therefore $p \equiv 1 \ (mod\ k)$.

2. For every i, $n - a_i - 2a_p \geq 0$, since we must have enough vertices in each $n - a_i$ to "match" the vertices of the largest part. Since $a_1 \leq a_2 \leq \ldots \leq a_p$, it is sufficient that $n - a_{p-1} - 2a_p \geq 0$, i.e. $2a_p + a_{p-1} \leq n$.

3. Since each M_i is to be a perfect matching, $(\sum_{i=1}^{p} a_i) - a_i$ must be even for all i; hence all a_i have same parity. The "divisibility" condition requires $\sum_{i=1}^{p} \frac{n-a_i}{2} = \frac{(p-1)n}{2}$ to be divisible by k. Therefore either all a_i are even, or if k is even, all a_i are odd and $p \equiv 1 \ (mod\ 2k)$ and if k is odd, all a_i are odd and p is odd.

4. None of the edges in M_p use vertices in A_p. Therefore there must be enough edges in $M_1 \cup M_2 \cup \ldots \cup M_{p-1}$ not intersecting A_p to satisfy the "divisibility" condition. So,

$$(k-1)\frac{n - a_p}{2} \leq \sum_{i=1}^{p-1} \frac{n - a_i - 2a_p}{2} = \frac{(p-2)n - (2p-3)a_p}{2},$$

which implies

$$(2p-k-2)a_p \leq (p-k-1)n$$

5. From the above we have $p-k-1>0$, i.e. $p>k+1$.

It would be very useful to have an analogue of Lemma 2.1 and Lemma 2.2 for k-divisible-matchings.

Lemma 3.2 *If $M=(M_1,M_2,\ldots,M_p)$ is a k-divisible-matching of type $(a_1,a_2,\ldots,a_p)$ and $N=(N_1,N_2,\ldots,N_p)$ is a k-divisible-matching of type $(b_1,b_2,\ldots,b_p)$, on disjoint vertex sets, then*

$M\cup N=(M_1\cup N_1,M_2\cup N_2,\ldots,M_p\cup N_p)$ is a k-divisible-matching of type

$(a_1+b_1,a_2+b_2,\ldots,a_p+b_p)$.

Proof: Since the number of times each edge appears in M and N is a multiple of k this is a k-divisible-matching. $\square$

Lemma 3.3 *If $M=(M_1,M_2,\ldots,M_p)$ is a k-divisible-matching of type $(a_1,a_2,\ldots,a_p)$ and $N=(N_1,N_2,\ldots,N_q)$ is a k-divisible-matching of type $(a_1,b_2,\ldots,b_q)$, then there exists a k-divisible-matching for $K(a_1,a_2,\ldots,a_p,b_2,b_3,\ldots,b_q)$ of type*

$(a_1,a_2,\ldots,a_p,b_2,b_3,\ldots,b_q)$.

Proof: This proof is very similar to the proof for even matchings. The k-divisible-matching of type $(a_1,a_2,\ldots,a_p,b_2,b_3,\ldots,b_q)$ on $K(a_1,a_2,\ldots,a_p,b_2,b_3,\ldots,b_q)$ is given by

$(\overline{M}_1,\overline{M}_2,\ldots,\overline{M}_p,\overline{N}_2,\overline{N}_3,\ldots,\overline{N}_q)$:

$\overline{M_1}=M_1\cup N_1$;

$\overline{M_i} = M_i \cup N_1, \quad 2 \leq i \leq p;$

$\overline{N_i} = N_i \cup M_1, \quad 2 \leq i \leq q.$

This uses the matchings $M_1, \ldots, M_p$ and $N_1, \ldots, N_q$ once, M_1 an additional $q-1$ times, and N_1 an additional $p-1$ times. Since $p, q \equiv 1 \pmod{k}$ the above is a k-divisible-matching. $\square$

The following are generalizations of Lemmas 2.4 and 2.5.

Lemma 3.4 *If $a_1, \ldots, a_p$ are even and $(a_1, \ldots, a_p)$ is an element of S_k, then $(0^{kn}, a_1, \ldots, a_p)$ is also an element of S_k for any integer n.*

Proof: Let $(M_1, \ldots, M_p)$ be a k-divisible-matching of type $(a_1, \ldots, a_p)$. Let N be any perfect matching on $K(a_1, \ldots, a_p)$. By Lemma 2.3, N exists. We construct a k-divisible-matching of $K(0^{kn}, a_1, \ldots, a_p)$ as follows.

$\overline{M_i} = N$ for $1 \leq i \leq kn$

$\overline{M_{j+kn}} = M_j$ for $1 \leq j \leq p$

Clearly this is a k-divisible-matching. $\square$

Lemma 3.5 *If $a_1, \ldots, a_p, b_2, \ldots, b_q$ are all even and $(a_1, \ldots, a_p)$, $(0, b_2, \ldots, b_q)$ are both elements of S_k, then $(b_2, \ldots, b_q, a_1, \ldots, a_p)$ is also an element of S_k.*

Proof: Let $(M_1, \ldots, M_p)$ be a k-divisible-matching of type $(a_1, \ldots, a_p)$ and $(N_1, \ldots, N_q)$ be a k-divisible-matchings of type $(0, b_2, \ldots, b_q)$. Let R be any perfect matching on $K(a_1, \ldots, a_p)$. By Lemma 2.3, R exists.

We construct a k-divisible-matching of $K(b_2, \ldots, b_q, a_1, \ldots, a_p)$ as follows.

$\overline{M_i} = N_{i+1} \cup R$ for $1 \leq i \leq q-1$

$\overline{M_{j+q-1}} = N_1 \cup M_j$ for $1 \leq j \leq p$

Since $p-1$ and $q-1$ are both divisible by k this is a k-divisible-matching. $\square$

3.2 3-divisible-matchings

Let us consider $k =$ 3-divisible-matchings. By condition 1, $p \equiv 1 \ (mod\ 3)$. We will first consider $p = 7$ and $p = 10$. Notice that when $p = 10$ all parts a_i must be of even size.

3.2.1 Any number of parts of the same size

Let us consider 3-divisible-matchings with all parts of the same size. It is sufficient to consider $(1^7) = (1,1,1,1,1,1,1)$, $(2^7) = (2,2,2,2,2,2,2)$ and

$(2^{10}) = (2,2,2,2,2,2,2,2,2,2)$. We can then use Lemma 3.2, as follows:

If a is odd: $(a^7) = (1^7) + \frac{a-1}{2}$ copies of (2^7).

If a is even: $(a^7) = \frac{a}{2}$ copies of (2^7).

$(a^{10}) = \frac{a}{2}$ copies of (2^{10}).

Also, to construct any 3-divisible matching (a^p) we use Lemma 3.3.

$(1^7) = (1,1,1,1,1,1,1)$:

For $2 \leq i \leq 8$:

M_{i-1}: $\{\{i, i+1\}, \{i+2, i+3\}, \{i+4, i+5\} | i \in \mathbb{Z}_7\}$

$(2^7) = (2,2,2,2,2,2,2)$:

M_1: $\{\{2,3\}, \{4,5\}, \{6,7\}, \{2',3'\}, \{4',5'\}, \{6',7'\}\}$

M_2: $\{\{1,7\}, \{3,4\}, \{5,6\}, \{1',7'\}, \{3',4'\}, \{5',6'\}\}$

M_3: $\{\{1,2\}, \{4,5\}, \{6,7\}, \{1',2'\}, \{4',5'\}, \{6',7'\}\}$

M_4: $\{\{1,7\}, \{2,3\}, \{5,6\}, \{1',7'\}, \{2',3'\}, \{5',6'\}\}$

M_5: $\{\{1,2\}, \{3,4\}, \{6,7\}, \{1',2'\}, \{3',4'\}, \{6',7'\}\}$

M_6: $\{\{1,7\}, \{2,3\}, \{4,5\}, \{1',7'\}, \{2',3'\}, \{4',5'\}\}$

M_7: $\{\{1,2\},\{3,4\},\{5,6\},\{1',2'\},\{3',4'\},\{5',6'\}\}$

$(2^{10}) = (2,2,2,2,2,2,2,2,2,2)$:

M_1: $\{\{2,3\},\{2',4\},\{3',4'\},\{5,6\},\{5',7\},\{6',7'\},\{8',10\},\{8,9\},\{9',10'\}\}$

M_2: $\{\{3,4\},\{3',5\},\{4',5'\},\{6,7\},\{6',8\},\{7',8'\},\{9',1\},\{9,10\},\{1',10'\}\}$

M_3: $\{\{4,5\},\{4',6\},\{5',6'\},\{7,8\},\{7',9\},\{8',9'\},\{10',2\},\{1,10\},\{1',2'\}\}$

M_4: $\{\{5,6\},\{5',7\},\{6',7'\},\{8,9\},\{8',10\},\{9',10'\},\{1',3\},\{1,2\},\{2',3'\}\}$

M_5: $\{\{6,7\},\{6',8\},\{7',8'\},\{9,10\},\{9',1\},\{1',10'\},\{2',4\},\{2,3\},\{3',4'\}\}$

M_6: $\{\{7,8\},\{7',9\},\{8',9'\},\{1,10\},\{10',2\},\{1',2'\},\{3',5\},\{3,4\},\{4',5'\}\}$

M_7: $\{\{8,9\},\{8',10\},\{9',10'\},\{1,2\},\{1',3\},\{2',3'\},\{4',6\},\{4,5\},\{5',6'\}\}$

M_8: $\{\{9,10\},\{9',1\},\{10',1'\},\{2,3\},\{2',4\},\{3',4'\},\{5',7\},\{5,6\},\{6',7'\}\}$

M_9: $\{\{1,10\},\{10',2\},\{1',2'\},\{3,4\},\{3',5\},\{4',5'\},\{6',8\},\{6,7\},\{7',8'\}\}$

M_{10}: $\{\{1,2\},\{1',3\},\{2',3'\},\{4,5\},\{4',6\},\{5',6'\},\{7',9\},\{7,8\},\{8',9'\}\}$

3.2.2 Seven parts of any size

In this Section we will give 3-divisible-matchings of type $K = (a_1, \ldots, a_7)$. Since $p = 7 \equiv 1 \ (mod\ 6)$ we can have all parts of even size or all parts of odd size. We will use a similar algorithm as for even matchings to construct a 3-divisible-matching $(a_1, \ldots, a_7)$ from $(0,0,0,2,\_,\_,\_)$ or $(1,1,1,1,1,\_,\_)$.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = a_2 = a_3 = 0$ and $a_4 = 0, 2$, $a_1 = \ldots = a_5 = 1$ or $K = (0,0,2,2,2,2,2)$ look up the matching in the list provided. Otherwise continue below.

2. For $0 \le i \le 3$ repeat the following steps.

3. Subtract $(0^i, 2, 0^{4-i}, 2, 2)$.

4. If necessary, rearrange the terms to ensure that the parts form a nondecreasing sequence.

5. Repeat steps 3-4 until you obtain $(0,0,0,2,\_,\_,\_)$ or $(1,1,1,1,\_,\_,\_)$. If at any point $K = (0,0,2,2,2,2,2)$ look it up.

6. The 3-divisible-matchings of type $(0,0,0,2,\_,\_,\_)$ are listed below.

7. Subtract $(0,0,0,0,2,2,2)$.

8. If necessary, rearrange the terms to ensure that the parts form a nondecreasing sequence.

9. Repeat until you obtain $(1,1,1,1,1,\_,\_)$.

10. The 3-divisible-matchings of type $(1,1,1,1,1,\_,\_)$ are listed below.

Let us consider the four necessary conditions during the first part of the "subtracting process." Notice that for $k = 3$ and $p = 7$ condition 4 becomes $3a_7 \leq n$.

1. $p = 7$ is not affected.

2. If $(a_1, \ldots, a_7)$ satisfies $2a_7 + a_6 \leq a_1 + \ldots + a_7$, then $(a_1 - 2, a_2, a_3, a_4, a_5, a_6 - 2, a_7 - 2)$ satisfies $2(a_7 - 2) + (a_6 - 2) \leq (a_1 - 2) + a_2 + a_3 + a_4 + a_5 + (a_6 - 2) + (a_7 - 2)$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_7 - 2$ is not the largest, but the second largest part, then $a_5 = a_7$. So we started with $(a_1, a_2, a_3, a_4, a_7, a_7, a_7)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_7 - 2, a_7 - 2, a_7)$. By Properties 2 and 4, $(a_1 - 2, a_2, a_3, a_4, a_7 -$

$2, a_7 - 2, a_7)$ is in S_3 as long as $6 \leq a_1 + \ldots + a_4$. Let us consider the case when $a_1 + \ldots + a_4 < 6$. We could have $(0,0,0,0,a_7,a_7,a_7)$, $(0,0,0,2,a_7,a_7,a_7)$, $(0,0,0,4,a_7,a_7,a_7)$, $(0,0,2,2,a_7,a_7,a_7)$ or $(1,1,1,1,a_7,a_7,a_7)$. Each one of those is discussed and shown to be in S_3.

Case 2: If after rearranging $a_7 - 2$ is not the largest or second largest part, then $a_4 = \ldots = a_7$. So we started with $(a_1, a_2, a_3, a_7, a_7, a_7, a_7)$ and now have $(a_1 - 2, a_2, a_3, a_7 - 2, a_7 - 2, a_7, a_7)$. By Properties 2 and 4, $(a_1 - 2, a_2, a_3, a_7 - 2, a_7 - 2, a_7, a_7)$ is in S_3 as long as $6 \leq a_1 + a_2 + a_3 + a_7$. Let us consider the case when $6 > a_1 + a_2 + a_3 + a_7$. We could have $(0,0,0,2,2,2,2)$, $(0,0,0,4,4,4,4)$, $(0,0,2,2,2,2,2)$ or $(1,1,1,1,1,1,1)$. Each one of those is covered by Case 1.

Case 3: If after rearranging $a_7 - 2$ is the largest, but $a_6 - 2$ is not the second largest part, then $a_5 = a_6$. So we started with $(a_1, a_2, a_3, a_4, a_6, a_6, a_7)$ and now have $(a_1 - 2, a_2, a_3, a_4, a_6 - 2, a_6, a_7 - 2)$. By Property 2, $(a_1 - 2, a_2, a_3, a_4, a_6 - 2, a_6, a_7 - 2)$ is in S_3 as long as $a_7 \neq a_1 + a_2 + a_3 + a_4 + a_5$. When $a_7 = a_1 + \ldots + a_5$ and $a_5 = a_6$ we have types of the form $(a_1, a_2, a_3, a_4, a_7 - (a_1 + \ldots + a_4), a_7 - (a_1 + \ldots + a_4), a_7)$. By Property 4, this implies $a_1 + \ldots + a_4 \leq 0$. Hence we must have $(0,0,0,0,a_6,a_6,a_7)$ and $a_7 \leq a_6 \leq a_7$. Types $(0,0,0,0,a_7,a_7,a_7)$ are discussed below as valid.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, \ldots, a_7)$ satisfies $3a_7 \leq (a_1 + \ldots + a_7)$, then $(a_1 - 2, a_2, a_3, a_4, a_5, a_6 - 2, a_7 - 2)$ satisfies $3(a_7 - 2) \leq ((a_1 - 2) + a_2 + a_3 + a_4 + a_5 + (a_6 - 2) + (a_7 - 2))$. Problems arising from rearranging of the terms are discussed under Property 2.

This is also true for the remaining parts of the algorithm. Therefore, each time we subtract we obtain an element of S_3. When we reach $(1, 1, 1, 1, 1, \_, \_)$ or $(0, 0, 0, 2, \_, \_, \_)$ all four conditions are satisfied.

Here are some "building blocks" we will use throughout this Section.

$(0, 0, 0, 0, 2, 2, 2)$:

M_1: $\{\{5, 6\}, \{5', 7\}, \{6', 7'\}\}$

M_2: $\{\{5, 7'\}, \{5', 6'\}, \{6, 7\}\}$

M_3: $\{\{5, 6\}, \{5', 7\}, \{6', 7'\}\}$

M_4: $\{\{5, 7'\}, \{5', 6'\}, \{6, 7\}\}$

M_5: $\{\{6, 7\}, \{6', 7'\}\}$

M_6: $\{\{5, 7'\}, \{5', 7\}\}$

M_7: $\{\{5, 6\}, \{5', 6'\}\}$

$(0, 0, 0, 2, 2, 2, 2)$:

M_1: $\{\{4, 7\}, \{4', 6\}, \{5', 6'\}, \{5, 7'\}\}$

M_2: $\{\{4, 5\}, \{4', 5'\}, \{6, 7\}, \{6', 7'\}\}$

M_3: $\{\{4, 5\}, \{4', 5'\}, \{6, 7\}, \{6', 7'\}\}$

M_4: $\{\{5, 7'\}, \{5', 6'\}, \{6, 7\}\}$

M_5: $\{\{4, 7\}, \{4', 6\}, \{6', 7'\}\}$

M_6: $\{\{4, 7\}, \{4', 5'\}, \{5, 7'\}\}$

M_7: $\{\{4, 5\}, \{4', 6\}, \{5', 6'\}\}$

$(0, 0, 0, 0, 0, 0, 0, 2, 2, 2)$:

M_1: $\{\{8, 10\}, \{9, 10'\}, \{8', 9'\}\}$

M_2: $\{\{8, 9'\}, \{8', 10'\}, \{9, 10\}\}$

M_3: $\{\{8,9'\},\{9,10'\},\{8',10\}\}$

M_4: $\{\{8,9'\},\{8',10'\},\{9,10\}\}$

M_5: $\{\{8,9\},\{8',10\},\{9',10'\}\}$

M_6: $\{\{8,9\},\{8',10\},\{9',10'\}\}$

M_7: $\{\{8,10\},\{8',9'\},\{9,10'\}\}$

M_8: $\{\{9,10\},\{9',10'\}\}$

M_9: $\{\{8,10\},\{8',10'\}\}$

M_{10}: $\{\{8,9\},\{8',9'\}\}$

Following is a list of 3-divisible-matchings of type $(1,1,1,1,1,\_,\_)$.

$(1,1,1,1,1,1,1)$ given in Section 3.2.1

$(1,1,1,1,1,1,3):$

M_1: $\{\{2,7'\},\{3,4\},\{5,7''\},\{6,7\}\}$

M_2: $\{\{1,7\},\{3,7''\},\{4,7'\},\{5,6\}\}$

M_3: $\{\{1,2\},\{4,7'\},\{5,7''\},\{6,7\}\}$

M_4: $\{\{1,7\},\{2,7'\},\{3,7''\},\{5,6\}\}$

M_5: $\{\{1,2\},\{3,7''\},\{4,7'\},\{6,7\}\}$

M_6: $\{\{1,7\},\{2,7'\},\{3,4\},\{5,7''\}\}$

M_7: $\{\{1,2\},\{3,4\},\{5,6\}\}$

$(1,1,1,1,1,3,3):$

M_1: $\{\{2,7'\},\{3,6''\},\{4,7''\},\{5,6\},\{6',7\}\}$

M_2: $\{\{1,6''\},\{3,7\},\{4,6'\},\{5,7'\},\{6,7''\}\}$

M_3: $\{\{1,6''\},\{2,7'\},\{4,7''\},\{5,6\},\{6',7\}\}$

M_4: $\{\{1,2\},\{3,6''\},\{5,7'\},\{6',7\},\{6,7''\}\}$

M_5: $\{\{1,6''\},\{2,7'\},\{3,7\},\{4,6'\},\{6,7''\}\}$

M_6: $\{\{1,2\},\{3,7\},\{4,7''\},\{5,7'\}\}$

M_7: $\{\{1,2\},\{3,6''\},\{4,6'\},\{5,6\}\}$

$(1,1,1,1,1,5,5)$:

M_1: $\{\{2,6'\},\{3,7''\},\{4,6'''\},\{5,7\},\{6,7'\},\{6'',7'''\},\{6^4,7^4\}\}$

M_2: $\{\{1,7^4\},\{3,7''\},\{4,6'''\},\{5,6^4\},\{6,7'\},\{6',7\},\{6'',7'''\}\}$

M_3: $\{\{1,6\},\{2,6'\},\{4,7'\},\{5,7\},\{6'',7'''\},\{6''',7''\},\{6^4,7^4\}\}$

M_4: $\{\{1,7^4\},\{2,7'''\},\{3,6''\},\{5,6^4\},\{6,7'\},\{6',7\},\{6''',7''\}\}$

M_5: $\{\{1,6\},\{2,7'''\},\{3,6''\},\{4,7'\},\{6',7\},\{6''',7''\},\{6^4,7^4\}\}$

M_6: $\{\{1,7^4\},\{2,7'''\},\{3,7''\},\{4,7'\},\{5,7\}\}$

M_7: $\{\{1,6\},\{2,6'\},\{3,6''\},\{4,6'''\},\{5,6^4\}\}$

Following is a list of 3-divisible-matchings of types $(0,0,0,2,\_,\_,\_)$ as well as $(0,0,2,2,2,2,2)$.

$(0,0,2,2,2,2,2)$:

M_1: $\{\{3,4\},\{3',7\},\{4',5'\},\{5,6\},\{6',7'\}\}$

M_2: $\{\{3,7'\},\{3',4'\},\{4,5\},\{5',6'\},\{6,7\}\}$

M_3: $\{\{4,5\},\{4',5'\},\{6',7'\},\{6,7\}\}$

M_4: $\{\{3,7'\},\{3',7\},\{5',6'\},\{5,6\}\}$

M_5: $\{\{3,4\},\{3',4'\},\{6',7'\},\{6,7\}\}$

M_6: $\{\{3,7'\},\{3',7\},\{4',5'\},\{4,5\}\}$

M_7: $\{\{3,4\},\{3',4'\},\{5',6'\},\{5,6\}\}$

$(0,0,0,0,2,2,2)$ given above

$(0,0,0,0,a_7,a_7,a_7)$=$\frac{a_7}{2}$ copies of $(0,0,0,0,2,2,2)$

By Property 2, a 3-divisible-matching of type $(0,0,0,2,a_5,a_6,a_7)$ exists if $a_5 \geq a_7 - 2$.

$(0,0,0,2,a_7,a_7,a_7)$ from $(0,0,0,2,2,2,2)$ and $(a_7-2)/2$ copies of $(0,0,0,0,2,2,2)$

$(0,0,0,2,a_7-2,a_7,a_7)$ from $(0,0,0,2,0,2,2)$ and $(a_7-2)/2$ copies of $(0,0,0,0,2,2,2)$

This gives the 3-divisible-matchings of type $(a_1,\ldots,a_7)$.

3.2.3 Ten parts of any size

Let us now consider 3-divisible-matchings of type $K = (a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9 \leq a_{10}$. Since $p = 10$ is even, all parts must be of even size. Similar to the case of seven parts, to find a 3-divisible-matching of type $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10})$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$ and $a_7 = 0, 2$ or $K = (0,0,0,0,0,2,2,2,2,2)$ look up the matching in list provided. Otherwise continue below.

2. For $0 \leq i \leq 6$ repeat the following.

3. If $5a_{10} = 2n$ skip down to the Special Case 1 section. If $a_8 = a_9$ and $a_{10} = a_1 + a_2 + a_3 + \ldots + a_8$ skip down to Special Case 2 section. Otherwise continue below.

4. Subtract $(0^i, 2, 0^{7-i}, 2, 2)$.

5. If necessary, rearrange the terms to ensure that the parts are in a nondecreasing sequence.

6. Repeat steps 3-5 until you obtain $(0^{i+1}, \_^{9-i})$ for $0 \leq i \leq 5$ and $(0^6, 2, \_^3)$ for $i = 6$. If at any point $K = (0,0,0,0,0,2,2,2,2,2)$ look it up.

7. Stop when you reach $(0^6, 2, \_, \_, \_)$. The 3-divisible-matchings of type $(0^6, 2, \_, \_, \_)$ are listed below.

Let us consider the four necessary conditions during steps 4-6 of the algorithm.

1. $p = 10$ is not affected.

2. If $(a_1, a_2, \ldots, a_{10})$ satisfies $2a_{10} + a_9 \leq a_1 + \ldots + a_{10}$, then

   $(a_1 - 2, a_2, \ldots, a_9 - 2, a_{10} - 2)$ satisfies

   $2(a_{10} - 2) + (a_9 - 2) \leq (a_1 - 2) + a_2 + \ldots + a_8 + (a_9 - 2) + (a_{10} - 2)$.

   We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_{10} - 2$ is not the largest, but the second largest part, then $a_8 = a_{10}$. So we started with $(a_1, a_2, \ldots, a_7, a_{10}, a_{10}, a_{10})$ and now have $(a_1 - 2, a_2, \ldots, a_7, a_{10} - 2, a_{10} - 2, a_{10})$. By Properties 2 and 4 $(a_1 - 2, a_2, \ldots, a_7, a_{10} - 2, a_{10} - 2, a_{10})$ is in S_3 as long as $4 \leq a_1 + \ldots + a_7$ and $12 - a_{10} \leq 2(a_1 + \ldots + a_7)$. Let us consider the case when $a_1 + \ldots + a_7 < 4$. We could have $(0,0,0,0,0,0,0,a_{10},a_{10},a_{10})$ or $(0,0,0,0,0,0,2,a_{10},a_{10},a_{10})$. Each one of those is discussed and shown to be in S_3 below. Let us consider the case when $12 - a_{10} > 2(a_1 + \cdots + a_7)$. We could have $(0,0,0,0,0,0,0,2,2,2)$, $(0,0,0,0,0,0,0,4,4,4)$, $\ldots$, $(0,0,0,0,0,0,0,10,10,10)$, $(0,0,0,0,0,0,2,2,2,2)$, $(0,0,0,0,0,0,2,4,4,4)$, $(0,0,0,0,0,0,2,6,6,6)$ or $(0,0,0,0,0,2,2,2,2,2)$. Each one of those is shown to be in S_3.

Case 2: If after rearranging $a_{10} - 2$ is not the largest or second largest part, then $a_7 = a_8 = a_{10}$. So we started with $(a_1, \ldots, a_6, a_{10}, a_{10}, a_{10}, a_{10})$ and now have $(a_1 - 2, a_2, \ldots, a_{10} - 2, a_{10} - 2, a_{10}, a_{10})$. By Properties 2 and 4 $(a_1 - 2, a_2, \ldots, a_{10} - 2, a_{10} - 2, a_{10}, a_{10})$ is in S_3 as long as $6 - a_{10} \leq a_1 + \ldots + a_6$ and $12 - 3a_{10} \leq 2(a_1 + \ldots + a_6)$. Let us consider the case when $6 - a_{10} > a_1 + \ldots + a_6$ or $12 - 3a_{10} > 2(a_1 + \ldots + a_6)$. We could have $(0, 0, 0, 0, 0, 0, 2, 2, 2, 2)$, $(0, 0, 0, 0, 0, 0, 4, 4, 4, 4)$ or $(0, 0, 0, 0, 0, 2, 2, 2, 2, 2)$. Each one of those is in S_3.

Case 3: If after rearranging $a_{10} - 2$ is the largest, but $a_9 - 2$ is not the second largest part, then $a_8 = a_9$. So we started with $(a_1, a_2, \ldots, a_7, a_9, a_9, a_{10})$ and now have $(a_1 - 2, a_2, \ldots, a_7, a_9 - 2, a_9, a_{10} - 2)$.

By Property 2, $(a_1 - 2, a_2, \ldots, a_7, a_9 - 2, a_9, a_{10} - 2)$ is in S_3 as long as $a_{10} \neq a_1 + \ldots + a_7 + a_9$. The case when $a_{10} = a_1 + \ldots + a_7 + a_9$ and $a_8 = a_9$ is discussed below as Special Case 2.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, a_2, \ldots, a_{10})$ satisfies $5a_{10} < 2(a_1 + \ldots + a_{10})$, then $(a_1 - 2, a_2, \ldots, a_9 - 2, a_{10} - 2)$ satisfies $5(a_{10} - 2) \leq 2((a_1 - 2) + a_2 + \ldots + (a_9 - 2) + (a_{10} - 2))$. If $5a_{10} = 2n$ we refer to Special Case 1. Problems arising from rearranging of the terms are discussed under Property 2.

Therefore, each time we repeat steps 2-6 we obtain an element of S_3. Notice that the above holds for remaining steps of the algorithm. So it is sufficient to give 3-divisible-matchings of type $(0, 0, 0, 0, 0, 0, 2, \_, \_, \_)$ and consider the special cases mentioned above.

Let us consider the **Special Case 1**.

We need to find 3-divisible-matchings for $(a_1, \ldots, a_{10})$ satisfying the four necessary conditions and $3a_{10} = 2(a_1 + \ldots + a_9)$. Let us refer to these as matching type $SP^3 1 \subseteq S_3$. Say we need $K = (a_1, a_2, \ldots, a_9, \frac{2(a_1+\ldots+a_9)}{3}) \in SP^3 1$. We will build it by induction.

We start by subtracting $(0, 0, 0, 0, 0, 0, 2, 2, 2, 4)$ to obtain $K' = (a_1, \ldots, a_6, a_7 - 2, a_8 - 2, a_9 - 2, \frac{2(a_1+\ldots+a_9)}{3} - 4)$. As long as no rearranging is necessary K' is in $SP^3 1$. If rearranging is necessary, we started with $K = (a_1, \ldots, a_5, a_9, \ldots, a_9, \frac{2(a_1+\ldots+a_9)}{3})$ and now have $K'' = (a_1, \ldots, a_5, a_9 - 2, a_9 - 2, a_9 - 2, a_9, \frac{2(a_1+\ldots+a_9)}{3} - 4)$. This is in $SP^3 1$ as long as $6 \leq a_1 + \ldots + a_5 + a_9$. The only time $6 > a_1 + \ldots + a_5 + a_9$ is for $K = (0^6, 2^3, 4)$ which is given. We need to consider if it is possible for $a_9 > \frac{2(a_1+\ldots+a_9)}{3} - 4$. This is equivalent to $8 \leq a_1 + \ldots + a_8$ which is true for all elements of $SP^3 1$ except $K = (0^6, 2^3, 4)$. By induction, this concludes Special Case 1. We take any element of $SP^3 1$ and subtract to get a smaller element of $SP^3 1$.

Now let us consider **Special Case 2**.

If $a_8 = a_9$ and $a_{10} = a_1 + \ldots + a_7 + a_9$ we have types of the form $(a_1, a_2, \ldots, a_7, a_{10} - (a_1 + \ldots + a_7), a_{10} - (a_1 + \ldots + a_7), a_{10})$. We will construct a 3-divisible-matching of such type as follows: $a_1/2$ copies of $(2, 0, 0, 0, 0, 0, 0, 2, 2, 4)$, $a_2/2$ copies of $(0, 2, 0, 0, 0, 0, 0, 2, 2, 4)$, $\ldots$, $a_7/2$ copies of $(0, 0, 0, 0, 0, 0, 2, 2, 2, 4)$ and $(a_{10} - 2(a_1 + \ldots + a_7))/2$ copies of $(0, 0, 0, 0, 0, 0, 0, 2, 2, 2)$. Notice that by Property 4 $a_{10} \geq 2(a_1 + \ldots + a_7)$. This concludes Special Case 2.

The following are 3-divisible-matchings of type $(0, 0, 0, 0, 0, 0, 2, \_, \_, \_)$ and $(0, 0, 0, 0, 0, 2, 2, 2, 2, 2)$.

$(0, 0, 0, 0, 0, 2, 2, 2, 2, 2)$:

M_1: $\{\{6,7\},\{8,9\},\{6',10\},\{7',8'\},\{9',10'\}\}$

M_2: $\{\{6,10'\},\{6',7'\},\{7,8\},\{8',9'\},\{9,10\}\}$

M_3: $\{\{6,7\},\{6',8\},\{7',8'\},\{9,10\},\{9',10'\}\}$

M_4: $\{\{6,7\},\{6',8\},\{7',8'\},\{9,10\},\{9',10'\}\}$

M_5: $\{\{6,7\},\{6',8\},\{7',8'\},\{9,10\},\{9',10'\}\}$

M_6: $\{\{7,8\},\{7',8'\},\{9,10\},\{9',10'\}\}$

M_7: $\{\{6,10'\},\{6',10\},\{8,9\},\{8',9'\}\}$

M_8: $\{\{6,7\},\{6',7'\},\{9,10\},\{9',10'\}\}$

M_9: $\{\{6,10'\},\{6',10\},\{7,8\},\{7',8'\}\}$

M_{10}: $\{\{6,7\},\{6',7'\},\{8,9\},\{8',9'\}\}$

$(0,0,0,0,0,0,0,2,2,2)$ was given in Section 3.2.2

By Property 2, a 3-divisible-matching of type $(0,0,0,0,0,0,0,a_8,a_9,a_{10})$ exists if $a_{10} \leq a_8$. Hence we must have $(0,0,0,0,0,0,0,a_{10},a_{10},a_{10})$.

$(0,0,0,0,0,0,0,a_{10},a_{10},a_{10})$ from $a_{10}/2$ copies of $(0,0,0,0,0,0,0,2,2,2)$

$(0,0,0,0,0,0,2,2,2,2)$:

M_1: $\{\{8,9\},\{7,9'\},\{8',10\},\{7',10'\}\}$

M_2: $\{\{7,8\},\{9',10'\},\{9,10\},\{7',8'\}\}$

M_3: $\{\{7,10\},\{7',9\},\{8,10'\},\{8',9'\}\}$

M_4: $\{\{7,9'\},\{7',10'\},\{8,9\},\{8',10\}\}$

M_5: $\{\{7,8\},\{7',8'\},\{9',10'\},\{9,10\}\}$

M_6: $\{\{7,10\},\{7',9\},\{8',9'\},\{8,10'\}\}$

M_7: $\{\{8,9\},\{8',10\},\{9',10'\}\}$

M_8: $\{\{7,9'\},\{9,10\},\{7',10'\}\}$

M_9: $\{\{7,10\},\{8,10'\},\{7',8'\}\}$

M_{10}: $\{\{7,8\},\{7',9\},\{8',9'\}\}$

$(0,0,0,0,0,0,2,2,2,4)$:

M_1: $\{\{8,10''\},\{7,10'''\},\{8',9'\},\{7',10'\},\{9,10\}\}$

M_2: $\{\{7,8\},\{9,10'\},\{9',10\},\{7',10''\},\{8',10'''\}\}$

M_3: $\{\{7,10\},\{7',9\},\{8,10''\},\{8',10'''\},\{9',10'\}\}$

M_4: $\{\{7,10'''\},\{7',10'\},\{8,10''\},\{8',9'\},\{9,10\}\}$

M_5: $\{\{7,8\},\{9,10'\},\{9',10\},\{7',10''\},\{8',10'''\}\}$

M_6: $\{\{7,10\},\{7',9\},\{8,10''\},\{8',10'''\},\{9',10'\}\}$

M_7: $\{\{8,10''\},\{8',10'''\},\{9',10'\},\{9,10\}\}$

M_8: $\{\{7,10'''\},\{9',10\},\{7',10''\},\{9,10'\}\}$

M_9: $\{\{7,10\},\{8,10''\},\{7',10'\},\{8',10'''\}\}$

M_{10}: $\{\{7,8\},\{7',9\},\{8',9'\}\}$

By Property 2, a 3-divisible-matching of type $(0,0,0,0,0,0,2,a_8,a_9,a_{10})$ exists if $a_8 \geq a_{10}-2$.

$(0,0,0,0,0,0,2,a_{10},a_{10},a_{10})$ from $(0,0,0,0,0,0,2,2,2,2)$ and $(a_{10}-2)/2$ copies of $(0,0,0,0,0,0,0,2,2,2)$

$(0,0,0,0,0,0,2,a_{10}-2,a_{10},a_{10})$ from $(0,0,0,0,0,0,2,0,2,2)$ and $(a_{10}-2)/2$ copies of $(0,0,0,0,0,0,0,2,2,2)$

Clearly $a_{10}-2 \geq 0$ since $a_{10} \geq 2$.

$(0,0,0,0,0,0,2,a_{10}-2,a_{10}-2,a_{10})$ from $(0,0,0,0,0,0,2,2,2,4)$ and $(a_{10}-4)/2$ copies of $(0,0,0,0,0,0,0,2,2,2)$

Similarly, $a_{10}-4 \geq 0$ since $a_{10}-2 \geq 2$.

Hence we have 3-divisible-matchings of type $(a_1, \ldots, a_{10})$.

3.2.4 $p \equiv 1 \ (mod \ \ 6)$ parts of any size

In this Section we will give a general construction of 3-divisible-matchings of type $(a_1, \ldots, a_{6h+1})$ for any positive integer $h \geq 2$. Since $p \equiv 1 \ (mod \ \ 6)$ we can have all parts of even size or all parts of odd size. We will need "building blocks" similar to the ones used before.

$(0^{6h-2}, 2, 2, 2)$: from Lemma 3.4

$(0^{6h-3}, 2, 2, 2, 2)$: from Lemma 3.4

Notice that when $p = 6h + 1$ condition 4 becomes $2ha_p \leq (2h-1)(a_1 + \ldots + a_{p-1})$ and condition 2 remains $a_p \leq a_1 + \ldots + a_{p-2}$. We can construct a 3-divisible-matching of type $(0, 0, 0, a_4, \ldots, a_p)$ inductively. We start with the 3-divisible-matching $(a_4, \ldots, a_p)$ (as long as it exists) and apply Lemma 3.4.

To find a 3-divisible-matching of type $(a_1, \ldots, a_{6h+1})$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = \ldots = a_{p-4} = 0$, $a_{p-3} = 2$ or $a_1 = \ldots = a_{p-2} = 1$ or $K = (0^{6h-4}, 2, 2, 2, 2, 2)$ or $K = (0^{6h-4}, 2, 2, 2, 2, 6)$ look up the matching in list provided. Otherwise continue below.

2. If $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-3$ skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h-1)$ section. Otherwise continue below.

3. Subtract $(2, 0^{6h-2}, 2, 2)$ to obtain $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$.

4. If necessary, rearrange the terms to ensure a nondecreasing sequence.

5. Repeat steps 2-4 until you obtain $(0, \_^{6h})$ or $(1, \_^{6h})$.

6. If $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-3$ skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h-1)$ section.

   Otherwise continue below.

7. For $6h - 3 \geq j \geq 1$ repeat the following steps.

8. Subtract $(0^{6h-2-j}, 2, 0^j, 2, 2)$.

9. If necessary, rearrange the terms to ensure the sequence is nondecreasing.

10. Stop when you obtain $(0^{6h-3}, 2, \_, \_, \_)$ or $(1^{6h-1}, \_, \_)$.

11. The 3-divisible-matchings of type $(0^{6h-3}, 2, \_, \_, \_)$ and $(1^{6h-1}, \_, \_)$ will need to be given.

Let us consider the four necessary conditions during steps 2-5 of the algorithm.

1. $p = 6h + 1$ is not affected.

2. If $(a_1, \ldots, a_p)$ satisfies $a_p \leq a_1 + \ldots + a_{p-2}$, then $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$ satisfies $a_p - 2 \leq a_1 - 2 + a_2 + \ldots + a_{p-2}$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_p - 2$ is not the largest, but the second largest part, then $a_{p-2} = a_p$. So we started with $(a_1, \ldots, a_{p-3}, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_p - 2, a_p - 2, a_p)$. By Properties 2 and 4, this is in S_3 as long as $4 \leq a_1 + \ldots + a_{p-3}$ and $12h - 6 - (2h-2)a_p \leq (2h-1)(a_1 + \ldots + a_{p-3})$, which is equivalent to $a_1 + \ldots + a_{p-3} \geq 6 - \frac{2h-2}{2h-1}a_p$. Let us consider the cases when $4 > a_1 + \ldots + a_{p-3}$ or $a_1 + \ldots + a_{p-3} < 6 - \frac{2h-2}{2h-1}a_p$. We could have $(0^{6h-2}, a_p, a_p, a_p)$, $(0^{6h-3}, 2, a_p, a_p, a_p)$ or $(0^{6h-4}, 2, 2, 2, 2, 2)$. Each one of those is in S_3.

Case 2: If after rearranging $a_p - 2$ is not the largest or second largest part, then $a_{p-3} = a_p$. So we started with $(a_1, \ldots, a_{p-4}, a_p, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-4}, a_p - 2, a_p - 2, a_p, a_p)$. By Properties 2 and 4, this is in S_3 as long as $6 - a_p \leq a_1 + \ldots + a_{p-4}$ and $6 - \frac{4h-3}{2h-1}a_p \leq a_1 + \ldots + a_{p-4}$. Let us consider the cases when $6 - a_p > a_1 + \ldots + a_{p-4}$ or $6 - \frac{4h-3}{2h-1}a_p > a_1 + \ldots + a_{p-4}$. We could have $(0^{6h-3}, 2, 2, 2, 2)$, $(0^{6h-3}, 4, 4, 4, 4)$ or $(0^{6h-4}, 2, 2, 2, 2, 2)$. Each one of those is in S_3.

Case 3: If after rearranging $a_p - 2$ is the largest, but $a_{p-1} - 2$ is not the second largest part, then $a_{p-2} = a_{p-1}$. So we started with $(a_1, \ldots, a_{p-3}, a_{p-1}, a_{p-1}, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_{p-1} - 2, a_{p-1}, a_p - 2)$. By Property 2, this is in S_3 as long as $a_p \neq a_1 + \ldots + a_{p-2}$. The case when $a_p = a_1 + \ldots + a_{p-2}$ and $a_{p-2} = a_{p-1}$ is discussed below as Special Case $(2h-1)$.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, \ldots, a_p)$ satisfies $2ha_p \leq (2h-1)(a_1 + \ldots + a_{p-1}) - 4(h-1)$, then $(a_1 - 2, a_2, \ldots, a_{p-2}, a_{p-1} - 2, a_p - 2)$ satisfies $2h(a_p - 2) \leq (2h-1)(a_1 + \ldots + a_{p-1} - 4)$. Each Special Case $(c+1)$ section covers the instances when $2ha_p = (2h-1)(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-3$. Problems arising from rearranging the terms are discussed under condition 2.

Therefore, each time we repeat steps 2-5 we obtain an element of S_3. Notice that the above holds for the remaining steps of the algorithm. So it is sufficient to give 3-divisible-matchings of type $(0^{6h-3}, 2, \_, \_, \_)$ and $(1^{6h-1}, \_, \_)$ and consider the special cases mentioned above.

Let us consider the **Special Case** $(c+1)$ for $0 \leq c \leq 2h-3$.

We need to find 3-divisible-matchings for $(a_1, \ldots, a_p)$ satisfying the four necessary conditions and $2ha_p = (2h-1)(a_1+\ldots+a_{p-1})-2c$. Let us refer to these as matching type $SP^3(c+1) \subseteq S_3$. Say we need $K = (a_1, a_2, \ldots, a_{p-1}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h}) \in SP^3(c+1)$. We will build it by induction.

We consider parts of odd size first. We start by subtracting $R^1 = (1^{4h+c}, 3^{2h-c}, 10h-5-2c)$ to obtain $K' = (a_1-1, \ldots, a_{4h+c}-1, a_{4h+c+1}-3, \ldots, a_{6h}-3, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (10h-5-2c))$. Since necessary conditions 2 and 4 were satisfied in K, they are still satisfied in K'. Also, $K' \in SP^31$. However, if rearranging is necessary and we obtain $K'' = (a_1-1, \ldots, a_{4h+c+1}-3, a_{4h+c+2}-3, \ldots, a_{6h}-3, a_{4h+c}-1, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (10h-5-2c))$ instead of K', we must have $a_{4h+c}-1 = a_{6h}-3+2$ or $a_{4h+c} = a_{6h}$. Hence we started with $K = (a_1, \ldots, a_{4h+c-1}, a_{6h}, \ldots, a_{6h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$. In this case instead of subtracting $R^1 = (1^{4h+c}, 3^{2h-c}, 10h-5-2c)$, we subtract $R^2 = (1^{2h+c}, 3^{4h-c}, 14h-7-2c)$ to obtain $K' = (a_1-1, \ldots, a_{2h+c}-1, a_{2h+c+1}-3, \ldots, a_{6h}-3, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (14h-7-2c))$. Notice that as long as no rearranging is necessary, $K' \in SP^31$. However, if rearranging is necessary and results in $K'' = (a_1-1, \ldots, a_{2h+c-1}-1, a_{2h+c+1}-3, \ldots, a_{6h}-3, a_{2h+c}-1, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (14h-7-2c))$ we must have $a_{2h+c} = a_{6h}$. So we started with $K = (a_1, \ldots, a_{2h+c-1}, a_{6h}, \ldots, a_{6h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$ and now have $K'' = (a_1 - 1, \ldots, a_{2h+c-1} - 1, a_{6h} - 3, \ldots, a_{6h} - 3, a_{6h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$. Satisfying the conditions 2 and 4, this K'' is always in $SP^3 1$. This shows that when we complete this step, we obtain an element of $SP^k 1$.

For parts of even size we start by subtracting $(0^{4h+c}, 2^{2h-c}, 4h - 2 - 2c)$ to obtain $K' = (a_1, \ldots, a_{4h+c}, a_{4h+c+1} - 2, \ldots, a_{6h} - 2, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (4h - 2 - 2c))$. As long as no rearranging is necessary K' is in $SP^3 1$. If rearranging is necessary, we started with $K = (a_1, \ldots, a_{4h+c-1}, a_{6h}, \ldots, a_{6h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h})$ and now have $K'' = (a_1, \ldots, a_{4h+c-1}, a_{6h} - 2, \ldots, a_{6h} - 2, a_{6h}, \frac{(2h-1)(a_1+\ldots+a_{p-1})-2c}{2h} - (4h - 2 - 2c))$. This is in $SP^3 1$. Notice that no matter what $SP^3(c + 1)$ we start with, after the first subtraction, we will continue the induction process with $SP^3 1$ (i.e. $c = 0$).

By induction, this concludes Special Case $(c+1)$. We take any element of $SP^3(c+1)$ and subtract to get a smaller element of $SP^3 1$.

Now let us consider **Special Case** $(2h - 1)$.

If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ we have types of the form $(a_1, a_2, \ldots, a_{p-3}, a_p - (a_1 + \ldots + a_{p-3}), a_p - (a_1 + \ldots + a_{p-3}), a_p)$. We will construct a 3-divisible-matching of such type as follows: $a_j/2$ copies of $(0^{j-1}, 2, 0^{6h-2-j}, 2, 2, 4)$ for each $1 \leq j \leq 6h - 2$, and $a_p - 2(a_1 + \ldots + a_{p-3})/2$ copies of $(0^{6h-2}, 2, 2, 2)$. Notice that by Property 4, $a_p \geq 2(a_1 + \ldots + a_{p-3})$. This concludes Special Case $(2h - 1)$.

The following are 3-divisible-matchings of type $(0^{6h-3}, 2, \_, \_, \_)$ and $(1^{6h-1}, \_, \_)$ and other matchings used in the above construction.

$(0^{6h-2}, 2, 2, 2)$: from Lemma 3.4

By Property 2, a 3-divisible-matching of type $(0^{6h-2}, a_{p-2}, a_{p-1}, a_p)$ exists if $a_p \leq a_{p-2}$. Hence we must have $(0^{6h-2}, a_p, a_p, a_p)$.

$(0^{6h-2}, a_p, a_p, a_p)$ from $a_p/2$ copies of $(0^{6h-2}, 2, 2, 2)$

$(0^{6h-3}, 2, 2, 2, 2)$: from Lemma 3.4

$(0^{6h-3}, 2, 2, 2, 4)$: from Lemma 3.4

By Property 2, a 3-divisible-matching of type $(0^{6h-3}, 2, a_{p-2}, a_{p-1}, a_p)$ exists if $a_{p-2} \geq a_p - 2$.

$(0^{6h-3}, 2, a_p, a_p, a_p)$ from $(0^{6h-3}, 2, 2, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{6h-2}, 2, 2, 2)$

$(0^{6h-3}, 2, a_p - 2, a_p, a_p)$ from $(0^{6h-3}, 2, 0, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{6h-2}, 2, 2, 2)$

Since $a_p \geq 2$, $a_p - 2 \geq 0$.

$(0^{6h-3}, 2, a_p - 2, a_p - 2, a_p)$ from $(0^{6h-3}, 2, 2, 2, 4)$ and $(a_p - 4)/2$ copies of $(0^{6h-2}, 2, 2, 2)$

Since $a_p - 2 \geq 2$, $a_p - 4 \geq 0$.

$(0^{6h-4}, 2, 2, 2, 2, 2)$: from Lemma 3.4

$(0^{6h-4}, 2, 2, 2, 2, 6)$: from Lemma 3.4

$(0^{4h}, 2^{2h}, 4h - 2)$: Needed; not yet found.

The remaining $(0^{4h+c}, 2^{2h-c}, 4h - 2 - 2c)$ for $0 < c \leq 2h - 3$ are obtained from Lemma 3.4

(1^{6h+1}): given in Section 3.2.1

$(1^{4h+c}, 3^{2h-c}, 10h - 5 - 2c)$: Needed; not yet found.

$(1^{2h+c}, 3^{4h-c}, 7^h, 14h - 7 - 2c)$: Needed; not yet found.

In the following families of matchings we use i, j odd.

For $3 \leq i \leq 6h - 3$, $(1^{6h}, i)$: Needed; not yet found.

For $3 \leq i \leq 6h - 1$ and $3 \leq j \leq i$, $(1^{6h-1}, j, i)$: Needed; not yet found.

Once the above missing matchings are found, we will have 3-divisible-matchings of type $(a_1, \ldots, a_{6h+1})$.

3.2.5 $p \equiv 4$ $(mod\ \ 6)$ parts of any size

In this Section we will give a general construction of 3-divisible-matchings of type $(a_1, \dots, a_{6h+4})$ for any positive integer $h \geq 2$. Since $p \equiv 4$ $(mod\ 6)$ we can only have all parts of even size. We will use "building blocks" similar to the ones above.

$(0^{6h+1}, 2, 2, 2)$: from Lemma 3.4

$(0^{6h}, 2, 2, 2, 2)$: from Lemma 3.4

Notice that when $p = 6h+4$ condition 4 becomes $(2h+1)a_p \leq 2h(a_1+\ldots+a_{p-1})$ and condition 2 remains $a_p \leq a_1+\ldots+a_{p-2}$. We can still construct a 3-divisible-matching of type $(0, 0, 0, a_3, \dots, a_p)$ inductively. We start with the 3-divisible-matching $(a_3, \dots, a_p)$ (as long as it exists) and apply Lemma 3.4.

To find a 3-divisible-matching of type $(a_1, \dots, a_{6h+4})$ we use the following algorithm.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = \ldots = a_{p-4} = 0$, $a_{p-3} = 2$ or $K = (0^{6h-1}, 2, 2, 2, 2, 2)$ or $K = (0^{6h-1}, 2, 2, 2, 2, 6)$ look up the matching in list provided. Otherwise continue below.

2. If $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$, for $0 \leq c \leq 2h-2$, skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h)$ section. Otherwise continue below.

3. Subtract $(2, 0^{6h+1}, 2, 2)$ to obtain $(a_1 - 2, a_2, \dots, a_{p-1} - 2, a_p - 2)$.

4. If necessary, rearrange the terms to ensure a nondecreasing sequence.

5. Repeat steps 2-4 until you obtain $(0, \_^{6h+3})$.

6. If $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h-2$ skip down to the Special Case $(c+1)$ section.

   If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ skip down to Special Case $(2h)$ section. Otherwise continue below.

7. For $6h \geq j \geq 1$ repeat the following steps.

8. Subtract $(0^{6h+1-j}, 2, 0^j, 2, 2)$.

9. If necessary, rearrange the terms to ensure the sequence is nondecreasing.

10. Stop when you obtain $(0^{6h}, 2, \_, \_, \_)$.

11. The 3-divisible-matchings of type $(0^{6h}, 2, \_, \_, \_)$ will need to be given.

Let us consider the four necessary conditions during steps 2-5 of the algorithm.

1. $p = 6h + 4$ is not affected.

2. If $(a_1, \ldots, a_p)$ satisfies $a_p \leq a_1 + \ldots + a_{p-2}$, then $(a_1 - 2, a_2, \ldots, a_{p-1} - 2, a_p - 2)$ satisfies $a_p - 2 \leq a_1 - 2 + a_2 + \ldots + a_{p-2}$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_p - 2$ is not the largest, but the second largest part, then $a_{p-2} = a_p$. So we started with $(a_1, \ldots, a_{p-3}, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_p - 2, a_p - 2, a_p)$. By Properties 2 and 4, this is in S_3 as long as $4 \leq a_1 + \ldots + a_{p-3}$ and $a_1 + \ldots + a_{p-3} \geq 6 - \frac{2h-1}{2h}a_p$. Let us consider the cases when $4 > a_1 + \ldots + a_{p-3}$ or $a_1 + \ldots + a_{p-3} < 6 - \frac{2h-1}{2h}a_p$. We could have $(0^{6h+1}, a_p, a_p, a_p)$, $(0^{6h}, 2, a_p, a_p, a_p)$ or $(0^{6h-1}, 2, 2, 2, 2, 2)$. Each one of those is in S_3.

Case 2: If after rearranging $a_p - 2$ is not the largest or second largest part, then $a_{p-3} = a_p$. So we started with $(a_1, \ldots, a_{p-4}, a_p, a_p, a_p, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-4}, a_p - 2, a_p - 2, a_p, a_p)$. By Properties 2 and 4, this is in S_3 as long as $6 - a_p \leq a_1 + \ldots + a_{p-4}$ and $6 - \frac{4h-1}{2h} a_p \leq a_1 + \ldots + a_{p-4}$. Let us consider the cases when $6 - a_p > a_1 + \ldots + a_{p-4}$ or $6 - \frac{4h-1}{2h} a_p > a_1 + \ldots + a_{p-4}$. We could have $(0^{6h}, 2, 2, 2, 2)$, $(0^{6h}, 4, 4, 4, 4)$ or $(0^{6h}, 2, 2, 2, 2, 2)$. Each one of those is in S_3.

Case 3: If after rearranging $a_p - 2$ is the largest, but $a_{p-1} - 2$ is not the second largest part, then $a_{p-2} = a_{p-1}$. So we started with $(a_1, \ldots, a_{p-3}, a_{p-1}, a_{p-1}, a_p)$ and now have $(a_1 - 2, a_2, \ldots, a_{p-3}, a_{p-1} - 2, a_{p-1}, a_p - 2)$. By Property 2, this is in S_3 as long as $a_p \neq a_1 + \ldots + a_{p-2}$. The case when $a_p = a_1 + \ldots + a_{p-2}$ and $a_{p-2} = a_{p-1}$ is discussed below as Special Case $(2h)$.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, \ldots, a_p)$ satisfies $(2h+1)a_p \leq 2h(a_1 + \ldots + a_{p-1}) - 2(2h-1)$, then $(a_1 - 2, a_2, \ldots, a_{p-2}, a_{p-1} - 2, a_p - 2)$ satisfies $(2h+1)(a_p - 2) \leq 2h(a_1 + \ldots + a_{p-1} - 4)$. Each Special Case $(c+1)$ section covers the instances when $(2h+1)a_p = 2h(a_1 + \ldots + a_{p-1}) - 2c$ for $0 \leq c \leq 2h - 2$. Problems arising from rearranging the terms are discussed under condition 2.

Therefore, each time we repeat steps 2-5 we obtain an element of S_3. Notice that the above holds for the remaining steps of the algorithm. So it is sufficient to give 3-divisible-matchings of type $(0^{6h}, 2, \_, \_, \_)$ and consider the special cases mentioned above.

Let us consider the **Special Case** $(c+1)$.

As with $p = 1 \ (mod \ \ 6)$ we define the set $\overline{SP^3(c+1)}$ as the set of all matching types satisfying the four necessary conditions and $(2h+1)a_p = 2h(a_1+\ldots+a_{p-1})-2c$. We construct a 3-divisible-matching of type $K = (a_1, \ldots, a_{6h+4}) \in \overline{SP^3(c+1)}$ by induction. We start by subtracting $(0^{4h+c+2}, 2^{2h-c+1}, 4h-2c)$ to obtain $K' = (a_1, \ldots, a_{4h+c+2}, a_{4h+c+3} - 2, \ldots, a_{6h+3} - 2, \frac{(2h)(a_1+\ldots+a_{p-1})-2c}{2h+1} - (4h-2c))$. As long as no rearranging is necessary K' is in $\overline{SP^3 1}$. If rearranging is necessary, we started with

$K = (a_1, \ldots, a_{4h+c+1}, a_{6h+3}, \ldots, a_{6h+3}, \frac{(2h)(a_1+\ldots+a_{p-1})-2c}{2h+1})$ and now have

$K'' = (a_1, \ldots, a_{4h+c+1}, a_{6h+3} - 2, \ldots, a_{6h+3} - 2, a_{6h+3}, \frac{(2h)(a_1+\ldots+a_{p-1})-2c}{2h+1} - (4h-2c))$.

This is always in $\overline{SP^3 1}$. Notice that no matter what $\overline{SP^3(c+1)}$ we start with, after the first subtraction, we will continue the induction process with $\overline{SP^3 1}$ (i.e. $c = 0$). This concludes the inductive argument, since each time we repeat this process we obtain a smaller element of $\overline{SP^3 1}$. Thus Special Case $(c+1)$ is solved.

Now let us consider **Special Case** $(2h)$.

If $a_{p-2} = a_{p-1}$ and $a_p = a_1 + \ldots + a_{p-2}$ we have types of the form $(a_1, a_2, \ldots, a_{p-3}, a_p - (a_1 + \ldots + a_{p-3}), a_p - (a_1 + \ldots + a_{p-3}), a_p)$. We will construct a 3-divisible-matching of such type as follows: $a_j/2$ copies of $(0^{j-1}, 2, 0^{6h+1-j}, 2, 2, 4)$ for each $1 \leq j \leq 6h+1$, and $a_p - 2(a_1 + \ldots + a_{p-3})/2$ copies of $(0^{6h+1}, 2, 2, 2)$. This concludes Special Case $(2h)$.

The following are 3-divisible-matchings of type $(0^{6h}, 2, \_, \_, \_)$ and other matchings used above.

$(0^{6h+1}, 2, 2, 2)$: from Lemma 3.4

By Property 2, a 3-divisible-matching of type $(0^{6h+1}, a_{p-2}, a_{p-1}, a_p)$ exists if $a_p \leq a_{p-2}$. Hence we must have $(0^{6h+1}, a_p, a_p, a_p)$.

$(0^{6h+1}, a_p, a_p, a_p)$ from $a_p/2$ copies of $(0^{6h+1}, 2, 2, 2)$

$(0^{6h}, 2, 2, 2, 2)$: from Lemma 3.4

$(0^{6h}, 2, 2, 2, 4)$: from Lemma 3.4

By Property 2, a 3-divisible-matching of type $(0^{6h}, 2, a_{p-2}, a_{p-1}, a_p)$ exists if $a_{p-2} \geq a_p - 2$.

$(0^{6h}, 2, a_p, a_p, a_p)$ from $(0^{6h}, 2, 2, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{6h+1}, 2, 2, 2)$

$(0^{6h}, 2, a_p - 2, a_p, a_p)$ from $(0^{6h}, 2, 0, 2, 2)$ and $(a_p - 2)/2$ copies of $(0^{6h+1}, 2, 2, 2)$

Since $a_p \geq 2$, $a_p - 2 \geq 0$.

$(0^{6h}, 2, a_p - 2, a_p - 2, a_p)$ from $(0^{6h}, 2, 2, 2, 4)$ and $(a_p - 4)/2$ copies of $(0^{6h+1}, 2, 2, 2)$

Since $a_p - 2 \geq 2$, $a_p - 4 \geq 0$.

$(0^{6h-1}, 2, 2, 2, 2, 2)$: from Lemma 3.4

$(0^{6h-1}, 2, 2, 2, 2, 6)$: from Lemma 3.4 For $c \geq 1$ we have $(0^{4h+c+2}, 2^{2h-c+1}, 4h - 2c)$ from Lemma 3.4

$(0^{4h+2}, 2^{2h+1}, 4h)$: Needed; not yet found.

Therefore we have 3-divisible-matchings for $p = 4$ $(mod\ 6)$. Once the missing matchings are found, this will prove the sufficiency of the necessary conditions listed in Section 3.1.

3.3 k-divisible-matchings

When $p \equiv 1$ $(mod\ 2k)$, say $p = 2kh + 1$, Property 4 is $2kha_p \leq (2kh - k)(a_1 + \ldots + a_{p-1})$ and when $p \equiv (1 + k)$ $(mod\ 2k)$, say $p = 2kh + k + 1$, it is $(2kh + k)a_p \leq (2kh)(a_1 + \ldots + a_{p-1})$.

3.3.1 Any number of parts of the same size

For the case of same sized parts we need to construct k-divisible-matchings of types (1^{2k+1}), (2^{2k+1}) and (2^{3k+1}). Following are those matchings.

(1^{2k+1}):

M_1: $\{\{2,3\},\{4,5\},\ldots,\{2k,2k+1\}\}$

For $1 \le i \le k$:

M_{2i}: $\{\{2i+1,2i+2\},\{2i+3,2i+4\},\ldots,\{2k+1,1\},\ldots,\{2i-2,2i-1\}\}$

For $1 \le i \le k-1$:

M_{2i+1}: $\{\{2i+2,2i+3\},\{2i+4,2i+5\},\ldots,\{2k,2k+1\},\ldots,\{2i-1,2i\}\}$

And finally,

M_{2k+1}: $\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}$

(2^{2k+1}):

M_1: $\{\{2,3\},\{4,5\},\ldots,\{2k,2k+1\}\ \{2',3'\},\{4',5'\},\ldots,\{(2k)',(2k+1)'\}\}$

For $1 \le i \le k$:

M_{2i}: $\{\{2i+1,2i+2\},\{2i+3,2i+4\},\ldots,\{2k+1,1\},\ldots,\{2i-2,2i-1\},$

$\{(2i+1)',(2i+2)'\},\{(2i+3)',(2i+4)'\},\ldots,\{(2k+1)',1'\},\ldots,\{(2i-2)',(2i-1)'\}\}$

For $1 \le i \le k-1$:

M_{2i+1}: $\{\{2i+2,2i+3\},\{2i+4,2i+5\},\ldots,\{2k,2k+1\},\{1,2\},\ldots,\{2i-1,2i\},$

$\{(2i+2)',(2i+3)'\},\{(2i+4)',(2i+5)'\},\ldots,\{(2k)',(2k+1)'\},\{1',2'\},\ldots,\{(2i-1)',(2i)'\}\}$

And finally,

M_{2k+1}: $\{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\},\{1',2'\},\{3',4'\},\ldots,\{(2k-1)',(2k)'\}\}$

It is clear that each edge shows up k times. Take an edge $(a,a+1)$ or $(a',(a+1)')$ it will only show up in M_i for all odd i or all even i. It will also not show up in M_a or M_{a+1}.

(2^{3k+1}):

M_1: $\{\{2,3\},\{2',4\},\{3',4'\},\ldots,\{3k-1,3k\},\{(3k-1)',3k+1\},\{(3k)',(3k+1)'\}\}$

M_2: $\{\{3,4\},\{3',5\},\{4',5'\},\ldots,\{3k,3k+1\},\{(3k)',1\},\{(3k+1)',1'\}\}$

M_3: $\{\{4,5\},\{4',6\},\{5',6'\},\ldots,\{3k+1,1\},\{(3k+1)',2\},\{1',2'\}\}$

For $2 \le i \le k$:

M_{3i-1}: $\{\{3i,3i+1\},\{(3i)',3i+2\},\{(3i+1)',(3i+2)'\},\ldots,\{3k,3k+1\},\{(3k)',1\},$

$\{(3k+1)',1'\},\{2,3\},\{2',4\},\{3',4'\},\ldots,\{3i-4,3i-3\},\{(3i-4)',3i-2\},$

$\{(3i-3)',(3i-2)'\}\}$

For $2 \le i \le k$:

M_{3i}: $\{\{3i+1,3i+2\},\{(3i+1)',3i+3\},\{(3i+2)',(3i+3)'\},\ldots,\{3k+1,1\},$

$\{(3k+1)',2\},\{1',2'\},\{3,4\},\{3',5\},\{4',5'\},\ldots,\{3i-3,3i-2\},\{(3i-3)',3i-1\},$

$\{(3i-2)',(3i-1)'\}\}$

For $1 \le i \le k-1$:

M_{3i+1}: $\{\{3i+2,3i+3\},\{(3i+2)',3i+4\},\{(3i+3)',(3i+4)'\},\ldots,\{3k-1,3k\},$

$\{(3k-1)',3k+1\},\{(3k)',(3k+1)'\},\{1,2\},\{1',3\},\{2',3'\},\ldots,\{3i-2,3i-1\},$

$\{(3i-2)',3i\},\{(3i-1)',(3i)'\}\}$

And finally,

M_{3k+1}: $\{\{1,2\},\{1',3\},\{2',3'\},\ldots,\{3k-2,3k-1\}\,\{(3k-2)',3k\},\{(3k-1)',(3k)'\}\}$

Each edge appears $\frac{3k+1-1}{3} = k$ times. Hence it is a k-divisible-matching.

3.3.2 $2k+1$ parts of any size

In this Section we will give k-divisible-matchings of type $K = (a_1, a_2, \ldots, a_{2k+1})$. We can have all parts of even size or all parts of odd size. We will use a similar algorithm as for even matchings to construct a k-divisible-matching from $(0^{2k-3}, 2, \_, \_, \_)$ or $(1^{2k-1}, \_, \_)$.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = \ldots = a_{2k-3} = 0$, and $a_{2k-2} = 0, 2$, $a_1 = \ldots = a_{2k-2} = 1$ or $K = (0^{2k-4}, 2, 2, 2, 2, 2)$ look up the matching in list provided. Otherwise continue below.

2. For $0 \le i \le 2k-3$ repeat the following steps.

3. Subtract $(0^i, 2, 0^{2k-2-i}, 2, 2)$ and rearrange terms when necessary until you obtain $(0^{2k-3}, 2, \_, \_, \_)$ or $(1^{2k-1}, \_, \_, \_)$. Look the appropriate ones up in list provided. If at any point $K = (0^{2k-4}, 2, 2, 2, 2, 2)$ look it up.

4. Subtract $(0^{2k-2}, 2, 2, 2)$ and rearrange terms when necessary until you obtain $(1^{2k-1}, \_, \_)$. Look it up.

Let us consider the four necessary conditions during the first part of the "subtracting process." Notice that for $p = 2k+1$ condition 4 becomes $3a_{2k+1} \le n$.

1. $p = 2k+1$ is not affected.

2. If $(a_1, a_2, \ldots, a_{2k+1})$ satisfies $2a_{2k+1} + a_{2k} \le a_1 + \ldots + a_{2k} + a_{2k+1}$, then $(a_1 - 2, a_2, \ldots, a_{2k} - 2, a_{2k+1} - 2)$ satisfies $2(a_{2k+1} - 2) + (a_{2k} - 2) \le (a_1 - 2) + a_2 + \ldots + (a_{2k} - 2) + (a_{2k+1} - 2)$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

Case 1: If after such rearranging $a_{2k+1}-2$ is not the largest, but the second largest part, then $a_{2k-1} = a_{2k+1}$. So we started with $(a_1, a_2, \ldots, a_{2k+1}, a_{2k+1}, a_{2k+1})$ and now have $(a_1 - 2, a_2, \ldots, a_{2k+1} - 2, a_{2k+1} - 2, a_{2k+1})$. By Properties 2 and 4 $(a_1 - 2, a_2, \ldots, a_{2k+1} - 2, a_{2k+1} - 2, a_{2k+1})$ is in S_k as long as $6 \leq a_1 + a_2 + \ldots + a_{2k-2}$. Let us consider the case when $a_1 + a_2 + \ldots + a_{2k-2} < 6$. We could have $(0^{2k-2}, a_{2k+1}, a_{2k+1}, a_{2k+1})$, $(0^{2k-3}, 2, a_{2k+1}, a_{2k+1}, a_{2k+1})$, $(0^{2k-3}, 4, a_{2k+1}, a_{2k+1}, a_{2k+1})$, or $(0^{2k-4}, 2, 2, a_{2k+1}, a_{2k+1}, a_{2k+1})$. Each one of those is discussed and shown to be in S_k below.

Case 2: If after rearranging $a_{2k+1}-2$ is not the largest or second largest part, then $a_{2k-2} = a_{2k-1} = a_{2k+1}$. So we started with $(a_1, a_2, \ldots, a_{2k+1}, a_{2k+1}, a_{2k+1}, a_{2k+1})$ and now have $(a_1-2, a_2, \ldots, a_{2k+1}-2, a_{2k+1}-2, a_{2k+1}, a_{2k+1})$. By Properties 2 and 4 $(a_1 - 2, a_2, \ldots, a_{2k+1} - 2, a_{2k+1} - 2, a_{2k+1}, a_{2k+1})$ is in S_k as long as $6 \leq a_1 + a_2 + \ldots + a_{2k-3} + a_{2k+1}$. Let us consider the case when $6 > a_1 + a_2 + \ldots + a_{2k-3} + a_{2k+1}$. We could have $(0^{2k-3}, 2, 2, 2, 2)$, $(0^{2k-3}, 4, 4, 4, 4)$, or $(0^{2k-4}, 2, 2, 2, 2, 2)$. Each one of those is covered by Case 1.

Case 3: If after rearranging $a_{2k+1} - 2$ is the largest, but $a_{2k} - 2$ is not the second largest part, then $a_{2k-1} = a_{2k}$. So we started with $(a_1, a_2, \ldots, a_{2k-2}, a_{2k}, a_{2k}, a_{2k+1})$ and now have $(a_1 - 2, a_2, \ldots, a_{2k-2}, a_{2k} - 2, a_{2k}, a_{2k+1} - 2)$. By Property 2, $(a_1 - 2, a_2, \ldots, a_{2k-2}, a_{2k} - 2, a_{2k}, a_{2k+1} - 2)$ is in S_k as long as $a_{2k+1} \neq a_1 + a_2 + \ldots + a_{2k-2} + a_{2k}$. When $a_{2k+1} = a_1 + a_2 + \ldots + a_{2k-2} + a_{2k}$ and $a_{2k-1} = a_{2k}$ we have types of the form $(a_1, a_2, \ldots, a_{2k-2}, a_{2k+1} - (a_1 + \ldots + a_{2k-2}), a_{2k+1} - (a_1 + \ldots + a_{2k-2}), a_{2k+1})$. By Property 4 this implies $a_1 + a_2 + \ldots + a_{2k-2} \leq 0$. Hence we must have

$(0^{2k-2}, a_{2k}, a_{2k}, a_{2k+1})$ and by Property 2, $a_{2k+1} \leq a_{2k} \leq a_{2k+1}$. Types $(0^{2k-2}, a_{2k+1}, a_{2k+1}, a_{2k+1})$ are discussed below as valid.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, a_2, \ldots, a_{2k}, a_{2k+1})$ satisfies $3a_{2k+1} \leq (a_1 + a_2 + \ldots + a_{2k+1})$, then $(a_1 - 2, a_2, \ldots, a_{2k} - 2, a_{2k+1} - 2)$ satisfies $3(a_{2k+1} - 2) \leq ((a_1 - 2) + a_2 + \ldots + (a_{2k} - 2) + (a_{2k+1} - 2))$. Problems arising from rearranging of the terms are discussed under Property 2.

This is also true for the remaining parts of the algorithm. Therefore, each time we subtract we obtain an element of S_k.

We will need the following "building blocks."

$(0^{2k-2}, 2, 2, 2)$:

For $1 \leq i \leq k-1$:

M_{2i-1}: $\{\{2k-1, 2k\}, \{(2k-1)', 2k+1\}, \{(2k)', (2k+1)'\}\}$

M_{2i}: $\{\{2k-1, (2k+1)'\}, \{(2k-1)', (2k)'\}, \{2k, 2k+1\}\}$

And finally,

M_{2k-1}: $\{\{2k, 2k+1\}, \{(2k)', (2k+1)'\}\}$

M_{2k}: $\{\{2k-1, (2k+1)'\}, \{(2k-1)', 2k+1\}\}$

M_{2k+1}: $\{\{2k-1, 2k\}, \{(2k-1)', (2k)'\}\}$

$(0^{2k-3}, 2, 2, 2, 2)$:

For $1 \leq i \leq k-1$:

M_{2i-1}: $\{\{2k-2, 2k-1\}, \{(2k-2)', (2k-1)'\}, \{2k, 2k+1\}, \{(2k)', (2k+1)'\}\}$

For $1 \leq i \leq k-2$:

M_{2i}: $\{\{2k-2, 2k+1\}, \{(2k-2)', 2k\}, \{(2k-1)', (2k)'\}, \{2k-1, (2k+1)'\}\}$

M_{2k-2}: $\{\{2k-1, (2k+1)'\}, \{(2k-1)', (2k)'\}, \{2k, 2k+1\}\}$

M_{2k-1}: $\{\{2k-2, 2k+1\}, \{(2k-2)', 2k\}, \{(2k)', (2k+1)'\}\}$

M_{2k}: $\{\{2k-2, 2k+1\}, \{(2k-2)', (2k-1)'\}, \{2k-1, (2k+1)'\}\}$

M_{2k+1}: $\{\{2k-2, 2k-1\}, \{(2k-2)', 2k\}, \{(2k-1)', (2k)'\}\}$

$(0^{3k-2}, 2, 2, 2)$:

For $1 \leq i \leq k-1$:

M_{3i-2}: $\{\{3k-1, 3k\}, \{(3k-1)', 3k+1\}, \{(3k)', (3k+1)'\}\}$

M_{3i-1}: $\{\{3k-1, 3k+1\}, \{(3k-1)', (3k)'\}, \{3k, (3k+1)'\}\}$

M_{3i}: $\{\{3k-1, (3k)'\}, \{(3k-1)', (3k+1)'\}, \{3k, 3k+1\}\}$

M_{3k-2}: $\{\{3k-1, (3k)'\}, \{(3k-1)', 3k+1\}, \{3k, (3k+1)'\}\}$

M_{3k-1}: $\{\{3k, 3k+1\}, \{(3k)', (3k+1)'\}\}$

M_{3k}: $\{\{3k-1, 3k+1\}, \{(3k-1)', (3k+1)'\}\}$

M_{3k+1}: $\{\{3k-1, 3k\}, \{(3k-1)', (3k)'\}\}$

Following is a list of k-divisible-matchings of type $(1^{2k-1}, \_, \_)$.

(1^{2k+1}) given in Section 3.3.1

$(1^{2k}, 3)$, $k \geq 3$:

M_1: $\{\{2, (2k+1)'\}, \{3, 4\}, \{5, (2k+1)''\}, \{6, 7\}, \ldots, \{2k, 2k+1\}\}$

M_2: $\{\{1, 2k+1\}, \{3, (2k+1)''\}, \{4, (2k+1)'\}, \{5, 6\}, \ldots, \{2k-1, 2k\}\}$

M_3: $\{\{1, 2\}, \{4, (2k+1)'\}, \{5, (2k+1)''\}, \{6, 7\}, \ldots, \{2k, 2k+1\}\}$

M_4: $\{\{1, 2k+1\}, \{2, (2k+1)'\}, \{3, (2k+1)''\}, \{5, 6\}, \ldots, \{2k-1, 2k\}\}$

M_5: $\{\{1, 2\}, \{3, (2k+1)''\}, \{4, (2k+1)'\}, \{6, 7\}, \ldots, \{2k, 2k+1\}\}$

M_6: $\{\{1, 2k+1\}, \{2, (2k+1)'\}, \{3, 4\}, \{5, (2k+1)''\}, \{7, 8\}, \ldots, \{2k-1, 2k\}\}$

M_7: $\{\{1,2\},\{3,(2k+1)''\},\{4,(2k+1)'\},\{5,6\},\{8,9\},\ldots,\{2k,2k+1\}\}$

M_8: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{6,7\},\{9,10\},\ldots,\{2k-1,2k\}\}$

For $4 \leq j \leq k$

M_{2j-1}: $\{\{1,2\},\{3,(2k+1)''\},\{4,(2k+1)'\},\{5,6\},\ldots,\{2j-3,2j-2\},\{2j,2j+1\},\ldots,$

$\{2k,2k+1\}\}$

For $4 \leq j \leq k-1$

M_{2j}: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{6,7\},\ldots,\{2j-2,2j-1\},$

$\{2j+1,2j+2\},\ldots,\{2k-1,2k\}\}$

And finally,

M_{2k}: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{6,7\},\ldots,\{2k-2,2k-1\}\}$

M_{2k+1}: $\{\{1,2\},\{3,4\},\{5,6\},\ldots,\{2k-1,2k\}\}$

$(1^{2k},5)$, $k \geq 5$:

M_1: $\{\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{6,(2k+1)'''\},\{7,8\},\{9,(2k+1)^4\},\{10,11\},\ldots,$

$\{2k,2k+1\}\}$

M_2: $\{\{1,2k+1\},\{3,(2k+1)''\},\{4,(2k+1)'\},\{5,6\},\{7,(2k+1)^4\},\{8,(2k+1)'''\},\{9,10\},\ldots,$

$\{2k-1,2k\}\}$

M_3: $\{\{1,2\},\{4,(2k+1)'\},\{5,(2k+1)''\},\{6,(2k+1)'''\},\{7,8\},\{9,(2k+1)^4\},\{10,11\},\ldots,$

$\{2k,2k+1\}\}$

M_4: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,(2k+1)''\},\{5,6\},\{7,(2k+1)^4\},\{8,(2k+1)'''\},\{9,10\},\ldots,$

$\{2k-1,2k\}\}$

M_5: $\{\{1,2\},\{3,(2k+1)''\},\{4,(2k+1)'\},\{6,(2k+1)'''\},\{7,8\},\{9,(2k+1)^4\},\{10,11\},\ldots,$

$\{2k,2k+1\}\}$

M_6: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{7,(2k+1)^4\},\{8,(2k+1)'''\},\{9,10\},\ldots,$

$\{2k-1, 2k\}\}$

M_7: $\{\{1,2\}, \{3,(2k{+}1)''\}, \{4,(2k{+}1)'\}, \{5,6\}, \{8,(2k{+}1)'''\}, \{9,(2k{+}1)^4\}, \{10,11\}, \ldots,$

$\{2k, 2k+1\}\}$

M_8: $\{\{1,2k{+}1\}, \{2,(2k{+}1)'\}, \{3,4\}, \{5,(2k{+}1)''\}, \{6,(2k{+}1)'''\}, \{7,(2k{+}1)^4\}, \{9,10\}, \ldots,$

$\{2k-1, 2k\}\}$

M_9: $\{\{1,2\}, \{3,(2k{+}1)''\}, \{4,(2k{+}1)'\}, \{5,6\}, \{7,(2k{+}1)^4\}, \{8,(2k{+}1)'''\}, \{10,11\}, \ldots,$

$\{2k, 2k+1\}\}$

M_{10}: $\{\{1,2k{+}1\}, \{2,(2k{+}1)'\}, \{3,4\}, \{5,(2k{+}1)''\}, \{6,(2k{+}1)'''\}, \{7,8\}, \{9,(2k{+}1)^4\},$

$\{11,12\}, \ldots, \{2k-1, 2k\}\}$

For $6 \le j \le k$

M_{2j-1}:

$\{\{1,2\}, \{3,(2k+1)''\}, \{4,(2k+1)'\}, \{5,6\}, \{7,(2k+1)^4\}, \{8,(2k+1)'''\}, \{9,10\}, \ldots,$

$\{2j-3, 2j-2\}, \{2j, 2j+1\}, \ldots, \{2k, 2k+1\}\}$

For $6 \le j \le k-1$

M_{2j}:

$\{\{1,2k+1\}, \{2,(2k+1)'\}, \{3,4\}, \{5,(2k+1)''\}, \{6,(2k+1)'''\}, \{7,8\}, \{9,(2k+1)^4\},$

$\{10,11\}, \ldots, \{2j-2, 2j-1\}, \{2j+1, 2j+2\}, \ldots, \{2k-1, 2k\}\}$

And finally,

M_{2k}:

$\{\{1,2k+1\}, \{2,(2k+1)'\}, \{3,4\}, \{5,(2k+1)''\}, \{6,(2k+1)'''\}, \{7,8\}, \{9,(2k+1)^4\},$

$\{10,11\}, \ldots, \{2k-2, 2k-1\}\}$

M_{2k+1}:

$\{\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2k-1, 2k\}\}$

$(1^{2k}, i)$, $3 \leq i \leq 2\lceil \frac{k}{2} \rceil - 1$:

M_1: $\{\{2, (2k+1)'\}, \{3, 4\}, \{5, (2k+1)''\}, \ldots, \{2i-4, (2k+1)^{i-2}\}, \{2i-3, 2i-2\},$

$\{2i-1, (2k+1)^{i-1}\}, \{2i, 2i+1\}, \ldots, \{2k, 2k+1\}\}$

M_2: $\{\{1, 2k+1\}, \{3, (2k+1)''\}, \{4, (2k+1)'\}, \{5, 6\}, \ldots, \{2i-3, (2k+1)^{i-1}\},$

$\{2i-2, (2k+1)^{i-2}\}, \{2i-1, 2i\}, \{2i+1, 2i+2\}, \ldots, \{2k-1, 2k\}\}$

M_3: $\{\{1, 2\}, \{4, (2k+1)'\}, \{5, (2k+1)''\}, \{6, (2k+1)'''\}, \{7, 8\}, \{9, (2k+1)^4\}, \ldots,$

$\{2i-4, (2k+1)^{i-2}\}, \{2i-3, 2i-2\}, \{2i-1, (2k+1)^{i-1}\}, \{2i, 2i+1\}, \ldots, \{2k, 2k+1\}\}$

M_4: $\{\{1, 2k+1\}, \{2, (2k+1)'\}, \{3, (2k+1)''\}, \{5, 6\}, \ldots, \{2i-3, (2k+1)^{i-1}\},$

$\{2i-2, (2k+1)^{i-2}\}, \{2i-1, 2i\}, \{2i+1, 2i+2\}, \ldots, \{2k-1, 2k\}\}$

M_5: $\{\{1, 2\}, \{3, (2k+1)''\}, \{4, (2k+1)'\}, \{6, (2k+1)'''\}, \{7, 8\}, \{9, (2k+1)^4\}, \ldots,$

$\{2i-4, (2k+1)^{i-2}\}, \{2i-3, 2i-2\}, \{2i-1, (2k+1)^{i-1}\}, \{2i, 2i+1\}, \ldots, \{2k, 2k+1\}\}$

M_6: $\{\{1, 2k+1\}, \{2, (2k+1)'\}, \{3, 4\}, \{5, (2k+1)''\}, \ldots, \{2i-3, (2k+1)^{i-1}\},$

$\{2i-2, (2k+1)^{i-2}\}, \{2i-1, 2i\}, \{2i+1, 2i+2\}, \ldots, \{2k-1, 2k\}\}$

M_7: $\{\{1, 2\}, \{3, (2k+1)''\}, \{4, (2k+1)'\}, \{5, 6\}, \{8, (2k+1)'''\}, \{9, (2k+1)^4\}, \ldots,$

$\{2i-4, (2k+1)^{i-2}\}, \{2i-3, 2i-2\}, \{2i-1, (2k+1)^{i-1}\}, \{2i, 2i+1\}, \ldots, \{2k, 2k+1\}\}$

M_8: $\{\{1, 2k+1\}, \{2, (2k+1)'\}, \{3, 4\}, \{5, (2k+1)''\}, \{6, (2k+1)'''\}, \{7, (2k+1)^4\},$

$\{9, 10\}, \ldots, \{2i-3, (2k+1)^{i-1}\}, \{2i-2, (2k+1)^{i-2}\}, \{2i-1, 2i\}, \{2i+1, 2i+2\}, \ldots,$

$\{2k-1, 2k\}\}$

For $6 \leq i \leq k-1$:

M_{2i-3}: $\{\{1, 2\}, \{3, (2k+1)''\}, \{4, (2k+1)'\}, \{5, 6\}, \{7, (2k+1)^4\}, \{8, (2k+1)'''\}, \ldots,$

$\{2i-5, 2i-4\}, \{2i-2, (2k+1)^{i-2}\}, \{2i-1, (2k+1)^{i-1}\}, \{2i, 2i+1\}, \{2i+2, 2i+3\}, \ldots,$

$\{2k, 2k+1\}\}$

M_{2i-2}: $\{\{1, 2k+1\}, \{2, (2k+1)'\}, \{3, 4\}, \{5, (2k+1)''\}, \{6, (2k+1)'''\}, \{7, 8\},$

$\{9,(2k+1)^4\},\ldots,\{2i-4,(2k+1)^{i-2}\},\{2i-3,(2k+1)^{i-1}\},\{2i-1,2i\},$

$\{2i+1,2i+2\},\ldots,\{2k-1,2k\}\}$

M_{2i-1}: $\{\{1,2\},\{3,(2k+1)''\},\{4,(2k+1)'\},\{5,6\},\{7,(2k+1)^4\},\{8,(2k+1)'''\},\ldots,$

$\{2i-5,2i-4\},\{2i-3,(2k+1)^{i-1}\},\{2i-2,(2k+1)^{i-2}\},\{2i,2i+1\},\{2i+2,2i+3\},\ldots,$

$\{2k,2k+1\}\}$

M_{2i}: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{6,(2k+1)'''\},\{7,8\},$

$\{9,(2k+1)^4\},\ldots,\{2i-4,(2k+1)^{i-2}\},\{2i-3,2i-2\},\{2i-1,(2k+1)^{i-1}\},$

$\{2i+1,2i+2\},\ldots,\{2k-1,2k\}\}$

M_{2i+1}: $\{\{1,2\},\{3,(2k+1)''\},\{4,(2k+1)'\},\{5,6\},\{7,(2k+1)^4\},\{8,(2k+1)'''\},\ldots,$

$\{2i-5,2i-4\},\{2i-3,(2k+1)^{i-1}\},\{2i-2,(2k+1)^{i-2}\},\{2i-1,2i\},\{2i+2,2i+3\},\ldots,$

$\{2k,2k+1\}\}$

M_{2i+2}: $\{\{1,2k+1\},\{2,(2k+1)'\},\{3,4\},\{5,(2k+1)''\},\{6,(2k+1)'''\},\{7,8\},$

$\{9,(2k+1)^4\},\ldots,\{2i-4,(2k+1)^{i-2}\},\{2i-3,2i-2\},\{2i-1,(2k+1)^{i-1}\},\{2i,2i+1\},$

$\{2i+3,2i+4\},\ldots,\{2k-1,2k\}\}$

And finally,

M_{2k+1}: $\{\{1,2\},\{3,4\},\{5,6\},\ldots,\{2k-1,2k\}\}$

$(1^{2k-1},3,3)$:

M_1: $\{\{2,(2k+1)'\},\{3,(2k)''\},\{4,(2k+1)''\},\{5,6\},\ldots,\{2k-1,2k\},\{(2k)',2k+1\}\}$

M_2: $\{\{1,(2k)''\},\{3,2k+1\},\{4,(2k)'\},\{5,(2k+1)'\},\{6,7\},\ldots,\{2k-2,2k-1\},$

$\{2k,(2k+1)''\}\}$

M_3: $\{\{1,(2k)''\},\{2,(2k+1)'\},\{4,(2k+1)''\},\{5,6\},\ldots,\{2k-1,2k\},\{(2k)',2k+1\}\}$

M_4: $\{\{1,2\},\{3,(2k)''\},\{5,(2k+1)'\},\{6,7\},\ldots,\{2k-2,2k-1\},\{2k,(2k+1)''\},$

$\{(2k)',2k+1\}\}$

M_5: $\{\{1,(2k)''\},\{2,(2k+1)'\},\{3,2k+1\},\{4,(2k)'\},\{6,7\},\ldots,\{2k,(2k+1)''\}\}$

M_6: $\{\{1,2\},\{3,(2k)''\},\{4,(2k{+}1)''\},\{5,(2k{+}1)'\},\{7,8\},\ldots,\{2k{-}1,2k\},\{(2k)',2k{+}1\}\}$

For $4 \leq j \leq k$:

M_{2j-1}: $\{\{1,(2k)''\},\{2,(2k+1)'\},\{3,2k+1\},\{4,(2k)'\},\{5,6\},\ldots,\{2j-3,2j-2\},$

$\{2j,2j+1\},\ldots,\{2k,(2k+1)''\}\}$

For $4 \leq j \leq k-1$:

M_{2j}: $\{\{1,2\},\{3,(2k)''\},\{4,(2k+1)''\},\{5,(2k+1)'\},\{6,7\},\ldots,\{2j-2,2j-1\},$

$\{2j+1,2j+2\},\ldots,\{2k-1,2k\},\{(2k)',2k+1\}\}$

And finally,

M_{2k}: $\{\{1,2\},\{3,2k+1\},\{4,(2k+1)''\},\{5,(2k+1)'\},\{6,7\},\ldots,\{2k-2,2k-1\}\}$

M_{2k+1}: $\{\{1,2\},\{3,(2k)''\},\{4,(2k)'\},\{5,6\},\ldots,\{2k-1,2k\}\}$

The following matchings are needed and are yet unfound.

$(1^{2k-1},3,i)$, for $5 \leq i \leq 2\lfloor \frac{k}{2} \rfloor + 1$

$(1^{2k-1},5,i)$, for $5 \leq i \leq 2\lceil \frac{k}{2} \rceil + 1$

$(1^{2k-1},7,i)$, for $7 \leq i \leq 2\lfloor \frac{k}{2} \rfloor + 3$

In general,

$(1^{2k-1},d,i)$, $d \leq i \leq 2\lfloor \frac{k}{2} \rfloor + \frac{d-1}{2}$ if $d \equiv 3 \ (mod 4)$

$(1^{2k-1},d,i)$, $d \leq i \leq 2\lceil \frac{k}{2} \rceil + \frac{d-3}{2}$ if $d \equiv 1 \ (mod 4)$

Following is a list of k-divisible-matchings of types $(0^{2k-3},2,\_,\_,\_)$ as well as $(0^{2k-4},2,2,2,2,2)$.

$(0^{2k-4},2,2,2,2,2)$:

For $1 \leq i \leq k-2$:

M_{2i-1}: $\{\{2k{-}3,2k{+}1\},\{(2k{-}3)',(2k{-}2)'\},\{2k{-}2,(2k{-}1)'\},\{2k{-}1,2k\},\{(2k)',(2k+$

$1)'\}\}$

M_{2i}: $\{\{2k{-}3, 2k{-}2\}, \{(2k{-}3)', (2k{+}1)'\}, \{(2k{-}2)', 2k{-}1\}, \{(2k{-}1)', (2k)'\}, \{2k, 2k{+}1\}\}$

Finally,

M_{2k-3}: $\{\{2k-2, (2k-1)'\}, \{(2k-2)', 2k-1\}, \{2k, 2k+1\}, \{(2k)', (2k+1)'\}\}$

M_{2k-2}: $\{\{2k-3, 2k+1\}, \{(2k-3)', (2k+1)'\}, \{(2k-1)', (2k)'\}, \{2k-1, 2k\}\}$

M_{2k-1}: $\{\{2k-3, 2k-2\}, \{(2k-3)', (2k-2)'\}, \{2k, 2k+1\}, \{(2k)', (2k+1)'\}\}$

M_{2k}: $\{\{2k-3, 2k+1\}, \{(2k-3)', (2k+1)'\}, \{(2k-2)', 2k-1\}, \{2k-2, (2k-1)'\}\}$

M_{2k+1}: $\{\{2k-3, 2k-2\}, \{(2k-3)', (2k-2)'\}, \{2k-1, 2k\}, \{(2k-1)', (2k)'\}\}$

$(0^{2k-2}, 2, 2, 2)$ given in Section 3.3.2

$(0^{2k-2}, d, d, d)$=$\frac{d}{2}$ copies of $(0^{2k-2}, 2, 2, 2)$

$(0^{2k-3}, 2, 2, 2, 2)$ given in Section 3.3.2

$(0^{2k-3}, 2, d, d, d)$ from $(0^{2k-3}, 2, 2, 2, 2)$ and $(d-2)/2$ copies of $(0^{2k-2}, 2, 2, 2)$

$(0^{2k-3}, 2, d, d+2, d+2)$ from $(0^{2k-3}, 2, 0, 2, 2)$ and $d/2$ copies of $(0^{2k-2}, 2, 2, 2)$

This gives the k-divisible-matchings with $2k+1$ parts.

3.3.3 $3k+1$ parts of any size

In this Section we will give k-divisible-matchings of type $K = (a_1, a_2, \ldots, a_{3k+1})$. We can only have all parts of even size. We will use a similar algorithm as above to construct a k-divisible-matching from $(0^{3k-3}, 2, \_, \_, \_)$.

1. Check if the four necessary conditions are satisfied. If not, the matching does not exist. If $a_1 = \ldots = a_{3k-3} = 0$, $a_{3k-2} = 0, 2$ or $K = (0^{3k-4}, 2, 2, 2, 2, 2)$ look up the matching in list provided. Otherwise continue below.

2. If $5a_{3k+1} = 2n$ skip down to the Special Case 1 section. If $a_{3k-1} = a_{3k}$ and $a_{3k+1} = a_1 + a_2 + a_3 + \ldots + a_{3k-1}$ skip down to Special Case 2 section. Otherwise continue below.

3. For $0 \le i \le 3k-3$ subtract $(0^i, 2, 0^{3k-2-i}, 2, 2)$ and rearrange terms when necessary until you obtain $(0^{3k-3}, 2, \_, \_, \_)$. Look those up in the list provided.

   If at any point $K = (0^{3k-4}, 2, 2, 2, 2, 2)$ look it up.

Let us consider the four necessary conditions during the first part of the "subtracting process." Notice that for $p = 3k+1$ condition 4 becomes $5a_{3k+1} \le 2n$.

1. $p = 3k+1$ is not affected.

2. If $(a_1, a_2, \ldots, a_{3k+1})$ satisfies $2a_{3k+1} + a_{3k} \le a_1 + \ldots + a_{3k} + a_{3k+1}$, then

   $(a_1 - 2, a_2, \ldots, a_{3k} - 2, a_{3k+1} - 2)$ satisfies

   $2(a_{3k+1} - 2) + (a_{3k} - 2) \le (a_1 - 2) + a_2 + \ldots + (a_{3k} - 2) + (a_{3k+1} - 2)$. We are, however, rearranging the terms to ensure a nondecreasing sequence. Let us consider the following cases:

   Case 1: If after such rearranging $a_{3k+1} - 2$ is not the largest, but the second largest part, then $a_{3k-1} = a_{3k+1}$. So we started with $(a_1, a_2, \ldots, a_{3k+1}, a_{3k+1}, a_{3k+1})$ and now have $(a_1 - 2, a_2, \ldots, a_{3k+1} - 2, a_{3k+1} - 2, a_{3k+1})$. By Properties 2 and 4 $(a_1 - 2, a_2, \ldots, a_{3k+1} - 2, a_{3k+1} - 2, a_{3k+1})$ is in S_k as long as $4 \le a_1 + a_2 + \ldots + a_{3k-2}$ and $12 - a_{3k+1} \le 2(a_1 + a_2 + \ldots + a_{3k-2})$. Let us consider the cases when $a_1 + a_2 + \ldots + a_{3k-2} < 4$ or

   $2(a_1 + a_2 + \ldots + a_{3k-2}) < 12 - a_{3k+1}$. We could have $(0^{3k-2}, a_{3k+1}, a_{3k+1}, a_{3k+1})$, $(0^{3k-3}, 2, a_{3k+1}, a_{3k+1}, a_{3k+1})$ or $(0^{3k-4}, 2, 2, 2, 2, 2)$. Each one of those is discussed and shown to be in S_k below.

Case 2: If after rearranging $a_{3k+1}-2$ is not the largest or second largest part, then $a_{3k-2} = a_{3k-1} = a_{3k+1}$. So we started with $(a_1, a_2, \ldots, a_{3k+1}, a_{3k+1}, a_{3k+1}, a_{3k+1})$ and now have $(a_1-2, a_2, \ldots, a_{3k+1}-2, a_{3k+1}-2, a_{3k+1}, a_{3k+1})$. By Properties 2 and 4 $(a_1 - 2, a_2, \ldots, a_{3k+1} - 2, a_{3k+1} - 2, a_{3k+1}, a_{3k+1})$ is in S_k as long as $6 - a_{3k+1} \leq a_1 + a_2 + \ldots + a_{3k-3}$ and

$12 - 3a_{3k+1} \leq 2(a_1 + a_2 + \ldots + a_{3k-3})$. Let us consider the case when

$6 - a_{3k+1} > a_1 + a_2 + \ldots + a_{3k-3}$ or $12 - 3a_{3k+1} > 2(a_1 + a_2 + \ldots + a_{3k-3})$. We could have $(0^{3k-3}, 2, 2, 2, 2)$, $(0^{3k-3}, 4, 4, 4, 4)$, or $(0^{3k-4}, 2, 2, 2, 2, 2)$. Each one of those is discussed below.

Case 3: If after rearranging $a_{3k+1} - 2$ is the largest, but $a_{3k} - 2$ is not the second largest part, then $a_{3k-1} = a_{3k}$. So we started with $(a_1, a_2, \ldots, a_{3k-2}, a_{3k}, a_{3k}, a_{3k+1})$ and now have $(a_1 - 2, a_2, \ldots, a_{3k-2}, a_{3k} - 2, a_{3k}, a_{3k+1} - 2)$. By Property 2,

$(a_1 - 2, a_2, \ldots, a_{3k-2}, a_{3k} - 2, a_{3k}, a_{3k+1} - 2)$ is in S_k as long as $a_{3k+1} \neq a_1 + a_2 + \ldots + a_{3k-2} + a_{3k}$. The case when $a_{3k-1} = a_{3k}$ and $a_{3k+1} = a_1 + a_2 + \ldots + a_{3k-2} + a_{3k}$ is discussed below as Special Case 2.

3. Since we are subtracting zeroes and twos, the parity of the parts is not affected. Neither is the number of parts.

4. If $(a_1, a_2, \ldots, a_{3k}, a_{3k+1})$ satisfies $5a_{3k+1} < 2(a_1 + a_2 + \ldots + a_{3k+1})$, then $(a_1 - 2, a_2, \ldots, a_{3k} - 2, a_{3k+1} - 2)$ satisfies

   $5(a_{3k+1} - 2) \leq 2((a_1 - 2) + a_2 + \ldots + (a_{3k} - 2) + (a_{3k+1} - 2))$. The case when $5a_{3k+1} = 2n$ is discussed below as Special Case 1.

This is also true for the remaining parts of the algorithm. Therefore, each time we subtract we obtain an element of S_k.

Let us consider the **Special Case 1**.

We need to find even matchings for $(a_1, \ldots, a_{3k+1})$ satisfying the four necessary conditions and $3a_{3k+1} = 2(a_1 + \ldots + a_{3k})$. Let us refer to these as matching type $SP^k1 \subseteq S_k$. Say we need $K = (a_1, a_2, \ldots, a_{3k}, \frac{2(a_1+\ldots+a_{3k})}{3}) \in SP^k1$. We will build it by induction.

We start by subtracting $(0^{3k-3}, 2, 2, 2, 4)$ to obtain $K' = (a_1, \ldots, a_{3k-3}, a_{3k-2} - 2, a_{3k-1} - 2, a_{3k} - 2, \frac{2(a_1+\ldots+a_{3k})}{3} - 4)$. As long as no rearranging is necessary K' is in SP^k1. If rearranging is necessary, we started with $K = (a_1, \ldots, a_{3k-4}, a_{3k}, \ldots, a_{3k}, \frac{2(a_1+\ldots+a_{3k})}{3})$ and now have $K'' = (a_1, \ldots, a_{3k-4}, a_{3k} - 2, a_{3k} - 2, a_{3k} - 2, a_{3k}, \frac{2(a_1+\ldots+a_{3k})}{3} - 4)$. This is in SP^k1 as long as $6 \leq a_1 + \ldots + a_{3k-4} + a_{3k}$. The only time $6 > a_1 + \ldots + a_{3k-4} + a_{3k}$ is for $K = (0^{3k-3}, 2^3, 4)$ which is given. We need to consider if it is possible for $a_{3k} > \frac{2(a_1+\ldots+a_{3k})}{3} - 4$. This is equivalent to $8 \leq a_1 + \ldots + a_{3k-1}$ which is true for all elements of SP^k1 except $K = (0^{3k-3}, 2^3, 4)$. By induction, this concludes Special Case 1. We take any element of SP^k1 and subtract to get a smaller element of SP^k1.

Now let us consider **Special Case 2**.

If $a_{3k-1} = a_{3k}$ and $a_{3k+1} = a_1 + \ldots + a_{3k-2} + a_{3k}$ we have types of the form $(a_1, a_2, \ldots, a_{3k-2}, a_{3k+1} - (a_1 + \ldots + a_{3k-2}), a_{3k+1} - (a_1 + \ldots + a_{3k-2}), a_{3k+1})$. We will construct a k-divisible-matching of such type as follows: $a_1/2$ copies of $(2, 0^{3k-3}, 2, 2, 4)$, $a_2/2$ copies of $(0, 2, 0^{3k-4}, 2, 2, 4)$, …, $a_{3k-2}/2$ copies of $(0^{3k-3}, 2, 2, 2, 4)$ and $(a_{3k+1} - 2(a_1 + \ldots + a_{3k-2}))/2$ copies of $(0^{3k-2}, 2, 2, 2)$. Notice that by Property 4 $a_{3k+1} \geq 2(a_1 + \ldots + a_{3k-2})$. This concludes Special Case 2.

Following is a list of k-divisible-matchings of types $(0^{3k-3}, 2, \_, \_, \_)$ as well as $(0^{3k-4}, 2, 2, 2, 2, 2)$.

$(0^{3k-4}, 2, 2, 2, 2, 2)$:

For $1 \leq i \leq k-2$:

M_{2i-1}:

$\{\{3k-3, 3k-2\}, \{(3k-3)', 3k+1\}, \{(3k-2)', (3k-1)'\}, \{3k-1, 3k\}, \{(3k)', (3k+1)'\}\}$

M_{2i}:

$\{\{3k-3, (3k+1)'\}, \{(3k-3)', (3k-2)'\}, \{3k-2, 3k-1\}, \{(3k-1)', (3k)'\}, \{3k, 3k+1\}\}$

For $k-1 \leq i \leq 2k-2$:

M_{k-2+i}:

$\{\{3k-3, 3k-2\}, \{(3k-3)', 3k-1\}, \{(3k-2)', (3k-1)'\}, \{(3k)', (3k+1)'\}, \{3k, 3k+1\}\}$

And finally,

M_{3k-3}:

$\{\{3k-2, 3k-1\}, \{(3k-2)', (3k-1)'\}, \{(3k)', (3k+1)'\}, \{3k, 3k+1\}\}$

M_{3k-2}:

$\{\{3k-3, (3k+1)'\}, \{(3k-3)', 3k+1\}, \{(3k-1)', (3k)'\}, \{3k-1, 3k\}\}$

M_{3k-1}:

$\{\{3k-3, 3k-2\}, \{(3k-3)', (3k-2)'\}, \{3k, 3k+1\}, \{(3k)', (3k+1)'\}\}$

M_{3k}:

$\{\{3k-3, (3k+1)'\}, \{(3k-3)', 3k+1\}, \{(3k-2)', (3k-1)'\}, \{3k-2, 3k-1\}\}$

M_{3k+1}:

$\{\{3k-3, 3k-2\}, \{(3k-3)', (3k-2)'\}, \{3k-1, 3k\}, \{(3k-1)', (3k)'\}\}$

$(0^{3k-2}, 2, 2, 2)$ given in Section 3.3.1

By Property 2, a k-divisible-matching of type $(0^{3k-2}, a_{3k-1}, a_{3k}, a_{3k+1})$ exists if $a_{3k+1} \leq a_{3k-1} \leq a_{3k+1}$.

$(0^{3k-2}, a_{3k+1}, a_{3k+1}, a_{3k+1})$=$\frac{a_{3k+1}}{2}$ copies of $(0^{3k-2}, 2, 2, 2)$

$(0^{3k-3}, 2, 2, 2, 2)$:

For $1 \leq i \leq k-1$:

M_{3i-2}: $\{\{3k-2, (3k)'\}, \{(3k-2)', (3k+1)'\}, \{3k-1, 3k\}, \{(3k-1)', 3k+1\}\}$

M_{3i-1}: $\{\{3k-2, 3k-1\}, \{(3k-2)', (3k-1)'\}, \{3k, 3k+1\}, \{(3k)', (3k+1)'\}\}$

M_{3i}: $\{\{3k-2, 3k+1\}, \{(3k-2)', 3k\}, \{3k-1, (3k+1)'\}, \{(3k-1)', (3k)'\}\}$

Finally,

M_{3k-2}: $\{\{3k-1, 3k\}, \{(3k-1)', 3k+1\}, \{(3k)', (3k+1)'\}\}$

M_{3k-1}: $\{\{3k-2, (3k)'\}, \{(3k-2)', (3k+1)'\}, \{3k, 3k+1\}\}$

M_{3k}: $\{\{3k-2, 3k+1\}, \{(3k-2)', (3k-1)'\}, \{3k-1, (3k+1)'\}\}$

M_{3k+1}: $\{\{3k-2, 3k-1\}, \{(3k-2)', 3k\}, \{(3k-n1)', (3k)'\}\}$

$(0^{3k-3}, 2, 2, 2, 4)$:

For $1 \leq i \leq k-1$:

M_{3i-2}:

$\{\{3k-2, (3k+1)'''\}, \{(3k-2)', (3k+1)'\}, \{3k-1, (3k+1)''\}, \{(3k-1)', (3k)'\}, \{3k, 3k+1\}\}$

M_{3i-1}:

$\{\{3k-2, 3k-1\}, \{(3k-2)', (3k+1)''\}, \{(3k-1)', (3k+1)'''\}, \{(3k)', 3k+1\}, \{3k, (3k+1)'\}\}$

M_{3i}:

$\{\{3k-2, 3k+1\}, \{(3k-2)', 3k\}, \{3k-1, (3k+1)''\}, \{(3k-1)', (3k+1)'''\}, \{(3k)', (3k+1)'\}\}$

Finally,

M_{3k-2}:

$\{\{3k-1,(3k+1)''\},\{(3k-1)',(3k+1)'''\},\{(3k)',(3k+1)'\},\{3k,3k+1\}\}$

M_{3k-1}:

$\{\{3k-2,(3k+1)'''\},\{(3k-2)',(3k+1)''\},\{3k,(3k+1)'\},\{(3k)',3k+1\}\}$

M_{3k}:

$\{\{3k-2,3k+1\},\{(3k-2)',(3k+1)'\},\{(3k-1)',(3k+1)'''\},\{3k-1,(3k+1)''\}\}$

M_{3k+1}:

$\{\{3k-2,3k-1\},\{(3k-2)',3k\},\{(3k-1)',(3k)'\}\}$

By Property 2, a k-divisible-matching of the type $(0^{3k-3},2,a_{3k-1},a_{3k},a_{3k+1})$ exists if $a_{3k+1} \leq a_{3k-1}+2$.

$(0^{3k-3},2,a_{3k-1},a_{3k-1},a_{3k-1})$ from $(0^{3k-3},2,2,2,2)$ and $(a_{3k-1}-2)/2$ copies of $(0^{3k-2},2,2,2)$

$(0^{3k-3},2,a_{3k-1},a_{3k-1},a_{3k-1}+2)$ from $(0^{3k-3},2,2,2,4)$ and $(a_{3k-1}-2)/2$ copies of $(0^{3k-2},2,2,2)$

$(0^{3k-3},2,a_{3k-1},a_{3k-1}+2,a_{3k-1}+2)$ from $(0^{3k-3},2,0,2,2)$ and $a_{3k-1}/2$ copies of $(0^{3k-2},2,2,2)$

This gives the k-divisible-matchings with $3k+1$ parts. It is left for future research to consider the generalization of k-divisible-matchings with $p=2kh+1$ and $p=2kh+k+1$ for $h \geq 2$.

Bibliography

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